Graphs and Graph polynomials

by

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A dissertation submitted to the

School of Mathematics

in fulfilment of the

requirements for the degree of

Master of Science

School of Mathematics

University of the Witwatersrand

October 2017

Johannesburg South Africa

DECLARATION

I declare that this Dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

(Signature of candidate)

12th day of JANUARY 2018 in JOHANNESBURG.

Abstract

In this work we study the k-defect polynomials of a graph G. The k defect polynomial is a function in λ that gives the number of improper colourings of a graph using λ colours. The k-defect polynomials generate the bad colouring polynomial which is equivalent to the Tutte polynomial, hence their importance in a more general graph theoretic setting. By setting up a one-to-one correspondence between triangular numbers and complete graphs, we use number theoretical methods to study certain characteristics of the k-defect polynomials of complete graphs. Specifically we are able to generate an expression for any k-defect polynomial of a complete graph, determine integer intervals for k on which the k-defect polynomials for complete graphs are equal to zero and also determine a formula to calculate the minimum number of k-defect polynomials that are equal to zero for any complete graph.

— Christo Kriel

Acknowledgements

I would like to thank my supervisor Eunice Mphako-Banda for introducing me to the world of graph colouring and associated polynomials and especially for all the effort spent to get me to understand the details involved in producing an extended mathematical argument.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

In this chapter we give a brief background to colouring problems, the general area in graph theory under which the problems discussed in this dissertation fall. Then we give an overview of the dissertation, followed by some basic definitions. We list a number of graph operations that we use throughout this dissertation as well as some definitions that follow from these operations, most notably the definition of a closed set. In Sections 1.5 and 1.6 we describe proper and improper colourings and the related chromatic and k-defect polynomials.

1.1 Background to colouring problems

> It would not be hard to present the history of graph theory as an account of the struggle to prove the four colour conjecture, or at least to find out why the problem is difficult.
>
> William T. Tutte (1967)

The subject matter of this dissertation falls broadly under the umbrella of graph colouring problems. The quotation by Tutte at the beginning of this section, quoted in [3], not only points to the importance of graph colouring problems in the area of Graph Theory, but specifically to what was, arguably, the first and most famous problem in graph colouring, the *four colour conjecture*. The short overview of the history of the four colour problem and how it influenced the subject matter of this work was compiled from [3, 6, 10, 12] and [13].

In 1852 Francis Guthrie noticed that it is possible to colour all the counties of England with just four colours, such that counties that share a contiguous border are always coloured differently. This led to a conjecture that it is possible to do so for all maps. Francis's brother, Frederick, brought the conjecture to the attention of Augustus de Morgan, professor at University College London. De Morgan was intrigued by the conjecture, but was unable to prove it. In 1879, Alfred Kempe put forward a proof of the conjecture, but this was shown to be erroneous by Percy Heawood in 1890. Thus followed almost a hundred years of attempts to prove or disprove the conjecture until 1976 when Wolfgang Haken and Kenneth Appel put forward a, at the time controversial, computer aided proof of the conjecture by using an unavoidable set of 1482 reducible configurations in maps.

How does a map colouring translate to a problem in graph theory? Suppose we draw a dot in each country and draw a line from the dot through the border shared by two adjoining countries to the dot in the adjacent country, then we end up with a graph in which none of the edges cross, that is, what is called a planar graph. The conjecture thus translates to it being possible to colour the vertices of a planar graph with no more than four colours in such a way that no two adjacent vertices are the same colour. This is called a proper colouring of a graph.

Most colouring problems in graph theory involve proper colourings, not just because of the famous four colour theorem, but also because of the many applications of proper colourings in problems that can be modeled by graphs. But, we don't just have proper vertex colouring problems in graphs. We also have edge colouring problems and many different types of vertex colourings that emerge if we relax some of the conditions of a proper colouring or add more constraints. See, for example, the section on graph colouring in [10] or the survey on (m, k)-colourings by Frick in [9].

As an example of problems in graph colouring see Section 5.6 of [10], where graph colouring is applied to four different types of timetabling problems. The automation of timetabling in educational institutions is an area that has seen much growth since the 1970's and still continues to be an area of great interest to graph theorists.

Back to the four colour problem. In 1912 George Birkhoff defined a function $\chi(M;\lambda)$ that gives the number of proper colourings of a map M for a positive integer λ. This is a polynomial in λ, called the chromatic polynomial of M. If it could be shown that $\chi(M;4) > 0$ for every map M, the truth of the four colour conjecture would be established. In 1932 Hassler Whitney generalised the chromatic polynomial of a map to the chromatic polynomial of an arbitrary graph, $\chi(G;\lambda)$, and established many results for this polynomial, see [6]. In 1968, Ronald Read aroused renewed interest in the subject of chromatic polynomials with his survey paper "An Introduction to Chromatic Polynomials", see [20]. Since then, the subject of chromatic polynomials has been widely studied. We refer to the monograph by Dong, Koh and Teo, see [6], for many of the latest results and open problems surrounding the chromatic polynomial.

The main object of study in this work is the k-defect polynomial of a graph, with specific reference to complete graphs. If we relax the condition that no two adjacent vertices in a graph colouring be the same colour and allow a certain number of such vertices to be coloured the same, we get an improper colouring of a graph. The k-defect polynomial is a function $\phi_k(G;\lambda)$ that gives the number of such improper colourings of a graph G for a positive integer λ. This is a generalisation of the chromatic polynomial which is then the 0-defect polynomial of a graph. We describe proper and improper colourings and their associated polynomials in more detail in Sections 1.5 and 1.6.

Tutte noticed certain properties of the chromatic polynomial that were similar to other functions on graphs. These observations eventually led to the definition of the Tutte polynomial, see the 1947 paper [22] as well as [23] for Tutte's own account of discovering the polynomial that is now named after him. The Tutte polynomial is a very important graph invariant that encodes many of the properties of a graph. It turns out that the Tutte polynomial is equivalent to the bad colouring polynomial. The bad colouring polynomial was originally defined and studied by Crapo, in 1969, as a generating function in S of the k-defect polynomials, see [4] and [19]. Hence, the study of the k-defect polynomials could potentially yield some important results with respect to both the Tutte and bad colouring polynomials. We describe the equivalence of the different polynomials studied in this dissertation in more detail in Chapter 2.

1.2 Overview

In Chapter 2, we introduce and define the concept of a polynomial of a graph G. We discuss some graph polynomials, specifically the k-defect, dichromatic, Tutte (dichromate) and bad colouring polynomials. We point out the importance of the Tutte polynomial and the relationship between the Tutte polynomial and the other three polynomials. In Sections 2.2 to 2.5 we define these different polynomials and give examples of how to calculate these polynomials from the given definition. Finally, we use alternative methods of calculating these polynomials and verify their equivalence found in the literature.

In Chapter 3, we start by defining triangular numbers and number partitions. Then we describe and define complete graphs and closed sets of a complete graph. By setting up a one-to-one correspondence between complete graphs and triangular numbers in Section 3.3, we are able to state and prove one of the main results of this dissertation on the relationship between sizes of closed sets and triangular number partitions in Section 3.5.

In Chapter 4, we start by looking at known methods in the literature for calculating k-defect polynomials of a graph. In Section 4.3 we give one of the main results where we use the concept of triangular number partitions and closed sets of complete graphs from Chapter 3 to develop an algorithm for calculating a k-defect polynomial of a complete graph, using triangular number partitions of k. In Subsection 4.3.2 we use this algorithm to generate an expression for any k defect polynomial of a complete graph, another of the main results of this dissertation.

In Chapter 5, we investigate k-defect polynomials of a graph that are equal to zero. We use known methods and formulae to find some of the values of such k. In Section 5.3 we state and prove one of the main results of this chapter by applying the theory of triangular number partitions to identify the values of such k for complete graphs and proving that the k-defect polynomials of complete graphs are zero on certain integer intervals of k. In Subsection 5.3.1 we give an algorithm to identify for which values of k the k-defect polynomial of a complete graph is equal to zero. In Subsection 5.3.2 we use known summation properties of the triangular numbers to determine a lower bound on the number of k-defect polynomials of a complete graph that are equal to zero.

In Chapter 6, we conclude this dissertation by pointing out some further problems that have emerged from this study and may be interesting to look at and merit further investigation.

1.3 Basic definitions

In this section we start by defining a graph G, the main structure of investigation, and some of its related concepts and properties which are useful to this work. For basic definitions and properties of graphs we will follow the details and notation as given in [3].

Definition 1.3.1. A *graph* G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of vertices called edges.

Graphically, we represent the vertices as dots or points and the edges are the lines that join them. That is, if two vertices are in the same 2-element subset, then the dots representing them are joined by a line. We refer to two such vertices as the *end points* of the edge. We say that these vertices are *adjacent* and will call two adjacent vertices *neighbours* of each other. If a vertex v is the endpoint of an edge e we say that v is *incident* on e. If $E(G)$ is a multiset, a set containing more than one copy of a 2-element subset, we have multiple edges between vertices and we call such edges *parallel edges.*

The *order* of a graph G is the number of vertices of G, denoted $|V(G)|$, and the *size* of a graph G is the number of edges of G, denoted $|E(G)|$. The degree of a vertex v, denoted $deg(v)$, is the number of edges incident on it.

Definition 1.3.2. Two graphs, G and H, are *isomorphic* if there is a bijective function $\psi : V(G) \to V(H)$, such that two vertices, u and v, are adjacent in G, if and only if $\psi(u)$ and $\psi(v)$ are adjacent in H. We denote the isomorphism of the two graphs with $G \cong H$.

Definition 1.3.3. Let G be a graph with multiple edges between vertices. Let H be a graph, without multiple edges, or at least fewer multiple edges than G, and $\psi : V(G) \to V(H)$ a bijective function such that two vertices, u and v, are adjacent in G, if and only if $\psi(u)$ and $\psi(v)$ are adjacent in H, then we say G and H are isomorphic up to *parallel class.* That is $|V(G)| = |V(H)|$ and $|E(G)| \neq |E(H)|$, but ψ preserves the adjacency of the vertices in the mapping.

Definition 1.3.4. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$. A *spanning* subgraph is a subgraph H of G such that $V(H) = V(G)$.

We will regularly encounter certain classes of graphs, that is, graphs that share certain properties by definition. In what follows, we define three such classes, namely paths, cycles and complete graphs. After defining connectedness we define a fourth class, that of trees.

Definition 1.3.5. A *path*, P_n, is a graph of order n and size $n-1$, in which the vertices can be labeled $v_1, v_2, \ldots, v_n$ and the edges are the pairs $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Definition 1.3.6. A *cycle*, C_n, is a graph of order n and size n, $n \geq 3$, in which the vertices can be labeled $v_1, v_2, \ldots, v_n$ and the edges are the pairs $v_n v_1$ and $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Definition 1.3.7. A *complete graph*, K_n, is graph of order n in which every pair of vertices is adjacent.

Definition 1.3.8. A graph G of order n is *connected* if, for every pair of vertices u and w, there is a subgraph $H \subseteq G$ such that $H \cong P_i$, a path on i vertices, $i \leq n$, with $u = v_1$ and $w = v_i$. A graph that is not connected is *disconnected.* A connected subgraph H of a graph G is called a *component* of G if it is not a proper subgraph of another connected subgraph of G. We denote by $k(G)$ the number of components or "connected pieces" of a graph G.

Definition 1.3.9. A *tree*, T_n, is a connected graph of order n and size $n-1$ that contains no subgraph isomorphic to a cycle. Note that a path, P_n, is a tree.

Definition 1.3.10. A *loop* is an edge with the same vertex as both end points.

Note that if we add a loop to a vertex, then the degree of the vertex increases by two, since v is incident on e twice.

Definition 1.3.11. A *bridge* is an edge, the removal of which will disconnect a connected graph G.

Definition 1.3.12. The *rank* of a graph G is

$$r(G) = |V(G)| - k(G),$$

where $|V(G)|$ is the order and $k(G)$ the number of components of G.

Definition 1.3.13. The *nullity* or *cycle rank* of a graph G is

$$\mu(G) = |E(G)| - |V(G)| + k(G),$$

where $|E(G)|$ is the size, $|V(G)|$ the order and $k(G)$ the number of components of G.

One can think of the nullity of a graph as the minimum number of edges that must be deleted in order to break all the cycles of the graph, see [10].

Example 1.3.14. The diagram in Figure 1.3 is a connected graph G with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$. G has size 5 and order 4. Edge e_1 is a loop and edge e_2 is a bridge. The rank of G is $r(G) = 4 - 1 = 3$ and the nullity of G is $\mu(G) = 5 - 4 + 1 = 2$. Finally, the degrees of the vertices of G are $deg(v_1) = deg(v_2) = 3$ and $deg(v_3) = deg(v_4) = 2$.

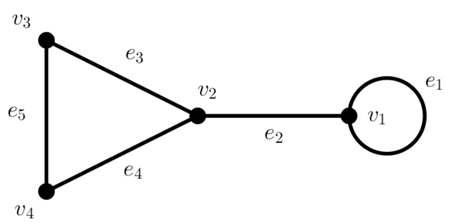

Figure 1.1: A graph G.

1.4 Graph operations

In this section we list some operations on graphs which allow us to create new graphs from old ones. The first two operations are on edges and will be used frequently in the subsequent calculations of different graph polynomials in this work, see [7].

1. *Deleting* an edge $e \in E(G)$ gives a graph $G \backslash e$ with the edge e removed.

2. *Contracting* an edge e means identifying the two endpoints of e followed by removal of e (see Figure 1.2). We denote the graph resulting from the contraction of edge e in G as G/e.

   Example 1.4.1. The diagrams in Figure 1.2 are graphs G, $G \backslash e$ and G/e.

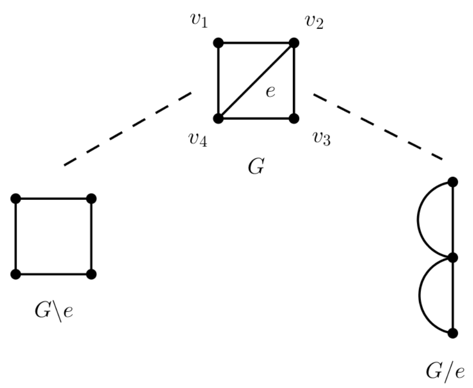

Figure 1.2: A graph G, $G \backslash e$ and G/e.

3. *Deleting* a vertex $v \in V(G)$ gives a graph $G \backslash v$ with v, and all edges incident with it, removed.

4. *The closure operation*: In a graph of order n, add an edge between two non-adjacent vertices u and v if $\deg u + \deg v \geq n$.

Example 1.4.2. The diagrams in Figure 1.3 illustrate the closure of a cycle on four vertices. Each of the vertices v_1, v_2, v_3 and v_4 are of degree 2 which means that we should add an edge between v_1 and v_3 and also between v_2 and v_4.

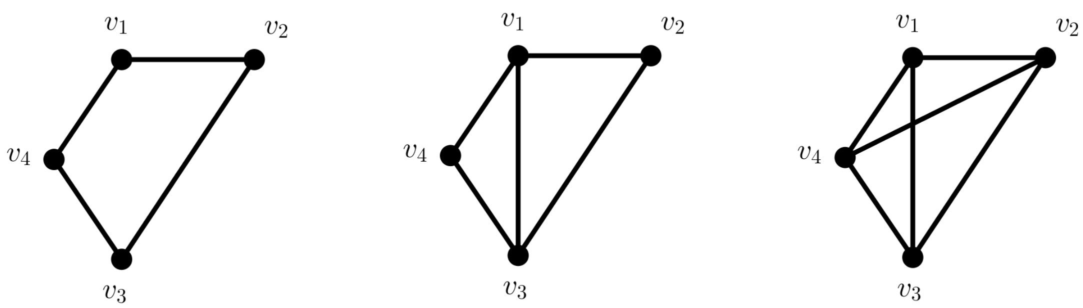

Figure 1.3: The closure operation.

5. The *join* of two graphs G and H, denoted by $G + H$, is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv | u \in V(G) \text{ and } v \in V(H)\}$.

Definition 1.4.3. A graph H is a *minor* of a graph G if $H \cong G$ or H can be obtained from G by a succession of edge contractions, edge deletions and vertex deletions. Each of the graphs $G \backslash e$ and G/e in Figure 1.2 are minors of the original graph G.

Definition 1.4.4. A set S of graphs is said to be *minor-closed* if for every graph G in S, every minor of G also belongs to S.

Definition 1.4.5. A *closed set* X of size k, is defined as the largest rank-r subgraph of $E(G)$ containing X, see [24].

Example 1.4.6. The diagrams in Figure 1.4 show the graph of K_6 with two of its subgraphs, G_1 and G_2, on the vertex set $\{v_1, v_2, v_5, v_6\}$, isomorphic to the cycle C_4 and the complete graph K_4, respectively. By Definition 1.3.12 the rank of G_1 is

$r(G_1) = 4 - 1 = 3$ and we say that G_1 is a subgraph of K_6 of rank 3. The edge set of this subgraph is not a closed set of edges in K_6, since G_1 is a subgraph of G_2, also of rank 3. Thus G_1 is not the largest rank-3 subgraph on the given vertex set in K_6.

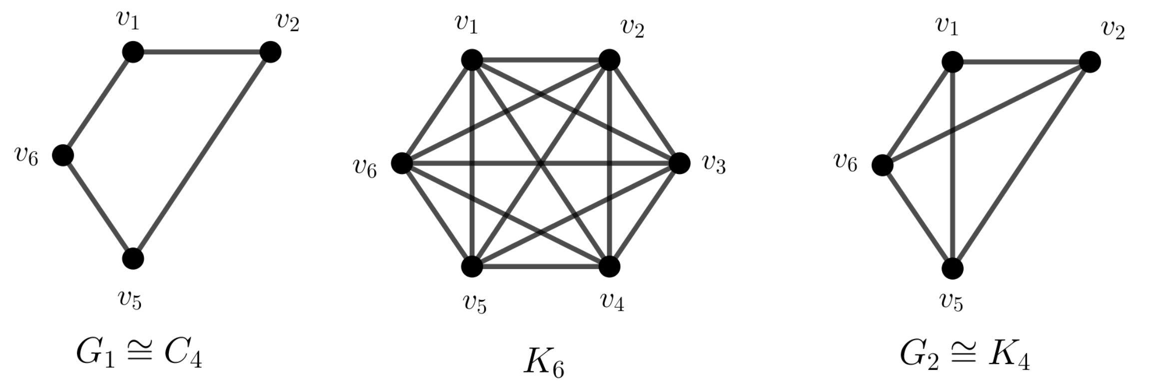

Figure 1.4: Closed and not closed sets in K_6.

Informally we can think of a closed set of edges in a graph G as a set of edges, $E_1 \subseteq E(G)$, with vertex set $V_1 \subseteq V(G)$; where the set of endpoints of the edges in E_1 is such that no further edges from $E(G)$ can be added to E_1 without adding more vertices from $V(G)$ to V_1.

1.5 Proper colouring and chromatic polynomials

A *proper colouring* of a graph G is a colouring of the vertices of G in such a way that no two adjacent vertices are coloured with the same colour. The minimum number of colours required for such a colouring is called the *chromatic number* of the graph, denoted $\chi(G)$. A colouring for which this is not true is called an *improper colouring.*

Definition 1.5.1. The number of distinct proper λ-colourings of G, $\lambda \in \mathbb{N}$, which we denote by $\chi(G; \lambda)$, is a polynomial in λ called the *chromatic polynomial.*

The smallest integer value k for which $\chi(G;\lambda) > 0$, is the chromatic number $\chi(G)$ and, by convention, $\chi(G;0) = 0$, see [3]. For more details on the proper colouring of a graph G and an introduction to chromatic polynomials, see [20], as well as the monograph, [6], by Dong, Koh and Teo for a detailed exposition on chromatic polynomials and chromaticity of graphs.

