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Research Article

H-Distributions: An Extension of *H*-Measures to an L^p-L^q Setting

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We use the continuity of Fourier multiplier operators on L^p to introduce the H-distributions—an extension of H-measures in the L^p framework. We apply the H-distributions to obtain an L^p version of the localisation principle and reprove the Murat L^p-$L^{p'}$ variant of div-curl lemma.

1. Introduction

In the study of partial differential equations, quite often it is of interest to determine whether some L^p weakly convergent sequence converges strongly. Various techniques and tools have been developed for that purpose (for the state of the art twenty years ago, see [1]); of more modern ones, we only mention the H-measure of Tartar [2], independently introduced by Gérard [3] under the name of *microlocal defect measures*. H-measures proved to be very powerful tool in a number of applications (see, e.g., [4–13] and references therein, which is surely an incomplete list). The main theorem on the existence of H-measures, in an equivalent form suitable for our purposes, reads as following:

Theorem 1.1. *If scalar sequences $u_n, v_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a complex Radon measure μ on $\mathbb{R}^d \times S^{d-1}$ such that, for every $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and every $\psi \in C(S^{d-1})$,*

$$\lim_{n'} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) \mid \varphi_2 v_{n'} \rangle = \lim_{n'} \int_{\mathbb{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} = \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle, \tag{1.1}$$

where $\mathcal{A}_\psi$ is the Fourier multiplier operator with the symbol ψ:

$$\mathcal{A}_\psi u := \overline{\mathcal{F}}(\psi \hat{u}). \tag{1.2}$$

We call the measure μ the *H-measure* corresponding to the sequence (u_n, v_n). In fact, it corresponds to the nondiagonal element of the corresponding 2×2 matrix Radon measure of the vector function (u_n, v_n) (cf. [14]).

Remark 1.2. After applying the Plancherel theorem, the term under the limit sign in Theorem 1.1 takes the form

$$\int_{\mathbb{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} \psi \, d\boldsymbol{\xi}, \tag{1.3}$$

where by $\hat{u}(\boldsymbol{\xi}) = (\mathcal{F}u)(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \boldsymbol{\xi}} u(\mathbf{x}) d\mathbf{x}$ we denote the Fourier transform on $\mathbb{R}^d$ (with the inverse $(\overline{\mathcal{F}}v)(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \boldsymbol{\xi}} v(\boldsymbol{\xi}) d\boldsymbol{\xi}$). Of course, ψ is extended by homogeneity (i.e., $\psi(\boldsymbol{\xi}) := \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$) to $\mathbb{R}^d \setminus \{0\}$.

In the particular case of $u_n = v_n$, μ roughly describes the loss of strong L^2 precompactness of sequence (u_n). Indeed, it is not difficult to see that if (u_n) is strongly convergent in L^2, then the corresponding H-measure is trivial; on the other hand, if the H-measure is trivial, then $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ (for the details in a similar situation see, [15]).

In order to explain how to apply this idea to L^p-weakly converging sequences when $p \neq 2$, consider the integral in (1.1). The Cauchy-Schwartz inequality and the Plancherel theorem imply (see, e.g., [2, page 198])

$$\left| \int_{\mathbb{R}^d} \mathcal{A}_\psi (\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} d\mathbf{x} \right| \leq C \|\psi\|_{C(S^{d-1})} \|\varphi_1 \overline{\varphi_2}\|_{C_0(\mathbb{R}^d)}, \tag{1.4}$$

where C depends on a uniform bound for $\|(u_n, v_n)\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)}$. In essence, this fact and the linearity of integral in (1.1) with respect to $\varphi_1 \overline{\varphi_2}$ and ψ enable us to state that the limit in (1.1) is a Radon measure (a bounded linear functional on $C_0(\mathbb{R}^d \times S^{d-1})$). Furthermore, the bound is obtained by a simple estimate $\|\mathcal{A}_\psi\|_{L^2 \to L^2} \leq \|\psi\|_{L^\infty(\mathbb{R}^d)}$ and the fact that (u_n, v_n) is a bounded sequence in $L^2(\mathbb{R}^d; \mathbb{R}^2)$.

In [3], the question whether it is possible to extend the notion of H-measures (or microlocal defect measures in the terminology used there) to the L^p framework is posed (see also [16, page 331]). We will consider only the case $p \in \langle 1, \infty \rangle$ (i.e., $1 < p < \infty$; its dual exponent we will consistently denote by p').

To answer that question, one necessarily needs precise bounds for the Fourier multiplier operator $\mathcal{A}_\psi$ as a mapping from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. The bounds are given by the famous Hörmander-Mikhlin theorem [17, 18].

