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Research Article

Knot Universes in Bianchi Type I Cosmology

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We investigate the trefoil and figure eight knot universes from Bianchi type I cosmology. In particular, we construct several concrete models describing the knot universes related to the cyclic universe and examine those cosmological features and properties in detail. Finally some examples of unknotted closed curves solutions (spiky and Mobius strip universes) are presented.

1. Introduction

Inflation is one of the most important phenomena in modern cosmology and has been confirmed by recent observations on cosmic microwave background (CMB) radiation [1–4]. Furthermore, it is suggested by the cosmological and astronomical observations of Type Ia supernovae [5, 6], CMB radiation [1–4], large scale structure (LSS) [7, 8], baryon acoustic oscillations (BAO) [9], and weak lensing [10] that the expansion of the current universe is accelerating. In order to explain the late time cosmic acceleration, we need to introduce so-called dark energy in the framework of general relativity or modify the gravitational theory, which can be regarded as a kind of geometrical dark energy (for reviews on dark energy, see, e.g., [11–16], and for reviews on modified gravity, see, e.g., [17–23]).

It is considered that there happened a Big Bang singularity in the early universe. In addition, at the dark energy dominated stage, the finite-time future singularities will occur [24–70]. There also exists the possibility that a Big Crunch singularity will happen. To avoid such cosmological singularities, there are various proposals such as the cyclic universe [71–80] (in other approach of the cyclic universe, see [81]), the ekpyrotic scenario [82–85], and the bouncing universe [86–97].

On the other hand, as a related theory to the cyclic universe, the trefoil and figure-eight knot universes have been explored in [98–103]. In the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) and the homogeneous and anisotropic

Bianchi-type I cosmologies, the geometrical description of these knot theories corresponds to oscillating solutions of the gravitational field equations. Note that the terms "the trefoil knot universe" and "the figure-eight knot universe" were introduced for the first time in [98–103]. Moreover, the Weierstrass $\wp(t)$, $\zeta(t)$, and $\sigma(t)$ functions and the Jacobian elliptic functions have been applied to solve several issues on astrophysics and cosmology [104–106]. In particular, very recently, by combining the reconstruction method in [17, 18, 66, 107, 108] with the Weierstrass and Jacobian elliptic functions, the equation of state (EoS) for the cyclic universes [109] and periodic generalizations of Chaplygin gas type models [110–112] for dark energy [113] have been examined. This procedure can be considered to be a novel approach to cosmological models in order to investigate the properties of dark energy.

In this paper, we explore the cosmological features and properties of the trefoil and figure-eight knot universes from Bianchi-type I cosmology in detail. In particular, we construct several concrete models describing the trefoil and figure-eight knot universes based on Bianchi-type I spacetime. In our previous work [98–103], the models of the knot universes from the homogeneous and isotropic FLRW spacetime were studied. By using the equivalent procedure, as continuous investigations, in this work we explicitly demonstrate that the knot universes can be constructed by Bianchi-type I spacetime. In other words, our purpose is to establish the formalism which can describe the knot universes.

It is significant to emphasize that according to the recent cosmological data analysis [1–4], it is implied that the universe is homogeneous and isotropic. In fact, however, recently the feature of anisotropy of cosmological phenomena such as anisotropic inflation [114, 115] has also been studied in the literature. In such a cosmological sense, it can be regarded as reasonable to consider the anisotropic universe including Bianchi-type I spacetime. The units of the gravitational constant $8\pi G = c = 1$ with G and c being the gravitational constant and the seed of light are used.

The organization of the paper is as follows. In Section 2, we explain the model and derive the basic equations. In Section 3, we investigate the trefoil knot universe. Next, we study the figure-eight knot universe in Section 4. In Section 5 we present some unknotted closed curve solutions of the model. Finally, we give conclusions in Section 6.

2. The Model

In this section we briefly review some basic facts about Einstein's field equation. We start from the standard gravitational action (chosen units are $c = 8\pi G = 1$)

$$S = \frac{1}{4}\int d^4x\sqrt{-g}(R - 2\Lambda + L_m), \tag{2.1}$$

where R is the Ricci scalar, Λ is the cosmological constant, and L_m is the matter Lagrangian. For a general metric $g_{\mu\nu}$, the line element is

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}, \quad (\mu, \nu = 0, 1, 2, 3). \tag{2.2}$$

The corresponding Einstein field equations are given by

$$R_{\mu\nu} + \left(\Lambda - \frac{1}{2}R\right)g_{\mu\nu} = -\kappa^2 T_{\mu\nu}, \tag{2.3}$$

where $R_{\mu\nu}$ is the Ricci tensor. This equation forms the mathematical basis of the theory of general relativity. In (2.3), $T_{\mu\nu}$ is the energy-momentum tensor of the matter field defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\mu\nu}}, \tag{2.4}$$

and satisfies the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.5}$$

where ∇_μ is the covariant derivative which is the relevant operator to smooth a tensor on a differentiable manifold. Equation (2.5) yields the conservations of energy and momentums, corresponding to the independent variables involved. The general Einstein Equation (2.3) is a set of nonlinear partial differential equations. We consider the Bianchi-I metric

$$ds^2 = -d\tau^2 + A^2 dx_1^2 + B^2 dx_2^2 + C^2 dx_3^2, \tag{2.6}$$

where we assume that $\tau = t/t_0$, $x_i = x_i'/x_{i0}$, A, B, C are dimensionless (usually we put $t_0 = x_{i0} = 1$). Here the metric potentials A, B, and C are functions of $\tau = t$ alone. This insures that the model is spatially homogeneous. The statistical volume for the anisotropic Bianchi type-I model can be written as

$$V = ABC. \tag{2.7}$$

The Ricci scalar is

$$R = g^{ij} R_{ij} = 2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} \right), \tag{2.8}$$

where $\dot{A} = dA/d\tau$ and so on. The nonvanishing components of Einstein tensor

$$G_{ij} = R_{ij} - 0.5 g_{ij} R \tag{2.9}$$

are

$$\begin{aligned} G_{00} &= \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}, \\ G_{AA} &= -A^2\left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} \right), \\ G_{BB} &= -B^2\left(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} \right), \\ G_{CC} &= -C^2\left(\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{B}\dot{A}}{BA} \right). \end{aligned} \tag{2.10}$$

We define $a = (ABC)^{1/3}$ as the average scale factor so that the average Hubble parameter may be defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \tag{2.11}$$

We write this average Hubble parameter H sometimes as

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{2.12}$$

where

$$H_1 = \frac{\dot{A}}{A}, \qquad H_2 = \frac{\dot{B}}{B}, \qquad H_3 = \frac{\dot{C}}{C} \tag{2.13}$$

are the directional Hubble parameters in the directions of x_1, x_2, and x_3, respectively. Hence we get the important relations

$$A = A_0 e^{\int H_1 dt}, \qquad B = B_0 e^{\int H_2 dt}, \qquad C = C_0 e^{\int H_3 dt}, \tag{2.14}$$

where A_0, B_0, C_0 are integration constants. The other important cosmological quantity is the deceleration parameter q, which for our model reads as

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{2.15}$$

Next, we assume that the energy-momentum tensor of fluid has the form

$$T_{ij} = \operatorname{diag}[T_{00}, T_{11}, T_{22}, T_{33}] = \operatorname{diag}[\rho, -p_1, -p_2, -p_3]. \tag{2.16}$$

Here p_i are the pressures along the x_i axes, recpectively, ρ is the proper density of energy. Then the Einstein equations (with gravitational units, $8\pi G = 1$ and $c = 1$) read as

$$R_{ij} - \frac{1}{2}R g_{ij} = -T_{ij}, \tag{2.17}$$

where we assumed $\Lambda = 0$. For the metric (2.6) these equations take the form

$$\begin{aligned} &\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \rho = 0, \\ &\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + p_1 = 0, \\ &\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} + p_2 = 0, \\ &\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + p_3 = 0. \end{aligned} \tag{2.18}$$

In terms of the Hubble parameters this system takes the form

$$\begin{aligned} &H_1H_2 + H_2H_3 + H_1H_3 - \rho = 0, \\ &\dot{H}_2 + \dot{H}_3 + H_2^2 + H_3^2 + H_2H_3 + p_1 = 0, \\ &\dot{H}_3 + \dot{H}_1 + H_3^2 + H_1^2 + H_3H_1 + p_2 = 0, \\ &\dot{H}_1 + \dot{H}_2 + H_1^2 + H_2^2 + H_1H_2 + p_3 = 0. \end{aligned} \tag{2.19}$$

Also we can introduce the three EoS parameters as

$$\omega_1 = \frac{p_1}{\rho}, \qquad \omega_2 = \frac{p_2}{\rho}, \qquad \omega_3 = \frac{p_3}{\rho} \tag{2.20}$$

and the deceleration parameters

$$q_1 = -\frac{\ddot{A}A}{\dot{A}^2}, \qquad q_2 = -\frac{\ddot{B}B}{\dot{B}^2}, \qquad q_3 = -\frac{\ddot{C}C}{\dot{C}^2}. \tag{2.21}$$

Finally we want to present

$$2\dot{H} + 6H^2 = \rho - p, \tag{2.22}$$

where

$$p = \frac{p_1 + p_2 + p_3}{3} \tag{2.23}$$

is the average pressure. Hence we can calculate the average parameter of the EoS as

$$\omega = \frac{p}{\rho} = \frac{\omega_1 + \omega_2 + \omega_3}{3}. \tag{2.24}$$

Let us also present the expression of R in terms of H_i. From (2.8) and (2.13) follows

$$R = 2\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1 H_2 + H_1 H_3 + H_2 H_3\right). \tag{2.25}$$

Now we want to present the knot and unknotted universe solutions of the system (2.18) or its equivalent (2.19). Consider some examples.

3. The Trefoil Knot Universe

Our aim in this section is to construct the simplest examples of the knot universes, namely, the trefoil knot universes. Consider the following examples.

3.1. Example 1

Let us assume that our universe is filled by the fluid with the following parametric EoS:

$$\begin{aligned} p_1 &= -\frac{D_1}{E_1}, \\ p_2 &= -\frac{D_2}{E_2}, \\ p_3 &= -\frac{D_3}{E_3}, \\ \rho &= \frac{D_0}{E_0}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} D_1 &= \left(-12\sin^2(3\tau) + 36\cos(3\tau) + 18\cos^2(3\tau)\right)\cos(2\tau) \\ &\quad - 49\sin(2\tau)\left(\frac{26}{49} + \cos(3\tau)\right)\sin(3\tau), \\ E_1 &= \sin(3\tau)(2 + \cos(3\tau))\sin(2\tau), \\ D_2 &= -18\sin(2\tau)\cos^2(3\tau) + (-49\sin(3\tau)\cos(2\tau) - 36\sin(2\tau))\cos(3\tau) \\ &\quad - 26\sin(3\tau)\cos(2\tau) + 12\sin^2(3\tau)\sin(2\tau), \\ E_2 &= \sin(3\tau)(2 + \cos(3\tau))\cos(2\tau), \\ D_3 &= -30\sin(3\tau)(2 + \cos(3\tau))\cos^2(2\tau) \\ &\quad - 38\sin(2\tau)\left(\cos^2(3\tau) - \left(\frac{27}{38}\right)\sin^2(3\tau) + \left(\frac{58}{19}\right)\cos(3\tau) + \frac{40}{19}\right)\cos(2\tau) \\ &\quad + 30\sin(3\tau)\sin^2(2\tau)(2 + \cos(3\tau)), \end{aligned}$$

$$
\begin{aligned}
E_3 &= (2+\cos(3\tau))^2\cos(2\tau)\sin(2\tau),\\
D_0 &= \left(6\cos^2(2\tau)-6\sin^2(2\tau)\right)\cos^3(3\tau)\\
&\quad+\left(24\cos^2(2\tau)\ -22\sin(3\tau)\sin(2\tau)\cos(2\tau)-24\sin^2(2\tau)\right)\cos^2(3\tau)\\
&\quad+\Big(\left(-6\sin^2(3\tau)+24\right)\cos^2(2\tau)-52\sin(3\tau)\sin(2\tau)\cos(2\tau)\\
&\qquad+\left(6\sin^2(3\tau)-24\right)\sin^2(2\tau)\Big)\cos(3\tau)\\
&\quad-\left(12\left(\cos(2\tau)-\left(\frac{3}{4}\right)\sin(3\tau)\sin(2\tau)\right)\right)\\
&\quad\times\left(\sin(3\tau)\cos(2\tau)+\left(\frac{4}{3}\right)\sin(2\tau)\right)\sin(3\tau),\\
E_0 &= (2+\cos(3\tau))^2\cos(2\tau)\sin(2\tau)\sin(3\tau).
\end{aligned}
\tag{3.2}
$$

Substituting these expressions for the pressures and density of energy into the system (2.18), we obtain the following solution:

$$
\begin{aligned}
A &= A_0+[2+\cos(3\tau)]\cos(2\tau),\\
B &= B_0+[2+\cos(3\tau)]\sin(2\tau),\\
C &= C_0+\sin(3\tau),
\end{aligned}
\tag{3.3}
$$

where A_0, B_0, C_0 are some real constants. We see that this solution describes the trefoil knot. In fact the solution (3.3) is the parametric equation of the trefoil knot. In Figure 1 we plot the trefoil knot for (3.3), where we assume

$$
A_0 = B_0 = C_0 = 0 \tag{3.4}
$$

and the initial conditions are $A(0) = 3$, $B(0) = C(0) = 0$. The Hubble parameters for the solution (3.3) with (3.4) read as

$$
\begin{aligned}
H_1 &= -2\tan(2\tau)-\frac{2\sin(3\tau)}{2+\cos(3\tau)},\\
H_2 &= -2\cot(2\tau)-\frac{2\sin(3\tau)}{2+\cos(3\tau)},\\
H_3 &= 3\cot(3\tau).
\end{aligned}
\tag{3.5}
$$

In Figure 2 we plot the evolution of H_i for thr solution (3.5) with (3.4). It is interesting to study the evolution of the volume of the trefoil knot universe. For our case it is given by

$$
V = [2+\cos(3\tau)]^2\cos(2\tau)\sin(2\tau)\sin(3\tau). \tag{3.6}
$$

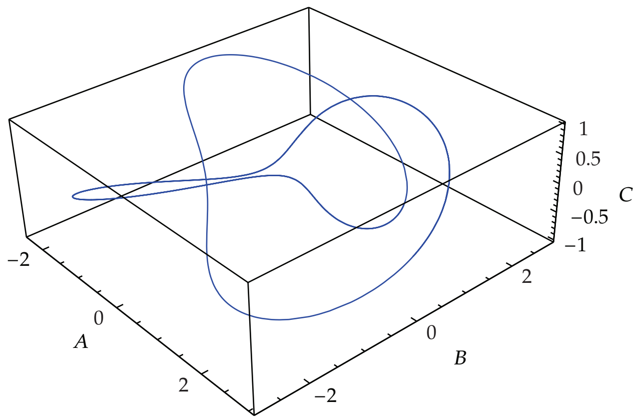

Figure 1: The trefoil knot for (3.3), where $A_0 = B_0 = C_0 = 0$.

In Figure 3 we plot the evolution of the volume of the trefoil knot universe with respect to the cosmic time τ for (3.6) with (3.4). To get $V \geq 0$, we must consider $A_0, B_0, C_0 > 0$, if exactly for example, as $A_0 > 3$, $B_0 > 3$, $C_0 > 1$. But below for simplicity we take the case (3.4). The other interesting quantity is the scalar curvature. For the trefoil knot solution (3.3), it has the form

$$\begin{aligned} R = \Big(6\Big(&12\sin^2(3\tau)\sin^2(2\tau) - 28\sin(3\tau)\cos(2\tau)\sin(2\tau) \\ &+ 3\sin^3(3\tau)\cos(2\tau)\sin(2\tau) \\ &- 12\sin^2(3\tau)\cos^2(2\tau) - 8\cos(3\tau)\sin^2(2\tau) \\ &- 8\cos^2(3\tau)\sin^2(2\tau) - 2\cos^3(3\tau)\sin^2(2\tau) \\ &+ 8\cos(3\tau)\cos^2(2\tau) + 8\cos^2(3\tau)\cos^2(2\tau) + 2\cos^3(3\tau)\cos^2(2\tau) \\ &- 52\sin(3\tau)\cos(2\tau)\cos(3\tau)\sin(2\tau) \\ &- 19\cos(2\tau)\sin(2\tau)\sin(3\tau)\cos^2(3\tau) \\ &+ 6\sin^2(2\tau)\sin^2(3\tau)\cos(3\tau) \\ &- 6\cos^2(2\tau)\sin^2(3\tau)\cos(3\tau)\Big)\Big) \\ \Big/\Big(&\sin(2\tau)\cos(2\tau)(2+\cos(3\tau))^2\sin(3\tau)\Big). \end{aligned} \tag{3.7}$$

In Figure 4 we plot the evolution of the R with respect of the cosmic time τ.

