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Research Article

A Study on Ricci Solitons in Kenmotsu Manifolds

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We study and obtain results on Ricci solitons in Kenmotsu manifolds satisfying $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \overline{P} = 0$, and $\overline{P}(\xi, X) \cdot S = 0$, where B and $\overline{P}$ are C-Bochner and pseudo-projective curvature tensor.

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field, and λ a real scalar such that

$$\mathscr{L}_V g + 2S + 2\lambda g = 0, \tag{1}$$

where S is a Ricci tensor of M and $\mathscr{L}_V$ denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive, respectively [1]. In this paper, we prove conditions for Ricci solitons in Kenmotsu manifolds to be shrinking, steady, and expanding.

In 1972, Kenmotsu [2] studied a class of contact Riemannian manifolds satisfying some special conditions and this manifold is known as Kenmotsu manifolds. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant. Kenmotsu proved that if in a Kenmotsu manifold the condition $R(X, Y) \cdot R = 0$ holds, then the manifold is of negative curvature -1, where R is the curvature tensor of type $(1, 3)$ and $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the tangent space.

The authors in [3–7] have studied Ricci solitons in contact and Lorentzian manifolds. The authors in [8] have obtained some results on Ricci solitons satisfying $R(\xi, X) \cdot \widetilde{C} = 0$, $P(\xi, X) \cdot \widetilde{C} = 0$, $H(\xi, X) \cdot S = 0$ and $\widetilde{C}(\xi, X) \cdot S = 0$ and now we extend the work to $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \overline{P} = 0$ and $\overline{P}(\xi, X) \cdot S = 0$.

2. Preliminaries

An n-dimensional differential manifold M is said to be an almost contact metric manifold [9] if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1, 1)$, a vector field ξ, a 1-form η, and a Riemannian metric g compatible with (ϕ, ξ, η, g) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$
$$\eta \circ \phi = 0, \qquad \phi\xi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \xi) = \eta(X), \tag{2}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold [2] if

$$(\nabla_X \phi) Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{3}$$

From (3), we have

$$\nabla_X \xi = X - \eta(X)\xi, \tag{4}$$

where ∇ denotes the Riemannian connection of g.

In an n-dimensional Kenmotsu manifold, we have

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{5}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{6}$$

$$R(\xi, X) Y = \eta(Y) X - g(X, Y) \xi, \tag{7}$$

$$R(\xi, X) \xi = X - \eta(X) \xi, \tag{8}$$

where R is the Riemannian curvature tensor.

Let (g, V, λ) be a Ricci soliton in an n-dimensional Kenmotsu manifold M. From (4) we have

$$\left(\mathscr{L}_{\xi} g\right)(X, Y) = 2\left[g(X, Y) - \eta(X) \eta(Y)\right]. \tag{9}$$

From (1) and (9) we get

$$S(X, Y) = -(\lambda + 1) g(X, Y) + \eta(X) \eta(Y). \tag{10}$$

The above equation yields that

$$QX = -(\lambda + 1) X + \eta(X) \xi, \tag{11}$$

$$S(X, \xi) = -\lambda \eta(X), \tag{12}$$

$$r = -\lambda n - (n - 1), \tag{13}$$

where S is the Ricci tensor, Q is the Ricci operator, and r is the scalar curvature on M.

2.1. Example for 3-Dimensional Kenmotsu Manifolds. We consider 3-dimensional manifold $M = \{(x, y, z) \in R^3; z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent given by

$$E_1 = z\frac{\partial}{\partial x}, \qquad E_2 = z\frac{\partial}{\partial y}, \qquad E_3 = -z\frac{\partial}{\partial z}. \tag{14}$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, where g is given by

$$g = \frac{1}{z^2}\left(dx \otimes dx + dy \otimes dy + dz \otimes dz\right). \tag{15}$$

The (ϕ, ξ, η) structure is given by

$$\begin{gathered} \eta = -\frac{1}{z} dz, \qquad \xi = E_3 = -z\frac{\partial}{\partial z}, \\ \phi E_1 = -E_2, \qquad \phi E_2 = E_1, \qquad \phi E_3 = 0. \end{gathered} \tag{16}$$