Example 1.5.2. We calculate $\chi(G;\lambda)$ where $G = K_3$. The diagram shown in Figure 1.5 is K_3 with labeled vertices. Using λ colours in a proper colouring of K_3, there are λ ways to colour vertex v_1, $\lambda-1$ ways to colour vertex v_2 and $\lambda-2$ ways to colour vertex v_3. Therefore, there are $\lambda(\lambda-1)(\lambda-2)$ ways to colour K_3 using λ colours. It follows that the chromatic polynomial of K_3 is $\lambda^3 - 3\lambda^2 + 2\lambda$. Any values of λ giving values such that $\chi(G;\lambda) \leq 0$ implies that there is no proper colouring of the graph G using that number of colours. Hence, we need at least $\chi(K_3) = 3$ colours for a proper colouring of K_3.

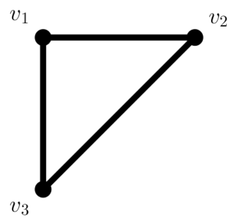

Figure 1.5: K_3 with labeled vertices

We note that we regard graphs as if their vertices were fixed in space, see [20]. In other words, a colouring with colours a, b, c of the labeled vertices in the diagram shown in Figure 1.5 given by the ordered pairs (v_1, a), (v_2, b), (v_3, c), is different to (v_1, c), (v_2, a), (v_3, b). The two colourings may be regarded as differing only by a cyclic permutation, and hence equivalent, but, since we regard the vertices as fixed in space, we will regard these colourings as different.

The following theorem, *of fundamental importance*, in Read's own words, leads to a method for computing the chromatic polynomial for any graph G. See [20] for the proof.

Theorem 1.5.3.

$$\chi(G;\lambda) = \chi(G+e;\lambda) + \chi(G/e;\lambda)$$

where $G+e$ is a graph obtained from G by adding an edge e between two non-adjacent vertices $u, v \in V(G)$ and G/e is obtained from G by identifying u and v.

By repeated application of Theorem 1.5.3 we reach a point where all the chromatic polynomials in the sum are the chromatic polynomials of complete graphs, see [20]. It is easy to calculate the chromatic polynomial of a complete graph. Simply follow the same procedure as in Example 1.5.2. It should be clear that for any K_n,

$$\chi(K_n;\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1).$$

Hence, it follows that $\chi(G;\lambda)$ is a sum of factorials and thus $\chi(G;\lambda)$ is a polynomial.

Note that Theorem 1.5.3 can be rewritten in the form

$$\chi(G+e;\lambda) = \chi(G;\lambda) - \chi(G/e;\lambda)$$

which is the form in which the theorem by R.M. Foster is written as mentioned in Section 2.1.

We summarise known formulas for chromatic polynomials of certain classes of graphs in the following proposition, see for example [6].

Proposition 1.5.4. *Let T_n be a tree, C_n a cycle and K_n a complete graph on n vertices, respectively. Then the chromatic polynomial,*

1. $\chi(T_n;\lambda) = \lambda(\lambda-1)^{n-1}$,

2. $\chi(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$,

3. $\chi(K_n;\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$.

Definition 1.5.5. The *falling factorial*

$$\lambda_{(n)} = \lambda(\lambda - 1) \dots (\lambda - n + 1).$$

We note that $\chi(K_n; \lambda) = \lambda_{(n)}$. Also, viewing K_2 as a complete graph or as a tree gives the same chromatic polynomial, $\lambda(\lambda - 1)$, and, similarly, the chromatic polynomial for C_3 and K_3 is $\lambda(\lambda-1)(\lambda-2)$ in both cases. This results from $K_2 \cong T_2$ and $K_3 \cong C_3$ as per Definition 1.3.2.

1.6 Improper colouring and k-defect polynomials

If we allow an improper colouring of a graph, we allow certain adjacent vertices to have the same colour.

Definition 1.6.1. In a colouring of a graph G, a *bad* edge is an edge $e \in E(G)$ with endpoints $u, v \in V(G)$ such that u and v are assigned the same colour. We denote the number of bad edges allowed in an improper colouring of a graph G with k, where k is a non-negative integer.

Definition 1.6.2. The *k-defect polynomial*, denoted $\phi_k(G, \lambda)$, of G is a polynomial that counts the number of λ colourings of the vertices of G with k bad edges.

Example 1.6.3. We calculate the k-defect polynomials of K_3.

Suppose we allow one bad edge in a colouring of K_3, that is, we let $k = 1$. In the diagram in Figure 1.6 the bad edges are represented with solid lines and the dotted lines represent edges incident on vertices coloured differently. There are three edges, so there are three ways in which we can choose one bad edge. This gives λ ways to colour the two adjacent vertices of the bad edge and $\lambda - 1$ ways to colour the remaining vertex. So the 1-defect polynomial for K_3 is $3\lambda(\lambda - 1) = 3\lambda^2 - 3\lambda$.

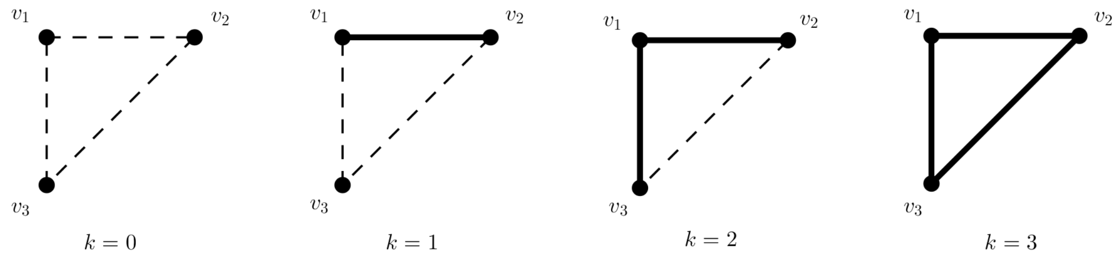

Figure 1.6: K_3 with 0, 1, 2 and 3 bad edges solid.

There is no way to choose two bad edges, since this would imply that all three vertices are the same colour. This would in turn imply that the third edge has to be bad too, because it joins two vertices of the same colour. Since we cannot choose two bad edges in K_3, the 2-defect polynomial must be 0.

The 3-defect polynomial for K_3 would then simply be λ. In general it should be clear that for any connected graph G of order n and size m, the m-defect polynomial must be λ, since, if all the edges are bad, then all the vertices are the same colour.

It is clear from Definition 1.6.2 that the chromatic polynomial is the 0-defect polynomial. Hence, the idea of k-defect polynomials is a generalisation of chromatic polynomials.

Chapter 2

The Tutte, Bad Colouring and k-defect Polynomials

2.1 Introduction

In this chapter we look at some graph polynomials, specifically the k-defect, dichromatic, Tutte (dichromate) and bad colouring polynomials of a graph G. We look at different ways in the literature of calculating these polynomials.

Following Ellis-Monaghan and Merino, in [8], we define a graph polynomial as follows:

Definition 2.1.1. A *graph polynomial* is an algebraic object associated with a graph that is usually invariant at least under graph isomorphism. As such it encodes information about the graph and enables algebraic methods for extracting this information.

One of the earliest graph polynomials, for example, is the edge-difference polynomial and was studied originally by Sylvester as well as Peterson in the late 1800s, see [8]. Another example is the chromatic polynomial described in Section 1.5. Due to its theoretical and applied importance, the chromatic polynomial has generated a large body of work, see [6], as well as the seminal work by Read, [20], which introduced many of the properties of the chromatic polynomial.

While doing PhD research, Tutte came across a theorem of R.M. Foster, that states that the chromatic polynomials satisfy the recursion

$$\chi(G, \lambda) = \chi(G \backslash e, \lambda) - \chi(G/e, \lambda)$$

and discovered a similar property for the Flow Polynomial,

$$F(G, \lambda) = F(G/e, \lambda) - F(G \backslash e, \lambda).$$

These observations, see [23], led to a paper in 1947, [22], in which a function f on graphs is discussed that would satisfy the rules

$$f(G) = f(G \backslash e) + f(G/e), \tag{2.1}$$

$$f(H \bigcup K) = f(H)f(K), \tag{2.2}$$

where e is any edge of the graph G. $H \bigcup K$ is a graph which is the union of two disjoint subgraphs H and K of G. Tutte's attempts to find a sum over subgraphs that satisfies both equations 2.1 and 2.2 succeeded with the dichromatic polynomial $Q(G; x, y)$.

The Tutte polynomial (or dichromate), $T(G; x, y)$, is a simplification of the dichromatic polynomial, as shown in [23]. It is an important two variable polynomial that encodes many characteristics of a graph. For an extensive treatment of the Tutte polynomial and its applications, see [7].

As an example of the power of the Tutte polynomial we give the following theorem from [7], illustrating how to extract information about the graph, by evaluating $T(G; x, y)$ at certain values of x and y.

Theorem 2.1.2. *If $G = (V, E)$ is a connected graph then:*

1. *$T(G, 1, 1) = \tau(G)$, the number of spanning trees of G.*

2. *$T(G; 2, 1)$ equals the number of spanning forests of G.*

3. $T(G,1,2)$ *equals the number of spanning connected subgraphs of* G*.*

4. $T(G;2,2) = 2^{|E|}$.

In [7], we also find the universality property of the Tutte polynomial as one of its most powerful aspects. This property says that any graph invariant that is multiplicative on disjoint unions and one-point joins of graphs and that has a deletion/contraction reduction must be an evaluation of the Tutte polynomial.

Recall Definition 1.6.2 and the description of the k-defect polynomial in Section 1.6. The *bad colouring polynomial* of a graph G, $B(G;\lambda,S)$, or co-boundary polynomial in matroids, was originally defined and studied by Crapo as a generating function in S, see [4, 19], where the polynomial coefficients of the S^k are the k-defect polynomials of a graph G. Since the bad colouring polynomial has a deletion/contraction reduction, see Proposition 2.5.3, it should not surprise us to find, given the universality property, that we can evaluate the bad colouring polynomial, $B(G;\lambda,S)$, through the Tutte polynomial and vice versa.

Hence, since we can generate $B(G;\lambda,S)$ from the k-defect polynomials and evaluate $B(G;\lambda,S)$ to find $T(G;x,y)$, it seems that it should be possible to identify certain properties of the Tutte polynomial from the k-defect polynomials of a graph G.

Given this close relationship between the dichromatic, Tutte (dichromate), bad colouring and k-defect polynomials of a graph G, we will define these polynomials in this chapter and use an example for each to illustrate their calculation by definition and other means.

For each of the polynomials discussed in this chapter, we will use the graph $K_4\backslash e$, shown in the diagram in Figure 2.1, the complete graph on four vertices with one edge deleted, as an example to calculate the respective polynomials.

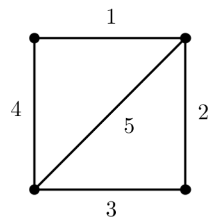

Figure 2.1: The graph $G = K_4 \backslash e$.

2.2 The k-defect Polynomial

Recall from Section 1.6 that an improper colouring of a graph is a colouring where we allow certain adjacent vertices to have the same colour and we call an edge "bad" if it joins two vertices of the same colour.

For convenience we restate Definition 1.6.2 here: The *k-defect polynomial*, denoted $\phi_k(G, \lambda)$, of a graph G is a polynomial that counts the number of λ colourings of the vertices of a graph with k bad edges.

In the rest of this section we calculate, as an example, the k-defect polynomials of the graph $G = K_4 \backslash e$ from Figure 2.1, using Definition 1.6.2.

Example 2.2.1. The k-defect polynomials for $G = K_4 \backslash e$.

In the diagram of Figure 2.2 the "good" edges are represented as dotted edges and the "bad" edges are the solid edges. Each row in the diagram shows the different choices for each $0 \leq k \leq 5$ edges to be bad. Hence, the vertices incident on those edges have the same colour. Where two bad edges are disjoint, their vertices are assigned different colours. We also have the λ number of colorings for the vertices.

So, for example, in row 2, we choose one bad edge. The first four choices all give the same number of λ colourings. That is, λ colours are available to colour the two vertices on the bad edge, $\lambda - 1$ colours are available for the third vertex and $\lambda - 2$ colours for the last. The fifth choice, however, gives a different colouring: λ colours are available for the two vertices on the bad edge, but, since the remaining two vertices are not adjacent, there are $\lambda - 1$ colours available to colour each of them.

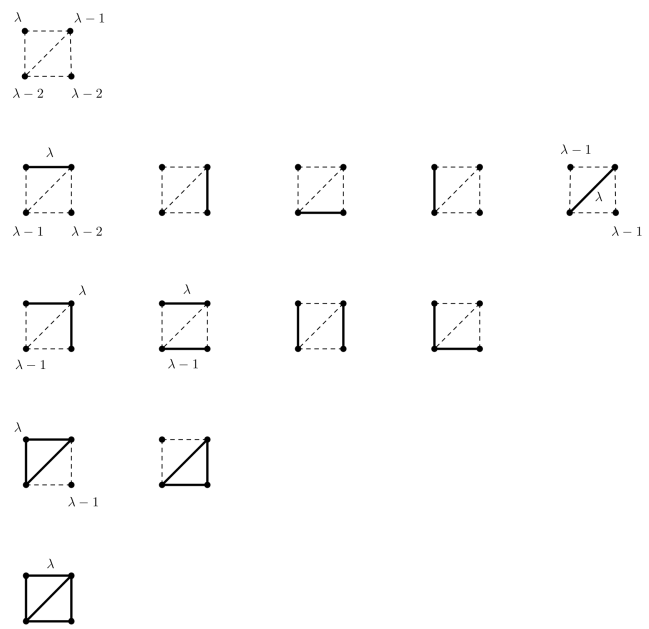

Figure 2.2: $G = K_4 \backslash e$ with all possible choices of bad edges.

Using the number of choices for bad edges as shown in Figure 2.2, we see that the k-defect polynomials of G are:

$$\begin{array}{rcl}
\phi_0(G,\lambda) & = & \lambda(\lambda-1)(\lambda-2)^2 & = & -4\lambda + 8\lambda^2 - 5\lambda^3 + \lambda^4 \\
\phi_1(G,\lambda) & = & 4\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)^2 & = & 9\lambda - 14\lambda^2 + 5\lambda^3 \\
\phi_2(G,\lambda) & = & 4\lambda(\lambda-1) & = & -4\lambda + 4\lambda^2 \\
\phi_3(G,\lambda) & = & 2\lambda(\lambda-1) & = & -2\lambda + 2\lambda^2 \\
\phi_4(G,\lambda) & = & 0 \\
\phi_5(G,\lambda) & = & \lambda
\end{array}$$

The 4-defect polynomial is equal to zero, since there is no way to choose four bad edges. Choosing four bad edges would mean four vertices the same colour and, hence, the fifth edge would also have to be bad.

In Section 2.5 the relationship between the k-defect polynomial and the other polynomials discussed in this chapter will be made explicit.

2.3 The Dichromatic Polynomial

In this section we define the dichromatic polynomial of a graph. As an example, we calculate the dichromatic polynomial of the graph $G = K_4 \backslash e$, the diagram in Figure 2.1.

Definition 2.3.1. The *dichromatic polynomial* of a graph G is

$$Q(G; x, y) = \sum_{S} x^{k(G:S)} y^{\mu(G:S)}$$

where S runs through the subsets of $E(G)$, $G : S$ denotes the spanning subgraph of G whose edges are the members of S and $k(G : S)$ and $\mu(G : S)$ denote the number of components and nullity of $G : S$ respectively, see Definitions 1.3.8 and 1.3.13.

Example 2.3.2. We calculate the dichromatic polynomial of the graph $G = K_4 \backslash e$ from Definition 2.3.1. For ease of reference the subgraphs, $G : S$, are shown in Figure 2.3 in order to confirm the number of components and nullity. The diagrams in Figure 2.3, show the number of vertices, edges and components of $G : S$.

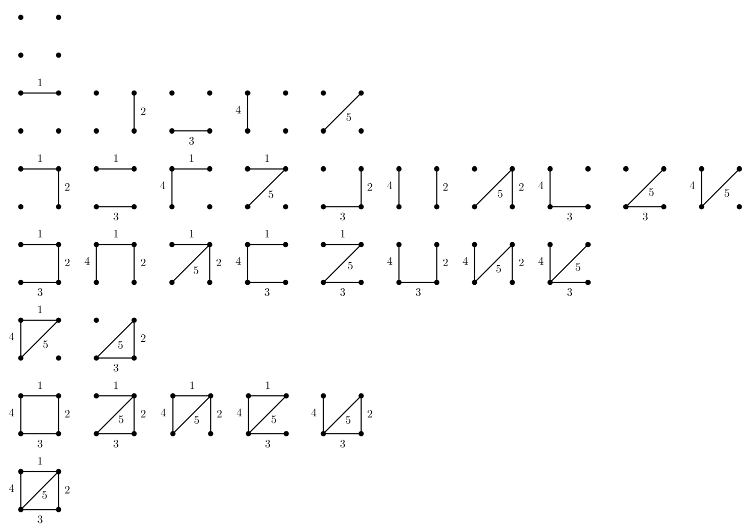

Figure 2.3: $G : S$ where $G = K_4 \backslash e$.

The subsets of $E(G)$ are given in Table 2.1, together with the corresponding values of $x^{k(G:S)}y^{\mu(G:S)}$.

Subsets of $E(G)$	$x^{k(G:S)}y^{\mu(G:S)}$
{}	x^4y^0
{1}, {2}, {3}, {4}, {5}	$5x^3y^0$
{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}	$10x^2y^0$
{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {2, 3, 4}, {2, 4, 5}, {3, 4, 5}	$8xy^0$
{1, 4, 5}, {2, 3, 5}	$2x^2y$
{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {2, 3, 4, 5}	$5xy$
{1, 2, 3, 4, 5}	xy^2

Table 2.1: The subsets of G and corresponding $x^{k(G:S)}y^{\mu(G:S)}$.

Therefore, the dichromatic polynomial for the given graph is $x^4 + 5x^3 + 10x^2 + 8x + 2x^2y + 5xy + xy^2$.

2.4 The Tutte Polynomial

In this section we define the Tutte polynomial of a graph G. We then calculate the Tutte polynomial of the graph $G = K_4\backslash e$, given in the diagram of Figure 2.1, using the definition. We verify the simplification of the dichromatic polynomial to the Tutte polynomial with an example. Finally, we demonstrate the computation of the Tutte polynomial using the most commonly used method of deletion and contraction.

Definition 2.4.1. The *Tutte polynomial* (dichromate) of a graph G is

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{\mu(A)}$$

where $r(E)$ denotes the rank of G and $r(A)$ the rank of the subgraph induced by the edge set A.

Example 2.4.2. We calculate the Tutte polynomial for the graph $G = K_4\backslash e$. Table 2.2 shows the subsets of the edge set and the variables making up the polynomial as per the definition.

Subsets of $E(G)$	$(x-1)^{r(E)-r(A)}(y-1)^{\mu(A)}$
{}	$(x-1)^3(y-1)^0$
{1}, {2}, {3}, {4}, {5}	$5(x-1)^2(y-1)^0$
{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}	$10(x-1)(y-1)^0$
{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {2, 3, 4}, {2, 4, 5}, {3, 4, 5}	$8(x-1)^0(y-1)^0$
{1, 4, 5}, {2, 3, 5}	$2(x-1)(y-1)$
{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {2, 3, 4, 5}	$5(x-1)^0(y-1)$
{1, 2, 3, 4, 5}	$(y-1)^2$

Table 2.2: Subsets of G and corresponding values of $(x-1)^{r(E)-r(A)}(y-1)^{\mu(A)}$.

Therefore, the Tutte polynomial for the given graph is $T(G; x, y) = x + 2x^2 + x^3 + y + 2xy + y^2$.

The following proposition is given by Tutte, see [23], as the simplification of the dichromatic polynomial to the dichromate, nowadays known as the Tutte polynomial.

Proposition 2.4.3. *The Tutte polynomial of a graph G, $T(G;x,y)$, is a two variable polynomial given by*

$$T(G;x,y) = (x-1)^{-k(G)} Q(G;x-1,y-1)$$

where $k(G)$ is the number of components of a graph G and $Q(G;x,y)$ is the dichromatic polynomial of G.

