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Research Article

Homotheties of a Class of Spherically Symmetric Space-Time Admitting G_3 as Maximal Isometry Group

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The homotheties of spherically symmetric space-time admitting G_4, G_6, and G_{10} as maximal isometry groups are already known, whereas, for the space-time admitting G_3 as isometry groups, the solution in the form of differential constraints on metric coefficients requires further classification. For a class of spherically symmetric space-time admitting G_3 as maximal isometry groups without imposing any restriction on the stress-energy tensor, the metrics along with their corresponding homotheties are found. In one case, the metric is found along with its homothety vector that satisfies an additional constraint and is illustrated with the help of an example of a metric. In another case, the metric and the corresponding homothety vector are found for a subclass of spherically symmetric space-time for which the differential constraint is reduced to separable form. Stress-energy tensor and related quantities of the metrics found are given in the relevant section.

1. Introduction

General relativity (GR) [1] formulates a physical problem in terms of differential equations as a geometric requirement that a space-time may correspond to a Riemannian manifold as the interaction of matter and gravitation. A relation between the geometry and the distribution of matter in the space-time is expressed by the following Einstein's field equations (EFEs):

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \kappa T_{ab}. \quad (1)$$

GR is expressed in terms of Pseudo-Riemannian Geometry. The torsion free space (V_4) is represented by a Riemannian manifold M of four dimensions having signature $(+,-,-,-)$ with metric tensor g_{ab} and symmetric stress-energy tensor T_{ab} $(a, b = 0, 1, 2, 3)$. The curvature of the space-time is represented by the Riemann tensor R_{abcd}, where $R_{ab} = R^c_{acb}$ is the contraction of Riemann curvature tensor, $R = R^a_a$, the Ricci scalar, $\kappa = 8\pi G/c^4$ and Λ, the cosmological constant. The value of Λ is observed to be negligible and usually taken to be zero. The Λ term is only of significance at cosmological scale. In case of nonvanishing Λ, the Λ-term is usually treated as part of the stress-energy tensor T_{ab}.

EFEs (1) break down into highly nonlinear, second-order coupled partial differential equations and are difficult to handle in general unless certain symmetries are assumed by the space-time. Exact solutions [2] of EFEs (1) may be found by requiring certain symmetry property of a space-time and they have played a significant role in the discussion of physical problems, for example, the Kerr and Schwarzschild solutions for the final collapsed state of massive bodies. The exact solutions mostly arose from highly idealized physical problems requiring high symmetry as has been compiled by Stephani et al. [2], for example, the well-known spherically symmetric solutions of Schwarzschild, Reissner and Nordstrom, Tolman, and Friedmann. The known exact solutions may be classified into (at least) four classes [2], namely, the algebraic classification of conformal curvature, physical characterization of the energy momentum tensor, existence, and structure of preferred vector fields and group of motions. The groups of symmetries are used to construct more general cosmologies. One of these symmetries called homotheties of a space-time are more restrictive than the isometries of the space-time. They are useful to find the solutions of EFEs and their properties and they can model the universe to find new facts related to cosmology and singularities [3]. Classical hydrodynamics also has benefitted from the similarity solutions assuming the models for physical systems having no intrinsic scale of length, mass, or time [3]. Cahill and Taub [4] analysed the homotheties of the spherically symmetric distribution of self-gravitating perfect fluid satisfying the homothety (4). Taub [5] studied the homotheties of plane symmetric space-time underlining the physical significance of homotheties in GR. Godfrey [6] constructed all homothetic Wyel space-time. Collinson and French [7], Katzin et al. [8], and Collinson [9] studied more general geometric symmetries. Maartens et al. [10] found the general solution of the conformal Killing equations for the static spherically symmetric space-time and that for nonconformally flat space-time; there are at the most two proper conformal motions including the regular conformally flat space-time with the conformal motion at the center. Bin Farid et al. [11] classified static plane symmetric space-time according to their Ricci collineations (RCs) and their relation with isometries of the space-time. Sharif and Aziz [12, 13] studied kinematic self-similar solutions of plane and cylindrically symmetric space-time for the perfect fluid and dust. Bokhari et al. [14] and Saifullah and Shair-E-Yazdan [15] studied conformal motions in the context of plane symmetric static space-time. Moopanar and Maharaj [16] investigated the conformal geometry of spherically symmetric distribution of mass, obtaining the general conformal Killing symmetry subject to a number of integrability conditions. They also found the rare space-time that admit the inheriting conformal symmetry given by their (8) for their sheering spherically symmetric space-time (1) through their general conformal symmetry (5)-(6). Shabbir and Khan [17] utilize the algebraic and direct integration techniques to find self-similar vector fields in static spherically symmetric space-time including the orthogonal, parallel, and nonparallel nontilted proper self-similar vector fields for a special choice of the metric functions. Dorst [18] explores the geometrical but computational way of working out conformal motions in 3D. Manjonjo et al. [19] studied the relationship between conformal symmetries and relativistic spheres in astrophysics and exploit the nonvanishing components of the Weyl tensor to classify the conformal symmetries in static spherical space-time. Banerjee et al. [20] explored the possibility of finding the static and spherically symmetric anisotropic compact stars in general relativity that admit conformal motions in the framework of $f(R)$ gravity theory.

