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Research Article

A Finite Element Variational Multiscale Method Based on Two Local Gauss Integrations for Stationary Conduction-Convection Problems

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A new finite element variational multiscale (VMS) method based on two local Gauss integrations is proposed and analyzed for the stationary conduction-convection problems. The valuable feature of our method is that the action of stabilization operators can be performed locally at the element level with minimal additional cost. The theory analysis shows that our method is stable and has a good precision. Finally, the numerical test agrees completely with the theoretical expectations and the " exact solution," which show that our method is highly efficient for the stationary conduction-convection problems.

1. Introduction

The conduction-convection problems constitute an important system of equations in atmospheric dynamics and dissipative nonlinear system of equations. Many authors have worked on these problems [1–8]. The governing equations couple viscous incompressible flow and heat transfer process [9], where the incompressible fluid is the Boussinesq approximation to the nonstationary Navier-Stokes equations. Christon et al. [10] summarized some relevant results for the fluid dynamics of thermally driven cavity. A multigrid (MG) technique was applied for the conduction-convection problems [11, 12]. Luo et al. [13] combined proper orthogonal decomposition (POD) with the Petrov-Galerkin least squares mixed finite element (PLSMFE) method for the problems. In [14], a Newton iterative mixed finite element method for the stationary conduction-convection problems was shown by Si et al. In [15], Si and He gave a defect-correction mixed finite element method for the stationary conduction-convection problems. In [3], an analysis of conduction natural convection conjugate heat transfer in the gap between concentric cylinders under solar irradiation was carried out. In [16], Boland and Layton gave an error analysis for finite element methods for steady natural convection problems. Variational multiscale (VMS) method which defines the large scales in a different way, namely, by a projection into appropriate subspaces, see Guermond [17], Hughes et al. [18–20] and Layton [21], and other literatures on VMS methods [22–24]. The new finite element VMS strategy requires edge-based data structure and a subdivision of grids into patches. It does not require a specification of mesh-dependent parameters and edge-based data structure, and it is completely local at the element level. Consequently, the new VMS method under consideration can be integrated in existing codes with very little additional coding effort.

For the conduction-convection problems, we establish such system that Ω be a bounded domain in R^d ($d = 2$ or 3), with Lipschitz-continuous boundary $\partial\Omega$. In this paper, we consider the stationary conduction-convection problem as follows:

$$\begin{gathered}
-2\nu\nabla \cdot D(u) + (u \cdot \nabla u) + \nabla p = \lambda jT, \quad x \in \Omega, \\
\nabla \cdot u = 0, \quad x \in \Omega, \\
-\Delta T + \lambda u \cdot \nabla T = 0, \quad x \in \Omega, \\
u = 0, \quad T = T_0, \quad x \in \partial\Omega,
\end{gathered} \tag{1.1}$$

where $D(u) = (\nabla u + \nabla u^T)/2$ is the velocity deformation tensor, $(u, p, T) \in X \times M \times W$, $\Omega \subset R^d$ is a bounded convex domain. $u = (u_1(x), u_2(x))^T$ represents the velocity vector, $p(x)$ the pressure, $T(x)$ the temperature, $\lambda > 0$ the Grashoff number, $j = (0,1)^T$ the two-dimensional vector and $\nu > 0$ the viscosity.

The study is organized as follows. In the next section, the finite element VMS method is given. In Section 3, we give the stability. The error analysis is given in Section 4. In Section 5, we show some numerical test. The last but not least is the conclusion given in Section 6.

2. Finite Element VMS Method

Here, we introduce some notations

$$X = H_0^1(\Omega)^d, \qquad M = L_0^2(\Omega) = \left\{\varphi \in L^2(\Omega); \int_\Omega \varphi dx = 0\right\}, \qquad W = H^1(\Omega). \tag{2.1}$$

For $h > 0$, finite-dimension subspace $(X_h, M_h, W_h) \subset (X, M, W)$ is introduced which is associated with Ω_e, a triangulation of Ω into triangles or quadrilaterals, assumed to be regular in the usual sense. In this study, the finite-element subspaces of personal preference are defined by setting the continuous piecewise (bi)linear velocity and pressure subspace, let τ_h be the regular triangulations or quadrilaterals of the domain Ω and define the mesh parameter $h = \max_{\Omega_e \in \tau_h}\{\text{diam}(\Omega_e)\}$,

$$\begin{aligned}
X_h &= \left\{ v \in X : v|_{\Omega_e} \in R_l(\Omega_e)^d \ \forall \Omega_e \in \tau_h \right\}, \\
M_h &= \left\{ q \in M : q|_{\Omega_e} \in R_l(\Omega_e) \ \forall \Omega_e \in \tau_h \right\}, \\
W_h &= \left\{ \phi \in M : \phi|_{\Omega_e} \in R_l(\Omega_e) \ \forall \Omega_e \in \tau_h \right\},
\end{aligned} \tag{2.2}$$

where $W_{0h} = W_h \cap H_0^1$, $l \geq 1$ is integers. $R_l(\Omega_e) = P_l(\Omega_e)$ if Ω_e is triangular and $R_l(\Omega_e) = Q_l(\Omega_e)$ if Ω_e is quadrilateral. Here (X_h, M_h) does not satisfy the discrete Ladyzhenskaya-Babuška-Brezzi (LBB) condition

$$\sup_{v_h \in X_h} \frac{d(v_h, p_h)}{\|\nabla v_h\|_0} \geq \beta \|p_h\|_0, \quad \forall p_h \in M_h. \tag{2.3}$$