Example 2.4.4. Let $G = K_4 \backslash e$ be the graph given in the diagram of Figure 2.1. Using the dichromatic polynomial calculated in Example 2.3.2 and the Tutte polynomial calculated in Example 2.4.2, we verify the relationship given in Proposition 2.4.3.

$$\begin{aligned}
(x-1)^{-1} Q(G;x-1,y-1) &= (x-1)^{-1}[(x-1)^4 + 5(x-1)^3 \\
&\quad + 10(x-1)^2 + 8(x-1) + 2(x-1)^2(y-1) \\
&\quad + 5(x-1)(y-1) + (x-1)(y-1)^2] \\
&= x + 2x^2 + x^3 + y + 2xy + y^2 = T(G;x,y)
\end{aligned}$$

In the literature we find a number of other ways of calculating the Tutte polynomial, see [7, 11, 16]. Of interest to this work will be the method of deletion and contraction summarised in the following proposition.

Proposition 2.4.5. *The* Tutte Polynomial *of a graph G is a two variable polynomial $T(G;x,y)$. If G is a graph and e is an edge, then*

$$T(G;x,y) = \begin{cases} yT(G\backslash e;x,y) & \text{if } e \text{ is a loop} \\ xT(G/e;x,y) & \text{if } e \text{ is a bridge} \\ T(G\backslash e;x,y) + T(G/e;x,y) & \text{if } e \text{ is neither} \\ x^i y^j & \text{if } G \text{ consists of } i \text{ bridges and } j \text{ loops.} \end{cases}$$

Example 2.4.6. In Figure 2.4 we illustrate the method of calculating the Tutte polynomial of $G = K_4 \backslash e$ using deletion and contraction. In the diagram we start with G with a dotted edge. We delete the dotted edge to get the graph to its left and contract the dotted edge to get the graph to its right. We repeat this process until we get bridges and loops only. We now apply Proposition 2.4.5 to collect the variables.

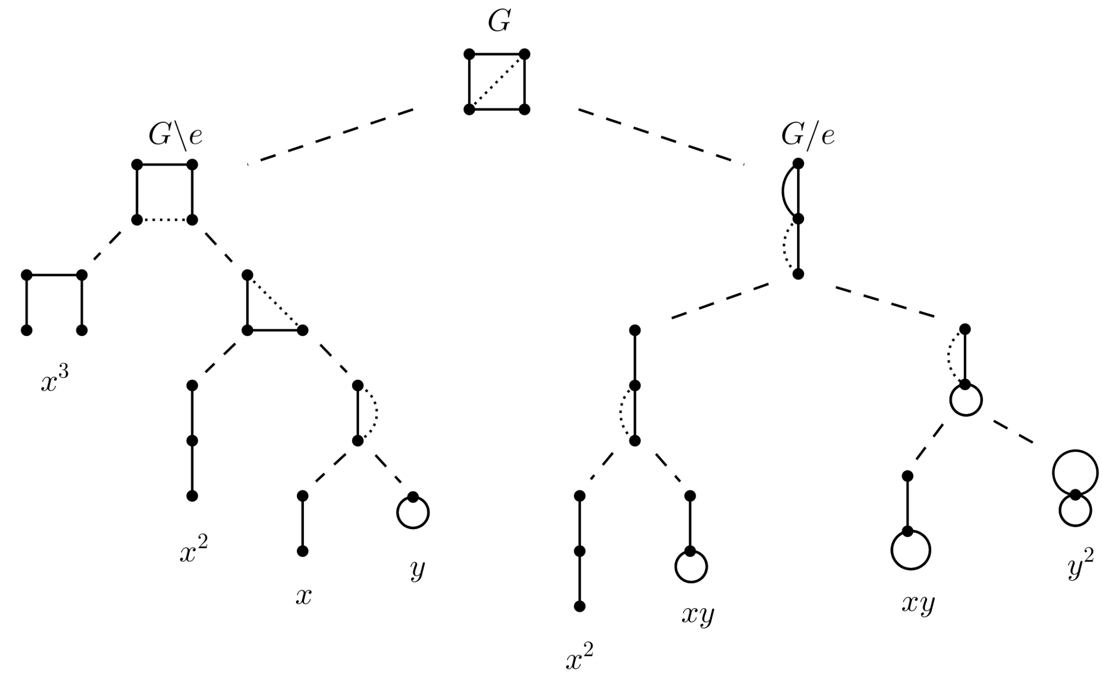

Figure 2.4: Deletion and contraction of G.

Collecting the variables at the end of the process we find $T(G; x, y) = x + 2x^2 + x^3 + y + 2xy + y^2$ as we did from the definition.

2.5 The Bad Colouring Polynomial

In this section we define the bad colouring polynomial, $B(G;\lambda,S)$, of a graph G. We calculate $B(G;\lambda,S)$ of $G = K_4\backslash e$ as an example and give the equivalence of the bad colouring and Tutte polynomials. Finally, we verify that the bad colouring polynomial is a generating function in S, where the coefficients of the S^k are the k-defect polynomials in λ of a graph G.

Definition 2.5.1. The *bad colouring polynomial* of a graph G is a polynomial in two independent variables and is defined as

$$B(G;\lambda,S) = \lambda^{k(G)} \sum_{A\subseteq E} (S-1)^{|A|}\lambda^{r(E)-r(A)}$$

where $|A|$ is the size of the subgraph induced by the subset A of edges.

Example 2.5.2. In Table 2.3 we have the subsets of $E(G)$. The subsets are the same as shown in Figure 2.3.

Subsets of $G(E)$	$(S-1)^{\lvert A\rvert}\lambda^{r(E)-r(A)}$
{}	$(S-1)^0\lambda^3$
{1}, {2}, {3}, {4}, {5}	$5(S-1)^1\lambda^2$
{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}	$10(S-1)^2\lambda^1$
{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {2, 3, 4}, {2, 4, 5}, {3, 4, 5}	$8(S-1)^3\lambda^0$
{1, 4, 5}, {2, 3, 5}	$2(S-1)^3\lambda^1$
{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {2, 3, 4, 5}	$5(S-1)^4\lambda^0$
{1, 2, 3, 4, 5}	$(S-1)^5\lambda^0$

Table 2.3: Subsets of $G = K_4\backslash e$ and the corresponding values of $(S-1)^{|A|}\lambda^{r(E)-r(A)}$.

Since G is connected, $k(G) = 1$. Hence, expanding and factoring out powers of S, we have

$$\begin{aligned} B(G;\lambda,S) &= (-4\lambda + 8\lambda^2 - 5\lambda^3 + \lambda^4) + (9\lambda - 14\lambda^2 + 5\lambda^3)S \\ &\quad + (-4\lambda + 4\lambda^2)S^2 + (-2\lambda + 2\lambda^2)S^3 + \lambda S^5. \end{aligned}$$

As shown in [7], we can also calculate $\lambda^{-k(G)}B(G;\lambda,S) = \bar{B}(G;\lambda,S)$, where $k(G)$ is the number of components of G, by the method of deletion and contraction as follows.

Proposition 2.5.3. *For a graph G and an edge e the* bad colouring *polynomial of G is given by*

$$B(G;\lambda,S) = \lambda^{k(G)}\bar{B}(G;\lambda,S),$$

where

$$\bar{B}(G;\lambda,S) = \begin{cases} S\bar{B}(G\backslash e;\lambda,S) & \text{if } e \text{ is a loop} \\ (S+\lambda-1)\bar{B}(G/e;\lambda,S) & \text{if } e \text{ is a bridge} \\ \bar{B}(G\backslash e;\lambda,S) + (S-1)\bar{B}(G/e;\lambda,S) & \text{if } e \text{ is neither.} \end{cases}$$

The Tutte polynomial and bad colouring polynomial of a graph are equivalent; we refer the reader to [18]. The following relationship between $T(G;x,y)$ and $B(G;\lambda,S)$ shows this equivalence.

$$B(G;\lambda,S) = \lambda(S-1)^r T(G;\frac{S+\lambda-1}{S-1},S),$$

where r is the rank of G. If G is connected $r = n-1$, where n is the order of G.

Vice versa, given the bad colouring polynomial of G, we can calculate the Tutte polynomial,

$$T(G;x,y) = \frac{1}{(x-1)(y-1)(y-1)^r}B(G;(x-1)(y-1),y).$$

As mentioned in Section 2.1, the *bad colouring polynomial* of a graph G was originally defined and studied by Crapo as a generating function in S, see [4, 19].

Proposition 2.5.4. *The* bad colouring polynomial*, $B(G;\lambda,S)$, of a graph G is given by*

$$B(G;\lambda,S) = \sum S^k\phi_k(G;\lambda),$$

where $\phi_k(G;\lambda)$ is the k-defect polynomial of the graph G.

Recall that the k-defect polynomials of $G = K_4 \backslash e$ calculated in Section 2.2 are

$$\begin{aligned}
\phi_0(G,\lambda) &= \lambda(\lambda-1)(\lambda-2)^2 &&= -4\lambda + 8\lambda^2 - 5\lambda^3 + \lambda^4 \\
\phi_1(G,\lambda) &= 4\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)^2 &&= 9\lambda - 14\lambda^2 + 5\lambda^3 \\
\phi_2(G,\lambda) &= 4\lambda(\lambda-1) &&= -4\lambda + 4\lambda^2 \\
\phi_3(G,\lambda) &= 2\lambda(\lambda-1) &&= -2\lambda + 2\lambda^2 \\
\phi_4(G,\lambda) &= 0 && \\
\phi_5(G,\lambda) &= \lambda. &&
\end{aligned}$$

We confirm that

$$\begin{aligned}
\sum S^k \phi_k(G;\lambda) &= (-4\lambda + 8\lambda^2 - 5\lambda^3 + \lambda^4)S^0 \\
&\quad +(9\lambda - 14\lambda^2 + 5\lambda^3)S^1 + (-4\lambda + 4\lambda^2)S^2 \\
&\quad +(-2\lambda + 2\lambda^2)S^3 + \lambda S^5 = B(G;\lambda,S)
\end{aligned}$$

as calculated from Table 2.3.

2.6 Conclusion

In this chapter we defined the k-defect, dichromatic, Tutte and bad colouring polynomials. We calculated, from the definitions and various other methods found in the literature, these polynomials for the graph $G = K_4 \backslash e$.

Using the polynomials from the examples, we verified the equivalence of the Tutte and bad colouring polynomials. We also verified that the bad colouring polynomial is a generating function for the k-defect polynomials of a graph G.

Given the importance of the Tutte polynomial in identifying many characteristics of a graph G, and the equivalences explored in this chapter, we motivate the further study of the k-defect polynomial in the rest of this work as a way of possibly casting new light on the properties of the Tutte polynomial.

Chapter 3

Integer partitions, Triangular numbers and Closed sets of complete graphs

3.1 Introduction

In this chapter we start by defining triangular numbers and number partitions. Then we describe and define complete graphs and closed sets of a complete graph. By setting up a one-to-one correspondence between complete graphs and triangular numbers, we are able to state and prove one of the main results of this dissertation on the relationship between sizes of closed sets and triangular number partitions. In Chapter 4 we will use this relationship to generate the k-defect polynomials of complete graphs.

3.2 Triangular number partitions

In this section, we give some well known definitions and theorems involving triangular number partitions which are relevant to this work. For further details, we refer the reader to [2, 14].

An *integer partition* is a way of splitting a number into integer parts. For example, we can write the number 4 as 1+1+1+1 or 2+1+1 or 2+2 or 3+1 or 4. The *partition* of a number into integer parts should not be confused with the *representations* of a number by integer parts, see [14]. Note that $2+1+1$, $1+2+1$ and $1+1+2$ are all representations of 4 stemming from the same partition $2+1+1$.

Definition 3.2.1. The nth *triangular number*, Δ_n, is $\frac{n(n-1)}{2}$ or $\binom{n}{2}$.

Of special interest in this work are integer partitions involving the triangular numbers. We denote a partition of an integer k into triangular numbers by $\pi_\Delta(k)$ and we will say that $\Delta_i \in \pi_\Delta(k)$ if we use a triangular number, Δ_i, in the partition. By definition, $\binom{n}{r} = 0$ for $n < r$. Hence, we note from Definition 3.2.1 that 0 is the first triangular number. This means that our definition differs slightly from the usual definition given in [2], that is, $\Delta_n = \frac{n(n+1)}{2}$, which implies that the first triangular number is 1. The need for this change in the definition will become clear once we set up the correspondence between triangular numbers and the sizes of complete graphs in Theorem 3.3.2. Thus, in this dissertation, the triangular numbers $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \ldots\}$ are $\{0, 1, 3, 6, \ldots\}$, respectively. It is clear from the definition that

$$\Delta_n + n = \binom{n}{2} + n = \frac{n(n-1)+2n}{2} = \frac{(n+1)n}{2} = \Delta_{n+1}.$$

Pictorially the triangular numbers can be viewed as the number of dots in triangles of increasing size. In Figure 3.1 we see the triangular numbers $1, 3, 6$ and 10 as dots in triangles. $\Delta_1 = 0$ is just a blank space.

The following theorem, which is stated without proof, is well known in the literature, sometimes as the Gauss "Eureka Theorem". We refer the reader to [1] and [5] for further discussion and proof.

Theorem 3.2.2 (Gauss Eureka)**.** *Every integer can be written as the sum of three triangular numbers.*

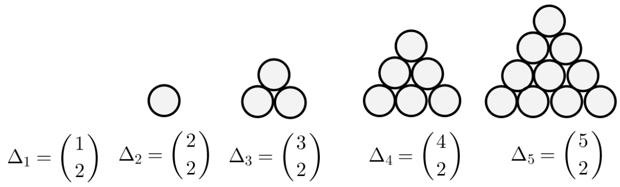

Figure 3.1: Triangular numbers. $\Delta_1 = 0$ is just a blank space.

Note that integers can also be written as the sum of more than three triangular numbers. For example, we have not only $8 = 6 + 1 + 1$, but also $8 = 3 + 3 + 1 + 1$ or $8 = 3 + 1 + 1 + 1 + 1 + 1$ and so on.

3.3 Complete Graphs

In this section we state a few known facts about complete graphs. Then we set up a one-to-one correspondence between complete graphs and triangular numbers.

Recall from Definition 1.3.7 that a *complete graph* is a graph in which every two distinct vertices are adjacent. We denote the complete graph of order n as K_n. Recall also from Chapter 1, Section 1.3, that the *order* of a graph is the number of vertices and the *size* of a graph is the number of edges. The following proposition summarises some of the properties of complete graphs well known in the literature, see for example in [3].

Proposition 3.3.1. *Let K_n be the complete graph of order n and size m. Then*

1. *the size of K_n is $\binom{n}{2}$,*

2. *K_n is $(n-1)$-regular, that is, every vertex has degree $n-1$,*

3. *the chromatic polynomial of K_n is $\lambda(\lambda-1)\ldots(\lambda-n+1)$,*

4. *the chromatic number of K_n is n.*

The graphs in the diagrams in Figure 3.2 are examples of K_1, K_2, …, K_6, with sizes $0, 1, 3, 6, 10$ and 15 respectively.

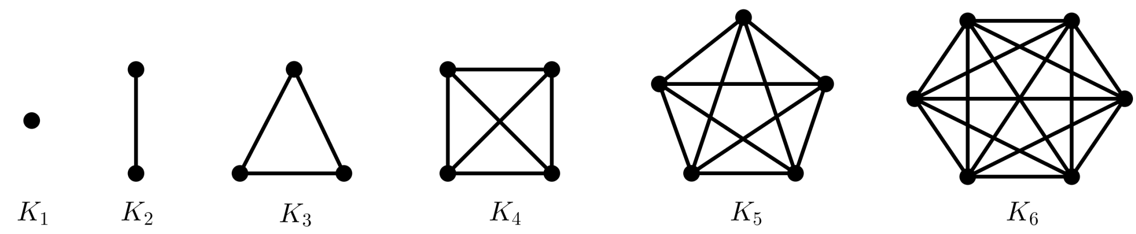

Figure 3.2: Complete graphs of order 1 to 6

The following theorem, one of the main results of this chapter, provides a link between graph theory and number theory that we will use extensively in the rest of this dissertation.

Theorem 3.3.2. *There is a one-to-one correspondence between triangular numbers and complete graphs.*

Proof. By Proposition 3.3.1 and Definition 3.2.1 we have, for every complete graph K_n, the size of K_n,

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2} = \Delta_n.$$

Thus, for every non-negative integer n, the complete graph K_n is mapped to exactly one triangular number by its size. By applying the quadratic formula to solve for n where $\frac{n(n-1)}{2} = \Delta_n$, we have, for every triangular number Δ_n,

$$\frac{1 + \sqrt{1 + 8\Delta_n}}{2} = n = |V(K_n)|.$$

Thus every triangular number Δ_n is mapped back to exactly one complete graph of order n for every non-negative integer n. □

3.4 Closed sets of size k of complete graphs

In this section we give some definitions and theorems on closed sets which are relevant to this work. We refer the reader to [25], one of the original papers describing sets of bad edges as *bonds* and, more recently, to [3] and [17].

Definition 3.4.1. A *vertex induced subgraph* G_i of a graph G has vertex set $V(G_i) \subseteq V(G)$. For each pair of vertices $u, v \in V(G_i)$, if u and v are adjacent in G, then they are adjacent in G_i.

Definition 3.4.2. An *edge induced subgraph* G_i of a graph G has edge set $E(G_i) \subseteq E(G)$. A vertex $v \in V(G_i)$, if v is incident with at least one edge in G_i.

In this work we denote induced subgraphs simply as G_i. However, if confusion is possible, we will refer to the subgraph induced by $S \subseteq V(G)$ as $G[S]$ or the subgraph induced by $X \subseteq E(G)$ as $G[X]$.

For example, in the diagram in Figure 3.3, G_1 is the edge induced subgraph of G on edge set $X = \{e_1, e_2, e_3, e_4\}$, while G_2 is the vertex induced subgraph of G with vertex set $S = \{1, 2, 3, 4\}$.

Lemma 3.4.3. *Let G be a complete graph of order n. Then every induced subgraph on m vertices of G is a complete graph K_m.*

Proof. Let G_i be an induced subgraph on m vertices of G. By Definition 3.4.1, each pair of vertices $u, v \in V(G_i)$ are adjacent in G_i if they are adjacent in G. But G is a complete graph. Hence every pair of vertices in G are adjacent. Therefore every pair of vertices in G_i are adjacent and G_i is a complete graph of order m. □

We state the following well-known proposition without proof, see [3].

Proposition 3.4.4. *Let G be a* connected *graph. Then there is a path between every pair of vertices u and v of G.*

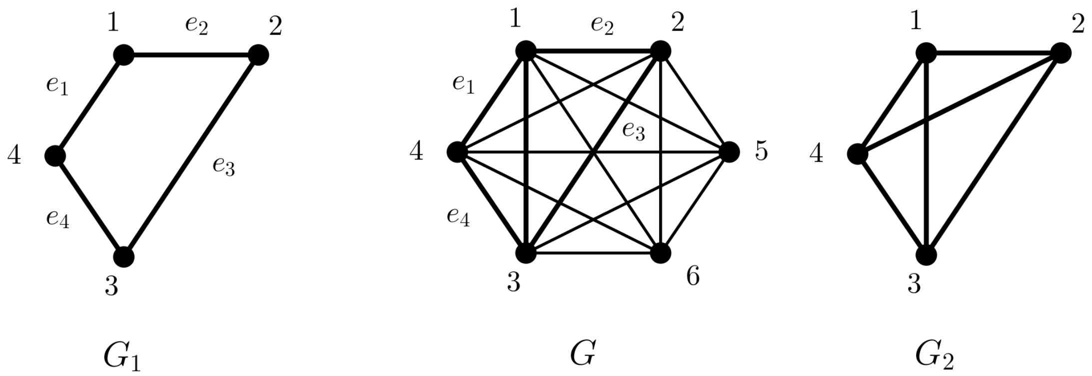

Figure 3.3: Edge and vertex induced subgraphs

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Recall from Chapter 1 Section 1.6 that a bad edge joins two vertices of the same colour in an improper colouring of a graph. Also from Sections 1.3 and 1.4 that a closed set X of size k, is the largest rank-r subgraph of $E(G)$ containing X, where the rank $r(X)$ of X is $|V(G[X])| - k(G[X])$.