Definition 1.3. Let $\phi : \mathbb{R}^d \to \mathbb{C}$ satisfy $(1 + |\mathbf{x}|^2)^{-k/2} \phi \in L^1(\mathbb{R}^d)$ for some $k \in \mathbb{N}_0$. Then, ϕ is called the *Fourier multiplier* on $L^p(\mathbb{R}^d)$, if $\overline{\mathcal{F}}(\phi \mathcal{F}(\theta)) \in L^p(\mathbb{R}^d)$ for any $\theta \in \mathcal{S}(\mathbb{R}^d)$, and

$$\mathcal{S}\left(\mathbb{R}^d\right) \ni \theta \longmapsto \overline{\mathcal{F}}(\phi \mathcal{F}(\theta)) \in L^p\left(\mathbb{R}^d\right) \tag{1.5}$$

can be extended to a continuous mapping $T_\phi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$. One calls operator T_ϕ the *L^p-multiplier operator* with *symbol* ϕ.

Theorem 1.4 (Hörmander-Mikhlin). *Let $\phi \in L^\infty(\mathbb{R}^d)$ have partial derivatives of order less than or equal to κ, where κ is the least integer strictly greater than $d/2$ (i.e., $\kappa = [d/2] + 1$). If, for some constant $k > 0$,*

$$(\forall r > 0)\ \left(\forall \alpha \in \mathbb{N}_0^d\right) \quad |\alpha| \leq \kappa \Longrightarrow \int_{r/2 \leq |\boldsymbol{\xi}| \leq r} \left| D_{\boldsymbol{\xi}}^\alpha \phi(\boldsymbol{\xi}) \right|^2 d\boldsymbol{\xi} \leq k^2 r^{d-2|\alpha|}, \tag{1.6}$$

then, for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator T_ϕ, there exists a constant C_d (depending only on the dimension d; see [18, page 367]) such that

$$\| T_\phi \|_{L^p \to L^p} \leq C_d \max\left\{ p, \frac{1}{p-1} \right\} (k + \|\phi\|_\infty). \tag{1.7}$$

Remark 1.5. It is important to notice that, according to [19, Section 3.2, Example 2], if the symbol of a multiplier is a C^κ function defined on the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$, then the constant k from Theorem 1.4 can be taken to be equal to $\|\phi\|_{C^\kappa(S^{d-1})}$.

By an application of Theorem 1.4, in Section 2 we are able to introduce H-distributions (see Theorem 2.1)—an extension of H-measures in the L^p-setting. Its proof is the main result of the paper and forms Section 3. We conclude in Section 4 by an L^p-variant of the localisation principle and a proof of an $(L^p, L^{p'})$-variant of the div-curl lemma.

Remark 1.6. Recently, variants of H-measures with a different scaling were introduced (the parabolic H-measures [15, 20] and the ultraparabolic H-measures [21]). We can apply the procedure from this paper to extend the notion of such H-measures to the L^p-setting in the same fashion as it is given here based on Theorem 1.1 for the classical H-measures.

2. A Generalisation of H-Measures

We have already seen (Remark 1.2) that an H-measure μ corresponding to a sequence (u_n) in $L^2(\mathbb{R}^d)$ can describe its loss of strong compactness. We would like to introduce a similar notion describing the loss (at least in L^1_{loc}) of strong compactness for sequences weakly converging in $L^p(\mathbb{R}^d)$.

Consider a sequence (u_n) weakly converging to zero in $L^p(\mathbb{R}^d)$ and satisfying the following sequence of differential equations:

$$\sum_{i=1}^d \partial_i (A_i(\mathbf{x}) u_n(\mathbf{x})) = f_n(\mathbf{x}), \tag{2.1}$$

where $A_i \in C_0(\mathbb{R}^d)$ and $f_n \to 0$ strongly in the Sobolev space $H^{-1}(\mathbb{R}^d)$. When dealing with the latter equation, it is standard to multiply (2.1) by $\mathcal{A}_{\psi/|\boldsymbol{\xi}|}(\phi u_n)$, for $\phi \in C_0(\mathbb{R}^d)$, where $\mathcal{A}_{\psi/|\boldsymbol{\xi}|}$ is the multiplier operator with symbol $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)/|\boldsymbol{\xi}|$, $\psi \in C(S^{d-1})$, and then pass to the limit (see, e.g., [14, 22]). If $u_n \in L^2(\mathbb{R}^d)$, then we can apply the classical H-measures to describe the defect of compactness for (u_n).

If we instead take $u_n \in L^p(\mathbb{R}^d)$, for $p < 2$, then we cannot apply the same tool. Here we propose the following replacement.

