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Research Article

Fekete-Szegö Problems for Quasi-Subordination Classes

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An analytic function f is quasi-subordinate to an analytic function g, in the open unit disk if there exist analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclasses are derived.

1. Introduction and Motivation

Let $\mathcal{A}$ be the class of analytic function f in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For two analytic functions f and g, the function f is *subordinate* to g, written as follows:

$$f(z) \prec g(z), \tag{1.1}$$

if there exists an analytic function w, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \tag{1.2}$$

where ϕ is an analytic function with positive real part in $\mathbb{D}$, $\phi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{S}^*(\phi)$ is called Ma-Minda starlike (with respect to ϕ). The class $\mathcal{C}(\phi)$ is the class of functions $f \in \mathcal{A}$ for which $1 + zf''(z)/f'(z) \prec \phi(z)$. The class $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ include several well-known subclasses of starlike and convex functions as special case.

In the year 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is *quasi-subordinate* to g, written as follows:

$$f(z) \prec_q g(z), \tag{1.3}$$

if there exist analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Observe that when $\varphi(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in $\mathbb{D}$. Also notice that if $w(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that f is *majorized* by g and written $f(z) \ll g(z)$ in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4–6] for works related to quasi-subordination.

Throughout this paper it is assumed that ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. Motivated by [2, 3], we define the following classes.

Definition 1.1. Let the class $\mathcal{S}_q^*(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$\frac{zf'(z)}{f(z)} - 1 \prec_q \phi(z) - 1. \tag{1.4}$$

Example 1.2. Since

$$\frac{zf'(z)}{f(z)} - 1 = z\left(\phi(z) - 1\right) \prec_q \phi(z) - 1, \tag{1.5}$$

the function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$f(z) = z \exp\left(-z + \int_0^z \phi(\xi)\,d\xi\right) \tag{1.6}$$

belongs to the class $\mathcal{S}_q^*(\phi)$.

Definition 1.3. Let the class $\mathcal{C}_q(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$\frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1. \tag{1.7}$$

Example 1.4. The function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$f(z) = \int_0^z \exp\left(-\zeta + \int_0^\zeta \phi(\xi)d\xi\right)d\zeta \tag{1.8}$$

belongs to the class $\mathcal{C}_q(\phi)$.

The classes $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$ are analogous to the Ma-Minda starlike and convex classes defined in the form of quasi-subordination.

Definition 1.5. Let the class $\mathcal{R}_q(\phi)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$f'(z) - 1 \prec_q \phi(z) - 1. \tag{1.9}$$

Example 1.6. The function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$f(z) = z - \frac{z^2}{2} + \exp\left(\int_0^z \phi(\xi)d\xi\right) \tag{1.10}$$

belongs to the class $\mathcal{R}_q(\phi)$.

It is known that a function $f \in \mathcal{A}$ with $\operatorname{Re} f'(z) > 0$ in $\mathbb{D}$ is univalent. The above class of functions defined in terms of the quasi-subordination is associated with the class of functions with positive real part.

Functions in the following classes, $\mathcal{M}_q(\alpha, \phi)$ and $\mathcal{L}_q(\alpha, \phi)$ are analogous to the α-convex functions of Miller et al. [7] and α-logarithmically convex functions introduced by Lewandowski et al. [8] (see also [9]), respectively.

Definition 1.7. Let the class $\mathcal{M}_q(\alpha, \phi)$, $(\alpha \geq 0)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \phi(z) - 1. \tag{1.11}$$

Example 1.8. The function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = z(\phi(z) - 1) \tag{1.12}$$

belongs to the class $\mathcal{M}_q(\phi)$.

Definition 1.9. Let the class $\mathcal{L}_q(\alpha,\phi)$, $(\alpha \geq 0)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \prec_q \phi(z) - 1. \tag{1.13}$$

Example 1.10. The function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 = z(\phi(z) - 1) \tag{1.14}$$

belongs to the class $\mathcal{L}_q(\phi)$.

It is well known (see [10]) that the n-th coefficient of a univalent function $f \in \mathcal{A}$ is bounded by n. The bounds for coefficient give information about various geometric properties of the function. Many authors have also investigated the bounds for the Fekete-Szegö coefficient for various classes [11–25]. In this paper, we obtain coefficient estimates for the functions in the above defined classes.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. We need the following lemma to prove our results.

