School of Mathematics

Masters Dissertation

Darboux Transformations on Sturm-Liouville Eigenvalue Problems with Eigenparameter Dependent Transmission Conditions

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg. In fulfillment of the requirements for the degree of Master of Science.

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Abstract

Sturm-Liouville eigenvalue problems arise prominently in mathematical physics. An innumerous amount of complexities have been encountered in solving these problems and a myriad of techniques have been explored over the century. In this work, we investigate one such technique, namely the Darboux-Crum transformation. This transformation transforms an existing problem into one that is readily solvable or displays properties that are better understood. In particular, we focus our attention on the effect the Darboux-Crum transformation has on the eigenparameter dependence of the transmission condition of our Sturm-Liouville eigenvalue problem.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Rakgwahla Jessica Phalafala

This 18th day of January 2018, at Johannesburg, South Africa.

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Contents

Chapter 1

Introduction

The branch of mathematical analysis in which differential operators are studied in great detail is called functional analysis. Historically, the core of functional analysis is the study of functions and, in particular, the study of spaces of functions. Today, it has become an extensive area of mathematics that can be described as the study of infinite-dimensional vector spaces endowed with a topology. This robust branch of mathematical analysis unifies various mathematical areas such as linear algebra and real/complex analysis.

Consider the Sturm-Liouville equation

$$\ell y := -y'' + qy = \lambda y, \text{ on } [-a, b],$$

in $L^2(-a, b)$, $a,\ b > 0$, for $q \in L^2(-a, b)$ with boundary conditions

$$y(-a) \cos \alpha = y'(-a) \sin \alpha, \tag{1.1}$$

$$y(b) \cos \beta = y'(b) \sin \beta, \tag{1.2}$$

where $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$, and transmission condition

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = M \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix},$$

where the entries of M may be eigenparameter dependent as Nevanlinna functions of the eigenparameter. Our main interest in this dissertation is to investigate the effect that the Darboux-Crum transformation has on the transmission matrix M. The effect of the Darboux-Crum transformation on the boundary conditions (1.1) and (1.2) is discussed in [3].

This dissertation is structured as follows. In this chapter we present an historical background, highlighting the origin of Sturm-Liouville boundary value problems and the inception of the Darboux-Crum transformation. We also explore the literature wherein the authors apply the Darboux-Crum transformation to Sturm-Liouville problems of a similar nature. Finally, we underline some real-world applications of these results.

In Chapter 2 we present an introduction to the theory of Herglotz-Nevanlinna functions in which we recall some basic definitions and properties that will better equip us to understand our transmission conditions. We discuss absolutely continuous functions and give a brief introduction to the theory of differential operators and their structure in a Hilbert space setting. We conclude the chapter with Sturm's two comparison theorems in our outline of Sturmian theory.

The focus of Chapter 3 is the computation of the effect of the forward Darboux transformation on the potential, q, and the transmission conditions. We begin by describing the effect of the transformation on the potential, therefore allowing us to conclude the effect of successive applications of the transformation on the potential. We show that, given an arbitrary initial transmission matrix with no restrictions on its entries, the forward transformation indeed increases the eigenparameter dependence of our transmission matrix. An increase in eigenparameter dependence is characterized by an increase in the number of poles and/or the presence of a non-trivial affine term in our transmission condition. The aforementioned result forms the basis of the rest of the chapter as it provides us with the formulae needed to conduct successive applications of the transformation, illustrating the eigenparameter dependence of our transmission matrix increases in half steps of Herglotz-Nevanlinna form. This result provides us with the structure of the hierarchy of Sturm-Liouville boundary value problems that is yielded by the forward transformation. The hierarchy is a sequence of Sturm-Liouville boundary value problems for which each step ascended in the hierarchy is characterised by an increased eigenparameter dependence of the transmission matrix.

In Chapter 4 we compute the inverse Darboux transformation and study its effect on the potential and transmission conditions of the boundary value problem. Similar to the case of Chapter 4, we give the effect of the inverse transformation on the potential of the boundary value problem after successive applications of the transformation. Given an arbitrary initial transformation, we show how the inverse transformation, like the forward transformation, increases the eigenparameter dependence of our transmission matrix. Using the aforementioned result, together with a particular choice of the transformation parameters, we then illustrate how the inverse transformation can decrease the eigenparameter dependence of the transmission matrix in half steps of Herglotz-Nevanlinna form. The above transformations provide a mapping that results in movement down the hierarchy of Sturm-Liouville boundary value problems with eigenparameter dependent transmission conditions to a Sturm-Liouville problem with eigenparameter independent transmission conditions.

In Chapter 5 we formulate the Sturm-Liouville boundary value problem with eigenparameter dependent transmission conditions first in differential equation form. Secondly, we use this formulation to pose these boundary value problems together with their transmission conditions in Pontryagin and Hilbert space settings by defining operators together with their respective domains for each class of the transmission conditions. We proceed to prove that the resulting operators in each case are symmetric.

Lastly, in Chapter 6 we discuss further work in this topic and a short description of what this work would entail.

1.1 Historical Background

The study of differential equations began in the late 17th century when it was discovered that various physical problems could be described and solved using equations that involved both a function and its derivatives. Isaac Newton was the first to classify these first order differential equations into three classes. The first two classes categorised ordinary differential equations and the third class involved what we now call partial differential equations. The search for general methods of solving various classes of differential equations proceeded for centuries with various classes proving more difficult to solve than others, [23].

The soliton theory originated in the study of non-linear waves and has interested mathematicians and physicists since the early nineteenth century. A soliton is a stable self-reinforcing wave that can be found in nature and has numerous scientific and technological applications. John Scott Russell was the first to describe the notion of a soliton after recording a sighting of a solitary water wave, or what he then named a Wave of Translation, along a canal in 1834, see [22], with mathematical approximations given by Boussinesq in [5] and Rayleigh in [19] in 1872 and 1876 respectively. In later developments, explicit solutions of nonlinear partial differential equations were found using methods from soliton theory, [15].

Nonlinear partial differential equations are common in scientific problems, however, there are few cases where the solutions can be expressed in explicitly. The inverse scattering method and Bäcklund transformation are the most popular methods for finding explicit solutions for soliton equations. However, these methods can only be employed where the nonlinear partial differential equation satisfies certain conditions. In the case of the inverse scattering method, explicit solutions cannot be derived if the kernel of the integral equation is not degenerate, [15]. Furthermore, a "nonlinear superposition formula" is generated in the Bäcklund transformation in order to replace the superposition principle of the linear case, [15]. It should be noted that this nonlinear superposition formula is generally difficult to derive. As a result, an additional class of transformations from the nineteenth century, namely, the Darboux transformations, were applied and found to also be effective for finding explicit solutions for many partial differential equations.

In 1882, Jean Gaston Darboux produced a study of Sturm-Liouville problems focused on the parametric dependence on a linear scalar parameter, [17]. It was in his 1882 paper, that the method of Darboux transformations was introduced. The importance of the Darboux transformation lies in the fact that one can produce a new solvable Sturm-Liouville equation after applying this transformation on an initial solvable Sturm-Liouville equation, [18]. This is possible due to the fact that Darboux transformations can be described as maps between solutions of linear equations.

A century after Darboux's study, it was discovered that the method introduced in his 1882 paper could be extended to some soliton equations. In his seminal paper that was published in 1955, Crum introduced Crum transformations by constructing iterated Darboux transformations expressed in Wronskian type determinants in his study of Sturm-Liouville problems with boundary conditions, [8]. The Wronskian determinant is defined in [17] as follows:

Definition 1.1.1. *Let $u_1,\ u_2, \cdots,\ u_n$ be n solutions of the homogeneous equation of degree n,*

$$\boldsymbol{L}(u) = 0,$$

then the most general solution or complete primitive of this equation is

$$u = C_1 u_1 + C_2 u_2 + \cdots + C_n u_n$$

provided that the solutions $u_1,\ u_2, \cdots,\ u_n$ are linearly independent. Then the Wronskian of the functions $u_1,\ u_2, \cdots,\ u_n$ is given by the following determinant

$$\varDelta(u_1,\ u_2, \cdots,\ u_n) \equiv \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}.$$

The Crum transformation was later used to develop multi-soliton solutions of integrable equations, [22]. The modification and generalisation of Darboux transformations and their usefulness in the study of Sturm-Liouville problems has subsequently been explored in [1] and [11]. These papers paid particular attention to the transformation of the regular Sturm-Liouville equation. As a result of these twentieth century findings, Darboux and Crum transformations are standard references in nonlinear science and they play an important role in physics and mathematics.

1.2 Literature Review

The effect of Darboux type transformations on boundary conditions has recently been explored in [3] and [4]. In particular, the authors consider the regular Sturm-Liouville equation

$$ly := -y'' + qy = \lambda y \text{ on } [0, 1] \tag{1.3}$$

with $q \in L^1[0, 1]$, subject to the boundary conditions

$$y(0) \cos \alpha = y'(0) \sin \alpha, \quad \alpha \in [0, \pi) \tag{1.4}$$

and

$$\frac{y'}{y}(1) = f(\lambda), \tag{1.5}$$

where $f(\lambda)$ is a rational function of the form

$$f(\lambda) = \eta\lambda + \zeta - \sum_{k=1}^{N} \frac{\beta_k}{\lambda - \gamma_k} \ . \tag{1.6}$$

Here, $\eta \geq 0$, $\beta_k > 0$, $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ and all the coefficients are real. It should be noted that the boundary condition (1.5) is rationally dependent on the eigenparameter λ as illustrated by (1.6). This dependence on the eigenparameter takes the form of a Herglotz-Nevanlinna type rational function which has been defined in [3] as follows:

Definition 1.2.1. *A function $g : \mathbb{C} \to \mathbb{C}$ such that $g(\bar{z}) = \overline{g(z)}$ and g maps the closed upper half-plane into itself is called a Herglotz-Nevanlinna function.*

In [3], Binding et al. use differential equation techniques to prove various properties of the eigenvalues and norming constants of this Sturm-Liouville boundary value problem given by (1.3) - (1.5). They also make use of the modified Darboux transformation and analyse its effect on the boundary conditions. In addition, they show that the application of the modified Darboux transformation to (1.3) - (1.5) produces a new spectrum that consists of the old eigenvalues (except possibly the least eigenvalue λ_0). It is then proven, by oscillation theory, that the transformation is isospectral and the new problem is a simplification of the original one. Finally, they use iterated transformations to study eigenvalue asymptotics.

In [4], the authors use given spectral data to recover q, α and f and they refer to this as the inverse spectral problem. The spectral data used consists of the real sequence of eigenvalues, $\lambda_0 < \lambda_1 < \cdots$, and the norming constants which correspond to the eigenfunctions. They then set up a Hilbert space structure and found that (1.3) - (1.5) is a standard eigenvalue problem for a self-adjoint operator with compact resolvent. A key tool in their analysis is the Darboux-Crum transformation. They use this transformation successively on (1.3) - (1.5) to transform it to a Sturm-Liouville problem with λ independent boundary conditions.

The mathematical analysis of scattering theory focuses on the scattering of particles and waves. It is a significant area of interest for both mathematicians and physicists. In [10], Currie et al. investigate the forward scattering for the differential equation

$$\ell y := -\frac{d^2 y}{dx^2} + q(x)y = \zeta^2 y, \text{ on } (-\infty, 0) \cup (0, \infty) \tag{1.7}$$

in $L^2(-\infty,0)\oplus L^2(0,\infty)=L^2(\mathbb{R})$ with the point transfer condition

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix} \tag{1.8}$$

where $M_{ij}\in\mathbb{R}$ for $i,j=1,2,$, $det(M_{ij})=1$ and $q\in L^2(\mathbb{R})$ is assumed to be real-valued and obeying the growth condition

$$\int_{-\infty}^{\infty}(1+|x|)|q(x)|dx<\infty. \tag{1.9}$$

Note that the above point transfer condition (or transmission condition) is not eigenparameter dependent and will form the first step of our hierarchy of problems. Transfer conditions of this form are characteristic of scattering problems.

The authors in, [10], define the scattering data of the problem given by (1.7) - (1.8) in terms of the Jost solutions of (1.7) and express these Jost solutions in terms of the classical Jost solutions where the matrix M is the identity matrix. The Jost solutions are defined as follows in [7, p.297].

Definition 1.2.2. *The Jost solutions $f_{+,M}(x,\zeta)$ and $f_{-,M}(x,\zeta)$ are the solutions of (1.7) and (1.8) with*

$$\lim_{x\to\infty} e^{-i\zeta x} f_{+,M}(x,\zeta)=1,$$
$$\lim_{x\to-\infty} e^{i\zeta x} f_{-,M}(x,\zeta)=1.$$

Consequently they could draw conclusions about the functional analytic aspects of the operator L in $L^2(\mathbb{R})$ defined by $Ly=\ell y$ with a suitably specified domain. One of these conclusions being the fact that, under (1.9), the operator L produces a spectrum which consists of a finite number of negative and simple eigenvalues and that $[0,\infty)$ is the continuous spectrum of L, [10].

1.3 Applications

As mentioned in the above section, in mathematical physics, scattering theory is the study of the distribution of radiation or waves. In particular, the *forward scattering problem* is the problem of inferring the distribution of scattered radiation or waves based on the properties of the object or scatterer. Whereas, the *inverse scattering problem* is the problem of inferring properties of the object based on the distribution of the radiation or waves scattered from it.

These problems arise in areas as diverse as echolocation, medical imaging, non-destructive testing or evaluation of materials, space exploration, military weapon design and quantum field theory. A simple example of an inverse scattering problem lies in one of our human senses. We obtain vision of the objects surrounding us by our brains' ability to infer the properties of the objects based on the distribution of the light that enters our eyes. In some cases, incomplete information obtained from scattering can be used to determine the properties of a body. One such case is the use of scattering of x-rays to establish the structure and characteristics of DNA [7].

Profound advances have been made in applications involving homogeneous media by scientists in this field, however the treatment of inhomogeneous bodies is yet to be fulfilled. For example, oil cavities could be detected using scattering theory but the inhomogeneous nature of the earth's surface has made a precise detection onerous. It is these numerous applications that have sparked the interest of many scientists in this field.

Chapter 2

Preliminaries

In this chapter we give the preliminary material that forms the foundation of our research.

2.1 Herglotz-Nevanlinna Functions

Recall that we defined a Herglotz-Nevanlinna function in Chapter 1 as a function $g : \mathbb{C} \to \mathbb{C}$ such that $g(\bar{z}) = \overline{g(z)}$ and g maps the closed upper half-plane into itself, [3]. These functions, sometimes referred to as R-functions, play a critical role in the study of the spectral properties of boundary value problems.

We now list some properties of Herglotz-Nevanlinna functions which can also be found in [9, pp.3].

(i) The reciprocal of a positive Herglotz-Nevanlinna function, $f(\lambda)$, is the negative of a Herglotz-Nevanlinna function, that is

$$\frac{1}{f(\lambda)} = -g(\lambda),$$

where $g(\lambda)$ is a Herglotz-Nevanlinna function.

(ii) If

$$f(\lambda) = \sigma - \sum_{i=1}^{n} \frac{\alpha_i}{\lambda - \delta_i},\ \alpha_i > 0,\ \sigma \neq 0,$$

then

$$\frac{1}{f(\lambda)} = \zeta - \sum_{i=1}^{n} \frac{\beta_i}{\lambda - \gamma_i},\ \beta_i < 0,\ \zeta \neq 0.$$

This follows from $\lim\limits_{\lambda\to\infty} \frac{1}{f(\lambda)} = \frac{1}{\sigma} \in \mathbb{C}\backslash\{0\}$ and that $-\frac{1}{f(\lambda)}$ is Herglotz-Nevanlinna and $f(\lambda)$ has n zeros so $-\frac{1}{f(\lambda)}$ has n poles.

(iii) If

$$f(\lambda) = \eta\lambda + \zeta - \sum_{i=1}^{n-1} \frac{\alpha_i}{\lambda - \delta_i},\ \eta, \alpha_i > 0,$$

then

$$-\frac{1}{f(\lambda)} = -\sum_{i=1}^{n} \frac{\beta_i}{\lambda - \gamma_i},\ \beta_i > 0.$$

This follows from $f(\lambda)$ having n zeros giving $-\frac{1}{f(\lambda)}$ n poles, $f(\lambda) \to \pm\infty$ as $\lambda \to \pm\infty$ giving $-\frac{1}{f(\lambda)} \to 0$ as $\lambda \to \infty$, and $-\frac{1}{f(\lambda)}$ being Herglotz-Nevanlinna.

Figure 2.1 illustrates the graph of a Herglotz-Nevanlinna function $f(\lambda) = \eta\lambda + \zeta - \sum\limits_{i=1}^{n-1} \frac{\alpha_i}{\lambda - \delta_i}$ and Figure 2.2 illustrates the graph of a Herglotz-Nevanlinna function of the form $-\frac{1}{f(\lambda)} = -\sum\limits_{i=1}^{n} \frac{\beta_i}{\lambda - \gamma_i}$.