Now, in order to stabilize the convective term appropriately for the higher Reynolds number and avoid the extra storage, we supply finite element VMS method that the local stabilization form of the difference between a consistent and an underintegrated mass matrices based on two local Gauss integrations at element level as the stabilize term

$$G(p_h, q_h) = \epsilon_d (a_k(p_h, q_h) - a_1(p_h, q_h)). \tag{2.4}$$

Here,

$$\begin{gathered}
a_k(p_h, q_h) = p_G^T M_k q_G, \qquad a_1(p_h, q_h) = p_G^T M_1 q_G, \\
p_G^T = [p_1, p_2 \ldots, p_N]^T, \qquad q_G = [q_1, q_2, \ldots, q_N], \\
M_{ij} = (\phi_i, \phi_j), \qquad p_h = \sum_{i=1}^{N} p_i \phi_i, \quad p_i = p_h(x_i),\ \forall p_h \in M_h,\ i = 1, 2, \ldots, N, \\
M_k = \left(M_{ij}^k\right)_{N\times N}, \qquad M_1 = \left(M_{ij}^1\right)_{N\times N},
\end{gathered} \tag{2.5}$$

the stabilization parameter ϵ_d ($\epsilon_d = o(h)$) in this scheme acts only on the small scales, ϕ_i is the basis function of the velocity on the domain Ω such that its value is one at node x_i and zero at other nodes, and N is the dimension of M_h. The symmetric and positive matrices M_{ij}^k, $k \geq 2$ and M_{ij}^1 are the stiffness matrices computed by using k-order and 1-order Gauss integrations at element level, respectively. p_i and q_i, $i = 1, 2, \ldots, N$ are the values of p_h and q_h at the node x_i. In detail, the stabilized term can be rewritten as

$$\begin{gathered}
G(p_h, q_h) = \epsilon_d \sum_{\Omega_e \in \tau_h} \left\{ \int_{\Omega_e, k} p_h q_h dx - \int_{\Omega_e, 1} p_h q_h dx \right\}, \quad \forall p_h, q_h \in M_h, \\
G(p, q) = \left(p - \coprod\nolimits_h p,\ q - \coprod\nolimits_h q\right).
\end{gathered} \tag{2.6}$$

L^2-projection operator $\coprod_h : L^2(\Omega) \to R_0$ with the following properties [25]:

$$\begin{gathered}
(p, q_h) = \left(\coprod\nolimits_h p, q_h\right), \quad \forall p \in M,\ q_h \in R_0; \\
\left\|\coprod\nolimits_h p\right\|_0 \leq c\|p\|_0, \quad \forall p \in M; \\
\left\|p - \coprod\nolimits_h p\right\|_0 \leq ch\|p\|_1, \quad \forall p \in H^1(\Omega) \cap M.
\end{gathered} \tag{2.7}$$

Lemma 2.1 (see [26]). *Let (X_h, M_h) be defined as above, then there exists a positive constant β independent of h, such that*

$$\left|B((u,p);(v,q))\right| \le c(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0)(u,p), \quad (v,q) \in (X, M),$$

$$\beta(\|u_h\|_1 + \|vp_h\|_0) \le \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\left|B((u_h,p_h);(v_h,q_h))\right|}{\|v\|_1 + \|q\|_0}, \quad \forall (u_h,p_h) \in (X_h, M_h), \tag{2.8}$$

$$\left|G(p,q)\right| \le C\|(I - II_h)p\|_0 \|(I - II_h)q\|_0, \quad \forall p,q \in M.$$

Using the above notations, the VMS variational formulation of problems (1.1) reads as follows.

Find (A_1) $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$ such that

$$a(u_h, v_h) - d(p_h, v_h) + d(q_h, u_h) + b(u_h, u_h, v_h) + G(p_h, q_h) = \lambda(jT_h, v_h), \quad \forall v_h \in X_h,\ \varphi_h \in M_h;$$

$$\overline{a}(T_h, \varphi_h) + \lambda \overline{b}(u_h, T_h, \varphi_h) = 0, \quad \forall \varphi_h \in W_{0h}. \tag{2.9}$$

Given (A_2) (u_h^{n-1}, T_h^{n-1}), find $(u_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$ such that

$$a(u_h^n, v_h) - d(p_h^n, v_h) + d(q_h, u_h^n) + b\left(u_h^n, u_h^{n-1}, v_h\right) + b\left(u_h^{n-1}, u_h^n, v_h\right) + G(p_h^n, q_h)$$

$$= b\left(u_h^{n-1}, u_h^{n-1}, v_h\right) + \lambda(jT_h^n, v_h), \quad \forall v_h \in X_h,\ \varphi_h \in M_h; \tag{2.10}$$

$$\overline{a}(T_h^n, \varphi_h) + \lambda \overline{b}\left(u_h^{n-1}, T_h^n, \varphi_h\right) = 0, \quad \forall \varphi_h \in W_{0h},$$

where $a(u,v) = \nu(\nabla u, \nabla v)$, $\overline{a}(T,\varphi) = (\nabla T, \nabla \varphi) d(q,v) = (q, \operatorname{div} v)$, and

$$b(u,v,w) = ((u \cdot \nabla v), w) + \frac{1}{2}((\operatorname{div} u)v, w) = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v),$$

$$\overline{b}(u,T,\varphi) = ((u \cdot \nabla T), w) + \frac{1}{2}((\operatorname{div} u)T, \varphi) = \frac{1}{2}((u \cdot \nabla)T, \varphi) - \frac{1}{2}((u \cdot \nabla)\varphi, T). \tag{2.11}$$