The linearity property of ϕ and g yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U) E_3$, $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$, for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_2] = 0, \qquad [E_1, E_3] = E_1, \qquad [E_2, E_3] = E_2. \tag{17}$$

Let ∇ be the Levi-Civita connection; with respect to above metric g is given by Koszula formula

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) = {} & X\left(g(Y, Z)\right) + Y\left(g(Z, X)\right) \\ & - Z\left(g(X, Y)\right) - g(X, [Y, Z]) \\ & - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{18}$$

and by virtue of it we have

$$\begin{array}{lll} \nabla_{E_1} E_3 = E_1, & \nabla_{E_2} E_3 = E_2, & \nabla_{E_3} E_3 = 0, \\ \nabla_{E_1} E_2 = 0, & \nabla_{E_2} E_2 = -E_3, & \nabla_{E_3} E_2 = 0, \\ \nabla_{E_1} E_1 = -E_3, & \nabla_{E_2} E_1 = 0, & \nabla_{E_3} E_1 = 0. \end{array} \tag{19}$$

Clearly (19) shows that (ϕ, ξ, η, g) satisfies (2), (3), and (4). Thus M is a Kenmotsu manifold.

It is known that

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{20}$$

With the help of (19) and (20), it can be easily verified that

$$\begin{gathered} R\left(E_1, E_2\right) E_2 = -E_1, \qquad R\left(E_1, E_3\right) E_3 = -E_1, \\ R\left(E_1, E_1\right) E_1 = 0, \qquad R\left(E_2, E_1\right) E_1 = -E_2, \\ R\left(E_2, E_3\right) E_3 = -E_2, \qquad R\left(E_2, E_2\right) E_2 = 0, \\ R\left(E_3, E_1\right) E_1 = -E_3, \qquad R\left(E_3, E_2\right) E_2 = -E_3, \\ R\left(E_3, E_3\right) E_3 = 0. \end{gathered} \tag{21}$$

From the above expression of the curvature tensor we obtain

$$\begin{aligned} S\left(E_1, E_1\right) = {} & g\left(R\left(E_1, E_2\right) E_2, E_1\right) \\ & + g\left(R\left(E_1, E_3\right) E_3, E_1\right) = -2. \end{aligned} \tag{22}$$

Similarly we have

$$\begin{gathered} S\left(E_2, E_2\right) = S\left(E_3, E_3\right) = -2, \\ \left(\mathscr{L}_{\xi} g\right)\left(E_i, E_i\right) = 2\left[g\left(E_i, E_i\right) - \eta\left(E_i\right) \eta\left(E_i\right)\right]. \end{gathered} \tag{23}$$

Now by $X = Y = E_i$, in (1), where $i = 1, 2, 3$ and by virtue of above equations we get the value of λ which is strictly greater than 0. Thus this is an example of expanding Ricci solitons in Kenmotsu manifolds.

3. Ricci Soliton in a Kenmotsu Manifold Satisfying $R(\xi, X) \cdot B = 0$

Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [10]. A geometric meaning of the Bochner curvature tensor is given by Blair in [11] by using the Boothby-Wang's fibration. In 1969, Matsumoto and Chūman [12] constructed the notion of C-Bochner curvature tensor in a Sasakian manifold and studied its several properties.

The C-Bochner curvature tensor [13] B in M is defined by

$$\begin{aligned} B(X,Y)Z = {} & R(X,Y)Z + \frac{1}{n+3} \\ & \times [g(X,Z)QY - S(Y,Z)X \\ & - g(Y,Z)QX + S(X,Z)Y \\ & + g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X \\ & - g(\phi Y,Z)Q\phi X + S(\phi X,Z)\phi Y \\ & + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z \\ & + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi \\ & + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] \\ & - \frac{D+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ & + 2g(\phi X,Y)\phi Z] \\ & + \frac{D}{n+3}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X \\ & + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi] \\ & - \frac{D-4}{n+3}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{24}$$

where $D = (r+n-1)/(n+1)$.