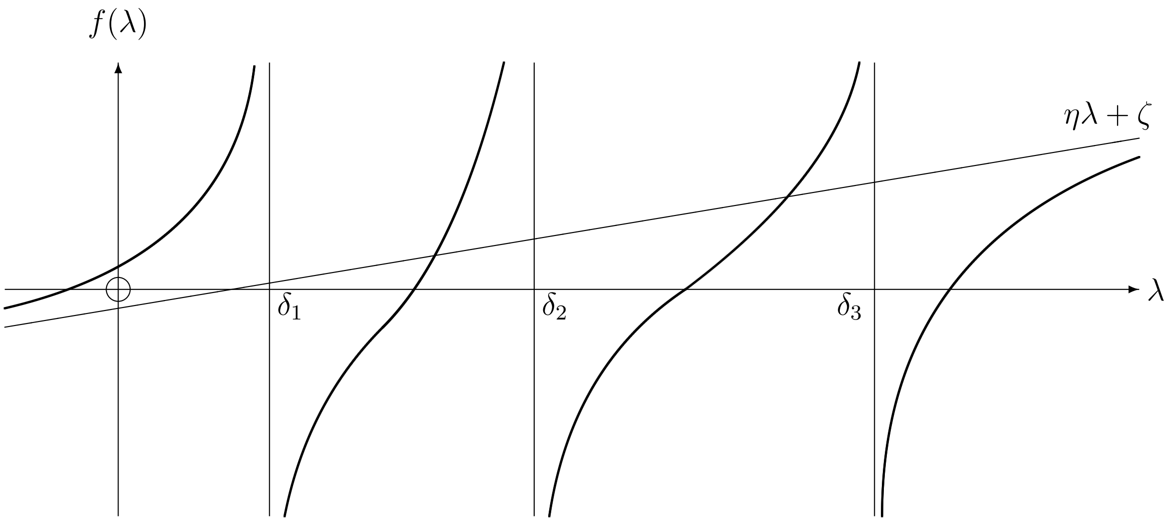

Figure 2.1: Graph of $f(\lambda)$

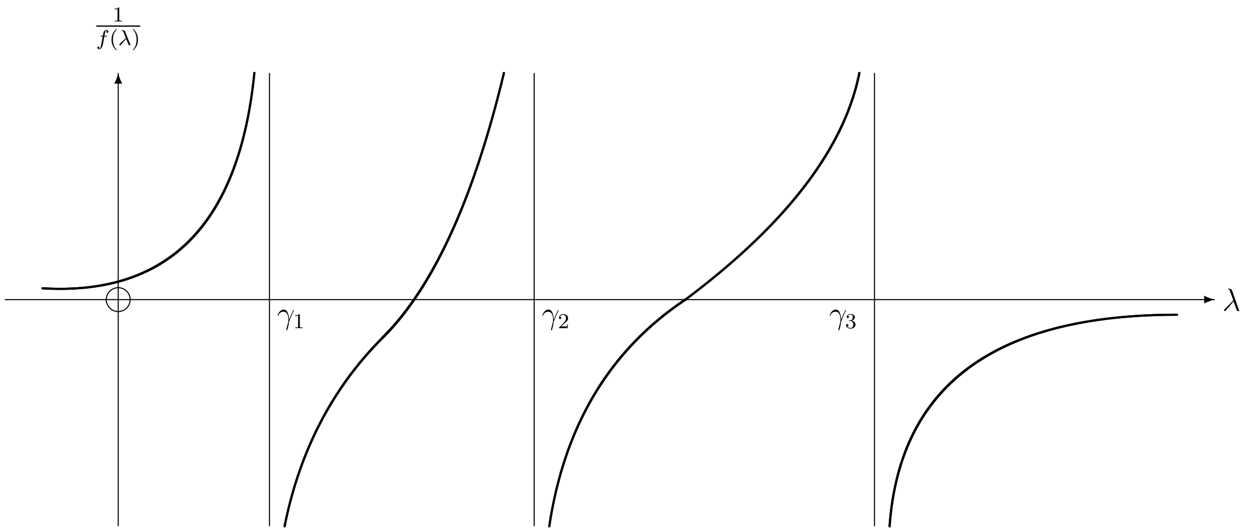

Figure 2.2: $-\frac{1}{f(\lambda)}$

Consider rational Herglotz-Nevanlinna functions f such as (1.6) where $\eta \geq 0$, $\beta_k > 0$, $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ and where all the coefficients are real. We will denote the class of such functions by $\mathcal{R}_N$.

Lemma 2.1.1. *[3, Lemma 2.1.] A rational function f with simple real poles is a Herglotz-Nevanlinna function if and only if $f \in \mathcal{R}_N$ for some N.*

We now denote the subclasses of $\mathcal{R}_N$ where $\eta > 0$ by $\mathcal{R}_N^+$ and where $\eta = 0$ by $\mathcal{R}_N^0$ respectively.

Lemma 2.1.2. *[3, Lemma 2.2.] Let $f \in \mathcal{R}_N$. Then*

(i) $f'(\lambda) > 0$ for each real λ, where $f(\lambda)$ is finite;

(ii) $\lim\limits_{\lambda \to \gamma_k \pm} f(\lambda) = \mp\infty$; and

(iii) if $f \in \mathcal{R}_N^+$, then $\lim\limits_{\lambda \to \pm\infty} f(\lambda) = \pm\infty$, while if $f \in \mathcal{R}_N^0$, then $f(\lambda) \to b$ from below (respectively, above) as $\lambda \to \infty$ (respectively, $-\infty$).

This leads us to the main result of this section on Herglotz-Nevanlinna functions. That is, given $f \in \mathcal{R}_N$ and $\delta < \gamma_1$, where δ is a constant, we define the function F as follows

$$F(\lambda) = \frac{\delta - \lambda}{f(\lambda) - f(\delta)} - f(\delta).$$

We can extend the definition of F by continuity such that $F(d_k) = -f(\delta)$, $1 \leq k \leq N$ and $F(\delta) = -f'(\delta)^{-1} - f(\delta)$. If $f \in \mathcal{R}_N$ then $F \in \mathcal{R}_M$, that is,

$$F(\lambda) = A\lambda + B - \sum_{k=1}^{M} \frac{C_k}{\lambda - D_k}, \tag{2.1}$$

where $M = N - 1$ or $M = N$ depending on a, [3], see below.

Theorem 2.1.3. *[3, Theorem 2.3] In the notation above,*

(i) if $f \in \mathcal{R}_N^+$, then $F \in \mathcal{R}_N^0$ and $\delta < \gamma_1 < D_1 < \gamma_2 < \cdots < \gamma_N < D_N$; and

(ii) if $f \in \mathcal{R}_N^0$*, then* $F \in \mathcal{R}_{N-1}^+$ *and* $\delta < \gamma_1 < D_1 < \gamma_2 < \cdots < D_{N-1} < \gamma_N$.

Remark 2.1.4. *Transformations such as (2.1) make it possible to transform eigenvalue problems with* λ*-dependent boundary conditions into eigenvalue problems with boundary conditions in* $\mathcal{R}_0^0$*. That is, they enable us to map* $\mathcal{R}_N$ *into* $\mathcal{R}_0^0$*.*

2.2 Absolutely Continuous Functions

Definition 2.2.1. *A function* f *is said to be* ***absolutely continuous*** *in an interval* $[a, b]$ *if, given* ϵ*, we can find* δ *such that for each* $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} |f(x_i + h_i) - f(x_i)| < \epsilon \tag{2.2}$$

for every set of mutually disjoint subintervals $(x_i, x_i + h_i)$*,* $i = 1, \ldots, n$*, of* $[a, b]$ *such that* $\sum_{i=1}^{n} h_i < \delta$*.*

An alternative form of this definition can be found in [25]. In [25], Titchmarsh defined absolutely continuous functions on the open interval (a, b) thus forgoing the strict inequalities in the sums in Definition 2.2.1. We will work on the closed compact interval $[a, b]$ in this dissertation.

Absolute continuity describes the smoothness of a function. It is important to note that if we were to modify the sum in (2.2) to consist of one term only we would obtain the definition of uniform continuity. Therefore, absolute continuity implies uniform continuity.

Lemma 2.2.2. *Let* f *and* g *be absolutely continuous functions on [a,b]. Then the following functions are absolutely continuous on* $[a, b]$

(i) $f + g$*;*

(ii) $f - g$*;*

(iii) fg*; and*

(iv) $\frac{f}{g}$ *if there exists a constant* $c > 0$ *such that* $|g(x)| \geq c$ *for all* $x \in [a, b]$.

Remark 2.2.3. *From the definition of absolutely continuous functions, we know that the total variation is at most* ϵ *over an interval of length* δ*. Therefore, absolutely continuous functions are of bounded variation with a total variation of at most* $(b - a)\epsilon/\delta$ *over the interval* $[a, b]$.

Theorem 2.2.4. *[25] A necessary and sufficient condition that a function should be an integral is that it is absolutely continuous.*

The proof is straightforward and can be found in [25, pp. 364].

2.3 Differential Operators

In this section we will develop an abstract theory of operators. Let X and Y be normed vector spaces. Let L be a mapping having domain, $D(L)$, and range, $R(L)$, a subset of Y.

Definition 2.3.1. *Let* $D(L)$ *be a dense linear subspace of* X*. An operator* $L : X \to Y$ *with domain* $D(L)$ *is called a* ***linear operator*** *if for every pair of functions* f, $g \in D(L)$ *and* $\alpha \in \mathbb{C}$ *we have*

(i) $L(f + g) = Lf + Lg$ *(linearity)*

(ii) $L(\alpha f) = \alpha Lf$ *(homogeneity)*

Definition 2.3.2. *A linear operator $L : X \to Y$ is said to be* ***bounded*** *if there is a constant $K \geq 0$ such that*

$$\|Lf\|_Y \leq K\|f\|_X \text{ for all } f \in X.$$

Note that

$$\left\| L\left(\frac{f}{\|f\|_X}\right)\right\|_Y = \left\|\frac{Lf}{\|f\|_X}\right\|_Y = \frac{\|Lf\|_Y}{\|f\|_X}$$

by the homogeneity of $\|\cdot\|_Y$ and the linearity of L. Hence, L is bounded if and only if

$$\sup_{\|f\|_X=1} \|Lf\|_Y \leq K.$$

We look at unbounded operators in the chapters that follow.

Definition 2.3.3. *[16] Let L and $\bar{L}$ be operators from X to Y. L and $\bar{L}$ are said to be* ***equal*** *if and only if $D(L) = D(\bar{L})$ and $Lf = \bar{L}f$ for all f in $D(L)$. $\bar{L}$ is said to be an* ***extension*** *of L (written $L \subset \bar{L}$), and L is said to be a* ***restriction*** *of $\bar{L}$, if and only if $D(L) \subset D(\bar{L})$ and $Lf = \bar{L}f$ for all $f \in D(L)$. The extension is described as* ***proper*** *if $D(\bar{L}) \neq D(L)$.*

Definition 2.3.4. *[14, pp.31] Let X be a vector space over the real or complex numbers. An inner product on X is a scalar-valued function $\langle\ ,\ \rangle$ defined on the Cartesian product $X \times X$ with the following properties.*

i. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

ii. $\langle x, y\rangle = \overline{\langle y, x\rangle}$*; that is,* $\langle x, y\rangle$ *is the complex conjugate of* $\langle y, x\rangle$

iii. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

iv. $\langle x, x\rangle > 0$ *whenever* $x \neq 0$.

X, together with an inner product, is called an ***inner-product space****.*

Definition 2.3.5. *[14, pp.34] A* ***Hilbert space*** *is an inner-product space which is also a Banach space with norm* $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$.

Definition 2.3.6. *[12, pp.4] By a* ***Krein̆ space*** *we mean an inner product space* $\mathfrak{h}$ *which can be expressed as an orthogonal direct sum*

$$\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

where $\mathfrak{h}_+$ *is a hilbert space and* $\mathfrak{h}_-$ *is the antispace of a Hilbert space.*

Definition 2.3.7. *A* ***Pontryagin space*** *is a Krein̆ space* $\mathfrak{h}$ *with* $ind_-\mathfrak{h} < \infty$.

Let X and Y be Hilbert spaces.

Definition 2.3.8. *An operator L is* ***densely defined*** *if* $D(L)$ *is a dense linear subspace of the Hilbert space* X.

Operators defined on the entire space X are also densely defined since the space X is dense in itself. As a result of the above definition, we note that unbounded operators are necessarily discontinuous at points of their domain. In particular, unbounded linear operators are discontinuous at all points of their domains of definition.

Definition 2.3.9. *An operator* $L : D(L) \to Y$ *is said to be a* ***closed operator****, if its graph*

$$\Gamma(L) = \{(f, Lf) \in X \times Y : f \in D(L)\}$$

is a closed subspace of $X \times Y$.

Example 2.3.10. *(i) The differentiation operator* $\frac{d}{dx} : C^1[0,1] \to C[0,1]$*, defined on the set of continuously differentiable functions into the space of all continuous functions on the unit interval* $0 \leq x \leq 1$*, is an example of an unbounded operator. The operator is not bounded as it maps the bounded set* $\{x \mapsto \cos(nx)\}_{n\in\mathbb{N}}$ *to the unbounded set* $\{x \mapsto -n\sin(nx)\}_{n\in\mathbb{N}}$.

We denote the *dual* (or *conjugate*) space of a Hilbert space H by H^*.

Theorem 2.3.11 (Riesz Representation Theorem)**.** *[21, pp.31] Let H be a Hilbert space and let $f \in H^*$. Then there is a unique $y \in H$ such that $f(x) = (x, y)$ for all $x \in H$. Moreover, $\|f\| \leq \|y\|$.*

The proof of which can be found in [14] and [21].

Definition 2.3.12. *Let $L : D(L) \to Y$ be a densely defined linear operator. We define the domain of the* ***adjoint*** *of L*

$$D(L^*) := \{g \in H \big| f \mapsto \langle Lf, g\rangle \text{ is a bounded linear functional on } D(L)\}.$$

For $g \in D(L^*)$ we define L^*g by the Riesz representation theorem to be $h \in H$ such that

$$\langle Lf, g\rangle = \langle f, h\rangle, \ \forall f \in D(L).$$

The denseness of $D(L)$ and the uniqueness established by the Riesz representation ensure that the adjoint operator is well defined, see [24].

Definition 2.3.13. *A densely defined linear operator $L : D(L) \to Y$ is* ***symmetric*** *if $\langle Lf, g\rangle = \langle f, Lg\rangle$ for all $f, g \in D(L)$. If L is symmetric and $D(L) = D(L^*)$ we say that L is* ***self-adjoint****.*

The above definition is equivalent to requiring that $L = L^*$. It also implies that L is closed. Therefore, if L is symmetric and has self-adjoint closure $\bar{L}$, we say that L is self-adjoint. The study of unbounded self-adjoint operators is important for spectral theory.

Remark 2.3.14. *Symmetry implies self-adjointness for bounded operators.*

Theorem 2.3.15. *Let H be a Hilbert space. If $L : H \to H$ is self-adjoint then its spectrum, $\sigma(L)$ is real.*

Proof. We prove by contradiction. Assume $\lambda \in \mathbb{C}$ is not real. Then

$$\begin{aligned}
0 &< |\lambda - \bar{\lambda}| \|f\|^2 \\
&= |([L - \lambda I]f, f) - ([L - \bar{\lambda} I]f, f)| \\
&= |([L - \lambda I]f, f) - (f, [L - \lambda I]f)| \\
&\leq 2\|[L - \lambda I](f)\| \|f\|.
\end{aligned}$$

That is,

$$\frac{|\lambda - \bar{\lambda}|}{2} \|f\| \leq \|[L - \lambda I](f)\|$$

for $f \in H$. The inequality also holds for the case where λ and $\bar{\lambda}$ are interchanged. Therefore $L - \lambda I$ and $L - \bar{\lambda} I$ have closed ranges and are injective. Suppose there is a $g \in H \backslash (L - \lambda I)(H)$ with $(g, (L - \lambda I)h) = 0$ for all $h \in H$. Therefore $((L - \bar{\lambda} I)g, h) = 0$ for all $h \in H$ and $(L - \bar{\lambda} I)g = 0$. Since $L - \bar{\lambda} I$ is injective, so $g = 0$. Thus $L - \lambda I$ is surjective and, therefore $\lambda \in \rho(L)$, where $\rho(L)$ is the resolvent set of L. $\square$

One may refer to Goldberg, [14], for a further study of unbounded operators and Hutson et al., [16], for a general study of linear operators.

2.4 Sturmian Theory

2.4.1 The Objective of Sturmian Theory

Ince [17, pp.223] considers an equation of the form

$$L(y) \equiv \frac{d}{dx}\left\{P \frac{dy}{dx}\right\} - Qy = 0, \tag{2.3}$$

where the coefficients P and Q are assumed to be continuous real functions of the real variable x in the closed interval $a \leq x \leq b$. In (2.3) P does not vanish therefore it may be assumed to be positive, and P' is continuous throughout the interval.

By the fundamental existence theorem, [17, pp.73] we know that this equation has precisely one continuously differentiable solution on (a, b) satisfying the conditions

$$y(c) = \gamma_0,\ y'(c) = \gamma_1,$$

for a given $c \in [a, b]$. The fundamental existence theorem only provides proof of the existence and uniqueness of a solution but does not provide information about the nature of the solution.

Sturm tackled this problem with the objective of finding the number of zeros that the solution has in the interval (a, b). Finding the number of zeros of the solution in the interval provides useful information for physical applications. The two *Theorems of Comparison*, which we present in this section, serve as a fundamental basis of work done on these type of problems.

2.4.2 Fundamental Theorems

Theorem 2.4.1. *[17, pp.223] No continuous solution of (2.3) can have an infinite number of zeros in (a,b) without being identically zero.*

Proof. We will prove by contradiction. Assume that there is a continuous solution of (2.3) having an infinite number of zeros in (a, b). By the Bolzano-Weierstrass theorem we know that these zeros would have at least one limit point $c \in [a, b]$. Continuity of y gives $y(c) = 0$. Suppose (x_n) is a sequence of zeros of y in (a, b) with limit point c. Then

$$0 = \frac{y(c) - y(x_n)}{c - x_n}$$

so

$$y'(c) = 0.$$

But the zero function is a solution of

$$L(y) = 0$$

with

$$y(c) = y'(c) = 0$$

and by the uniqueness $y \equiv 0$. □

This leads us to a classical theorem which is commonly referred to as the *Sturm Separation Theorem.*

Theorem 2.4.2 (The Separation Theorem)**.** *[17, pp.224] The zeros of two real linearly-distinct solutions of the linear differential equation (2.3) separate one another.*

Proof. Let y_0 and y_1 be any two real linearly independent solutions of (2.3). Suppose y_0 has at least two zeros in the interval (a, b) and let x_0 and x_1 be consecutive zeros of y_0 in the interval. Then if y_1 had a zero at x_1 or x_2 then y_1 would be a multiple of y_0. Suppose, on the contrary, that y_1 has no zeros on the interval $[x_0, x_1]$. Then we know that the function $\frac{y_0}{y_1}$ has zeros at the endpoints of the interval $[x_0, x_1]$ and that it is continuous and has continuous derivative on the interval. Therefore, by Rolle's theorem, the derivative must vanish at some $c \in (x_0, x_1)$. However,

$$\frac{d}{dx}\left\{\frac{y_0}{y_1}\right\} = \frac{y_1 y_0' - y_0 y_1'}{y_1^2} = \frac{W(y_0, y_1)}{y_1^2},$$

giving that $(y_0(c), y_0'(c))$ and $(y_1(c), y_1'(c))$ are linearly dependent making y_0 and y_1 linearly dependent which contradicts the assumption of linear independence of y_0 and y_1. Therefore, y_1 vanishes at least once on (x_0, x_1). □

It should be noted that the *Sturm Separation Theorem* only holds for real solutions.

Definition 2.4.3. *Consider two functions of x, y_0 and y_1, continuous on the interval (a,b). Suppose that y_1 has more zeros on the interval than y_0, then we say that y_1 oscillates more rapidly than y_0.*

With this understanding, we can restate the *Separation Theorem* as follows.