(B_1) There exists a constant C which only depends on Ω, such that

(i) $\|u\|_0 \le C\|\nabla u\|_0$, $\|u\|_{0,4} \le C\|\nabla u\|_0$, for all $u \in H_0^1(\Omega)^d$ (or $H_0^1(\Omega)$),

(ii) $\|u\|_{0,4} \le C\|u\|_1$, for all $u \in H^1(\Omega)^d$

(iii) $\|u\|_{0,4} \le 2^{1/2}\|\nabla u\|_0^{1/2}\|u\|_0^{1/2}$, for all $u \in H_0^1(\Omega)^d$ (or $H_0^1(\Omega)$)

(B_2) Assuming $\partial\Omega \in C^{k,\alpha}$ $(k \ge 0,\ \alpha > 0)$, then, for $T_0 \in C^{k,\alpha}(\partial\Omega)$, there exists an extension T_0 in $C_0^{k,\alpha}(R^d)$, such that

$$\|T_0\|_{k,q} \le \varepsilon, \quad k \ge 0,\ 1 \le q \le \infty, \tag{2.12}$$

where ε is an arbitrary positive constant.

(B_3) $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ have the following properties.

(i) For all $u \in X,\ v,w \in X,\ T\cdot\varphi \in H_0^1(\Omega)$, there holds that

$$b(u,v,w) = -b(u,w,v), \qquad \overline{b}(u,T,\psi) = -\overline{b}(u,\psi,T). \tag{2.13}$$

(ii) For all $u \in X,\ v \in H^1(\Omega)^d,\ T \in H^1(\Omega),$ for all $w \in X$ (or $\varphi \in H_0^1(\Omega)$), there holds that

$$\begin{aligned} |b(u,v,w)| &\le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0, \\ \left|\overline{b}(u,T,\varphi)\right| &\le \overline{N}\|\nabla u\|_0\|\nabla T\|_0\|\nabla\varphi\|_0, \end{aligned} \tag{2.14}$$

where $N = \sup_{u,v,w}|b(u,v,w)|/\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0$, $\overline{N} = \sup_{u,v,w}|\overline{b}(u,T,\varphi)|/\|\nabla u\|_0\|\nabla T\|_0$ $\|\nabla\varphi\|_0$.

3. Stability Analysis

Lemma 3.1. *The trilinear form b satisfies the following estimate:*

$$|b(u_h,v_h,w)| + |b(v_h,u_h,w)| + |b(w,u_h,v_h)| \le C\left|\log h\right|^{1/2}\|\nabla u_h\|_0\|\nabla v_h\|_0\|w\|_0. \tag{3.1}$$

Theorem 3.2. *Suppose that* (B_1)–(B3) *are valid and ε is a positive constant number, such that*

$$\frac{64C^2N\lambda\varepsilon}{3\nu^2} < 1, \qquad \frac{16C^2\lambda^2\overline{N}\varepsilon}{3\nu} < 1, \qquad \|\nabla T_0\|_0 \le \frac{\varepsilon}{4}, \qquad \|T_0\|_0 \le \frac{C\varepsilon}{4}. \tag{3.2}$$

Then (u_h^m, T_h^m) defined by (A_2) *satisfies*

$$\left\|\nabla u_h^m\right\|_0 \le \frac{8C^2\lambda\varepsilon}{3\nu}, \quad \left\|\nabla T_h^m\right\|_0 \le \varepsilon. \tag{3.3}$$

Proof. We prove this theorem by the inductive method. For $m = 1$, (3.3) holds obviously. Assuming that (3.3) holds for $m = n-1$, we want to prove that it holds for $m = n$. We estimate $\|\Delta u_h^n\|$ firstly. Letting $v_h = u_h^n,\ q_h = 0$ in the first equation of (2.10) and using (2.13), we get

$$a\left(u_h^n,u_h^n\right) + b\left(u_h^n,u_h^{n-1},u_h^n\right) = b\left(u_h^{n-1},u_h^{n-1},u_h^n\right) + \lambda\left(jT_h^n,u_h^n\right). \tag{3.4}$$

Setting $T_h^{n-1} = k_h^{n-1} + T_0$ and using (2.14), we have

$$\nu\left\|\nabla u_h^n\right\|_0 \le N\left\|\nabla u_h^n\right\|_0\left\|\nabla u_h^{n-1}\right\|_0 + N\left\|\nabla u_h^{n-1}\right\|_0^2 + C^2\lambda\left\|\nabla k_h^{n-1}\right\|_0 + C\lambda\|\nabla T_0\|_0. \tag{3.5}$$

Letting $T_h^n = k_h^n + T_0,\ \psi = k_h^n$ in the second equation of (2.10), we can obtain

$$\overline{a}(k_h^n, k_h^n) = -\lambda\overline{b}(u_h^n, T_0, k_h^n) - \overline{a}(T_0, k_h^n). \tag{3.6}$$