Taking $Z = \xi$ in (24) and using (6), (10), (11), we get

$$B(X,Y)\xi = \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right][\eta(X)Y - \eta(Y)X]. \tag{25}$$

Similarly using (5), (10), (11), (12) in (24), we get

$$\begin{aligned} \eta(B(X,Y)Z) = {} & \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right] \\ & \times [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \end{aligned} \tag{26}$$

We assume that the condition $R(\xi, X) \cdot B = 0$, then we have

$$\begin{aligned} & R(\xi,X)B(Y,Z)W - B(R(\xi,X)Y,Z)W \\ & - B(Y,R(\xi,X)Z)W - B(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{27}$$

Using (7) in (27), we get

$$\begin{aligned} & \eta(B(Y,Z)W)X - g(B(Y,Z)W,X)\xi + g(X,Y)B(\xi,Z)W \\ & - \eta(Y)B(X,Z)W + g(X,Z)B(Y,\xi)W \\ & - \eta(Z)B(Y,X)W + g(X,W)B(Y,Z)\xi \\ & - \eta(W)B(Y,Z)X = 0. \end{aligned} \tag{28}$$

By taking an inner product with ξ, we have

$$\begin{aligned} & \eta(B(Y,Z)W)\eta(X) - g(B(Y,Z)W,X) \\ & + g(X,Y)\eta(B(\xi,Z)W) - \eta(Y)\eta(B(X,Z)W) \\ & + g(X,Z)\eta(B(Y,\xi)W) - \eta(Z)\eta(B(Y,X)W) \\ & + g(X,W)\eta(B(Y,Z)\xi) - \eta(W)\eta(B(Y,Z)X) = 0. \end{aligned} \tag{29}$$

By using (25), (26) in (29), we have

$$\begin{aligned} & \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right][g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] \\ & - g(B(Y,Z)W,X) = 0. \end{aligned} \tag{30}$$

In view of (24) in (30), then we have

$$\begin{aligned} & \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right][g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] \\ & - g(R(Y,Z)W,X) \\ & - \frac{1}{n+3}[g(Y,W)S(Z,X) - S(Z,W)g(Y,X) \\ & - g(Z,W)S(Y,X) + S(Y,W)g(Z,X) \\ & + g(\phi Y,W)S(\phi Z,X) - S(\phi Z,W)g(\phi Y,X) \\ & - g(\phi Z,W)S(\phi Y,X) + S(\phi Y,W)g(\phi Z,X) \\ & + 2S(\phi Y,Z)g(X,\phi W) + 2g(\phi Y,Z)S(X,\phi W) \\ & + \eta(W)\eta(Z)S(Y,X) - \eta(X)\eta(Z)S(Y,W) \\ & + \eta(Y)\eta(X)S(Z,W) - \eta(W)\eta(Y)S(Z,X)] \\ & - \frac{D}{n+3}[\eta(X)\eta(Z)g(Y,W) - \eta(W)\eta(Z)g(Y,X) \\ & + \eta(W)\eta(Y)g(Z,X) - \eta(Y)\eta(X)g(Z,W)] \\ & + \frac{D+n-1}{n+3}[g(\phi Y,W)g(\phi Z,X) \\ & - g(\phi Z,W)g(\phi Y,X) \\ & + 2g(\phi Y,Z)g(X,\phi W)] \\ & + \frac{D-4}{n+3}[g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] = 0. \end{aligned} \tag{31}$$

Taking $X = Y = e_i$ in (31) and summing over $i = 1, 2, \ldots, n$. By virtue of (10), (11), (12), and on simplification, we get

$$S(Z, W) = \left[\frac{-(n+4)\lambda - 2n - 3}{n+3} + \frac{-n^2 - 6n + 8}{n+3} - \frac{r}{n+3}\right] g(W, Z) + \left[\frac{(\lambda+1)(n+4) - 2 + n}{n+3} + \frac{r + 4(n-1)}{n+3}\right] \eta(W)\eta(Z). \tag{32}$$

Putting $Z = W = \xi$ in (32) and by virtue of (10) and (13), we have

$$\lambda = (n-1). \tag{33}$$

Therefore, λ positive that is, the Ricci soliton in Kenmotsu manifold is expanding.