Corollary 2.4.4. *The zeros of all real linearly-distinct solutions of a second order linear differential equation oscillate equally rapidly. This implies that the number of zeros of any solution of the equation in a subinterval of (a,b) cannot exceed the number of zeros of any other linearly-distinct solution in the same subinterval by more than one.*

Further details on oscillation theory can be found in [17, pp.224-251].

2.4.3 Theorems of Comparison

Let u be a solution of the equation

$$\frac{d}{dx}\left\{P_1\frac{du}{dx}\right\} - Q_1u = 0 \tag{2.4}$$

satisfying the initial conditions

$$u(a) = \gamma_1,\ u'(a) = \gamma_1'. \tag{2.5}$$

Let v be a solution of the equation

$$\frac{d}{dx}\left\{P_2\frac{dv}{dx}\right\} - Q_2v = 0 \tag{2.6}$$

satisfying the initial conditions

$$v(a) = \gamma_2,\ v'(a) = \gamma_2'. \tag{2.7}$$

Assume that

$$P_1(x) \geq P_2(x) > 0,\ Q_1(x) \geq Q_2(x). \tag{2.8}$$

for all $x \in (a,b)$, $|\gamma_i| + |\gamma_i'| > 0$ for $i = 1, 2$, and that

(i) if $\gamma_1 \neq 0$, then $\gamma_2 \neq 0$ and

$$\frac{P_1(a)\gamma_1'}{\gamma_1} \geq \frac{P_2(a)\gamma_2'}{\gamma_2},$$

(ii) the identity $Q_1 \equiv Q_2 \equiv 0$ does not hold in any non empty subinterval of (a,b).

Sturm's first comparison theorem aims to compare the distribution of the zeros of $u(x)$ and $v(x)$ as defined aboved.

Theorem 2.4.5 (The First Comparison Theorem)**.** *[17, pp.228] Assume that conditions (2.8), (i) and (ii) are satisfied. If $u(x)$ is the solution of (2.4), with initial condition (2.5) and $u(x)$ has m zeros in $(a,b]$, then $v(x)$, the solution of (2.6) with initial condition (2.7), has at least m zeros in $(a,b]$, and the i^{th} zero of $v(x)$ is less than the i^{th} zero of $u(x)$.*

Proof. Let $x_1 < x_2 < \cdots < x_m$ denote the zeros of $u(x)$ in the interval $(a,b]$. By Theorem 2.4.2, there exists at least one zero of $v(x)$ between each pair x_i and x_{i+1}. It suffices to show that there is at least one zero of $v(x)$ between a and x_1.

Suppose $u(x)$ has a zero at a, that is $u(a) = \gamma_1 = 0$, then $v(x)$ has a zero in (a, x_1). Now suppose that $\gamma_1 \neq 0$. Since $v(a) = \gamma_2 \neq 0$ the Picone formula given by Ince in [17, pp.225] gives

$$\left[u^2\left(P_1\frac{u'}{u} - P_2\frac{v'}{v}\right)\right]_a^{x_1} = \int_a^{x_1}(Q_1 - Q_2)u^2dx + \int_a^{x_1}(P_1 - P_2)u'^2dx$$

$$+\int_a^{x_1} P_2\frac{(u'v-uv')^2}{v^2}dx. \tag{2.9}$$

Here the right hand side is positive. We now evaluate the left hand side and suppose, on the contrary, that $v(x)$ has no zero in (a, x_1). This gives

$$\begin{aligned}\left[u^2\left(P_1\frac{u'}{u}-P_2\frac{v'}{v}\right)\right]_a^{x_1} &= u^2(x_1)\left(P_1(x_1)\frac{u'(x_1)}{u(x_1)}-P_2(x_1)\frac{v'(x_1)}{v(x_1)}\right)\\ &\quad - u^2(a)\left(P_1(a)\frac{u'(a)}{u(a)}-P_2(a)\frac{v'(a)}{v(a)}\right)\\ &= -u^2(a)\left(P_1(a)\frac{\gamma_1'}{\gamma_1}-P_2(a)\frac{\gamma_2'}{\gamma_2}\right)\end{aligned}$$

which, by assumption (i), is negative or zero. Thus, we have proved by contradiction that $v(x)$ has at least one zero in (a, x_1). □

Theorem 2.4.6 (The Second Comparison Theorem)**.** *[17, pp.229] Assume that conditions (2.8), (i) and (ii) are satisfied. Let c be an interior point of the interval (a, b) which is not a zero of $u(x)$, the solution of (2.4) with initial condition (2.5), or of $v(x)$, the solution of (2.6) with initial condition (2.7). If c is such that $u(x)$ and $v(x)$ have the same number of zeros in the interval $a < x < c$, then*

$$\frac{P_1(c)u'(c)}{u(c)} > \frac{P_2(c)v'(c)}{v(c)}.$$

Proof. Let $x_1 < x_2 < \cdots < x_m$ denote the zeros of $u(x)$ in the interval $(a, b]$. Let x_i be the greatest zero in the interval (a, c). Then x_i is a zero of $u(x)$ since the interval (a, x_i) has exactly i zeros of $v(x)$, by the first comparison theorem and by supposition. The result follows from the application of (2.9) between the limits x_i and c, that is

$$\left[u^2\left(\frac{P_1u'}{u}-\frac{P_2v'}{v}\right)\right]_{x_i}^{c} > 0.$$

Similarly, if $u(x)$ and $v(x)$ have no zeros in the interval (a, c) then the Picone formula, (2.9) taken between the limits a and c yields the same result. □

Further work on Sturmian theory and ordinary differential equations may be found in [20], and many other places.

Chapter 3

Forward Transformation

In this chapter and the chapters to follow we will consider the Sturm-Liouville equation

$$\ell y := -y'' + qy = \lambda y, \text{ on } [-a, b], \tag{3.1}$$

in $L^2(-a, b)$, $a,\ b > 0$, for $q \in L^2(-a, b)$ with the boundary conditions

$$y(-a)\cos\alpha = y'(-a)\sin\alpha, \tag{3.2}$$

$$y(b)\cos\beta = y'(b)\sin\beta, \tag{3.3}$$

where $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$, and the transmission conditions

$$y(0^+) = r(\lambda) \triangle y', \tag{3.4}$$

$$y'(0^-) = s(\lambda) \triangle y. \tag{3.5}$$

Here

$$\triangle y = y(0^+) - y(0^-),$$

$$\triangle y' = y'(0^+) - y'(0^-).$$

In addition, in (3.4), (3.5) we will consider the following two possibilities for $r(\lambda)$ and $s(\lambda)$

Class 1:

$$r(\lambda) = \zeta + \sum_{j=1}^{M} \frac{\beta_j^2}{\lambda - \gamma_j}, \tag{3.6}$$

$$s(\lambda) = \sigma - \sum_{i=1}^{N} \frac{\alpha_i^2}{\lambda - \delta_i}, \tag{3.7}$$

where

$$\delta_1 < \delta_2 < \cdots < \delta_N,$$

$$\gamma_1 < \gamma_2 < \cdots < \gamma_M,$$

and $\alpha_i,\ \beta_j > 0$ for $i = 1, \cdots, N$, and $j = 1, \cdots, M$.

Class 2:

$$r(\lambda) = -\sum_{i=1}^{N} \frac{\alpha_i^2}{\lambda - \delta_i}, \tag{3.8}$$

$$s(\lambda) = -(\zeta + \eta\lambda) + \sum_{j=1}^{M} \frac{\beta_j^2}{\lambda - \gamma_j}, \tag{3.9}$$

where

$$\delta_1 < \delta_2 < \cdots < \delta_N,$$

$$\gamma_1 < \gamma_2 < \cdots < \gamma_M,$$

and $\alpha_i,\ \beta_j > 0$ for $i = 1, \cdots, N$, and $j = 1, \cdots, M$.

The transmission condition (3.4) - (3.5) can be rewritten in terms of a transmission matrix $M^{[0]}$ as follows

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = M^{[0]} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix} \tag{3.10}$$

where

$$M^{[0]} = \begin{bmatrix} M_{11}^{[0]} & M_{12}^{[0]} \\ M_{21}^{[0]} & M_{22}^{[0]} \end{bmatrix}. \tag{3.11}$$

We will compute the concrete forward Darboux-Crum transformation of the Sturm-Liouville equation given in (3.1), (3.4) and (3.5). We will formulate the transformation and compute $n+1$ iterations so as to analyse its effect on the transmission condition of our problem in each step. The aim is to illustrate how this transformation increases the eigenparameter dependence of our transmission condition in half steps of Herglotz-Nevanlinna functions. We will use the notation from [3] that was introduced in Section 2.1 to denote the subclasses of the transmission conditions yielded by the transformation.

Remark 3.0.1. *We will begin by assuming that the entries of our initial transmission matrix $M^{[0]}$ in (3.11) are all constant, that is, $M_{ij}^{[0]} \in \mathbb{R}$. Thus, using the notation of the subclasses of $\mathcal{R}_N$ introduced in Section 2.1, where the case of $\eta_n > 0$ is denoted by $\mathcal{R}_N^+$ and the case of $\eta_n = 0$ is denoted by $\mathcal{R}_N^0$, we note that the transmission matrix $M^{[0]}$ can be expressed as*

$$M^{[0]} = \begin{bmatrix} M_{11}^{[0]} & r(\lambda) \\ s(\lambda) & M_{22}^{[0]} \end{bmatrix}, \tag{3.12}$$

where $r(\lambda) \in \mathcal{R}_0^0$ and $s(\lambda) \in \mathcal{R}_0^0$.

3.1 Concrete Transformation

We denote by $q_n \in L^2(-a,b)$ the potential corresponding to the Sturm-Liouville equation resulting from the *nth* iteration of the forward transformation.

Theorem 3.1.1. *Let $\lambda_1 < \lambda_2 < \cdots \in \mathbb{R}$. Define*

$$u_1 := y' - \frac{z_1'}{z_1} y,$$

where z_1 *is a solution of (3.1) for* $\lambda = \lambda_1$ *with no zeros on* $[-a, b]$, *then* u_1 *obeys (3.1) with* q *replaced by* $q_1 = q - 2\left(\frac{z_1'}{z_1}\right)'$.

Proof. Let z_1 be a solution of (3.1) for $\lambda = \lambda_1$ that is,

$$-z_1'' + qz_1 = \lambda_1 z_1.$$

Then z_1 never vanishes in $[-a, b]$. Let $w_1 = \frac{z_1'}{z_1}$ and note that

$$w_1' = q - \lambda_1 - w_1^2. \tag{3.13}$$

Let $u_0 := y$ and define

$$u_1 = u_0' - w_1 u_0, \tag{3.14}$$

thus, by the product rule, we get

$$u_1' = (\lambda_1 - \lambda)u_0 - w_1 u_1. \tag{3.15}$$

Moreover, (3.13) - (3.15) gives

$$\begin{aligned} u_1'' &= (\lambda_1 - \lambda)(u_1 + w_1 u_0) - w_1' u_1 - w_1(\lambda_1 - \lambda)u_0 + w_1^2 u_1 \\ &= (\lambda_1 - \lambda)u_1 + (2w_1^2 - q + \lambda_1)u_1 \\ &= (\lambda_1 - \lambda)u_1 + (2q - 2\lambda_1 - 2w_1' - q + \lambda_1)u_1 \\ &= (\lambda_1 - \lambda)u_1 + (q - \lambda_1 - 2w_1')u_1 \\ &= -\lambda u_1 + (q - 2w_1')u_1 \end{aligned}$$

so u_1 satisfies (3.1) with potential $q_1 = q - 2w_1'$. □

If we do this procedure $n+1$ successive times i.e. let $w_{n+1} = \frac{z_{n+1}'}{z_{n+1}}$ where z_{n+1} is the eigenfunction corresponding to the least eigenvalue, λ_{n+1}, of the n^{th} transformed boundary value problem and define

$$u_{n+1} = u_n' - w_{n+1} u_n,$$

then

$$u''_{n+1} = \lambda u_{n+1} + (q_n - 2w'_{n+1})u_{n+1}.$$

I.e. u_{n+1} obeys (3.1) with the potential q replaced by $q_{n+1} = q_n - 2w'_{n+1}$ in the $(n+1)^{th}$ iteration of the forward transformation.

The focus of this paper is the effect that the above transformation has on the transmission condition. It should be noted that Binding et al. study the effect of this transformation on the boundary conditions in [3]. The authors use oscillation theory to show that applying this transformation to a Sturm-Liouville boundary value problem results in a new boundary value problem whose spectrum contains all the original eigenvalues excluding the first eigenvalue.

Note that (3.14) can be expressed using the Wronskian of u_0 and z_1 as follows:

$$T_{z_1(u_0)} = \frac{u'_0 z_1 - z'_1 u_0}{z_1} = \frac{W[y, z_1]}{z_1}.$$

We will use this notation for the forward transformation for the remainder of the dissertation.

Now let

$$\begin{aligned} z_n(0^-) &= a_n, \\ z'_n(0^-) &= b_n, \\ z_n(0^+) &= c_n, \\ z'_n(0^+) &= d_n, \end{aligned}$$

where z_n is a solution corresponding to the $(n-1)^{th}$ transformed equation. Here

$a_n,\ b_n,\ c_n,\ d_n \in \mathbb{R}$ and n denotes the iteration number.

In addition, let

$$A_n = \frac{d_n}{c_n}\left(M_{11}^{[n-1]} + \frac{b_n}{a_n}M_{12}^{[n-1]}\right) - \left(M_{21}^{[n-1]} + \frac{b_n}{a_n}M_{22}^{[n-1]}\right), \tag{3.16}$$

$$B_n = \frac{d_n}{c_n}M_{12}^{[n-1]} - M_{22}^{[n-1]}, \tag{3.17}$$

$$C_n = M_{11}^{[n-1]} + \frac{b_n}{a_n}M_{12}^{[n-1]}, \tag{3.18}$$

where $M_{ij}^{[n-1]}$ for $i,\ j = 1,\ 2$ are entries of the transmission matrix $M^{[n-1]}$ i.e. from the $(n-1)^{th}$ transformed bounday value problem. The case of $n = 1$ is considered in the theorem below.

Theorem 3.1.2. *The transmission condition (3.10) of the boundary value problem given by (3.1) - (3.5) transforms under*

$$T_{z_1}(u_0) = \frac{W[u_0, z_1]}{z_1},$$

where $u_0 := y$, *to a transmission condition,*

$$\begin{bmatrix} u_1(0^+) \\ u_1'(0^+) \end{bmatrix} = M^{[1]} \begin{bmatrix} u_1(0^-) \\ u_1'(0^-) \end{bmatrix},$$

where

$$M^{[1]} = \begin{bmatrix} \frac{b_1 A_1}{a_1(\lambda-\lambda_1)} - B_1 & \frac{A_1}{\lambda-\lambda_1} \\ -\frac{b_1 d_1 A_1}{a_1 c_1(\lambda-\lambda_1)} + \frac{d_1}{c_1}B_1 + \frac{b_1}{a_1}C_1 - (\lambda-\lambda_1)M_{12} & \frac{-d_1 A_1}{c_1(\lambda-\lambda_1)} + C_1 \end{bmatrix}. \tag{3.19}$$

Here $a_1,\ b_1,\ c_1,\ d_1,\ A_1,\ B_1$ *and* $C_1 \in \mathbb{R}$ *and are as given above.*

Proof. By (3.14) and (3.15) we have

$$u_1(0^+) = u_0'(0^+) - \frac{d_1}{c_1}u_0(0^+)$$

and

$$\begin{aligned}
u_1'(0^+) &= (\lambda_1 - \lambda)u_0(0^+) - \frac{d_1}{c_1}u_1(0^+) \\
&= (\lambda_1 - \lambda)u_0(0^+) - \frac{d_1}{c_1}\big(u_0'(0^+) - \frac{d_1}{c_1}u_0(0^+)\big) \\
&= \left(\lambda_1 - \lambda + \left(\frac{d_1}{c_1}\right)^2\right)u_0(0^+) - \frac{d_1}{c_1}u_0'(0^+).
\end{aligned}$$

Similarly for $u_1(0^-)$ we have

$$u_1(0^-) = u_0'(0^-) - \frac{b_1}{a_1}u_0(0^-)$$

and

$$\begin{aligned}
u_1'(0^-) &= (\lambda_1 - \lambda)u_0(0^-) - \frac{b_1}{a_1}u_1(0^-) \\
&= (\lambda_1 - \lambda)u_0(0^-) - \frac{b_1}{a_1}\big(u_0'(0^-) - \frac{b_1}{a_1}u_0(0^-)\big) \\
&= \left(\lambda_1 - \lambda + \left(\frac{b_1}{a_1}\right)^2\right)u_0(0^-) - \frac{b_1}{a_1}u_0'(0^-).
\end{aligned}$$

Expressing the above system of equations in matrix form gives

$$\begin{bmatrix} u_1(0^+) \\ u_1'(0^+) \end{bmatrix} = \begin{bmatrix} -\frac{d_1}{c_1} & 1 \\ \lambda_1 - \lambda + \left(\frac{d_1}{c_1}\right)^2 & -\frac{d_1}{c_1} \end{bmatrix} \begin{bmatrix} u_0(0^+) \\ u_0'(0^+) \end{bmatrix} \tag{3.20}$$

and

$$\begin{bmatrix} u_1(0^-) \\ u_1'(0^-) \end{bmatrix} = \begin{bmatrix} -\frac{b_1}{a_1} & 1 \\ \lambda_1 - \lambda + \left(\frac{b_1}{a_1}\right)^2 & -\frac{b_1}{a_1} \end{bmatrix} \begin{bmatrix} u_0(0^-) \\ u_0'(0^-) \end{bmatrix}. \tag{3.21}$$