So we have shown that the universe can live in the trefoil knot orbit according to the solution (3.3). It is interesting to note that this trefoil knot solution admits infinite number accelerated and decelerated expansion phases of the universe. To show this, as an example let us consider the solution for C from (3.3) that is $C = C_0 + \sin(3\tau)$. In this case we have $\ddot{C} = -9\sin(3\tau)$ so that $\ddot{C} > 0$ (accelerating phase) as $\tau \in ((\pi/3) + (2n\pi/3), (2\pi/3) + (2n\pi/3))$

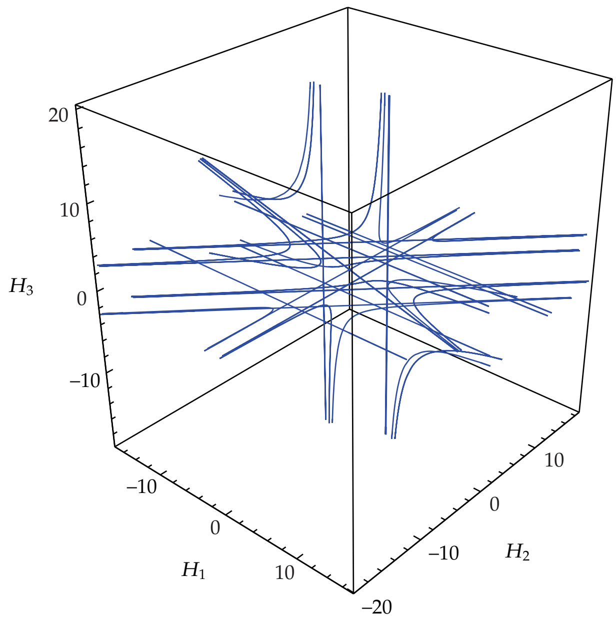

Figure 2: The evolution of the Hubble parameters for (3.5).

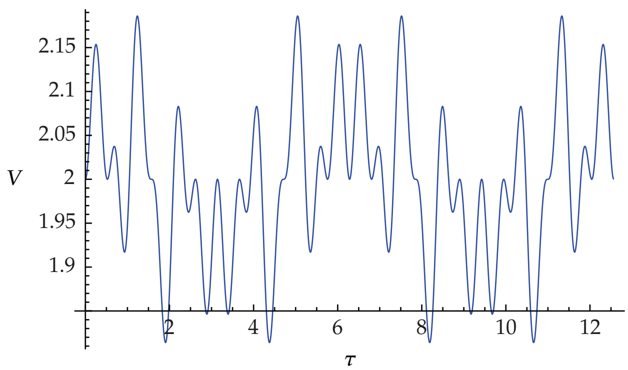

Figure 3: The evolution of the volume of the trefoil knot universe with respect to the cosmic time τ for (3.6).

and $\ddot{C} < 0$ (decelerating phase) as $\tau \in ((2n\pi/3), (\pi/3) + (2n\pi/3))$ with the transion points $\dot{C} = 3\cos(3\tau_i) = 0$ as $\tau_i = (0.5\pi + n\pi)/3$, where n is integer that is $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

3.2. Example 2

Now we consider the following parametric EoS:

$$p_1 = -\frac{D_1}{E_1},$$
$$p_2 = -\frac{D_2}{E_2},$$

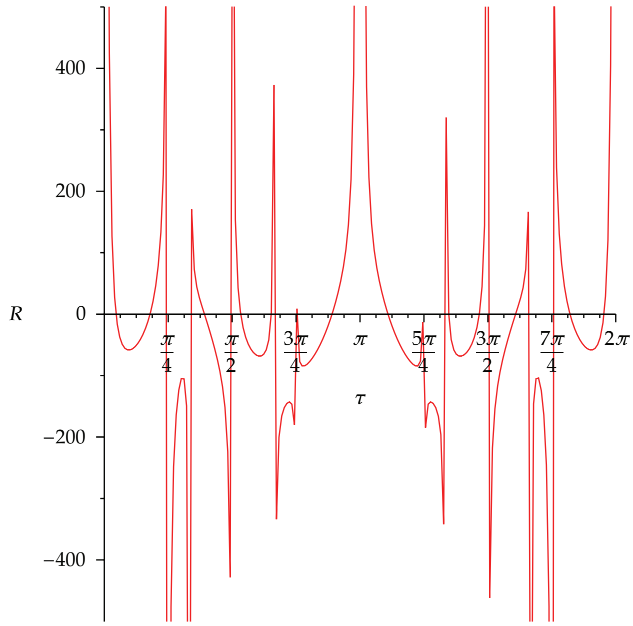

Figure 4: The evolution of the R with respect of the cosmic time τ for (3.7).

$$
\begin{aligned}
p_3 &= -\frac{D_3}{E_3}, \\
\rho &= \frac{D_0}{E_0},
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
D_1 = & -\sin^2(2\tau)\cos^2(3\tau) \\
& + \Big(-2\cos(2\tau) - 4\sin^2(2\tau) - 3 - \sin(3\tau)\sin(2\tau)\Big)\cos(3\tau) \\
& + \sin(3\tau)\sin(2\tau) - 4\cos(2\tau) \\
& - \sin^2(3\tau) - 4\sin^2(2\tau), \\
E_1 = & \ 1, \\
D_2 = & -\Big(2+\cos^2(3\tau)\Big)\cos^2(2\tau) - \sin(3\tau)(-1+\cos(3\tau))\cos(2\tau) \\
& + (-3 + 2\sin(2\tau))\cos(3\tau) + 4\sin(2\tau) - \sin^2(3\tau), \\
E_2 = & \ 1, \\
D_3 = & -\Big(2+\cos^2(3\tau)\Big)\cos^2(2\tau)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-\sin(2\tau)\cos^2(3\tau)+(-4\sin(2\tau)-2)\cos(3\tau)\\
&\qquad+3\sin(3\tau)-4-4\sin(2\tau)\Big)\cos(2\tau)+2\sin(2\tau)\cos(3\tau)\\
&+(4+3\sin(3\tau))\sin(2\tau)-\sin^2(3\tau),\\
E_3 &= 1,\\
D_0 &= (2+\cos(3\tau))\\
&\times(((2+\cos(3\tau))\sin(2\tau)+\sin(3\tau))\cos(2\tau)+\sin(3\tau)\sin(2\tau)),\\
E_0 &= 1.
\end{aligned} \tag{3.9}
$$

Substituting these expressions for the pressures and density of energy into the system (3.8), we obtain the following solution:

$$
\begin{aligned}
H_1 &= [2+\cos(3\tau)]\cos(2\tau)\\
&= 2\cos(2\tau)+0.5[\cos(5\tau)+\cos(\tau)],\\
H_2 &= [2+\cos(3\tau)]\sin(2\tau)\\
&= 2\sin(2\tau)+0.5[\sin(5\tau)-\sin(\tau)],\\
H_3 &= \sin(3\tau).
\end{aligned} \tag{3.10}
$$

We see that this solution again describes the trefoil knot but for the "coordinates" H_i. Note that the scale factors we can recovered from (2.14). We get

$$
\begin{aligned}
A &= A_0 e^{\sin(2\tau)+0.1\sin(5\tau)+0.5\sin(\tau)},\\
B &= B_0 e^{-[\cos(2\tau)+0.1\cos(5\tau)-0.5\cos(\tau)]},\\
C &= C_0 e^{-(1/3)\cos(3\tau)},
\end{aligned} \tag{3.11}
$$

where A_0, B_0, C_0 are some real constants. In Figure 5 we plot the evolution of A, B, C accordingly to (3.11) and for the initial conditions $A(0)=1$, $B(0)=e^{-0.6}$, $C(0)=e^{-1/3}$, where we assume that $A_0=B_0=C_0=1$. For this example, the volume of the universe is given by

$$
V = V_0 e^{\{\sin(2\tau)+0.1\sin(5\tau)+0.5\sin(\tau)-[\cos(2\tau)+0.1\cos(5\tau)-0.5\cos(\tau)]-(1/3)\cos(3\tau)\}}. \tag{3.12}
$$

The evolution of the volume for (3.12) is presented in Figure 6 for $A_0=B_0=C_0=V_0=1$ and for the intial condition $V(0)=e^{-14/15}$.

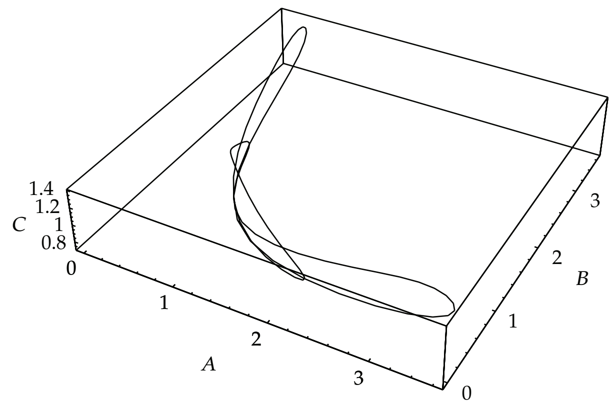

Figure 5: The evolution of A, B, C accordingly to (3.11), $t \in [0,\ 2\pi]$.

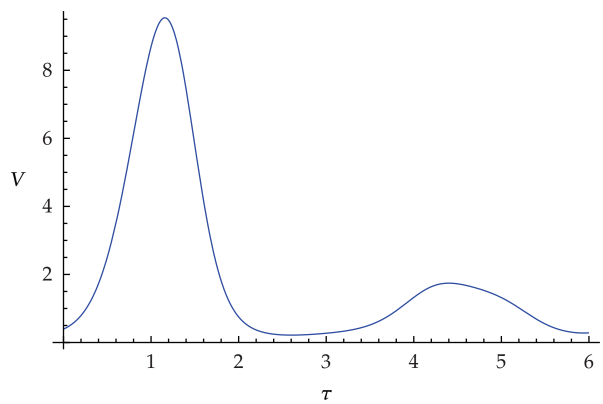

Figure 6: The evolution of the volume for (3.12) with $A_0 = B_0 = C_0 = V_0 = 1$.

The scalar curvature has the form

$$\begin{aligned} R = {} & \left(2\cos^2(2\tau) + 2\sin^2(2\tau) + 2\cos(2\tau)\sin(2\tau)\right)\cos^2(3\tau) \\ & + \Big(8\cos^2(2\tau) + (2\sin(3\tau) + 4 + 8\sin(2\tau))\cos(2\tau) \\ & \quad + 6 + 8\sin^2(2\tau) + (-4 + 2\sin(3\tau))\sin(2\tau)\Big)\cos(3\tau) \\ & + 8\cos^2(2\tau) + (-2\sin(3\tau) + 8 + 8\sin(2\tau))\cos(2\tau) \\ & + 8\sin^2(2\tau) + (-8 - 2\sin(3\tau))\sin(2\tau) + 2\sin^2(3\tau). \end{aligned} \tag{3.13}$$

In Figure 7 we plot the evolution of the R with respect of the cosmic time τ. Finally we conclude that the Einstein equations for the Bianchi I type metric admit the trefoil knot solution of the form (3.10) or (3.11). These solutions describe the accelerated and decelerated phases of the expansion of the universe.

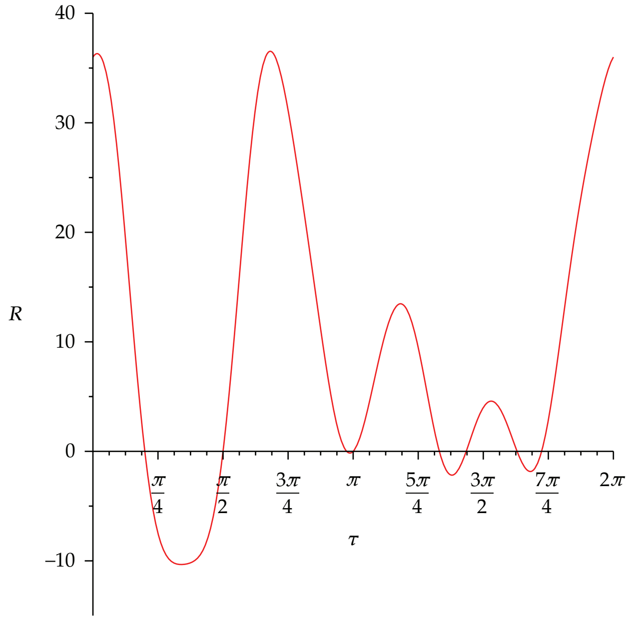

Figure 7: The evolution of the R with respect of the cosmic time τ for (3.13).

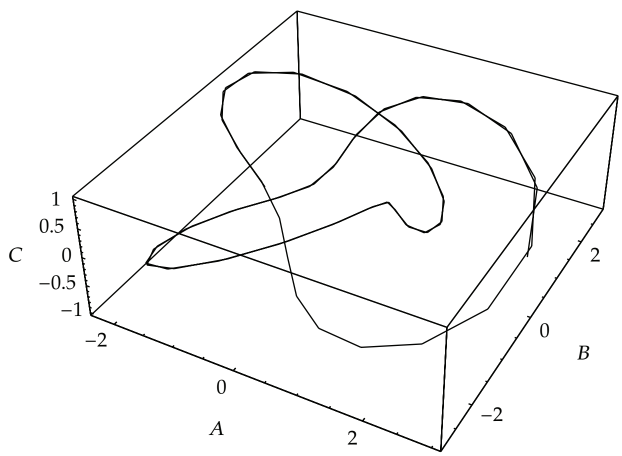

Figure 8: The knotted closed curve corresponding to the solution (3.14) with (3.4), $t \in [0, 4\pi]$, $k = 1/3$.

3.3. *Example 3*

Now we present a new kind of the trefoil knot universes. Let the system (2.18) has the solution

$$A = A_0 + [2 + \mathrm{cn}(3\tau)]\mathrm{cn}(2\tau),$$
$$B = B_0 + [2 + \mathrm{cn}(3\tau)]\mathrm{sn}(2\tau), \quad (3.14)$$
$$C = C_0 + \mathrm{sn}(3\tau),$$

where $\mathrm{cn}(t) \equiv \mathrm{cn}(t,k)$ and $\mathrm{sn}(t) \equiv \mathrm{sn}(t,k)$ are the Jacobian elliptic functions which are doubly periodic functions, and k is the elliptic modulus. Figure 8 shows the knotted closed curve corresponding to the solution (3.14) with (3.4). Substituting (3.14) into the system (2.18) we get the corresponding expressions for ρ and p_i that gives us the parametric EoS. This parametric EoS reads as

$$\begin{aligned} p_1 &= -\frac{D_1}{E_1}, \\ p_2 &= -\frac{D_2}{E_2}, \\ p_3 &= -\frac{D_3}{E_3}, \\ \rho &= \frac{D_0}{E_0}, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
D_1 = {} & 9k^2\mathrm{cn}(3\tau,k)\mathrm{sn}^3(3\tau,k)\mathrm{sn}(2\tau,k) \\
& - 12\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\mathrm{sn}^2(3\tau,k)\mathrm{cn}(2\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k) \\
& - 4\mathrm{sn}(2\tau,k)\Bigg(\left(\frac{9}{4}\right)\mathrm{cn}^3(3\tau,k)k^2 + \left(\frac{9}{2}\right)\mathrm{cn}^2(3\tau,k)k^2 \\
& \qquad + \left(\left(\frac{45}{4}\right)\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k) + \mathrm{cn}^2(2\tau,k)k^2 + \frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k)\right)\mathrm{cn}(3\tau,k) \\
& \qquad + 2\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k) + \left(\frac{9}{2}\right)\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k) + 2\mathrm{cn}^2(2\tau,k)k^2\Bigg)\mathrm{sn}(3\tau,k) \\
& + 18\mathrm{cn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)(2+\mathrm{cn}(3\tau,k))\mathrm{cn}(2\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k), \\
E_1 = {} & (2+\mathrm{cn}(3\tau,k))\mathrm{sn}(2\tau,k)\mathrm{sn}(3\tau,k), \\
D_2 = {} & 9k^2\mathrm{cn}(3\tau,k)\mathrm{sn}^3(3\tau,k)\mathrm{cn}(2\tau,k) + 12\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\mathrm{sn}^2(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k) \\
& - 9\mathrm{cn}(2\tau,k)\mathrm{cn}^3(3\tau,k)k^2 + 2\mathrm{cn}^2(3\tau,k)k^2 \\
& + \left(\left(\frac{4}{9}\right)\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k) + 5\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k) - \left(\frac{4}{9}\right)k^2\mathrm{sn}^2(2\tau,k)\right)\mathrm{cn}(3\tau,k) \\
& + \left(\frac{8}{9}\right)\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k) + 2\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k) - \left(\frac{8}{9}\right)k^2\mathrm{sn}^2(2\tau,k)\mathrm{sn}(3\tau,k) \\
& - 18\mathrm{cn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)(2+\mathrm{cn}(3\tau,k))\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k), \\
E_2 = {} & \mathrm{sn}(3\tau,k)(2+\mathrm{cn}(3\tau,k))\mathrm{cn}(2\tau,k), \\
D_3 = {} & -4k^2\mathrm{sn}(2\tau,k)(2+\mathrm{cn}(3\tau,k))^2\mathrm{cn}^3(2\tau,k) \\
& - 30\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)(2+\mathrm{cn}(3\tau,k))\mathrm{cn}^2(2\tau,k) + 4\mathrm{sn}(2\tau,k)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(k^2\left(2+\mathrm{cn}^2(3\tau,k)\right)\mathrm{sn}^2(2\tau,k)\right.\\
&\quad + \left(-\left(\frac{9}{2}\right)\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k) + \left(\frac{9}{2}\right)k^2\mathrm{sn}^2(3\tau,k) - 5\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k)\right)\mathrm{cn}^2(3\tau,k)\\
&\quad + \left(-20\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k) + 9k^2\mathrm{sn}^2(3\tau,k) - 9\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k)\right)\mathrm{cn}(3\tau,k)\\
&\quad \left. + \left(\frac{27}{4}\right)\mathrm{sn}^2(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)^2 - 20\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k)\right)\mathrm{cn}(2\tau,k)\\
&+ 30\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\mathrm{sn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}^2(2\tau,k)(2+\mathrm{cn}(3\tau,k)),\\
E_3 =&\ (2+\mathrm{cn}(3\tau,k))^2\mathrm{cn}(2\tau,k)\mathrm{sn}(2\tau,k),\\
D_0 =&\ -4\mathrm{sn}(3\tau,k)\mathrm{sn}(2\tau,k)\mathrm{cn}(2\tau,k)(2+\mathrm{cn}(3\tau,k))^2\frac{\partial}{\partial\tau}\mathrm{am}^2(2\tau,k)\\
&+ 6\frac{\partial}{\partial\tau}\mathrm{am}(\tau,k)(2+\mathrm{cn}(3\tau,k))\\
&\times \left(\mathrm{cn}^2(3\tau,k) + 2\mathrm{cn}(3\tau,k) - \mathrm{sn}^2(3\tau,k)\right)(\mathrm{cn}(2\tau,k) - \mathrm{sn}(2\tau,k))\\
&\times (\mathrm{cn}(2\tau,k) + \mathrm{sn}(2\tau,k))\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k) - 18\mathrm{sn}(3\tau,k)\mathrm{sn}(2\tau,k)\\
&\times \left(-\left(\frac{1}{2}\right)\mathrm{sn}^2(3\tau,k) + \mathrm{cn}^2(3\tau,k) + 2\mathrm{cn}(3\tau,k)\right)\mathrm{cn}(2\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}^2(3\tau,k),\\
E_0 =&\ \left(2+\mathrm{cn}^2(3\tau,k)\right)\mathrm{cn}(2\tau,k)\mathrm{sn}(2\tau,k)\mathrm{sn}(3\tau,k).
\end{aligned}
\tag{3.16}
$$