As mentioned above, one of such restrictions could be to allow space-time to admit certain symmetry properties. These symmetry properties lead space-time to obey a certain Lie group or an isometry group. The isometry group G_m of (M, g) is the Lie group of smooth maps of M into itself, leaving g invariant. The subscript m is equal to the number of generators or isometries of the group. It is the Lie algebra of continuously differentiable transformations $K = K^a(\partial/\partial x^a)$, where $K^a = K^a(x^b)$ are the components of the vector field K, known as a Killing vector (KV) field. A Killing vector field K is a field along which the Lie derivative of the metric tensor g is zero. That is,

$$\underset{K}{\mathscr{L}} g_{ab} = 0, \tag{2}$$

where $\mathscr{L}$ denotes the Lie derivative. Besides isometries, there are symmetries called the self-similar solutions of space-time which are more restrictive. These symmetry properties require space-time to admit a Lie group, for example, the conformal motions, homothetic motions, Ricci collineations, curvature collineations, and affine collineations. A homothety vector field $H = H^a(\partial/\partial x^a)$ is a field along which the Lie derivative of a metric tensor of space-time remains invariant up to a scale, given by

$$\underset{H}{\mathscr{L}} g_{ab} = 2\phi_0 g_{ab}, \tag{3}$$

where ϕ_0 is a scalar parameter, called the homothetic constant and H the homothetic vector field. Equation (3) may be rewritten in the component form given as follows:

$$H^c \nabla_c g_{ab} + g_{ac} \nabla_b H^c + g_{bc} \nabla_a H^c = 2\phi_0 g_{ab}. \tag{4}$$

The corresponding homothety group is denoted by H_r and the subscript r is the number of generators of the group. For $\phi_0 = 0$, the homotheties become motions but the converse may not be true.

It is well known that, for a Riemannian space V_n, the maximal group of motions is of the order less than or equal to $n(n+1)/2$. Fubini [21] has proved that a Riemannian manifold V_n cannot admit a maximal group of the order $n(n+1)/2 - 1$. Yegorov [22] proved a result for Lorentzian manifolds, according to which the maximum group of mobility cannot be of the order $n(n+1)/2 - 2$. It is well known [3] that, for a Riemannian manifold with metric g_{ab} and admitting G_m as the maximal group of isometries, H_r could be at the most of the order $r = m + 1$. Thus for a V_n, H_r could be at the most of the order $r = n(n+1)/2 + 1$. The results of Fubini and Yegorov show that space-time V_n cannot admit a homothety group H_r with $r = n(n+1)/2 - i$, where $i = 0, 1$. A detailed discussion of general relationship between isometries and homothetic motions can be seen in the work [23, 24]. It is known that, for V_n, there could be at the most $[n(n+1)/2] + 1$ homothetic motions. For the spherically symmetric space-time,

$$ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - e^{x(t,r)} d\Omega^2, \tag{5}$$

$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\phi^2$, it is known [25] that the spherically symmetric space-time cannot admit G_5 as the maximal group of motions. Qadir and Ziad [26] proved that the spherically symmetric space-time allow isometry group G_m of dimension $m = 3, 4, 6, 7, 10$. Therefore, these kinds of space-time could admit homothety groups H_r of dimension $r = 4, 5, 7, 8, 11$. To find which space-time admits a nontrivial homothety, the authors [27] solved the homothety equations for the spherically symmetric space-time for all the possible cases within the aforementioned class of space-time. It came out that $r \neq 8$. Thus, for spherically symmetric space-time, the possible maximal homothety groups H_r could be of the order $r = 4, 5, 7, 11$. The solution of the homothetic (4), as discussed in [27], results in the possible metrics along with the homothety groups H_5, H_7, and H_{11}. However, for the space-time admitting G_3 as the maximal isometry group, the solution of the homothety (4) is provided in the form of derivatives of unknown metric coefficients, which then requires a further classification for the case of homothety group H_4.