We label the coefficient matrices in (3.20) and (3.21) H^+ and H^- respectively. Thus,

$$\begin{aligned}
(H^+)^{-1}\begin{bmatrix} u_1(0^+) \\ u_1'(0^+) \end{bmatrix} &= \begin{bmatrix} u_0(0^+) \\ u_0'(0^+) \end{bmatrix} \\
&= M^{[0]}\begin{bmatrix} u_0(0^-) \\ u_0'(0^-) \end{bmatrix}
\end{aligned}$$

$$= M^{[0]}(H^-)^{-1}\begin{bmatrix} u_1(0^-) \\ u_1'(0^-) \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} u_1(0^+) \\ u_1'(0^+) \end{bmatrix} = H^+M^{[0]}(H^-)^{-1}\begin{bmatrix} u_1(0^-) \\ u_1'(0^-) \end{bmatrix},$$

where $\det(H^-) = \left(\frac{b_1}{a_1}\right)^2 - \lambda_1 + \lambda - \left(\frac{b_1}{a_1}\right)^2 = \lambda - \lambda_1$. Now, let $M^{[1]} = H^+M^{[0]}(H^-)^{-1}$ which yields a transmission matrix of the type given in (3.10) where $M^{[0]}$ is defined as it is in (3.11), therefore

$$M^{[1]} = \begin{bmatrix} -\frac{d_1}{c_1} & 1 \\ \lambda_1 - \lambda + \left(-\frac{d_1}{c_1}\right)^2 & -\frac{d_1}{c_1} \end{bmatrix}\begin{bmatrix} M_{11}^{[0]} & M_{12}^{[0]} \\ M_{21}^{[0]} & M_{22}^{[0]} \end{bmatrix}\begin{bmatrix} -\frac{b_1}{a_1(\lambda-\lambda_1)} & -\frac{1}{\lambda-\lambda_1} \\ \frac{\lambda-\lambda_1-(\frac{b_1}{a_1})^2}{\lambda-\lambda_1} & -\frac{b_1}{a_1(\lambda-\lambda_1)} \end{bmatrix}$$

The resultant matrix multiplication gives the following entries for the matrix $M^{[1]}$

$$\begin{aligned} M_{11}^{[1]} &= \frac{-\frac{b_1}{a_1}\left(-\frac{d_1}{c_1}M_{11}^{[0]} + M_{21}^{[0]}\right) + \left(\lambda - \lambda_1 - \left(\frac{b_1}{a_1}\right)^2\right)\left(-\frac{d_1}{c_1}M_{12}^{[0]} + M_{22}^{[0]}\right)}{\lambda - \lambda_1} \\ &= \frac{\frac{b_1d_1}{a_1c_1}\left(M_{11}^{[0]} + \frac{b_1}{a_1}M_{12}^{[0]}\right) - \frac{b_1}{a_1}\left(M_{21}^{[0]} + \frac{b_1}{a_1}M_{22}^{[0]}\right)}{\lambda - \lambda_1} - \frac{d_1}{c_1}M_{12}^{[0]} + M_{22}^{[0]} \\ &= \frac{\frac{b_1}{a_1}\left[\frac{d_1}{c_1}\left(M_{11}^{[0]} + \frac{b_1}{a_1}M_{12}^{[0]}\right) - \left(M_{21}^{[0]} + \frac{b_1}{a_1}M_{22}^{[0]}\right)\right]}{\lambda - \lambda_1} - \left(\frac{d_1}{c_1}M_{12}^{[0]} - M_{22}^{[0]}\right) \end{aligned}$$

$$\begin{aligned} M_{12}^{[1]} &= \frac{\frac{d_1}{c_1}M_{11}^{[0]} - M_{21}^{[0]} + \frac{b_1}{a_1}\left(\frac{d_1}{c_1}M_{12}^{[0]} - M_{22}^{[0]}\right)}{\lambda - \lambda_1} \\ &= \frac{\frac{d_1}{c_1}\left(M_{11}^{[0]} + \frac{b_1}{a_1}M_{12}^{[0]}\right) - \left(M_{21}^{[0]} + \frac{b_1}{a_1}M_{22}^{[0]}\right)}{\lambda - \lambda_1} \end{aligned}$$

$$M_{21}^{[1]} = \frac{-\frac{b_1}{a_1}\left(\left(\lambda_1 - \lambda + \left(\frac{d_1}{c_1}\right)^2\right)M_{11}^{[0]} - \frac{d_1}{c_1}M_{21}^{[0]}\right)}{\lambda - \lambda_1}$$

$$
+\frac{\left(\lambda-\lambda_1-\left(\frac{b_1}{a_1}\right)^2\right)\left(\left(\lambda_1-\lambda+\left(\frac{d_1}{c_1}\right)^2\right)M_{12}^{[0]}-\frac{d_1}{c_1}M_{22}^{[0]}\right)}{\lambda-\lambda_1}
$$

$$
=\frac{-\frac{b_1}{a_1}(\lambda_1-\lambda)M_{11}^{[0]}-\frac{b_1}{a_1}\left(\frac{d_1}{c_1}\right)^2M_{11}^{[0]}+\frac{b_1d_1}{a_1c_1}M_{21}^{[0]}}{\lambda-\lambda_1}
$$

$$
+\frac{\left(\lambda-\lambda_1-\left(\frac{b_1}{a_1}\right)^2\right)\left((\lambda_1-\lambda)M_{12}^{[0]}+\left(\frac{d_1}{c_1}\right)^2M_{12}^{[0]}-\frac{d_1}{c_1}M_{22}^{[0]}\right)}{\lambda-\lambda_1}
$$

$$
=\frac{-\frac{b_1d_1}{a_1c_1}\left[\frac{d_1}{c_1}\left(M_{11}^{[0]}+\frac{b_1}{a_1}M_{12}^{[0]}\right)-\left(M_{21}^{[0]}+\frac{b_1}{a_1}M_{22}^{[0]}\right)\right]}{\lambda-\lambda_1}+\frac{b_1}{a_1}\left(M_{11}^{[0]}+\frac{b_1}{a_1}M_{12}^{[0]}\right)
$$

$$
+\frac{d_1}{c_1}\left(\frac{d_1}{c_1}M_{12}^{[0]}-M_{22}^{[0]}\right)+(\lambda_1-\lambda)M_{12}^{[0]}
$$

$$
M_{22}^{[1]}=\frac{\frac{d_1}{c_1}M_{21}^{[0]}-\left(\lambda_1-\lambda+\left(\frac{d_1}{c_1}\right)^2\right)M_{11}^{[0]}-\frac{b_1}{a_1}\left(\left(\lambda_1-\lambda+\left(\frac{d_1}{c_1}\right)^2\right)M_{12}^{[0]}-\frac{d_1}{c_1}M_{22}^{[0]}\right)}{\lambda-\lambda_1}
$$

$$
=\frac{\frac{d_1}{c_1}M_{21}^{[0]}-(\lambda_1-\lambda)M_{11}^{[0]}-\left(\frac{d_1}{c_1}\right)^2M_{11}^{[0]}+\frac{b_1d_1}{a_1c_1}M_{22}^{[0]}-\frac{b_1}{a_1}(\lambda_1-\lambda)M_{12}^{[0]}-\frac{b_1}{a_1}\left(\frac{d_1}{c_1}\right)^2M_{12}^{[0]}}{\lambda-\lambda_1}
$$

$$
=\frac{-\frac{d_1}{c_1}\left[\frac{d_1}{c_1}\left(M_{11}^{[0]}+\frac{b_1}{a_1}M_{12}^{[0]}\right)-\left(M_{21}^{[0]}+\frac{b_1}{a_1}M_{22}^{[0]}\right)\right]}{\lambda-\lambda_1}+\left(M_{11}^{[0]}+\frac{b_1}{a_1}M_{12}^{[0]}\right).
$$

Now

$$
A_1=\frac{d_1}{c_1}\left(M_{11}^{[0]}+\frac{b_1}{a_1}M_{12}^{[0]}\right)-\left(M_{21}^{[0]}+\frac{b_1}{a_1}M_{22}^{[0]}\right),
$$

$$
B_1=\frac{d_1}{c_1}M_{12}^{[0]}-M_{22}^{[0]},
$$

$$
C_1=M_{11}^{[0]}+\frac{b_1}{a_1}M_{12}^{[0]}.
$$

Thus

$$
M_{11}^{[1]}=\frac{b_1A_1}{a_1(\lambda-\lambda_1)}-B_1,
$$

$$M_{12}^{[1]} = \frac{A_1}{\lambda - \lambda_1},$$

$$M_{21}^{[1]} = \frac{-b_1 d_1 A_1}{a_1 c_1(\lambda - \lambda_1)} + \frac{b_1}{a_1} C_1 + \frac{d_1}{c_1} B_1 - (\lambda - \lambda_1) M_{12}^{[0]},$$

$$M_{22}^{[1]} = \frac{-d_1 A_1}{c_1(\lambda - \lambda_1)} + C_1.$$

These are the entries of the matrix given in (3.19) therefore proving our result. □

The forward transformation has increased the eigenparameter dependence of the transmission condition. In order for us to identify the form of λ-dependence that is gained in each iteration of the transformation and establish whether there is a distinct manner in which the dependence increases, we would need to compute further iterations of the forward transformation.

The aim of computing multiple iterations is to inductively establish the n^{th} transmission condition yielded by n iterations of the forward transformation. This n^{th} transmission condition will embody all the properties gained in each step including the nature of the increase in the λ-dependence of the transmission condition. We will assume, without loss of generality, that $z_n'(0^-) = b_n = 0 = z_n'(0^+) = d_n$. Therefore, the matrix $M^{[1]}$ in Theorem 3.1.2 takes the form

$$M^{[1]} = \begin{bmatrix} M_{22}^{[0]} & -\frac{\alpha_{1,1}}{\lambda - \lambda_1} \\ -(\zeta_1 + \eta_1 \lambda) & M_{11}^{[0]} \end{bmatrix}, \tag{3.22}$$

where $\alpha_{1,1} = M_{21}^{[0]}$, $\zeta_1 = -\lambda_1 M_{12}^{[0]}$ and $\eta_1 = M_{12}^{[0]}$. Note that $M^{[n]}$ will denote the transmission matrix yielded by the nth iteration. For $\alpha_{n,m}$, n denotes the nth iteration and m will be a summation index. We suppose that $\zeta_n < 0$ and $\eta_n > 0$.

Remark 3.1.3. *We note that the transmission matrix $M^{[1]}$ can be expressed as*

$$M^{[1]} = \begin{bmatrix} M_{22}^{[0]} & r_1(\lambda) \\ -s_1(\lambda) & M_{11}^{[0]} \end{bmatrix},$$

where $r_1(\lambda) \in \mathcal{R}_1^0$ and $s_1(\lambda) \in \mathcal{R}_0^+$.

For the case of $n = 2$ we have the following Corollary.

Corollary 3.1.4. *The transmission condition given in Theorem 3.1.2 transforms under*

$$T_{z_2}(u_1) = \frac{W[u_1, z_2]}{z_2} \tag{3.23}$$

to a transmission condition given by

$$\begin{bmatrix} u_2(0^+) \\ u_2'(0^+) \end{bmatrix} = M^{[2]} \begin{bmatrix} u_2(0^-) \\ u_2'(0^-) \end{bmatrix},$$

where

$$M^{[2]} = \begin{bmatrix} M_{11}^{[0]} & \zeta_2 + \frac{\beta_{2,2}}{\lambda - \lambda_2} \\ \sigma_2 - \frac{\alpha_{2,1}}{\lambda - \lambda_1} & M_{22}^{[0]} \end{bmatrix}. \tag{3.24}$$

Here

$$\zeta_2 = \eta_1,$$
$$\beta_{2,2} = \zeta_1 + \lambda_2 \eta_1,$$
$$\sigma_2 = \alpha_{1,1},$$

and

$$\alpha_{2,1} = \alpha_{1,1}\lambda_2 - \alpha_{1,1}\lambda_1.$$

Proof. Let z_2 be the eigenfunction corresponding to the eigenvalue λ_2 satisfying $-z_2'' + qz_2 = \lambda_2 z_2$ and recall

$$z_2(0^-) = a_2$$

$$
\begin{aligned}
z_2'(0^-) &= b_2 \\
z_2(0^+) &= c_2 \\
z_2'(0^+) &= d_2.
\end{aligned}
$$

Applying the forward Darboux transformation, (3.23) (recall $b_2 = d_2 = 0$) gives

$$
\begin{aligned}
A_2 &= \frac{d_2}{c_2}\left(M_{11}^{[1]} + \frac{b_2}{a_2}M_{12}^{[1]}\right) - \left(M_{21}^{[1]} + \frac{b_2}{a_2}M_{22}^{[1]}\right) \\
&= -M_{21}^{[1]} \\
&= \zeta_1 + \eta_1\lambda,
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \frac{d_2}{c_2}M_{12}^{[1]} - M_{22}^{[1]} \\
&= -M_{22}^{[1]} \\
&= -M_{11}^{[0]},
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= M_{11}^{[1]} + \frac{b_2}{a_2}M_{12}^{[1]} \\
&= M_{11}^{[1]} \\
&= M_{22}^{[0]}.
\end{aligned}
$$

Substituting A_2, B_2 and C_2 into the matrix entries given in (3.19) (where all the 1's are replaced with 2's) and gathering constant terms gives

$$
\begin{aligned}
M_{11}^{[2]} &= -B_2 \\
&= M_{11}^{[0]},
\end{aligned}
$$

$$
\begin{aligned}
M_{12}^{[2]} &= \frac{A_2}{\lambda - \lambda_2} \\
&= \frac{\zeta_1 + \eta_1\lambda}{\lambda - \lambda_2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\zeta_1}{\lambda - \lambda_2} + \eta_1 + \frac{\lambda_2 \eta_1}{\lambda - \lambda_2} \\
&=: \zeta_2 + \frac{\beta_{2,2}}{\lambda - \lambda_2},
\end{aligned}
$$

$$
\begin{aligned}
M_{21}^{[2]} &= -(\lambda - \lambda_2) M_{12}^{[1]} \\
&= -(\lambda - \lambda_2)\left(- \frac{\alpha_{1,1}}{\lambda - \lambda_1} \right) \\
&= \alpha_{1,1} + \frac{\alpha_{1,1}\lambda_1}{\lambda - \lambda_1} - \frac{\alpha_{1,1}\lambda_2}{\lambda - \lambda_1} \\
&=: \sigma_2 - \frac{\alpha_{2,1}}{\lambda - \lambda_1},
\end{aligned}
$$

$$
\begin{aligned}
M_{22}^{[2]} &= C_2 \\
&= M_{22}^{[0]}.
\end{aligned}
$$

Therefore, the second iteration has moved us up the hierarchy and yielded a new transmission condition with increased λ-dependence of the form (3.24).

□

Remark 3.1.5. *We note that the transmission matrix $M^{[2]}$ can be expressed as*

$$
M^{[2]} = \begin{bmatrix} M_{11}^{[0]} & -r_2(\lambda) \\ s_2(\lambda) & M_{22}^{[0]} \end{bmatrix},
$$

where $r_2(\lambda) \in \mathcal{R}_1^0$ and $s_2(\lambda) \in \mathcal{R}_1^0$.

We consider the transmission matrices given by (3.22) and (3.24) as the base steps for our induction. We summarise our observations, thus far, in the remark below.

Remark 3.1.6. *As we move up the hierarchy by means of iterated forward transformations the following changes take place in the transmission matrix:*

(i) The main diagonal entries interchange in each iteration.

(ii) The off-diagonal entries interchange and increase in half steps of Herglotz-Nevanlinna form in each iteration.

We now need to split our considerations into two cases, namely, whether we have done an odd number of iterations or an even number of iterations. Clearly, if n is odd then the next iteration, $n+1$, will be an even number and vice versa. Thus we need only perform the following steps in our induction. Consider n odd (with base case (3.22)) and consequently $n+1$ even (with base case (3.24)).

Theorem 3.1.7. *The transmission matrix*

$$M^{[n-1]} = \begin{bmatrix} M_{11}^{[0]} & \zeta_{n-1} + \sum_{j=1,j\ even}^{n-1} \frac{\beta_{n-1,j}}{\lambda - \lambda_j} \\ \sigma_{n-1} - \sum_{i=1,i\ odd}^{n-1} \frac{\alpha_{n-1,i}}{\lambda - \lambda_i} & M_{22}^{[0]} \end{bmatrix} \tag{3.25}$$

transforms under

$$T_{z_n}(u_{n-1}) = \frac{W[u_{n-1}, z_n]}{z_n}$$

to a transmission condition given by

$$\begin{bmatrix} u_n(0^+) \\ u_n'(0^+) \end{bmatrix} = M^{[n]} \begin{bmatrix} u_n(0^-) \\ u_n'(0^-) \end{bmatrix},$$

where

$$M^{[n]} = \begin{bmatrix} M_{22}^{[0]} & -\sum_{i=1,i\ odd}^{n} \frac{\alpha_{n,i}}{\lambda - \lambda_i} \\ -(\zeta_n + \eta_n \lambda) + \sum_{j=1,j\ even}^{n-1} \frac{\beta_{n,j}}{\lambda - \lambda_j} & M_{11}^{[0]} \end{bmatrix} \tag{3.26}$$

and $n \in \mathbb{Z}$ odd.