Using (2.12), (2.14), and (3.2), we get

$$\begin{aligned}
\left\|\nabla k_h^{n-1}\right\|_0 &\le \lambda\overline{N}\left\|\nabla u_h^{n-1}\right\|_0 \|\nabla T_0\|_0 + \|\nabla T_0\|_0 \\
&\le \frac{\lambda\overline{N}\varepsilon}{4}\left\|\nabla u_h^{n-1}\right\|_0 + \|\nabla T_0\|_0 \le \frac{3\varepsilon}{8} \le \frac{3\varepsilon}{4}, \\
\left(\nu - N\left\|\nabla u_h^{n-1}\right\|_0\right)\|\nabla u_h^n\|_0 &\le N\left\|\nabla u_h^{n-1}\right\|_0^2 + C^2\lambda\varepsilon \\
&\le C^2\lambda\varepsilon + \frac{64C^4N}{9\nu^2}\lambda^2\varepsilon^2 \le \frac{4C^2\lambda\varepsilon}{3}.
\end{aligned} \tag{3.7}$$

Using (3.2), we have $\nu - N\|\nabla u_h^{n-1}\|_0 \ge 7\nu/8$. Then,

$$\|\nabla u_h^n\|_0 \le \frac{8C^2\lambda\varepsilon}{3\nu}. \tag{3.8}$$

Combining (2.12), (2.14), (3.2), and (3.6), we arrive at

$$\|\nabla k_h^n\|_0 \le \lambda\overline{N}\|\nabla u_h^n\|_0 \|\nabla T_0\|_0 + \|\nabla T_0\|_0 \le \frac{3\varepsilon}{4}, \tag{3.9}$$

$$\|\nabla T_h^n\|_0 \le \|\nabla k_h^n\|_0 + \|\nabla T_0\|_0 \le \varepsilon. \tag{3.10}$$

Therefore, we finish the proof. □

4. Error Analysis

In this section, we establish the H^1-bound of the error $u_h^n - u$, $T_h^n - T$ and L^2-bounds of the error $p_h^n - p$. Setting $(e^n, \mu^n, \eta^n) = (u_h^n - u_h,\ p_h^n - p_h,\ T_h^n - T_h)$. Firstly, we give some Lemmas.

Lemma 4.1. *In [4], If* B_1-B_3 *hold,* $(u, p, T) \in H^{m+1}(\Omega)\times H^m(\Omega)\times H^{m+1}(\Omega)$ *and* $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$ *are the solution of problem* (A_1) *and* (A_2), *respectively, then there holds that*

$$\|\nabla(u - u_h)\|_0 + \|p - p_h\|_0 + \|\nabla(T - T_h)\|_0 \le Ch^m\left(\|u\|_{m+1} + \|p\|_m + \|T\|_{m+1}\right). \tag{4.1}$$

Lemma 4.2. *Under the assumptions of Theorem 3.2,* (A_2) *has a unique solution* $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$, *such that* $T|_{\partial\Omega} = T_0$ *and*

$$\|\nabla u_h\|_0 \le \frac{8C^2\lambda\varepsilon}{3\nu}, \qquad \|\nabla T_h\|_0 \le \varepsilon. \tag{4.2}$$

The detail proof we can see [4, 13, 14].

Theorem 4.3. *Under the assumption of Theorem 3.2, there holds*

$$\|\nabla e^n\|_0 \le \frac{C^2\lambda\varepsilon}{2^{n-3}3\nu}, \qquad \|\nabla \eta^n\|_0 \le \frac{\varepsilon}{2^{n+1}},$$
$$\|\mu^n\|_0 \le \beta^{-1}\begin{cases} \dfrac{\nu\varepsilon}{2} + \dfrac{4C^2\lambda\varepsilon}{3}, & n = 1 \\ (\nu + 2N\varepsilon)\dfrac{C^2\lambda\varepsilon}{2^{n-3}3\nu} + N\left(\dfrac{C^2\lambda\varepsilon}{2^{n-4}3\nu}\right)^2 + \dfrac{C^2\lambda\varepsilon}{2^n}, & n \ge 2. \end{cases} \tag{4.3}$$

Proof. Subtracting (2.10) from (2.9), we get the following error equations, namely (e^n, μ^n, η^n) satisfies

$$\begin{aligned} a(e^n, v_h) - d(\mu^n, v_h) + d(q_h, e^n) + b\left(e^n, u_h^{n-1}, v_h\right) + b\left(u_h^{n-1}, e^n, v_h\right) + G(\mu^n, q_h) \\ = b\left(e^{n-1}, e^{n-1}, v_h\right) + \lambda(j\eta^n, v_h), \end{aligned} \tag{4.4}$$

$$\overline{a}(\eta^n, \psi_h) + \lambda\overline{b}(e^n, T_h^n, \psi_h) + \lambda\overline{b}\left(u_h^{n-1}, \eta^n, q_h\right) = 0. \tag{4.5}$$

Here, let $\psi_h = \eta^n$, in (4.5), then we have

$$\overline{a}(\eta^n, \eta^n) + \lambda\overline{b}(e^n, T_h^n, \eta^n) = 0. \tag{4.6}$$

By using (2.14), we get

$$\|\nabla \eta^n\|_0 \le \lambda\overline{N}\varepsilon\|\nabla e^n\|_0. \tag{4.7}$$

