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Estimations of power difference mean by Heron mean

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Abstract

In this report, we discuss estimations of power difference mean by Heron mean. We obtain the greatest value $\alpha=\alpha(q)$ and the least value $\beta=\beta(q)$ such that the double inequality

$$K_{\alpha}(a,b)<J_{q}(a,b)<K_{\beta}(a,b)$$

holds for any $a,b>0$ and $q\in\mathbb{R}$, where $J_{q}(a,b)=\frac{q}{q+1}\frac{a^{q+1}-b^{q+1}}{a^{q}-b^{q}}$ is the power difference mean and $K_{q}(a,b)=(1-q)\sqrt{ab}+q\frac{a+b}{2}$ is the Heron mean. We also get similar inequalities for bounded linear operators on a Hilbert space.

1 Introduction

This report is based on [10]. For two positive real numbers a and b, the arithmetic mean, the geometric mean, the harmonic mean and the logarithmic mean are as follows:

$$A(a,b)=\frac{a+b}{2}\ \text{(arithmetic mean)},\quad G(a,b)=\sqrt{ab}\ \text{(geometric mean)},$$
$$H(a,b)=\frac{2ab}{a+b}\ \text{(harmonic mean)},\quad L(a,b)=\frac{a-b}{\log a-\log b}\ \text{(logarithmic mean)}.$$

We remark that these means are symmetric, that is, $A(a,b)=A(b,a)$, $G(a,b)=G(b,a)$ and so on. It is well known that the inequality $H(a,b)\leq G(a,b)\leq L(a,b)\leq A(a,b)$ always holds.

As one parameter extensions of above means, the following are known.

$$M_{q}(a,b)=\begin{cases}\left(\dfrac{a^{q}+b^{q}}{2}\right)^{\frac{1}{q}} & \text{if } q\neq 0,\\ \sqrt{ab} & \text{if } q=0,\end{cases}\quad \text{(power mean)},$$

$$J_{q}(a,b)=\begin{cases}\dfrac{q}{q+1}\dfrac{a^{q+1}-b^{q+1}}{a^{q}-b^{q}} & \text{if } q\neq 0,-1,\\ \dfrac{a-b}{\log a-\log b} & \text{if } q=0,\\ \dfrac{ab(\log a-\log b)}{a-b} & \text{if } q=-1,\end{cases}\quad \text{(power difference mean)},$$

$$K_{q}(a,b)=(1-q)\sqrt{ab}+q\frac{a+b}{2}\quad \text{(Heron mean)},$$

$$HZ_q(a,b) = \frac{a^{1-q}b^q + a^q b^{1-q}}{2} \quad \text{(Heinz mean)}.$$

We note that $J_q(a,a) \equiv \lim_{b\to a} J_q(a,b) = a$, and also these means are also symmetric. Relations among these means are as follows:

$$\begin{aligned}
&M_1(a,b) = J_1(a,b) = K_1(a,b) = HZ_0(a,b) = HZ_1(a,b) = A(a,b),\\
&J_0(a,b) = L(a,b),\\
&M_0(a,b) = J_{\frac{-1}{2}}(a,b) = K_0(a,b) = HZ_{\frac{1}{2}}(a,b) = G(a,b),\\
&M_{-1}(a,b) = J_{-2}(a,b) = H(a,b),
\end{aligned}$$

and also $M_q(a,b)$, $J_q(a,b)$ and $K_q(a,b)$ are monotone increasing on $q \in \mathbb{R}$.

Our purpose in this report is to give estimations of power difference mean by Heron mean. In section 2, we state former results on inequalities estimating above means. In section 3, we obtain the greatest value $\alpha = \alpha(q)$ and the least value $\beta = \beta(q)$ such that the double inequality

$$K_\alpha(a,b) < J_q(a,b) < K_\beta(a,b)$$

holds for any $a, b > 0$ and $q \in \mathbb{R}$. In section 4, we have similar inequalities to those in section 3 for bounded linear operators on a Hilbert space.