In order to find the space-time along with their respective homothety groups, one needs to solve (4) for the spherically symmetric space-time (5); for details and background, we refer the reader to [27]. For $x(t, r) = \mu(t, r) + 2\ln r$, the solution of the homothety (4) for the spherically symmetric space-time (5) (dot and dash below denote derivatives with respect to t and r, respectively) is given by

$$H = H^0 \frac{\partial}{\partial t} + H^1 \frac{\partial}{\partial r} + H^2 \frac{\partial}{\partial \vartheta} + H^3 \frac{\partial}{\partial \varphi}, \tag{6}$$

where

$$\begin{aligned} H^0 &= -r^2 e^{\mu-\nu} \left(\sin\vartheta\left(\dot{g}_1 \sin\varphi - \dot{g}_2 \cos\varphi\right) + \dot{g}_3 \cos\varphi\right) \\ &\quad + g_4, \\ H^1 &= r^2 e^{\mu-\lambda} \left(\sin\vartheta\left({g'}_1 \sin\varphi - {g'}_2 \cos\varphi\right) + {g'}_3 \cos\vartheta\right) \\ &\quad + g_5, \\ H^2 &= -\cos\vartheta\left(g_1 \sin\varphi - g_2 \cos\varphi\right) + g_3 \sin\vartheta \\ &\quad + \left(c_1 \sin\varphi - c_2 \cos\varphi\right), \\ H^3 &= -\cos ec\vartheta\left(g_1 \cos\varphi + g_2 \sin\varphi\right) \\ &\quad + \cot\vartheta\left(c_1 \cos\varphi + c_2 \sin\varphi\right) + c_3, \end{aligned} \tag{7}$$

where c_j $(j = 1, 2, 3)$ correspond to the generators of $SO(3)$ and g_k for $k = 1, 2, 3, 4, 5$ are functions of t and r, and they satisfy the following constraints:

$$-\dot{x} e^{x-\nu} \dot{g}_j + x' e^{x-\lambda} {g'}_j + 2 g_j = 0, \tag{8}$$

$$2\ddot{g}_j + (2\dot{x} - \dot{\nu}) \dot{g}_j - \nu' e^{\nu-\lambda} + 2{g'}_j = 0, \tag{9}$$

$$2\dot{g}'_j + \left(x' - \nu'\right) \dot{g}_j + \left(\dot{x} - \dot{\lambda}\right) {g'}_j = 0, \tag{10}$$

$$2{g''}_j + \left(2x' - \lambda'\right) {g'}_j - e^{\lambda-\nu} \dot{g}_j = 0, \tag{11}$$

$$\dot{x} g_4 + x' g_5 = 2\varphi_0, \tag{12}$$

$$2\dot{g}_4 + \dot{\nu} g_4 + \nu' g_5 = 2\varphi_0, \tag{13}$$

$$e^{\nu} {g'}_4 - e^{\lambda} \dot{g}_5 = 0, \tag{14}$$

$$2{g'}_5 + \lambda' g_5 + \dot{\lambda} g_4 = 2\varphi_0. \tag{15}$$

A complete solution of above (8)–(15) provides all possible metrics with their homotheties admitting homothety groups H_5, H_7, H_{11} except for homothety group H_4 for which the space-time should admit additional differential constraints. In [27], it is found that, for $r = 5$, there are three kinds of space-time (2.1)–(2.3) admitting $G_4 \equiv SO(3) \otimes R$ where R is time-like, space-like, and null, respectively, including all static spherically symmetric space-time and the Bertotti-Robinson metrics; for $r = 7$ there are four kinds of space-time (Robertson-Walker (RW) space-time (3.1) with (3.5) and a RW-like space-time (3.3) with (3.9)) which admit $G_6 \equiv SO(4), SO(3) \otimes R^3, SO(1,3)$ as the maximal isometry groups; for $r = 11$, the only space-time is Minkowski space-time, for which the maximal group of isometries is $SO(1,3) \otimes R^4$; for $r = 4$, H_4 as the maximal group of homotheties of the space-time admitting $G_3 \equiv SO(3)$ as the maximal isometry group satisfy additional differential constraints (4.3)–(4.7), which require further classification according to different types of stress-energy tensor as done by, for example, Cahill and Taub [4].