Proposition 3.4.5. *Let X_k be the set of k bad edges in an improper colouring of a graph G. If G_i is a connected edge induced subgraph of G on X_k, then all the vertices of G_i are the same colour.*

Proof. Label a vertex of G_i as v. Since X_k is the edge set of G_i, all the edges incident with v are bad edges in G, and it follows that every vertex adjacent to v is the same colour as v in the bad colouring of G. Similarly, every vertex adjacent to a neighbour of v has the same colour as v. Since G_i is connected there is a path between every two vertices. Thus all the vertices on every path have the same colour, that is, the same colour as v. □

Proposition 3.4.6. *Let X_k be the set of k bad edges in an improper colouring of a graph G. If S is the vertex set of G_i, the edge induced subgraph of G on X_k, then all the edges of $G[S]$ are bad.*

Proof. By Proposition 3.4.5, if G_i is connected then all the vertices of S are the same colour. But, by Definition 3.4.1, this means that any other edges of $G[S]$ must also be bad. By extension, if G_i is not connected then the vertices of each of its components with vertex sets $S_1, S_2, \ldots, S_i$ must be the same colour, which implies that all the edges of $G[S_1], G[S_1], \ldots, G[S_i]$ must be bad. □

Proposition 3.4.7. *Let X_k be the set of k bad edges in an improper colouring of a graph G. Then X_k is a closed subgraph or the disjoint union of closed subgraphs of G.*

Proof. Let $X_k = \bigcup X_i$, be the disjoint union of i sets of bad edges. Note that $(i-1)$ of the X_i may be empty if $G[X_k]$ is connected. If any of the X_i is not closed in G, then, by the definition of a closed subgraph, there is some X_1, say, such that $X_1 \subset E(G_1)$, where G_1 is a subgraph of G with same vertex set and rank as $G[X_1]$, but $|X_1| < |E(G_1)|$. But, by Proposition 3.4.5 all the vertices of $G[X_1]$, and hence G_1, are the same colour. Therefore, by Proposition 3.4.6 all the edges of G_1 are bad. But this would imply an improper colouring of G with more than k edges, contradicting the assumption that $|X_k| = k$. □

Corollary 3.4.8. *Let X_k be the set of k bad edges in an improper colouring of a graph G. Then the vertex set $S = V(G[X_k])$ is a set of vertices such that $|E(G[S])| = k$ or $|E(G[S_1])| + |E(G[S_2])| + \ldots |E(G[S_i])| = k$, where $S = \bigcup S_i$.*

We now state and prove the main results of this section.

Proposition 3.4.9. *All closed sets of K_n are complete graphs or disjoint unions of complete graphs.*

Proof. Let X_k be a closed set in K_n. By definition of a closed set $G[X_k] \not\subset G_i$, where G_i is an induced subgraph of G with the same vertex set as $G[X_k]$. But, by Lemma 3.4.3, every induced subgraph on m vertices of $G = K_n$ is a complete graph K_m. Therefore, $G[X_k]$ must be a complete graph, otherwise $r(G_i) > r(G[X_k])$. The argument is the same if $G[X_k]$ is not connected and hence a disjoint union of complete graphs. $\square$

Theorem 3.4.10. *Let X_k be the set of k bad edges in an improper colouring of a complete graph K_n. Then X_k partitions K_n into a disjoint union of i complete subgraphs, K_i, such that $|\bigcup E(K_i)| = k$.*

Proof. By Proposition 3.4.7, X_k is closed or the disjoint union of closed sets. Therefore, by Proposition 3.4.9, $G[X_k]$ is a complete graph or the disjoint union of complete graphs. Let $S = V(G[X_k])$, then, by Corollary 3.4.8, S is a set of vertices such that $|E(G[S])| = k$ or $|E(G[S_1])| + |E(G[S_2])| + \ldots |E(G[S_i])| = k$, where $S = \bigcup S_i$. Therefore, if $S = V(G)$, G is partitioned into a set of disjoint complete graphs. If $S \neq V(G)$, the remaining vertices in the partition of G cannot contribute any edges to X_k and hence must all be isomorphic to K_1. $\square$

3.5 The relationship between triangular number partitions and closed sets of complete graphs

In this section we will state and prove one of the main results of this dissertation, namely the correspondence between the triangular number partitions of an integer and a disjoint union of complete graphs. In conjunction with the results from Section 3.4, we will show a correspondence between closed sets of k bad edges in complete graphs and triangular number partitions of k. This will place us in a position to put forward a new algorithm for calculating the k-defect polynomials of a complete graph in Chapter 4.

Proposition 3.5.1. *There is a one-to-one correspondence between every partition of an integer k into triangular numbers and a disjoint union of complete graphs such that the sum of the edges of the graphs is k.*

Proof. By Theorem 3.3.2, there is a one-to-one correspondence between complete graphs and triangular numbers. Thus, by Proposition 3.4.9 and Theorem 3.4.10, a partition of an integer k into triangular numbers gives a corresponding disjoint union of complete graphs such that the sum of the edges of the graphs is k. □

In the following three propositions we explore some of the implications of Proposition 3.5.1 and develop formulae that will be used in the subsequent chapters of this work. Of particular importance will be Proposition 3.5.3.

Recall that $\pi_\Delta(k)$ is a partition of an integer k into triangular numbers and that $\Delta_i \in \pi_\Delta(k)$ means that the triangular number Δ_i is used in the partition.

Proposition 3.5.2. *Let $\pi_\Delta(k)$ be a partition of an integer k into triangular numbers. Then*

$$\sum_{\Delta_i \in \pi_\Delta(k)} \frac{1+\sqrt{1+8\Delta_i}}{2}$$

corresponds to the order of a complete graph or disjoint union of complete graphs with size k.

Proof. By Proposition 3.5.1, there is a one-to-one correspondence between every partition of an integer k into triangular numbers and a disjoint union of complete graphs of size k. By Theorem 3.3.2 every triangular number Δ_n is mapped to a complete graph K_n of order n by $\frac{1+\sqrt{1+8\Delta_n}}{2} = n = |V(K_n)|$. Thus, the sum over all the $\Delta_i \in \pi_\Delta(k)$ gives the order of a complete graph or disjoint union of complete graphs with size k, since each of the Δ_i is mapped to a complete graph of order i and the edges of the partition sum to k. □

Proposition 3.5.3. *Let $p_\Delta(k)$ be the* number of partitions*, $\pi_\Delta(k)$, of an integer k into triangular numbers Δ_i, such that*

$$\sum_{\Delta_i \in \pi_\Delta(k)} \frac{1+\sqrt{1+8\Delta_i}}{2} \leq n,$$

where n is an integer and $k \leq \binom{n}{2}$. Then $p_\Delta(k)$ is also the number of non-isomorphic complete subgraphs or disjoint union of complete subgraphs of K_n with size k.

Proof. By Proposition 3.5.2, $\sum_{\Delta_i \in \pi_\Delta(k)} \frac{1+\sqrt{1+8\Delta_i}}{2}$ corresponds to the order of a complete graph or disjoint union of complete graphs with size k. Thus, every partition of k into triangular numbers is a partition of a complete graph K_n into complete subgraphs with edge sum k, as long as the sum of the order of the subgraphs making up the partition is less than or equal to n, the order of K_n. Hence, the number of partitions $\pi_\Delta(k)$ is also the number of non-isomorphic complete subgraphs or union of disjoint subgraphs of K_n. □

We note that we can have integer partitions of k where $\sum_{\Delta_i \in \pi_\Delta(k)} \frac{1+\sqrt{1+8\Delta_i}}{2} > n$, see Example 3.5.5, and hence we exclude these partitions in our definition of $p_\Delta(k)$.

With a slight abuse of notation we will say $\pi_\Delta(k) \in p_\Delta(k)$ if $\pi_\Delta(k)$ is counted by $p_\Delta(k)$.

From the definition of a closed set, Lemma 3.4.3, Corollary 3.4.8 and Proposition 3.4.9 as well as Theorem 3.4.10, we have that a closed set of k edges, $cl(X_k)$, in a complete graph is

$$cl(X_k) = \bigcup_{i \leq n} K_i,\ K_i \subseteq K_n$$

such that $|E(cl(X_k))| = k$ and $|V(cl(X_k))| \leq n$ and denote by $|cl(X_k)|$ the number of such non-isomorphic closed sets.

Proposition 3.5.4. *Let $cl(X_k)$ be a closed set of K_n, then*

$$|cl(X_k)| = p_\Delta(k).$$

Proof. By Proposition 3.5.1 and Proposition 3.5.2, every $\pi_\Delta(k)$ gives a partition of a complete graph into a disjoint union of complete subgraphs. By Proposition 3.4.9 all closed sets of K_n are complete graphs or disjoint unions of complete graphs. Thus, by Proposition 3.5.3, $p_\Delta(k)$ is the number of size k closed sets of K_n, as long as the restriction on the order is respected. $\square$

We now demonstrate Propositions 3.5.3 and 3.5.4 with an example.

Example 3.5.5. We will partition $k = 6$ using the triangular numbers $\{0, 1, 3, 6\}$.

1. $6 = 1 + 1 + 1 + 1 + 1 + 1$,

2. $6 = 3 + 1 + 1 + 1$,

3. $6 = 3 + 3$,

4. $6 = 6$.

Computing $\sum_{\Delta_i \in \pi_\Delta(k)} \frac{1+\sqrt{1+8\Delta_i}}{2}$ for each of the $\pi_\Delta(6)$ above we have

1. $2 + 2 + 2 + 2 + 2 + 2 = 12$,

2. $3 + 2 + 2 + 2 = 9$,

3. $3 + 3 = 6$,

4. $4 = 4$.

We see that $p_\Delta(6)$ is 4 if $n \geq 12$, 3 if $9 \leq n < 12$, 2 if $6 \leq n < 9$ and 1 if $4 \leq n < 6$. Clearly if $n < 4$ our graph will not have enough edges to compose a closed set of 6 bad edges.

Following we list the corresponding $cl(X_6)$ and note that $|cl(X_6)| = p_\Delta(6)$ given the restrictions on n.

1. Disjoint union of six K_2's,

2. Disjoint union of one K_3 and three K_2's,

3. Disjoint union of two K_3's,

4. One K_4.

3.6 Conclusion

In this chapter we defined triangular numbers and triangular number partitions. We set up a one-to-one correspondence between complete graphs and triangular numbers in Section 3.3.

We showed that a set X_k of k bad edges in an improper colouring of a graph G, is a closed subgraph or the disjoint union of closed subgraphs of G. Furthermore, X_k partitions K_n into a disjoint union of i complete subgraphs, K_i, such that $|\bigcup E(K_i)| = k$.

In Section 3.5 we were able to prove the correspondence between the triangular number partitions of an integer k and a disjoint union of complete graphs of size k. We also proved the correspondence between closed sets of size k of a complete graph and the triangular number partitions of k.

We concluded, therefore, that the triangular number partitions of an integer k correspond to partitions of a complete graph into disjoint unions of complete subgraphs such that the size of each union is k.

We are now in a position to put forward an algorithm for calculating the k-defect polynomials of a complete graph in Chapter 4.

Chapter 4

Calculating the k-defect polynomial of a complete graph

4.1 Introduction

In this chapter we use the relationship between triangular number partitions and closed sets of complete graphs discussed in Chapter 3 to develop an algorithm for calculating the k-defect polynomial of a complete graph. We give an example on using the algorithm to generate an expression for any k-defect polynomial of a complete graph.

4.2 Methods for calculating k-defect polynomials

There are several known methods for calculating the k-defect polynomial of a graph. In this section we discuss two of these methods and give examples on how to use them to calculate the k-defect polynomials of a graph G.

4.2.1 Method 1

Recall from Chapter 2, the definition of the Tutte and bad colouring polynomials and their equivalence.

$$B(G;\lambda,S)=\lambda(S-1)^rT(G;\frac{S+\lambda-1}{S-1},S),$$

where r is the rank of G. If G is connected $r=n-1$, where n is the order of G.

Vice versa, given the bad colouring polynomial of G, we can calculate the Tutte polynomial,

$$T(G;x,y)=\frac{1}{(x-1)(y-1)(y-1)^r}B(G;(x-1)(y-1),y).$$

We also recall from Proposition 2.5.4 that

$$B(G;\lambda,S)=\sum S^k\phi_k(G;\lambda),$$

where $\phi_k(G;\lambda)$ is the k-defect polynomial of the graph G.

Thus, it is possible to read off the required k-defect polynomial of G from the bad colouring polynomial. Given the equivalence of the Tutte and bad colouring polynomials we can either obtain the bad colouring polynomial of G from the definition, or by first calculating the Tutte polynomial of G .

Example 4.2.1. In this example we calculate the 6-defect polynomial of the complete graph K_6.

By Definition 2.5.1, the bad colouring polynomial is

$$B(G;\lambda,S)=\lambda^{k(G)}\sum_{A\subseteq E}(S-1)^{|A|}\lambda^{r(E)-r(A)}$$

where $|A|$ is the size of the subgraph induced by the subset A of edges.

K_6 has $\binom{6}{2}=15$ edges. Thus, for $|A|=6$, we would need to look at at least $\binom{15}{6}$ subsets $A\subset E(K_6)$ and calculate $\lambda^{r(E)-r(A)}$ for all non-isomorphic such sets. This is tedious. Clearly finding the Tutte Polynomial by deletion and contraction for K_6 would be equally laborious.

Hence we use Mathematica to calculate the Tutte polynomial.

$$\begin{aligned}
T(K_6;x,y) &= 24x+50x^2+35x^3+10x^4+x^5+24y+106xy+90x^2y+20x^3y+80y^2 \\
&+ 145xy^2+45x^2y^2+120y^3+105xy^3+15x^2y^3+120y^4+60xy^4+96y^5 \\
&+ 24xy^5+64y^6+6xy^6+35y^7+15y^8+5y^9+y^{10}
\end{aligned}$$

and

$$\begin{aligned}
B(K_6;\lambda,S) &= \lambda(S-1)^5T(K_6;\frac{S+\lambda-1}{S-1},S) \\
&= -120\lambda+274\lambda^2-225\lambda^3+85\lambda^4-15\lambda^5+\lambda^6 \\
&+ (360\lambda-750\lambda^2+525\lambda^3-150\lambda^4+15\lambda^5)S \\
&+ (-270\lambda+495\lambda^2-270\lambda^3+45\lambda^4)S^2+(-90\lambda+175\lambda^2-105\lambda^3+20\lambda^4)S^3 \\
&+ (120\lambda-180\lambda^2+60\lambda^3)S^4+(20\lambda-35\lambda^2+15\lambda^3)S^6 \\
&+ (-15\lambda+15\lambda^2)S^7+(-6\lambda+6\lambda^2)S^{10}+\lambda S^{15}.
\end{aligned}$$

Reading off the 6-defect polynomial from the generating function, $B(K_6;\lambda,S)$, we have

$$\phi_6(\lambda)=20\lambda-35\lambda^2+15\lambda^3.$$

4.2.2 Method 2

An alternative known method for calculating the k-defect polynomial of a graph, is summing the chromatic polynomials of the minors obtained by contracting all closed sets of size k, as described by Mphako in [17].

We recall from Definition 1.4.3 that a graph H is a *minor* of a graph G if $H \cong G$ or H can be obtained from G by a succession of edge contractions, edge deletions and vertex deletions.

Proposition 4.2.2.

$$\phi_k(G;\lambda) = \sum_{X \in L(G), |X|=k} \chi(G/X;\lambda)$$

where $L(G)$ is the set of all closed sets of G and G has at least one closed set of size k. G/X is the minor obtained by contracting the closed set X and $\chi(G/X;\lambda)$ is the chromatic polynomial of the minor.

Proof. Let X_k be a set of k bad edges in an improper colouring of a graph G and let S be the vertex set of $G[X_k]$. As noted in Proposition 3.4.5, all the vertices in a closed set of bad edges are the same colour and by Proposition 3.4.6 all the edges of $G[S]$ are bad. Any adjacent vertices in the rest of the bad colouring must be coloured differently. Otherwise we have more than k bad edges. Since all the vertices of $G[S]$ are connected (a similar argument holds if X_k is a disjoint union of bad edges), contracting the edges of the closed set means that all the vertices of $G[S]$ are identified. Deleting any parallel edges leaves us with a minor of the graph G all of whose adjacent vertices are coloured differently.

Hence, to calculate the k-defect polynomial of G we calculate the number of ways of choosing isomorphic closed subsets of size k and multiply by the chromatic polynomial of the minor. We then sum the polynomials obtained from non-isomorphic closed sets of size k. □

Before applying Method 2, we state the following lemma and illustrate the process of contraction in a complete graph. Recall from Section 1.3 that two graphs G and H are isomorphic up to parallel class if $|V(G)| = |V(H)|$ and $|E(G)| \neq |E(H)|$, but there is a bijective function $\psi : V(G) \to V(H)$ that preserves the adjacency of vertices in the mapping.

Lemma 4.2.3. *Let X_k be a closed set of a complete graph $G = K_n$. The minor obtained by contracting the edges of X_k is a complete graph up to parallel class.*

Proof. By Proposition 3.4.9, $G[X_k]$ is a complete subgraph of G such that the size of $G[X_k] = k$. Since $G[X_k]$ is a complete graph, k is a triangular number.

Let $k = \binom{r}{2}$, for some integer $r \leq n$, then $|V(G[X_k])| = r$ and we let $n = m + r$.

Recall from Section 1.4 that the operation of contracting an edge e means identifying the two endpoints of e followed by removal of e.

Thus, identifying the r vertices of $G[X_k]$ and deleting the edges of X_k leaves a single vertex u. But, since G is a complete graph, each of the r vertices in $G[X_k]$ is adjacent to each of the m vertices not in $V(G[X_k])$. These edges are not deleted by the operation of contraction, and, hence, u is adjacent to all m remaining vertices in $V(G)$. By Lemma 3.4.3 every induced subgraph on m vertices of G is a complete graph K_m. Hence, $G[V(K_m) + u]$ is also a complete graph, K_{m+1}, with extra parallel edges remaining after contracting X_k. $\square$

Example 4.2.4. Without loss of generality, we illustrate the proof using the diagrams in Figure 4.1, where $k = 6$ and $n = 6$.

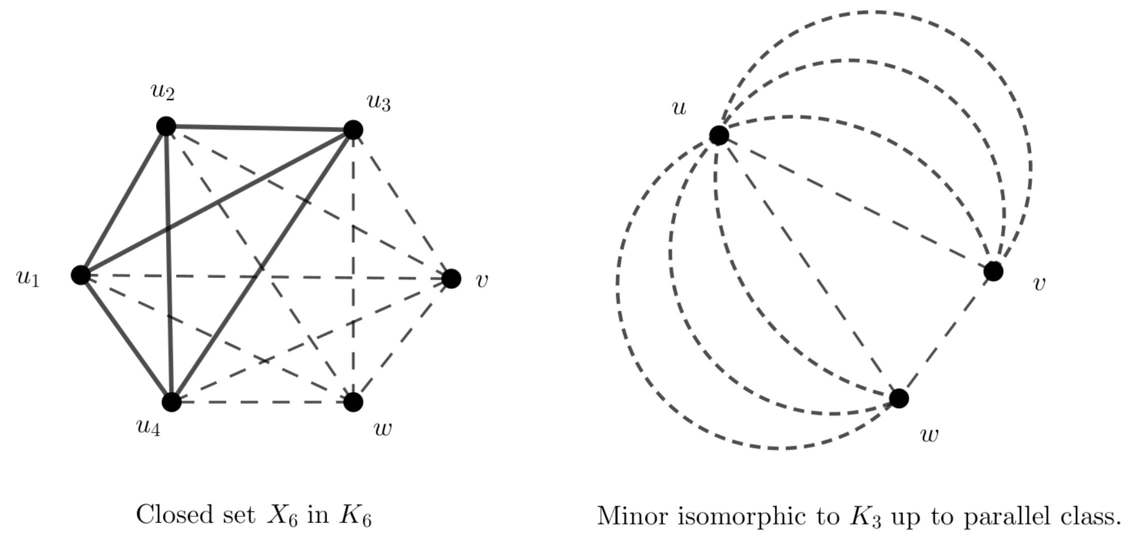

Figure 4.1: Minor isomorphic to K_3 up to parallel class.