The volume of the universe for the solution (3.14) with (3.4) looks like

$$
V = [2+\ \mathrm{cn}(3\tau)]^2\ \mathrm{cn}(2\tau)\mathrm{sn}(2\tau)\mathrm{sn}(3\tau). \tag{3.17}
$$

The evolution of the volume for (3.17) is presented in Figure 9 The scalar curvature has the form

$$
\begin{aligned}
R = \Big(&-8\mathrm{sn}(2\tau,k)\mathrm{sn}(3\tau,k)k^2(2+\mathrm{cn}(3\tau,k))^2\mathrm{cn}^3(2\tau,k)\\
&+ 12\mathrm{dn}(2\tau,k)\mathrm{dn}(3\tau,k)(2+\mathrm{cn}(3\tau,k))\\
&\times \left(\mathrm{cn}^2(3\tau,k) + 2\mathrm{cn}(3\tau,k) - 3\mathrm{sn}^2(3\tau,k)\right)\mathrm{cn}^2(2\tau,k) - 18\mathrm{sn}(3\tau,k)
\end{aligned}
$$

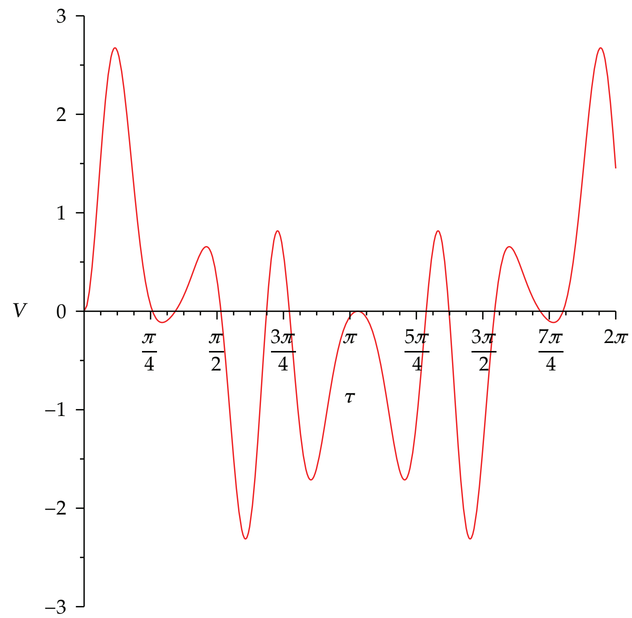

Figure 9: The evolution of the volume of the trefoil knot universe with respect to the cosmic time τ for (3.17).

$$
\begin{aligned}
&\times\Bigg(-\left(\frac{4}{9}\right)k^2(2+\mathrm{cn}(3\tau,k))^2\mathrm{sn}^2(2\tau,k)\\
&\quad+\Big(-4k^2\mathrm{cn}(3\tau,k)-2\mathrm{cn}^2(3\tau,k)k^2-\mathrm{dn}^2(3\tau,k)\Big)\mathrm{sn}^2(3\tau,k)\\
&\quad+(2+\mathrm{cn}(3\tau,k))\\
&\quad\times\Bigg(\mathrm{cn}^3(3\tau,k)k^2+2\mathrm{cn}^2(3\tau,k)k^2\\
&\qquad+\left(5\mathrm{dn}^2(3\tau,k)+\left(\frac{4}{3}\right)\mathrm{dn}^2(2\tau,k)\right)\mathrm{cn}(3\tau,k)\\
&\qquad+\left(\frac{8}{3}\right)\mathrm{dn}^2(2\tau,k)+2\mathrm{dn}^2(3\tau,k)\Bigg)\Bigg)\mathrm{sn}(2\tau,k)\mathrm{cn}(2\tau,k)\\
&-12\mathrm{dn}(2\tau,k)\mathrm{sn}^2(2\tau,k)\mathrm{dn}(3\tau,k)(2+\mathrm{cn}(3\tau,k))\\
&\times\Big(\mathrm{cn}^2(3\tau,k)+2\mathrm{cn}(3\tau,k)-3\mathrm{sn}^2(3\tau,k)\Big)\Big)\\
&/\Big(\mathrm{cn}(2\tau,k)\mathrm{sn}(2\tau,k)(2+\mathrm{cn}(3\tau,k))^2\mathrm{sn}(3\tau,k)\Big).
\end{aligned}
\tag{3.18}
$$

In Figure 10 we plot the evolution of the R with respect of the cosmic time τ.

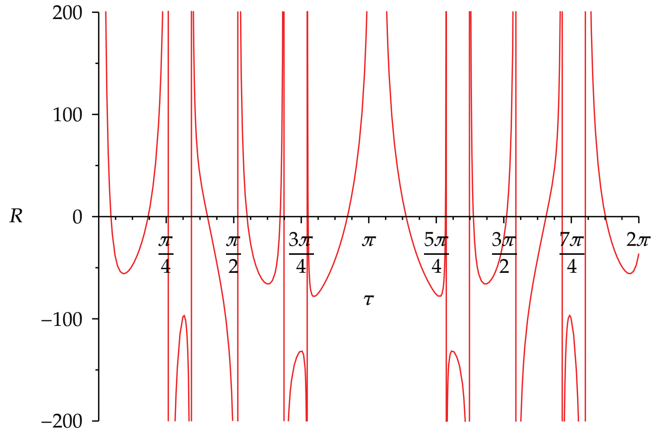

Figure 10: The evolution of the R with respect of the cosmic time τ for (3.18).

3.4. Example 4

Our fourth example is given by

$$\begin{aligned} H_1 &= [2+\mathrm{cn}(3\tau)]\mathrm{cn}(2\tau), \\ H_2 &= [2+\mathrm{cn}(3\tau)]\mathrm{sn}(2\tau), \\ H_3 &= \mathrm{sn}(3\tau), \end{aligned} \tag{3.19}$$

Which again the knotted closed curve in Figure 8 but for the "coordinates" H_i. Note that the corresponding parametric EoS looks like

$$\begin{aligned} p_1 &= -\frac{D_1}{E_1}, \\ p_2 &= -\frac{D_2}{E_2}, \\ p_3 &= -\frac{D_3}{E_3}, \\ \rho &= \frac{D_0}{E_0}, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} D_1 = &-(2+\mathrm{cn}(3\tau,k))^2\mathrm{sn}^2(2\tau,k) - \mathrm{sn}(3\tau,k)\left(-3\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k) + 2 + \mathrm{cn}(3\tau,k)\right)\mathrm{sn}(2\tau,k) \\ &+ \left(-2\mathrm{cn}(2\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k) - 3\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\right)\mathrm{cn}(3\tau,k) - 4\mathrm{cn}(2\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k) \\ &- \mathrm{sn}^2(3\tau,k), \\ E_1 = &\ 1, \end{aligned}$$

$$
\begin{aligned}
D_2 = &-(2+\mathrm{cn}(3\tau,k))^2\mathrm{cn}^2(2\tau,k) - \mathrm{sn}(3\tau,k)\left(-3\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)+2+\mathrm{cn}(3\tau,k)\right)\mathrm{cn}(2\tau,k)\\
&+\left(-3\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)+2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k)\right)\mathrm{cn}(3\tau,k)+4\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k)\\
&-\mathrm{sn}^2(3\tau,k),\\
E_2 = &\ 1,\\
D_3 = &-(2+\mathrm{cn}(3\tau,k))^2\mathrm{cn}^2(2\tau,k)\\
&+\left(-\mathrm{sn}(2\tau,k)\mathrm{cn}^2(3\tau,k)+\left(-4\mathrm{sn}(2\tau,k)-2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\right)\mathrm{cn}(3\tau,k)\right.\\
&\quad\left.+3\mathrm{sn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)-4\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)-4\mathrm{sn}(2\tau,k)\right)\\
&\times\mathrm{cn}(2\tau,k)+2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k)\mathrm{cn}(3\tau,k)\\
&+\left(4\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)+3\mathrm{sn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\right)\mathrm{sn}(2\tau,k)\\
&-\mathrm{sn}^2(3\tau,k),\\
E_3 = &\ 1,\\
D_0 = &\ (((2+\mathrm{cn}(3\tau,k))\mathrm{sn}(2\tau,k)+\mathrm{sn}(3\tau,k))\mathrm{cn}(2\tau,k)+\mathrm{sn}(2\tau,k)\mathrm{sn}(3\tau,k))\\
&\times(2+\mathrm{cn}(3\tau,k)),\\
E_0 = &\ 1.
\end{aligned}
\tag{3.21}
$$

In Figure 11 we plot the evolution of p_i, ρ for (3.20). The scalar curvature has the form

$$
\begin{aligned}
R = &\ 2(2+\mathrm{cn}(3\tau,k))^2\mathrm{cn}^2(2\tau,k)\\
&+\left(2\mathrm{sn}(2\tau,k)\mathrm{cn}^2(3\tau,k)\right.\\
&\quad+(8\mathrm{sn}(2\tau,k)+4\mathrm{dn}(2\tau,k)+2\mathrm{sn}(3\tau,k))\mathrm{cn}(3\tau,k)\\
&\quad+8\mathrm{sn}(2\tau,k)+(4-6\mathrm{dn}(3\tau,k))\mathrm{sn}(3\tau,k)\\
&\quad\left.+8\mathrm{dn}(2\tau,k)\right)\mathrm{cn}(2\tau,k)+2\mathrm{sn}^2(2\tau,k)\mathrm{cn}^2(3\tau,k)\\
&+\left(8\mathrm{sn}^2(2\tau,k)+(-4\mathrm{dn}(2\tau,k)+2\mathrm{sn}(3\tau,k))\mathrm{sn}(2\tau,k)+6\mathrm{dn}(3\tau,k)\right)\\
&\times\mathrm{cn}(3\tau,k)+8\mathrm{sn}^2(2\tau,k)+((4-6\mathrm{dn}(3\tau,k))\mathrm{sn}(3\tau,k)-8\mathrm{dn}(2\tau,k))\\
&\times\mathrm{sn}(2\tau,k)+2\mathrm{sn}\ (3\tau,k)^2,
\end{aligned}
\tag{3.22}
$$

In Figure 12 we plot the evolution of the R with respect of the cosmic time τ.

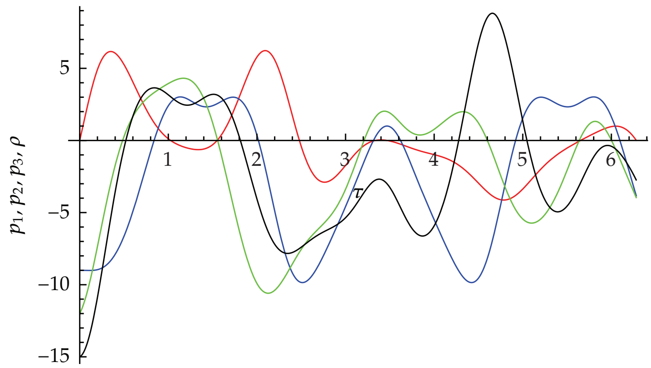

Figure 11: The evolution of p_i, ρ for (3.20), $t \in [0, 2\pi]$, $k = 1/3$, ρ (red), p_1 (blue), p_2 (green), p_3 (black).

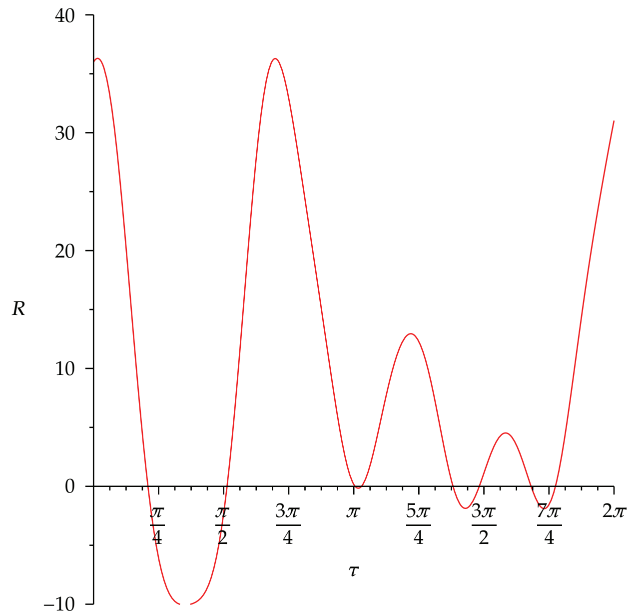

Figure 12: The evolution of the R with respect of the cosmic time τ for (3.22).

4. The Figure-Eight Knot Universe

Our aim in this section is to demonstrate some examples of the figure-eight knot universes for the Bianchi type I metric (2.6). We give some particular figure-eight knot universe models.

4.1. Example 1

Again, let us assume that our universe is filled by the fluid with the following parametric EoS:

$$\rho = \frac{D_8}{E_8},$$

$$p_1 = -\frac{D_9}{E_9},$$

$$p_2 = -\frac{D_{10}}{E_{10}},$$
$$p_3 = -\frac{D_{11}}{E_{11}}, \tag{4.1}$$

where

$$\begin{aligned}
D_8 =& \ (-2\sin(2\tau)\cos(3\tau) - (3(2+\cos(2\tau)))\sin(3\tau)) \\
&\times (-2\sin(2\tau)\sin(3\tau) + (3(2+\cos(2\tau)))\cos(3\tau)) \\
&\times \sin(4\tau) + 12\cos(3\tau) \\
&\times \left((2+\cos(2\tau))\cos(3\tau) - \left(\frac{2}{3}\right)\sin(2\tau)\sin(3\tau)\right) \\
&\times (2+\cos(2\tau))\cos(4\tau) - (12(2+\cos(2\tau)))\cos(4\tau) \\
&* \left((2+\cos(2\tau))\sin(3\tau) + \left(\frac{2}{3}\right)\sin(2\tau)\cos(3\tau)\right)\sin(3\tau), \\
E_8 =& \ (2+\cos(2\tau))^2\cos(3\tau)\sin(3\tau)\sin(4\tau), \\
D_9 =& \ ((72+36\cos(2\tau))\cos(4\tau) - 12\sin(4\tau)\sin(2\tau))\cos(3\tau) \\
&- \left(29\left(\left(\frac{24}{29}\right) * \cos(4\tau)\sin(2\tau) + \left(\cos(2\tau) + \frac{50}{29}\right)\sin(4\tau)\right)\right)\sin(3\tau), \\
E_9 =& \ \sin(4\tau)(2+\cos(2\tau))\sin(3\tau), \\
D_{10} =& \ (-24\cos(4\tau)\sin(2\tau) + \sin(4\tau)(-29\cos(2\tau) - 50))\cos(3\tau) \\
&- \left(36\left((2+\cos(2\tau))\cos(4\tau) - \left(\frac{1}{3}\right)\sin(4\tau)\sin(2\tau)\right)\right)\sin(3\tau), \\
E_{10} =& \ \sin(4\tau)(2+\cos(2\tau))\cos(3\tau), \\
D_{11} =& \ -(30(2+\cos(2\tau)))\sin(2\tau)\cos(3\tau)^2 + \sin(3\tau) \\
&\times \left(12\sin(2\tau)^2 - 196\cos(2\tau) - 180 - 53\cos(2\tau)^2\right)\cos(3\tau) \\
&+ 30\sin(2\tau)\sin(3\tau)^2(2+\cos(2\tau)), \\
E_{11} =& \ (2+\cos(2\tau))^2\cos(3\tau)\sin(3\tau).
\end{aligned} \tag{4.2}$$

Substituting these expressions for the pressuries and the density of energy into the system (2.18), we obtain the following its solution [98–103]:

$$\begin{aligned}
A &= A_0 + [2+\cos(2\tau)]\cos(3\tau), \\
B &= B_0 + [2+\cos(2\tau)]\sin(3\tau), \\
C &= C_0 + \sin(4\tau).
\end{aligned} \tag{4.3}$$

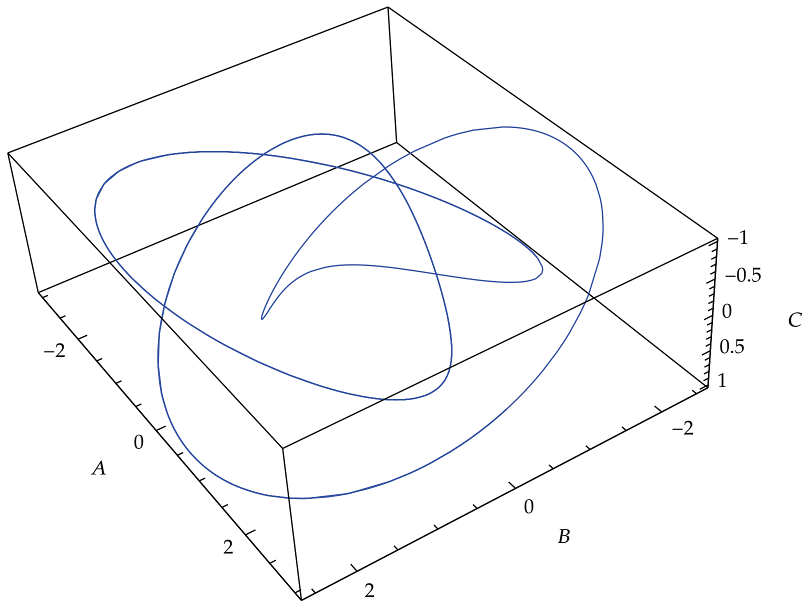

Figure 13: The figure-eight knot for (4.3) with (3.4).