Theorem 2.1. *If $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $v_n \overset{*}{\rightharpoonup} v$ in $L^q(\mathbb{R}^d)$ for some $q \geq \max\{p', 2\}$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued distribution $\mu \in \mathfrak{D}'(\mathbb{R}^d \times S^{d-1})$ of order not more than $\kappa = [d/2] + 1$ in ξ, such that, for every $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$ one has:*

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} = \lim_{n' \to \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x}$$
$$= \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle, \tag{2.2}$$

where $\mathcal{A}_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a multiplier operator with symbol $\psi \in C^\kappa(S^{d-1})$.

We call the functional μ the H-distribution corresponding to (a subsequence of) (u_n) and (v_n). Of course, for $q \in \langle 1, \infty \rangle$, the weak $*$ convergence coincides with the weak convergence.

If we are given sequences (u_n) and (v_n) defined on an open set $\Omega \subseteq \mathbb{R}^d$, then we can extend the functions by zero to $\mathbb{R}^d$, preserving the convergence, and then apply Theorem 2.1 in the above form. The resulting H-distribution will be supported on $\mathrm{Cl}\,\Omega \times S^{d-1}$, as it can easily be seen by taking test functions φ_1 and φ_2 supported within the complement of the closure $\mathrm{Cl}\,\Omega$.

Remark 2.2. Notice that, unlike what was the case with H-measures, it is not possible to write (2.2) in a form similar to (1.3) since, according to the Hausdorff-Young inequality, $\|\mathcal{F}(u)\|_{L^{p'}(\mathbb{R}^d)} \leq C\|u\|_{L^p(\mathbb{R}^d)}$ only if $1 < p \leq 2$. This means that we are not able to estimate $\|\mathcal{F}(\varphi_2 v_n)\|_{L^{q'}(\mathbb{R}^d)}$, for $q > 2$, which would appear from (2.2) when rewriting it in a form similar to (1.3).

Remark 2.3. In Theorem 2.1, we clearly distinguish between $u_n \in L^p(\mathbb{R}^d)$ and $v_n \in L^q(\mathbb{R}^d)$. For $p \geq 2$, $p' \leq 2$ and we can take $q \geq 2$; in particular, this covers the classical L^2 case (including $u_n = v_n$). Even more, in this case ($p \geq 2$), the assumptions of Theorem 2.1 imply that $u_n, v_n \rightharpoonup 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and we can again use a classical framework, resulting in a distribution μ of order zero (a Radon measure, not necessary bounded), instead of a more general distribution of order κ. The real improvement in Theorem 2.1 is for the case $p < 2$.

Remark 2.4. For applications, it might be of interest to extend the result to vector-valued functions. In the case when $\mathsf{u}_n \in L^p(\mathbb{R}^d)^k$ and $\mathsf{v}_n \in L^q(\mathbb{R}^d)^l$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, where $i \in 1, \ldots, t$ and $j \in 1, \ldots, l$.

It should be noted that, in contrast to what is done with H-measures, in general we cannot consider H-distributions corresponding to the same sequence, but only to a pair of sequences, and H-distribution would correspond to nondiagonal blocks for H-measures [14] (see also the example at the beginning of Section 4).

3. Proof of Theorem 2.1

In order to prove the theorem, we need a consequence of Tartar's first commutation lemma [2, Lemma 1.7]. First, for $\psi \in C^{\kappa}(S^{d-1})$ and $b \in C_0(\mathbb{R}^d)$, define the Fourier multiplier operator $\mathcal{A}_\psi$ and the operator of multiplication B on $L^p(\mathbb{R}^d)$, by the formulae

$$\mathcal{F}(\mathcal{A}_\psi u)(\xi) = \psi\left(\frac{\xi}{|\xi|}\right)\mathcal{F}(u)(\xi),$$
$$Bu(\mathbf{x}) = b(\mathbf{x})u(\mathbf{x}). \tag{3.1}$$

Notice that ψ satisfies the conditions of the Hörmander-Mikhlin theorem (see Remark 1.5). Therefore, $\mathcal{A}_\psi$ and B are bounded operators on $L^p(\mathbb{R}^d)$, for any $p \in \langle 1, \infty\rangle$. We are interested in the properties of their commutator, $C = \mathcal{A}_\psi B - B\mathcal{A}_\psi$.

Lemma 3.1. *Let (v_n) be bounded in both $L^2(\mathbb{R}^d)$ and $L^r(\mathbb{R}^d)$, for some $r \in \langle 2, \infty]$, and such that $v_n \rightharpoonup 0$ in the sense of distributions. Then the sequence (Cv_n) strongly converges to zero in $L^q(\mathbb{R}^d)$, for any $q \in [2, r] \setminus \{\infty\}$.*

Proof. If $r < \infty$, then we can apply the classical interpolation inequality:

$$\|Cv_n\|_q \le \|Cv_n\|_2^{\alpha}\|Cv_n\|_r^{1-\alpha}, \tag{3.2}$$

for $\alpha \in \langle 0, 1\rangle$ such that $1/q = \alpha/2 + (1-\alpha)/r$. As C is a compact operator on $L^2(\mathbb{R}^d)$ by the first commutation lemma, while C is bounded on $L^r(\mathbb{R}^d)$, from (3.2) we get the claim.

