Title	UNBOUNDED STRONGLY IRREDUCIBLE OPERATORS AND TRANSITIVE REPRESENTATIONS OF QUIVERS ON INFINITE-DIMENSIONAL HILBERT SPACES (The research of geometric structures in quantum information based on Operator Theory and related topics)
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Citation	数理解析研究所講究録 = RIMS Kokyuroku (2017), 2033: 147-157
Issue Date	2017-06
URL	http://hdl.handle.net/2433/236780
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第2033巻 2017年 147-157

UNBOUNDED STRONGLY IRREDUCIBLE OPERATORS AND TRANSITIVE REPRESENTATIONS OF QUIVERS ON INFINITE-DIMENSIONAL HILBERT SPACES

MASATOSHI ENOMOTO AND YASUO WATATANI

This is a joint work with Yasuo Watatani([EW3]).

1.Introduction and motivation from history.

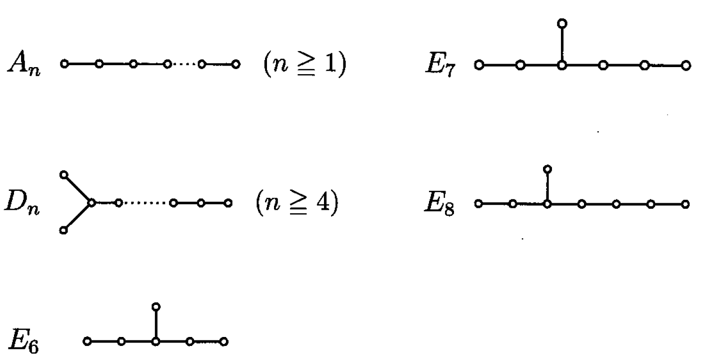
Weierstrass (1868) and Jordan (1870) classified finite dimensional operators up to similarity. That is, any finite dimensional operator T is represented as a direct sum of Jordan blocks. Kronecker (1890) generalized their results to pairs of finite dimensional operators A, B from a finite-dimensional space H to a finite dimensional space K. The result is as follows.

Any finite-dimensional indecomposable representation of pairs of finite dimensional operators is one of the following up to isomorphism:

(1) $H = K = \mathbb{C}^n$,
$A = \lambda I_n + J_n$, (Jordan block) , $B = I_n, \lambda \in \mathbb{C}, n \geq 1$.
(2) $H = K = \mathbb{C}^n$,
$A = I_n$, $B = \lambda I_n + J_n$, (Jordan block) $, \lambda \in \mathbb{C}$, $n \geq 1$.
(3) $H = \mathbb{C}^{n+1}, K = \mathbb{C}^n$, $A = [I_n, 0], B = [0, I_n], n \geq 0$.
(4) $H = \mathbb{C}^n, K = \mathbb{C}^{n+1}$, $A = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, n \geq 0$.

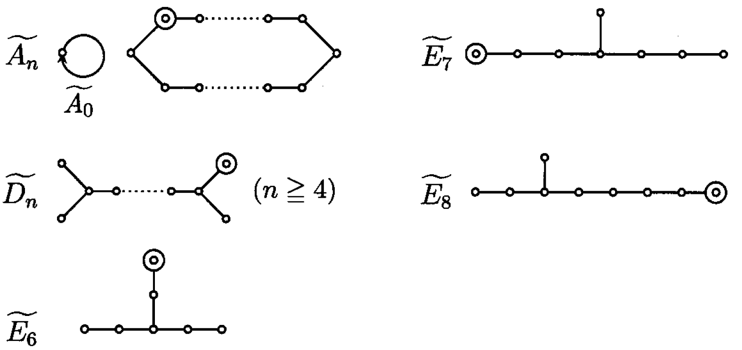
Nazarova (1967) and Gelfand-Ponomarev (1970) classified finite - dimensional indecomposable representations of four subspaces. After Yoshii (1956) and Jans (1957), Gabriel(1972) introduced representations of quivers and showed that only Dynkin quivers have finite numbers of finite-dimensional indecomposable representations.

The following is the Dynkin diagram.



Bernstein - Gelfand - Ponomarev (1973) gave another proof of Gabriel result by using reflection functors. For extended Dynkin quivers there exist infinite number of finite- dimensional indecomposable representations.

The following is the Extended Dynkin diagram.