In (4.4), we take $v_h = e^n \in X_h,\ q_h = \mu^n$, then

$$a(e^n, e^n) + b\left(e^n, u_h^{n-1}, e^n\right) + b\left(u_h^{n-1}, e^n, e^n\right) + G(\mu^n, \mu^n) = b\left(e^{n-1}, e^{n-1}, e^n\right) + \lambda(j\eta^n, e^n). \tag{4.8}$$

Using (2.13) and (2.14), we have

$$\nu\|\nabla e^n\|_0 + G(\mu^n, \mu^n) \le N\|\nabla e^n\|_0\left\|\nabla u_h^{n-1}\right\|_0 + N\left\|\nabla e^{n-1}\right\|_0^2 + C^2\lambda\|\nabla \eta^n\|_0, \tag{4.9}$$

then, we obtain

$$\left(\nu - N\left\|\nabla u_h^{n-1}\right\|_0\right)\|\nabla e^n\|_0 \le N\left\|\nabla e^{n-1}\right\|_0^2 + C^2\lambda\|\nabla \eta^n\|_0. \tag{4.10}$$

By using $\nu - N\|\nabla u_h^{n-1}\|_0 \geq 7\nu/8$. Equations (3.3) and (4.2), we get

$$\begin{aligned}\frac{7}{8}\nu\|\nabla e^n\|_0 &\leq N\left\|\nabla e^{n-1}\right\|_0^2 + C^2\lambda\|\nabla \eta^n\|_0 \\ &\leq \left(N\left\|\nabla u_h^{n-1}\right\|_0 + N\|\nabla u_h\|_0 + C^2\lambda^2\overline{N}\varepsilon\right)\left\|\nabla e^{n-1}\right\|_0 \\ &\leq \left(\frac{16NC^2\lambda\varepsilon}{3\nu} + C^2\lambda^2\overline{N}\varepsilon\right)\left\|\nabla e^{n-1}\right\|_0 = \frac{7\nu}{16}\left\|\nabla e^{n-1}\right\|_0, \\ \|\nabla e^n\|_0 &\leq \frac{1}{2}\left\|\nabla e^{n-1}\right\|_0.\end{aligned} \tag{4.11}$$

From the inductive method, we know, for $n = 1$, subtracting (2.10) from (2.9), we can get

$$a\left(e^1, v_h\right) - d\left(\mu^1, v_h\right) + d\left(q_h, e^1\right) + b(u_h, u_h, v_h) + G\left(\mu^1, q_h\right) = \lambda(jT^n, v_h). \tag{4.12}$$

Letting $v_h = e^1, q_h = \mu^1$ in (4.12) and using (2.14), we have

$$\begin{aligned}\left\|\nabla e^1\right\|_0 + G\left(\mu^1, \mu^1\right) &\leq \nu^{-1}N\|\nabla u_h\|_0^2 + \nu^{-1}C^2\lambda\|\nabla T_h\|_0 \\ &\leq \frac{64C^4\lambda^2N\varepsilon^2}{9\nu^3} + \frac{C^2\lambda\varepsilon}{\nu} \leq \frac{4C^2\lambda\varepsilon}{3\nu},\end{aligned} \tag{4.13}$$

then

$$\left\|\nabla e^1\right\|_0 \leq \frac{4C^2\lambda\varepsilon}{3\nu}. \tag{4.14}$$

By (4.7), we have

$$\left\|\nabla \eta^1\right\|_0 \leq \lambda\overline{N}\varepsilon\|\nabla e^n\|_0 \leq \lambda\overline{N}\frac{4C^2\lambda\varepsilon^2}{3\nu} \leq \frac{\varepsilon}{4}. \tag{4.15}$$

Letting $q_h = 0$ in (4.12), (2.14), and (3.9), using Lemma 2.1, we get

$$\beta\left\|\mu^1\right\|_0 \leq \nu\left\|\nabla e^1\right\|_0 + N\|\nabla u_h\|_0^2 + C\lambda\|T_h\|_0 \leq \frac{\nu\varepsilon}{2} + \frac{4C^2\lambda\varepsilon}{3\nu}. \tag{4.16}$$

Assuming that (4.3) is true for $n = k - 1$, using (4.7) and (4.11), we know that both of them are valid for $n = k$. Using (4.7) holds for $n = k$, we let $q_h = 0$ in (4.4) and using Lemma 2.1, (4.5), and (3.3), we have

$$\begin{aligned}\beta\|\mu^n\|_0 &\leq (\nu + 2N\varepsilon)\|\nabla e^n\|_0 + N\left\|\nabla e^{n-1}\right\|_0^2 + C^2\lambda\left\|\nabla \eta^{n-1}\right\|_0 \\ &\leq (\nu + 2N\varepsilon)\frac{C^2\lambda\varepsilon}{2^{n-3}3\nu} + N\left(\frac{C^2\lambda\varepsilon}{2^{n-4}3\nu}\right)^2 + \frac{C^2\lambda\varepsilon}{2^n}.\end{aligned} \tag{4.17}$$