In the case $r = \infty$, notice that we do not have the boundedness of $\mathcal{A}_\psi$ on L^∞, but only on L^p, for $p < \infty$. Therefore, we take $p \in \langle q, \infty\rangle$ and by the interpolation inequality conclude that (v_n) is bounded in L^p. Now, we can proceed as above, with r replaced by p. ☐

Proof of Theorem 2.1. The first equality from (2.2) follows from the fact that the adjoint operator $\mathcal{A}_\psi^*$ corresponding to $\mathcal{A}_\psi$ is actually the multiplier operator $\mathcal{A}_{\overline{\psi}}$ (see [17, Theorem 7.4.3]). This means that (we take the duality product to be sesquilinear, i.e., antilinear in the second variable, in order to get the scalar product when $p = p' = 2$)

$${}_{L^p}\langle \mathcal{A}_\psi(\varphi_1 u_{n'}), \varphi_2 v_{n'}\rangle_{L^{p'}} = {}_{L^p}\langle \varphi_1 u_{n'}, \mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})\rangle_{L^{p'}}, \tag{3.3}$$

which is exactly what we need. We can now concentrate our attention on the second equality in (2.2).

Since $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, while for $v \in L^q(\mathbb{R}^d)$ one has $\varphi_1\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v)} \in L^{p'}(\mathbb{R}^d)$, according to the Hörmander-Mikhlin theorem for any $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$, it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v)}\,d\mathbf{x} = 0. \tag{3.4}$$

We can write $\mathbb{R}^d = \bigcup_{l\in\mathbb{N}} K_l$, where K_l form an increasing family of compact sets (e.g., closed balls around the origin of radius l); therefore supp $\varphi_2 \subseteq K_l$ for some $l \in \mathbb{N}$. One has

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi_1 u_n\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)}dx &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi_1 u_n\overline{\mathcal{A}_{\overline{\psi}}[\varphi_2\chi_l(v_n-v)]}d\mathbf{x}\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi_1\overline{\varphi}_2 u_n\overline{\mathcal{A}_{\overline{\psi}}(\chi_l(v_n-v))}d\mathbf{x}\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi_1\overline{\varphi}_2 u_n\overline{\mathcal{A}_{\overline{\psi}}(\chi_l v_n)}d\mathbf{x},\end{aligned} \tag{3.5}$$

where χ_l is the characteristic function of K_l. In the second equality, one has used Lemma 3.1.

This allows us to express the above integrals as bilinear functionals, after denoting $\varphi = \varphi_1\overline{\varphi}_2$:

$$\mu_{n,l}(\varphi,\psi) = \int_{\mathbb{R}^d}\varphi u_n\overline{\mathcal{A}_{\overline{\psi}}(\chi_l v_n)}d\mathbf{x}. \tag{3.6}$$

Furthermore, $\mu_{n,l}$ is bounded by $\widetilde{C}\|\varphi\|_{C_0(\mathbb{R}^d)}\|\psi\|_{C^\kappa(S^{d-1})}$, as according to the Hölder inequality, Theorem 1.4 and Remark 1.5:

$$\left|\mu_{n,l}(\varphi,\psi)\right| \le \|\varphi u_n\|_p\left\|\mathcal{A}_{\overline{\psi}}(\chi_l v_n)\right\|_{p'} \le \widetilde{C}\|\psi\|_{C^\kappa(S^{d-1})}\|\varphi\|_{C_0(\mathbb{R}^d)}, \tag{3.7}$$

where the constant $\widetilde{C}$ depends on $L^p(K_l)$-norm and $L^{p'}(K_l)$-norm of the sequences (u_n) and (v_n), respectively.

For each $l \in \mathbb{N}$, we can apply Lemma 3.2 (actually, the operators are defined in its proof) to obtain operators $B^l \in \mathcal{L}(C_{K_l}(\mathbb{R}^d); (C^\kappa(S^{d-1}))')$. Furthermore, for the construction of B^l, we can start with a defining subsequence for B^{l-1}, so that the convergence will remain valid on $C_{K_{l-1}}(\mathbb{R}^d)$, in such a way obtaining that B^l is an extension of B^{l-1}.

This allows us to define the operator B on $C_c(\mathbb{R}^d)$: for, $\varphi \in C_c(\mathbb{R}^d)$, we take $l \in \mathbb{N}$ such that supp $\varphi \subseteq K_l$, and set $B\varphi := B^l\varphi$. Because of the above-mentioned extension property, this definition is good, and one has a bounded operator:

$$\|B\varphi\|_{(C^\kappa(S^{d-1}))'} \le \widetilde{C}\|\varphi\|_{C_0(\mathbb{R}^d)}. \tag{3.8}$$

In such a way one has got a bounded linear operator B on the space $C_c(\mathbb{R}^d)$ equipped with the uniform norm; the operator can be extended to its completion, the Banach space $C_0(\mathbb{R}^d)$.