In the first diagram in Figure 4.1 the set X_k of bad edges are the 6 solid edges and $G[X_k]$ is the complete subgraph K_4 with vertex set $\{u_1, \ldots, u_4\}$.

In the second diagram in Figure 4.1 we identify the vertices $\{u_1, \dots, u_4\}$ by contracting the six solid edges of the closed set of size 6 in K_6. We obtain a minor isomorphic to K_3 up to parallel class.

Clearly none of the parallel edges are "bad" edges since the three vertices in the minor are all coloured differently and, hence, the chromatic polynomial is the same as that of K_3. That is, the "extra" edges do not contribute anything to the chromatic polynomial and we can safely ignore them. Since the edge set that we are contracting is closed, we also have no loops and hence the chromatic polynomial is not zero.

Example 4.2.5. In this example we use Method 2 to calculate the 6-defect polynomial of K_6 as outlined in Section 4.2.2.

There are only two partitions of the vertex set of K_6 that will yield closed sets of size 6. That is, a subgraph isomorphic to K_4 or two disjoint subgraphs isomorphic to K_3, see the diagrams in Figure 4.2. The minors obtained by contraction and deletion are isomorphic to K_3 and K_2 respectively.

Recall from Section 1.5 that the chromatic polynomials of the two minors are $\chi(K_3, \lambda) = \lambda(\lambda-1)(\lambda-2)$ and $\chi(K_2, \lambda) = \lambda(\lambda-1)$. Also, by Lemma 3.4.3, every induced subgraph on m vertices of K_6 is a complete graph K_m. Thus there are $\binom{6}{4}$ subgraphs of K_6 isomorphic to K_4. Similarly, there are $\frac{\binom{6}{3}\binom{3}{3}}{2}$ subgraphs of K_6 isomorphic to two disjoint copies of K_3.

Thus

$$\begin{aligned}
\phi_6(K_6; \lambda) &= \sum_{X \in L(K_6), |X|=6} \chi(K_6/X; \lambda) \\
&= \binom{6}{4}\lambda(\lambda-1)(\lambda-2) + \frac{\binom{6}{3}\binom{3}{3}}{2}\lambda(\lambda-1) \\
&= 15\lambda(\lambda-1)(\lambda-2) + 10\lambda(\lambda-1) = \lambda(\lambda-1)(15\lambda-20) \\
&= 20\lambda - 35\lambda^2 + 15\lambda^3,
\end{aligned}$$

as calculated using Method 1.

It should be clear that identifying all the closed sets of size k is fairly straight

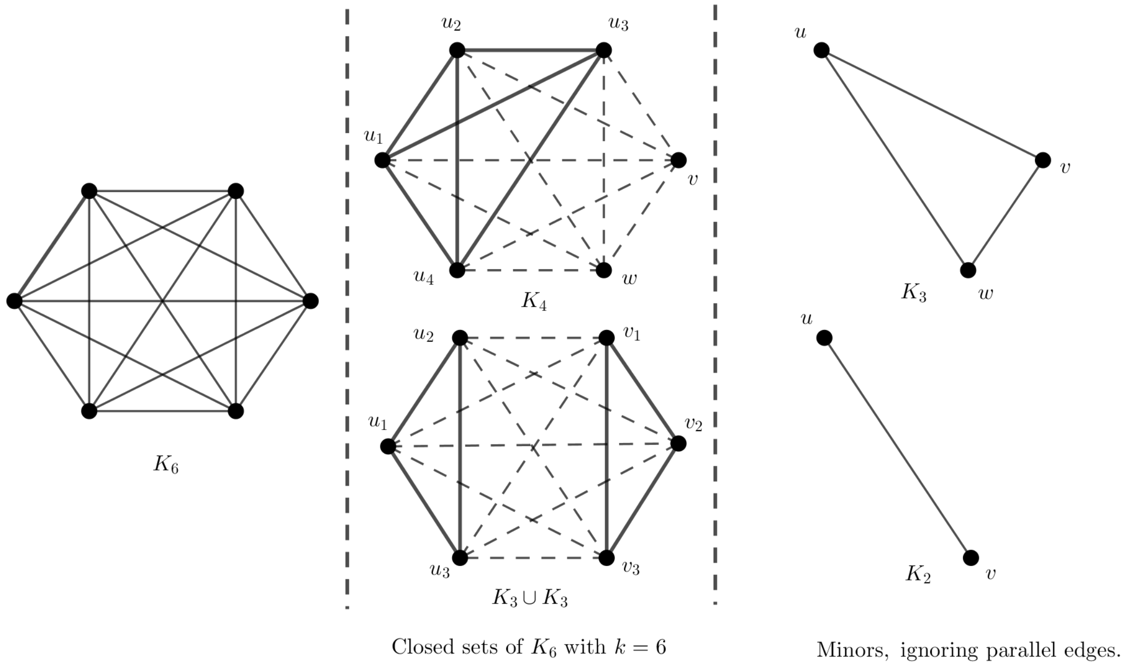

Figure 4.2: Closed sets of K_6 with $k = 6$

forward in complete graphs of small order. The task becomes more laborious, however, when dealing with complete graphs of large order and greater values of k. Hence the effort in the next section to find an easy way of calculating the k-defect polynomial using an algorithm. We use the algorithm to calculate the 6-defect polynomial of K_6, which gives a straight forward method to guarantee that we have identified *all* possible non-isomorphic closed sets of K_n of size k.

4.3 An algorithm for finding the k-defect polynomial of a complete graph

In this section, we give a step by step procedure on how to calculate a k-defect polynomial of a complete graph using the partitions of k into triangular numbers.

Recall that $\pi_\Delta(k)$ is a partition of an integer k into triangular numbers. We also need the following definition of a *block* with reference to set partitions, see [15]. This should not be confused with the graph theoretical term used when referring to a block of G as a maximal nonseparable subgraph of G, see [3].

Definition 4.3.1. A set partition π of a set S is a collection B_1, B_2, $\ldots$, B_k of nonempty disjoint subsets of S such that $\bigcup_{i=1}^{k} B_i = S$. The elements of a set partition are called *blocks* and the size of a *block* B is given by $|B|$, the number of elements in B.

We will write, for example, the vertex partitions of K_6 yielding $k = 6$ bad edges used in Method 2 of Subsection 4.2.2 as 4/1/1 and 3/3 respectively, using the block notation for set partitions from [15]. Since every choice of m vertices in a complete graph G induces a complete subgraph, the block notation is equivalent to writing $K_4 \bigcup K_1 \bigcup K_1$ and $K_3 \bigcup K_3$, where $\bigcup$ is the disjoint union of the subgraphs induced by the vertices in the respective blocks. Unless confusion may result, we may also refer to each of the triangular numbers in a triangular number partition of k as a block.

4.3.1 Eight step algorithm

S-i. Find all partitions $\pi_\Delta(k)$.

S-ii. Translate each of the partitions $\pi_\Delta(k)$ into a vertex partition, $\pi(v)$, of K_n by applying $v(\Delta_i) = \frac{1+\sqrt{1+8\Delta_i}}{2}$ to each $\Delta_i \in \pi_\Delta(k)$ using the one-one correspondence stated in Theorem 3.3.2.

S-iii. Consider all $\pi_\Delta(k)$ such that $\pi_\Delta(k) \in p_\Delta(k)$. Recall from Proposition 3.5.3 that $p_\Delta(k)$ is the number of triangular number partitions of k with corresponding vertex sum less than n. Then $p_\Delta(k)$ is the number of non-isomorphic complete subgraphs or disjoint union of complete subgraphs of K_n with size k. Hence, $p_\Delta(k)$ only counts those partitions of k that have a corresponding vertex count less than or equal to n, where n is the order of our complete graph.

S-iv. If the total number of vertices $|\pi(v)| < n$, add 1's to the partition until $|\pi(v)| = n$. This step will ensure that when we assign colours to the vertex partition blocks we take into account all vertices and adding 1 to a vertex partition implies adding 0 to an edge partition, since the size of K_1 is 0.

S-v. Calculate the number of ways to choose the vertices of each block $\pi(v)$.

S-vi. The vertices in each block of a $\pi(v)$ are assigned the same colour and each block a different colour, thus bad edges are the edges of the complete graph corresponding to each block.

S-vii. Since blocks have different colours, the first block will be coloured with λ colours, the second block with $(\lambda - 1)$ colours, and so on. This gives a polynomial in λ.

S-viii. Finally, add all the polynomials generated above to get the k-defect polynomial since each gives a way of colouring the graph with k bad edges using different closed set configurations.

The following example illustrates the procedure of calculating the k-defect polynomial.

Example 4.3.2. Let G be the complete graph K_6 and we consider $k = 6$. Using the triangular numbers $\{0, 1, 3, 6\}$ to partition 6 we will calculate the 6-defect polynomial for G.

S-i. We list all $\pi_\Delta(6)$.

1. $\pi_\Delta(6)_1 = 1 + 1 + 1 + 1 + 1 + 1$,
2. $\pi_\Delta(6)_2 = 3 + 1 + 1 + 1$,
3. $\pi_\Delta(6)_3 = 3 + 3$,
4. $\pi_\Delta(6)_4 = 6$.

S-ii. The corresponding partition of vertices and complete subgraphs looks as follows:

1. $\pi(v)_1 = 2/2/2/2/2/2$, the disjoint union of six complete subgraphs, K_2, of K_n. This is only possible in a complete graph where $n \geq 12$.
2. $\pi(v)_2 = 3/2/2/2$, the disjoint union of one K_3 and three K_2 subgraphs of K_n. This is only possible in a complete graph where $n \geq 9$.
3. $\pi(v)_3 = 3/3$, the disjoint union of two K_3 subgraphs of K_n. This is possible in a complete graph where $n \geq 6$.
4. $\pi(v)_4 = 4$, one K_4 subgraph of K_n. This is possible in a complete graph where $n \geq 4$. We will take care of the remaining two vertices in S-iv.

S-iii. We are calculating the 6-defect polynomial for K_6 with $n = 6$. Hence, only $\pi_\Delta(6)_3$, $\pi_\Delta(6)_4 \in p_\Delta(6)$. Thus we will apply S-v to S-viii to these two partitions only.

S-iv. $\pi_\Delta(6)_4 = 6$ gives a corresponding vertex partition with $|\pi(v)| = 4 < 6$. Thus we add the remaining two vertices to the partition as disjoint copies of K_1, thereby adding no extra edges. Now we have $\pi(v)_4 = 4/1/1$.

S-v. We need to look at the number of ways of choosing four of the six vertices, inducing a K_4 subgraph and thus the other two vertices will induce K_1 subgraphs respectively; or two sets of three vertices, inducing two disjoint K_3 subgraphs, as shown in the diagram in Figure 4.3. That is, how many ways are there to partition the six vertices into blocks $****/*/*$ or $***/***$, where $*$ represents a vertex.

1. $\pi_\Delta(6)_4$: There are $\binom{6}{4} = 15$ different subgraphs of K_6 isomorphic to K_4.
2. $\pi_\Delta(6)_3$: There are $\binom{6}{3} \times \binom{3}{3} = 20$ ways to choose two sets of three vertices each from the six available in K_6. But the disjoint unions of two K_3 subgraphs are

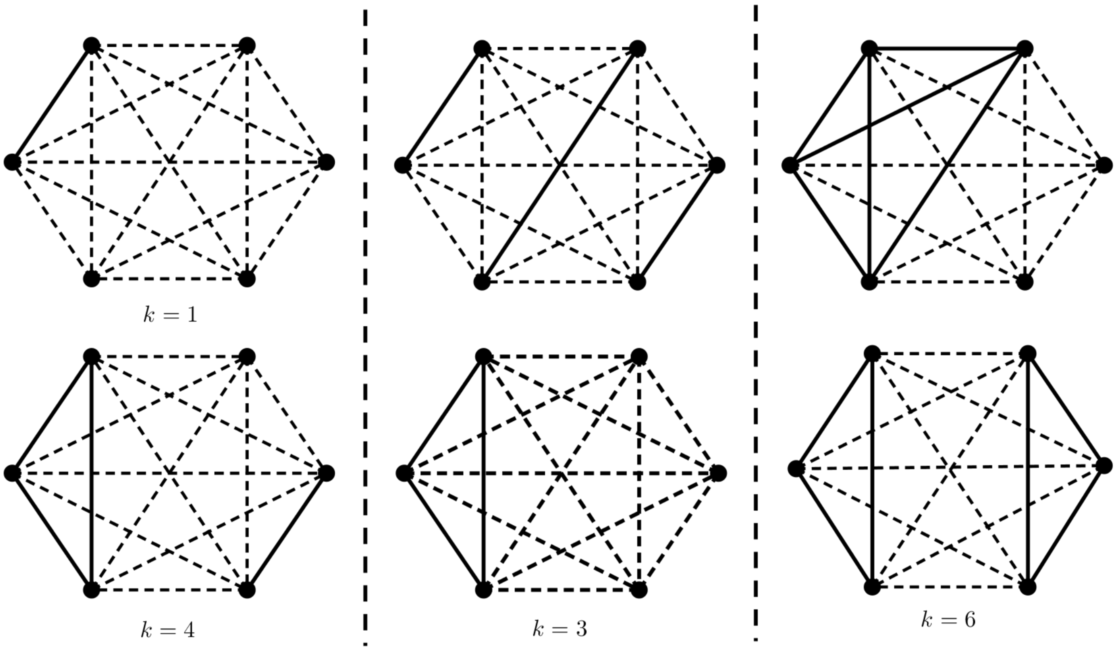

Figure 4.3: K_6 with $k = 1, 3, 4$ and 6 bad edges solid.

not different in all 20 cases. We are counting double, since, given a vertex set {1, 2, 3, 4, 5, 6}, for example, the partitions 123/456 and 456/123 would give the same two subgraph partitions. So, we need to divide by two and thus we have $\frac{\binom{6}{3}\times\binom{3}{3}}{2} = 10$ ways to choose two disjoint K_3's.

S-vi. 1. $\pi_\Delta(6)_4$: We have three blocks. The first block has six bad edges, the remaining two have no edges.

2. $\pi_\Delta(6)_3$: We have two blocks. The first block has three bad edges and the second has three bad edges.

S-vii. 1. $\pi_\Delta(6)_4$: We have three blocks so the 6-defect polynomial for this subgraph choice would be $15\lambda_{(3)} = 15\lambda(\lambda-1)(\lambda-2) = 15\lambda^3 - 45\lambda^2 + 30\lambda$.

2. $\pi_\Delta(6)_3$: We have two blocks so the 6-defect polynomial for this subgraph choice would be $10\lambda_{(2)} = 10\lambda(\lambda-1) = 10\lambda^2 - 10\lambda$ as our 6-defect polynomial for this choice of subgraphs.

S-viii. We can have six bad edges by choosing one K_4 *or* two K_3's and thus, adding the two polynomials we have the 6-defect polynomial for K_6 as $15\lambda^3 - 35\lambda^2 + 20\lambda$. This verifies our result as given in Section 4.2.

4.3.2 Expression for the k-defect polynomial in any complete graph K_n

In order to generate the k-defect polynomial we follow steps (i) to (iv) in the algorithm outlined in Subsection 4.3.1. Thereafter we generate the k-defect polynomial as outlined in Theorem 4.3.3, one of the main results of this chapter.

For ease of reference we recall Theorem 3.4.10 here:

Let X_k be the set of k bad edges in an improper colouring of a complete graph K_n. Then X_k partitions K_n into a disjoint union of i complete subgraphs, K_i, such that $|\bigcup E(K_i)| = k$.

Theorem 3.4.10 implies that if $\pi(v)$ is a vertex partition of $V(K_n)$ generated by X_k, where X_k is the set of k bad edges in an improper colouring of K_n, then $\pi(v)$ is a disjoint union of complete subgraphs of K_n.

Also, from Definition 4.3.1, recall that the elements of a set partition are called *blocks* and the size of a *block* B is given by $|B|$, the number of elements in B.

To ease notation, we label each block in the vertex partition $\pi(v)$ as i_j, where i is the number of vertices in the block and j the number of the block. We list the blocks in decreasing size and increasing values of j. We let $|j_i|$ be the number of blocks in the partition that have the same number of vertices i. We will denote the total number of blocks in the partition as $maxj$ and recall that $\lambda_{(maxj)}$ is the falling factorial in λ.

Thus, for example, we will label the blocks in the vertex partition $\pi(v) = 4/1/1$ of K_6, encountered in Section 4.3, as $4_1/1_2/1_3$ with $|j_1| = 2$ and the partition $\pi(v) = 3/3$ as $3_1/3_2$ with $|j_3| = 2$.

Theorem 4.3.3. *Let X_k be the set of k bad edges in an improper colouring of a complete graph K_n and let $\pi(v)$ be a vertex partition of $V(K_n)$ generated by X_k. We label each block in the partition and set it equal to the number of vertices it contains, that is, we let $i_j = i$.*

Then the k-defect polynomial of K_n is

$$\phi_k(K_n;\lambda) = \sum_{\pi(v)} \left[\prod_{all\, j\, s.t.\, i>1} \binom{n - \sum_{r=1}^{j-1} i_r}{i_j} \Big/ \prod |j_i|! \right] \lambda_{(max j)}.$$

Proof. The proof is simply by construction as outlined in steps $i - viii$ in Subsection 4.3.1. $\square$

Example 4.3.4. We illustrate the use of the notation by applying Theorem 4.3.3 to the two partitions $\pi(v) = 4/1/1 = 4_1/1_2/1_3$ and $\pi(v) = 3/3 = 3_1/3_2$ of K_6 and verifying the 6-defect polynomial as calculated in Subsection 4.3.1.

$$\begin{aligned}
\phi_6(K_6;\lambda) &= \sum_{\pi(v)_4, \pi(v)_3} \left[\prod_{\text{all}\, j\, \text{s.t.}\, i>1} \binom{6 - \sum_{r=1}^{j-1} i_r}{i_j} \Big/ \prod |j_i|! \right] \lambda_{(\max j)} \\
&= \left[\binom{6 - \sum_{r=1}^{0} 4_r}{4_1} \Big/ \prod |j_4|! \right] \lambda_{(3)} \\
&+ \left[\binom{6 - \sum_{r=1}^{0} 3_r}{3_1} \binom{6 - \sum_{r=1}^{1} 3_r}{3_2} \Big/ \prod |j_3|! \right] \lambda_{(2)} \\
&= \frac{\binom{6}{4}}{1!} \lambda_{(3)} + \frac{\binom{6-0}{3}\binom{6-3}{3}}{2!} \lambda_{(2)} \\
&= 15\lambda(\lambda-1)(\lambda-2) + 10\lambda(\lambda-1).
\end{aligned}$$

In the following example we use Theorem 4.3.3 to calculate the k-defect polynomial of a complete graph much larger than we have done thus far and for a relatively large value of k. We show that it is possible to calculate $\phi_k(K_n)$ without first having to calculate the Tutte or bad colouring polynomials as in Method 1 in Subsection 4.2.1 and that we are certain that we have included all the minors corresponding to X_k, as we would need in Method 2 of Subsection 4.2.2.

Example 4.3.5. We apply Theorem 4.3.3 to calculate the 10-defect polynomial of K_{18}.

Table 4.1 shows the triangular number partitions of 10, the minimum number of vertices required to have a corresponding closed set partition, the corresponding vertex partitions and the number of blocks in the partition. Note that the exponent used in the notation refers to addition as the repeated operation, not multiplication. For example, we write 1^4 to mean $1+1+1+1$.

$\pi_\Delta(10)$	$\min(v)$	$\pi(v)$	number of blocks
10	5	5/1/1/1/1/1/1/1/1/1/1/1/1/1	14
$6+3+1$	9	4/3/2/1/1/1/1/1/1/1/1/1	12
$6+1^4$	12	4/2/2/2/2/1/1/1/1/1/1	11
3^3+1	11	3/3/3/2/1/1/1/1/1/1/1	11
3^2+1^4	14	3/3/2/2/2/2/1/1/1/1	10
$3+1^7$	17	3/2/2/2/2/2/2/2/1	9
1^{10}	20	2/2/2/2/2/2/2/2/2/2	10

Table 4.1: Triangular number partitions of 10 and corresponding vertex partitions of closed sets.