□

Theorem 4.4. *Under the assumptions of Theorem 4.3, then there holds that*

$$\lim_{n\to\infty}\left(\|u_h^n - u_h^{n-1}\|_0 + \|\nabla(u_h^n - u_h^{n-1})\|_0\right) = 0,$$
$$\|\nabla e^n\|_0 + \|\mu^n\|_0 + \|\nabla\eta^n\|_0 \le F|\log h|^{1/2}\|\nabla(u_h^n - u_h^{n-1})\|_0\|u_h^n - u_h^{n-1}\|_0 + \frac{H\varepsilon}{2^{n+1}}, \tag{4.18}$$

where F *and* H *are two positive constants.*

Proof. By using (B_1) and triangle inequality, we have

$$\left\|u_h^n - u_h^{n-1}\right\|_0 + \left\|\nabla(u_h^n - u_h^{n-1})\right\|_0 \le (C+1)\left(\|\nabla e^n\|_0 + \left\|\nabla e^{n-1}\right\|_0\right). \tag{4.19}$$

Using Theorem 4.3, letting $n \to \infty$, we obtain (4.18). Taking $v_h = e^n$, $q_h = \mu^n$ in (4.4) and using (2.14), we get

$$a(e^n, e^n) + b\left(e^n, u_h^{n-1}, e^n\right) + G(\mu^n, \mu^n) = -b\left(u_h^n - u_h^{n-1}, u_h^n - u_h^{n-1}, e^n\right) + \lambda(j\eta^n, e^n). \tag{4.20}$$

By (2.14) and Lemma 3.1, we deduce

$$\left(\nu - N\left\|u_h^{n-1}\right\|_0\right)\|\nabla e^n\|_0 + G(\mu^n, \mu^n) \le F|\log h|^{1/2}\left\|\nabla\left(u_h^n - u_h^{n-1}\right)\right\|_0\left\|u_h^n - u_h^{n-1}\right\|_0 + F^2\lambda\|\nabla\eta^n\|_0. \tag{4.21}$$

Combining (3.3) and (4.7), we obtain

$$\left(\nu - \frac{8N\varepsilon}{3\nu}\right)\|\nabla e^n\|_0 \le F|\log h|^{1/2}\left\|\nabla\left(u_h^n - u_h^{n-1}\right)\right\|_0\left\|u_h^n - u_h^{n-1}\right\|_0 + F^2\lambda^2\overline{N}\varepsilon\left\|\nabla e^{n-1}\right\|_0. \tag{4.22}$$

Using (3.2), we get

$$\|\nabla e^n\|_0 \le F|\log h|^{1/2}\left\|\nabla(u_h^n - u_h^{n-1})\right\|_0\left\|u_h^n - u_h^{n-1}\right\|_0 + \frac{H\varepsilon}{2^{n+1}}. \tag{4.23}$$

Combining (3.2), (4.7), and (4.17), we get

$$\|\nabla\eta^n\|_0 \le F|\log h|^{1/2}\left\|\nabla(u_h^n - u_h^{n-1})\right\|_0\left\|u_h^n - u_h^{n-1}\right\|_0 + \frac{H\varepsilon}{2^{n+1}},$$
$$\|\nabla\mu^n\|_0 \le F|\log h|^{1/2}\left\|\nabla(u_h^n - u_h^{n-1})\right\|_0\left\|u_h^n - u_h^{n-1}\right\|_0 + \frac{H\varepsilon}{2^{n+1}}. \tag{4.24}$$

Here, we complete the proof. □

Theorem 4.5. *Under the assumptions of Theorem 4.3, the following inequality:*

$$\begin{aligned}\left\|\nabla(u-u_h^n)\right\|_0+\left\|p-p_h^n\right\|_0+\left\|\nabla(T-T_h^n)\right\|_0 &\le F_1h^m(\|u\|_{m+1}+\|p\|_m+\|T\|_{m+1})\\ &+F\left|\log h\right|^{1/2}\left\|\nabla\left(u_h^n-u_h^{n-1}\right)\right\|_0\left\|u_h^n-u_h^{n-1}\right\|_0+\frac{H\varepsilon}{2^{n+1}},\end{aligned} \tag{4.25}$$

holds, where F_1 and H are the positive constants.

Proof. By Lemma 4.1, Theorem 4.4, and the triangle inequality, this theorem is obviously true. □

5. Numerical Test

This section presents the numerical results that complement the theoretical analysis.

5.1. Convergence Analysis

In our experiment, $\Omega = [0,1] \times [0,1]$ is the unit square in R^2. Let $T_0 = 0$ on left and lower boundary of the cavity, $\partial T/\partial n = 0$ on upper boundary of the cavity, and $T_0 = 4y(1-y)$ on right boundary of the cavity (see Figure 1). Physics model of the cavity flows: $t = 0$, that is, $n = 0$ initial values on boundary. In general, we cannot know the exact solution of the stationary conduction-convection equations. In order to get the exact solution, we design the procedure as follows. Firstly, solving the stationary conduction-convection equations by using the P_2-P_1-P_2 finite element pair, which holds stability, on the finer mesh, we take the solution as the exact solution. Secondly, the absolute error is obtained by comparing the exact solution and the finite element solutions with VMS methods. Finally, we can easily obtain errors and convergence rates.