Hence we state the following theorem:

Theorem 1. *A Ricci soliton in a Kenmotsu manifold satisfying $R(\xi, X) \cdot B = 0$ is expanding.*

4. Ricci Soliton in a Kenmotsu Manifolds Satisfying $B(\xi, X) \cdot S = 0$

The condition $B(\xi, X) \cdot S = 0$ implies that

$$S(B(\xi, X)Y, Z) + S(Y, B(\xi, X)Z) = 0. \tag{34}$$

By using (10) in (34), we have

$$\eta(Z)\eta(B(\xi, X)Y) - (\lambda+1) g(B(\xi, X)Y, Z) - (\lambda+1) g(Y, B(\xi, X)Z) + \eta(Y)\eta(B(\xi, X)Z) = 0, \tag{35}$$

the above equation implies that

$$[\eta(Z)\eta(B(\xi, X)Y) + \eta(Y)\eta(B(\xi, X)Z)] = (\lambda+1)[g(B(\xi, X)Y, Z) + g(Y, B(\xi, X)Z)]. \tag{36}$$

By using (24) and (26) in (36), we have

$$2\eta(X)\eta(Y)\eta(Z)\left[1 - \frac{\lambda}{(n+3)} + \frac{4}{(n+3)}\right] - \left[1 - \frac{\lambda}{(n+3)} + \frac{4}{(n+3)}\right] \times [g(X, Z)\eta(Y) + g(X, Y)\eta(Z)] = 0. \tag{37}$$

Put $X = Y = \xi$ in (37) then the equation is identically satisfied and we do not get the value for λ. So, we proceed as follows: Taking $X = Y = e_i$ in (37) and summing over $i = 1, 2, \ldots, n$ and by virtue of (13) and $\eta(Z) \neq 0$ conditions, we obtain

$$\lambda = n + 7. \tag{38}$$

Therefore, λ is positive that is Ricci soliton in Kenmotsu manifolds satisfying $B(\xi, X) \cdot S = 0$ is expanding.

Hence we can state the following theorem.

Theorem 2. *A Ricci soliton in a Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is expanding.*

5. Ricci Soliton in a Kenmotsu Manifold Satisfying $S(\xi, X) \cdot R = 0$

Using the following equations:

$$S((X,\xi) \cdot R)(U, V)W = ((X \wedge_S \xi) \cdot R)(U, V)W = (X \wedge_S \xi) R(U, V)W + R((X \wedge_S \xi)U, V)W + R(U, (X \wedge_S \xi)V)W + R(U, V)(X \wedge_S \xi)W, \tag{39}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y, \tag{40}$$

we have

$$\begin{aligned} S((X,\xi) \cdot R)(U, V)W &= S(\xi, R(U, V)W)X - S(X, R(U, V)W)\xi \\ &\quad + S(\xi, U)R(X, V)W - S(X, U)R(\xi, V)W \\ &\quad + S(\xi, V)R(U, X)W - S(X, V)R(U, \xi)W \\ &\quad + S(\xi, W)R(U, V)X - S(X, W)R(U, V)\xi. \end{aligned} \tag{41}$$

By using the condition $S(\xi, X) \cdot R = 0$, and by virtue of (10), (12), we have

$$\begin{aligned} &-\lambda\eta(R(U, V)W)X \\ &\quad - [-(\lambda+1) g(X, R(U, V)W) + \eta(X)\eta(R(U, V)W)]\xi \\ &\quad - \lambda\eta(U)R(X, V)W \\ &\quad - [-(\lambda+1) g(X, U) + \eta(X)\eta(U)]R(\xi, V)W \\ &\quad - \lambda\eta(V)R(U, X)W \\ &\quad - [-(\lambda+1) g(X, V) + \eta(X)\eta(V)]R(U, \xi)W \\ &\quad - \lambda\eta(W)R(U, V)X \\ &\quad - [-(\lambda+1) g(X, W) + \eta(X)\eta(W)]R(U, V)\xi = 0. \end{aligned} \tag{42}$$