$\tilde{A}_1$(Kronecker case) and $\tilde{D}_4$(four subspace case)are the cases of extended Dynkin diagram. Among Kronecker cases, any representation in (3),(4), n=1 of (1) or n = 1 of (2) is transitive, that is,algebras of endomorphisms of representations $End(H, f) = \mathbb{C}I$. All finite dimensional indecomposable representations of Dynkin quivers are transitive.

But finite dimensional indecomposable representations of extended Dynkin quivers are not necessarily transitive. Therefore it is interesting to find infinite-dimensional indecomposable (transitive) Hilbert representations of the extended Dynkin quivers. For this purpose it is important to use unbounded strongly irreducible operators and transitive operators. The related research area of operator theory is subspace lattice, transitive lattice and strongly irreducible operators.

2.Definition of Hilbert representations of quivers.

A quiver $\Gamma = (V, E, s, r)$ is a quadruple consisting of the set V of vertices, the set E of arrows, and two maps $s, r : E \to V$ which associate with each arrow $\alpha \in E$ its support $s(\alpha)$ and range $r(\alpha)$. A quiver Γ is said to be finite if both V and E are finite sets.

We list up some examples of Quivers. The Jordan quiver L is a quiver $L = (V, E, s, r)$ such that $V = \{1\}$ and $E = \{\alpha\}$ and $s(\alpha) = r(\alpha) = 1$. The quiver $C_n(n \geq 2)$ is a quiver such that $V = \{1, 2, \cdots, n\}$ and $E = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $s(\alpha_1) = 1, r(\alpha_1) = 2, s(\alpha_2) = 2, r(\alpha_2) = 3, \cdots, s(\alpha_n) = n, r(\alpha_n) = 1$. The quivers L and $C_n(n \geq 2)$ are called the oriented cyclic quivers. The Kronecker quiver Q is a quiver with two vertices $\{1, 2\}$ and two paralleled arrows $\{\alpha, \beta\}$:

$$Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that (H, f) is a Hilbert representation of Γ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_\alpha)_{\alpha \in E}$ is a family of bounded linear operators $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$.

Definition Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (H, f) and (K, g) be Hilbert representations of Γ. A homomorphism $T : (H, f) \to (K, g)$ is a family $T = (T_v)_{v \in V}$ of bounded operators $T_v : H_v \to K_v$ satisfying for any arrow $\alpha \in E$,

$$T_{r(\alpha)} f_\alpha = g_\alpha T_{s(\alpha)}.$$

We denote by $\mathrm{Hom}((H, f), (K, g))$,the set of homomorphisms $T : (H, f) \to (K, g)$. We denote by $End(H, f) := Hom((H, f), (H, f))$ the set of endomorphisms. We say that (H, f) and (K, g) are isomorphic, denoted by $(H, f) \cong (K, g)$, if there exists an isomorphism $\varphi : (H, f) \to (K, g)$, that is, there exists a family $\varphi = (\varphi_v)_{v \in V}$ of bounded invertible operators $\varphi_v \in B(H_v, K_v)$ such that ,for any arrow $\alpha \in E$,

$$\varphi_{r(\alpha)} f_\alpha = g_\alpha \varphi_{s(\alpha)}.$$

We say that (H, f) is a finite dimensional representation if H_v is finite dimensional for all $v \in V$. And (H, f) is an infinite dimensional representation if H_v is infinite-dimensional for some $v \in V$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (K, g) and (K', g') be Hilbert representations of Γ. We define the direct sum $(H, f) = (K, g) \oplus (K', g')$ by $H_v = K_v \oplus K'_v$(for $v \in V$) and $f_\alpha = g_\alpha \oplus g'_\alpha$(for $\alpha \in E$).

It is said that a Hilbert representation (H, f) is zero, denoted by $(H, f) = 0$ if $H_v = 0$ for any $v \in V$.

Definition. A Hilbert representation (H, f) of Γ is said to be decomposable if (H, f) is isomorphic to a direct sum of two non-zero Hilbert representations. A non-zero Hilbert representation (H, f) of Γ is called indecomposable if it is not decomposable.

The following result is useful to study indecomposable representations.

Lemma 1. *Let (H, f) be a Hilbert representation of a quiver Γ. Then the following conditions are equivalent:*

(1) *(H, f) is indecomposable.*

(2) *$Idem(H, f)$ = the set of idempotents in $End(H, f) = \{0, I\}$.*

Definition. A Hilbert representation (H, f) of a quiver Γ is called *transitive* if $End(H, f) = \mathbb{C}I$. If a Hilbert representation (H, f) of Γ is transitive , then (H, f) is indecomposable.