5.2. Driven Cavity

In this experiment, $\Omega = [0,1] \times [0,1]$ is the unit square in R^2. Let $T_0 = 0$ on left and lower boundary of the cavity, $\partial T/\partial n = 0$ on upper boundary of the cavity, and $T_0 = 4y(1-y)$ on right boundary of the cavity (see Figure 1). Physics model of the cavity flows: $t = 0$, that is, $n = 0$ initial values on boundary. Solving the stationary conduction-convection equations by using the P_2-P_1-P_2 finite element pair, which holds stability results, on the finer mesh, we take the solution as the exact solution. From Figures 1 and 2, we know that the solution of finite element VMS using P_1-P_1-P_1 element agree completely with the "exact solution." In Figure 3, we choose Re = 2000, divide the cavity into $M \times N = 100 \times 100$, from left to right shows the numerical streamline, the numerical isobar, and the numerical isotherms. In Figure 4, we choose Re = 3000, divide the cavity into $M \times N = 100 \times 100$, from left to right shows the numerical streamline, the numerical isobar, and the numerical isotherms.

Remark 5.1. Our VMS finite element method based on two local Gauss integrations and $\varepsilon_d = 0.1\,h$ is suitable for the Sobolev space. Throughout the paper, our analysis and numerical tests are all carried out for the P_1-P_1-P_1 element (see Tables 1 and 2).

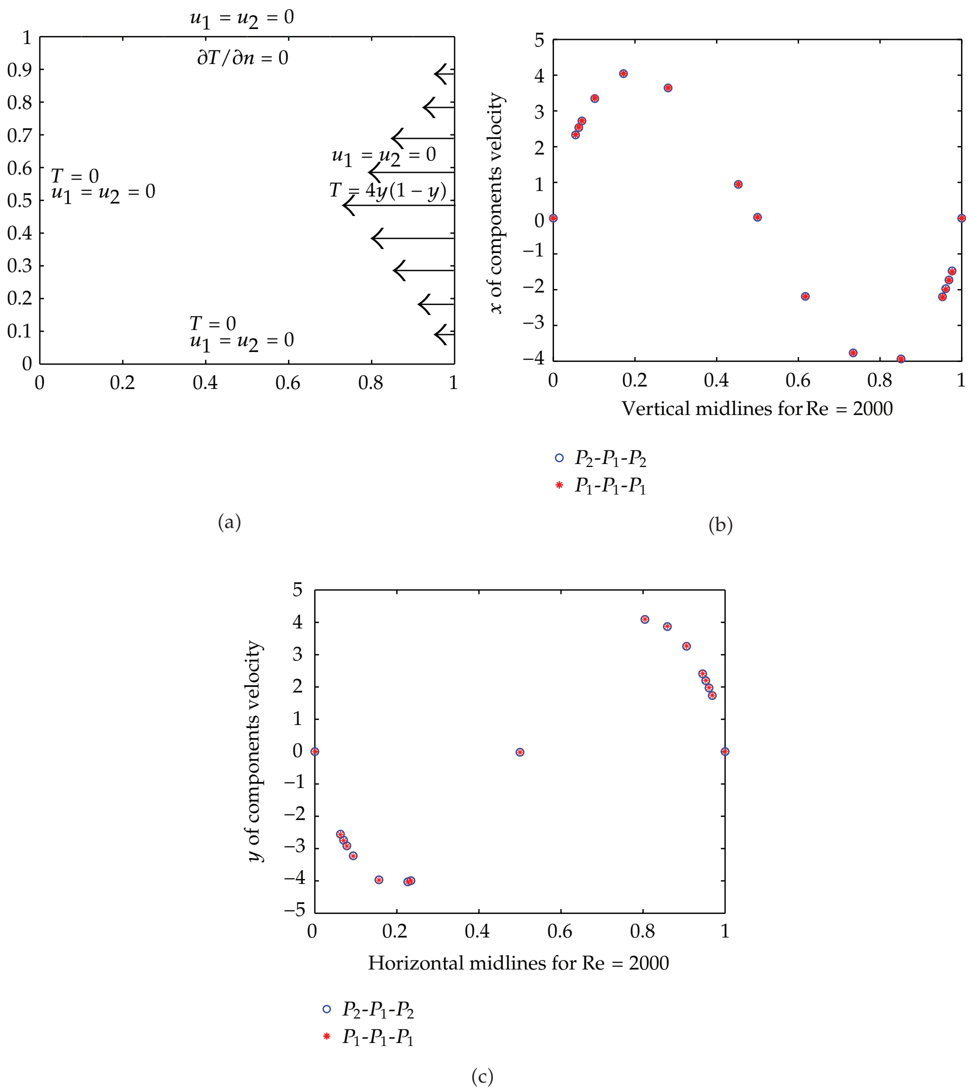
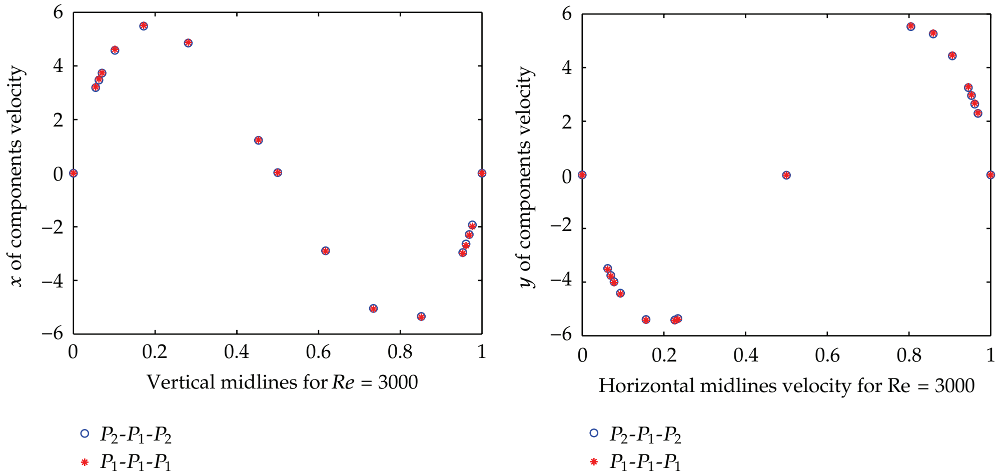