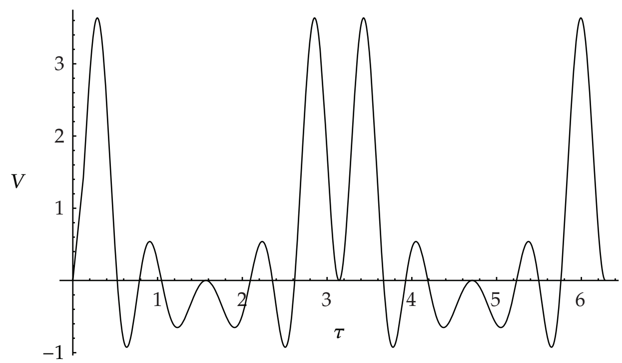

Figure 14: The evolution of the volume for the solution (4.3) with (3.4), $t \in [0, 2\pi]$.

This solution is nothing but the parametric equation of the figure-eight knot as we can see from Figure 13, where we assume that $A_0 = B_0 = C_0 = 0$ and the initial conditions have the form $A(0) = 3, B(0) = 0$, $C(0) = 0$. And for that reason in [98–103] we called such models as the figure-eight knot universes. Note that the "coordinates" A, B, C with (3.4) satisfy the equation

$$4(h-2)^4 - 4(h-2)^2 + z^2 = 0, \tag{4.4}$$

where $h = 2 + \cos(2\tau)$. Let us calculate the volume of the universe. For our case it is given by

$$V = [2 + \cos(2\tau)]^2 \cos(3\tau) \sin(3\tau) \sin(4\tau), \tag{4.5}$$

where we used (3.4). In Figure 14 we present the evolution of the volume for the solution (4.3) with (3.4). The scalar curvature has the form

$$\begin{aligned}
R = {} & \Bigg(\Bigg(24\left(\cos(4\tau)\cos(2\tau) - \left(\frac{3}{2}\right)\sin(4\tau)\sin(2\tau) + 2\cos(4\tau)\right)\Bigg) \\
& \times (2+\cos(2\tau))\cos^2(3\tau) - 102\sin(3\tau) \\
& \times \left(\sin(4\tau)\cos^2(2\tau) + \left(\left(\frac{188}{51}\right)\sin(4\tau) + \left(\frac{16}{51}\right)\cos(4\tau)\sin(2\tau)\right)\cos(2\tau)\right. \\
& \quad + \left(\frac{32}{51}\right)\cos(4\tau)\sin(2\tau) + \left(\frac{172}{51}\right)\sin(4\tau) \\
& \quad \left. - \left(\frac{4}{51}\right)\sin^2(2\tau)\sin(4\tau)\right)\cos(3\tau) - 24\sin^2(3\tau) \\
& \times \left(\cos(4\tau)\cos(2\tau) - \left(\frac{3}{2}\right)\sin(4\tau)\sin(2\tau) + 2\cos(4\tau)\right)(2+\cos(2\tau))\Bigg) \\
& /\left(\sin(3\tau)\cos(3\tau) * (2+\cos(2\tau))^2\sin(4\tau)\right).
\end{aligned} \tag{4.6}$$

In Figure 15 we plot the evolution of the R with respect of the cosmic time τ. So we found the figure-eight knot solution of the Einstein equations which again describe the accelerated and decelerated expansion phases of the universe.

4.2. Example 2

Now we consider the system (2.19). Its solution is given by

$$\begin{aligned}
H_1 &= [2+\cos(2\tau)]\cos(3\tau) = 2\cos(3\tau) + \cos(5\tau) + \cos(\tau), \\
H_2 &= [2+\cos(2\tau)]\sin(3\tau) = 2\sin(3\tau) + \sin(\tau) + \sin(5\tau), \\
H_3 &= \sin(4\tau).
\end{aligned} \tag{4.7}$$

Then the coorresponding scale factors read as

$$\begin{aligned}
A &= A_0 e^{(2/3)\sin(3\tau)+0.2\sin(5\tau)+\sin(\tau)}, \\
B &= B_0 e^{-[(2/3)\cos(3\tau)+0.2\cos(5\tau)+\cos(\tau)]}, \\
C &= C_0 e^{-0.25\cos(4\tau)}.
\end{aligned} \tag{4.8}$$

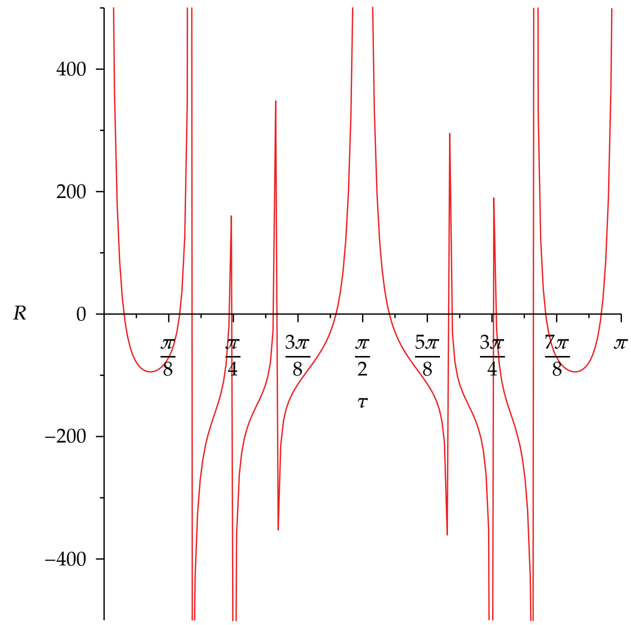

Figure 15: The evolution of the R with respect of the cosmic time τ for (4.6).

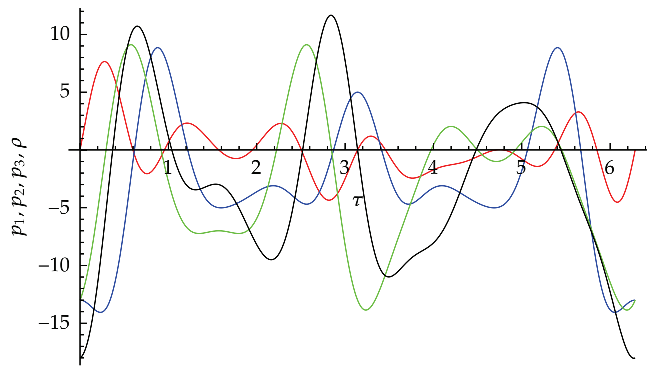

Figure 16: The plot of the EoS (4.9), $t \in [0, 2\pi]$, ρ (red), p_1 (blue), p_2 (green), p_3 (black).

For this solution the parametric EoS looks like

$$\rho = \frac{D_0}{E_0},$$
$$p_1 = -\frac{D_1}{E_1},$$
$$p_2 = -\frac{D_2}{E_2}, \tag{4.9}$$
$$p_3 = -\frac{D_3}{E_3},$$

where

$$
\begin{aligned}
D_0 &= (((2+\cos(2\tau))\sin(3\tau)+\sin(4\tau))\cos(3\tau)+\sin(3\tau)\sin(4\tau))(2+\cos(2\tau)),\\
E_0 &= 1,\\
D_1 &= -(2+\cos(2\tau))^2\sin^2(3\tau)+(2\sin(2\tau)-2\sin(4\tau)-\sin(4\tau)\cos(2\tau))\sin(3\tau)\\
&\quad -6\cos(3\tau)-3\cos(3\tau)\cos(2\tau)-4\cos(4\tau)-\sin^2(4\tau),\\
E_1 &= 1,\\
D_2 &= -(2+\cos(2\tau))^2\cos^2(3\tau)\\
&\quad +(2\sin(2\tau)-2\sin(4\tau)-\sin(4\tau)\cos(2\tau))\cos(3\tau)\\
&\quad -4\cos(4\tau)+6\sin(3\tau)+3\sin(3\tau)\cos(2\tau)-\sin^2(4\tau),\\
E_2 &= 1,\\
D_3 &= -3\sin(\tau)-64\sin(\tau)\cos^9(\tau)+36\sin(\tau)\cos^5(\tau)\\
&\quad +40\sin(\tau)\cos^4(\tau)+4\sin(\tau)\cos^3(\tau)\\
&\quad -6\sin(\tau)\cos^2(\tau)-3\sin(\tau)\cos(\tau)-25\cos^2(\tau)\\
&\quad +5\cos(\tau)-40\cos^5(\tau)-64\cos^10(\tau)\\
&\quad +96\cos^8(\tau)-84\cos^6(\tau)+68\cos^4(\tau)+26\cos^3(\tau),\\
E_3 &= 1.
\end{aligned}
\tag{4.10}
$$

In Figure 16 we plot the EoS (4.9). For this example, the evolution of the volume of the universe is given by

$$
V = V_0 e^{(2/3)\sin(3\tau)+0.2\sin(5\tau)+\sin(\tau)-(2/3)\cos(3\tau)-0.2\cos(5\tau)-\cos(\tau)-0.25\cos(4\tau)}. \tag{4.11}
$$

The evolution of the volume is presented in Figure 17 for $A_0 = B_0 = C_0 = V_0 = 1$ and for the intial condition $V(0) = e^{-127/60}$. The scalar curvature has the form

$$
\begin{aligned}
R = {} & 2(2+\cos(2\tau))^2\cos^2(3\tau)\\
& + \Big(2(2+\cos(2\tau))^2\sin(3\tau)+(6+2\sin(4\tau))\cos(2\tau)\\
& \qquad + 12-4\sin(2\tau)+4\sin(4\tau)\Big)\cos(3\tau)\\
& + 2(2+\cos(2\tau))^2\sin^2(3\tau)\\
& + ((-6+2\sin(4\tau))\cos(2\tau)+4\sin(4\tau)-4\sin(2\tau)-12)\\
& * \sin(3\tau)+2\sin^2(4\tau)+8\cos(4\tau).
\end{aligned}
\tag{4.12}
$$

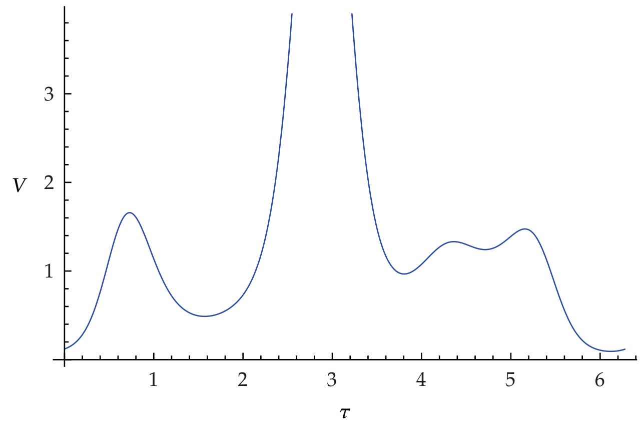

Figure 17: The evolution of the volume for the expression (4.11) with $V_0 = 1$, $t \in [0, 2\pi]$.

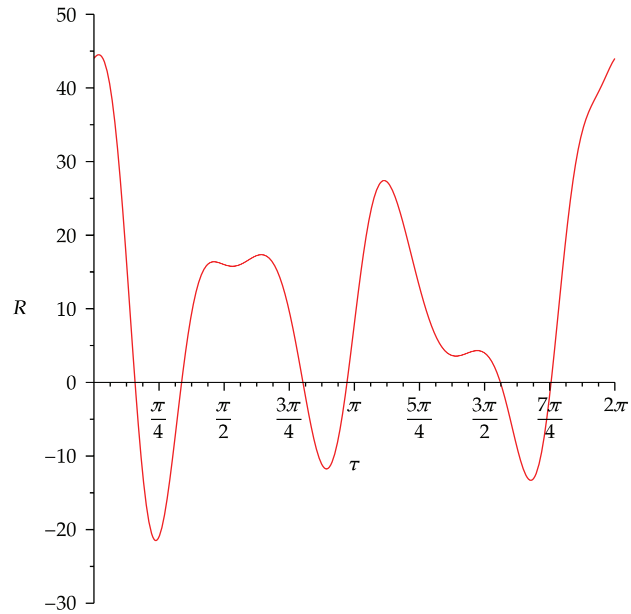

Figure 18: The evolution of the R with respect of the cosmic time τ for (4.12).

In Figure 18 we plot the evolution of the R with respect of the cosmic time τ. Again we have shown that the Einstein equations admit the figure-eight knot solution and it again describe the accelerated and decelerated expansion phases of the universe.

4.3. Example 3

Now we present the figure-eight knot universe induced by the Jacobian elliptic functions. Let the system (2.18) have the solution

$$\begin{aligned} A &= A_0 + [2 + \mathrm{cn}(2\tau)]\mathrm{cn}(3\tau), \\ B &= B_0 + [2 + \mathrm{cn}(2\tau)]\mathrm{sn}(3\tau), \\ C &= C_0 + \mathrm{sn}(4\tau). \end{aligned} \tag{4.13}$$

Note that $\mathrm{cn}(t)$ and $\mathrm{sn}(t)$ are the doubly periodic Jacobian elliptic functions. Figure 19 shows the knotted closed curve corresponding to the solution (4.13) with (3.4). Substituting the formulas (4.13) into the system (2.18) we get the corresponding expressions for ρ and p_i that gives us the parametric EoS. The evolution of the volume of the universe for (3.4) reads as

$$V = [2 + \mathrm{cn}(2\tau)]^2 \mathrm{cn}(3\tau)\mathrm{sn}(3\tau)\mathrm{sn}(4\tau). \tag{4.14}$$

The scalar curvature has the form

$$\begin{aligned}
R = \Bigg(& -18\mathrm{sn}(3\tau,k)\mathrm{sn}(4\tau,k)k^2(2+\mathrm{cn}(2\tau,k))^2\mathrm{cn}^3(3\tau,k) \\
& + \left(24\left(-\left(\frac{3}{2}\right)\mathrm{sn}(2\tau,k)\mathrm{dn}(2\tau,k)\mathrm{sn}(4\tau,k)\right.\right. \\
& \qquad \left.\left. +\mathrm{cn}(4\tau,k)\mathrm{dn}(4\tau,k)(2+\mathrm{cn}(2\tau,k))\right)\right) \\
& \times (2+\mathrm{cn}(2\tau,k))\mathrm{dn}(3\tau,k)\mathrm{cn}^2(3\tau,k) \\
& - \left(32\left(-\left(\frac{9}{16}\right)\mathrm{sn}(4\tau,k)k^2(2+\mathrm{cn}(2\tau,k))^2\mathrm{sn}^2(3\tau,k)\right.\right. \\
& \qquad + \left(\left(\mathrm{cn}^2(4\tau,k)k^2 + \left(\frac{27}{16}\right)\mathrm{dn}^2(3\tau,k) + \mathrm{dn}^2(4\tau,k)\right.\right. \\
& \qquad\qquad \left. - \left(\frac{1}{2}\right)k^2\mathrm{sn}^2(2\tau,k) + \left(\frac{1}{2}\right)\mathrm{dn}^2(2\tau,k)\right)\mathrm{cn}^2(2\tau,k) \\
& \qquad\quad + \left(-k^2\mathrm{sn}^2(2\tau,k)\left(\frac{27}{4}\right)\mathrm{dn}^2(3\tau,k) + \mathrm{dn}^2(2\tau,k)\right. \\
& \qquad\qquad \left. +4\mathrm{dn}^2(4\tau,k) + 4\mathrm{cn}^2(4\tau,k)k^2\right) \\
& \qquad\quad \times \mathrm{cn}(2\tau,k) + 4\,\mathrm{dn}^2(4\tau,k) \\
& \qquad\quad \left. + \left(\frac{27}{4}\right)\mathrm{dn}^2(3\tau,k) - \left(\frac{1}{4}\right)\mathrm{dn}^2(2\tau,k)\mathrm{sn}^2(2\tau,k) + 4\mathrm{cn}^2(4\tau,k)k^2\right) \\
& \qquad \times \mathrm{sn}(4\tau,k) + \mathrm{cn}(4\tau,k)\mathrm{dn} \\
& \qquad \left.\left. \times\ (4\tau,k)\mathrm{dn}(2\tau,k)\mathrm{sn}(2\tau,k)(2+\mathrm{cn}(2\tau,k))\right)\right)\mathrm{sn}(3\tau,k)\mathrm{cn}(3\tau,k) \\
& - \left(24\left(-\left(\frac{3}{2}\right)\mathrm{sn}(2\tau,k)\mathrm{dn}(2\tau,k)\mathrm{sn}(4\tau,k) + \mathrm{cn}(4\tau,k)\mathrm{dn}(4\tau,k)(2+\mathrm{cn}(2\tau,k))\right)\right) \\
& \left. \times \left(2+\mathrm{cn}(2\tau,k)\right)\mathrm{sn}^2(3\tau,k)\mathrm{dn}(3\tau,k)\right) \\
& / \left(\mathrm{cn}(3\tau,k)\mathrm{sn}(3\tau,k)(2+\mathrm{cn}(2\tau,k))^2\mathrm{sn}(4\tau,k)\right).
\end{aligned} \tag{4.15}$$

In Figure 20 we plot the evolution of the R with respect of the cosmic time τ.