Lemma 1.11 (see [26]). *If $w \in \Omega$, then for any complex number t*

$$\left|w_2 - tw_1^2\right| \leq \max\{1; |t|\}. \tag{1.15}$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

2. Main Results

Although Theorems 2.1 and 2.4 are contained in the corresponding results for the classes $\mathcal{M}_q(\alpha,\phi)$ and $\mathcal{L}_q(\alpha,\phi)$, they are stated and proved separately here because of the importance of the classes.

Throughout, let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, $\varphi(z) = c_0 + c_1z + c_2z^2 + c_3z^3 + \cdots$, $B_1 \in \mathbb{R}$ and $B_1 > 0$.

Theorem 2.1. *If $f \in \mathcal{A}$ belongs to $\mathcal{S}_q^*(\phi)$, then*

$$|a_2| \leq B_1,$$
$$|a_3| \leq \frac{1}{2}\left(B_1 + \max\left\{B_1, B_1^2 + |B_2|\right\}\right), \tag{2.1}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2}\left(B_1 + \max\left\{B_1, |1-2\mu|B_1^2 + |B_2|\right\}\right). \tag{2.2}$$

Proof. If $f \in \mathcal{S}_q^*(\phi)$, then there exist analytic functions φ and w, with $|\varphi(z)| \le 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} - 1 = \varphi(z)\big(\phi(w(z)) - 1\big). \tag{2.3}$$

Since

$$\begin{aligned} \frac{zf'(z)}{f(z)} - 1 &= a_2 z + \left(-a_2^2 + 2a_3\right)z^2 + \cdots, \\ \phi(w(z)) - 1 &= B_1 w_1 z + \left(B_1 w_2 + B_2 w_1^2\right)z^2 + \cdots, \end{aligned} \tag{2.4}$$

$$\varphi(z)\big(\phi(w(z)) - 1\big) = B_1 c_0 w_1 z + \left(B_1 c_1 w_1 + c_0\left(B_1 w_2 + B_2 w_1^2\right)\right)z^2 + \cdots, \tag{2.5}$$

it follows from (2.3) that

$$\begin{aligned} a_2 &= B_1 c_0 w_1 \\ a_3 &= \frac{1}{2}\left(B_1 c_1 w_1 + B_1 c_0 w_2 + c_0\left(B_2 + B_1^2 c_0\right)w_1^2\right). \end{aligned} \tag{2.6}$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{D}$, we have [27, page 172]

$$|c_n| \le 1 - |c_0|^2 \le 1 \quad (n > 0). \tag{2.7}$$

By using this fact and the well-known inequality, $|w_1| \le 1$, we get

$$|a_2| \le B_1. \tag{2.8}$$

Further,

$$a_3 - \mu a_2^2 = \frac{1}{2}\left(B_1 c_1 w_1 + c_0\left(B_1 w_2 + \left(B_2 + B_1^2 c_0 - 2\mu B_1^2 c_0\right)w_1^2\right)\right). \tag{2.9}$$

Then

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2}\left(|B_1 c_1 w_1| + \left|B_1 c_0\left(w_2 - \left(2\mu B_1 c_0 - B_1 c_0 - \frac{B_2}{B_1}\right)w_1^2\right)\right|\right). \tag{2.10}$$

Again applying $|c_n| \le 1$ and $|w_1| \le 1$, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{2}\left(1 + \left|w_2 - \left(-(1 - 2\mu)B_1 c_0 - \frac{B_2}{B_1}\right)w_1^2\right|\right). \tag{2.11}$$

Applying Lemma 1.11 to

$$\left|w_2 - \left(-(1-2\mu)B_1c_0 - \frac{B_2}{B_1}\right)w_1^2\right| \tag{2.12}$$

yields

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{2}\left(1 + \max\left\{1, \left|-(1-2\mu)B_1c_0 - \frac{B_2}{B_1}\right|\right\}\right). \tag{2.13}$$

Observe that

$$\left|-(1-2\mu)B_1c_0 - \frac{B_2}{B_1}\right| \le B_1|c_0|\left|1-2\mu\right| + \left|\frac{B_2}{B_1}\right|, \tag{2.14}$$

and hence we can conclude that

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{2}\left(1 + \max\left\{1, B_1\left|1-2\mu\right| + \left|\frac{B_2}{B_1}\right|\right\}\right). \tag{2.15}$$

For $\mu = 0$, the above will reduce to the estimate of $|a_3|$. □

Remark 2.2. For $\varphi(z) \equiv 1$, Theorem 2.1 gives a particular case of the estimates in [13, Theorem 1] for $p = 1$ and [14, Theorem 2.1] for $k = 1$.