In this paper, we find homotheties of a class of the spherically symmetric space-time (5) admitting $G_3 \equiv SO(3)$ as the maximal isometry group for $x(t, r) = 2\ln r$, imposing no restriction on the stress-energy tensor. We accomplish this task in Section 2. This gives rise to the two cases that either $\dot{\lambda} = 0$ or $\dot{\lambda} \neq 0$. In the former case, the metric found is given by (26) along with its homothety vector (27). In particular for $\nu(t, r) = t/r$, $h(t) = t$, the corresponding metric and the homothety vector are given by (30) and (31), respectively. In the latter case, the metric and the homothety vector are given by (46) and (47), for a subclass of spherically symmetric space-time (5) for $\nu = \nu(r)$ for which one of the constraint equations is reduced to separable form. Furthermore, we have included Ricci tensor components R_{ab}, Ricci scalar R, and the stress-energy tensor T_{ab} of space-time (26), (30), and (46) in the relevant section. The results and remarks are presented in Section 3.

2. Spherically Symmetric Space-Time Admitting H_4 as the Homothety Group

For the space-time (5) to have $SO(3)$ as the maximal isometry group, one must have $g_j(t, r) = 0$, for which (8)–(11) are satisfied, where $j = 1, 2, 3$. However, for space-time to have homothetic motion, g_4 and g_5 satisfying (12)–(15) should include only one arbitrary constant, ϕ_0 corresponding to the scale parameter of the homothety. For details, we refer the reader to the [27]. Let us suppose that

$$\begin{aligned} g_4 &= \phi_0 h(t, r), \\ g_5 &= \phi_0 g(t, r). \end{aligned} \tag{16}$$

For $g_j(t, r) = 0$ along with g_4 and g_5 as given above, (12)–(15) reduce to

$$\begin{aligned} \dot{x} h + \dot{x} g &= 2, \\ 2\dot{h} + \dot{\nu} h + \nu' g &= 2, \\ e^{\nu} h' - e^{\lambda} \dot{g} &= 0, \\ 2g' + \lambda' g + \dot{\lambda} h &= 2. \end{aligned} \tag{17}$$

Corresponding homothety vector (6) in this case reduces to

$$H = g_4 \frac{\partial}{\partial t} + g_5 \frac{\partial}{\partial r} + (c_1 \sin\varphi - c_2 \cos\varphi) \frac{\partial}{\partial \vartheta} + (\cot\vartheta (c_1 \cos\varphi + c_2 \sin\varphi) + c_3) \frac{\partial}{\partial\varphi}. \tag{18}$$

As mentioned above, an attempt to solve (17), without imposing any restriction on stress-energy tensor T_{ab} or line element, produces a solution in the form of differential constraints. However, for $x(t,r) = 2\ln r$ these differential constraints are reduced to meaningful expressions which then produce the metrics admitting the above-mentioned homothety groups. For $x = 2\ln r$, the line element (5) comes out to be

$$ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - r^2 d\Omega^2, \tag{19}$$

and (17) yield

$$g = r, \tag{20}$$

$$2\dot{h} + \dot{\nu} h + \nu' r = 2, \tag{21}$$

$$h' = 0, \quad e^{\nu} \neq 0, \tag{22}$$

$$\lambda' r + \dot{\lambda} h = 0. \tag{23}$$

From (23), we find

$$h(t,r) = -\frac{\lambda' r}{\dot{\lambda}}. \tag{24}$$

For $\dot{\lambda} \neq 0$, (24) along with (22) gives us

$$\left(\frac{\lambda' r}{\dot{\lambda}}\right)' = 0, \quad \dot{\lambda} \neq 0, \ \lambda = \lambda(t,r). \tag{25}$$

Here two cases arise, either $\dot{\lambda} = 0$ or $\dot{\lambda} \neq 0$:

Case 1. For $\dot{\lambda} = 0$, (23) gives us $\lambda' = 0$ as $r \neq 0$, which is possible only when $\lambda = 0$. Thus $\nu = \nu(t,r)$ and $\lambda = 0$ reduce (19) to the metric

$$ds^2 = e^{\nu(t,r)} dt^2 - dr^2 - r^2 d\Omega^2, \tag{26}$$

that satisfies (20), (22), and (23) with an additional constraint on $\nu(t,r)$ given by (21), where $h = h(t)$. The corresponding homothety vector H (18) in this case reduces to

$$H = \phi_0 \left(h(t) \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \right) + (c_1 \sin\varphi - c_2 \cos\varphi) \frac{\partial}{\partial\vartheta} + (\cot\vartheta (c_1 \cos\varphi + c_2 \sin\varphi) + c_3) \frac{\partial}{\partial\varphi}. \tag{27}$$

For the space-time (26), Ricci scalar and the independent nonzero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= \frac{1}{4r}\left(4\nu' + r\nu'^2 + 2r\nu''\right) e^{\nu(t,r)}, \\ R_{11} &= -\frac{1}{4}\left(\nu'^2 + 2\nu''\right), \\ R_{22} &= \frac{1}{2} r\nu', \\ R_{33} &= \frac{1}{2} r\nu' \sin^2\theta, \\ R &= \frac{1}{2r}\left(4\nu' + r\nu'^2 + 2r\nu''\right), \end{aligned} \tag{28}$$

and the components of stress-energy tensor and the stress-energy tensor T for the above space-time (26) are given as follows:

$$\begin{aligned} \kappa T_{00} &= 0, \\ \kappa T_{11} &= \frac{\nu'}{r}, \\ \kappa T_{22} &= \frac{1}{4} r \left(\nu'\left(r\nu' + 2\right) + 2r\nu''\right), \\ \kappa T_{33} &= \kappa T_{22} \sin^2\theta, \\ \kappa T &= -\frac{\nu'\left(r\nu' + 4\right)}{2r} - \nu''. \end{aligned} \tag{29}$$

Thus the case for $\dot{\lambda} = 0$ results in a metric (26), which seems to be uninteresting as the energy density of the corresponding self-gravitating system given by (29) is zero which is not possible for a physical system. This means that every temporal metric coefficient $\nu(t,r)$ for $\dot{\lambda} = 0$ leads to nonphysical zones of the space-time. For example, the space-time (26) for $\nu(t,r) = t/r$ and $h(t) = t$ reduces to

$$ds^2 = e^{t/r} dt^2 - dr^2 - r^2 d\Omega^2. \tag{30}$$

The above space-time (30) satisfies the additional constraint (21). In this case, the homothety vector H (27) of the above space-time (30) is given by

$$H = \phi_0 \left(t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \right) + (c_1 \sin\varphi - c_2 \cos\varphi) \frac{\partial}{\partial\vartheta} + (\cot\vartheta (c_1 \cos\varphi + c_2 \sin\varphi) + c_3) \frac{\partial}{\partial\varphi}, \tag{31}$$

clearly showing that the space-time (30) admits four homotheties. For the space-time (30), Ricci scalar and the independent nonzero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= \frac{t^2}{4r^4} e^{t/r}, \\ R_{11} &= -\frac{t}{r^4}\left(r + \frac{t}{4}\right), \\ R_{22} &= \frac{t}{2r}, \\ R_{33} &= R_{22} \sin^2\theta, \\ R &= \frac{t^2}{2r^4}. \end{aligned} \tag{32}$$

Using EFEs (1) (without cosmological constant), the components of stress-energy tensor and the stress-energy tensor T for the above space-time (30) are given by the following:

$$\begin{aligned} \kappa T_{00} &= 0, \\ \kappa T_{11} &= -\frac{t}{r^3}, \\ \kappa T_{22} &= \frac{t(2r+t)}{4r^2}, \\ \kappa T_{33} &= \kappa T_{22}\sin^2\theta, \\ \kappa T &= -\frac{t^2}{2r^4}. \end{aligned} \tag{33}$$

As mentioned above, the case for $\dot{\lambda} = 0$ results in a metric for which the energy density of the spherically symmetric space-time is zero. In particular, for example, for $\nu(t,r) = t/r$, the metric (26) reduces to the metric (30) with stress-energy tensor components (33) corresponding to a self-gravitating-source with zero-energy density, a characteristic of the metric that every choice of the temporal metric coefficient leads to some nonphysical zones of the space-time.