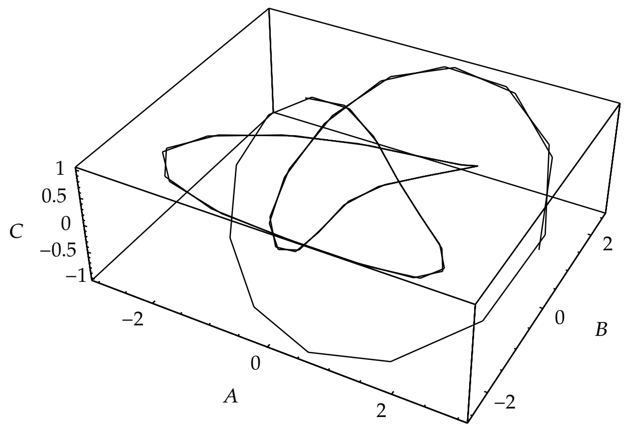

Figure 19: The knotted closed curve corresponding to the solution (4.13) with (3.4), $t \in [0, 4\pi]$, $k = 1/3$.

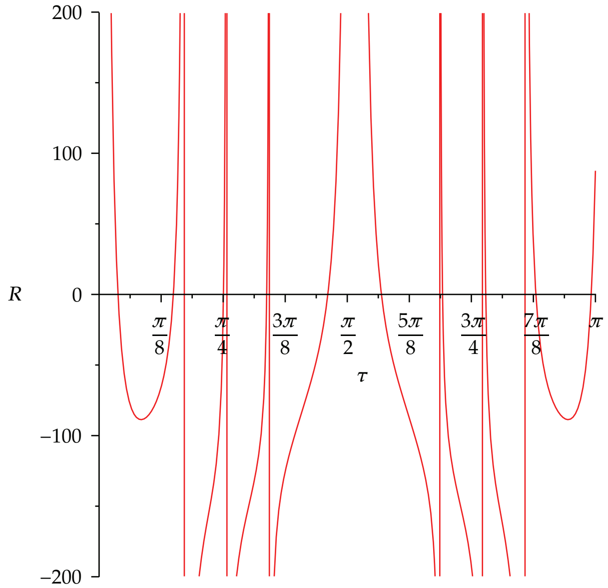

Figure 20: The evolution of the R with respect of the cosmic time τ for (4.15).

4.4. Example 4

We now consider the following solution of the system (2.19):

$$\begin{aligned} H_1 &= [2 + \text{cn}(2\tau)]\text{cn}(3\tau), \\ H_2 &= [2 + \text{cn}(2\tau)]\text{sn}(3\tau), \\ H_3 &= \text{sn}(4\tau), \end{aligned} \tag{4.16}$$

which again the trefoil knot universe as shown in Figure 19 but for the "coordinates" H_i. The corresponding parametric EoS reads as

$$\rho = \frac{D_0}{E_0}, \qquad p_1 = -\frac{D_1}{E_1}, \qquad p_2 = -\frac{D_2}{E_2}, \qquad p_3 = -\frac{D_3}{E_3}, \tag{4.17}$$

where

$$\begin{aligned}
D_0 &= (((2+\mathrm{cn}(2\tau,k))\mathrm{sn}(3\tau,k)+\mathrm{sn}(4\tau,k))\mathrm{cn}(3\tau,k)+\mathrm{sn}(3\tau,k)\mathrm{sn}(4\tau,k)) \\
&\quad \times (2+\mathrm{cn}(2\tau,k)), \\
E_0 &= 1, \\
D_1 &= 2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k)\mathrm{sn}(3\tau,k)-(3(2+\mathrm{cn}(2\tau,k)))\mathrm{cn}(3\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k) \\
&\quad -4\mathrm{cn}(4\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(4\tau,k)-(2+\mathrm{cn}(2\tau,k))^2\mathrm{sn}(3\tau,k)^2-\mathrm{sn}(4\tau,k)^2 \\
&\quad -(2+\mathrm{cn}(2\tau,k))\mathrm{sn}(3\tau,k)\mathrm{sn}(4\tau,k), \\
E_1 &= 1, \\
D_2 &= -4\mathrm{cn}(4\tau,k)\frac{\partial}{\partial\tau}\mathrm{am}(4\tau,k)+2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k)\mathrm{cn}(3\tau,k) \\
&\quad +(3(2+\mathrm{cn}(2\tau,k)))\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\mathrm{sn}(3\tau,k)-\mathrm{sn}^2(4\tau,k)-(2+\mathrm{cn}(2\tau,k))^2\mathrm{cn}^2(3\tau,k) \\
&\quad -(2+\mathrm{cn}(2\tau,k))\mathrm{cn}(3\tau,k)\mathrm{sn}(4\tau,k), \\
E_2 &= 1, \\
D_3 &= -(2+\mathrm{cn}(2\tau,k))^2\mathrm{cn}^2(3\tau,k) \\
&\quad +\Bigg(-\mathrm{sn}(3\tau,k)\mathrm{cn}^2(2\tau,k) \\
&\quad\quad +\left(-4\mathrm{sn}(3\tau,k)-3\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\right)\mathrm{cn}(2\tau,k)+2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k) \\
&\quad\quad -6\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)-4\mathrm{sn}(3\tau,k)\Bigg) \\
&\quad \times \mathrm{cn}(3\tau,k)+3\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)\mathrm{sn}(3\tau,k)\mathrm{cn}(2\tau,k) \\
&\quad +\left(6\frac{\partial}{\partial\tau}\mathrm{am}(3\tau,k)+2\frac{\partial}{\partial\tau}\mathrm{am}(2\tau,k)\mathrm{sn}(2\tau,k)\right) \\
&\quad \times \mathrm{sn}(3\tau,k)-\mathrm{sn}^2(4\tau,k), \\
E_3 &= 1.
\end{aligned} \tag{4.18}$$

Its plot we give in Figure 21.

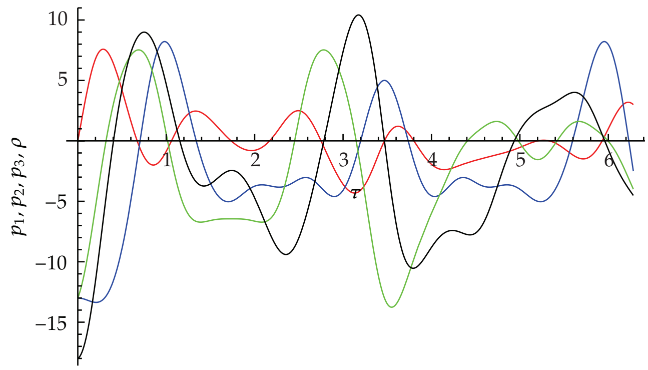

Figure 21: The plot of the EoS (4.17), $t \in [0, 2\pi]$, $k = 1/3$, ρ (red), p_1 (blue), p_2 (green), p_3 (black).

The scalar curvature has the form

$$\begin{aligned}
R = {} & 2(2+\mathrm{cn}(2\tau,k))^2\mathrm{cn}(3\tau,k)^2 \\
& + \Big(2(2+\mathrm{cn}(2\tau,k))^2\mathrm{sn}(3\tau,k) + (6\mathrm{dn}(3\tau,k) + 2\mathrm{sn}(4\tau,k))\mathrm{cn}(2\tau,k) \\
& \quad + 12\mathrm{dn}(3\tau,k) - 4\mathrm{dn}(2\tau,k)\mathrm{sn}(2\tau,k) + 4\mathrm{sn}(4\tau,k)\Big) \\
& \times \mathrm{cn}(3\tau,k) + 2(2+\mathrm{cn}(2\tau,k))^2\mathrm{sn}(3\tau,k)^2 \\
& + ((-6\mathrm{dn}(3\tau,k) + 2\mathrm{sn}(4\tau,k))\mathrm{cn}(2\tau,k) \\
& \quad + 4\mathrm{sn}(4\tau,k) - 4\mathrm{dn}(2\tau,k)\mathrm{sn}(2\tau,k) - 12\mathrm{dn}(3\tau,k)) \\
& * \mathrm{sn}(3\tau,k) + 2\mathrm{sn}(4\tau,k)^2 + 8\mathrm{cn}(4\tau,k)\mathrm{dn}(4\tau,k).
\end{aligned} \tag{4.19}$$

In Figure 22 we plot the evolution of the R with respect of the cosmic time τ.

5. Other Unknotted Models of the Universe

In this section we would like to present some unknotted but closed curve solutions of the Einstein equation for the Bianchi I type metric. As an examples we consider the spiky and Mobious strip universe solutions.

5.1. Spiky Universe Solutions

Our aim in this subsection is to present some unknotted closed curve solutions namely the spiky universe solutions.

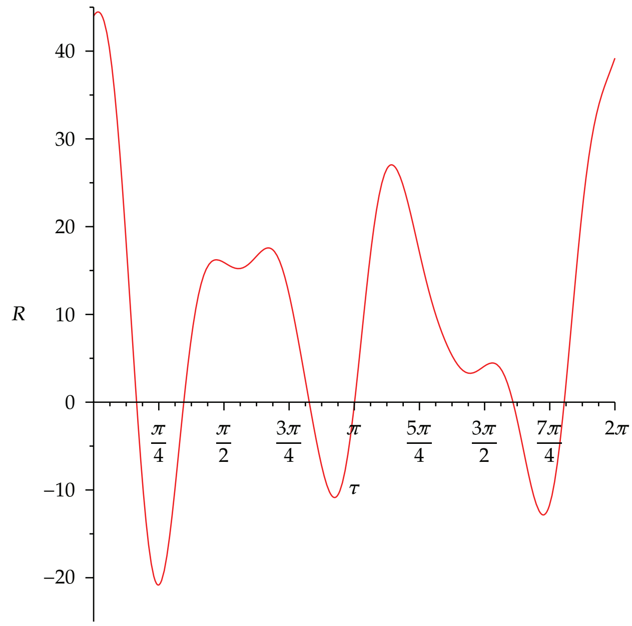

Figure 22: The evolution of the R with respect of the cosmic time τ for (4.19).

5.1.1. Example 1

Let our universe be filled by the fluid with the following parametric EoS:

$$\rho = \frac{D_8}{E_8},$$
$$p_1 = -\frac{D_9}{E_9},$$
$$p_2 = -\frac{D_{10}}{E_{10}},$$
$$p_3 = -\frac{D_{11}}{E_{11}}, \tag{5.1}$$

where

$$\begin{aligned} D_8 = &-[\alpha\sin((n-1)\tau)(n-1)+\alpha(n-1)\sin(\tau)] \\ &\times[\alpha\cos((n-1)\tau)(n-1)-\alpha(n-1)\cos(\tau)]\sin(\tau) \\ &+[\alpha\cos((n-1)\tau)(n-1)-\alpha(n-1)\cos(\tau)]\cos(\tau) \\ &\times[\alpha\cos((n-1)\tau)+\alpha(n-1)\cos(\tau)] \\ &-[\alpha\sin((n-1)\tau)(n-1)+\alpha(n-1)\sin(\tau)] \\ &\times[\alpha\sin((n-1)\tau)-\alpha(n-1)\sin(\tau)]\cos(\tau), \end{aligned}$$

$$
\begin{aligned}
E_8 &= [\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)] \\
&\quad \times [\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)]\sin(\tau), \\
D_9 &= \left[-\alpha\sin((n-1)\tau)(n-1)^2 + \alpha(n-1)\sin(\tau)\right]\sin(\tau) \\
&\quad - [\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)]\sin(\tau) \\
&\quad + [\alpha\cos((n-1)\tau)(n-1) - \alpha(n-1)\cos(\tau)]\cos(\tau), \\
E_9 &= [\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)]\sin(\tau), \\
D_{10} &= [\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)]\sin(\tau) + \sin(\tau) \\
&\quad \times \left[\alpha\cos((n-1)\tau)(n-1)^2 + \alpha(n-1)\cos(\tau)\right] \\
&\quad + [\alpha\sin((n-1)\tau)(n-1) + \alpha(n-1)\sin(\tau)]\cos(\tau), \\
E_{10} &= -\,[\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)]\sin(\tau), \\
D_{11} &= [\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)] \\
&\quad \times \left[-\alpha\cos((n-1)\tau)(n-1)^2 - \alpha(n-1)\cos(\tau)\right] \\
&\quad + [\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)] \\
&\quad \times \left[-\alpha\sin((n-1)\tau)(n-1)^2 + \alpha(n-1)\sin(\tau)\right] \\
&\quad - [\alpha\sin((n-1)\tau)(n-1) + \alpha(n-1)\sin(\tau)] \\
&\quad \times [\alpha\cos((n-1)\tau)(n-1) - \alpha(n-1)\cos(\tau)], \\
E_{11} &= [\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)] \\
&\quad \times [\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)].
\end{aligned} \tag{5.2}
$$

Substituting these expressions for the pressuries and the density of energy into the system (2.18), we obtain the following solution:

$$
\begin{gathered}
A = \alpha\cos[(n-1)\tau] + \alpha(n-1)\cos[\tau], \\
B = \alpha\sin[(n-1)\tau] - \alpha(n-1)\sin[\tau], \\
C = \sin(\tau).
\end{gathered} \tag{5.3}
$$

It is the spiky-like solution so that such solutions we call the spike universe. Its plot is presented in Figure 23 for the initial conditions $A(0) = \alpha n = 10$, $B(0) = 0$, $C(0) = 0$. Let us calculate the volume of this universe. It is given by

$$
V = \alpha^2[\cos[(n-1)\tau] + (n-1)\cos[\tau]][\sin[(n-1)\tau] - (n-1)\sin[\tau]]\sin(\tau). \tag{5.4}
$$

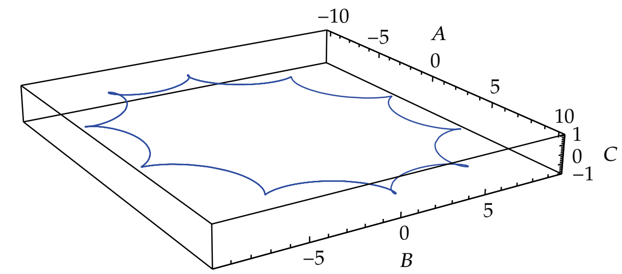

Figure 23: The spiky universe for (5.3), $n = 10$, $\alpha = 1$.

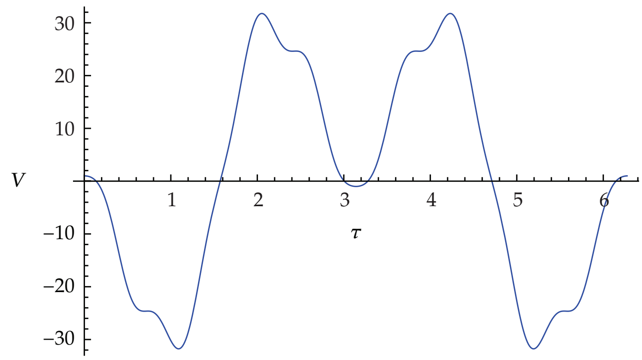

Figure 24: The evolution of the volume for (5.4), $n = 10$, $\alpha = 1$.

In Figure 24 the evolution of the volume for (5.4) is shown, $n = 10$, $\alpha = 1$. The scalar curvature has the form

$$\begin{aligned} R = \Big(& -2\cos(\tau)(n-1)\cos^2((n-1)\tau) \\ & + \left(\left(6\left(\frac{4}{3} - 2n + n^2\right)\right)\sin(\tau)\sin((n-1)\tau)\right. \\ & \quad \left. - \left(2\left((n-2)\cos(\tau)^2 + \sin^2(\tau)\left(n^2 - 3n + 4\right)\right)\right)(n-1)\right)\cos((n-1)\tau) \\ & + 2\cos(\tau)\Big(\sin^2((n-1)\tau) + \sin(\tau)\left(n^2 - 4n + 6\right)\sin((n-1)\tau) \\ & \qquad + \left(\cos(\tau)^2 - 5\sin^2(\tau)\right)(n-1)\Big)(n-1)\Big) \\ & / ((\cos((n-1)\tau) + \cos(\tau)(n-1))(-\sin((n-1)\tau) + (n-1)\sin(\tau))\sin(\tau)). \end{aligned} \tag{5.5}$$

In Figure 25 we plot the evolution of the R with respect of the cosmic time τ. In this example, we have shown that the Einstein equations admit the spike-like solution. We can show that this solution describes the accelerated and decelerated expansion phases of the universe.