2 Former results

Many researchers investigate inequalities comparing two means. For example,

- If $q \in (-2, \frac{-1}{2}) \cup (1, \infty)$, then $M_{\frac{1+2q}{3}}(a,b) < J_q(a,b)$, and if $q \in (-\infty, -2) \cup (\frac{-1}{2}, 1)$, then $J_q(a,b) < M_{\frac{1+2q}{3}}(a,b)$ for all $a, b > 0$ with $a \neq b$. The parameter $\frac{1+2q}{3}$ is best possible. (Xia, Wang, Chu, Hou [14])
- If $q \in [0,1]$, then $HZ_q(a,b) \leq K_{(1-2q)^2}(a,b)$ for all $a, b > 0$. (Bhatia [1])

Here, we pay attention to the following result.

Proposition 2.A ([1]). *The inequality $L(a,b) \leq K_\alpha(a,b)$ holds for all $a, b > 0$ if and only if $\alpha \geq \frac{1}{3}$.*

The optimal inequality $L(a,b) \leq K_{\frac{1}{3}}(a,b)$ is well known as the classical Pólya inequality. As related results to Proposition 2.A, for example, they obtain matrix norm inequalities in [1, 5, 8], and also operator inequalities for bounded linear operators on a Hilbert space in [3, 4]. In [1], Bhatia proved Proposition 2.A by using Taylor expansion. Here, we give a proof by this way for readers' sake.

Proof of Proposition 2.A. Since $L(a,b) \le K_\alpha(a,b)$ for all $a,b>0$ is equivalent to

$$\frac{x-1}{\log x} \le (1-\alpha)\sqrt{x} + \alpha\frac{x+1}{2} \tag{2.1}$$

for all $x>0$ by replacing x by $\frac{a}{b}$, we have only to show that (2.1) holds for all $x>0$ if and only if $\alpha \ge \frac{1}{3}$.

Put $x = e^{2t}$ in (2.1). Then (2.1) holds for all $x>0$ if and only if

$$\frac{e^t - e^{-t}}{2t} \le (1-\alpha) + \alpha\frac{e^t+e^{-t}}{2}, \quad \text{that is,} \quad \frac{1}{t}\sinh t \le (1-\alpha) + \alpha\cosh t \tag{2.2}$$

for all $t \in \mathbb{R}$. By Taylor expansion, (2.2) can be written by

$$\frac{t^2}{3!} + \frac{t^4}{5!} + \cdots + \frac{t^{2k-2}}{(2k-1)!} + \cdots \le \alpha\left(\frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2k-2}}{(2k-2)!} + \cdots\right),$$

so that (2.2) holds for all $t \in \mathbb{R}$ if and only if $\alpha \ge \frac{1}{3}$. Hence the proof is complete. □

Noting that $\frac{1}{(2k-1)!} \le \alpha\frac{1}{(2k-2)!}$ if and only if $\alpha \ge \frac{1}{2k-1}$ for $k = 2,3,\ldots$ in the proof of Proposition 2.A, we obtain that the reverse inequality $K_\alpha(a,b) \le L(a,b)$ holds for all $a,b>0$ if and only if $\alpha \le 0$.

We have many related numerical inequalities to those in this section, see [7, 13] and so on. In [13], Xia, Hou, Wang and Chu obtained optimal inequalities between $J_q(a,b)$ and $K_q(a,b)$.

Theorem 2.B ([13]). *For all $a,b>0$ with $a \ne b$, we have the following inequalities.*

(i) *If $\alpha \in (0,\frac{2}{3})$, then $J_{\frac{3\alpha-1}{2}}(a,b) < K_\alpha(a,b) < J_{\frac{\alpha}{2-\alpha}}(a,b)$.*

(ii) *If $\alpha \in (\frac{2}{3},1)$, then $J_{\frac{\alpha}{2-\alpha}}(a,b) < K_\alpha(a,b) < J_{\frac{3\alpha-1}{2}}(a,b)$.*

The given parameters $\frac{3\alpha-1}{2}$ and $\frac{\alpha}{2-\alpha}$ in either case are best possible.

In Theorem 2.B, they obtain the greatest value $p = p(\alpha)$ and the least value $q = q(\alpha)$ such that the double inequality

$$J_p(a,b) < K_\alpha(a,b) < J_q(a,b)$$

holds for any $\alpha \in (0,1)$. We remark that Theorem 2.B implies Proposition 2.A by putting $\alpha = \frac{1}{3}$. Theorem 2.B can be written by the following Theorem 2.B' as the result estimating power difference mean by Heron mean.

Theorem 2.B'. *For all $a, b > 0$ with $a \neq b$, we have the following inequalities.*

(i) *If $q \in (0, \frac{1}{2})$, then* $K_{\frac{2q}{q+1}}(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b)$.