Now, we can define $\mu(\varphi,\psi) := \langle B\varphi,\psi\rangle$, which satisfies (2.2).

We can restrict B to an operator $\widetilde{B}$ defined only on $C_c^\infty(\mathbb{R}^d)$; as the topology on $C_c^\infty(\mathbb{R}^d)$ is stronger than the one inherited from $C_0(\mathbb{R}^d)$, the restriction remains continuous. Furthermore, $(C^\kappa(S^{d-1}))'$ is the space of distributions of order κ, which is a subspace of $\mathfrak{D}'(S^{d-1})$. In such a way, one has a continuous operator from $C_c^\infty(\mathbb{R}^d)$ to $\mathfrak{D}'(S^{d-1})$, which by the Schwartz kernel theorem can be identified to a distribution from $\mathfrak{D}'(\mathbb{R}^d \times S^{d-1})$ (for details cf. [23, Chapter VI]). □

We conclude this section by a simple lemma and its proof, which was used in the proof of Theorem 2.1.

Lemma 3.2. *Let E and F be separable Banach spaces and (b_n) an equibounded sequence of bilinear forms on $E \times F$ (more precisely, there is a constant C such that, for each $n \in \mathbb{N}$ one has $|b_n(\varphi, \psi)| \leq C\|\varphi\|_E \|\psi\|_F$).*

Then, there exists a subsequence (b_{n_k}) and a bilinear form b (with the same bound C) such that

$$(\forall \varphi \in E) \ (\forall \psi \in F) \quad \lim_k b_{n_k}(\varphi, \psi) = b(\varphi, \psi). \tag{3.9}$$

Proof. To each b_n, we associate a bounded linear operator $B_n : E \longrightarrow F'$ by

$$_{F'}\langle B_n \varphi, \psi \rangle_F := b_n(\varphi, \psi). \tag{3.10}$$

The above expression clearly defines a function (i.e., $B_n \varphi \in F'$ is uniquely determined); it is linear in φ and bounded:

$$\|B_n \varphi\|_{F'} = \sup_{\psi \neq 0} \frac{|b_n(\varphi, \psi)|}{\|\psi\|_F} \leq C\|\varphi\|_E. \tag{3.11}$$

Let $\mathcal{G} \subseteq E$ be a countable dense subset; for each $\varphi \in \mathcal{G}$, the sequence $(B_n \varphi)$ is bounded in F', so by the Banach theorem there is a subsequence such that

$$B_{n_1} \varphi \overset{*}{\rightharpoonup} \beta_1 =: B(\varphi). \tag{3.12}$$

By repeating this construction countably many times and then applying the Cantor diagonal procedure, we get a subsequence

$$(\forall \varphi \in \mathcal{G}) \quad B_{n_k} \varphi \overset{*}{\rightharpoonup} B(\varphi), \tag{3.13}$$

such that $\|B(\varphi)\|_{F'} \leq C\|\varphi\|_E$.

Then, it is standard to extend B to a bounded linear operator on the whole space E. Clearly,

$$b(\varphi, \psi) := {}_{F'}\langle B\varphi, \psi \rangle_F = \lim_k {}_{F'}\langle B_{n_k} \varphi, \psi \rangle_F = \lim_k b_{n_k}(\varphi, \psi). \tag{3.14}$$

□

4. Some Applications

It is well-known that weak convergences are ill behaved under nonlinear transformations (in contrast to their good behaviour under linear transformations). Only in some particular cases of compensation, it is even possible to pass to the limit in a product of two weakly converging sequences.

The prototype of this compensation effect is Tartar-Murat's div-curl lemma (cf. [24, Theorem 7.1]).

For simplicity, consider two vector-valued sequences, (u_n^1, u_n^2) and (v_n^1, v_n^2), converging to zero weakly in $L^2(\mathbb{R}^2)$, such that $(\partial_x u_n^1 + \partial_y u_n^2)$ and $(\partial_y v_n^1 - \partial_x v_n^2)$ are both contained in a compact set of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ (which then implies that they converge to zero strongly in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$).