5.1.2. Example 2

The system (2.19) admits the following solution:

$$\begin{gathered}H_1 = \alpha\cos[(n-1)\tau] + \alpha(n-1)\cos[\tau],\\ H_2 = \alpha\sin[(n-1)\tau] - \alpha(n-1)\sin[\tau],\\ H_3 = \sin(\tau).\end{gathered} \tag{5.6}$$

The corresponding EoS takes the form

$$\begin{gathered}\rho = \frac{D_{12}}{E_{12}}, \qquad p_1 = -\frac{D_{13}}{E_{13}}\\ p_2 = -\frac{D_{14}}{E_{14}}, \qquad p_3 = -\frac{D_{15}}{E_{15}},\end{gathered} \tag{5.7}$$

where

$$\begin{aligned}D_{12} =& \left[\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)\right]\\ &\times\left[\alpha\sin((n-1)\tau) + [1-\alpha(n-1)]\sin(\tau)\right]\\ &+\left[\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)\right]\sin(\tau),\\ E_{12} =& 1,\\ D_{13} =& \alpha(n-1)\left[\cos((n-1)\tau) - \cos(\tau)\right] + \cos(\tau)\\ &+\left[\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)\right]^2\\ &+\left[\alpha\sin((n-1)\tau) + [1-\alpha(n-1)]\sin(\tau)\right]\sin(\tau),\\ E_{13} =& 1,\\ D_{14} =& -\alpha\sin((n-1)\tau)(n-1) - \alpha(n-1)\sin(\tau) + \cos(\tau)\\ &+\left[\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)\right]^2\\ &+\sin(\tau)^2 + \left[\alpha\cos((n-1)\tau) + \alpha * (n-1)\cos(\tau)\right]\sin(\tau),\\ E_{14} =& 1,\\ D_{15} =& \alpha(n-1)\left[\cos((n-1)\tau) - \cos(\tau) - \sin((n-1)\tau) - \sin(\tau)\right]\\ &+\left[\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)\right]^2\\ &+\left[\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)\right]^2\\ &+\left[\alpha\cos((n-1)\tau) + \alpha(n-1)\cos(\tau)\right]\\ &\times\left[\alpha\sin((n-1)\tau) - \alpha(n-1)\sin(\tau)\right],\\ E_{15} =& 1.\end{aligned} \tag{5.8}$$

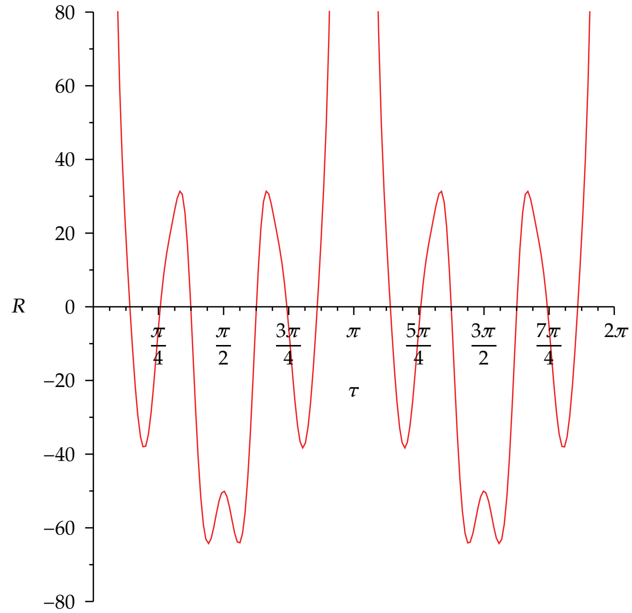

Figure 25: The evolution of the R with respect of the cosmic time τ for (5.5).

The scalar curvature has the form

$$\begin{aligned} R &= 2\alpha^2 \cos\left((n-1)\tau\right)^2 \\ &\quad + 4\alpha\left(\left(\frac{1}{2}\right)\alpha \sin((n-1)\tau) + \left(\frac{1}{2} + \left(-\left(\frac{1}{2}\right)n + \frac{1}{2}\right)\alpha\right)\sin(\tau)\right. \\ &\qquad \left. + (n-1)\left(\alpha\cos(\tau) + \frac{1}{2}\right)\right)\cos((n-1)\tau) + 2\alpha^2 \sin\left((n-1)\tau\right)^2 \\ &\quad + 2\alpha((1+(2-2n)\alpha)\sin(\tau) + (\alpha\cos(\tau)-1)(n-1))\sin((n-1)\tau) \\ &\quad + \left(2 + 2\alpha^2(n-1)^2 + (2-2n)\alpha\right)\sin(\tau)^2 \\ &\quad - (2(n-1))\alpha * (1 + (-1+\alpha(n-1))\cos(\tau))\sin(\tau) \\ &\quad + \left(2\left(2\alpha^2(n-1)^2\cos(\tau) + 1 + \alpha(-n+1)\right)\right)\cos(\tau). \end{aligned} \tag{5.9}$$

In Figure 26 we plot the evolution of the R with respect of the cosmic time τ.

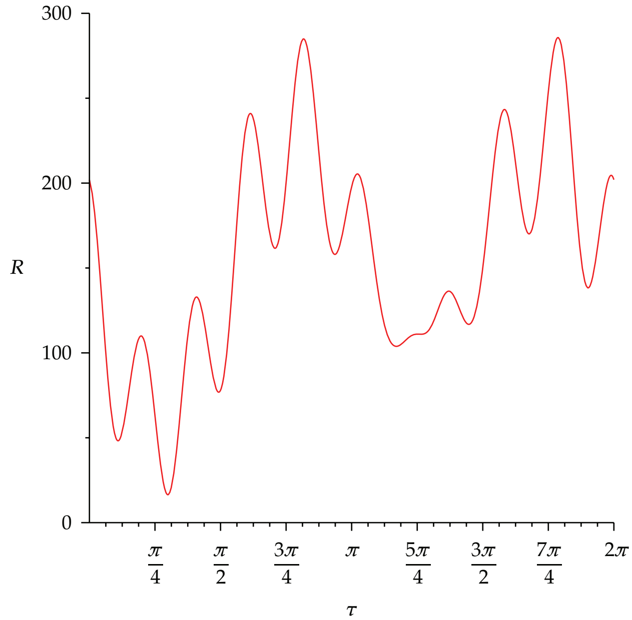

Figure 26: The evolution of the R with respect of the cosmic time τ for (5.9).

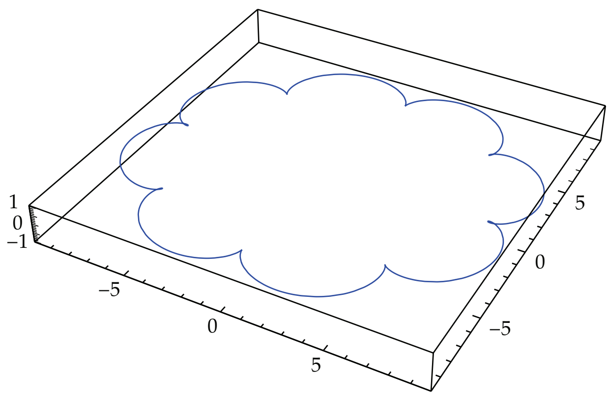

Figure 27: The evolution of the spiky type solution (5.10) with $n = 10$, $\alpha = 1$.

5.1.3. Example 3

Our next solution for the system (2.19) is given by

$$
\begin{aligned}
H_1 &= \alpha \cos[(n-1)\tau] - \alpha(n-1)\cos[\tau], \\
H_2 &= \alpha \sin[(n-1)\tau] - \alpha(n-1)\sin[\tau], \\
H_3 &= \sin(\tau).
\end{aligned}
\tag{5.10}
$$

In Figure 27 we plot this spiky type solution. The corresponding EoS takes the form

$$\begin{aligned}
\rho &= \frac{D_{16}}{E_{16}}, \\
p_1 &= -\frac{D_{17}}{E_{17}}, \\
p_2 &= -\frac{D_{18}}{E_{18}}, \\
p_3 &= -\frac{D_{19}}{E_{19}},
\end{aligned} \tag{5.11}$$

where

$$\begin{aligned}
D_{16} =\ & [\alpha \cos((n-1)\tau) - \alpha(n-1)\cos(\tau)] \\
& \times [\alpha \sin((n-1)\tau) + [1-\alpha(n-1)]\sin(\tau)] \\
& + [\alpha \sin((n-1)\tau) - \alpha(n-1)\sin(\tau)]\sin(\tau), \\
E_{16} =\ & 1, \\
D_{17} =\ & \alpha(n-1)[\cos((n-1)\tau) - \cos(\tau)] \\
& + \cos(\tau) + [\alpha \sin((n-1)\tau) - \alpha(n-1)\sin(\tau)]^2 \\
& + [\alpha \sin((n-1)\tau) + [1-\alpha(n-1)]\sin(\tau)]\sin(\tau), \\
E_{17} =\ & 1, \\
D_{18} =\ & -\alpha \sin((n-1)\tau)(n-1) + \alpha(n-1)\sin(\tau) \\
& + \cos(\tau) + [\alpha \cos((n-1)\tau) - \alpha(n-1)\cos(\tau)]^2 \\
& + \sin(\tau)^2 + [\alpha \cos((n-1)\tau) - \alpha * (n-1)\cos(\tau)]\sin(\tau), \\
E_{18} =\ & 1, \\
D_{19} =\ & \alpha(n-1)[\cos((n-1)\tau) - \cos(\tau) - \sin((n-1)\tau) + \sin(\tau)] \\
& + [\alpha \sin((n-1)\tau) - \alpha(n-1)\sin(\tau)]^2 \\
& + [\alpha \cos((n-1)\tau) - \alpha(n-1)\cos(\tau)]^2 \\
& + [\alpha \cos((n-1)\tau) - \alpha(n-1)\cos(\tau)] \\
& \times [\alpha \sin((n-1)\tau) - \alpha(n-1)\sin(\tau)], \\
E_{19} =\ & 1.
\end{aligned} \tag{5.12}$$

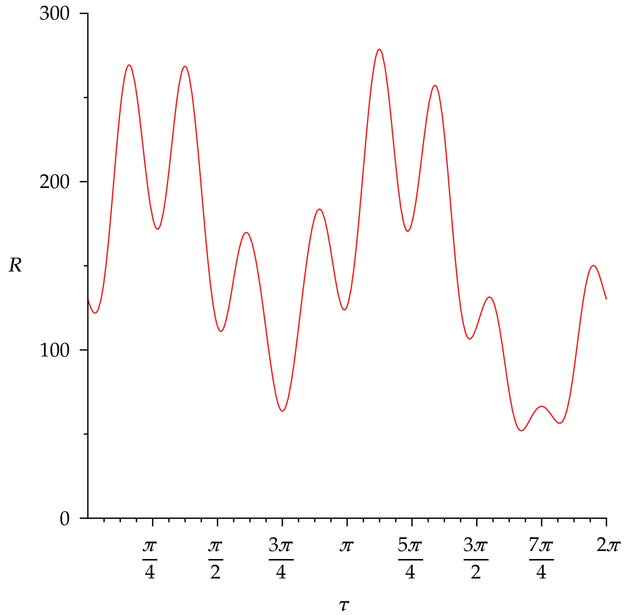

Figure 28: The evolution of the R with respect of the cosmic time τ for (5.13).

The scalar curvature has the form

$$\begin{aligned}
R = {} & 2\alpha^2 \cos((n-1)\tau)^2 \\
& - \left(4\left(-\left(\frac{1}{2}\right)\alpha \sin((n-1)\tau) + \left(-\frac{1}{2} + \left(\left(\frac{1}{2}\right)n - \frac{1}{2}\right)\alpha\right)\sin(\tau)\right.\right. \\
& \qquad \left.\left. + (n-1)\left(-\frac{1}{2} + \alpha\cos(\tau)\right)\right)\right)\alpha\cos((n-1)\tau) \\
& + 2\alpha^2 \sin((n-1)\tau)^2 \\
& - (2((-1 + (-2+2n)\alpha)\sin(\tau) + (\alpha\cos(\tau)+1)(n-1)))\alpha\sin((n-1)\tau) \\
& + \left(2 + 2\alpha^2(n-1)^2 + (-2n+2)\alpha\right)\sin(\tau)^2 + (2(n-1)) \\
& \times (1 + (-1 + \alpha(n-1)) * \cos(\tau))\alpha\sin(\tau) \\
& + 2\cos(\tau)\left(\alpha^2(n-1)^2\cos(\tau) + 1 + (1-n)\alpha\right).
\end{aligned} \tag{5.13}$$

In Figure 28 we plot the evolution of the R with respect of the cosmic time τ.

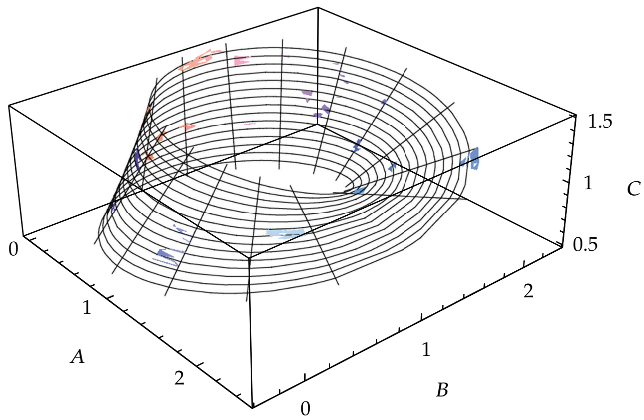

Figure 29: The plot of the Möbius strip universe for (5.16) with (3.4) and $\tau = 0 \to 2\pi$ and $\Lambda = [-1.1]$.

5.2. Möbius Strip Universe Solutions

If we consider the model with the "cosmological constant", then the systems (2.18) and (2.19) take the form, respectively,

$$\begin{aligned} &\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \rho - \Lambda = 0, \\ &\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + p_1 - \Lambda = 0, \\ &\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} + p_2 - \Lambda = 0, \\ &\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + p_3 - \Lambda = 0, \end{aligned} \tag{5.14}$$

$$\begin{aligned} &H_1H_2 + H_2H_3 + H_1H_3 - \rho - \Lambda = 0, \\ &\dot{H}_2 + \dot{H}_3 + H_2^2 + H_3^2 + H_2H_3 + p_1 - \Lambda = 0, \\ &\dot{H}_3 + \dot{H}_1 + H_3^2 + H_1^2 + H_3H_1 + p_2 - \Lambda = 0, \\ &\dot{H}_1 + \dot{H}_2 + H_1^2 + H_2^2 + H_1H_2 + p_3 - \Lambda = 0. \end{aligned} \tag{5.15}$$

Now we want to present some solutions of these systems. Consider the following examples.

5.2.1. Example 1

One of the simplest solutions of (5.14) is given by

$$\begin{aligned} A &= A_0 + \left(1 + \frac{1}{2}\Lambda \cos\frac{\tau}{2}\right)\cos\tau, \\ B &= B_0 + \left(1 + \frac{1}{2}\Lambda \cos\frac{\tau}{2}\right)\sin\tau, \\ C &= C_0 + \frac{1}{2}\Lambda \sin\frac{\tau}{2}. \end{aligned} \tag{5.16}$$

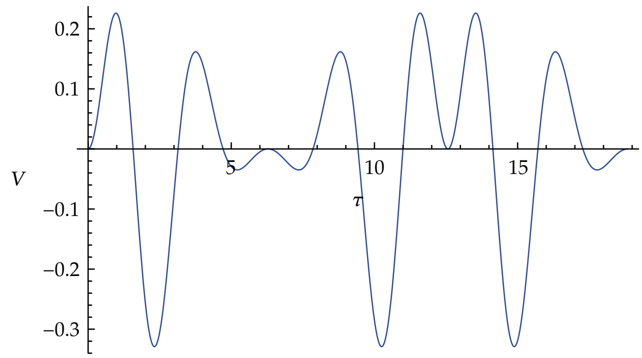

Figure 30: The evolution of the volume of the Möbius strip universe for (5.16) with (3.4) and $\alpha = \Lambda = 1$.

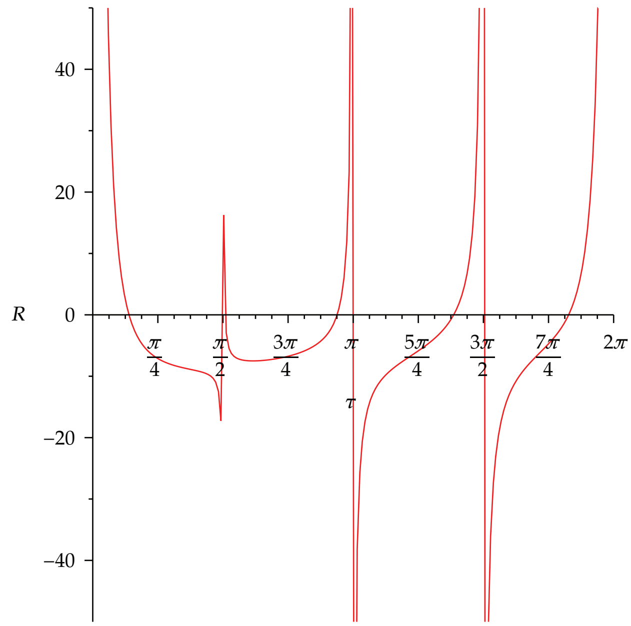

Figure 31: The evolution of the R with respect of the cosmic time τ for (5.20).