By taking an inner product with ξ and by virtue of (5), (6), (7), and (8), we have

$$\begin{aligned} &-(\lambda+1)\eta(X)[g(U, W)\eta(V) - g(V, W)\eta(U)] \\ &\quad + \lambda[g(V, W)\eta(X)\eta(U) - g(U, W)\eta(X)\eta(V) \\ &\qquad - g(U, X)\eta(V)\eta(W) + g(V, X)\eta(U)\eta(W)] \\ &\quad + (\lambda+1) g(X, R(U, V)W) \\ &\quad + (\lambda+1) g(X, U)[\eta(W)\eta(V) - g(V, W)] \\ &\quad + (\lambda+1) g(X, V)[g(U, W) - \eta(W)\eta(U)] \\ &\quad + g(V, W)\eta(X)\eta(U) - g(U, W)\eta(X)\eta(V) = 0. \end{aligned} \tag{43}$$

Taking $X = U = e_i$ and summing over $i = 1, 2, \ldots, n$, we obtain

$$2(\lambda+1)[g(V,W)-\eta(W)\eta(V)] + (\lambda+1)S(V,W) - (\lambda+1)(n-1)g(V,W) + (n-1)\eta(V)\eta(W) = 0. \quad (44)$$

Taking $V = W = \xi$ in (44) and by virtue of (12), (13), we obtain

$$-\lambda(\lambda+n) = 0. \quad (45)$$

This implies either

$$\lambda = 0 \quad \text{or} \quad \lambda = -n. \quad (46)$$

Therefore for any $\lambda = 0$ or $\lambda = -n$ the Ricci soliton in Kenmotsu manifolds satisfying $S(\xi, X) \cdot R = 0$ is either steady or shrinking.

Hence we can state the following theorem.

Theorem 3. *A Ricci soliton in a Kenmotsu manifold satisfying $S(\xi, X) \cdot R = 0$ is either steady or shrinking.*

6. Ricci Soliton in a Kenmotsu Manifolds Satisfying $R(\xi, X) \cdot \overline{P} = 0$

The Pseudo-projective curvature tensor $\overline{P}$ is defined by

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X - g(X,Z)Y], \quad (47)$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (47) and using (6), (10), (11), we get

$$\overline{P}(X,Y)\xi = \left[a + b\lambda + \frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][\eta(X)Y - \eta(Y)X]. \quad (48)$$

Similarly using (5), (10), (11), (12) in (47), we get

$$\eta\left(\overline{P}(X,Y)Z\right) = \left[a + b(\lambda+1) + \frac{r}{n}\left(\frac{a}{n-1}+b\right)\right] \times [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \quad (49)$$

We assume that the condition $R(\xi, X) \cdot \overline{P} = 0$, then we have

$$R(\xi,X)\overline{P}(U,V)W - \overline{P}(R(\xi,X)U,V)W - \overline{P}(U,R(\xi,X)V)W - \overline{P}(U,V)R(\xi,X)W = 0. \quad (50)$$

Using (7) in (50), we find

$$\eta\left(\overline{P}(U,V)W\right)X - g\left(X,\overline{P}(U,V)W\right)\xi - \eta(U)\overline{P}(X,V)W + g(X,U)\overline{P}(\xi,V)W - \eta(V)\overline{P}(U,X)W + g(X,V)\overline{P}(U,\xi)W - \eta(W)\overline{P}(U,V)X + g(X,W)\overline{P}(U,V)\xi = 0. \quad (51)$$

By taking an inner product with ξ then we get

$$\eta\left(\overline{P}(U,V)W\right)\eta(X) - g\left(X,\overline{P}(U,V)W\right) - \eta(U)\eta\left(\overline{P}(X,V)W\right) + g(X,U)\eta\left(\overline{P}(\xi,V)W\right) - \eta(V)\eta\left(\overline{P}(U,X)W\right) + g(X,V)\eta\left(\overline{P}(U,\xi)W\right) - \eta(W)\eta\left(\overline{P}(U,V)X\right) + g(X,W)\eta\left(\overline{P}(U,V)\xi\right) = 0. \quad (52)$$

By using (48), (49) in (52), we have

$$-g\left(X,\overline{P}(U,V)W\right) + \left[a + b(\lambda+1) + \frac{r}{n}\left[\frac{a}{n+1}+b\right]\right] \times [g(X,V)g(U,W) - g(X,U)g(V,W)] = 0. \quad (53)$$