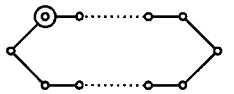
3.Transitive Hilbert representations of extended Dynkin quivers.

We consider transitive Hilbert representations of quivers whose underlying undirected graph is an extended Dynkin diagram $\tilde{A}_n (n \geq 0)$.

We have no infinite dimensional transitive Hilbert representations of the Jordan quiver L whose underlying undirected graph is an extended Dynkin diagram $\tilde{A}_0$.

Next we consider transitive Hilbert representations of quivers whose underlying undirected graph is an extended Dynkin diagram $\tilde{A}_n (n \geq 1)$.

$\widetilde{A_n}$



For $\tilde{A}_1$ case, the quivers are the oriented cyclic quiver C_2 and the Kronecker quiver Q.

The oriented cyclic quiver C_2 is the quiver such that $V = \{1, 2\}, E = \{\alpha_1, \alpha_2\}, s(\alpha_1) = r(\alpha_2) = 1, r(\alpha_1) = s(\alpha_2) = 2,$

the Kronecker quiver Q is the quiver such that $V = \{1, 2\}, E = \{\alpha_1, \alpha_2\}, s(\alpha_1) = s(\alpha_2) = 1, r(\alpha_1) = r(\alpha_2) = 2.$

For the Kronecker quiver, we [EW2] have the following :

Theorem 1. *Let Q be the Kronecker quiver. Then there exists an infinite-dimensional transitive Hilbert representation of Q.*

Using the above theorem we have the following:

Theorem 2. *Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_n}, (n \geq 0)$.*

If Γ is not the oriented cyclic quiver, then there exists an infinite-dimensional transitive Hilbert representation of Γ.

Thus remaining case of the existence problem of infinite dimensional transitive Hilbert representations of quivers whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_n}, (n \geq 1)$ is the oriented cyclic quiver.

We shall consider transitive Hilbert representations of C_2 .

Let (H, f) be a Hilbert representation of C_2.

We put $A_1 = f_{\alpha_1}$ and $A_2 = f_{\alpha_2}$.

Lemma 2. *([EW3]) Let (H, f) be a Hilbert representation of C_2. Then (H, f) is transitive if and only if one of the following conditions holds.*

(1) $H_1 = \mathbb{C}, H_2 = 0, A_1 = 0$ *and* $A_2 = 0$,

(2) $H_1 = 0, H_2 = \mathbb{C}, A_1 = 0$ *and* $A_2 = 0$,

(3) $H_1 = \mathbb{C}$ *and* $H_2 = \mathbb{C}$ *and* ($A_1 \neq 0$ *or* $A_2 \neq 0$).

In order to prove the necessary condition of this lemma, it is sufficient to assume $\dim H_1 \neq 0$ and $\dim H_2 \neq 0$. Furthermore we may assume that $\dim H_1 \leq \dim H_2$ and H_1 is a subspace of H_2.

We define

$$T = (T_1, T_2) = (A_2 A_1, A_1 A_2).$$

Then $T \in End(H, f)$. In fact

$$A_1 T_1 = A_1(A_2 A_1) = (A_1 A_2) A_1 = T_2 A_1$$

and

$$T_1 A_2 = (A_2 A_1) A_2 = A_2(A_1 A_2) = A_2 T_2.$$

By the assumption of transitivity for (H, f),

$$(T_1, T_2) \in \mathbb{C} = \{(\mu I_{H_1}, \mu I_{H_2}) | \mu \in \mathbb{C}\}.$$

Hence

$$T_1 = A_2 A_1 = \mu I_{H_1}, T_2 = A_1 A_2 = \mu I_{H_2},$$

for some $\mu \in \mathbb{C}$.

We define $E_1 \in B(H_1, H_2)$ by the embedding map from H_1 into H_2 and $E_2 \in B(H_2, H_1)$ by the projection map from H_2 onto H_1.