The last partition requires more than 18 vertices so we will ignore it. Then the first six partitions give us the following 10-defect polynomial for K_{18}:

$$\begin{aligned}
\phi_{10}(K_{18};\lambda) &= \binom{18}{5}\lambda_{(14)} + \binom{18}{4}\binom{14}{3}\binom{11}{2}\lambda_{(12)} + \frac{\binom{18}{4}\binom{14}{2}\binom{12}{2}\binom{10}{2}\binom{8}{2}}{4!}\lambda_{(11)} \\
&+ \frac{\binom{18}{3}\binom{15}{3}\binom{12}{3}\binom{9}{2}}{3!}\lambda_{(11)} + \frac{\binom{18}{3}\binom{15}{3}\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{8}{2}}{2!4!}\lambda_{(10)} \\
&+ \frac{\binom{18}{3}\binom{15}{2}\binom{13}{2}\binom{11}{2}\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}}{7!}\lambda_{(9)}.
\end{aligned}$$

Simplifying the expression gives the 10-defect polynomial for K_{18}:

$$\begin{aligned}
\phi_{10}(K_{18};\lambda) &= -653575649126400\lambda + 1813956914085120\lambda^2 - 2037012613953408\lambda^3 \\
&+ 1258088633566128\lambda^4 - 493084834049808\lambda^5 + 136327261618548\lambda^6 \\
&- 29080916468604\lambda^7 + 4980518406864\lambda^8 - 656225073504\lambda^9 \\
&+ 60187291164\lambda^{10} - 3368753388\lambda^{11} + 93228408\lambda^{12} - 779688\lambda^{13} \\
&+ 8568\lambda^{14}.
\end{aligned}$$

4.4 Conclusion

In this chapter, after looking at some known methods for calculating k-defect polynomials of graphs, we showed, using the results from Chapter 3 and by means of an algorithm, that it is possible to find the k-defect polynomial of a complete graph by using only the triangular number partitions of k.

We then used the algorithm to find an expression for any k-defect polynomial of a complete graph K_n. Although this is not a closed form expression, it is the only such expression that we are aware of at the time of writing.

In Chapter 5 we build on the algorithm set out in this chapter in conjunction with the ideas formulated in Chapter 3, to determine which k-defect polynomials of a complete graph are equal to zero and we also calculate a lower bound for the number of such k-defect polynomials.

Chapter 5

Zero k-defect polynomials of complete graphs

5.1 Introduction

In this chapter we identify the values of k such that the k-defect polynomial of a graph G is equal to zero. We use known methods and formulae to find some of the values of such k for any graph G. Then we apply the theory of triangular number partitions to find the values of such k for complete graphs. In addition we determine a lower bound on the number of k-defect polynomials that are equal to zero in a complete graph.

We note from the literature that the bad colouring polynomial is equivalent to the Potts Partition function. Since it forms the denominator in $Pr(\sigma)$, that gives the Boltzmann entropy distribution of a system, the zeros of $B(G, \lambda, S)$ are very important. Although we do not touch on the zeros of $B(G; \lambda, S)$ in this dissertation, the intervals of k and the number of k where the S^k terms of this polynomial are zero may cast a different perspective on the structure of the Partition function.

5.2 Known facts about k-defect polynomials equal to zero.

In this section we state some results on the k-defect polynomials of graphs that are equal to zero. We also give some classes of graphs in which there exist an integer k such that the k-defect polynomial is equal to zero.

Definition 5.2.1. A *cut-set*, or edge-cut, of a connected graph G, is a minimal edge subset, $X \subset E(G)$, such that $G \backslash X$ has at least two different components. We call $|X|$ the *cut-set number* of G.

The following proposition was proved in [17] for matroids. We state it here, without proof, in terms of graphs.

Proposition 5.2.2. *Let G be a graph with edge set E and h be the cut set number of G. Then for all k such that*

$$|E| - h < k < |E|,$$

$\phi_k(G; \lambda) = 0$.

Proposition 5.2.3. *Let G be a tree (or forest), then there is no integer value for $0 \leq k \leq (n-1)$ such that the coefficient of S^k in $B(G; \lambda, S)$ is zero. That is, for every $0 \leq k \leq (n-1)$ we have a non-zero k-defect polynomial.*

Proof. Let T_n be a tree on n vertices, then $|E(T_n)| = n - 1$. The cut-set number, h, of T_n is 1, since we only have to delete one edge to disconnect a leaf from the tree to give a disconnected graph. Since n and k are both integers, there is no integer on the interval $|E| - h < k < |E|$ and hence there is no value for k corresponding to Proposition 5.2.2.

Now start with a proper colouring of T_n and choose an arbitrary vertex $u \in V(T_n)$ with colour a. Choosing any neighbour v of u and colouring it the same gives one bad edge. We can continue to add bad edges in this fashion without restriction, since there are no cycles in T_n. That is, we can add one bad edge at a time by colouring any neighbour of a vertex already coloured a with the same colour. We continue until we have coloured all n vertices the same colour. Hence, we can choose k bad edges for every value of $0 \leq k \leq (n-1)$, which implies that none of the k-defect polynomials generated by $B(G; \lambda, S)$ will be zero. $\square$

Proposition 5.2.4. *Let G be a cycle, C_n, on n vertices and $m = n$ edges, then the only value for k where the k-defect polynomial is equal to zero is $k = n - 1$.*

Proof. The cut-set number, h, for any cycle, C_n, is two. That is, we have to delete at least two edges to disconnect the graph. Then $|E| - h = n - 2$ and, since n and k are both integers, there is only one integer on the interval $|E| - h < k < |E|$, that is, $k = n - 1$.

Now start with a proper colouring of C_n. We label an arbitrary vertex u_1. Label either neighbour of u_1 as u_2 and the unlabeled neighbour of u_2 as u_3, and so on. We end up with a labeling of the vertices of C_n, $\{u_1, u_2, \ldots, u_n\}$, where we can choose successive vertices on the cycle by simply counting along their index number. We start by colouring u_2 the same as u_1, giving one bad edge. Clearly we can continue along the circuit, adding one bad edge at a time, until we have coloured u_{n-1} the same colour. This gives a colouring of C_n where only u_n is a different colour and the two edges connecting it to u_1 and u_{n-1} are not bad. Thus we will have k bad edges for all values $0 \leq k \leq n-2$, which implies that the k-defect polynomials for these values of k are non-zero. We know from Proposition 5.2.2 that $\phi_{n-1}(C_n; \lambda) = 0$. We also know that we can choose all edges to be bad by colouring all n vertices the same colour, that is $\phi_n(C_n; \lambda) \neq 0$. $\square$

Proposition 5.2.5. *Let G be a graph where at least one edge is a loop. Then $\chi(G; \lambda)$, the chromatic or 0-defect polynomial of G is zero.*

Proof. Let C be a proper colouring of G, then no two adjacent vertices are the same colour. But, if e is a loop on vertex v, then v is always adjacent to a vertex of the same colour, that is, itself, since e has v as both end points. Hence there can be no value of λ that will give a value $\chi(G;\lambda) > 0$ and hence the chromatic polynomial of G is zero. □

Analogous to the methods of computing the k-defect polynomial by first generating the Tutte and the bad colouring polynomial, shown in Section 4.2, we can find values for k where S^k does not appear in the polynomial, implying that the k defect polynomial is equal to zero. Alternatively, using the method of contracting closed sets of size k and summing the chromatic polynomials of these minors, we find that the k-defect polynomials for those values of k where it is not possible to have a closed set of size k, X_k, are equal to zero.

We recall in this regard the closed sets of K_3 as illustrated in the diagrams in Figure 1.6. It was noted at the time that it is not possible to colour the vertices in such a way as to yield 2 bad edges. That is, for K_3, there is no closed set of edges $X_2 \subset E(K_3)$ with $|X_2| = 2$. Thus, the k-defect polynomials of K_3 are:

$$\begin{aligned}\phi_3 &= \lambda\\ \phi_2 &= 0\\ \phi_1 &= 3\lambda^2 - 3\lambda\\ \phi_0 &= \lambda^3 - 3\lambda^2 + 2\lambda\end{aligned}$$

The 2-defect polynomial of K_3 is equal to 0.

It is interesting to note from Propositions 5.2.3 and 5.2.4 that when there are no cycles in a graph, there is no k-defect polynomial equal to zero. On the other hand, with a cycle there is one k-defect polynomial equal to zero. An obvious question to ask at this point is whether the number of k-defect polynomials that are equal to zero is related to the number of cycles in the graph. In the next section we will look at those k-defect polynomials of complete graphs that are equal to zero. Note that in a complete graph *every* choice of three vertices induces a cycle.

5.3 Some k-defect polynomials of complete graphs that are equal to zero

In this section we will characterize the k-defect polynomials of complete graphs that are equal to zero. Applying the relationship between closed sets of complete graphs and partitions of integers into triangular numbers, we derive some exact integer values of k such that the k-defect polynomial is equal to zero.

The following two propositions follow trivially from the work done in Chapters 3 and 4.

Proposition 5.3.1. *Let K_n be a complete graph of order n and $k \leq \binom{n}{2}$ be a triangular number. Then the k-defect polynomial of K_n is non-zero.*

Proof. There is a one-to-one correspondence between triangular numbers and complete graphs. Thus, if $k \leq \binom{n}{2}$ is a triangular number, it maps to a complete subgraph, K_i say, of K_n, which is a closed set by Proposition 3.4.9. Thus we can colour K_n with k bad edges corresponding to $K_i \subseteq K_n$ where $\binom{i}{2} = k$. □

Proposition 5.3.2. *Let G be a complete graph of order at least $2k$. Then the k-defect polynomial of G is non-zero.*

Proof. If we have $2k$ vertices, we can partition the graph into k disjoint copies of K_2, which are closed sets. Thus we can colour each pair of vertices of the respective K_2 subgraphs with a distinct colour, giving k bad edges. Hence, the k-defect polynomial is non-zero. □

We need the following Proposition 5.3.3 and Lemmas 5.3.5 and 5.3.6 to state and prove Theorem 5.3.7, the main result of this chapter.

Proposition 5.3.3. *The k-defect polynomials of K_n for $\binom{n-1}{2} < k < \binom{n}{2}$ and $\binom{n-2}{2} + 1 < k < \binom{n-1}{2}$ are all equal to zero.*

Proof. Without loss of generality, we illustrate the proof using the diagrams in Figure 5.1, where bad edges are represented by the solid edges and $n = 6$.

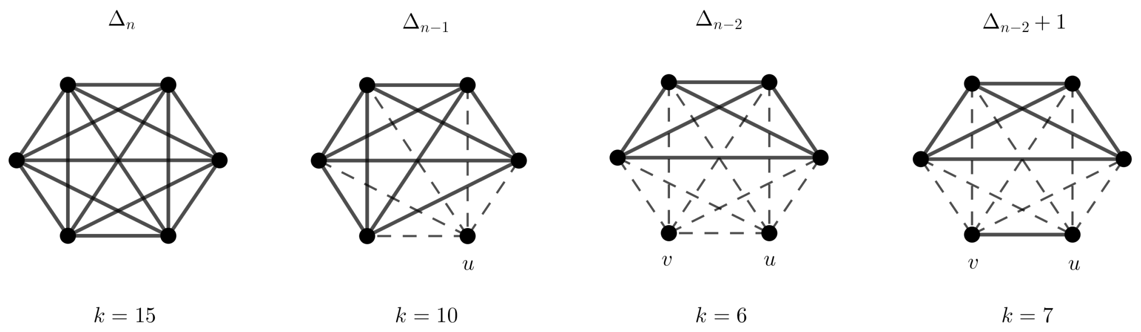

Figure 5.1: Illustrating the process of adding good edges using $n = 6$.

Choose all $\binom{n}{2}$ edges to be bad, that is, all vertices are the same colour as shown in the diagram represented by Δ_n. Now make one edge "good". In order to do this we need to change the colour of one vertex u, as in the diagram represented by Δ_{n-1}. But this means that all $(n-1)$ edges incident with u will be good, so the next most bad edges possible after choosing all to be bad is

$$\begin{aligned}\binom{n}{2} - (n-1) &= \frac{n!}{2!(n-2)!} - (n-1) = \frac{n! - 2(n-1)(n-2)!}{2!(n-2)!} \\ &= \frac{n! - 2(n-1)!}{2!(n-2)!} = \frac{(n-1)!(n-2)}{2!(n-2)!} \\ &= \frac{(n-1)!}{2!(n-3)!} = \binom{n-1}{2}.\end{aligned}$$

That is, it is not possible to have k bad edges for $\binom{n-1}{2} < k < \binom{n}{2}$ and hence the k-defect polynomials for these integers are zero. We note that this is the interval from Proposition 5.2.2 applied to complete graphs, since the cut-set number of a complete graph, K_n, is $(n-1)$.

We continue and add another good edge to those from the previous step, see the diagram represented by Δ_{n-2}. We need to choose another vertex v to be a different colour. But this implies that all edges incident with v must be good, which adds $(n-2)$ good edges, since uv is already good. Thus we have

$$\begin{aligned}
\binom{n}{2} - (n-1) - (n-2) &= \binom{n-1}{2} - (n-2) = \frac{(n-1)! - 2!(n-2)(n-3)!}{2!(n-3)!} \\
&= \frac{(n-1)! - 2!(n-2)!}{2!(n-3)!} = \frac{(n-2)!(n-3)}{2!(n-3)!} \\
&= \frac{(n-2)!}{2!(n-4)!} = \binom{n-2}{2}
\end{aligned}$$

bad edges. On the other hand, we can colour u and v the same colour as in the diagram represented by $\Delta_{n-2}+1$. This gives $\binom{n-2}{2}+1$ bad edges. That is, it is not possible to choose k bad edges for $\binom{n-2}{2}+1 < k < \binom{n-1}{2}$ and hence the k-defect polynomials for these values of k are equal to zero. $\square$

Example 5.3.4. We look at the k-defect polynomials of K_6.

Applying Proposition 5.3.3, $n = 6$ and, hence, $\binom{n-2}{2}+1 = \binom{4}{2}+1 = 6+1 = 7$, $\binom{n-1}{2} = \binom{5}{2} = 10$ and $\binom{n}{2} = \binom{6}{2} = 15$. This gives the intervals for k on which the k-defect polynomials are equal to zero as $7 < k < 10$ and $10 < k < 15$.

The bad colouring polynomial of K_6, calculated using Mathematica, is

$$\begin{aligned}
B(K_6;\lambda,S) &= (-120\lambda + 274\lambda^2 - 225\lambda^3 + 85\lambda^4 - 15\lambda^5 + \lambda^6) \\
&+ (360\lambda - 750\lambda^2 + 525\lambda^3 - 150\lambda^4 + 15\lambda^5)S \\
&+ (-270\lambda + 495\lambda^2 - 270\lambda^3 + 45\lambda^4)S^2 \\
&+ (-90\lambda + 175\lambda^2 - 105\lambda^3 + 20\lambda^4)S^3 + (120\lambda - 180\lambda^2 + 60\lambda^3)S^4 \\
&+ (20\lambda - 35\lambda^2 + 15\lambda^3)S^6 + (-15\lambda + 15\lambda^2)S^7 \\
&+ (-6\lambda + 6\lambda^2)S^{10} + \lambda S^{15}.
\end{aligned}$$

Note that the k-defect polynomials for $k = 8$, 9 and $11, \ldots, 14$ are equal to zero, thus verifying our result. This can also be confirmed from the diagrams in Figure 5.1.

We note that the 5-defect polynomial is equal to zero. However, 5 does not fall in the intervals identified in Proposition 5.3.3. This is an indication that there are other values of k, for which the k-defect polynomial is equal to zero, that we have not identified. That is, there are values of such k that do not fall in the intervals identified in Proposition 5.3.3, nor in the intervals identified in the more general Theorem 5.3.7. We will have more to say on this in Subsection 5.3.1 and 5.3.2.

Lemma 5.3.5. *For integers n and p, if*

$$\frac{p^2+p+4}{2} \leq n$$

then there is at least one integer between $\binom{n-p}{2} + \binom{p}{2}$ and $\binom{n-p+1}{2}$.

Proof. If $\frac{p^2+p+4}{2} \leq n$, then

$$\begin{aligned}
p^2+p+4 &\leq 2n \\
n^2-2np-p+p^2+p^2+p+4-n &\leq n+n^2-2np-p+p^2 \\
n^2-2np-n+p+p^2+p^2-p+4 &\leq n^2-2np+n-p+p^2 \\
(n-p)(n-p-1)+p(p-1)+4 &\leq (n-p+1)(n-p) \\
\frac{(n-p)!}{(n-p-2)!}+\frac{p!}{(p-2)!}+4 &\leq \frac{(n-p+1)!}{(n-p-1)!} \\
\frac{(n-p)!}{2!(n-p-2)!}+\frac{p!}{2!(p-2)!}+2 &\leq \frac{(n-p+1)!}{2!(n-p-1)!} \\
\binom{n-p}{2}+\binom{p}{2}+2 &\leq \binom{n-p+1}{2}.
\end{aligned}$$

The left hand side of the last line is an integer as is the right hand side. Furthermore, $\binom{n-p}{2}$ is a triangular number and $\binom{n-p+1}{2}$ is the next triangular number, so $\binom{n-p+1}{2} > \binom{n-p}{2}$.

If the difference between $\binom{n-p}{2} + \binom{p}{2}$ and $\binom{n-p+1}{2}$ is 1, then $\binom{n-p+1}{2}$ is the integer after $\binom{n-p}{2} + \binom{p}{2}$. Thus, since the difference is greater than or equal to 2, we must have at least one integer between the two integers $\binom{n-p}{2} + \binom{p}{2}$ and $\binom{n-p+1}{2}$. $\square$

From the inequality on p and n in Lemma 5.3.5, given n, we can calculate the values of p as

$$p \leq \lfloor \frac{-1+\sqrt{8n-15}}{2} \rfloor.$$

The following Lemma 5.3.6, listed in [21], is straight forward to prove from the formula of a triangular number. Since the usual definition of a triangular number does not include zero and our definition of a triangular number *does* include zero as the first triangular number, we include a proof.

Lemma 5.3.6.

$$\Delta_{n-i} = \Delta_n + \Delta_i - i(n-1).$$

Proof.

$$\begin{aligned}
\Delta_n + \Delta_i - i(n-1) &= \binom{n}{2} + \binom{i}{2} - i(n-1) = \frac{n!}{2!(n-2)!} + \frac{i!}{2!(i-2)!} - i(n-1) \\
&= \frac{n(n-1)}{2} + \frac{i(i-1)}{2} - i(n-1) \\
&= \frac{n(n-1) + i(i-1) - 2i(n-1)}{2} \\
&= \frac{n^2 - n + i^2 - i - 2ni + 2i}{2} \\
&= \frac{(n-i)^2 - n + i}{2} = \frac{(n-i)^2 - (n-i)}{2} = \frac{(n-i)(n-i-1)}{2} \\
&= \frac{(n-i)!}{2!(n-i-2)!} = \binom{n-i}{2} = \Delta_{n-i}
\end{aligned}$$

□

We are now in a position to state and prove the main result of this chapter. Recall that, by definition, $\binom{n}{r} = 0$ for $n < r$. Thus the statement in Theorem 5.3.7 is the same as Proposition 5.3.3 for $p = 1$ and 2.

Theorem 5.3.7. *Let K_n be a complete graph of order n. For integers p and k, $p \geq 1$ and $\frac{p^2+p+4}{2} \leq n$, let*

$$\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}.$$

Then the k-defect polynomial of K_n is equal to zero.

Proof. Recall from Theorem 3.4.10 that a set X_k of k bad edges in an improper colouring of a complete graph K_n partitions K_n into a disjoint union of i complete subgraphs, K_i, such that $|\bigcup E(K_i)| = k$.