Proof. For $n \in \mathbb{Z}$ odd and $b_n,\ d_n = 0$ we have from (3.16), (3.17) and (3.18)

$$\begin{aligned} A_n &= -M_{21}^{[n-1]} = -\sigma_{n-1} + \sum_{i=1, i \ odd}^{n-2} \frac{\alpha_{n-1,i}}{\lambda - \lambda_i}, \\ B_n &= -M_{22}^{[n-1]} = -M_{22}^{[0]}, \\ C_n &= M_{11}^{[n-1]} = M_{11}^{[0]}. \end{aligned}$$

We now consider the entries of $M^{[n]}$ individually with $b_n = 0 = d_n$. Using formula (3.19) for $M^{[n]}$ (i.e. with n replacing 1 throughout) we obtain

$$\begin{aligned} M_{11}^{[n]} &= \frac{b_n A_n}{a_n(\lambda - \lambda_n)} - B_n \\ &= -B_n \\ &= M_{22}^{[0]}, \end{aligned}$$

$$\begin{aligned} M_{12}^{[n]} &= \frac{A_n}{\lambda - \lambda_n} \\ &= \frac{1}{\lambda - \lambda_n}\Bigg(- \sigma_{n-1} + \sum_{i=1, i \ odd}^{n-2} \frac{\alpha_{n-1,i}}{\lambda - \lambda_i}\Bigg) \\ &= -\frac{\sigma_{n-1}}{\lambda - \lambda_n} + \sum_{i=1, i \ odd}^{n-2} \frac{\alpha_{n-1,i}}{(\lambda - \lambda_i)(\lambda - \lambda_n)} \\ &= -\frac{\sigma_{n-1}}{\lambda - \lambda_n} + \Bigg(\frac{\alpha_{n-1,1}}{(\lambda - \lambda_1)(\lambda - \lambda_n)} + \cdots + \frac{\alpha_{n-1,n-2}}{(\lambda - \lambda_{n-2})(\lambda - \lambda_n)} \Bigg) \\ &= -\frac{\sigma_{n-1}}{\lambda - \lambda_n} + \Bigg[\Bigg(\frac{\frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1}}{\lambda - \lambda_n} - \frac{\frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1}}{\lambda - \lambda_1} \Bigg) + \Bigg(\frac{\frac{\tilde{\beta}_{n,3}}{\lambda_n - \lambda_3}}{\lambda - \lambda_n} - \frac{\frac{\tilde{\beta}_{n,3}}{\lambda_n - \lambda_3}}{\lambda - \lambda_3} \Bigg) \\ &\quad + \cdots + \Bigg(\frac{\frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}}}{\lambda - \lambda_n} - \frac{\frac{\tilde{\beta}_{n,n-2}}{\lambda - \lambda_{n-2}}}{\lambda - \lambda_{n-2}} \Bigg) \Bigg] \\ &= - \sum_{i=1, i \ odd}^{n} \frac{\alpha_{n,i}}{\lambda - \lambda_i} \end{aligned}$$

where

$$\alpha_{n,i} = \frac{\tilde{\beta}_{n,i}}{\lambda_n - \lambda_i}$$

$$= \frac{-\alpha_{n-1,i}}{\lambda_n - \lambda_i}$$

for n odd, $i = 1, 3, \cdots, n-2$ odd, and,

$$\begin{aligned}
\alpha_{n,n} &= \sigma_{n-1} - \frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1} - \frac{\tilde{\beta}_{n,3}}{\lambda_n - \lambda_3} - \cdots - \frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}} \\
&= \sigma_{n-1} - \sum_{i=1, i \ odd}^{n-2} \frac{\alpha_{n-1,i}}{\lambda_n - \lambda_i}.
\end{aligned}$$

$$\begin{aligned}
M_{21}^{[n]} &= \frac{-b_n d_n A_n}{a_n c_n(\lambda - \lambda_n)} + \frac{b_n}{a_n} C_n + \frac{d_n}{c_n} B_n - (\lambda - \lambda_n) M_{12}^{[n-1]} \\
&= -(\lambda - \lambda_n) M_{12}^{[n-1]} \\
&= -(\lambda - \lambda_n)\Bigg(\zeta_{n-1} + \sum_{j=1, j \ even}^{n-1} \frac{\beta_{n-1,j}}{\lambda - \lambda_j}\Bigg) \\
&= -(\lambda - \lambda_n)\Bigg(\zeta_{n-1} + \Bigg(\frac{\beta_{n-1,2}}{\lambda - \lambda_2} + \frac{\beta_{n-1,4}}{\lambda - \lambda_4} + \cdots + \frac{\beta_{n-1,n-1}}{\lambda - \lambda_{n-1}}\Bigg)\Bigg) \\
&= -\zeta_{n-1}\lambda + \zeta_{n-1}\lambda_n - (\lambda - \lambda_n)\Bigg(\frac{\beta_{n-1,2}}{\lambda - \lambda_2} + \frac{\beta_{n-1,4}}{\lambda - \lambda_4} + \cdots + \frac{\beta_{n-1,n-1}}{\lambda - \lambda_{n-1}}\Bigg) \\
&= -\zeta_{n-1}\lambda + \zeta_{n-1}\lambda_n - \Bigg(\beta_{n-1,2} + \frac{\lambda_2\beta_{n-1,2} - \lambda_n\beta_{n-1,2}}{\lambda - \lambda_2} + \beta_{n-1,4} + \frac{\lambda_4\beta_{n-1,4} - \lambda_n\beta_{n-1,4}}{\lambda - \lambda_4} + \cdots \\
&\quad + \beta_{n-1,n-1} + \frac{\lambda_{n-1}\beta_{n-1,n-1} - \lambda_n\beta_{n-1,n-1}}{\lambda - \lambda_{n-1}}\Bigg).
\end{aligned}$$

However, since $\lambda_1 < \lambda_2 < \cdots$ we know that $\lambda_j\beta_{n-1,j} - \lambda_n\beta_{n-1,j} < 0$ for each $j = 1, \cdots, n$ even. Therefore,

$$M_{21}^{[n]} = -(\zeta_n + \eta_n\lambda) + \sum_{j=1, j \ even}^{n-1} \frac{\beta_{n,j}}{\lambda - \lambda_j},$$

where

$$\begin{aligned}
\beta_{n,j} &= \beta_{n-1,j}(\lambda_n - \lambda_j) \\
\zeta_n &= -\zeta_{n-1}\lambda_n + \sum_{j=1, j \ even}^{n-1} \beta_{n-1,j} \\
\eta_n &= \zeta_{n-1}
\end{aligned}$$

for n odd and $j = 2, 4, \cdots, n-1$ even.

Lastly,

$$\begin{aligned} M_{22}^{[n]} &= -\frac{d_n A_n}{c_n(\lambda - \lambda_n)} + C_n \\ &= C_n \\ &= M_{11}^{[0]}. \end{aligned}$$

Thus, applying the forward transformation to $M^{[n-1]}$ yields the following transmission matrix

$$M^{[n]} = \begin{bmatrix} M_{22}^{[0]} & -\sum_{i=1, i\ odd}^{n} \frac{\alpha_{n,i}}{\lambda - \lambda_i} \\ -(\zeta_n + \eta_n \lambda) + \sum_{j=1, j\ even}^{n-1} \frac{\beta_{n,j}}{\lambda - \lambda_j} & M_{11}^{[0]} \end{bmatrix}$$

where n is odd, $\eta_n > 0$ and $\zeta_n < 0$.

□

Remark 3.1.8. *We note that the transmission matrix $M^{[n]}$ can be expressed as*

$$M^{[n]} = \begin{bmatrix} M_{22}^{[0]} & r_n(\lambda) \\ -s_n(\lambda) & M_{11}^{[0]} \end{bmatrix},$$

where $r_n(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$, $s_n(\lambda) \in \mathcal{R}^+_{\frac{n-1}{2}}$ and n is odd.

To complete the results of this section we must consider the $(n+1)^{th}$ iteration which would then be even with base case (3.24).

Theorem 3.1.9. *The transmission matrix $M^{[n]}$ in (3.26) transforms under*

$$T_{z_{n+1}}(u_n) = \frac{W[u_n, z_{n+1}]}{z_{n+1}}$$

to a transmission condition given by

$$\begin{bmatrix} u_{n+1}(0^+) \\ u'_{n+1}(0^+) \end{bmatrix} = M^{[n+1]} \begin{bmatrix} u_{n+1}(0^-) \\ u'_{n+1}(0^-) \end{bmatrix},$$

where

$$M^{[n+1]} = \begin{bmatrix} M_{11}^{[0]} & \zeta_{n+1} + \sum_{j=1, j \ \textit{even}}^{n+1} \dfrac{\beta_{n+1,j}}{\lambda - \lambda_j} \\ \sigma_{n+1} - \sum_{i=1, i \ \textit{odd}}^{n+1} \dfrac{\alpha_{n+1,i}}{\lambda - \lambda_i} & M_{22}^{[0]} \end{bmatrix} \tag{3.27}$$

and $n+1 \in \mathbb{Z}$ *is even.*

Proof. Let $M^{[n]}$ be as given in (3.26). For $n+1 \in \mathbb{Z}$ even and $b_{n+1},\ d_{n+1} = 0$ we have from (3.16), (3.17) and (3.18)

$$\begin{aligned} A_{n+1} &= -M_{21}^{[n]} = \zeta_n + \eta_n \lambda - \sum_{j=1, j \ \textit{even}}^{n-1} \frac{\beta_{n,j}}{\lambda - \lambda_j}, \\ B_{n+1} &= -M_{22}^{[n]} = -M_{11}^{[0]}, \\ C_{n+1} &= M_{11}^{[n]} = M_{22}^{[0]}. \end{aligned}$$

We now look at each of the entries of $M^{[n+1]}$ by using formula (3.24) for $M^{[n+1]}$ (i.e. where 2 is now replaced by $n+1$) to obtain

$$\begin{aligned} M_{11}^{[n+1]} &= -B_{n+1} \\ &= M_{11}^{[0]}, \end{aligned}$$

$$\begin{aligned} M_{12}^{[n+1]} &= \frac{A_{n+1}}{\lambda - \lambda_{n+1}} \\ &= \frac{\eta_n(\lambda - \lambda_{n+1}) + \eta_n \lambda_{n+1}}{\lambda - \lambda_{n+1}} + \frac{\zeta_n}{\lambda - \lambda_{n+1}} - \sum_{j=1, j \ \textit{even}}^{n-1} \frac{\beta_{n,j}}{(\lambda - \lambda_j)(\lambda - \lambda_{n+1})} \\ &= \eta_n + \frac{\eta_n \lambda_{n+1}}{\lambda - \lambda_{n+1}} + \frac{\zeta_n}{\lambda - \lambda_{n+1}} - \left(\frac{\beta_{n,2}}{(\lambda - \lambda_2)(\lambda - \lambda_{n+1})} + \frac{\beta_{n,4}}{(\lambda - \lambda_4)(\lambda - \lambda_{n+1})} + \cdots \right. \end{aligned}$$

$$+\frac{\beta_{n,n-1}}{(\lambda-\lambda_{n-1})(\lambda-\lambda_{n+1})}\Bigg)$$

$$= \eta_n + \frac{\eta_n\lambda_{n+1}}{\lambda-\lambda_{n+1}} + \frac{\zeta_n}{\lambda-\lambda_{n+1}} - \left[\left(\frac{\frac{\tilde{\beta}_{n+1,2}}{\lambda_{n+1}-\lambda_2}}{\lambda-\lambda_{n+1}} - \frac{\frac{\tilde{\beta}_{n+1,2}}{\lambda_{n+1}-\lambda_2}}{\lambda-\lambda_2}\right) + \left(\frac{\frac{\tilde{\beta}_{n+1,4}}{\lambda_{n+1}-\lambda_4}}{\lambda-\lambda_{n+1}} - \frac{\frac{\tilde{\beta}_{n+1,4}}{\lambda_{n+1}-\lambda_4}}{\lambda-\lambda_4}\right)\right.$$

$$\left. + \cdots + \left(\frac{\frac{\tilde{\beta}_{n+1,n-1}}{\lambda_{n+1}-\lambda_{n-1}}}{\lambda-\lambda_{n+1}} - \frac{\frac{\tilde{\beta}_{n+1,n-1}}{\lambda_{n+1}-\lambda_{n-1}}}{\lambda-\lambda_{n-1}}\right)\right]$$

$$= \zeta_{n+1} + \sum_{j=1, j \ even}^{n+1} \frac{\beta_{n+1,j}}{\lambda-\lambda_j}$$

where

$$\beta_{n+1,j} = \frac{\tilde{\beta}_{n+1,j}}{\lambda_{n+1}-\lambda_j} = \frac{\beta_{n,i}}{\lambda_{n+1}-\lambda_i},$$

and

$$\beta_{n+1,n+1} = \zeta_n + \eta_n\lambda_{n+1} - \frac{\tilde{\beta}_{n+1,2}}{\lambda_{n+1}-\lambda_2} - \frac{\tilde{\beta}_{n+1,4}}{\lambda_{n+1}-\lambda_4} - \cdots - \frac{\tilde{\beta}_{n+1,n-1}}{\lambda_{n+1}-\lambda_{n-1}}$$
$$= \zeta_n + \eta_n\lambda_{n+1} - \sum_{j=1, j \ even}^{n-1} \frac{\beta_{n+1}}{\lambda_{n+1}-\lambda_i}$$

with

$$\zeta_{n+1} = \eta_n,$$

for $n+1$ even and $j = 2, 4, \ldots, n-1$ even.

Finally,

$$M_{21}^{[n+1]} = -(\lambda-\lambda_{n+1})M_{12}^{[n]}$$
$$= -(\lambda-\lambda_{n+1})\Bigg(-\sum_{i=1, i \ odd}^{n} \frac{\alpha_{n,i}}{\lambda-\lambda_i}\Bigg)$$
$$= (\lambda-\lambda_{n+1})\Bigg(\frac{\alpha_{n,1}}{\lambda-\lambda_1} + \frac{\alpha_{n,3}}{\lambda-\lambda_3} + \cdots + \frac{\alpha_{n,n}}{\lambda-\lambda_n}\Bigg)$$

$$= \left(\alpha_{n,1} + \frac{\lambda_1 \alpha_{n,1} - \lambda_{n+1}\alpha_{n,1}}{\lambda - \lambda_1} + \alpha_{n,3} + \frac{\lambda_3 \alpha_{n,3} - \lambda_{n+1}\alpha_{n,3}}{\lambda - \lambda_3} + \cdots \right.$$
$$\left. + \alpha_{n,n} + \frac{\lambda_n \alpha_{n,n} - \lambda_{n+1}\alpha_{n,n}}{\lambda - \lambda_n} \right).$$

However, since $\lambda_1 < \lambda_2 < \cdots$ we know that $\lambda_i \alpha_{n,i} - \lambda_{n+1}\alpha_{n,i} < 0$ for each $i = 1, \cdots, n$ odd. Therefore,

$$M_{21}^{[n+1]} = \sigma_{n+1} - \sum_{i=1, i \ odd}^{n} \frac{\alpha_{n+1,i}}{\lambda - \lambda_i},$$

where

$$\alpha_{n+1,i} = \alpha_{n,i}(\lambda_{n+1} - \lambda_i),$$
$$\sigma_{n+1} = \sum_{i=1.i \ odd}^{n} \alpha_{n,i}$$

for $n+1$ even and $i = 1, 3, \cdots, n$ odd.

Finally,

$$M_{22}^{[n+1]} = C_{n+1}$$
$$= M_{22}^{[0]}.$$

Thus, applying the forward transformation to $M^{[n]}$ yields the following transmission matrix

$$M^{[n+1]} = \begin{bmatrix} M_{11}^{[0]} & \zeta_{n+1} + \sum_{j=1, j \ even}^{n+1} \frac{\beta_{n+1,j}}{\lambda - \lambda_j} \\ \sigma_{n+1} - \sum_{i=1, i \ odd}^{n} \frac{\alpha_{n+1,i}}{\lambda - \lambda_i} & M_{22}^{[0]} \end{bmatrix}$$

where $n+1$ is even. □

Remark 3.1.10. *We note that the transmission matrix $M^{[n+1]}$ can be expressed as*

$$M^{[n+1]} = \begin{bmatrix} M_{11}^{[0]} & -r_{n+1}(\lambda) \\ s_{n+1}(\lambda) & M_{22}^{[0]} \end{bmatrix},$$

where $r_{n+1}(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$ *and* $s_{n+1}(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$.

Thus, to summarise the results of Chapter 3, we have shown that the forward transformation yields two classes of the transmission matrix.

(i) **Class 1:** m even

$$M^{[m]} = \begin{bmatrix} M^{[0]}_{11} & -r_m(\lambda) \\ s_m(\lambda) & M^{[0]}_{22} \end{bmatrix}$$

where $r_m(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$ and $s_m(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$.

(ii) **Class 2:** n odd

$$M^{[n]} = \begin{bmatrix} M^{[0]}_{22} & r_n(\lambda) \\ -s_n(\lambda) & M^{[0]}_{11} \end{bmatrix}$$

where $r_n(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$ and $s_n(\lambda) \in \mathcal{R}^+_{\frac{n-1}{2}}$.

That is, the transmission matrix alternates between these two forms as we successively apply the forward Darboux-Crum transformation.

Chapter 4

Inverse Transformation

In this chapter we will compute the concrete inverse transformation. The aim is to illustrate how the inverse transformation combined with the correct choice of parameters can reverse the results of the forward transformation and map our transmission matrix, $M^{[n+1]}$, back to the initial transmission matrix, $M^{[0]}$. We will also observe how this transformation allows us to move back down the hierarchy as it strips the transmission condition of its λ-dependence in each step.

4.1 Concrete Transformation

Theorem 4.1.1. *Let $\lambda_1 < \lambda_2 < \cdots \in \mathbb{R}$. If*

$$u_1^- := y' - \frac{z_1^{-\prime}}{z_1^-} y, \tag{4.1}$$

where z_1^- is a solution of (3.1) for $\lambda = \lambda_1$. Then u_1^- obeys (3.1) with q replaced by $q_1^- = q + 2\left(\frac{z_1^{-\prime}}{z_1^-}\right)'$.

Proof. Let z_1^- be a solution of $-z_1^{-\prime\prime} + qz_1^- = \lambda_1 z_1^-$. We can define $w_1^- = -\frac{z_1^{-\prime}}{z_1^-}$ and note that

$$w_1^{-\prime} = \lambda_1 - q + w_1^{2^-}. \tag{4.2}$$

Define $u_0 := y$ and let

$$u_1^- = u_0' + w_1^- u_0. \tag{4.3}$$

By the product rule, we get

$$u_1^{-\prime} = (\lambda_1 - \lambda)u_0 + w_1^- u_1^-. \tag{4.4}$$

Moreover, (4.2) - (4.4) gives

$$\begin{aligned} u_1^{-\prime\prime} &= (\lambda_1 - \lambda)(u_1^- - w_1^- u_0) + w_1^{-\prime} u_1^- + w_1^-(\lambda_1 - \lambda)u_0 + w_1^{2^-} u_1^- \\ &= (\lambda_1 - \lambda)u_1^- + (\lambda_1 - q + 2w_1^{2^-})u_1^- \\ &= (\lambda_1 - \lambda)u_1^- + (2w_1^{-\prime} - 2\lambda_1 + 2q + \lambda_1 - q)u_1^- \\ &= (2w_1^{-\prime} + q - \lambda)u_1^- \end{aligned}$$

so u_1^- satisfies (3.1) with q replaced by $q_1^- = q + 2w_1^{-\prime}$. □

If we repeat this procedure $n+1$ successive times, i.e. let $w_{n+1}^- = \frac{z_{n+1}^{-\prime}}{z_{n+1}^-}$ where z_{n+1}^- is a solution of the nth equation and define

$$u_{n+1}^- = u_n^{-\prime} + w_{n+1}^- u_n^-, \tag{4.5}$$

then

$$u_{n+1}^{-\prime\prime} = (2w_{n+1}^{-\prime} + q_n^- - \lambda)u_{n+1}^-.$$

Thus u_{n+1}^- obeys (3.1) with q replaced by $q_{n+1}^- = q_n^- + 2w_{n+1}^{-\prime}$ in the $(n+1)^{th}$ iteration of the inverse transformation.

Binding et al. have shown in [4] that this inverse transformation applied to a Sturm-Liouville problem with boundary conditions dependent on the eigenparameter combined with a suitable choice of transformation parameters yields a new boundary value problem whose spectrum contains all the same eigenvalues and in addition a new least eigenvalue.