We define

$$T^{\{1\}} = (T_1^{\{1\}}, T_2^{\{1\}}) = (A_2 E_1, E_1 A_2).$$

Then $T^{\{1\}} \in End(H, f)$. In fact

$$\begin{aligned} A_1 T_1^{\{1\}} &= A_1(A_2 E_1) = (A_1 A_2) E_1 = (\mu I_{H_2}) E_1 \\ &= \mu E_1 = E_1(\mu I_{H_1}) = E_1(A_2 A_1) = (E_1 A_2) A_1 = T_2^{\{1\}} A_1. \end{aligned}$$

$$T_1^{\{1\}} A_2 = (A_2 E_1) A_2 = A_2(E_1 A_2) = A_2 T_2^{\{1\}}.$$

Thus $T^{\{1\}} \in End(H, f)$. Since (H, f) is transitive, there exists a constant $\mu^{\{1\}} \in \mathbb{C}$ such that $A_2 E_1 = \mu^{\{1\}} I_{H_1}$ and $E_1 A_2 = \mu^{\{1\}} I_{H_2}$.

We define

$$T^{\{2\}} = (T_1^{\{2\}}, T_2^{\{2\}}) = (E_2 A_1, A_1 E_2).$$

Then $T^{\{2\}} \in End(H, f)$. In fact,

$$A_1 T_1^{\{2\}} = A_1(E_2 A_1) = (A_1 E_2) A_1 = T_2^{\{2\}} A_1.$$

$$\begin{aligned} T_1^{\{2\}} A_2 &= (E_2 A_1) A_2 = E_2(\mu I_{H_2}) = \mu E_2 \\ &= \mu I_{H_1} E_2 = A_2(A_1 E_2) = A_2 T_2^{\{2\}}. \end{aligned}$$

Since (H, f) is transitive, there exists a constant $\mu^{\{2\}} \in \mathbb{C}$ such that $E_2 A_1 = \mu^{\{2\}} I_{H_1}$ and $A_1 E_2 = \mu^{\{2\}} I_{H_2}$.

We define

$$T^{\{1,2\}} = (T_1^{\{1,2\}}, T_2^{\{1,2\}}) = (E_2 E_1, E_1 E_2).$$

Then $T^{\{1,2\}} \in End(H, f)$. In fact,

$$\begin{aligned} A_1 T_1^{\{1,2\}} &= A_1(E_2 E_1) = (A_1 E_2) E_1 = (\mu^{\{2\}} I_{H_2}) E_1 = \mu^{\{2\}} E_1 = \\ &= \mu^{\{2\}} E_1 = E_1(\mu^{\{2\}} I_{H_1}) = E_1(E_2 A_1) = T_2^{\{1,2\}} A_1, \end{aligned}$$

Also we have

$$\begin{aligned} T_1^{\{1,2\}} A_2 &= (E_2 E_1) A_2 = E_2(E_1 A_2) = E_2(\mu^{\{1\}} I_{H_2}) = \mu^{\{1\}} E_2 = \\ &= \mu^{\{1\}} E_2 = (\mu^{\{1\}} I_{H_1}) E_2 = (A_2 E_1) E_2 = A_2(E_1 E_2) = A_2 T_2^{\{1,2\}}. \end{aligned}$$

Since (H, f) is transitive, there exists a constant $\mu^{\{1,2\}} \in \mathbb{C}$ such that

$$E_2 E_1 = \mu^{\{1,2\}} I_{H_1} \text{ and } E_1 E_2 = \mu^{\{1,2\}} I_{H_2}.$$

For $x (\neq 0) \in H_1$, we have $E_2 E_1 x = x$.

Since $E_2E_1 = \mu^{\{1,2\}}I_{H_1}$, we have $x = \mu^{\{1,2\}}x$,so $\mu^{\{1,2\}} = 1$.

If $H_1 \neq H_2$,then $H_1^{\perp} \cap H_2 \neq 0$. Take $x(\neq 0) \in H_1^{\perp} \cap H_2$. Then $E_1E_2x = \mu^{\{1,2\}}I_{H_2}x$. Hence $0 = x$. This is a contradiction.

Thus $H_1 = H_2$ and $E_1 = E_2$. Since $A_1E_2 = \mu^{\{2\}}I_{H_2}$,

$$A_1 = \mu^{\{2\}}I_{H_1}.$$

Since $E_1A_2 = \mu^{\{1\}}I_{H_2}$,

$$A_2 = \mu^{\{1\}}I_{H_1}.$$

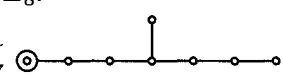
Since (H, f) is transitive, $A_1 \neq 0$ or $A_2 \neq 0$.