We can define $\mathsf{U}_n := \begin{bmatrix} \mathsf{u}_n \\ \mathsf{v}_n \end{bmatrix}$, which (on a subsequence) defines a 4×4 H-measure $\boldsymbol{\mu}$. By the localisation principle [2, Theorem 1.6] and [14, Theorem 2], as the above relations can be written in the form ($\mathbf{A}^1, \mathbf{A}^2$ are 4×4 constant matrices with all entries zero except $A^1_{11} = A^2_{12} = A^2_{33} = 1$ and $A^1_{34} = -1$)

$$\mathbf{A}^1 \partial_1 \mathsf{U}_n + \mathbf{A}^2 \partial_2 \mathsf{U}_n \longrightarrow 0 \quad \text{strongly in } H^{-1}_{\mathrm{loc}}\left(\mathbb{R}^2\right)^4, \tag{4.1}$$

the corresponding H-measure satisfies $(\xi_1 \mathbf{A}^1 + \xi_2 \mathbf{A}^2)\boldsymbol{\mu} = \mathbf{0}$. After straightforward calculations this shows that $u_n^1 v_n^1 + u_n^2 v_n^2 \rightharpoonup 0$ weak $*$ in the sense of Radon measures (and therefore in the sense of distributions as well).

For the above, one has used only the nondiagonal blocks $\boldsymbol{\mu}_{12} = \boldsymbol{\mu}^*_{21}$ of

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix}, \tag{4.2}$$

corresponding to products of u_n^i and v_n^j; in fact, the calculation shows that $\mu_{12}^{11} + \mu_{12}^{22} = 0$, which gives the above result.

In order to get a similar result using H-distributions, we first show that the following localisation principle holds.

Theorem 4.1. *Assume that $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $f_n \longrightarrow 0$ in $W^{-1,q}(\mathbb{R}^d)$, for some $q \in \langle 1, d \rangle$, such that they satisfy*

$$\sum_{i=1}^{d} \partial_i (A_i(\mathbf{x}) u_n(\mathbf{x})) = f_n(\mathbf{x}). \tag{4.3}$$

Take an arbitrary sequence (v_n) bounded in $L^\infty(\mathbb{R}^d)$, and by μ denote the H-distribution corresponding to some subsequences of sequences (u_n) and (v_n). Then,

$$\sum_{i=1}^{d} A_i(\mathbf{x}) \xi_i \mu(\mathbf{x}, \boldsymbol{\xi}) = 0 \tag{4.4}$$

in the sense of distributions on $\mathbb{R}^d \times S^{d-1}$, the function $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \sum_{i=1}^{d} A_i(\mathbf{x}) \xi_i$ being the symbol of the linear partial differential operator with C_0^κ coefficients.

Proof. In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\xi|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\xi_j/i|\xi|}$ [19, V.1,2]. We note that [id.,V.2.3]

$$\int I_1(\phi)\partial_j g = \int (R_j\phi)g, \quad g \in \mathcal{S}\left(\mathbb{R}^d\right). \tag{4.5}$$

From here, using the density argument and the fact that R_j is bounded from $L^p(\mathbb{R}^d)$ to itself, we conclude that $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in L^p(\mathbb{R}^d)$.

We should prove that the H-distribution corresponding to (the chosen subsequences of) (u_n) and (v_n) satisfies (4.4). To this end, take the following sequence of test functions:

$$\phi_n := \varphi_1\left(I_1 \circ \mathcal{A}_{\psi(\xi/|\xi|)}\right)(\varphi_2 v_n), \tag{4.6}$$

where $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$, $\kappa = [d/2] + 1$. Then, apply the right-hand side of (4.3), which converges strongly to 0 in $W^{-1,q}(\mathbb{R}^d)$ by the assumption, to a weakly converging sequence (ϕ_n) in the dual space $W^{1,q'}(\mathbb{R}^d)$.

We can do that since (ϕ_n) is a bounded sequence in $W^{1,r}(\mathbb{R}^d)$ for any $r \in \langle 1, \infty\rangle$.

Indeed, $\mathcal{A}_\psi(\varphi_2 v_n)$ is bounded in any $L^r(\mathbb{R}^d)$ $(r > 1)$. By the well-known fact [19, Theorem V.1] that I_1 is bounded from $L^q(\mathbb{R}^d)$ to $L^{q^*}(\mathbb{R}^d)$, for $q \in \langle 1, d\rangle$ and $1/q^* = 1/q - 1/d$, ϕ_n is bounded in $L^{q^*}(\mathbb{R}^d)$ for all sufficiently large q^*. Then, take $q^* \geq r$ and due to the compact support of φ_1, one has that L^{q^*} boundedness implies the same in L^r. On the other hand, R_j is bounded from $L^r(\mathbb{R}^d)$ to itself, for any $r \in \langle 1, \infty\rangle$, thus; $\partial_j(\varphi_1(I_1 \circ \mathcal{A}_{\psi(\xi/|\xi|)})(\varphi_2 v_n))$ is bounded in $L^r(\mathbb{R}^d)$.