Theorem 2.3. *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} - 1 \ll \phi(z) - 1, \tag{2.16}$$

then the following inequalities hold:

$$\begin{aligned} |a_2| &\le B_1, \\ |a_3| &\le \frac{1}{2}\left(B_1 + B_1^2 + |B_2|\right), \end{aligned} \tag{2.17}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2}\left(B_1 + \left|1-2\mu\right|B_1^2 + |B_2|\right). \tag{2.18}$$

Proof. The result follows by taking $w(z) = z$ in the proof of Theorem 2.1. □

Theorem 2.4. *If $f \in \mathcal{A}$ belongs to $\mathcal{C}_q(\phi)$, then*

$$|a_2| \le \frac{B_1}{2},$$
$$|a_3| \le \frac{1}{6}\left(B_1 + \max\left\{B_1, B_1^2 + |B_2|\right\}\right), \tag{2.19}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{6}\left(B_1 + \max\left\{B_1, \left|1 - \frac{3}{2}\mu\right| B_1^2 + |B_2|\right\}\right). \tag{2.20}$$

Proof. Observe that when $zf' \in \mathcal{S}_q^*$, equality (2.3) becomes

$$\frac{z\left(zf'(z)\right)'}{zf'(z)} - 1 = \varphi(z)\left(\phi(w(z)) - 1\right), \tag{2.21}$$

or equally

$$\frac{zf''(z)}{f'(z)} \prec \phi(w(z)) - 1, \tag{2.22}$$

and the converse can be verified easily. By the Alexander relation, that is $f \in \mathcal{C}_q$ if and only if $zf' \in \mathcal{S}_q^*$, we can obtain the required estimates. □

Theorem 2.5. *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} \ll \phi(z) - 1, \tag{2.23}$$

then the following inequalities hold:

$$|a_2| \le \frac{B_1}{2},$$
$$|a_3| \le \frac{1}{6}\left(B_1 + B_1^2 + |B_2|\right), \tag{2.24}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{6}\left(B_1 + \left|1 - \frac{3}{2}\mu\right| B_1^2 + |B_2|\right). \tag{2.25}$$

Theorem 2.6. *If $f \in \mathcal{A}$ belongs to $\mathcal{R}_q(\phi)$, then*

$$|a_2| \le \frac{B_1}{2}, \qquad |a_3| \le \frac{1}{3}(B_1 + \max\{B_1, |B_2|\}), \tag{2.26}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3}\left(B_1 + \max\left\{B_1, \frac{3}{4}\left|\mu\right|B_1^2 + |B_2|\right\}\right). \tag{2.27}$$

Proof. For $f \in \mathcal{R}_q(\phi)$, we know that by Definition 1.5 there exist analytic functions φ and w, with $|\varphi(z)| \le 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$f'(z) - 1 = \varphi(z)\big(\phi(w(z)) - 1\big). \tag{2.28}$$

Since

$$f'(z) - 1 = 2a_2 z + 3a_3 z^2 + \cdots, \tag{2.29}$$

it follows from (2.28) and (2.5) that

$$a_2 = \frac{1}{2}B_1 c_0 w_1, \qquad a_3 = \frac{1}{3}\left(B_1 c_1 w_1 + c_0\left(B_1 w_2 + B_2 w_1^2\right)\right). \tag{2.30}$$

Following the same argument as in Theorem 2.1, where $|c_0| \le 1$ and $|c_1| \le 1$, we can deduce that

$$|a_2| \le \frac{B_1}{2}, \qquad \left|a_3 - \mu a_2^2\right| \le \frac{B_1}{3}\left(1 + \left|w_2 - \left(\frac{3B_1 c_0}{4}\mu - \frac{B_2}{B_1}\right)w_1^2\right|\right). \tag{2.31}$$

Applying Lemma 1.11, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{3}\left(1 + \max\left\{1, \left|\frac{3B_1 c_0}{4}\mu - \frac{B_2}{B_1}\right|\right\}\right). \tag{2.32}$$