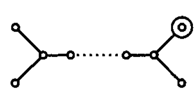
Also we have $H_1 = H_2 = \mathbb{C}$. Thus (H, f) is in the case (3).

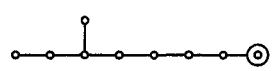
Generalizing these argument to C_n case, we have

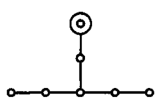
Theorem 3. *([EW3]) Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_n}, (n \geq 0)$. Then there exists an infinite dimensional transitive Hilbert representation of Γ if and only if Γ is not the oriented cyclic quiver.*

Thefore remaining cases are $\widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$.

$\widetilde{E_7}$ 

$\widetilde{D_n}$  $(n \geqq 4)$

$\widetilde{E_8}$ 

$\widetilde{E_6}$ 

In order to consider the remaining cases $\widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$, we need more preparations. The following theorem is one of the tools.

Theorem 4. *([EW3]) Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. If a Hilbert representation (H, f) of Γ is co-full at v, then $\Phi_v^- : End(H, f) \to End(\Phi_v^-(H, f))$ is an isomorphism as $\mathbb{C}$-algebras.*

Reflection functors are defined as follows([EW1]):

Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that a vertex $v \in V$ is a *sink* if $v \neq s(\alpha)$ for any $\alpha \in E$. Put $E^v = \{\alpha \in E;\ r(\alpha) = v\}$. We denote by $\overline{E}$ the set of all formally reversed new arrows $\overline{\alpha}$ for $\alpha \in E$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we construct a new quiver $\sigma_v^+(\Gamma) = (\sigma_v^+(V), \sigma_v^+(E), s, r)$ as follows: All the arrows of Γ having v as range are reversed and all the other arrows remain unchanged. That is,

$$\sigma_v^+(V) = V, \quad \sigma_v^+(E) = (E \setminus E^v) \cup \overline{E^v},$$

where $\overline{E^v} = \{\overline{\alpha};\ \alpha \in E^v\}$.

Definition. (reflection functor Φ_v^+.) Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we define a *reflection functor* at v

$$\Phi_v^+ : HRep(\Gamma) \to HRep(\sigma_v^+(\Gamma))$$

between the categories of Hilbert representations of Γ and $\sigma_v^+(\Gamma)$ as follows:

For a Hilbert representation (H, f) of Γ, we define a Hilbert representation $(K, g) = \Phi_v^+(H, f)$ of $\sigma_v^+(\Gamma)$. Let

$$h_v : \oplus_{\alpha \in E^v} H_{s(\alpha)} \to H_v$$

be a bounded linear operator defined by

$$h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha).$$

We shall define

$$K_v := Ker(h_v) = \{(x_\alpha)_{\alpha \in E^v} \in \oplus_{\alpha \in E^v} H_{s(\alpha)}; \sum_{\alpha \in E^v} f_\alpha(x_\alpha) = 0\}.$$

We also consider the canonical inclusion map $i_v : K_v \to \oplus_{\alpha \in E^v} H_{s(\alpha)}$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

For $\beta \in E^v$, let

$$P_\beta : \oplus_{\alpha \in E^v} H_{s(\alpha)} \to H_{s(\beta)}$$

be the canonical projection. Then we shall define

$$g_{\overline{\beta}} : K_{s(\overline{\beta})} = K_v \to K_{r(\overline{\beta})} = H_{s(\beta)} \quad \text{by} \quad g_{\overline{\beta}} = P_\beta \circ i_v$$

that is, $g_{\overline{\beta}}((x_\alpha)_{\alpha \in E^v}) = x_\beta$.

For $\beta \notin E^v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H, f) \to (H', f')$, we define a homomorphism

$$S = (S_u)_{u \in V} = \Phi_v^+(T) : (K, g) = \Phi_v^+(H, f) \to (K', g') = \Phi_v^+(H', f')$$

If $u = v$, a bounded operator $S_v : K_v \to K'_v$ is given by

$$S_v((x_\alpha)_{\alpha \in E^v}) = (T_{s(\alpha)}(x_\alpha))_{\alpha \in E^v}.$$

For other $u \in V$ with $u \neq v$, put

$$S_u = T_u : K_u = H_u \to K'_u = H'_u.$$

Definition. (reflection functor Φ_v^-.)

Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we shall define a *reflection functor* at v

$$\Phi_v^- : HRep(\Gamma) \to HRep(\sigma_v^-(\Gamma))$$

between the categories of Hilbert representations of Γ and $\sigma_v^-(\Gamma)$ as follows: For a Hilbert representation (H, f) of Γ, we define a Hilbert representation $(K, g) = \Phi_v^-(H, f)$ of $\sigma_v^-(\Gamma)$. Let

$$\hat{h}_v : H_v \to \oplus_{\alpha \in E_v} H_{r(\alpha)}$$

be a bounded linear operator defined by

$$\hat{h}_v(x) = (f_\alpha(x))_{\alpha\in E_v} \quad \text{for } x \in H_v.$$

We shall define

$$K_v := (Im(\hat{h}_v))^\perp = Ker\hat{h}_v^* \subset \oplus_{\alpha\in E_v} H_{r(\alpha)},$$

where $\hat{h}_v^* : \oplus_{\alpha\in E_v} H_{r(\alpha)} \to H_v$ is given $\hat{h}_v^*((x_\alpha)_{\alpha\in E_v}) = \sum f_\alpha^*(x_\alpha)$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

Let $Q_v : \oplus_{\alpha\in E_v} H_{r(\alpha)} \to K_v$ be the canonical projection. For $\beta \in E_v$, let

$$j_\beta : H_{r(\beta)} \to \oplus_{\alpha\in E_v} H_{r(\alpha)}$$

be the canonical inclusion.

We shall define

$$g_{\overline{\beta}} : K_{s(\overline{\beta})} = H_{r(\beta)} \to K_{r(\overline{\beta})} = K_v \quad \text{by } g_{\overline{\beta}} = Q_v \circ j_\beta.$$

For $\beta \notin E_v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H, f) \to (H', f')$, we shall define a homomorphism

$$S = (S_u)_{u\in V} = \Phi_v^-(T) : (K, g) = \Phi_v^-(H, f) \to (K', g') = \Phi_v^-(H', f').$$

For $u = v$, a bounded operator $S_v : K_v \to K_v'$ is given by

$$S_v((x_\alpha)_{\alpha\in E_v}) = Q_v'((T_{r(\alpha)}(x_\alpha))_{\alpha\in E_v}),$$

where $Q_v' : \oplus_{\alpha\in E_v} H_{r(\alpha)}' \to K_v'$ be the canonical projection. For other $u \in V$ with $u \neq v$, put

$$S_u = T_u : K_u = H_u \to K_u' = H_u'.$$

The following theorem is the main result of [EW3].

Theorem 5. *([EW3]) Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram. Then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver.*

We shall give a sketch of proof of theorem 5. For the remaining cases $\widetilde{D}_n$, $\widetilde{E}_6$,$\widetilde{E}_7$,$\widetilde{E}_8$, at first, we construct a transitive Hilbert representation for subspace quiver by using transitive operators. This transitive Hilbert representation has a nice property. By this nice property we can use theorem 4 about endomorphism algebras. Thus we can change an orientation of the subspace quiver. Hence we have a transitive Hilbert representation for quivers with any orientation.

4.Transitive operators and transitive representations. To consider transitive representations, we need unbounded strongly irreducible operators and transitive operators.

Definition.([EW3]) An unbounded closed operator A is said to be strongly irreducible if A satisfies the following condition: For any idempotent $E \in B(H)$, if E is in the commutant $\{A\}'$, then $E = 0$ or $E = I$.

An unbounded closed operator A is said to be transitive if A satisfies the following condition: For any $T \in B(H)$, if T is in the commutant $\{A\}'$, then T is a scalar operator.

We show that there is a non-zero surjective algebra homomorphism of the endomorphism algebra of a Hilbert representation of the Kronecker quiver to the endomorphism algebra of a four-subspace system.

Theorem 6. *([EW3]) Let $K \neq 0$ be a Hilbert space and $A, B \in B(K)$. Let (H, f) be a Hilbert representation of the Kronecker quiver Q such that*

$$H_1 = H_2 = K, f_\alpha = A \text{ and } f_\beta = B.$$

Let $\mathcal{S} = (E_0; E_1, E_2, E_3, E_4)$ be a four-subspace system such that

$$E_0 = K \oplus K, E_1 = K \oplus 0, E_2 = 0 \oplus K,$$

$$E_3 = \{(Ax, Bx); x \in K\}, E_4 = \{(x, x); x \in K\}.$$

Assume that E_3 is closed. Then there exists a non-zero surjective algebra homomorphism Φ of $End(H, f)$ to $End(\mathcal{S})$. Moreover, if $\ker A \cap \ker B = 0$*, then Φ is one to one.*

Under a certain condition we have a correspondence between transitive Hilbert representations of the Kronecker quiver and transitive operators.