Remark 4.1.2. *The negative superscripts are to indicate that we are working with the inverse tranformation. This, notation will allow us to observe how the inverse transformation maps the transmission matrix given in (3.27) back to* $M^{[0]}$*.*

Now let

$$
\begin{aligned}
z_n^-(0^-) &= a_n^-,\\
z_n^{-\prime}(0^-) &= -b_n^-,\\
z_n^-(0^+) &= c_n^-,\\
z_n^{-\prime}(0^+) &= -d_n^-,
\end{aligned}
$$

where z_n^- is a solution to the $(n+1)^{th}$ transformed equation. Let $a_n^-, b_n^-, c_n^-, d_n^- \in \mathbb{R}$. In addition, let

$$A_n^- = \frac{d_n^-}{c_n^-}\left(M_{11}^{[n+1]^-} - \frac{b_n^-}{a_n^-}M_{12}^{[n+1]^-}\right) + \left(M_{21}^{[n+1]^-} - \frac{b_n^-}{a_n^-}M_{22}^{[n+1]^-}\right), \tag{4.6}$$

$$B_n^- = \frac{d_n^-}{c_n^-}M_{12}^{[n+1]^-} + M_{22}^{[n+1]^-}, \tag{4.7}$$

$$C_n^- = M_{11}^{[n+1]^-} - \frac{b_n^-}{a_n^-}M_{12}^{[n+1]^-}, \tag{4.8}$$

where $M_{ij}^{[n+1]^-}$ for $i, j = 1, 2$ are entries of the transmission matrix $M^{[n+1]}$. The case of $n = -1$ is considered in the theorem below.

Theorem 4.1.3. *The transmission condition (3.10) of the boundary value problem given by (3.1) - (3.5) transforms under*

$$u_1^- = u_0' - \frac{z_1'^-}{z_1^-}u_0,$$

where $u_0 := y$, to a transmission condition, N, such that

$$\begin{bmatrix} u_1^-(0^+) \\ u_1^{-\prime}(0^+) \end{bmatrix} = N \begin{bmatrix} u_1^-(0^-) \\ u_1^{-\prime}(0^-) \end{bmatrix}$$

where

$$N = \begin{bmatrix} \frac{b_1^- A_1^-}{a_1^-(\lambda-\lambda_1)} + B_1^- & \frac{-A_1^-}{\lambda-\lambda_1} \\ \frac{b_1^- d_1^- A_1^-}{a_1^- c_1^-(\lambda-\lambda_1)} - \frac{b_1^-}{a_1^-}C_1^- + \frac{d_1^-}{c_1^-}B_1^- - (\lambda-\lambda_1)M_{12} & \frac{-d_1^- A_1^-}{c_1^-(\lambda-\lambda_1)} + C_1^- \end{bmatrix}. \qquad (4.9)$$

Here $a_1^-,\ b_1^-,\ c_1^-,\ d_1^-,\ A_1^-,\ B_1^-$ and $C_1^- \in \mathbb{R}$ and are as given above.

Proof. By (4.3) and (4.4) we have

$$u_1^-(0^+) = u_0'(0^+) + \frac{d_1^-}{c_1^-}u_0(0^+)$$

and

$$\begin{aligned} u_1^{-\prime}(0^+) &= (\lambda_1 - \lambda)u_0(0^+) + \frac{d_1^-}{c_1^-}u_1^-(0^+) \\ &= (\lambda_1 - \lambda)u_0(0^+) + \frac{d_1^-}{c_1^-}\big(u_0'(0^+) + \frac{d_1^-}{c_1^-}u_0(0^+)\big) \\ &= \left(\lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2\right)u_0(0^+) + \frac{d_1^-}{c_1^-}u_0'(0^+). \end{aligned}$$

Similarly for $u_1^-(0^-)$ we have

$$u_1^-(0^-) = u_0'(0^-) + \frac{b_1^-}{a_1^-}y(0^-)$$

and

$$\begin{aligned} u_1^{-\prime}(0^-) &= (\lambda_1 - \lambda)u_0(0^-) + \frac{b_1^-}{a_1^-}u_1^-(0^-) \\ &= (\lambda_1 - \lambda)u_0(0^-) + \frac{b_1^-}{a_1^-}\big(u_0'(0^-) + \frac{b_1^-}{a_1^-}u_0(0^-)\big) \\ &= \left(\lambda_1 - \lambda + \left(\frac{b_1^-}{a_1^-}\right)^2\right)u_0(0^-) + \frac{b_1^-}{a_1^-}u_0'(0^-). \end{aligned}$$

Expressing the above system of equations in matrix form gives

$$\begin{bmatrix} u_1^-(0^+) \\ u_1^{-\prime}(0^+) \end{bmatrix} = \begin{bmatrix} \frac{d_1^-}{c_1^-} & 1 \\ \lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2 & \frac{d_1^-}{c_1^-} \end{bmatrix} \begin{bmatrix} u_0(0^+) \\ u_0'(0^+) \end{bmatrix} \tag{4.10}$$

and

$$\begin{bmatrix} u_1^-(0^-) \\ u_1^{-\prime}(0^-) \end{bmatrix} = \begin{bmatrix} \frac{b_1^-}{a_1^-} & 1 \\ \lambda_1 - \lambda + \left(\frac{b_1^-}{a_1^-}\right)^2 & \frac{b_1^-}{a_1^-} \end{bmatrix} \begin{bmatrix} u_0(0^-) \\ u_0'(0^-) \end{bmatrix}. \tag{4.11}$$

We label the coefficient matrices in (4.10) and (4.11) K^+ and K^- respectively. Thus,

$$\begin{aligned} (K^+)^{-1} \begin{bmatrix} u_1^-(0^+) \\ u_1^{-\prime}(0^+) \end{bmatrix} &= \begin{bmatrix} u_0(0^+) \\ u_0'(0^+) \end{bmatrix} \\ &= M^{[0]} \begin{bmatrix} u_0(0^-) \\ u_0'(0^-) \end{bmatrix} \\ &= M^{[0]}(K^-)^{-1} \begin{bmatrix} u_1^-(0^-) \\ u_1^{-\prime}(0^-) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{bmatrix} u_1^-(0^+) \\ u_1^{-\prime}(0^+) \end{bmatrix} = K^+ M^{[0]} (K^-)^{-1} \begin{bmatrix} u_1^-(0^-) \\ u_1^{-\prime}(0^-) \end{bmatrix}$$

where $\det(K^-) = \left(\frac{b_1^-}{a_1^-}\right)^2 - \lambda_1 + \lambda - \left(\frac{b_1^-}{a_1^-}\right)^2 = \lambda - \lambda_1$. Now, let $N = K^+ M^{[0]} (K^-)^{-1}$ which yields a transmission matrix of the form given in (3.10), where $M^{[0]}$ is defined as it is in (3.11), therefore

$$N = \begin{bmatrix} \frac{d_1^-}{c_1^-} & 1 \\ \lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2 & \frac{d_1^-}{c_1^-} \end{bmatrix} \begin{bmatrix} M_{11}^{[0]} & M_{12}^{[0]} \\ M_{21}^{[0]} & M_{22}^{[0]} \end{bmatrix} \begin{bmatrix} \frac{b_1^-}{a_1^-(\lambda-\lambda_1)} & -\frac{1}{\lambda-\lambda_1} \\ \frac{\lambda-\lambda_1-\left(\frac{b_1^-}{a_1^-}\right)^2}{\lambda-\lambda_1} & \frac{b_1^-}{a_1^-(\lambda-\lambda_1)}. \end{bmatrix}$$

Matrix multiplication gives the following entries for the matrix N

$$N_{11} = \frac{\frac{b_1^-}{a_1^-}\left(\frac{d_1^-}{c_1^-}M_{11}^{[0]} + M_{21}^{[0]}\right) + \left(\lambda - \lambda_1 - \left(\frac{b_1^-}{a_1^-}\right)^2\right)\left(\frac{d_-}{c_1^-}M_{12}^{[0]} + M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$= \frac{\frac{b_1^- d_1^-}{a_1^- c_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right) + \frac{b_1^-}{a_1^-}\left(M_{21}^{[0]} - \frac{b_1^-}{a_1^-}M_{22}^{[0]}\right)}{\lambda - \lambda_1} + \frac{d_1^-}{c_1^-}M_{12}^{[0]} + M_{22}^{[0]}$$

$$= \frac{\frac{b_1^-}{a_1^-}\left[\frac{d_1^-}{c_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right) + \left(M_{21}^{[0]} - \frac{b_1^-}{a_1^-}M_{22}^{[0]}\right)\right]}{\lambda - \lambda_1} + \left(\frac{d_1^-}{c_1^-}M_{12}^{[0]} + M_{22}^{[0]}\right)$$

$$N_{12} = \frac{-\frac{d_1^-}{c_1^-}M_{11}^{[0]} - M_{21}^{[0]} + \frac{b_1^-}{a_1^-}\left(\frac{d_1^-}{c_1^-}M_{12}^{[0]} + M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$= \frac{-\frac{d_1^-}{c_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right) - \left(M_{21}^{[0]} - \frac{b_1^-}{a_1^-}M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$N_{21} = \frac{\frac{b_1^-}{a_1^-}\left(\left(\lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2\right)M_{11}^{[0]} + \frac{d_1^-}{c_1^-}M_{21}^{[0]}\right)}{\lambda - \lambda_1}$$

$$+ \frac{\left(\lambda - \lambda_1 - \left(\frac{b_1^-}{a_1^-}\right)^2\right)\left(\left(\lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2\right)M_{12}^{[0]} + \frac{d_1^-}{c_1^-}M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$= \frac{\frac{b_1^-}{a_1^-}(\lambda_1 - \lambda)M_{11}^{[0]} + \frac{b_1^-}{a_1^-}\left(\frac{d_1^-}{c_1^-}\right)^2 M_{11}^{[0]} + \frac{b_1^- d_1^-}{a_1^- c_1^-}M_{21}^{[0]}}{\lambda - \lambda_1}$$

$$+ \frac{\left(\lambda - \lambda_1\right)\left((\lambda_1 - \lambda)M_{12}^{[0]} + \left(\frac{d_1^-}{c_1^-}\right)^2 M_{12}^{[0]} + \frac{d_1^-}{c_1^-}M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$- \frac{\left(\frac{b_1^-}{a_1^-}\right)^2\left((\lambda_1 - \lambda)M_{12}^{[0]} + \left(\frac{d_1^-}{c_1^-}\right)^2 M_{12}^{[0]} + \frac{d_1^-}{c_1^-}M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$= \frac{\frac{b_1^- d_1^-}{a_1^- c_1^-}\left[\frac{d_1^-}{c_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right) + \left(M_{21}^{[0]} - \frac{b_1^-}{a_1^-}M_{22}^{[0]}\right)\right]}{\lambda - \lambda_1} - \frac{b_1^-}{a_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right)$$

$$+ \frac{d_1^-}{c_1^-}\left(\frac{d_1^-}{c_1^-}M_{12}^{[0]} + M_{22}^{[0]}\right) + (\lambda_1 - \lambda)M_{12}^{[0]}$$

$$N_{22} = \frac{-\frac{d_1^-}{c_1^-}M_{21}^{[0]} - \left(\lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2\right)M_{11}^{[0]} + \frac{b_1^-}{a_1^-}\left(\left(\lambda_1 - \lambda + \left(\frac{d_1^-}{c_1^-}\right)^2\right)M_{12}^{[0]} + \frac{d_1^-}{c_1^-}M_{22}^{[0]}\right)}{\lambda - \lambda_1}$$

$$= \frac{-\frac{d_1^-}{c_1^-}M_{21}^{[0]} - (\lambda_1 - \lambda)M_{11}^{[0]} - \left(\frac{d_1^-}{c_1^-}\right)^2 M_{11}^{[0]} + \frac{b_1^- d_1^-}{a_1^- c_1^-}M_{22}^{[0]} + \frac{b_1^-}{a_1^-}(\lambda_1 - \lambda)M_{12}^{[0]} + \frac{b_1^-}{a_1^-}\left(\frac{d_1^-}{c_1^-}\right)^2 M_{12}^{[0]}}{\lambda - \lambda_1}$$

$$= \frac{-\frac{d_1^-}{c_1^-}\left[\frac{d_1^-}{c_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right) + \left(M_{21}^{[0]} - \frac{b_1^-}{a_1^-}M_{22}^{[0]}\right)\right]}{\lambda - \lambda_1} + \left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right)$$

Now

$$A_1^- = \frac{d_1^-}{c_1^-}\left(M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}\right) + \left(M_{21}^{[0]} - \frac{b_1^-}{a_1^-}M_{22}^{[0]}\right),$$

$$B_1^- = \frac{d_1^-}{c_1^-}M_{12}^{[0]} + M_{22}^{[0]},$$

$$C_1^- = M_{11}^{[0]} - \frac{b_1^-}{a_1^-}M_{12}^{[0]}.$$

Thus

$$N_{11} = \frac{b_1^- A_1^-}{a_1^-(\lambda - \lambda_1)} + B_1^-,$$

$$N_{12} = \frac{-A_1^-}{\lambda - \lambda_1},$$

$$N_{21} = \frac{b_1^- d_1^- A_1^-}{a_1^- c_1^-(\lambda - \lambda_1)} - \frac{b_1^-}{a_1^-}C_1^- + \frac{d_1^-}{c_1^-}B_1^- - (\lambda - \lambda_1)M_{12},$$

$$N_{22} = \frac{-d_1^- A_1^-}{c_1^-(\lambda - \lambda_1)} + C_1^-.$$

These are the entries of the matrix given in (4.9) therefore proving our result. $\square$

We will assume, without loss of generality, that $z_n^{-\prime}(0^-) = b_n^- = 0 = d_n^- = z_n^{-\prime}(0^+)$ for all $n \in \mathbb{N}$. Therefore, the matrix N in (4.9) takes the form

$$N = \begin{bmatrix} M_{22}^{[0]} & -\frac{M_{21}^{[0]}}{\lambda - \lambda_1} \\ -(\lambda - \lambda_1)M_{12}^{[0]} & M_{11}^{[0]} \end{bmatrix}. \tag{4.12}$$

Note that, like the forward transformation, the inverse transformation has increased the λ-dependence of the transmission condition. By making suitable choices for our coefficients $\alpha_{n,i}$ and $\beta_{n,i}$ (from our corresponding transmission matrices in Chapter 3) we can ensure that the inverse transformation acts as an inverse mapping of the forward transformation, therefore, decreasing the λ-dependence of the transmission condition in each step.

Remark 4.1.4. *We will begin by assuming that our initial transmission matrix is* $M^{[n+1]^-} = M^{[n+1]}$. *Using the notation of the subclasses of* $\mathcal{R}_N$ *introduced in Section 2.1, where the case of* $\eta_n > 0$ *is denoted by* $\mathcal{R}_N^+$ *and the case of* $\eta_n = 0$ *is denoted by* $\mathcal{R}_N^0$, *we note that the transmission matrix* $M^{[n+1]^-}$ *can be expressed as*

$$M^{[n+1]^-} = \begin{bmatrix} M_{11}^{[0]} & -r_{n+1}^-(\lambda) \\ s_{n+1}^-(\lambda) & M_{22}^{[0]} \end{bmatrix},$$

where $r_{n+1}^-(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$ *and* $s_{n+1}^-(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$.

Theorem 4.1.5. *The transmission matrix* $M^{[n+1]}$ *in (3.27) transforms under*

$$T_{z_1^-}(u_{n+1}^-) = \frac{u_{n+1}^{-\prime} z_1^- - z_1^{-\prime} u_{n+1}^-}{z_1^-}$$

to a transmission condition given by

$$\begin{bmatrix} u_n^-(0^+) \\ u_n^{-\prime}(0^+) \end{bmatrix} = M^{[n]^-} \begin{bmatrix} u_n^-(0^-) \\ u_n^{-\prime}(0^-) \end{bmatrix},$$

where $M^{[n]^-}$ *is precisely* $M^{[n]}$ *as given by (3.26) and* $n+1 \in \mathbb{N}$ *is even. Here*

$$\alpha_{n+1,i} = \alpha_{n,i}(\lambda_{n+1} - \lambda_i) \tag{4.13}$$

where $n+1$ *is even and* $i = 1, 3, \cdots, n$ *is odd and*

$$\beta_{n+1,j} = \frac{\beta_{n,j}}{\lambda_{n+1} - \lambda_j},$$

$$\beta_{n+1,n+1} = \zeta_n + \eta_n \lambda_{n+1} - \sum_{j=1, j \text{ even}}^{n-1} \frac{\beta_{n,j}}{\lambda_{n+1} - \lambda_j} \tag{4.14}$$

where $n+1$ *is even and* $j = 2, 4, \cdots, n-1$.