Since

$$\left|\frac{3B_1c_0}{4}\mu-\frac{B_2}{B_1}\right|\leq\frac{3B_1}{4}\left|\mu\right|\left|c_0\right|+\left|\frac{B_2}{B_1}\right|, \tag{2.33}$$

and $|c_0|\leq 1$ we can conclude the hypothesis. □

Theorem 2.7. *If $f\in\mathcal{A}$ satisfies*

$$f'(z)-1\ll\phi(z)-1, \tag{2.34}$$

then the following inequalities hold:

$$\begin{aligned}|a_2|&\leq\frac{B_1}{2},\\ |a_3|&\leq\frac{1}{3}(B_1+|B_2|),\end{aligned} \tag{2.35}$$

and, for any complex number μ,

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{3}\left(B_1+\frac{3}{4}\left|\mu\right|B_1^2+|B_2|\right). \tag{2.36}$$

Let the class $\mathcal{R}_q^{\rho}(\phi)$ consist of functions $f\in\mathcal{A}$ satisfying the quasi-subordination

$$\frac{1}{\rho}(f'(z)-1)\prec_q\phi(z)-1, \tag{2.37}$$

where $\rho\in\mathbb{C}\setminus\{0\}$. The following corollary gives the results for $f\in\mathcal{R}_q^{\rho}(\phi)$.

Corollary 2.8. *Let $\rho\in\mathbb{C}\setminus\{0\}$. If $f\in\mathcal{A}$ belongs to $\mathcal{R}_q^{\rho}(\phi)$, then*

$$\begin{aligned}|a_2|&\leq\frac{|\rho|}{2}B_1,\\ |a_3|&\leq\frac{|\rho|}{3}(B_1+\max\{B_1,|B_2|\}),\end{aligned} \tag{2.38}$$

and, for any complex number μ,

$$\left|a_3-\mu a_2^2\right|\leq\frac{|\rho|}{3}\left(B_1+\max\left\{B_1,\frac{3}{4}\left|\mu\rho\right|B_1^2+|B_2|\right\}\right). \tag{2.39}$$

Remark 2.9. (1) For $\varphi(z) \equiv 1$, Corollary 2.8 gives a particular case of the estimates in [13, Theorem 3] for $p = 1$ and [14, Theorem 2.3] for $k = 1$.

(2) For $\varphi(z) \equiv 1$ and $\phi(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$, Corollary 2.8 reduces to the results in [19, Theorem 4].

Theorem 2.10. *Let $\alpha \geq 0$. If $f \in \mathcal{A}$ belongs to $\mathcal{M}_q(\alpha, \phi)$, then*

$$|a_2| \leq \frac{B_1}{1+\alpha},$$
$$|a_3| \leq \frac{1}{2(1+2\alpha)}\left(B_1 + \max\left\{B_1, \frac{1+3\alpha}{(1+\alpha)^2}B_1^2 + |B_2|\right\}\right), \tag{2.40}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2(1+2\alpha)}\left(B_1 + \max\left\{B_1, \frac{|2\mu(1+2\alpha) - (1+3\alpha)|}{(1+\alpha)^2}B_1^2 + |B_2|\right\}\right). \tag{2.41}$$

Proof. If $f \in \mathcal{M}_q(\alpha, \phi)$, for $\alpha \geq 0$ then there are analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = \varphi(z)(\phi(w(z)) - 1). \tag{2.42}$$

A computation shows that

$$(1-\alpha)\frac{zf'(z)}{f(z)} = (1-\alpha) + (1-\alpha)a_2 z + (1-\alpha)\left(-a_2^2 + 2a_3\right)z^2 + \cdots,$$
$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \alpha + 2\alpha a_2 z + 2\alpha\left(-2a_2^2 + 3a_3\right)z^2 + \cdots. \tag{2.43}$$

Hence from (2.43), we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = (1+\alpha)a_2 z + \left(-(1+3\alpha)a_2^2 + 2(1+2\alpha)a_3\right)z^2 + \cdots, \tag{2.44}$$

It then follows from relation (2.42) and (2.5) that

$$a_2 = \frac{B_1 c_0 w_1}{1+\alpha},$$
$$a_3 = \frac{1}{2(1+2\alpha)}\left(B_1 c_1 w_1 + B_1 c_0 w_2 + \left(B_2 c_0 + \frac{1+3\alpha}{(1+\alpha)^2}B_1^2 c_0^2\right)w_1^2\right). \tag{2.45}$$

We can then conclude the proof by proceeding similarly as previous theorems. □

Remark 2.11. (1) When $\alpha = 0$, Theorem 2.10 reduces to Theorem 2.1.

(2) When $\alpha = 1$, Theorem 2.10 reduces to Theorem 2.4.

(3) For $\varphi(z) \equiv 1$, Theorem 2.10 gives a particular case of the estimates in [14, Theorem 2.9] for $k = 1$.