(ii) *If $q \in (\frac{1}{2}, 1)$, then* $K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_{\frac{2q}{q+1}}(a,b)$.

(iii) *If $q \in (\frac{-1}{2}, 0]$, then* $J_q(a,b) < K_{\frac{2q+1}{3}}(a,b)$.

The given parameters $\frac{2q+1}{3}$ and $\frac{2q}{q+1}$ in either case are best possible.

3 Main result

Theorem 2.B' in the previous section seems to be a partial result since the range of q is restricted, so we obtain estimations of power difference mean by Heron mean for all $q \in \mathbb{R}$ as an extension of Theorem 2.B'. In other words, we get the greatest value $\alpha = \alpha(q)$ and the least value $\beta = \beta(q)$ such that the double inequality

$$K_\alpha(a,b) < J_q(a,b) < K_\beta(a,b)$$

holds for any $q \in \mathbb{R}$.

Theorem 3.1. *For all $a, b > 0$ with $a \neq b$, we have the following.*

(i) *Let $q \in (0, \frac{1}{2}) \cup (1, \infty)$. Then*

$$K_{\frac{2q}{q+1}}(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b).$$

(ii) *Let $q \in (\frac{1}{2}, 1)$. Then*

$$K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_{\frac{2q}{q+1}}(a,b).$$

(iii) *Let $q \in (\frac{-1}{2}, 0]$. Then*

$$G(a,b) = K_0(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b).$$

(iv) *Let $q \in (-\infty, \frac{-1}{2})$. Then*

$$K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_0(a,b) = G(a,b).$$

The given parameters of $K_\alpha(a,b)$ in each case are best possible.

Equalities hold between $J_q(a,b)$ and $K_\alpha(a,b)$ in the following cases.

$$J_q(a,b) = K_{\frac{2q+1}{3}}(a,b) = K_{\frac{2q}{q+1}}(a,b) \quad \text{for } q = \tfrac{1}{2}, 1.$$
$$J_q(a,b) = K_{\frac{2q+1}{3}}(a,b) = K_0(a,b) \quad \text{for } q = \tfrac{-1}{2}.$$

We shall show Theorem 3.1 by using Taylor expansion, which is the different way from [13]. To prove Theorem 3.1, we have only to show the following propositions. By putting $x = \frac{a}{b}$ in Propositions 3.2 and 3.3, we immediately obtain Theorem 3.1.

Proposition 3.2. *The following statements hold:*

(i) *Let* $q \in (\frac{-1}{2}, \frac{1}{2}) \cup (1, \infty)$. *Then*

$$J_q(x,1) < K_\alpha(x,1) \textit{ for all } x > 0 \textit{ with } x \neq 1 \textit{ if and only if } \alpha \geq \tfrac{2q+1}{3}.$$

(ii) *Let* $q \in (-\infty, \frac{-1}{2}) \cup (\frac{1}{2}, 1)$. *Then*

$$J_q(x,1) > K_\alpha(x,1) \textit{ for all } x > 0 \textit{ with } x \neq 1 \textit{ if and only if } \alpha \leq \tfrac{2q+1}{3}.$$

Proposition 3.3. *The following statements hold:*

(i-1) *Let* $q \in (0, \frac{1}{2}) \cup (1, \infty)$. *Then*

$$J_q(x,1) > K_\alpha(x,1) \textit{ for all } x > 0 \textit{ with } x \neq 1 \textit{ if and only if } \alpha \leq \tfrac{2q}{q+1}.$$

(i-2) *Let* $q \in (\frac{-1}{2}, 0]$. *Then*

$$J_q(x,1) > K_\alpha(x,1) \textit{ for all } x > 0 \textit{ with } x \neq 1 \textit{ if and only if } \alpha \leq 0.$$

(ii-1) *Let* $q \in (\frac{1}{2}, 1)$. *Then*

$$J_q(x,1) < K_\alpha(x,1) \textit{ for all } x > 0 \textit{ with } x \neq 1 \textit{ if and only if } \alpha \geq \tfrac{2q}{q+1}.$$

(ii-2) *Let* $q \in (-\infty, \frac{-1}{2})$. *Then*

$$J_q(x,1) < K_\alpha(x,1) \textit{ for all } x > 0 \textit{ with } x \neq 1 \textit{ if and only if } \alpha \geq 0.$$