Case 2. Now for $\dot{\lambda} \neq 0$, (25) suggests that

$$\frac{\lambda' r}{\dot{\lambda}} = \alpha(t), \tag{34}$$

where $\alpha(t)$ depends on t only. Equation (24) along with (34) yields

$$h(t,r) = -\alpha(t). \tag{35}$$

Substituting (35) in (21) yields

$$-2\dot{\alpha}(t) + \dot{\nu}h + \nu' r = 2. \tag{36}$$

Note that (36) is separable for $\dot{\nu} = 0$. Thus, we may rewrite above equation as

$$\begin{aligned} \dot{\alpha}(t) &= \frac{\nu' r - 2}{2}, \\ \dot{\nu} &= 0. \end{aligned} \tag{37}$$

In (37), the LHS is function of time t only whereas RHS is function of r only, which is possible only when

$$\dot{\alpha}(t) = \frac{\nu' r - 2}{2} = \alpha_1, \tag{38}$$

where α_1 is separation constant and (38) can be solved by separating it into two parts. We get

$$\alpha(t) = \alpha_1 t + \alpha_2, \tag{39}$$

where α_2 is a constant of integration and

$$\nu(r) = 2(\alpha_1 + 1)\ln r. \tag{40}$$

Equations (16) along with (20), (35), and (39) reduce to

$$\begin{aligned} g_4 &= -\phi_0(\alpha_1 t + \alpha_2), \\ g_5 &= \phi_0 r. \end{aligned} \tag{41}$$

We find now $\lambda(t,r)$. Equations (34) and (39) together may be written as

$$\lambda' r = \dot{\lambda}(\alpha_1 t + \alpha_2). \tag{42}$$

By the same argument used for (38), (42) is possible only when

$$\lambda' r = \dot{\lambda}(\alpha_1 t + \alpha_2) = m, \tag{43}$$

where m is a constant. Thus, from (43) we get

$$\begin{aligned} \lambda(r) &= m\ln r, \\ \lambda(t) &= \frac{m}{\alpha_1}\ln(\alpha_1 t + \alpha_2). \end{aligned} \tag{44}$$

Equation (44) implies that

$$\lambda(t,r) = m\ln r + \frac{m}{\alpha_1}\ln(\alpha_1 t + \alpha_2), \quad \alpha_1 \neq 0. \tag{45}$$

Equation (19) along with (40) and (45) is given by the following metric:

$$ds^2 = r^{2(\alpha_1+1)}dt^2 - r^m(\alpha_1 t + \alpha_2)^{m/\alpha_1}dr^2 - r^2 d\Omega^2. \tag{46}$$

Thus the corresponding homothety vector H, (18) is given by

$$\begin{aligned} H = &-\phi_0(\alpha_1 t + \alpha_2)\frac{\partial}{\partial t} + (\phi_0 r)\frac{\partial}{\partial r} \\ &+ (c_1\sin\varphi - c_2\cos\varphi)\frac{\partial}{\partial\vartheta} \\ &+ (\cot\vartheta(c_1\cos\varphi + c_2\sin\varphi) + c_3)\frac{\partial}{\partial\varphi}. \end{aligned} \tag{47}$$

Here the arbitrary constants are ϕ_0, c_1, c_2, c_3 whereas α_1 and α_2 are constants on metric. So that the generators of homothety group H_4 are

$$\begin{aligned} X^0 &= -(\alpha_1 t + \alpha_2)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \\ X^1 &= \sin\varphi\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi}, \\ X^2 &= -\cos\varphi\frac{\partial}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi}, \\ X^3 &= \frac{\partial}{\partial\varphi}, \end{aligned} \tag{48}$$

where $[X^0, X^i] = 0$ and $[X^i, X^j] = X^k$ for $i, j, k = 0, 1, 2, 3, i \neq j \neq k$, showing that there are four homothety vectors given by (48). For the space-time (46), the independent nonzero components of the Ricci tensor and the Ricci scalar are as follows:

$$R_{00}=\alpha^{-2}\Big[-1+\frac{m\alpha_1}{2}+(1+\alpha_1)\Big(3r^{-2(\alpha_1-1)}\alpha^{2(\alpha_1-1)/\alpha_1} - r^{2\alpha_1-1}(1+\alpha_1)\alpha^{(2\alpha_1-1)/\alpha_1} + r^{2\alpha_1-m}(1-m+2\alpha_1)\alpha^{(2\alpha_1-m)/\alpha_1}\Big)\Big],$$