It is the parametric equation of the Möbius strip and, hence, such model we call the Möbius strip universe. Its plot was presented in Figure 29. The evolution of the volume of the Möbius strip universe for (5.16) with (3.4) reads as

$$V = 0.5\Lambda\left(1 + \frac{1}{2}\Lambda\cos\frac{\tau}{2}\right)^2 \cos\tau\sin\tau\sin\frac{\tau}{2}. \tag{5.17}$$

The evolution of the volume with (3.4) and $\alpha = \Lambda = 1$ is presented in Figure 30.

The corresponding EoS takes the form

$$\rho = \frac{D_{20}}{E_{20}},$$
$$p_1 = -\frac{D_{21}}{E_{21}},$$
$$p_2 = -\frac{D_{22}}{E_{22}}, \tag{5.18}$$
$$p_3 = -\frac{D_{23}}{E_{23}},$$

where

$$\begin{aligned}
D_{20} = {} & \left[\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau) + \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right] \\
& \times \left[\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\sin(\tau) - \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right] \\
& \times \left[C_0 + \frac{1}{2}\Lambda\sin\left(\frac{\tau}{2}\right)\right] \\
& + \frac{\Lambda}{4}\left[-\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\sin(\tau) + \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right]\cos\left(\frac{\tau}{2}\right) \\
& \times \left[A_0 + \left(1 + \frac{1}{2}\right)\Lambda\cos\left(\frac{\tau}{2}\right)\cos(\tau)\right] \\
& + \frac{\Lambda}{4}\left[-\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau) - \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\cos\left(\frac{\tau}{2}\right) \\
& \times \left[B_0 + \left(1 + \frac{1}{2}\right)\Lambda\cos\left(\frac{\tau}{2}\right)\sin(\tau)\right] \\
& - \Lambda\left[A_0 + \left(1 + \frac{1}{2}\right)\Lambda\cos\left(\frac{\tau}{2}\right)\cos(\tau)\right] \\
& \times \left[B_0 + \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right] \\
& \times \left[C_0 + \frac{1}{2}\Lambda\sin\left(\frac{\tau}{2}\right)\right], \\
E_{20} = {} & \left[A_0 + \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right] \\
& \times \left[B_0 + \left(1 + \frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\left[C_0 + \frac{1}{2}\Lambda\sin\left(\frac{\tau}{2}\right)\right],
\end{aligned}$$

$$\begin{aligned}
D_{21} = {} & \left[C_0 + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\right] \\
& \times \left[-\frac{1}{8}\sin(\tau)\Lambda \cos\left(\frac{\tau}{2}\right) - \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau) - \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right] \\
& - \frac{\Lambda}{8}\left[B_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\sin\left(\frac{\tau}{2}\right) \\
& + \frac{\Lambda}{4}\left[-\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\sin(\tau) + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right) \times \cos(\tau)\right]\cos\left(\frac{\tau}{2}\right) \\
& - \Lambda\left[B_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\left[C_0 + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\right], \\
E_{21} = {} & \left[B_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\left[C_0 + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\right], \\
D_{22} = {} & -\frac{\Lambda}{8}\left[A_0 + \left(1 + \frac{1}{2}\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right]\Lambda \sin\left(\frac{\tau}{2}\right) + \left[C_0 + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\right] \\
& \times \left[-\frac{\Lambda}{8}\cos(\tau)\cos\left(\frac{\tau}{2}\right) + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\sin(\tau) - \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right] \\
& + \frac{1}{4}\left[-\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau) - \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\Lambda \cos\left(\frac{\tau}{2}\right) \\
& - \Lambda\left[A_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right] \\
& \times \left[C_0 + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\right], \\
E_{22} = {} & \left[A_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right]\left[C_0 + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\right], \\
D_{23} = {} & \left[B_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right] \\
& \times \left[-\frac{\Lambda}{8}\cos(\tau)\cos\left(\frac{\tau}{2}\right) + \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\sin(\tau)\right. \\
& \qquad \left. - \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right] \\
& + \left[A_0 + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right] \\
& \times \left[-\frac{\Lambda}{8}\sin(\tau)\cos\left(\frac{\tau}{2}\right) - \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau)\right. \\
& \qquad \left. - \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right] \\
& - \left[\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau) + \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]
\end{aligned}$$

$$
\begin{aligned}
&\times\left[-\frac{1}{4}\Lambda\sin\left(\frac{\tau}{2}\right)\sin(\tau)+\left(1+\frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right]\\
&-\Lambda\left[A_0+\left(1+\frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right]\\
&\times\left[B_0+\left(1+\frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right],\\
E_{23}={}&\left[B_0+\left(1+\frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\sin(\tau)\right]\left[A_0+\left(1+\frac{1}{2}\Lambda\cos\left(\frac{\tau}{2}\right)\right)\cos(\tau)\right].
\end{aligned}
\tag{5.19}
$$

The scalar curvature has the form

$$
\begin{aligned}
R={}&\Bigg(\left(-2\sin(\tau)^2\Lambda^2+2\cos(\tau)^2\Lambda^2\right)\cos\left(\left(\frac{1}{2}\right)\tau\right)^3\\
&+\left(-8\sin(\tau)^2\Lambda-17\cos(\tau)\sin(\tau)\sin\left(\left(\frac{1}{2}\right)\tau\right)\Lambda^2+8\cos(\tau)^2\Lambda\right)\cos\left(\left(\frac{1}{2}\right)\tau\right)^2\\
&+\Bigg(\left(6\sin(\tau)^2\Lambda^2-6\cos(\tau)^2\Lambda^2\right)\sin\left(\left(\frac{1}{2}\right)\tau\right)^2-60\cos(\tau)\sin(\tau)\sin\left(\left(\frac{1}{2}\right)\tau\right)\Lambda\\
&\qquad+8\cos(\tau)^2-8\sin(\tau)^2\Bigg)\cos\left(\left(\frac{1}{2}\right)\tau\right)\\
&+\Bigg(\sin\left(\left(\frac{1}{2}\right)\tau\right)^2\Lambda^2\cos(\tau)\sin(\tau)\\
&\qquad+\left(12\sin(\tau)^2\Lambda-12\cos(\tau)^2\Lambda\right)\sin\left(\left(\frac{1}{2}\right)\tau\right)\\
&\qquad-52\cos(\tau)\sin(\tau)\Bigg)\sin\left(\left(\frac{1}{2}\right)\tau\right)\Bigg)\\
&\Big/\left(\sin(\tau)\cos(\tau)\left(2+\Lambda\cos\left(\left(\frac{1}{2}\right)\tau\right)\right)^2\sin\left(\left(\frac{1}{2}\right)\tau\right)\right).
\end{aligned}
\tag{5.20}
$$

In Figure 31 we plot the evolution of the R with respect of the cosmic time τ. In this subsubsection, we have shown that the Einstein equations have the Möbius strip universe solution. Again we can show that this solution describes the accelerated and decelerated expansion phases of the universe.

5.2.2. Example 2

For the system (5.15) the Möbious solution reads as

$$H_1 = \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\cos(\tau),$$
$$H_2 = \left(1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right)\sin(\tau), \tag{5.21}$$
$$H_3 = \frac{1}{2}\Lambda \sin\left(\frac{\tau}{2}\right).$$

The corresponding EoS takes the form

$$\rho = \frac{D_{24}}{E_{24}}, \quad p_1 = -\frac{D_{25}}{E_{25}},$$
$$p_2 = -\frac{D_{26}}{E_{26}}, \quad p_3 = -\frac{D_{27}}{E_{27}}, \tag{5.22}$$

where

$$D_{24} = \left[1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right]^2 \cos(\tau)\sin(\tau) + \frac{\Lambda}{2}\left[1 + \frac{\Lambda}{2}\cos\left(\frac{\tau}{2}\right)\right]$$
$$\times [\sin(\tau) + \cos(\tau)]\sin\left(\frac{\tau}{2}\right) - \Lambda,$$
$$E_{24} = 1,$$
$$D_{25} = -\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\sin(\tau) + \left[1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right]$$
$$\times \left[\cos(\tau) + \frac{\Lambda}{2}\sin(\tau)\sin\left(\frac{\tau}{2}\right)\right] + \frac{1}{4}\Lambda \cos\left(\frac{\tau}{2}\right)$$
$$+ \left[1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right]^2 \sin^2(\tau) + \frac{\Lambda^2}{4}\sin^2\left(\frac{\tau}{2}\right) - \Lambda,$$
$$E_{25} = 1,$$
$$D_{26} = -\frac{1}{4}\Lambda \sin\left(\frac{\tau}{2}\right)\cos(\tau) - \left[1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right]$$
$$\times \left[\sin(\tau) + \frac{\Lambda}{2}\cos(\tau)\sin\left(\frac{\tau}{2}\right)\right] + \frac{1}{4}\Lambda \cos\left(\frac{\tau}{2}\right)$$
$$+ \left[1 + \frac{1}{2}\Lambda \cos\left(\frac{\tau}{2}\right)\right]^2 \cos^2(\tau) + \frac{\Lambda^2}{4}\sin^2\left(\frac{\tau}{2}\right) - \Lambda,$$

$$
\begin{aligned}
E_{26} &= 1,\\
D_{27} &= \left[1 + \frac{\Lambda}{2}\cos\left(\frac{\tau}{2}\right) - \frac{\Lambda}{4}\sin\left(\frac{\tau}{2}\right)\right][\cos(\tau) + \sin(\tau)]\\
&\quad + \left[1 + \frac{\Lambda}{2}\cos\left(\frac{\tau}{2}\right)\right]^2[1 + \cos(\tau)\sin(\tau)] - \Lambda,\\
E_{27} &= 1.
\end{aligned}
\tag{5.23}
$$

The scalar curvature has the form

$$
\begin{aligned}
R = {} & \left(\frac{1}{2}\right)\Lambda^2\left(\cos^2(\tau) + \sin^2(\tau) + \cos(\tau)\sin(\tau)\right)\cos^2\left(\left(\frac{1}{2}\right)\tau\right)\\
& + \left(\frac{1}{2\left(\mathfrak{A} + 1 - 2\sin(\tau) + 4\sin^2(\tau)\right)}\right)\\
& \times \Lambda\cos\left(\left(\frac{1}{2}\right)\tau\right) + \left(\frac{1}{2}\right)\Lambda^2\sin^2\left(\left(\frac{1}{2}\right)\tau\right)\\
& + \left(\frac{1}{2(\cos(\tau) + \sin(\tau))}\right)\Lambda\sin\left(\left(\frac{1}{2}\right)\tau\right)\\
& + 2\cos^2(\tau) + \left(\frac{1}{2(4 + 4\sin(\tau))}\right)\\
& \times \cos(\tau) - 2\sin(\tau) + 2\sin^2(\tau),
\end{aligned}
\tag{5.24}
$$

where $\mathfrak{A}$ denotes $(\cos(\tau) + \sin(\tau))\Lambda\sin((1/2)\tau) + 4\cos^2(\tau) + (2 + 4\sin(\tau))\cos(\tau)$.

In Figure 32 we plot the evolution of the R with respect of the cosmic time τ.

5.3. Other Examples of Möbius Strip Like Universes Induced by Jacobian Elliptic Functions

5.3.1. Example 1

Now we want to present some solutions in terms of the Jacobian elliptic functions. In fact, the system (5.14) has the following particular solution:

$$
\begin{aligned}
A &= A_0 + \left(1 + \frac{1}{2}\Lambda\mathrm{cn}\,\frac{\tau}{2}\right)\mathrm{cn}\,\tau,\\
B &= B_0 + \left(1 + \frac{1}{2}\Lambda\mathrm{cn}\,\frac{\tau}{2}\right)\mathrm{sn}\,\tau,\\
C &= C_0 + \frac{1}{2}\Lambda\mathrm{sn}\frac{\tau}{2}
\end{aligned}
\tag{5.25}
$$

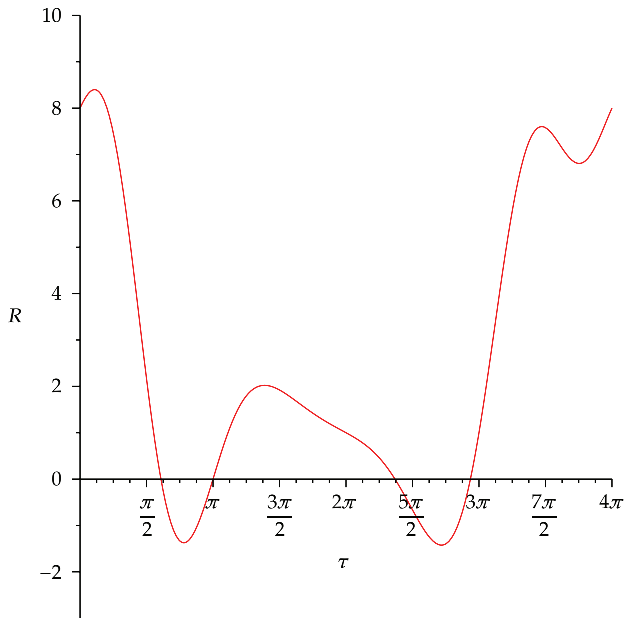

Figure 32: The evolution of the R with respect of the cosmic time τ for (5.24).

The corresponding EoS takes the form

$$\rho = \frac{D_{28}}{E_{28}}, \qquad p_1 = -\frac{D_{29}}{E_{29}}, \qquad p_2 = -\frac{D_{30}}{E_{30}}, \qquad p_3 = -\frac{D_{31}}{E_{31}}, \tag{5.26}$$

where

$$\begin{aligned} D_{28} = {} & \left[\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\mathrm{sn}\frac{\tau}{2}\mathrm{cn}\,\tau + \left(1+\frac{1}{2}\Lambda\mathrm{cn}\frac{\tau}{2}\right)\mathrm{dn}\,\tau\,\mathrm{sn}\,\tau\right] \\ & \times \left[\frac{1}{4}\Lambda\mathrm{dn}\frac{\tau}{2}\mathrm{sn}\frac{\tau}{2}\mathrm{sn}\,\tau - \left(1+\frac{1}{2}\Lambda\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\right] \\ & \times \left[C_0 + \frac{1}{2}\Lambda\mathrm{sn}\frac{\tau}{2}\right] \\ & + \frac{\Lambda}{4}\left[-\frac{1}{4}\Lambda\mathrm{dn}\frac{\tau}{2}\mathrm{sn}\frac{\tau}{2}\mathrm{sn}\,\tau + \left(1+\frac{1}{2}\Lambda\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\right] \\ & \times \mathrm{cn}\frac{\tau}{2}\mathrm{dn}\frac{\tau}{2} \times \left[A_0 + \left(1+\frac{1}{2}\Lambda\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}\left[\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\mathrm{sn}\,\frac{\tau}{2}\mathrm{cn}\,\tau+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{dn}\,\tau\,\mathrm{sn}\,\tau\right]\Lambda\mathrm{cn}\frac{\tau}{2}\mathrm{dn}\,\frac{\tau}{2}\\
&\times\left[B_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]-\Lambda\left[A_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\\
&\times\left[B_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\times\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
E_{28}=&\left[A_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\\
&\times\left[B_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
D_{29}=&\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right]\\
&\times\left[\frac{1}{8}\Lambda\,\mathrm{cn}\frac{\tau}{2}\mathrm{sn}^2\frac{\tau}{2}\mathrm{sn}\,\tau-\frac{1}{8}\Lambda\,\mathrm{dn}^2\frac{\tau}{2}\,\mathrm{cn}\frac{\tau}{2}\mathrm{sn}\,\tau-\frac{1}{2}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\right.\\
&\quad\left.-\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\,\frac{\tau}{2}\right)\mathrm{dn}^2\tau\,\mathrm{sn}\,\tau-\left[1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right]\mathrm{cn}^2\tau\,\mathrm{sn}\,\tau\right]\\
&+\left[B_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\\
&\times\left[-\frac{1}{8}\Lambda\,\mathrm{dn}^2\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}-\frac{1}{8}\Lambda\,\mathrm{cn}^2\frac{\tau}{2}\mathrm{sn}\frac{\tau}{2}\right]\\
&-\frac{\Lambda}{4}\left[\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\mathrm{sn}\,\tau-\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\right]\\
&\times\mathrm{cn}\frac{\tau}{2}\mathrm{dn}\frac{\tau}{2}-\Lambda\left[B_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\\
&\times\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right],
\end{aligned}
$$

$$
E_{29}=\left[B_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right],
$$

$$
\begin{aligned}
D_{30}=&-\left[A_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\\
&\times\left[\frac{1}{8}\Lambda\,\mathrm{dn}^2\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}+\frac{1}{8}\Lambda\,\mathrm{cn}^2\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\right]+\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right]\\
&\times\left[\frac{1}{8}\Lambda\mathrm{cn}\frac{\tau}{2}\,\mathrm{sn}^2\frac{\tau}{2}\,\mathrm{cn}\,\tau-\frac{1}{8}\Lambda\,\mathrm{dn}^2\frac{\tau}{2}\,\mathrm{cn}\frac{\tau}{2}\mathrm{cn}\,\tau+\frac{1}{2}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{dn}\,\tau\,\mathrm{sn}\,\tau\right.\\
&\quad\left.+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\left(\mathrm{cn}\,\tau\,\mathrm{sn}^2\tau-\mathrm{dn}^2\tau\,\mathrm{cn}\,\tau\right)\right]\\
&-\frac{\Lambda}{4}\left[\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{cn}\,\tau+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\times\mathrm{dn}\,\tau\,\mathrm{sn}\,\tau\right]\mathrm{cn}\frac{\tau}{2}\,\mathrm{dn}\frac{\tau}{2}\\
&-\Lambda\left[A_0+\left(1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\left[C_0+\frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
E_{30} &= \left[A_0 + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\left[C_0 + \frac{1}{2}\Lambda\,\mathrm{sn}\frac{\tau}{2}\right],\\
D_{31} &= \left[B_0 + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\\
&\quad\times\left[\frac{1}{8}\Lambda\,\mathrm{cn}\frac{\tau}{2}\,\mathrm{sn}^2\frac{\tau}{2}\,\mathrm{cn}\,\tau - \frac{1}{8}\Lambda\,\mathrm{dn}^2\frac{\tau}{2}\,\mathrm{cn}\frac{\tau}{2}\,\mathrm{cn}\tau\right.\\
&\qquad + \frac{1}{2}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{dn}\,\tau\,\mathrm{sn}\,\tau\\
&\qquad \left. + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\left(\mathrm{sn}^2\tau - \mathrm{dn}^2\tau\right)\mathrm{cn}\,\tau\right]\\
&\quad + \left[A_0 + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\\
&\quad\times\left[\frac{1}{8}\Lambda\,\mathrm{cn}\frac{\tau}{2}\,\mathrm{sn}\,\tau\left(\mathrm{sn}^2\frac{\tau}{2} - \mathrm{dn}^2\tau\right)\right.\\
&\qquad - \frac{1}{2}\Lambda\,\mathrm{dn}\frac{\tau}{2}\mathrm{sn}\frac{\tau}{2}\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\\
&\qquad - \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{dn}^2\tau\,\mathrm{sn}\,\tau\\
&\qquad \left. - \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}^2\tau\,\mathrm{sn}\,\tau\right]\\
&\quad + \left[\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{cn}\,\tau + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{dn}\,\tau\,\mathrm{sn}\,\tau\right]\\
&\quad\times\left[\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{sn}\,\tau - \left(1 + \frac{1}{2}\Lambda\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\right]\\
&\qquad - \Lambda\left[A_0 + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right]\\
&\qquad\times\left[B_0 + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right],\\
E_{31} &= \left[B_0 + \left(1 + \frac{1}{2}\Lambda\mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\,\tau\right]\left[A_0 + \left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\,\tau\right].
\end{aligned}
\tag{5.27}
$$