Theorem 7. *([EW3]) Let K be a Hilbert space and $A, B \in B(K)$. Assume that*

$$\ker A = 0 \text{ and } A^*(K) + B^*(K) \text{ is closed in } K.$$

Let (H, f) be a Hilbert representation of the Kronecker quiver Q such that

$$H_1 = H_2 = K, f_\alpha = A \text{ and } f_\beta = B.$$

Then BA^{-1} is transitive if and only if (H, f) is transitive.

We shall give some examples of transitive operators.

Example I.([EW3])

Let Q be the Kronecker quiver. Let S be the unilateral shift on $H = \ell^2(\mathbb{N})$ with a canonical basis $\{e_1, e_2, ...\}$.

For a bounded weight vector $\lambda = (\lambda_1, \lambda_2, ...) \in \ell^\infty(\mathbb{N})$ we associate with a diagonal operator $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, ...)$, so that SD_λ is a weighted shift operator.

We assume that

$$\lambda_i \neq \lambda_j \text{ if } i \neq j.$$

Take a vector $\overline{w} = (\overline{w_n})_n \in \ell^2(\mathbb{N})$ such that

$$w_n \neq 0 \text{ for any } n \in \mathbb{N}.$$

Put

$$A = SD_\lambda + \theta_{e_1, \overline{w}} \text{ and } B = S.$$

Define a Hilbert representation (H^λ, f^λ) of the Kronecker quiver Q by

$$H_1^\lambda = H_2^\lambda = H, f_\alpha^\lambda = A \text{ and } f_\beta^\lambda = B.$$

Then $\ker A = 0$ and the quotient BA^{-1} is a transitive operator.
Furthermore,the operator BA^{-1} is densely defined
if and only if

$$\lambda_k \neq 0 \text{ for each } k \in \mathbb{N} \text{ and } (\frac{w_k}{\lambda_k})_k \notin \ell^2(\mathbb{N}).$$

Example II.([EW3])
Let Q be the Kronecker quiver and $H = \ell^2(\mathbb{Z})$. Let $a = (a(n))_{n\in\mathbb{Z}}, b = (b(n))_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ such that

$$a(n) \neq 0, b(n) \neq 0 \text{ for any } n \in \mathbb{Z}.$$

We put

$$w_m = \frac{b(m)}{a(m)}, m \in \mathbb{Z}.$$

We put

$$M_k(m,n) := \frac{w_m w_{m+1} \cdots w_{m+k-1}}{w_n w_{n+1} \cdots w_{n+k-1}} \text{ for } m, n \in \mathbb{Z}, k \geq 1.$$

Assume that for any $m \neq n$, $(M_k(m,n))_k$ is an unbounded sequence.

Let D_a be a diagonal operator with $a = (a(n))_n$ as diagonal coefficients and D_b be a diagonal operator with $b = (b(n))_n$ as diagonal coefficients. Let U be the bilateral forward shift. Put

$$A = D_a \text{ and } B = UD_b.$$

Define a Hilbert representation (H, f) of the Kronecker quiver Q by

$$H_1 = H_2 = H, f_\alpha = A \text{ and } f_\beta = B.$$

Then the Hilbert representation (H, f) is transitive. We also have $\ker A = 0$ and $\ker B = 0$. And the operator BA^{-1} is a densely defined transitive operator.

The following two sequences a and b satisfy the condition of the example II. Fix a positive constant $\lambda > 1$. Consider two sequences $a = (a(n))_{n\in\mathbb{Z}}$ and $b = (b(n))_{n\in\mathbb{Z}}$ by

$$a(n) = \begin{cases} e^{-\lambda^n} & (n \geq 1, n \text{ is even }), \\ 1 & (otherwise), \end{cases} \quad b(n) = \begin{cases} e^{-\lambda^n} & (n \geq 1, n \text{ is odd }), \\ 1 & (otherwise). \end{cases}$$

The concept of transitive operators are useful because certain transitive Hilbert representations of a quiver are given in terms of transitive operators.

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