Here, we give a proof of Proposition 3.2. Proposition 3.3 can be shown by the similar argument. As lemmas to prove these propositions, we show two properties of functions $g_k(q)$ for $k = 2, 3, \ldots$ and $q \in \mathbb{R}$ defined by

$$g_k(q) \equiv \frac{(q+1)^{2(k-1)} - q^{2(k-1)}}{\sum_{i=2}^{k} \binom{2k-1}{2(i-1)} q^{2(k-i)}} \tag{3.1}$$

and $g_k(0) \equiv \dfrac{1}{2k-1}$ for convenience' sake. Here, $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$ is a binomial coefficient for nonnegative integers n and r such that $0 \leq r \leq n$. We remark that $g_2(q) = \dfrac{2q+1}{3}$ in particular.

Lemma 3.4. *The limit* $g_\infty(q) \equiv \lim_{k\to\infty} g_k(q)$ *exists and* $g_\infty(q) = \begin{cases} \dfrac{2q}{q+1} & (q>0), \\ 0 & (q \leq 0). \end{cases}$

Proof. Firstly, we state the following relation (3.2) which is important to prove results in this paper. By putting $j = k - i$,

$$\begin{aligned} &2q\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} = 2\sum_{j=0}^{k-2}\binom{2k-1}{2(k-j-1)}q^{2j+1} \\ &= 2\sum_{j=0}^{k-1}\binom{2k-1}{2j+1}q^{2j+1} - 2q^{2k-1} = (q+1)^{2k-1} + (q-1)^{2k-1} - 2q^{2k-1}. \end{aligned} \tag{3.2}$$

If $q \neq 0$, the following holds by (3.2).

$$\begin{aligned} g_k(q) &= \frac{(q+1)^{2(k-1)} - q^{2(k-1)}}{\frac{1}{2q}\left\{(q+1)^{2k-1} + (q-1)^{2k-1} - 2q^{2k-1}\right\}} \\ &= \frac{2q\left\{1 - \left(\frac{q}{q+1}\right)^{2(k-1)}\right\}}{q+1+(q-1)\left(\frac{q-1}{q+1}\right)^{2k-2} - 2q\left(\frac{q}{q+1}\right)^{2k-2}} \quad (\text{if } q \neq -1) \qquad (3.3) \\ &= \frac{2q\left\{\left(\frac{q+1}{q}\right)^{2(k-1)} - 1\right\}}{(q+1)\left(\frac{q+1}{q}\right)^{2k-2} + (q-1)\left(\frac{q-1}{q}\right)^{2k-2} - 2q}. \qquad (3.4) \end{aligned}$$

Now we divide the range of q into four cases.

(Case 1) If $q > 0$, then $-1 < \frac{q-1}{q+1} < 1$ and $0 < \frac{q}{q+1} < 1$. Therefore (3.3) implies $g_\infty(q) = \frac{2q}{q+1}$.

(Case 2) If $\frac{-1}{2} < q < 0$, then $\frac{q-1}{q+1} < -1$ and $-1 < \frac{q}{q+1} < 0$, so that we have $g_\infty(q) = 0$.

(Case 3) If $q < \frac{-1}{2}$, then $-1 < \frac{q+1}{q} < 1$ and $\frac{q-1}{q} > 1$. Therefore (3.4) implies $g_\infty(q) = 0$.

(Case 4) If $q = 0$, then $g_k(0) = \frac{1}{2k-1} \to 0$ as $k \to \infty$.

Hence the proof is complete. □

Lemma 3.5. *Let $g_k(q)$ for $q \in \mathbb{R}$ as in* (3.1). *Then the following assertions hold:*

(i) *If $k \geq 3$, then*

$$g_2(q) - g_k(q) = \frac{2(q-1)(2q+1)(2q-1)}{3\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \sum_{\substack{u,v,w \geq 0 \\ u+v+w=k-3}} (q+1)^{2u}(q-1)^{2v}q^{2w}.$$

(ii) *If $k \geq 2$ and $q > 0$, then*

$$g_k(q) - g_\infty(q) = \frac{(q-1)(2q-1)}{(q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \sum_{\substack{v,w \geq 0 \\ v+w=k-2}} (q-1)^{2v}q^{2w}.$$