Therefore, one has (the sequence is bounded and 0 is the only accumulation point, so the whole sequence converges to 0)

$$\lim_{n\to\infty} {}_{W^{-1,q}(IR^d)}\langle f_n, \phi_n\rangle_{W^{1,q'}(\mathbb{R}^d)} = 0. \tag{4.7}$$

Concerning the left-hand side of (4.3), according to (4.5), one has

$$\begin{aligned} {}_{W^{-1,q}(\mathbb{R}^d)}\left\langle \sum_{j=1}^d \partial_j(A_j u_n), \phi_n \right\rangle_{W^{1,q'}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \sum_{j=1}^d \overline{\varphi}_1 A_j u_n \overline{\mathcal{A}_{(\xi_j/|\xi|)\psi(\xi/|\xi|)}(\varphi_2 u_n)}\, d\mathbf{x} \\ &\quad - \int_{\mathbb{R}^d} \partial_j \overline{\varphi}_1 \sum_{j=1}^d A_j u_n \overline{\left(I_1 \circ \mathcal{A}_{\psi(\xi/|\xi|)}\right)(\varphi_2 v_n)}\, d\mathbf{x}. \end{aligned} \tag{4.8}$$

The first term on the right is of the form of the right-hand side of (2.2). The integrand in the second term is supported in a fixed compact and weakly converging to 0 in L^p, so strongly in $W^{-1,r'}$, where r is such that $p = r^*$ (i.e., $r = dp/(d-p)$). Of course, the argument giving the boundedness of ϕ_n in $W^{1,q'}(\mathbb{R}^d)$ above applies also to r instead of q'.

Therefore, from (4.7) and (4.8), one concludes (4.4). □

Remark 4.2. Notice that the assumption of the strong convergence of f_n in $W^{-1,q}(\mathbb{R}^d)$ can be relaxed to local convergence, because in the proof we used a cutoff function φ_1.

Let us return to the simple example from the beginning of this section; consider two vector-valued sequences (u_n^1, u_n^2) and (v_n^1, v_n^2), this time converging to zero weakly in $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$, respectively. Assume that the sequence $(\partial_1 u_n^1 + \partial_2 u_n^2)$ is bounded in $L^p(\mathbb{R}^2)$, and $(\partial_2 v_n^1 - \partial_1 v_n^2)$ in $L^{p'}(\mathbb{R}^2)$ (thus precompact in $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$, and $W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^2)$, resp.).

Then, the sequence $(u_n^1 v_n^1 + u_n^2 v_n^2)$ is bounded in $L^1(\mathbb{R}^2)$ so also in $\mathcal{M}_b$ (Radon measures) and by weak $*$ compactness it has a weakly converging subsequence. However, we can say more—the whole sequence converges to zero.

Denote by μ^{ij} the H-distribution corresponding to (some sub)sequences (of) (u_n^1, u_n^2) and (v_n^1, v_n^2).

Since $(\partial_1 u_n^1 + \partial_2 u_n^2)$ is bounded in $L^p(\mathbb{R}^2)$, and $(\partial_2 v_n^1 - \partial_1 v_n^2)$ is bounded in $L^{p'}(\mathbb{R}^2)$, they are weakly precompact, while the only possible limit is zero, so

$$\partial_1 u_n^1 + \partial_2 u_n^2 \rightharpoonup 0 \quad \text{in } L^p, \qquad \partial_2 v_n^1 - \partial_1 v_n^2 \rightharpoonup 0 \quad \text{in } L^{p'}. \tag{4.9}$$

Now, from the compactness properties of the Riesz potential I_1 (see the proof of previous theorem), we conclude that, for every $\varphi \in C_c(\mathbb{R}^2)$ and $\psi \in C^\kappa(S^{d-1})$, the following limit holds in $L^p(\mathbb{R}^2)$:

$$\mathcal{A}_{\psi(\xi/|\xi|)(\xi_1/|\xi|)}\left(\varphi u_n^1\right) + \mathcal{A}_{\psi(\xi/|\xi|)(\xi_2/|\xi|)}\left(\varphi u_n^2\right) = \mathcal{A}_{\psi(\xi/|\xi|)/|\xi|}\left(\partial_1\left(\varphi u_n^1\right) + \partial_2\left(\varphi u_n^2\right)\right) \longrightarrow 0. \tag{4.10}$$

Multiplying (4.10) first by φv_n^1 and then by φv_n^2, integrating over $\mathbb{R}^2$ and passing to the limit, we conclude from (2.2), due to the arbitrariness of ψ and φ:

$$\xi_1 \mu^{11} + \xi_2 \mu^{21} = 0, \qquad \xi_1 \mu^{12} + \xi_2 \mu^{22} = 0. \tag{4.11}$$