In view of (47) in (53), we have

$$-ag(X,R(U,V)W) - b[(\lambda+1)\{g(V,X)g(U,W) - g(V,W)g(U,X)\} + \eta(V)\eta(W)g(U,X) - g(V,X)\eta(U)\eta(W)] + [a + b(\lambda+1)] \times [g(X,V)g(U,W) - g(X,U)g(V,W)] = 0. \quad (54)$$

Taking $X = U = e_i$ in (54) and summing over $i = 1, 2, \ldots, n$, and on simplification, we get

$$aS(V,W) = -a(n-1)g(V,W) - b(n-1)\eta(V)\eta(W). \quad (55)$$

Putting $V = W = \xi$ in (55) and by virtue of (12), (13), we get the following equation:

$$\lambda = \frac{(n-1)(a+b)}{a}. \quad (56)$$

Since $(a+b)/a \neq 0$ implies that $\lambda > 0$, that is, the Ricci soliton in Kenmotsu manifold satisfying $R(\xi, X) \cdot \overline{P} = 0$ is expanding, hence we state the following theorem.

Theorem 4. *A Ricci soliton in a Kenmotsu manifold satisfying $R(\xi, X) \cdot \overline{P} = 0$ is expanding.*

7. Ricci Soliton in a Kenmotsu Manifolds Satisfying $\overline{P}(\xi, X) \cdot S = 0$

The condition $\overline{P}(\xi, X) \cdot S = 0$ implies that

$$S\left(\overline{P}(\xi,X)Y, Z\right) + S\left(Y, \overline{P}(\xi,X)Z\right) = 0. \quad (57)$$

By using (10) in (57), we have

$$\eta(Z)\eta\left(\overline{P}(\xi,X)Y\right)-(\lambda+1)g\left(\overline{P}(\xi,X)Y,Z\right)-(\lambda+1)g\left(Y,\overline{P}(\xi,X)Z\right)+\eta(Y)\eta\left(\overline{P}(\xi,X)Z\right)=0, \tag{58}$$

that is,

$$\begin{aligned}&\left[\eta(Z)\eta\left(\overline{P}(\xi,X)Y\right)+\eta(Y)\eta\left(\overline{P}(\xi,X)Z\right)\right]\\&\quad=(\lambda+1)\left[g\left(\overline{P}(\xi,X)Y,Z\right)+g\left(Y,\overline{P}(\xi,X)Z\right)\right].\end{aligned} \tag{59}$$

By using (47) and (48) in (59), we have

$$\begin{aligned}&\left[a+\frac{r}{n}\left[\frac{a}{n-1}+b\right]\right]\\&\quad\times\left[2\eta(X)\eta(Y)\eta(Z)\right.\\&\quad\quad\left.-g(X,Z)\eta(Y)-g(X,Y)\eta(Z)\right]=0.\end{aligned} \tag{60}$$

Put $X=Y=\xi$ in (60); then the equation is identically satisfied and we do not get the value for λ. So, we proceed as follows: taking $X=Y=e_i$, summing over $i=1,2,\dots,n$, and by virtue of (13) and $\eta(Z)\neq 0$ conditions we obtain

$$\lambda=\frac{(n-1)^2(a-b)}{n\left[a+b(n-1)\right]}. \tag{61}$$

Therefore, if $a=b$ in (61) then $\lambda=0$; that is, Ricci soliton in Kenmotsu manifolds satisfying $\overline{P}(\xi,X)\cdot S=0$ is steady. If $a\neq b$ then either $\lambda>0$ for $a>b$ or $\lambda<0$ for $a<b$, that is, the Ricci soliton in Kenmotsu manifold satisfying $\overline{P}(\xi,X)\cdot S=0$ is expanding or shrinking.

Hence we can state the following theorem.

Theorem 5. *A Ricci soliton in a Kenmotsu manifolds satisfying $\overline{P}(\xi,X)\cdot S=0$ is steady for $a=b$, expanding for $a>b$ and shrinking for $a<b$.*

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