Proof. Here we show (i) only. We consider the case $q \neq 0$ since the case $q = 0$ holds by

$$g_2(0) - g_k(0) = \frac{1}{3} - \frac{1}{2k-1} = \frac{2(k-2)}{3(2k-1)}.$$

Since we get

$$\begin{aligned} g_2(q) - g_k(q) &= \frac{2q+1}{3} - \frac{(q+1)^{2(k-1)} - q^{2(k-1)}}{\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \\ &= \frac{(2q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} - 3\left\{(q+1)^{2(k-1)} - q^{2(k-1)}\right\}}{3\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}}, \end{aligned}$$

we have only to show

$$\begin{aligned} h_1(q) &\equiv (2q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} - 3\{(q+1)^{2(k-1)} - q^{2(k-1)}\} \\ &= 2(q-1)(2q+1)(2q-1) \sum_{\substack{u,v,w \geq 0 \\ u+v+w=k-3}} (q+1)^{2u}(q-1)^{2v}q^{2w}. \end{aligned} \tag{3.5}$$

By (3.2), the equation (3.5) holds since

$$
\begin{aligned}
h_1(q) &= (2q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} - 3\{(q+1)^{2(k-1)} - q^{2(k-1)}\}\\
&= \frac{2q+1}{2q}\left\{(q+1)^{2k-1} + (q-1)^{2k-1} - 2q^{2k-1}\right\} - 3\{(q+1)^{2(k-1)} - q^{2(k-1)}\}\\
&= \frac{1}{2q}\left\{(2q+1)(q+1)(q+1)^{2k-2} + (2q+1)(q-1)(q-1)^{2k-2} - 2q(2q+1)q^{2k-2}\right.\\
&\qquad \left. -6q(q+1)^{2k-2} + 6q\cdot q^{2k-2}\right\}\\
&= \frac{1}{2q}\left\{(2q-1)(q-1)(q+1)^{2k-2} + (2q+1)(q-1)(q-1)^{2k-2} - 4q(q-1)q^{2k-2}\right\}\\
&= \frac{q-1}{2q}\left[(2q-1)\left\{(q+1)^{2(k-1)} - q^{2(k-1)}\right\} - (2q+1)\left\{q^{2(k-1)} - (q-1)^{2(k-1)}\right\}\right]\\
&\overset{(*)}{=} 2(q-1)(2q+1)(2q-1)\sum_{\substack{u,v,w\geq 0\\ u+v+w=k-3}}(q+1)^{2u}(q-1)^{2v}q^{2w},
\end{aligned}
$$

and the last equality $(*)$ holds since

$$
\begin{aligned}
&(2q-1)\left\{(q+1)^{2(k-1)} - q^{2(k-1)}\right\} - (2q+1)\left\{q^{2(k-1)} - (q-1)^{2(k-1)}\right\}\\
&= (2q-1)\{(q+1)^2 - q^2\}\left\{(q+1)^{2(k-2)} + (q+1)^{2(k-3)}q^2 + \cdots + (q+1)^2q^{2(k-3)} + q^{2(k-2)}\right\}\\
&\quad -(2q+1)\{q^2 - (q-1)^2\}\left\{q^{2(k-2)} + q^{2(k-3)}(q-1)^2 + \cdots + q^2(q-1)^{2(k-3)} + (q-1)^{2(k-2)}\right\}\\
&= (2q+1)(2q-1)\sum_{i=1}^{k-2}\left\{(q+1)^{2i} - (q-1)^{2i}\right\}q^{2(k-2-i)}\\
&= (2q+1)(2q-1)\sum_{i=1}^{k-2}\left\{(q+1)^2 - (q-1)^2\right\}\\
&\quad \times\left\{(q+1)^{2(i-1)} + (q+1)^{2(i-2)}(q-1)^2 + \cdots + (q-1)^{2(i-1)}\right\}q^{2(k-2-i)}\\
&= 4q(2q+1)(2q-1)\sum_{i=1}^{k-2}\left\{\sum_{j=0}^{i-1}(q+1)^{2j}(q-1)^{2(i-1-j)}\right\}q^{2(k-2-i)}\\
&= 4q(2q+1)(2q-1)\sum_{\substack{u,v,w\geq 0\\ u+v+w=k-3}}(q+1)^{2u}(q-1)^{2v}q^{2w}.
\end{aligned}
$$

Therefore the desired result holds. □

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. (i) Let $q \in (\frac{-1}{2}, \frac{1}{2}) \cup (1, \infty)$. Firstly we show that $\alpha \geq \frac{2q+1}{3}$ ensures

$$
J_q(x,1) = \frac{q}{q+1}\frac{x^{q+1}-1}{x^q-1} < (1-\alpha)\sqrt{x} + \alpha\frac{x+1}{2} = K_\alpha(x,1) \tag{3.6}
$$

for all $x > 0$ with $x \neq 1$.