Theorem 2.12. *Let $\alpha \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \ll \phi(z) - 1, \tag{2.46}$$

then the following inequalities hold:

$$\begin{gathered} |a_2| \leq \frac{B_1}{1+\alpha}, \\ |a_3| \leq \frac{1}{2(1+2\alpha)}\left(B_1 + \frac{1+3\alpha}{(1+\alpha)^2}B_1^2 + |B_2|\right), \end{gathered} \tag{2.47}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2(1+2\alpha)}\left(B_1 + \frac{|2\mu(1+2\alpha) - (1+3\alpha)|}{(1+\alpha)^2}B_1^2 + |B_2|\right). \tag{2.48}$$

Theorem 2.13. *Let $\alpha \geq 0$ and $\beta = 1 - \alpha$. If $f \in \mathcal{A}$ belongs to $\mathcal{L}_q(\alpha, \phi)$, then*

$$\begin{gathered} |a_2| \leq \frac{B_1}{|\alpha + 2\beta|}, \\ |a_3| \leq \frac{1}{2|\alpha + 3\beta|}\left(B_1 + \max\left\{B_1, \frac{\left|(\alpha+2\beta)^2 - 3(\alpha+4\beta)\right|}{2(\alpha+2\beta)^2}B_1^2 + |B_2|\right\}\right), \end{gathered} \tag{2.49}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2|\alpha + 3\beta|}\left(B_1 + \max\left\{B_1, \frac{\left|(\alpha+2\beta)^2 - 3(\alpha+4\beta) - 4\mu(\alpha+3\beta)\right|}{2(\alpha+2\beta)^2}B_1^2 + |B_2|\right\}\right). \tag{2.50}$$

Proof. If $f \in \mathcal{L}_q(\alpha, \phi)$, for $\alpha \geq 0$ and $\beta = 1 - \alpha$ then there are analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} - 1 = \varphi(z)(\phi(w(z)) - 1). \tag{2.51}$$

A computation shows that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1 + \alpha a_2 z + \frac{1}{2}\left(\left(\alpha^2 - 3\alpha\right)a_2^2 + 4\alpha a_3\right)z^2 + \cdots,$$
$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} = 1 + 2\beta a_2 z + \left(2\left(\beta^2 - 3\beta\right)a_2^2 + 6\beta a_3\right)z^2 + \cdots. \tag{2.52}$$

Thus (2.52) give

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} - 1$$
$$= (\alpha + 2\beta)a_2 z + \frac{1}{2}\left(\left((\alpha + 2\beta)^2 - 3(\alpha + 4\beta)\right)a_2^2 + 4(\alpha + 3\beta)a_3\right)z^2 + \cdots, \tag{2.53}$$

By using the above equation and (2.5) in (2.51) we have

$$a_2 = \frac{B_1 c_0 w_1}{\alpha + 2\beta}$$
$$a_3 = \frac{B_1}{2(\alpha + 3\beta)}\left(B_1 c_1 w_1 + B_1 c_0 w_2 + \left(B_2 c_0 - \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)}{2(\alpha + 2\beta)^2}B_1^2 c_0^2\right)w_1^2\right). \tag{2.54}$$

We can proceed similarly as previous theorems and proof the hypothesis. □

Remark 2.14. (1) When $\alpha = 0$, Theorem 2.13 reduces to Theorem 2.4.

(2) When $\alpha = 1$, Theorem 2.13 reduces to Theorem 2.1.

(3) For $\varphi(z) \equiv 1$, Theorem 2.13 gives a particular case of the estimates in [14, Theorem 2.7] for $k = 1$.

Theorem 2.15. *Let $\alpha \geq 0$ and $\beta = 1 - \alpha$. If $f \in \mathcal{A}$ satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \ll \phi(z) - 1, \tag{2.55}$$

then the following inequalities hold:

$$|a_2| \leq \frac{B_1}{|\alpha + 2\beta|},$$
$$|a_3| \leq \frac{1}{2|\alpha + 3\beta|}\left(B_1 + \frac{\left|(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)\right|}{2(\alpha + 2\beta)^2}B_1^2 + |B_2|\right), \tag{2.56}$$

and, for any complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2\left|\alpha + 3\beta\right|}\left(B_1 + \frac{\left|(\alpha + 2\beta)^2 - 3(\alpha + 4\beta) - 4\mu(\alpha + 3\beta)\right|}{2(\alpha + 2\beta)^2}B_1^2 + |B_2|\right). \tag{2.57}$$

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