$$R_{01}=\frac{2}{r\alpha},$$

$$R_{11}=r^{-2}\Big[2+2(1+\alpha_1)-(1+\alpha_1)^2 - r^{-2(\alpha_1-1)}\alpha^{-2(\alpha_1-1)/\alpha_1}+\frac{1}{2}mr^{m-2\alpha_1}(m-\alpha_1)\cdot\alpha^{(m-2\alpha_1)/\alpha_1}\Big],$$

$$R_{22}=r^{-3}\Big[r^3-r\alpha_1\alpha^{-2/\alpha_1}+(m-1)r^{3-m}\alpha^{-m/\alpha_1} + \alpha^{-3/\alpha_1}\big(1-3r\alpha^{1/\alpha_1}\big)\Big],\tag{49}$$

$$R_{33}=\left(R_{22}+\frac{1-r\alpha^{\alpha_1^{-1}}}{r^4\alpha^{4\alpha_1^{-1}}}\right)\sin^2\theta,$$

$$R=r^{-6}\Big[-\frac{1}{2}m^2r^{4-2\alpha_1}\alpha^{-2}+\alpha^{-m/\alpha_1}r^{4-m}\big(3\alpha_1^2-3m - \alpha_1 m+3\alpha_1\big)+\alpha_1 m r^{4-2\alpha_1}\alpha^{-2} + r^{-2\alpha_1-m+6}\alpha^{(-2\alpha_1-m+2)/\alpha_1}-r^{4-2\alpha_1}\alpha^{-2}-2r^4 - r^3\alpha^{-1/\alpha_1}\big(\alpha_1^2+2\alpha_1+1\big)+r^2\alpha^{-2/\alpha_1}(5\alpha_1+9) - r\alpha^{-3/\alpha_1}-\alpha^{-4/\alpha_1}\Big].$$

The stress-energy tensor for the above space-time (30) from EFEs (1) (without cosmological constant) comes out to be

$$\kappa T_{00}=\frac{1}{4}\Big[\big(m^2-2\big)\alpha^2-2\alpha^{(2-m)/\alpha_1-2}r^{2-m} + \alpha^{-(m+2)/\alpha_1}\Big(2\alpha^{2/\alpha_1}r^2\big({\alpha_1}^2-m\alpha_1+3\alpha_1+m+2\big) - 2\alpha^{m/\alpha_1}r^m\big(\alpha^{1/\alpha_1}(\alpha_1+1)^2r-\alpha_1\big)\Big)r^{2\alpha_1-m-2} + 2\alpha^{-4/\alpha_1}\big(2\alpha^{4/\alpha_1}r^4-3\alpha^{2/\alpha_1}r^2+\alpha^{1/\alpha_1}r+1\big)\cdot r^{2\alpha_1-4}\Big],\tag{50}$$

$$\kappa T_{01}=\frac{2}{\alpha r},\tag{51}$$

$$\kappa T_{11}=\frac{1}{4}\Big[\big(m^2-2\big)\alpha^{(m-2\alpha_1)/\alpha_1}r^{-2\alpha_1+m-2} - 2\alpha^{(m-4)/\alpha_1}r^{m-6}-2\alpha^{(m-3)/\alpha_1}r^{m-5} + 18r^{m-4}\alpha^{(m-2)/\alpha_1}-2\alpha^{(m-1)/\alpha_1}r^{m-3}-4\alpha^{m/\alpha_1}r^{m-2} + 2\alpha_1^2r^{-3}\big(r-\alpha^{(m-1)/\alpha_1}r^m\big)-6r^{-2}(m-2) - 2\alpha_1\big(2\alpha^{(m-1)/\alpha_1}r^{m+1}-5\alpha^{(m-2)/\alpha_1}r^m+mr^2 - 3r^2\big)r^{-4}-2\alpha^{2/\alpha_1-2}r^{-2\alpha_1}\Big],\tag{52}$$

$$\kappa T_{22}=\frac{1}{2}\Big[3r^{-m}\alpha^{-m/\alpha_1}+\alpha^{-2/\alpha_1}r^{-2}\big(3-2r\alpha^{1/\alpha_1}\big) + m\big(r^{-2\alpha_1}\alpha^{-2}-r^{-m}\alpha^{-m/\alpha_1}\big)-2^{-1}\big(2+m^2\big) \cdot r^{-2\alpha_1}\alpha^{-2}+r^{2-m-2\alpha_1}\alpha^{-2+(2-m)/\alpha_1}-\alpha^{-4/\alpha_1}\big(-1 + r\alpha^{1/\alpha_1}+3r^2\alpha^{2/\alpha_1}-r^3\alpha^{3/\alpha_1}\big)r^{-4}-(2+m) \cdot r^{-m}\alpha^{-m/\alpha_1}+\alpha_1-\big(\alpha^{-1/\alpha_1}r^{-1}-3r^{-m}\alpha^{-m/\alpha_1}\big)\alpha_1^2\Big],\tag{53}$$