Figure 1: From (a) to (c): physics model of the cavity flows, vertical midlines for Re = 2000, $h = 1/100$, horizontal midlines for Re = 2000, $h = 1/100$.

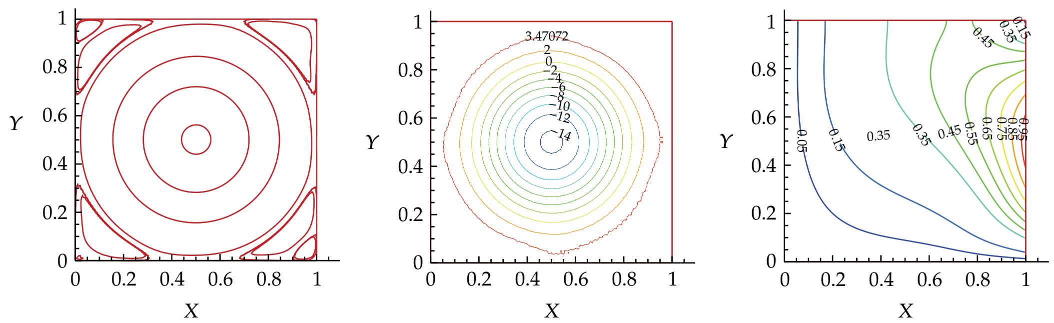
6. Conclusion

In this paper, we studied a finite element VMS algorithm based on two local Gauss integrations to solve the stationary conduction-convection problem. From Figures 1 and 2, we see that the solution of VMS using P_1-P_1-P_1 and $\varepsilon_d = 0.1\,h$ agrees completely with the “exact solution,” which shows that our method is highly efficient for the stationary conduction-convection problems. Numerical tests tell us that VMS finite element method based on two local Gauss integrations is very effective.



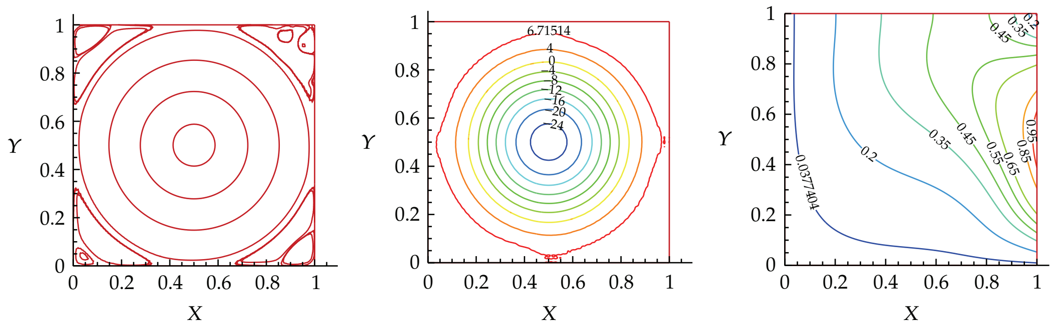
(a) (b)

Figure 2: From (a) to (b): vertical midlines for Re = 3000, $h = 1/100$, horizontal midlines for Re = 3000, $h = 1/100$.



(a) (b) (c)

Figure 3: For Re = 2000, $h = 1/100$, from (a) to (c): velocity streamlines, the pressure level lines, numerical isotherms.



(a) (b) (c)

Figure 4: For Re = 3000, $h = 1/100$, from (a) to (c): velocity streamlines, the pressure level lines, numerical isotherms.

Table 1: VMS: P_1-P_1-P_1 element.

$1/h$	$\|u-u_h\|_0$	$\|u-u_h\|_1$	$\|T-T_h\|_0$	$\|T-T_h\|_1$	$\|p-p_h\|_0$
10	0.000194122	0.00493006	0.00740049	0.277241	0.00506075
20	$4.91998e-005$	0.00252269	0.00208824	0.153561	0.00308312
40	$1.21288e-005$	0.00126459	0.0005746	0.0838877	0.00180539
60	$5.35135e-006$	0.000842093	0.000266991	0.0583954	0.00131444
80	$2.98429e-006$	0.000630808	0.000154418	0.0457979	0.00105007

Table 2: VMS: P_1-P_1-P_1 element.

$1/h$	u_{L^2} rate	u_{H^1} rate	T_{L^2} rate	T_{H^1} rate	p_{L^2} rate
10	/	/	/	/	/
20	1.9802	0.9666	1.8253	0.8523	0.7150
40	2.0202	0.9963	1.8617	0.8723	0.7721
60	2.0180	1.0028	1.8903	0.8934	0.7827
80	2.0300	1.0042	1.9033	0.8447	0.7806

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