Thus, we need to show that there is no disjoint union of complete subgraphs of K_n with size k and order n. In other words, we need to show that once we have partitioned K_n into two subgraphs K_{n-p} and K_p, it is not possible to get more than $\binom{n-p}{2} + \binom{p}{2}$ bad edges from any other partition until we choose K_{n-p+1} as a subgraph, thus giving us $\binom{n-p+1}{2}$ bad edges. The proof is in two parts. For ease of reference we will write the two parts as separate propositions. Theorem 5.3.7 follows directly from Propositions 5.3.8 and 5.3.9. □

We recall that our use of *block* as per Definition 4.3.1 refers to the elements of a set partition and the size of a *block* B is given by $|B|$, the number of elements in B.

Proposition 5.3.8. *Let K_n be a complete graph and p an integer such that $p \geq 1$ and $\frac{p^2+p+4}{2} \leq n$. Then there is no partition of K_n into two complete subgraphs such that there will be more than $\binom{n-p}{2} + \binom{p}{2}$ and less than $\binom{n-p+1}{2}$ bad edges.*

Proof. We have $p \geq 1$ and $\frac{p^2+p+4}{2} \leq n$. The latter inequality guarantees by Lemma 5.3.5 that the interval $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$ is not empty.

Suppose there are two subgraphs K_{n-r} and K_r such that $k = \binom{n-r}{2} + \binom{r}{2}$ and $r \neq p$ such that $\binom{n-r}{2} + \binom{r}{2} > \binom{n-p}{2} + \binom{p}{2}$. We use all n vertices in the supposed partition in order to maximise the number of bad edges. We must have $(n-r) < (n-p)$, otherwise $\binom{n-r}{2} \geq \binom{n-p+1}{2}$, giving a value for k outside our proposition statement. Hence we have $r > p$. Also, $r, p \leq \lfloor \frac{n}{2} \rfloor$ otherwise the two parts in each partition simply reverse their places and our proof is the same by symmetry. We use the identified inequalities and Lemma 5.3.6 to prove that $\binom{n-p}{2} + \binom{p}{2} > \binom{n-r}{2} + \binom{r}{2}$. Using the correspondence between complete graphs and triangular numbers

$$\binom{n-p}{2} + \binom{p}{2} > \binom{n-r}{2} + \binom{r}{2} \Rightarrow \Delta_{n-p} + \Delta_p > \Delta_{n-r} + \Delta_r.$$

Hence, we need to show that

$$(\Delta_{n-p} + \Delta_p) - (\Delta_{n-r} + \Delta_r) > 0.$$

By Lemma 5.3.6

$$\begin{aligned}
(\Delta_{n-p} + \Delta_p) - (\Delta_{n-r} + \Delta_r) &= (\Delta_n + \Delta_p - p(n-1) + \Delta_p) \\
&\quad -(\Delta_n + \Delta_r - r(n-1) + \Delta_r) \\
&= \Delta_n + 2\Delta_p - p(n-1) - (\Delta_n + 2\Delta_r - r(n-1)) \\
&= 2\Delta_p - p(n-1) - (2\Delta_r - r(n-1)) \\
&= 2\binom{p}{2} - pn + p - (2\binom{r}{2} - rn + r) \\
&= 2\frac{p(p-1)}{2} - pn + p - (2\frac{r(r-1)}{2} - rn + r) \\
&= p^2 - p - pn + p - (r^2 - r - rn + r) \\
&= p^2 - pn - r^2 + rn = p^2 - r^2 + rn - pn \\
&= (p-r)(p+r) + n(r-p) = (p-r)(p+r) - n(p-r) \\
&= (p-r)(p+r-n).
\end{aligned}$$

Since $r > p$, $(p - r) < 0$. Also, $r, p \leq \lfloor \frac{n}{2} \rfloor$. But $p < r$, so $p \leq \lfloor \frac{n}{2} \rfloor - 1$.

Hence,

$$\begin{aligned}
r + p &\leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - 1 \\
&\leq \frac{n}{2} + \frac{n}{2} - 1 \leq n - 1 \\
&< n.
\end{aligned}$$

Thus, $(r+p-n)<0$ and $(p-r)(p+r-n)>0$.

We conclude that there is no partition of k into two triangular numbers such that $\binom{n-p}{2}+\binom{p}{2}<k<\binom{n-p+1}{2}$, given the bounds identified, and therefore there is no partition of K_n into two complete subgraphs such that there will be more than $\binom{n-p}{2}+\binom{p}{2}$ and less than $\binom{n-p+1}{2}$ bad edges. $\square$

Proposition 5.3.9. *Let K_n be a complete graph and p an integer such that $p \geq 1$ and $\frac{p^2+p+4}{2} \leq n$. Then there is no partition of K_n into three or more complete subgraphs such that there will be more than $\binom{n-p}{2}+\binom{p}{2}$ and less than $\binom{n-p+1}{2}$ bad edges.*

Proof. We proceed to prove that there is no disjoint union of three or more complete subgraphs that will give us more bad edges on n vertices than we get from the partition $\{K_{n-p}, K_p\}$ and fewer bad edges than when we choose K_{n-p+1} as the induced closed subgraph with $\binom{n-p+1}{2}$ edges.

There is a total of $\binom{n}{2}$ edges in K_n. Choosing a closed set of edges as bad edges partitions the edge set, one part of the total set of edges will be bad and the remainder will be "good". Hence, in order to prove $\binom{n-p}{2}+\binom{p}{2}$ yields more bad edges than any partition of K_n into three (or more) complete subgraphs, it is sufficient to show that the number of "good" edges in the partition with larger number of blocks is greater than the number of good edges in the partition $\{K_{n-p}, K_p\}$, given certain bounds which arise naturally from the proposition statement.

We start with two complete subgraphs K_{n-p} and K_p. We have $(n-p)p$ good edges, call these red edges. We partition K_{n-p} into two complete subgraphs K_{n-r} and K_{n-q}. This gives the partition I in Figure 5.2. There are $(n-q)p$ and $(n-r)p$ good edges between K_p and K_{n-q} and K_{n-r} respectively and $(n-q)(n-r)$ good edges between K_{n-q} and K_{n-r}. Call the good edges between the latter two graphs green edges. Clearly we have more good edges in our partition $\{K_{n-r}, K_{n-q}, K_p\}$ than in our partition $\{K_{n-p}, K_p\}$. Also $(n-q)p+(n-r)p=(n-p)p$ since the good edges between K_p and the other two graphs remain constant no matter how we split the vertices in K_{n-p} (recall that these are red edges). Similarly we partition K_p into two subgraphs K_q and K_{p-q} as in partition II in Figure 5.2. What should be clear is that the "red" edges remain the same as in the original $\{K_{n-p}, K_p\}$ partition and the "green" edges resulting from our partitions I and II are extra. So we have

$$(n-q)(n-r)+(n-q)p+(n-r)p > (n-p)p \text{ in Partition } I$$

and

$$(p-q)q+(n-p)(p-q)+(n-p)q > (n-p)p \text{ in Partition } II.$$

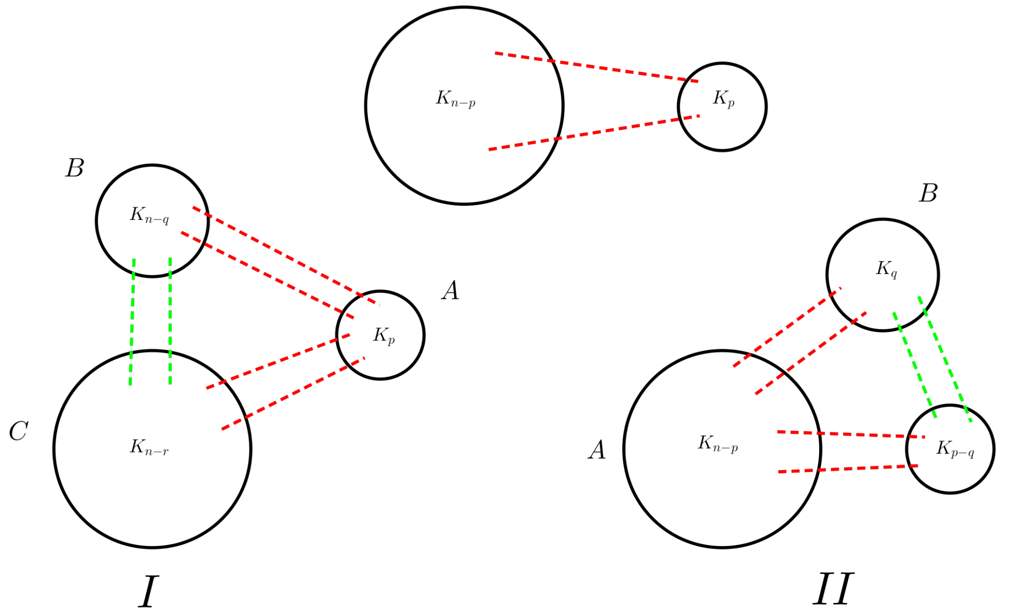

Figure 5.2: We divide K_{n-p} into two subgraphs (I) and K_p into two subgraphs (II).

Now partition K_n into three complete subgraphs. We use all the n vertices in the partition in order to get as many bad edges as possible. Then we proceed to maximise the number of bad edges by minimising the number of good edges. We show that the result is either a partition like I or II in Figure 5.2, in which case the proposition holds, or a partition of K_n into two complete subgraphs and we have the same case as Proposition 5.3.8.

Label the vertex partitions A, B and C. The induced subgraphs are all complete graphs and the edges in each of the partitions are bad edges and the vertices all the same colour. The edges between the partitions are good edges. If $|A|$, $|B|$ or $|C| = n-p$ or p then we have partition I or II and the proposition holds, so we will assume that $|A|$, $|B|$ or $|C| \neq n-p$ nor p. We must have $|A|$, $|B|$ and $|C| \leq n-p$ otherwise we have at least $\binom{n-p+1}{2}$ bad edges and this falls outside the proposition statement. Since we are assuming $|A|$, $|B|$ or $|C| \neq n-p$, this implies that $|A|$, $|B|$ and $|C| < n-p$.

We minimise the good edges by moving vertices from one block to another. Move a vertex by changing its colour to that of another block. Without loss of generality, we move a vertex u, say, from A to B. The bad edges incident on u in A now become good edges, and the good edges incident on u from B become bad edges. The good edges incident on u from C remain good edges since the vertices in C do not change colour and remain the same in number. Hence, in order to reduce the number of good edges we should always move a vertex from a smaller block to a larger or equal block.

It should be clear that reducing good edges in this manner leads to three possible outcomes.

1. $|A|$, $|B|$ and $|C| < n-p$, so it is possible to move vertices between blocks until one of them has $(n-p)$ vertices, in which case we are either back at our original partition, $\{K_{n-p}, K_p\}$, or we have a partition as in II with relabeling of the blocks and the inequality holds.

2. It may be that $|A|$, $|B|$ or $|C| < p$, so it is possible to move vertices between blocks until one of them has p vertices in which case we are either back at our original partition, $\{K_{n-p}, K_p\}$, or we have a partition as in I with relabeling of the blocks and the inequality holds. It is possible to move on from this configuration to get the configuration as in case 1.

3. We move vertices between blocks until one of the blocks is completely empty and the remaining two blocks each has less than $(n-p)$ vertices. But here we have the same case as in Proposition 5.3.8 and we know that we have more good edges than in the partition $\{K_{n-p},\, K_p\}$, so the inequality holds.

□

We use a similar argument to show that the proposition holds even if we are using a larger number of blocks in our partitions.

5.3.1 An algorithm for finding the k-defect polynomials equal to zero in K_n using the triangular numbers.

In this section we use triangular number partitions of an integer k in order to determine for which complete graphs the k-defect polynomial is equal to zero. We set out the method as an algorithm and use the algorithm to generate a table of ordered pairs, (k, n), with n the order of the smallest complete graph in which the k-defect polynomial will be non-zero, implying that for all K_i such that $i < n$, the k-defect polynomial will be equal to zero.

In Sections 3.3 and 3.4 we set up a correspondence between the triangular number partitions of an integer k and the closed sets in a complete graph that will give k bad edges in a bad colouring. For each triangular number in a given partition we can determine the corresponding number of vertices of the complete subgraph, using the formula $n = \frac{1+\sqrt{1+8\Delta_i}}{2}$ from Theorem 3.3.2. In this case n is the number of vertices of the complete subgraph corresponding to the triangular number in the partition block.

Thus, given a triangular number partition of an integer k, we can determine the number of vertices of the complete subgraphs making up a partition of a complete graph such that there will be k bad edges. By comparing the number of vertices from different partitions of k, we can find a minimum number of vertices. The minimum number of vertices translates to the smallest complete graph in which there will be k

bad edges. Call this minimum the *minimum vertex number* associated with k. Thus, given a complete graph K_n, if the minimum vertex number for a particular integer k is greater than n, then the k-defect polynomial for K_n will be zero.

Example 5.3.10. We recall the bad colouring of K_3 from Section 1.6.

Since K_3 has three edges we can have 0, 1, 2, or 3 bad edges. The triangular number partition for 2 is $1 + 1$. This corresponds to a vertex partition $2/2$, that is, two disjoint copies of K_2. In other words, we would need a complete graph with a minimum of four vertices in order to have two bad edges. Since there are only three vertices in K_3 the 2-defect polynomial for K_3 is zero.

To sum up, determining the number of vertices corresponding to each partition of an integer k enables us to find a minimum number of vertices for a complete graph to have k bad edges. If the minimum number of vertices is greater than n, the k-defect polynomial will be equal to zero in K_n.

Alternatively, we can write out an algorithm to determine for which complete graphs a certain k-defect polynomial will be equal zero.

S-i. Find all triangular number partitions of k using triangular numbers less than or equal to k.

S-ii. For each block in the number partition, determine the corresponding number of vertices using the formula $i = \frac{1+\sqrt{1+8\Delta_i}}{2}$. This is the number of vertices of a complete subgraph, K_i of some larger complete graph K_n.

S-iii. Sum the vertices over the blocks for each partition.

S-iv. If the minimum number of vertices corresponding to the partitions of k is greater than n, then that k-defect polynomial will be zero in K_n.

Example 5.3.11. We illustrate the use of this algorithm to find the smallest complete graph in order to have the 18-defect polynomial non-zero. The *Mathematica* code showing the calculation for $k = 18$ can be found in Appendix A.

S-i. Partition 18 using the triangular numbers $\{1, 3, 6, 10, 15\}$. We use the exponent as a shorthand for repeated addition as we did in Table 4.1.

$$\begin{aligned} 18 = \quad & 15+3;\ 15+1^3;\ 10+6+1^2;\ 10+3^2+1^2;\ 10+3+1^5;\ 10+1^8; \\ & 6^3;\ 6^2+3^2;\ 6^2+3+1^3;\ 6^2+1^6;\ 6+3^4;\ 6+3^3+1^3;\ 6+3^2+1^6; \\ & 6+3+1^9;\ 6+1^{12};\ 3^6;\ 3^5+1^3;\ 3^4+1^6;\ 3^3+1^9;\ 3^2+1^{12}; \\ & 3+1^{15};\ 1^{18} \end{aligned}$$

S-ii. The corresponding number of vertices per block in each partition.

$$\begin{aligned} \pi(v) = \quad & 6/3;\ 6/2/2/2; \\ & 5/4/2/2;\ 5/3/3/2/2;\ 5/3/2/2/2/2/2; \\ & 5/2/2/2/2/2/2/2/2; \\ & 4/4/4;\ 4/4/3/3;\ 4/4/3/2/2/2;\ 4/4/2/2/2/2/2/2; \\ & 4/3/3/3/3;\ 4/3/3/3/2/2/2; \\ & 4/3/3/2/2/2/2/2/2;\ 4/3/2/2/2/2/2/2/2/2/2; \\ & 4/2/2/2/2/2/2/2/2/2/2/2/2; \\ & 3/3/3/3/3/3;\ 3/3/3/3/3/2/2/2; \\ & 3/3/3/3/2/2/2/2/2/2; \\ & 3/3/3/2/2/2/2/2/2/2/2/2; \\ & 3/3/2/2/2/2/2/2/2/2/2/2/2/2; \\ & 3/2/2/2/2/2/2/2/2/2/2/2/2/2/2/2; \\ & 2/2/2/2/2/2/2/2/2/2/2/2/2/2/2/2/2/2 \end{aligned}$$

S-iii. The corresponding vertex sums are:

$\{9, 12, 13, 15, 18, 21, 12, 14, 17, 20, 16, 19, 22, 25, 28, 18, 21, 24, 27, 30, 33, 36\}$

S-iv. The minimum number of vertices is 9 and we can confirm that the 18-defect polynomial for K_8 is 0, but that it does appear in the bad colouring polynomial for K_9. This of course implies that the 18-defect polynomial is also equal to zero for all K_n where $n \leq 8$ and non-zero for all K_n such that $n \geq 9$, since K_9 is a subgraph of all complete graphs K_n where $n \geq 9$.

Table 5.1 shows ordered pairs (k, n) for integers k and their associated minimum vertex number as calculated in Example 5.3.11. We used Mathematica to generate the first 153 such ordered pairs. The code is included in Appendix B.

We can use Table 5.1 to find k-defect polynomials equal to zero for all complete graphs K_n where $n \leq 18$, since $|E(K_{18})| = \binom{18}{2} = 153$.

Example 5.3.12. We identify the k-defect polynomials of K_6 that are equal to zero.

Since K_6 has $\binom{6}{2} = 15$ edges, we look at the first fifteen order pairs. We note that for $k = \{5, 8, 9, 11, 12, 13, 14\}$ we have $n > 6$ and hence the k-defect polynomials for these values of k are equal to zero, as shown in Section 5.3.

For interest, we display the information from Table 5.1 graphically in Appendix C. The chart shows all values of k for which the k-defect polynomials are equal to zero for a particular K_n, $1 \leq n \leq 18$.

(1,2),	(2,4),	(3,3),	(4,5),	(5,7),	(6,4),	(7,6),	(8,8),
(9,7),	(10,5),	(11,7),	(12,8),	(13,8),	(14,10),	(15,6),	(16,8),
(17,10),	(18,9),	(19,11),	(20,10),	(21,7),	(22,9),	(23,11),	(24,10),
(25,11),	(26,13),	(27,11),	(28,8),	(29,10),	(30,12),	(31,11),	(32,13),
(33,15),	(34,12),	(35,14),	(36,9),	(37,11),	(38,13),	(39,12),	(40,14),
(41,16),	(42,13),	(43,14),	(44,16),	(45,10),	(46,12),	(47,14),	(48,13),
(49,15),	(50,17),	(51,14),	(52,16),	(53,18),	(54,17),	55,11),	(56,13),
(57,15),	(58,14),	(59,16),	(60,16),	(61,15),	(62,17),	(63,19),	(64,17),
(65,16),	(66,12),	(67,14),	(68,16),	(69,15),	(70,17),	(71,19),	(72,16),
(73,18),	(74,20),	(75,19),	(76,17),	(77,19),	(78,13),	(79,15),	(80,17),
(81,16),	(82,18),	(83,19),	(84,17),	(85,19),	(86,21),	(87,19),	**(88,18)**,
(89,20),	*(90,20)*,	**(91,14)**,	(92,16),	(93,18),	(94,17),	(95,19),	(96,21),
(97,18),	*(98,20)*,	*(99,20)*,	*(100,21)*,	*(101,19)*,	*(102,21)*,	*(103,22)*,	*(104,22)*,
(105,15),	(106,17),	(107,19),	**(108,18)**,	*(109,20)*,	*(110,22)*,	*(111,19)*,	*(112,21)*,
(113,23),	*(114,22)*,	*(115,20)*,	*(116,22)*,	*(117,23)*,	*(118,23)*,	*(119,22)*,	**(120,16)**,
(121,18),	*(122,20)*,	*(123,19)*,	*(124,21)*,	*(125,23)*,	*(126,20)*,	*(127,22)*,	*(128,24)*,
(129,23),	*(130,21)*,	*(131,23)*,	*(132,24)*,	*(133,23)*,	*(134,25)*,	*(135,22)*,	**(136,17)**,
(137,19),	*(138,21)*,	*(139,20)*,	*(140,22)*,	*(141,23)*,	*(142,21)*,	*(143,23)*,	*(144,25)*,
(145,24),	*(146,22)*,	*(147,24)*,	*(148,24)*,	*(149,25)*,	*(150,25)*,	*(151,23)*,	*(152,25)*,
(153,18)							

Table 5.1: Ordered pairs of integers (k, n).