If $q \neq 0$, by putting $x = e^{2t}$, (3.6) holds if and only if

$$\frac{q}{q+1}\frac{e^{(q+1)t}-e^{-(q+1)t}}{e^{qt}-e^{-qt}} < (1-\alpha)+\alpha\frac{e^t+e^{-t}}{2} \quad \text{for all } t \in \mathbb{R}\setminus\{0\}. \tag{3.7}$$

Since both sides of (3.7) are even functions, we have only to consider the case $t > 0$. Then, since $\frac{e^{qt}-e^{-qt}}{q} > 0$, (3.7) for $t > 0$ is equivalent to

$$\begin{aligned} f(t) &\equiv \frac{e^{qt}-e^{-qt}}{q}\left\{(1-\alpha)+\alpha\frac{e^t+e^{-t}}{2}\right\} - \frac{e^{(q+1)t}-e^{-(q+1)t}}{q+1} \\ &= \frac{2}{q}\sinh(qt)\{(1-\alpha)+\alpha\cosh t\} - \frac{2}{q+1}\sinh((q+1)t) > 0 \quad \text{for all } t > 0. \end{aligned} \tag{3.8}$$

Therefore we prove (3.8). By Taylor expansion, we have

$$\begin{aligned} f(t) &= \frac{2}{q}\left(qt+\frac{q^3t^3}{3!}+\frac{q^5t^5}{5!}+\cdots\right)\left\{(1-\alpha)+\alpha\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right)\right\} \\ &\quad - \frac{2}{q+1}\left\{(q+1)t+\frac{(q+1)^3t^3}{3!}+\frac{(q+1)^5t^5}{5!}+\cdots\right\} \\ &= 2\left(t+\frac{q^2}{3!}t^3+\frac{q^4}{5!}t^5+\cdots\right)\left(1+\frac{\alpha}{2!}t^2+\frac{\alpha}{4!}t^4+\cdots\right) \\ &\quad - 2\left\{t+\frac{(q+1)^2}{3!}t^3+\frac{(q+1)^4}{5!}t^5+\cdots\right\} \\ &= 2\sum_{k=2}^{\infty}\left\{\frac{q^{2(k-1)}}{(2k-1)!}+\sum_{i=2}^{k}\frac{q^{2(k-i)}\alpha}{(2i-2)!(2k+1-2i)!}-\frac{(q+1)^{2(k-1)}}{(2k-1)!}\right\}t^{2k-1} \\ &= 2\sum_{k=2}^{\infty}\phi_{k,q}(\alpha)\,t^{2k-1}, \end{aligned}$$

where

$$\phi_{k,q}(\alpha) \equiv \frac{q^{2(k-1)}}{(2k-1)!}+\sum_{i=2}^{k}\frac{q^{2(k-i)}\alpha}{(2i-2)!(2k+1-2i)!}-\frac{(q+1)^{2(k-1)}}{(2k-1)!} \quad \text{for } k=2,3,\ldots.$$

Then $\phi_{k,q}(\alpha) > 0$ if and only if

$$\alpha > \frac{(q+1)^{2(k-1)}-q^{2(k-1)}}{\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} = g_k(q).$$

If $q = 0$, by the similar argument, we can get

$$\phi_{k,0}(\alpha) \equiv \frac{\alpha}{(2k-2)!}-\frac{1}{(2k-1)!} \quad \text{for } k=2,3,\ldots,$$

so that $\phi_{k,0}(\alpha) > 0$ if and only if $\alpha > \frac{1}{2k-1} = g_k(0)$.

By (i) in Lemma 3.5, $q \in (\frac{-1}{2}, \frac{1}{2}) \cup (1, \infty)$ ensures that $g_2(q) > g_k(q)$ for all $k \geq 3$. Therefore, if $\alpha \geq \frac{2q+1}{3} = g_2(q)$, then $\phi_{2,q}(\alpha) \geq 0$ and $\phi_{k,q}(\alpha) > 0$ for all $k \geq 3$, that is, (3.8) holds.