Proof. For $n+1 \in \mathbb{N}$ even and $b_1^- = d_1^- = 0$ we have from (4.6), (4.7) and (4.8)

$$\begin{aligned} A_1^- &= M_{21}^{[n+1]^-} = \sigma_{n+1} - \sum_{i=1, i\ odd}^{n} \frac{\alpha_{n+1,i}}{\lambda - \lambda_i}, \\ B_1^- &= M_{22}^{[n+1]^-} = M_{22}^{[0]}, \\ C_1^- &= M_{11}^{[n+1]^-} = M_{11}^{[0]}. \end{aligned}$$

We now consider the entries of $M^{[n]^-}$ individually with $b_1^- = d_1^- = 0$. Using the approach in Theorem 4.1.3, we obtain

$$\begin{aligned} M_{11}^{[n]^-} &= B_1^- \\ &= M_{22}^{[0]} \end{aligned}$$

and

$$\begin{aligned} M_{12}^{[n]^-} &= -\frac{A_1^-}{\lambda - \lambda_{n+1}} \\ &= -\frac{1}{\lambda - \lambda_{n+1}} \left[\sigma_{n+1} - \frac{\alpha_{n+1,1}}{\lambda - \lambda_1} - \frac{\alpha_{n+1,3}}{\lambda - \lambda_3} - \cdots - \frac{\alpha_{n+1,n}}{\lambda - \lambda_n} \right] \\ &= -\frac{\sigma_{n+1}}{\lambda - \lambda_{n+1}} + \frac{\alpha_{n+1,1}}{(\lambda - \lambda_1)(\lambda - \lambda_{n+1})} + \frac{\alpha_{n+1,3}}{(\lambda - \lambda_3)(\lambda - \lambda_{n+1})} + \cdots + \frac{\alpha_{n+1,n}}{(\lambda - \lambda_n)(\lambda - \lambda_{n+1})}. \end{aligned}$$

Using (4.13) yields,

$$\begin{aligned} M_{12}^{[n]^-} &= -\frac{\alpha_{n,1} + \alpha_{n,3} + \cdots + \alpha_{n,n}}{\lambda - \lambda_{n+1}} + \frac{\alpha_{n,1}(\lambda_{n+1} - \lambda_1)}{(\lambda - \lambda_1)(\lambda - \lambda_{n+1})} + \frac{\alpha_{n,3}(\lambda_{n+1} - \lambda_3)}{(\lambda - \lambda_3)(\lambda - \lambda_{n+1})} \\ &\quad + \cdots + \frac{\alpha_{n,n}(\lambda_{n+1} - \lambda_n)}{(\lambda - \lambda_n)(\lambda - \lambda_{n+1})} \\ &= -\frac{\alpha_{n,1} + \alpha_{n,3} + \cdots + \alpha_{n,n}}{\lambda - \lambda_{n+1}} + \frac{\alpha_{n,1}(\lambda - \lambda_1) - \alpha_{n,1}\lambda + \alpha_{n,1}\lambda_{n+1}}{(\lambda - \lambda_1)(\lambda - \lambda_{n+1})} \\ &\quad + \frac{\alpha_{n,3}(\lambda - \lambda_3) - \alpha_{n,3}\lambda + \alpha_{n,3}\lambda_{n+1}}{(\lambda - \lambda_3)(\lambda - \lambda_{n+1})} + \cdots + \frac{\alpha_{n,n}(\lambda - \lambda_n) - \alpha_{n,n}\lambda + \alpha_{n,n}\lambda_{n+1}}{(\lambda - \lambda_n)(\lambda - \lambda_{n+1})} \\ &= -\frac{\alpha_{n,1}}{\lambda - \lambda_1} - \frac{\alpha_{n,3}}{\lambda - \lambda_3} - \cdots - \frac{\alpha_{n,n}}{\lambda - \lambda_n} \\ &= -\sum_{i=1, i\ odd}^{n} \frac{\alpha_{n,i}}{\lambda - \lambda_i}. \end{aligned}$$

In addition,

$$
\begin{aligned}
M_{21}^{[n]^-} &= -(\lambda-\lambda_{n+1})M_{12}^{[n+1]^-}\\
&= -(\lambda-\lambda_{n+1})\left[\zeta_{n+1}+\frac{\beta_{n+1,2}}{\lambda-\lambda_2}+\cdots+\frac{\beta_{n+1,n-1}}{\lambda-\lambda_{n-1}}+\frac{\beta_{n+1,n+1}}{\lambda-\lambda_{n+1}}\right]\\
&= -\zeta_{n+1}\lambda+\zeta_{n+1}\lambda_{n+1}-\frac{\beta_{n+1,2}(\lambda-\lambda_{n+1})}{\lambda-\lambda_2}-\cdots-\frac{\beta_{n+1,n-1}(\lambda-\lambda_{n+1})}{\lambda-\lambda_{n-1}}-\beta_{n+1,n+1}.
\end{aligned}
$$

By (4.14) we obtain

$$
\begin{aligned}
M_{21}^{[-n]} &= -\eta_n\lambda+\eta_n\lambda_{n+1}-\frac{\beta_{n+1,2}(\lambda-\lambda_2)+\beta_{n+1,2}\lambda_2-\beta_{n+1,2}\lambda_{n+1}}{\lambda-\lambda_2}\\
&\quad -\cdots-\frac{\beta_{n+1,n-1}(\lambda-\lambda_{n-1})+\beta_{n+1,n-1}\lambda_{n-1}-\beta_{n+1,n-1}\lambda_{n+1}}{\lambda-\lambda_{n-1}}-\zeta_n-\eta_n\lambda_{n+1}\\
&\quad +\frac{\tilde{\beta}_{n+1,2}}{\lambda_{n+1}-\lambda_2}+\frac{\tilde{\beta}_{n+1,4}}{\lambda_{n+1}-\lambda_4}+\cdots+\frac{\tilde{\beta}_{n+1,n-1}}{\lambda_{n+1}-\lambda_{n-1}}\\
&= -\zeta_n-\eta_n\lambda-\left[\frac{\tilde{\beta}_{n+1,2}}{\lambda_{n+1}-\lambda_2}-\frac{\tilde{\beta}_{n+1,2}}{\lambda_{n+1}-\lambda_2}+\frac{\tilde{\beta}_{n+1,4}}{\lambda_{n+1}-\lambda_4}-\frac{\tilde{\beta}_{n+1,4}}{\lambda_{n+1}-\lambda_4}\right.\\
&\quad \left.+\cdots+\frac{\tilde{\beta}_{n+1,n-1}}{\lambda_{n+1}-\lambda_{n-1}}-\frac{\tilde{\beta}_{n+1,n-1}}{\lambda_{n+1}-\lambda_{n-1}}\right]+\frac{\beta_{n+1,2}(\lambda_{n+1}-\lambda_2)}{\lambda-\lambda_2}+\frac{\beta_{n+1,4}(\lambda_{n+1}-\lambda_4)}{\lambda-\lambda_4}\\
&\quad +\cdots+\frac{\beta_{n+1,n-1}(\lambda_{n+1}-\lambda_{n-1})}{\lambda-\lambda_{n-1}}\\
&= -\zeta_n-\eta_n\lambda+\frac{\tilde{\beta}_{n+1,2}}{\lambda-\lambda_2}+\frac{\tilde{\beta}_{n+1,4}}{\lambda-\lambda_4}+\cdots+\frac{\tilde{\beta}_{n+1,n-1}}{\lambda-\lambda_{n-1}}\\
&= -\zeta_n-\eta_n\lambda+\frac{\beta_{n,2}}{\lambda-\lambda_2}+\frac{\beta_{n,4}}{\lambda-\lambda_4}+\cdots+\frac{\beta_{n,n-1}}{\lambda-\lambda_{n-1}}\\
&= -(\zeta_n+\eta_n)+\sum_{j=1,j\ even}^{n-1}\frac{\beta_{n,j}}{\lambda-\lambda_j}.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
M_{22}^{[n]^-} &= C_1^-\\
&= M_{11}^{[0]}.
\end{aligned}
$$

Thus, the inverse transformation yields the transmission matrix given in (3.26), that is,

$$M^{[n]^-} = \begin{bmatrix} M_{22}^{[0]} & -\sum_{i=1,i\ odd}^{n} \frac{\alpha_{n,i}}{\lambda-\lambda_i} \\ -(\zeta_n+\eta_n\lambda)+\sum_{j=1,j\ even}^{n-1} \frac{\beta_{n,j}}{\lambda-\lambda_j} & M_{11}^{[0]} \end{bmatrix} = M^{[n]}$$

where n is odd. □

Remark 4.1.6. *We note that the transmission matrix $M^{[n]^-}$ can be expressed as*

$$M^{[n]^-} = \begin{bmatrix} M_{22}^{[0]} & r_n^-(\lambda) \\ -s_n^-(\lambda) & M_{11}^{[0]} \end{bmatrix},$$

where $r(\lambda) \in \mathcal{R}^0_{\frac{n+1}{2}}$ and $s(\lambda) \in \mathcal{R}^+_{\frac{n-1}{2}}$.

Theorem 4.1.7. *The the transmission matrix $M^{[n]}$ in (3.26) transforms under*

$$T_{z_2^-}(u_n^-) = \frac{u_n^{-\prime} z_2^- - z_2^{-\prime} u_n^-}{z_2^-}$$

to a transmission condition given by

$$\begin{bmatrix} u_{n-1}^-(0^+) \\ u_{n-1}^{-\prime}(0^+) \end{bmatrix} = M^{[n-1]^-} \begin{bmatrix} u_{n-1}^-(0^-) \\ u_{n-1}^{-\prime}(0^-) \end{bmatrix},$$

where $M^{[n-1]^-}$ is given by (3.25) and $n \in \mathbb{N}$ is odd. Here we assume that

$$\begin{aligned} \alpha_{n,i} &= \frac{-\alpha_{n-1,i}}{\lambda_n - \lambda_i}, \\ \alpha_{n,n} &= \sigma_{n-1} - \sum_{i=1,i\ odd}^{n-2} \frac{\alpha_{n-1,i}}{\lambda_n - \lambda_i} \end{aligned} \tag{4.15}$$

where n is odd and $i = 1, 3, \cdots, n-2$ is odd and

$$\beta_{n,j} = \beta_{n-1,j}(\lambda_n - \lambda_j) \tag{4.16}$$

where n is odd and $j = 2, 4, \cdots, n-1$ is even.

Proof. Let $M^{[n]^-} = M^{[n]}$ be as given in (3.26). For $n \in \mathbb{N}$ and $b_2^- = d_2^- = 0$ we have from (4.6), (4.7) and (4.8)

$$\begin{aligned}
A_2^- &= M_{21}^{[n]^-} = -(\zeta_n + \eta_n) + \sum_{j=1, j\ even}^{n-1} \frac{\beta_{n,j}}{\lambda - \lambda_j}, \\
B_2^- &= M_{22}^{[n]^-} = M_{11}^{[0]}, \\
C_2^- &= M_{11}^{[n]^-} = M_{22}^{[0]}.
\end{aligned}$$

We now look at each of the entries of $M^{[n-1]^-}$ by using the approach in Theorem 4.1.3 we get

$$\begin{aligned}
M_{11}^{[n-1]^-} &= B_2^- \\
&= M_{11}^{[0]},
\end{aligned}$$

$$\begin{aligned}
M_{12}^{[n-1]^-} &= -\frac{A_2^-}{\lambda - \lambda_n} \\
&= -\frac{1}{\lambda - \lambda_n}\left[-\zeta_n - \eta_n\lambda + \left(\frac{\beta_{n,2}}{\lambda - \lambda_2} + \frac{\beta_{n,4}}{\lambda - \lambda_4} + \cdots + \frac{\beta_{n,n-1}}{\lambda - \lambda_{n-1}} \right) \right] \\
&= \frac{\zeta_n}{\lambda - \lambda_n} + \frac{\eta_n \lambda}{\lambda - \lambda_n} - \frac{\beta_{n,2}}{(\lambda - \lambda_2)(\lambda - \lambda_n)} - \frac{\beta_{n,4}}{(\lambda - \lambda_4)(\lambda - \lambda_n)} - \cdots \\
&\quad - \frac{\beta_{n,n-1}}{(\lambda - \lambda_{n-1})(\lambda - \lambda_n)}.
\end{aligned}$$

Using (4.16) yields,

$$\begin{aligned}
M_{12}^{[n-1]^-} &= \frac{-\zeta_{n-1}\lambda_n + \beta_{n-1,2} + \beta_{n-1,4} + \cdots + \beta_{n-1,n-1}}{\lambda - \lambda_n} + \frac{\zeta_{n-1}(\lambda - \lambda_n) + \zeta_{n-1}\lambda_n}{\lambda - \lambda_n} \\
&\quad - \frac{\beta_{n-1,2}(\lambda_n - \lambda_2)}{(\lambda - \lambda_2)(\lambda - \lambda_n)} - \frac{\beta_{n-1,4}(\lambda_n - \lambda_4)}{(\lambda - \lambda_4)(\lambda - \lambda_n)} - \cdots - \frac{\beta_{n-1,n-1}(\lambda_n - \lambda_{n-1})}{(\lambda - \lambda_{n-1})(\lambda - \lambda_n)} \\
&= \zeta_{n-1} + \frac{\beta_{n-1,2}(\lambda - \lambda_2) - \beta_{n-1,2}(\lambda_n - \lambda_2)}{(\lambda - \lambda_2)(\lambda - \lambda_n)} + \frac{\beta_{n-1,4}(\lambda - \lambda_4) - \beta_{n-1,4}(\lambda_n - \lambda_4)}{(\lambda - \lambda_4)(\lambda - \lambda_n)} \\
&\quad - \cdots - \frac{\beta_{n-1,n-1}(\lambda - \lambda_{n-1}) - \beta_{n-1,n-1}(\lambda_n - \lambda_{n-1})}{(\lambda - \lambda_{n-1})(\lambda - \lambda_n)} \\
&= \zeta_{n-1} + \frac{\beta_{n-1,2}}{\lambda - \lambda_2} + \frac{\beta_{n-1,4}}{\lambda - \lambda_4} + \cdots + \frac{\beta_{n-1,n-1}}{\lambda - \lambda_{n-1}}
\end{aligned}$$

$$= \zeta_{n-1} + \sum_{j=1, j\ even}^{n-1} \frac{\beta_{n-1,j}}{\lambda - \lambda_j}.$$

Also,

$$\begin{aligned} M_{21}^{[n-1]^-} &= -(\lambda - \lambda_n) M_{12}^{[n]^-} \\ &= -(\lambda - \lambda_n)\Big[- \frac{\alpha_{n,1}}{\lambda - \lambda_1} - \cdots - \frac{\alpha_{n,n-2}}{\lambda - \lambda_{n-2}} - \frac{\alpha_{n,n}}{\lambda - \lambda_n}\Big] \\ &= \frac{\alpha_{n,1}(\lambda - \lambda_n)}{\lambda - \lambda_1} + \cdots + \frac{\alpha_{n,n-2}(\lambda - \lambda_n)}{\lambda - \lambda_{n-2}} + \alpha_{n,n}. \end{aligned}$$

By (4.15) this results in

$$\begin{aligned} M_{21}^{[n-1]^-} &= \frac{\frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1}(\lambda - \lambda_n)}{\lambda - \lambda_1} + \cdots + \frac{\frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}}(\lambda - \lambda_n)}{\lambda - \lambda_{n-2}} + \sigma_{n-1} - \frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1} - \frac{\tilde{\beta}_{n,3}}{\lambda_n - \lambda_3} \\ &\quad - \cdots - \frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}} \\ &= \frac{\frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1}(\lambda - \lambda_1) + \frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1}(\lambda_1 - \lambda_n)}{\lambda - \lambda_1} + \cdots + \frac{\frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}}(\lambda - \lambda_{n-2}) + \frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}}(\lambda_{n-2} - \lambda_n)}{\lambda - \lambda_{n-2}} \\ &\quad + \sigma_{n-1} - \frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1} - \frac{\tilde{\beta}_{n,3}}{\lambda_n - \lambda_3} - \cdots - \frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}} \\ &= \sigma_{n-1} - \frac{\frac{\tilde{\beta}_{n,1}}{\lambda_n - \lambda_1}(\lambda_n - \lambda_1)}{\lambda - \lambda_1} - \cdots - \frac{\frac{\tilde{\beta}_{n,n-2}}{\lambda_n - \lambda_{n-2}}(\lambda_n - \lambda_{n-2})}{\lambda - \lambda_{n-2}} \\ &= \sigma_{n-1} - \sum_{i=1, i\ odd}^{n-2} \frac{\alpha_{n-1,i}}{\lambda - \lambda_i}. \end{aligned}$$

Lastly,

$$\begin{aligned} M_{22}^{[n-1]^-} &= C_2^- \\ &= M_{11}^{[0]}. \end{aligned}$$

Thus, the inverse transformation yields the transmission matrix given in (3.25), that is,

$$M^{[n-1]^-} = \begin{bmatrix} M_{11} & \zeta_{n-1} + \displaystyle\sum_{j=1,j\ even}^{n-1} \frac{\beta_{n-1,j}}{\lambda - \lambda_j} \\ \sigma_{n-1} - \displaystyle\sum_{i=1,i\ odd}^{n-2} \frac{\alpha_{n-1,i}}{\lambda - \lambda_i} & M_{22} \end{bmatrix} = M^{[n-1]}.$$

□

Remark 4.1.8. *We note that the transmission matrix $M^{[n-1]^-}$ can be expressed as*

$$M^{[n-1]^-} = \begin{bmatrix} M_{11}^{[0]} & -r_{n-1}^-(\lambda) \\ s_{n-1}^-(\lambda) & M_{22}^{[0]} \end{bmatrix},$$

where $r_{n-1}^-(\lambda) \in \mathcal{R}^0_{\frac{n-1}{2}}$ and $s_{n-1}^-(\lambda) \in \mathcal{R}^0_{\frac{n-1}{2}}$.

Remark 4.1.9. *As we move down the hierarchy by means of repeatedly applying the inverse transformation the following changes take place in the transmission matrix:*

- *(i) The main diagonal entries interchange in each iteration.*
- *(ii) The off-diagonal entries interchange and decrease in half steps of Herglotz-Nevanlinna form in each iteration.*

The n^{th} case (i.e. what happens after applying the inverse transformation n times) is considered in the theorem below.

Theorem 4.1.10. *The transmission matrix $M^{[2]}$ in (3.24) transforms under*

$$T_{z_n^-}(u_2^-) = \frac{{u_2^-}' z_n^- - {z_n^-}' u_2^-}{z_n^-}$$

to a transmission condition given by

$$\begin{bmatrix} u_1^-(0^+) \\ {u_1^-}'(0^+) \end{bmatrix} = M^{[1]^-} \begin{bmatrix} u_1^-(0^-) \\ {u_1^-}'(0^-) \end{bmatrix},$$

where $M^{[1]^-}$ is given by (3.22). Here we assume

$$\alpha_{1,1} = M_{21}^{[0]},$$
$$\zeta_1 = -\lambda M_{12}^{[0]},$$
$$\eta_1 = M_{12}^{[0]}.$$

Proof. Let $b_n = d_n = 0$ then

$$\begin{aligned} A_n^- &= M_{21}^{[2]^-} = \sigma_2 - \frac{\alpha_{2,1}}{\lambda - \lambda_1}, \\ B_n^- &= M_{22}^{[2]^-} = M_{22}^{[0]}, \\ C_n^- &= M_{11}^{[2]^-} = M_{11}^{[0]}. \end{aligned}$$

Using the approach in Theorem 4.1.3 we obtain

$$\begin{aligned} M_{11}^{[1]^-} &= B_n^- \\ &= M_{22}^{[0]}, \end{aligned}$$

$$\begin{aligned} M_{12}^{[1]^-} &= -\frac{A_n^-}{\lambda - \lambda_2} \\ &= -\frac{1}{\lambda - \lambda_2}\left[\sigma_2 - \frac{\alpha_{2,1}}{\lambda - \lambda_1}\right] \\ &= -\frac{\sigma_2}{\lambda - \lambda_2} + \frac{\alpha_{2,1}}{(\lambda - \lambda_1)(\lambda - \lambda_2)} \\ &= -\frac{\alpha_{1,1}}{\lambda - \lambda_2} + \frac{\alpha_{1,1}(\lambda_1 - \lambda_2)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} \\ &= -\frac{\alpha_{1,1}(\lambda - \lambda_2)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} \\ &= -\frac{\alpha_{1,1}}{\lambda - \lambda_1}, \end{aligned}$$

$$\begin{aligned} M_{21}^{[1]^-} &= -(\lambda - \lambda_2) M_{12}^{[2]^-} \\ &= -(\lambda - \lambda_2)\left[\zeta_2 + \frac{\beta_{2,2}}{\lambda - \lambda_2}\right] \end{aligned}$$

$$= -\zeta_2\lambda + \zeta_2\lambda_2 - \beta_{2,2}$$
$$= -\eta_1\lambda + \eta_1\lambda_2 - \zeta_1 - \eta_1\lambda_2$$
$$= -(\zeta_1 + \eta_1\lambda)$$

and

$$\begin{aligned} M_{22}^{[1]^-} &= C_n^- \\ &= M_{11}^{[0]}. \end{aligned}$$

Thus, the inverse transformation has moved us down the hierarchy and yielded the transmission matrix in (3.22) after n iterations, that is,

$$M^{[1]^-} = \begin{bmatrix} M_{22}^{[0]} & -\frac{\alpha_{1,1}}{\lambda-\lambda_1} \\ -(\zeta_1 + \eta_1\lambda) & M_{11}^{[0]} \end{bmatrix} = M^{[1]}.$$

□

Remark 4.1.11. *We note that the transmission matrix $M^{[1]^-}$ can be expressed as*

$$M^{[1]^-} = \begin{bmatrix} M_{22}^{[0]} & r_1^-(\lambda) \\ -s_1^-(\lambda) & M_{11}^{[0]} \end{bmatrix},$$

where $r_1^-(\lambda) \in \mathcal{R}_1^0$ and $s_1^-(\lambda) \in \mathcal{R}_0^+$.