Next, take

$$w_n^j = \varphi \mathcal{A}_{\psi(\xi/|\xi|)/|\xi|}\left(\varphi u_n^j\right) \in W^{1,p'}\left(\mathbb{R}^d\right), \quad j = 1, 2. \tag{4.12}$$

From (4.9), we get

$$\left\langle \left(\varphi v_n^1, -\varphi v_n^2\right), \nabla w_n^j \right\rangle = -\left\langle \operatorname{curl}\left(\varphi v_n^1, \varphi v_n^2\right), w_n^j \right\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{4.13}$$

for $j = 1, 2$. Rewriting it in the integral formulation, we obtain, from (2.2),

$$\xi_2 \mu^{11} - \xi_1 \mu^{12} = 0, \qquad \xi_2 \mu^{21} - \xi_1 \mu^{22} = 0. \tag{4.14}$$

From the algebraic relations (4.11) and (4.14), we can easily conclude

$$\xi_1\left(\mu^{11} + \mu^{22}\right) = 0, \qquad \xi_2\left(\mu^{11} + \mu^{22}\right) = 0, \tag{4.15}$$

implying that the distribution $\mu^{11} + \mu^{22}$ is supported on the set $\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap P = \emptyset$, which implies $\mu^{11} + \mu^{22} \equiv 0$.

After inserting $\psi \equiv 1$ in the definition of H-distribution (2.2), we immediately reach the conclusion. This proof is similar to the L^2 case, but it should be noted that there we had used only a nondiagonal block of 4×4 H-measure, which corresponds to the only available 2×2 H-distribution.

There is no reason to limit oneself to two dimensions; take (u_n) and (v_n) converging weakly to zero in $L^p(\mathbb{R}^d)^d$ and $L^{p'}(\mathbb{R}^d)^d$, and by $\boldsymbol{\mu}$ denote $d \times d$ matrix H-distribution corresponding to some chosen subsequences of (u_n) and (v_n).

From $\operatorname{div} \mathsf{u}_n \to 0$ strongly in $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, for $\varphi_1 \in C_c(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$, one has as in (4.10) that

$$\mathcal{A}_{\psi(\xi/|\xi|)(\xi/|\xi|)}\left(\varphi_1 \mathsf{u}_n\right) = \mathcal{A}_{\psi(\xi/|\xi|)/|\xi|}\operatorname{div}\left(\varphi_1 \mathsf{u}_n\right) \longrightarrow 0 \quad \text{strongly in } L^p\left(\mathbb{R}^d\right). \tag{4.16}$$

After forming a product with $\varphi_2 v_n$, integrating and passing to the limit, we conclude that

$$\boldsymbol{\xi}^{\mathsf{T}} \boldsymbol{\mu} = 0, \tag{4.17}$$

namely, that the columns of $\boldsymbol{\mu}$ are perpendicular to $\boldsymbol{\xi}$.

On the other hand, from $\operatorname{curl} \mathsf{v}_n \to \mathbf{0}$ strongly in $W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^d)^{d\times d}$, in an analogous way, we conclude that, for each row (denoted by $\boldsymbol{\mu}^i$) of $\boldsymbol{\mu}$, for all j, k, one has

$$\xi_j \mu^{ik} - \xi_k \mu^{ij} = 0, \tag{4.18}$$

so the rows of $\boldsymbol{\mu}$ are proportional to $\boldsymbol{\xi}$ and $\boldsymbol{\mu} = \boldsymbol{\lambda} \otimes \boldsymbol{\xi}$ (a rank-one matrix), λ_i being the *constants* of proportionality. So, the columns of $\boldsymbol{\mu}$ are proportional to $\boldsymbol{\lambda}$, while earlier we showed that they are perpendicular to $\boldsymbol{\xi}$. Thus, $\operatorname{tr} \boldsymbol{\mu} = \boldsymbol{\lambda} \cdot \boldsymbol{\xi} = 0$, which implies the convergence $\mathsf{u}_n \cdot \mathsf{v}_n \rightharpoonup 0$, as in the two-dimensional situation.

The above result is the well-known Murat's div-curl lemma in the $(L^p, L^{p'})$-setting [24, 25], which we state as a theorem.

Theorem 4.3. *Let (u_n) and (v_n) be vector-valued sequences converging to zero weakly in $L^p(\mathbb{R}^d)^d$ and $L^{p'}(\mathbb{R}^d)^d$, respectively. Assume that the sequence $(\operatorname{div} \mathsf{u}_n)$ is bounded in $L^p(\mathbb{R}^d)$ and the sequence $(\operatorname{curl} \mathsf{v}_n)$ is bounded in $L^{p'}(\mathbb{R}^d)^{d\times d}$.*

Then, the sequence $(\mathsf{u}_n \cdot \mathsf{v}_n)$ converges to zero in the sense of distributions (or vaguely in the sense of Radon measures).

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