On the other hand, if $\alpha < \frac{2q+1}{3} = g_2(q)$, then $\phi_{2,q}(\alpha) < 0$ holds, that is, $f(t) < 0$ for sufficiently small $t > 0$. Therefore (3.8) assures $\alpha \geq \frac{2q+1}{3}$.

We can prove (ii) similarly, so the proof is complete. □

4 Operator inequalities

Here, an operator means a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. We denote the set of all positive operators by $\mathcal{B}^+(\mathcal{H})$. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if

$$A \leq B \text{ implies } f(A) \leq f(B)$$

for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leq B$ means $B - A \geq 0$.

For two positive invertible operators A and B, the arithmetic mean $A \nabla B$, the geometric mean $A \sharp B$ and the harmonic mean $A \,!\, B$ are defined as follows:

$$A \nabla B = \frac{A+B}{2}, \quad A \sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}} \quad \text{and} \quad A \,!\, B = \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1}.$$

Kubo and Ando [11] constructed the general theory of operator means. A binary operation $(A, B) \in \mathcal{B}^+(\mathcal{H}) \times \mathcal{B}^+(\mathcal{H}) \to A \,\sigma\, B \in \mathcal{B}^+(\mathcal{H})$ in the cone of positive operators on $\mathcal{H}$ is called an operator mean if the following conditions are satisfied:

(i) $A \leq C$ and $B \leq D$ imply $A \,\sigma\, B \leq C \,\sigma\, D$ (monotonicity),

(ii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$ (upper semicontinuity),

(iii) $T^*(A \,\sigma\, B)T \leq (T^*AT) \,\sigma\, (T^*BT)$ for every operator T (transformer inequality),

(iv) $I \,\sigma\, I = I$ (normalized condition).

A binary operation σ is said to be an operator connection if (i), (ii) and (iii) are satisfied. We remark that an operator connection σ satisfies the transformer equality $T^*(A \,\sigma\, B)T = (T^*AT) \,\sigma\, (T^*BT)$ if T is invertible.

Moreover, they obtained in [11] that there exists a one-to-one correspondence between an operator mean σ and an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$.

We remark that f is called the representing function of σ, and also an operator mean σ can be defined by

$$A \sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{4.1}$$

if $A > 0$ and $B \geq 0$.

In this report, we use the notation like $\mathfrak{M}(A, B)$ for an operator mean $A \sigma B$. For $A, B > 0$, the representing functions of the arithmetic mean $\mathfrak{A}(A, B)$, the geometric mean $\mathfrak{G}(A, B)$, the harmonic mean $\mathfrak{H}(A, B)$ and the logarithmic mean $\mathfrak{L}(A, B)$ are

$$A(1, x) = \frac{x+1}{2}, \quad G(1, x) = \sqrt{x}, \quad H(1, x) = \frac{2x}{x+1} \quad \text{and} \quad L(1, x) = \frac{x-1}{\log x}.$$

Now it is permitted to consider binary operations given by (4.1) for general real-valued functions. The power difference mean $\mathfrak{J}_q(A, B)$ and the Heron mean $\mathfrak{K}_q(A, B)$ are given by $J_q(1, x)$ and $K_q(1, x)$, respectively. For $-2 \leq q \leq 1$, it is known in [2, 6, 9, 12] that $\mathfrak{J}_q(A, B)$ is increasing on q and $\mathfrak{J}_q(A, B)$ is an operator mean. Obviously $\mathfrak{K}_q(A, B)$ is an operator mean for $0 \leq q \leq 1$.

As an estimation of Heron mean for positive operators, Fujii, Furuichi and Nakamoto [3] showed the following result.

Proposition 4.A ([3]). *Let A and B be positive invertible operators and $r \in \mathbb{R}$. Then the following inequalities hold:*

(i) *If $r \geq 2$, then $rA \natural B + (1-r) A \nabla B \leq A \,!\, B$.*

(ii) *If $r \leq 1$, then $rA \natural B + (1-r) A \nabla B \geq A \,!\, B$.*

The conditions on r is optimal, that is,

$$\inf\{r \mid rA \natural B + (1-r) A \nabla B \leq A \,!\, B\} = 2 \quad \textit{and}$$
$$\sup\{r \mid rA \natural B + (1-r) A \nabla B \geq A \,!\, B\} = 1.$$

By Propositions 3.2 and 3.3, we can obtain an extension of Proposition 4.A immediately.