The evolution of the volume of the universe for (3.4) reads as ($A_0 = B_0 = C_0 = 0$)

$$
V = \frac{1}{2}\Lambda\left(1 + \frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right)^2\mathrm{cn}\,\tau\,\mathrm{sn}\,\tau\,\mathrm{sn}\frac{\tau}{2}. \tag{5.28}
$$

The evolution of the volume with (3.4) and $\Lambda = 1$ is presented in Figure 33.

The scalar curvature has the form

$$
\begin{aligned}
R = \Bigg(& 2\Lambda\left(2k^2\operatorname{cn}\left(\left(\frac{1}{2}\right)\tau,k\right)+\Lambda k^2\operatorname{cn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)\right.\\
&\qquad\left.+\left(\frac{1}{2}\right)\Lambda\operatorname{dn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)\right)\operatorname{cn}(\tau,k)\operatorname{sn}(\tau,k)\operatorname{sn}\left(\left(\frac{1}{2}\right)\tau,k\right)^3\\
&-6\Lambda\operatorname{dn}(\tau,k)\operatorname{dn}\left(\left(\frac{1}{2}\right)\tau,k\right)(\operatorname{cn}\ (\tau,k)-\operatorname{sn}(\tau,k))\\
&\times(\operatorname{cn}(\tau,k)+\operatorname{sn}(\tau,k))\\
&\times\left(2+\Lambda\operatorname{cn}\left(\left(\frac{1}{2}\right)\ \tau,k\right)\right)\operatorname{sn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)\\
&-\left(4\left(2+\Lambda\operatorname{cn}\left(\left(\frac{1}{2}\right)\tau,k\right)\right)\right)\\
&\times\left(\left(\frac{1}{4}\right)\operatorname{cn}^3\left(\left(\frac{1}{2}\right)\tau,k\right)k^2\Lambda+\left(\frac{1}{2}\right)\operatorname{cn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)k^2\right.\\
&\qquad+\Lambda\left(3\operatorname{dn}^2(\tau,k)+\operatorname{cn}^2(\tau,k)k^2\right.\\
&\qquad\qquad\left.+\left(\frac{5}{4}\right)\operatorname{dn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)-\operatorname{sn}^2(\tau,k)k^2\right)\\
&\qquad\times\operatorname{cn}\left(\left(\frac{1}{2}\right)\tau,k\right)+6\operatorname{dn}^2(\tau,k)+\left(\frac{1}{2}\right)\operatorname{dn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)\\
&\qquad\left.-2\operatorname{sn}^2(\tau,k)k^2+2\operatorname{cn}^2(\tau,k)k^2\right)\\
&\quad\times\operatorname{cn}(\tau,k)\operatorname{sn}(\tau,k)\operatorname{sn}\left(\left(\frac{1}{2}\right)\tau,k\right)\\
&\quad+2\operatorname{cn}\left(\left(\frac{1}{2}\right)\tau,k\right)\operatorname{dn}\left(\left(\frac{1}{2}\right)\tau,k\right)\operatorname{dn}(\tau,k)(\operatorname{cn}(\tau,k)-\operatorname{sn}(\tau,k))\\
&\quad\times(\operatorname{cn}(\tau,k)+\operatorname{sn}(\tau,k))\\
&\quad\times\left.\left(2+\Lambda\operatorname{cn}\left(\left(\frac{1}{2}\right)\tau,k\right)\right)^2\right)\\
/\Bigg(&\operatorname{cn}(\tau,k)\operatorname{sn}(\tau,k)\\
&\quad*\left.\left(2+\Lambda\operatorname{cn}^2\left(\left(\frac{1}{2}\right)\tau,k\right)\right)\operatorname{sn}\left(\left(\frac{1}{2}\right)\tau,k\right)\right).
\end{aligned}
\tag{5.29}
$$

In Figure 34 we plot the evolution of the R with respect of the cosmic time τ.

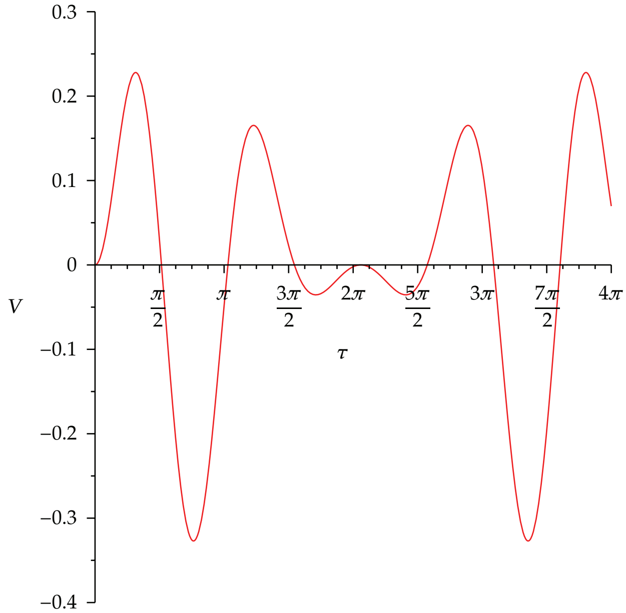

Figure 33: The evolution of the volume of the trefoil knot universe with respect to the cosmic time τ for (5.28).

5.3.2. *Example* 2

Similarly, we can show that the system (5.15) has the following solution:

$$H_1 = \left(1 + \frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right)\mathrm{cn}\tau,$$
$$H_2 = \left(1 + \frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right)\mathrm{sn}\tau, \tag{5.30}$$
$$H_3 = \frac{1}{2}\Lambda \mathrm{sn}\frac{\tau}{2}.$$

The corresponding EoS takes the form

$$\rho = \frac{D_{32}}{E_{32}},$$
$$p_1 = -\frac{D_{33}}{E_{33}},$$
$$p_2 = -\frac{D_{34}}{E_{34}}, \tag{5.31}$$
$$p_3 = -\frac{D_{35}}{E_{35}},$$

where

$$
\begin{aligned}
D_{32} &= \left[1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right]^2 \mathrm{cn}\,\tau\,\mathrm{sn}\,\tau + \frac{\Lambda}{2} \\
&\quad \times \left[1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right][\mathrm{sn}\,\tau + \mathrm{cn}\,\tau]\mathrm{sn}\frac{\tau}{2} - \Lambda, \\
E_{32} &= 1, \\
D_{33} &= \frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}[1-\ \mathrm{sn}\,\tau] + \left[1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right] \\
&\quad \times \left[\ \mathrm{cn}\,\tau\,\mathrm{dn}\,\tau + \frac{\Lambda}{2}\mathrm{sn}\,\tau\,\mathrm{sn}\frac{\tau}{2}\right] \\
&\quad + \left[1+\frac{1}{2}\Lambda\,\mathrm{cn}\frac{\tau}{2}\right]^2 \mathrm{sn}^2\tau + \frac{1}{4}\Lambda^2\mathrm{sn}^2\frac{\tau}{2} - \Lambda, \\
E_{33} &= 1, \\
D_{34} &= -\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\,\mathrm{sn}\frac{\tau}{2}\,\mathrm{cn}\,\tau - \left(1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right) \\
&\quad \times \left[\ \mathrm{dn}\,\tau\,\mathrm{sn}\,\tau + \frac{\Lambda}{2}\mathrm{cn}\tau\mathrm{sn}\frac{\tau}{2}\right] \\
&\quad + \frac{1}{4}\Lambda\,\mathrm{cn}\frac{\tau}{2}\mathrm{dn}\frac{\tau}{2} + \left[1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right]^2 \mathrm{cn}^2\tau + \frac{1}{4}\Lambda^2\mathrm{sn}^2\frac{\tau}{2} - \Lambda, \\
E_{34} &= 1, \\
D_{35} &= -\frac{1}{4}\Lambda\,\mathrm{dn}\frac{\tau}{2}\mathrm{sn}\ \frac{\tau}{2}[\mathrm{cn}\,\tau + \mathrm{sn}\,\tau] \\
&\quad + \left[1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right][\mathrm{cn}\,\tau - \mathrm{sn}\,\tau]\mathrm{dn}\tau \\
&\quad + \left[1+\frac{1}{2}\Lambda \mathrm{cn}\frac{\tau}{2}\right]^2 \left[\mathrm{sn}^2\tau + \mathrm{cn}^2\tau + \mathrm{cn}\,\tau\,\mathrm{sn}\,\tau\right] - \Lambda, \\
E_{35} &= 1.
\end{aligned}
\tag{5.32}
$$

The scalar curvature has the form

$$
\begin{aligned}
R = {} & \left(\frac{1}{2}\right)\Lambda^2\left(\mathrm{cn}^2(\tau,k) +\ \mathrm{sn}^2(\tau,k) + \mathrm{cn}(\tau,k)\mathrm{sn}(\tau,k)\right)\mathrm{cn}^2\left(\left(\frac{1}{2}\right)\tau,k\right) \\
& + \left(\frac{1}{2}\left(\Lambda(\mathrm{sn}(\tau,k)+\mathrm{cn}(\tau,k))\mathrm{sn}\left(\left(\frac{1}{2}\right)\tau,k\right) + \mathrm{dn}\left(\left(\frac{1}{2}\right)\tau,k\right) + 4\mathrm{cn}^2(\tau,k)\right.\right. \\
& \qquad + (4\mathrm{sn}(\tau,k) + 2\mathrm{dn}(\tau,k))\mathrm{cn}(\tau,k) + 4\mathrm{sn}^2(\tau,k)
\end{aligned}
$$

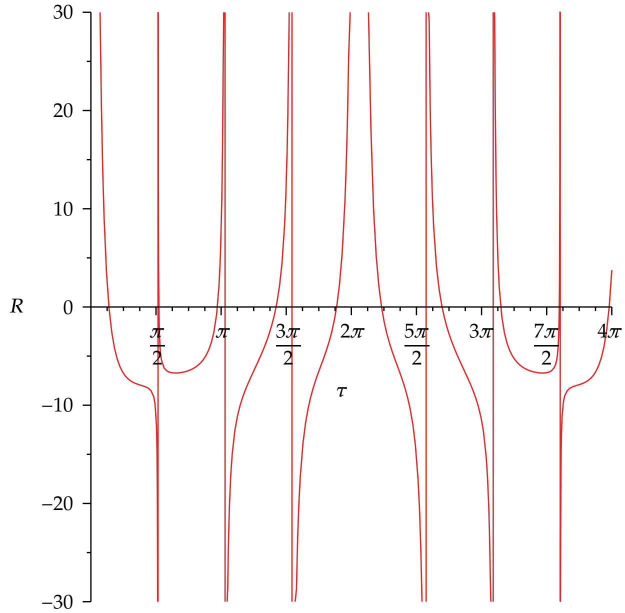

Figure 34: The evolution of the R with respect of the cosmic time τ for (5.29).

$$
\begin{aligned}
&- 2\mathrm{dn}(\tau,k)\mathrm{sn}(\tau,k)\Big)\Big)\Lambda\mathrm{cn}\left(\left(\frac{1}{2}\right)\tau,k\right) \\
&+ \left(\frac{1}{2}\right)\Lambda^2\mathrm{sn}^2\left(\left(\frac{1}{2}\right)\tau,k\right) - \left(\frac{1}{2}\right)\Lambda\left(\mathrm{dn}\left(\left(\frac{1}{2}\right)\tau,k\right) - 2\right) \\
&\times (\mathrm{sn}(\tau,k) + \mathrm{cn}(\tau,k))\mathrm{sn}\left(\left(\frac{1}{2}\right)\tau,k\right) \\
&+ 2\mathrm{cn}^2(\tau,k) + \left(\frac{1}{2(4\mathrm{dn}(\tau,k) + 4\mathrm{sn}(\tau,k))}\right)\mathrm{cn}(\tau,k) \\
&- 2\mathrm{sn}(\tau,k)(-\,\mathrm{sn}(\tau,k) + \mathrm{dn}(\tau,k)).
\end{aligned} \tag{5.33}
$$

In Figure 35 we plot the evolution of the R with respect of the cosmic time τ.

6. Conclusion

In the present paper, we have constructed several concrete models describing the trefoil and figure-eight knot universes from Bianchi-type I cosmology and examined the cosmological features and properties in detail.

To realize the cyclic universes, it is necessary to a noncanonical scalar field with ill-defined vacuum in the context of the quantum field theory or extended gravity, for example, with adding higher order derivative terms and $f(R)$ gravity [79]. Indeed, however, these modified gravity theories have to satisfy the tests on the solar system scale as well as cosmological constraints so that those can be alternative gravitational theories to general

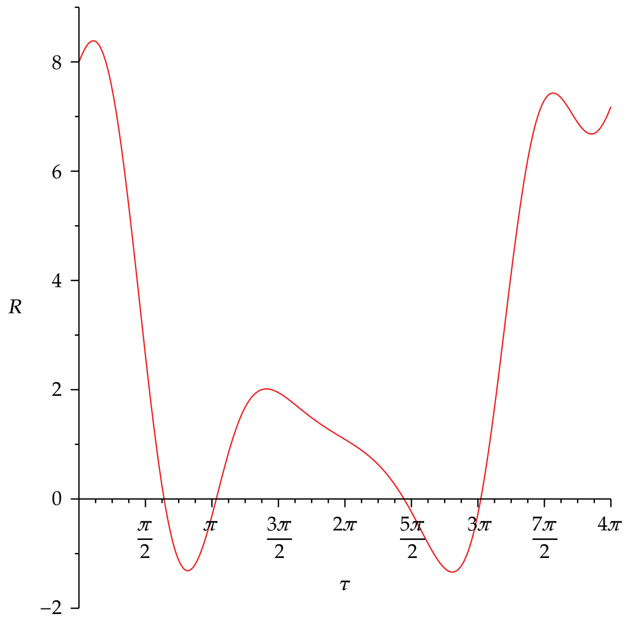

Figure 35: The evolution of the R with respect of the cosmic time τ for (5.33).

relativity. The significant cosmological consequence of our approach is that we have shown the possibility to obtain the knot universes related to the cyclic universes from Bianchi-type I spacetime within general relativity.

Furthermore, recently it has been pointed out that the asymmetry of the EoS for the universe can lead to cosmological hysteresis [80]. On the other hand, Bianchi-type I spacetime describes the spatially anisotropic cosmology and hence the EoS for the universe has the asymmetry in the oscillating process through the expanding and contracting behaviors. Accordingly, it is considered that in the constructed models of the knot universes cosmological hysteresis could occur. The observation of this phenomenon in our models is one of our future works on the knot universes.

Finally, it should be remarked that by summarizing the results of our previous [98–101, 103] and this works, the knot universes describing the cyclic universes can be realized from the homogeneous and isotropic FLRW spacetime as well as the homogeneous and anisotropic Bianchi-type I cosmology. In these series of works, the formulations of model construction method of the knot universes have been established. Thus, it can be expected that the presented formalism is useful to realize the universes with other features from both the isotropic and anisotropic spacetimes.

Finally we would like to note that all solutions presented above describe the accelerated and decelerated expansion phases of the universe.

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