Example 5.3.13. We use Table 5.1 to verify Theorem 5.3.7 for K_{18}.

According to Theorem 5.3.7 the k-defect polynomials of a complete graph, K_n, are all equal to zero if

$$\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2},$$

where $p \geq 1$ and $\frac{p^2+p+4}{2} \leq n$.

Recall the inequality following Lemma 5.3.5 that $\frac{p^2+p+4}{2} \leq n$ implies that

$$p \leq \lfloor \frac{-1+\sqrt{8(18)-15}}{2} \rfloor = \lfloor \frac{-1+\sqrt{129}}{2} \rfloor = \lfloor \frac{-1+11,35\ldots}{2} \rfloor = 5.$$

For ease of reference we include Table 5.2, listing the intervals for the values of $1 \leq p \leq 5$ on which the k-defect polynomials are equal to zero for K_{18}.

p	$\binom{n-p}{2} + \binom{p}{2}$	$\binom{n-p+1}{2}$
1	136	153
2	121	136
3	108	120
4	97	105
5	88	91

Table 5.2: Values of p and intervals on which the k-defect polynomials are equal to zero.

The endpoints of the intervals from Table 5.2 are in bold in Table 5.1 and the ordered pairs for which the minimum vertex number is greater than 18, that is where the k-defect polynomials are zero on these intervals, are italicised. Note that we have *not* italicised those pairs for values of k that fall outside the intervals determined in Theorem 5.3.7, but for which the k-defect polynomials are equal to zero for K_{18}.

Thus, for $88 < k < 91$, the associated minimum vertex numbers in the ordered pairs, $(89, 20)$, $(90, 20)$, are both greater than 18. For $97 < k < 105$ the ordered pairs are $(98, 20)$, $(99, 20)$, $(100, 21)$, $(101, 19)$, $(102, 21)$, $(103, 22)$, and $(104, 22)$, all with minimum vertex numbers greater than 18. Continuing in this manner we can confirm that the k-defect polynomials for

$$\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2} \quad , \quad \frac{p^2+p+4}{2} \leq n$$

are all zero, where $1 \leq p \leq 5$ and $n = 18$.

Similarly, we can use Table 5.1 to verify Theorem 5.3.7 for all K_n where $n \leq 18$.

5.3.2 A lower bound on the number of k-defect polynomials that are equal to zero.

Finally, we state and prove a lower bound for the number of k-defect polynomials which are equal to zero. We need the following two identities on triangular numbers which we state as lemmas. We note that Lemma 5.3.14 has been known since at least 1261, see [5], and Lemma 5.3.6, listed in [21], was proved in Section 5.3. As mentioned before, the identities in the cited sources are slightly different, since the usual definition of a triangular number does not include zero. Since our definition of a triangular number *does* include zero as the first triangular number, we include here a proof of Lemma 5.3.14 as well.

Lemma 5.3.14.

$$\sum_{i=1}^{p} \Delta_i = \frac{(p-1)p(p+1)}{6}$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{p} \Delta_i &= \sum_{i=1}^{p} \binom{i}{2} = \sum_{i=1}^{p} \frac{i!}{2!(i-2)!} = \sum_{i=1}^{p} \frac{i(i-1)}{2} = \sum_{i=1}^{p} \frac{i^2-i}{2} \\
&= \frac{1}{2}\left[\sum_{i=1}^{p} i^2 - \sum_{i=1}^{p} i\right] = \frac{1}{2}\left[\frac{p(p+1)(2p+1)}{6} - \frac{p(p+1)}{2}\right] \\
&= \frac{1}{2}\left[\frac{p(p+1)(2p+1)-3p(p+1)}{6}\right] = \frac{1}{2}\left[\frac{p(p+1)(2p+1-3)}{6}\right] \\
&= \frac{1}{2}\left[\frac{p(p+1)(2p-2)}{6}\right] = \frac{(p-1)p(p+1)}{6}
\end{aligned}$$

□

For ease of reference we restate Lemma 5.3.6.

$$\Delta_{n-i} = \Delta_n + \Delta_i - i(n-1).$$

Theorem 5.3.15. *The number of k-defect polynomials of K_n that are equal to zero is at least $p(n-1) - \frac{p(p+1)(p+2)}{6}$, where $p = max\{1 \leq p \leq \lfloor \frac{-1+\sqrt{8n-15}}{2} \rfloor\}$.*

Proof. We know from Theorem 5.3.7 that the k-defect polynomials for all integers $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$ for $1 \leq p \leq \lfloor \frac{-1+\sqrt{8n-15}}{2} \rfloor$ are equal to zero, so we will count this number of integers over all p on the interval.

There are $\binom{n-p+1}{2} - \left(\binom{n-p}{2} + \binom{p}{2}\right) - 1$ integers between $\binom{n-p+1}{2}$ and $\binom{n-p}{2} + \binom{p}{2}$ so the total number of integers is given by

$$\sum_{i=1}^{p} \left(\binom{n-i+1}{2} - \binom{n-i}{2} - \binom{i}{2} - 1 \right).$$

By our definition of triangular numbers this is equivalent to

$$\sum_{i=1}^{p} (\Delta_{n+1-i} - \Delta_{n-i} - \Delta_i - 1),$$

where Δ_i is the i-th triangular number.

We recall from Definition 3.2.1 that $\Delta_{n+1} = \Delta_n + n$ and use the identities from Lemmas 5.3.14 and 5.3.6 to evaluate the sum.

$$\begin{aligned}
&\sum_{i=1}^{p} (\Delta_{n+1-i} - \Delta_{n-i} - \Delta_i - 1) \\
&\qquad = \sum_{i=1}^{p} (\Delta_{n+1} + \Delta_i - in - (\Delta_n + \Delta_i - i(n-1)) - \Delta_i - 1) \\
&\qquad = \sum_{i=1}^{p} (\Delta_{n+1} - \Delta_n - \Delta_i - i - 1) \\
&\qquad = \sum_{i=1}^{p} (n - \Delta_i - i - 1) \\
&\qquad = pn - \frac{(p-1)p(p+1)}{6} - \frac{p(p+1)}{2} - p \\
&\qquad = p(n-1) - \frac{(p)(p+1)(p+2)}{6}.
\end{aligned}$$

In addition, we know that there are some k-defect polynomials outside the intervals identified in Theorem 5.3.7 that are equal to zero. Without loss of generality, let $n = 6$ then $p = 2$. By Theorem 5.3.7 this means that all the k-defect polynomials for $7 < k < 10$ as well as $10 < k < 15$ are equal to zero. Evaluating $p(n-1) - \frac{p(p+1)(p+2)}{6}$ this gives 6 k-defect polynomials that are equal to zero. However, the 5-defect polynomial in K_6 is also 0, as shown in Example 5.3.12, but 5 falls outside the intervals determined by Theorem 5.3.7. Thus $p(n-1) - \frac{p(p+1)(p+2)}{6}$ gives a lower bound on the number of k-defect polynomials that are equal to zero in K_n.

□

Example 5.3.16. We look at the number of k-defect polynomials of K_{18} that are equal to zero.

Since $n = 18$,

$$p = \lfloor \frac{-1 + \sqrt{8(18) - 15}}{2} \rfloor = \lfloor \frac{-1 + \sqrt{129}}{2} \rfloor = \lfloor \frac{-1 + 11,35\ldots}{2} \rfloor = 5.$$

Theorem 5.3.15 states that at least $5(18-1) - \frac{5(5+1)(5+2)}{6} = 50$ of the k-defect polynomials are equal to zero. Recall that the italicised ordered pairs in Table 5.1 correspond to the k-defect polynomials equal to zero for all k on the intervals identified in Theorem 5.3.7. We can verify by counting that there are indeed 50 italicised ordered pairs in the table. We also note from the table that there are a further twelve k-defect polynomials that are equal to zero, namely those for $k = \{63, 71, 74, 75, 77, 83, 85, 86, 87, 95, 96, 107\}$.

5.4 Conclusion

In this chapter we identified some values of k such that the k-defect polynomial of a graph G is equal to zero. By applying the relationship between closed sets of complete graphs and partitions of integers into triangular numbers, we identified intervals of integers on which the k-defect polynomials are equal to zero for complete graphs, one of the main results of this chapter. We then developed an algorithm to calculate the minimum order for a complete graph to have a k-defect polynomial not equal to zero for a given value of k. In addition, using known summation formulae for triangular numbers, we determined a lower bound on the number of k-defect polynomials that are equal to zero in a complete graph.

Chapter 6

Conclusion

In this dissertation we set out to look at improper colourings of graphs and noted that the much studied chromatic polynomial, giving the number of proper λ colourings of a graph, is generalised by the bad colouring or k-defect polynomials. That is, if $\phi_k(\lambda)$ is the polynomial that gives us the λ colouring of a graph with k bad edges, then $\phi_0(\lambda)$ is the chromatic polynomial. This is not just generalisation for generalisation's sake, however, since the k-defect polynomials of a graph generate the bad colouring polynomial which is equivalent to the Tutte polynomial. The Tutte polynomial is a very important graph invariant that encodes a large amount of information about a graph. Furthermore, the Potts Partition function, a very important function in statistical mechanics, is an evaluation of the Tutte polynomial equivalent to the bad colouring polynomial. Thus we looked specifically at the definitions of the dichromatic, Tutte (dichromate), bad colouring and k-defect polynomials of a graph G. We looked at different ways in the literature of calculating these polynomials and pointed out their equivalent evaluations.

In Chapter 3 we showed that sets of bad edges are closed sets of a graph and specifically that closed sets of a complete graph are complete subgraphs or disjoint unions of complete subgraphs. By setting up a one-to-one correspondence between complete graphs and triangular numbers, we were able to state and prove one of the main results of this dissertation on the relationship between sizes of closed sets of complete graphs and triangular number partitions of an integer k. We note in passing that the underlying complete graph has a specific meaning with respect to the mean field in the Potts model mentioned before.

The relationship between triangular number partitions and complete graphs enabled us in Chapter 4 to develop an algorithm for calculating a k-defect polynomial of a complete graph, using triangular number partitions of k. We used this algorithm to generate an expression for any k defect polynomial of a complete graph.

Finally, in Chapter 5, we identified some intervals on which the k-defect polynomial of a complete graph is equal to zero, using the theory of triangular number partitions developed in the previous two chapters. In addition we determined a lower bound on the number of k for the k-defect polynomials that are equal to zero using well known summation formulae for the triangular numbers.

From this study has also emerged some further problems that may be interesting to look at and merit further investigation.

The intervals identified in Chapter 5 are not *all* the intervals on which the k-defect polynomials of a complete graph are equal to zero. Hence, the formula developed gives only a lower bound. An interesting next step would be to try to determine where all the k-defect polynomials equal to zero of a complete graph lie and to find an exact number. Looking at the distribution of these values of k for the first 20 complete graphs we can see a further pattern of zero's emerging as n increases, but that there are also still zero's that do not fit even this pattern.

We also noted that in a tree none of the k-defect polynomials are equal to zero, while in a cycle we have exactly one such value for k. The matter seems to become more and more complex as we add cycles on three vertices, until we maximise this situation in complete graphs where every subset of three vertices induces a cycle. An interesting problem would be to determine whether there is a correspondence between the number of such cycles in a graph and the number k-defect polynomials that are equal to zero, or whether there are other structural properties of a graph that influence the number of zero k-defects. Do the intervals on which these k occur in complete graphs give an indication where they might occur in other classes of graphs? Also, since the Tutte and bad colouring polynomials are equivalent, do the zero k-defects of a graph give any information as to the coefficients of the Tutte polynomial?

Lastly, there are classes of graphs for which we have closed form expressions for the chromatic polynomial, such as trees, cycles, wheels and complete graphs, to name but four. For trees and cycles a closed form expression for the k-defect polynomial has been found, but these are the only classes of graphs. For complete graphs we can generate such an expression for $k = 1$, but after that it becomes complicated. The nearest we have come in this dissertation is the expression developed in Chapter 4. A closed form expression for the k-defect polynomials of different classes of graphs may lead us to new insights on the zero's of the bad colouring polynomial and hence the Potts Partition function.

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Appendix A

Mathematica code: calculating minimum vertices for 18 bad edges

Below is the *Mathematica* code showing the calculation of the minimum number of vertices, that is, the smallest order of a complete graph, necessary for $k = 18$ bad edges.

In[15]:= f[n] := (1 + Sqrt[1 + 8 n])/2

In[16]:= a = Range[2, 18]

Out[16]= {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}

In[17]:= b = Binomial[a, 2]

Out[17]= {1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153}

In[18]:= e = 18

Out[18]= 18

In[19]:= c = IntegerPartitions[e, All, b]

Out[19]= {{15, 3}, {15, 1, 1, 1}, {10, 6, 1, 1}, {10, 3, 3, 1, 1}, {10, 3, 1, 1, 1, 1, 1}, {10, 1, 1, 1, 1, 1, 1, 1, 1}, {6, 6, 6}, {6, 6, 3, 3}, {6, 6, 3, 1, 1, 1}, {6, 6, 1, 1, 1, 1, 1, 1}, {6, 3, 3, 3, 3}, {6, 3, 3, 3, 1, 1, 1}, {6, 3, 3, 1, 1, 1, 1, 1, 1}, {6, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 3, 3, 3}, {3, 3, 3, 3, 3, 1, 1, 1}, {3, 3, 3, 3, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

In[20]:= f[c]

Out[20]= {{6, 3}, {6, 2, 2, 2}, {5, 4, 2, 2}, {5, 3, 3, 2, 2}, {5, 3, 2, 2, 2, 2, 2}, {5, 2, 2, 2, 2, 2, 2, 2, 2}, {4, 4, 4}, {4, 4, 3, 3}, {4, 4, 3, 2, 2, 2}, {4, 4, 2, 2, 2, 2, 2, 2}, {4, 3, 3, 3, 3}, {4, 3, 3, 3, 2, 2, 2}, {4, 3, 3, 2, 2, 2, 2, 2, 2}, {4, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2}, {4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}, {3, 3, 3, 3, 3, 3}, {3, 3, 3, 3, 3, 2, 2, 2}, {3, 3, 3, 3, 2, 2, 2, 2, 2, 2}, {3, 3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2}, {3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}, {3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}, {2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}}

In[21]:= Total /@ f[c]

Out[21]= {9, 12, 13, 15, 18, 21, 12, 14, 17, 20, 16, 19, 22, 25, 28, 18, 21, 24, 27, 30, 33, 36}

In[22]:= d = Min[%]

Out[22]= 9

Appendix B

Mathematica code: Ordered pairs for k and n

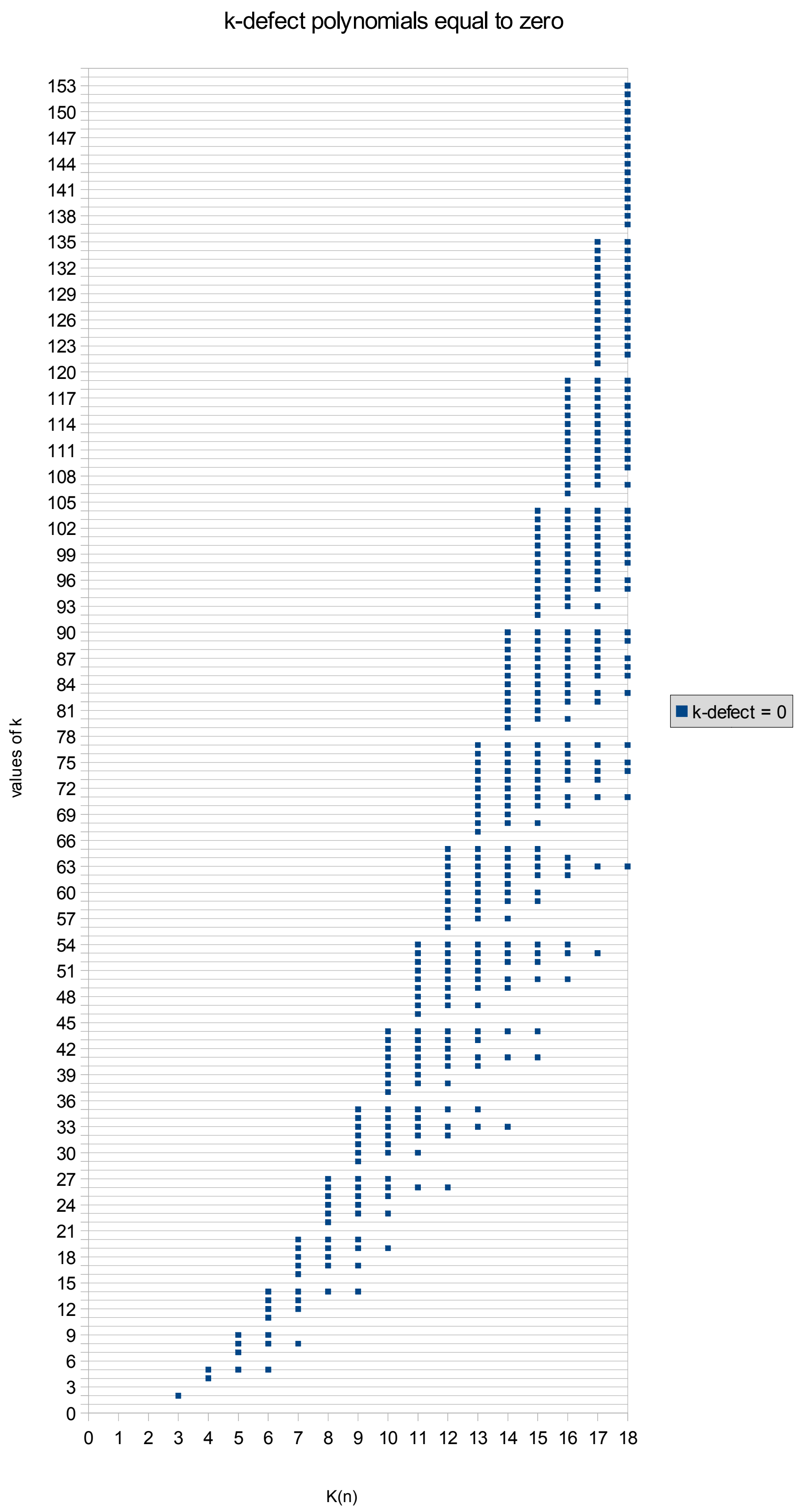
Below is the *Mathematica* code showing the calculation of ordered pairs (k, n) where n is the minimum number of vertices necessary for a complete graph to have the k-defect polynomial non-zero.

```
In[4]:= f[n_] := (1 + Sqrt[1 + 8 n])/2
In[5]:= a = Range[2, 18]
Out[5]= {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
In[6]:= b = Binomial[a, 2]
Out[6]= {1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153}
In[7]:= g[n_] := IntegerPartitions[n, All, b]
In[8]:= h[n_] := f[g[n]]
In[9]:= j[n_] := Total /@ h[n]
In[10]:= k[n_] := Min[j[n]]
In[11]:= Do[Print["(", n, ",", k[n], ")"], {n, 1, 153}]
```

Appendix C

k-defect polynomials of K_n equal to zero for $1 \leq n \leq 18$

The chart on the next page shows all integer values for $0 \leq k \leq 153$ and complete graphs K_n for $1 \leq n \leq 18$. The values for k where the k-defect polynomial for a particular complete graph is equal to zero is marked with a dot.


k-defect polynomials equal to zero
values of k
K(n)
k-defect = 0
0 3 6 9 12 15 18 21 24 27 30 33 36 39 42 45 48 51 54 57 60 63 66 69 72 75 78 81 84 87 90 93 96 99 102 105 108 111 114 117 120 123 126 129 132 135 138 141 144 147 150 153
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18