Theorem 4.1. *Let A and B be positive invertible operators.*

(i) *Let $q \in (0, \frac{1}{2}) \cup (1, \infty)$. Then*

$$\mathfrak{K}_{\frac{2q}{q+1}}(A, B) \leq \mathfrak{J}_q(A, B) \leq \mathfrak{K}_{\frac{2q+1}{3}}(A, B).$$

(ii) *Let* $q \in (\frac{1}{2}, 1)$. *Then*

$$\mathfrak{K}_{\frac{2q+1}{3}}(A,B) \le \mathfrak{J}_q(A,B) \le \mathfrak{K}_{\frac{2q}{q+1}}(A,B).$$

(iii) *Let* $q \in (\frac{-1}{2}, 0]$. *Then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_0(A,B) \le \mathfrak{J}_q(A,B) \le \mathfrak{K}_{\frac{2q+1}{3}}(A,B).$$

(iv) *Let* $q \in (-\infty, \frac{-1}{2})$. *Then*

$$\mathfrak{K}_{\frac{2q+1}{3}}(A,B) \le \mathfrak{J}_q(A,B) \le \mathfrak{K}_0(A,B) = \mathfrak{G}(A,B).$$

The given parameters of $\mathfrak{K}_\alpha(A,B)$ *in each case are best possible.*

Theorem 4.1 implies the following Corollary 4.2 by putting $q = 0, -2$. In Corollary 4.2, the second inequality in (i) is an operator version of Proposition 2.A, and also (ii) is just Proposition 4.A.

Corollary 4.2. *Let* A *and* B *be positive invertible operators. Then the following hold.*

(i) $\mathfrak{G}(A,B) = \mathfrak{K}_0(A,B) \le \mathfrak{L}(A,B) \le \mathfrak{K}_{\frac{1}{3}}(A,B)$.

(ii) $\mathfrak{K}_{-1}(A,B) \le \mathfrak{H}(a,b) \le \mathfrak{K}_0(A,B) = \mathfrak{G}(A,B)$.

The given parameters of $\mathfrak{K}_\alpha(A,B)$ *in each case are best possible.*

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References

[1] R. BHATIA, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra Appl., **413** (2006), 355–363.

[2] J. I. FUJII AND Y. SEO, *On parameterized operator means dominated by power ones*, Sci. Math., **1** (1998), 301–306.

[3] M. FUJII, S. FURUICHI AND R. NAKAMOTO, *Estimations of Heron means for positive operators*, J. Math. Inequal., **10** (2016), 19–30.

[4] S. FURUICHI, *Operator inequalities among arithmetic mean, geometric mean and harmonic mean*, J. Math. Inequal., **8** (2014), 669–672.

[5] S. FURUICHI AND K. YANAGI, *Bounds of the logarithmic mean*, J. Inequal. Appl. **2013**, 2013:535, 11 pp.

[6] T. FURUTA, *Concrete examples of operator monotone functions obtained by an elementary method without appearing to Löwner integral representation*, Linear Algebra Appl., **429** (2008), 972–980.

[7] Z.-Y. HE, M.-K. WANG AND Y.-M. CHU, *Optimal one-parameter mean bounds for the convex combination of arithmetic and logarithmic means*, J. Math. Inequal., **9** (2015), 699–707.

[8] F. HIAI AND H. KOSAKI, *Comparison of various means for operators*, J. Funct. Anal., **163** (1999), 300–323.

[9] F. HIAI AND H. KOSAKI, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J., **48** (1999), 899–936.

[10] M. ITO, *Estimations of power difference mean by Heron mean*, to appear in J. Math. Inequal..

[11] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[12] Y. UDAGAWA, S. WADA, T. YAMAZAKI AND M. YANAGIDA, *On a family of operator means involving the power difference means*, Linear Algebra Appl., **485** (2015), 124–131.

[13] W.-F. XIA, S.-W. HOU, G.-D. WANG AND Y.-M. CHU, *Optimal one-parameter mean bounds for the convex combination of arithmetic and geometric means*, J. Appl. Anal., **18** (2012), 197–207.

[14] W.-F. XIA, G.-D. WANG, Y.-M. CHU AND S.-W. HOU, *Sharp inequalities between one-parameter and power means*, Adv. Math. (China), **42** (2013), 713–722.

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