$$\kappa T_{33}=\Big[\kappa T_{22}+\alpha^{-4/\alpha_1}r^{-4}\big(1-r\alpha^{1/\alpha_1}\big)\Big]\sin^2\theta,\tag{54}$$

$$\kappa T=r^{-6}\Big[\frac{1}{2}m^2r^{4-2\alpha_1}\alpha^{-2}-\alpha^{-m/\alpha_1}r^{4-m}\big(3\alpha_1^2-3m - \alpha_1 m+3\alpha_1\big)-\alpha_1 m r^{4-2\alpha_1}\alpha^{-2} - r^{-2\alpha_1-m+6}\alpha^{(-2\alpha_1-m+2)/\alpha_1}+r^{4-2\alpha_1}\alpha^{-2}+2r^4 + r^3\alpha^{-1/\alpha_1}\big(\alpha_1^2+2\alpha_1+1\big)-r^2\alpha^{-2/\alpha_1}(5\alpha_1+9) + r\alpha^{-3/\alpha_1}+\alpha^{-4/\alpha_1}\Big].\tag{55}$$

The expression for the stress-energy tensor shows that the symmetries assumed correspond to more complicated dynamics in the case of H_4 for $\dot{\lambda}\neq 0$ (see (25)). The metric in this case (46) can be further analysed for different types of stress-energy tensor for which the corresponding homothety vector is given by (47) and the stress-energy tensor and its components by (50) to (54) and (55). For example, the presence of the term T_{01} suggests the heat dissipation effects in the evolution of self-similar space-time. The role of heat dissipation has utmost relevance in the study of one of the interesting phenomena of the stellar structures, that is, the behavior of spherically symmetric gravitational collapse of stellar masses. This process is highly dissipative in nature [28]. In order to see the role of radiations in the gravitational implosion, one must has nonzero values of T_{01}. One can see from (51) that the quantity $\alpha(t)=\alpha_1 t+\alpha_2$ given by (39) is controlling the effects of heat dissipation in the evolution of self-gravitating system. It can also be seen that $\alpha(t)=\alpha_1 t+\alpha_2$ (see (39)) depends further on two integration constants, that is, α_1 and α_2. The presence of these two constants α_1 and α_2 in (51) indicates the dissipation effects in our analysis. It is interesting to mention that one cannot see the dynamics of nonradiating system from our results, as α_1 and α_2 cannot be zero simultaneously (as such a choice will assign undefined values to T_{01}). However, for $\alpha_2 = 0$, κT_{00} reduces to $2/rt\alpha_1$, thereby indicating the dependence of radial metric coefficient on both t and r.

3. Conclusions

We have found the homotheties and corresponding metrics for a class of spherically symmetric space-time (5) admit G_3 as the maximal group of motions for $x(t,r) = 2\ln r$. The motivation behind this is the classification of spherically symmetric space-time according to their homotheties without imposing any restriction on the stress-energy tensor. The homotheties and the corresponding metrics are already known for the space-time admitting G_4, G_6, and G_{10} as maximal isometry groups whereas for the space-time admitting G_3 as the isometry group the solution is known in the form of differential constraints which needs further consideration. We found the homotheties and the corresponding metrics admitting G_3 as the maximal group of isometries for a class of spherically symmetric space-time (5), as mentioned above. For a subclass of spherically symmetric space-time, for which $\dot{\lambda} = 0$, the metric is given by (26) and the corresponding homothety vector is given by (27) subject to the additional constraint (21) in terms of derivatives of the metric coefficients. In particular for $\nu(t,r) = t/r$, $h(t) = t$, the metric satisfying the additional constraint (21) and the corresponding homothety vector are given by the (30) and (31), respectively, whereas, for $\dot{\lambda} \neq 0$, $\dot{\nu} = 0$ the metric (5) reduces to the metric (46) and the corresponding homothety vector is (47). Stress-energy tensor for the above space-time is given by (33) and (50)–(55). It might be interesting to see these results according to different types of stress-energy tensor.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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