Theorem 4.1.12. *The transmission matrix $M^{[1]^-}$ in (3.22) transforms under*

$$T_{z_{n+1}^-}(u_1^-) = \frac{u_1^{-\prime} z_{n+1}^- - z_{n+1}^{-\prime} u_1^-}{z_{n+1}^-} \tag{4.17}$$

to a transmission condition given by

$$\begin{bmatrix} u_0(0^+) \\ u_0'(0^+) \end{bmatrix} = M^{[0]^-} \begin{bmatrix} u_0(0^-) \\ u_0'(0^-) \end{bmatrix} \tag{4.18}$$

where

$$M^{[0]^-} = \begin{bmatrix} M_{11}^{[0]} & M_{12}^{[0]} \\ M_{21}^{[0]} & M_{22}^{[0]} \end{bmatrix} \tag{4.19}$$

and each $M_{ij}^{[0]}$, $i, j = 1,\ 2$, *is constant. Here we assume*

$$\alpha_{1,1} = M_{21}^{[0]},$$
$$\zeta_1 = -\lambda_1 M_{12}^{[0]},$$
$$\eta_1 = M_{12}^{[0]}.$$

Proof. Let $b_{n+1}^{-}, d_{n+1}^{-} = 0$ then

$$A_{n+1}^{-} = M_{21}^{[1]^{-}} = -(\zeta_1 + \eta_1 \lambda)$$
$$B_{n+1}^{-} = M_{22}^{[1]^{-}} = M_{11}^{[0]}$$
$$C_{n+1}^{-} = M_{11}^{[1]^{-}} = M_{22}^{[0]}.$$

We now consider the entries of $M^{[0]^{-}}$ using the approach of Theorem 4.1.3.

$$\begin{aligned} M_{11}^{[0]^{-}} &= B_{n+1}^{-} \\ &= M_{11}^{[0]}, \end{aligned}$$

$$\begin{aligned} M_{12}^{[0]^{-}} &= -\frac{A_{n+1}^{-}}{\lambda - \lambda_1} \\ &= \frac{\zeta_1 + \eta_1 \lambda}{\lambda - \lambda_1} \\ &= \frac{\zeta_1 + \eta_1 \lambda_1}{\lambda - \lambda_1} + \eta_1 \\ &= \frac{M_{12}\lambda_1 - M_{12}\lambda_1}{\lambda - \lambda_1} + M_{12} \\ &= M_{12}^{[0]}, \end{aligned}$$

$$\begin{aligned} M_{21}^{[0]^{-}} &= -(\lambda - \lambda_1) M_{12}^{[1]^{-}} \\ &= -(\lambda - \lambda_1)\left[-\frac{\alpha_{1,1}}{\lambda - \lambda_1} \right] \\ &= \alpha_{1,1} \\ &= M_{21}^{[0]}, \end{aligned}$$

$$\begin{aligned} M_{22}^{[0]^-} &= C_{n+1}^- \\ &= M_{22}^{[0]}. \end{aligned}$$

Thus, giving the result. Hence, $n+1$ iterations of the inverse transformation yields the following transmission matrix

$$M^{[0]^-} = M^{[0]} = \begin{bmatrix} M_{11}^{[0]} & M_{12}^{[0]} \\ M_{21}^{[0]} & M_{22}^{[0]} \end{bmatrix} \tag{4.20}$$

where each $M_{ij}^{[0]}$, $i, j = 1,\ 2$, is constant. □

Remark 4.1.13. *We note that the transmission matrix $M^{[0]}$ can be expressed as*

$$M^{[0]} = \begin{bmatrix} M_{11}^{[0]} & r(\lambda) \\ s(\lambda) & M_{22}^{[0]} \end{bmatrix},$$

where $r(\lambda) \in \mathcal{R}_0^0$ and $s(\lambda) \in \mathcal{R}_0^0$.

The above result proves that the inverse transformation is indeed an inverse mapping of the forward transformation discussed in Chapter 3. This inverse mapping, with a suitable choice of the transmission condition parameters, decreases the λ-dependence of our transmission condition and allows us to move down the hierarchy to the initial λ-independent transmission condition.

Chapter 5

Problem Formulation

In this chapter a formulation of the Sturm-Liouville eigenvalue problem with eigenparameter dependent transmission conditions will be developed in differential equation form. We will then complete the first step of posing our boundary value probem in a functional analytic framework by providing the Hilbert and Pontryagin space settings which give a symmetric operator for each class of our transmission conditions.

Once again, we consider the Strum-Liouville equation (3.1) with the boundary conditions (3.2) - (3.3) and the transmission conditions (3.4) - (3.5), where $r(\lambda)$ and $s(\lambda)$ are of the form (3.6), (3.7) or (3.8), (3.9).

5.1 Pontryagin Space Formulation

We now pose our boundary value problem (3.1) - (3.3) together with the first class of the transmission condition in the Pontryagin space, Π_{M+N}, by defining an operator together with its domain. In addition we prove the operator is, in fact, symmetric.

Class 1: Let

$$Y = \begin{bmatrix} y \\ y^1 \\ y^2 \end{bmatrix},$$

where $y^1 = \begin{bmatrix} y^1_1 \\ y^1_2 \\ \vdots \\ y^1_M \end{bmatrix}$ and $y^2 = \begin{bmatrix} y^2_1 \\ y^2_2 \\ \vdots \\ y^2_N \end{bmatrix}$. The Pontryagin space, Π_{M+N}, has inner product defined as follows

$$\langle Y, Z \rangle = \int_{-a}^{b} y\bar{z} + \sum_{j=1}^{M} y^1_j \overline{z^1_j} + \sum_{i=1}^{N} y^2_i \overline{z^2_i}.$$

We consider the operator corresponding to the transmission condition given by (3.4) - (3.5) and (**??**) - (**??**). Combining (3.4) and (**??**) gives

$$y(0^+) = \zeta \bigtriangleup y' + \sum_{j=1}^{M} \frac{\beta_j^2}{\lambda - \gamma_j} \bigtriangleup y'.$$

Now let

$$y^1_j = \frac{\beta_j}{\lambda - \gamma_j} \bigtriangleup y',$$

then we have

$$y^1 \cdot \beta = \sum_{j=1}^{M} \frac{\beta_j^2}{\lambda - \gamma_j} \bigtriangleup y'$$
$$\sum_{j=1}^{M} y^1_j \beta_j = \sum_{j=1}^{M} \frac{\beta_j^2}{\lambda - \gamma_j} \bigtriangleup y'$$

$$
\begin{aligned}
y_j^1 \beta_j &= \frac{\beta_j^2}{\lambda - \gamma_j} \bigtriangleup y' \\
(\lambda - \gamma_j) y_j^1 &= \beta_j \bigtriangleup y' \\
\lambda y_j^1 &= \gamma_j y_j^1 + \beta_j \bigtriangleup y'.
\end{aligned}
$$

Similarly, by letting $y_i^2 = \frac{\alpha_i}{\lambda - \delta_i} \bigtriangleup y$ and combining (3.5) and (**??**) we get

$$
\lambda y_i^2 = \delta_i y_i^2 + \alpha_i \bigtriangleup y.
$$

Hence, we define the operator corresponding to Class 1 as follows

$$
L_1 Y := \begin{bmatrix} \ell y \\ (\gamma_j y_j^1 + \beta_j \bigtriangleup y')_{j=1}^M \\ (\delta_i y_i^2 + \alpha_i \bigtriangleup y)_{i=1}^N \end{bmatrix},
$$

with domain

$$
D(L_1) = \left\{ Y = \begin{bmatrix} y \\ y^1 \\ y^2 \end{bmatrix} \;\middle|\; \begin{array}{l} y|_{(-a,0)},\ y|'_{(-a,0)},\ \ell y|_{(-a,0)} \in L^2(-a,0) \\ y|_{(0,b)},\ y|'_{(0,b)},\ \ell y|_{(0,b)} \in L^2(0,b) \\ y \text{ obeys (3.2) and (3.3)} \\ y(0^+) = \left(\zeta + \sum_{j=1}^M \frac{\beta_j^2}{\lambda - \gamma_j}\right) \bigtriangleup y' = \zeta \bigtriangleup y' + y^1 \cdot \beta \\ y'(0^-) = \left(\sigma - \sum_{i=1}^N \frac{\alpha_i^2}{\lambda - \delta_i}\right) \bigtriangleup y = \sigma \bigtriangleup y - y^2 \cdot \alpha \end{array} \right\}.
$$

Here $\beta := \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_M \end{bmatrix}$ and $\alpha := \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$.

Theorem 5.1.1. *The operator L_1 is symmetric in the Pontryagin space Π_{M+N}.*

Proof. Let $Y,\ Z \in D(L_1)$ with $y^{(k)}(-a) = 0 = y^{(k)}(b)$ for $k = 0,\ 1$ and $\ell y =$

$-y'' + qy$. Then the functional components y and z of Y and Z give

$$\langle L_1 Y, Z\rangle - \langle Y, L_1 Z\rangle = \left\langle \begin{bmatrix} \ell y \\ (\gamma_j y_j^1 + \beta_j \triangle y')_{j=1}^M \\ (\delta_i y_i^2 + \alpha_i \triangle y)_{i=1}^N \end{bmatrix}, \begin{bmatrix} z \\ z^1 \\ z^2 \end{bmatrix} \right\rangle - \left\langle \begin{bmatrix} y \\ y^1 \\ y^2 \end{bmatrix}, \begin{bmatrix} \ell z \\ (\gamma_j z_j^1 + \beta_j \triangle z')_{j=1}^M \\ (\delta_i z_i^2 + \alpha_i \triangle z)_{i=1}^N \end{bmatrix} \right\rangle$$

$$= \int_{-a}^{0} (-y''z + yz'') + \int_0^b (-y''z + yz'') - \sum_{j=1}^{M} \left\{ (\gamma_j y_j^1 + \beta_j \triangle y')\overline{z_j^1} - y_j^1 \overline{(\gamma_j z_j^1 + \beta_j \triangle z')} \right\}$$

$$- \sum_{i=1}^{N} \left\{ (\delta_i y_i^2 + \alpha_i \triangle y)\overline{z_i^2} - y_i^2 \overline{(\delta_i z_i^2 + \alpha_i \triangle z)} \right\}$$

$$= \big[-y'z + yz' \big]_{-a}^{0} + \big[-y'z + yz' \big]_0^b - \sum_{j=1}^{M} \beta_j \triangle y' \overline{z_j^1} + \sum_{j=1}^{M} y_j^1 \overline{\beta_j \triangle z'} - \sum_{i=1}^{N} \alpha_i \triangle y \overline{z_i^2}$$

$$+ \sum_{i=1}^{N} y_i^2 \overline{\alpha_i \triangle z}$$

$$= -y'z(0^-) + yz'(0^-) + y'z(0^+) - yz'(0^+) - \sum_{j=1}^{M} \beta_j \triangle y' \overline{z_j^1} + \sum_{j=1}^{M} y_j^1 \overline{\beta_j \triangle z'} - \sum_{i=1}^{N} \alpha_i \triangle y \overline{z_i^2}$$

$$+ \sum_{i=1}^{N} y_i^2 \overline{\alpha_i \triangle z}.$$

Using the domain of the operator, we get

$$\begin{aligned} \langle L_1 Y, Z\rangle - \langle Y, L_1 Z\rangle &= -y'z(0^-) + yz'(0^-) + y'z(0^+) - yz'(0^+) - \triangle y'(z(0^+) - \zeta \triangle z') \\ &\quad + \triangle z'(y(0^+) - \zeta \triangle y') - \triangle y(\sigma \triangle z - z'(0^-)) + \triangle z(\sigma \triangle y - y'(0^-)) \\ &= -y'(0^-)z(0^-) + y(0^-)z'(0^-) + y'(0^+)z(0^+) - y(0^+)z'(0^+) \\ &\quad - (y'(0^+) - y'(0^-))z(0^+) + (z'(0^+) - z'(0^-))y(0^+) \\ &\quad + (y(0^+) - y(0^-))z'(0^-) - (z(0^+) - z(0^-))y'(0^-) \\ &= 0 \end{aligned}$$

Thus L_1 is symmetric. □

5.2 Hilbert Space Setting

We now pose the boundary value problem (3.1) - (3.3) together with the the second class of the transmission condition in a Hilbert space, $H = L^2(-a,b)\oplus\mathbb{C}^N\oplus\mathbb{C}^M\oplus\mathbb{C}$, by defining an operator together with its domain. In addition we prove the operator symmetric.

Class 2: Let

$$Y = \begin{bmatrix} y \\ y^1 \\ y^2 \\ y_0 \end{bmatrix},$$

where $y^1 = \begin{bmatrix} y_1^1 \\ y_2^1 \\ \vdots \\ y_N^1 \end{bmatrix}$, $y^2 = \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_M^2 \end{bmatrix}$ and $y_0 \in \mathbb{C}$. Suppose $\eta > 0$, in (**??**), we see that

$$\langle Y, Z\rangle = \int_{-a}^{b} y\bar{z} + \sum_{i=1}^{N} y_i^1\overline{z_i^1} + \sum_{i=1}^{M} y_j^2\overline{z_j^2} + \frac{y_0\overline{z_0}}{\eta}$$

defines a Hilbert space inner product on H.

Now we consider the operator corresponding to the transmission condition given by (3.4) - (3.5) and (**??**) - (**??**). Combining (3.4) and (**??**), we get

$$y(0^+) = -\sum_{i=1}^{N} \frac{\alpha_i^2}{\lambda - \delta_i} \triangle y'.$$

Now let

$$y_i^1 = \frac{\alpha_i}{\lambda - \delta_i} \triangle y'$$

then

$$
\begin{aligned}
y^1 \cdot \alpha &= \sum_{i=1}^{N} \frac{\alpha_i^2}{\lambda - \delta_i} \triangle y' \\
\sum_{i=1}^{N} y_i^1 \alpha_i &= \sum_{i=1}^{N} \frac{\alpha_i^2}{\lambda - \delta_i} \triangle y' \\
y_i^1 \alpha_i &= \frac{\alpha_i^2}{\lambda - \delta_i} \triangle y' \\
(\lambda - \delta_i) y_i^1 &= \alpha_i \triangle y' \\
\lambda y_i^1 &= \delta_i y_i^1 + \alpha_i \triangle y'.
\end{aligned}
$$

Similarly, by letting $y_j^2 = \frac{\beta_j}{\lambda - \gamma_j} \triangle y$ and combining (3.5) and (**??**) we get

$$\lambda y_j^2 = \gamma_j y_j^2 + \beta_j \triangle y$$

and

$$y'(0^-) = -(\eta\lambda + \zeta) \triangle y + \sum_{j=1}^{M} \frac{\beta_j^2}{\lambda - \gamma_j} \triangle y,$$

which gives

$$\eta\lambda \triangle y = -y'(0^-) - \zeta \triangle y + \sum_{j=1}^{M} \beta_j y_j^2.$$

Therefore, we define the operator as follows

$$L_2 Y := \begin{bmatrix} \ell y \\ (\delta_i y_i^1 + \alpha_i \triangle y')_{i=1}^N \\ (\gamma_j y_j^2 + \beta_j \triangle y)_{j=1}^M \\ -y'(0^-) - \zeta \triangle y + \sum_{j=1}^{M} \bar{\beta}_j y_j^2 \end{bmatrix},$$

with domain

$$D(L_2) = \left\{ Y = \begin{bmatrix} y \\ y^1 \\ y^2 \\ y_0 \end{bmatrix} \left| \begin{array}{l} y|_{(-a,0)},\ y|'_{(-a,0)},\ \ell y|_{(-a,0)} \in L^2(-a,0) \\ y|_{(0,b)},\ y|'_{(0,b)},\ \ell y|_{(0,b)} \in L^2(0,b) \\ y \text{ obeys (3.2) and (3.3)} \\ -y(0^+) = \sum_{i=1}^{N} \bar{\alpha}_i y_i^1 = \langle y^1, \alpha \rangle \\ y_0 = \eta \triangle y \end{array} \right. \right\}.$$

Here $\beta := \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_M \end{bmatrix}$ and $\alpha := \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$.

Theorem 5.2.1. *The operator L_2 is symmetric in $H = L^2(-a,b) \oplus \mathbb{C}^N \oplus \mathbb{C}^M \oplus \mathbb{C}$.*

Proof. Similar to Theorem 3.2.1, we let $Y,\ Z \in D(L_2)$ with $y^{(k)}(-a) = 0 = y^{(k)}(b)$ for $k = 0,\ 1$ and $\ell y = -y'' + qy$. Then the functional components y and z of Y and Z give

$$\langle L_2 Y, Z \rangle - \langle Y, L_2 Z \rangle =$$

$$\left\langle \begin{bmatrix} \ell y \\ (\delta_i y_i^1 + \alpha_i \triangle y')_{i=1}^N \\ (\gamma_j y_j^2 + \beta_j \triangle y)_{j=1}^M \\ -y'(0^-) - \zeta \triangle y + \sum_{j=1}^M \bar{\beta}_j y_j^2 \end{bmatrix}, \begin{bmatrix} z \\ z^1 \\ z^2 \\ z_0 \end{bmatrix} \right\rangle - \left\langle \begin{bmatrix} y \\ y^1 \\ y^2 \\ y_0 \end{bmatrix}, \begin{bmatrix} \ell z \\ (\delta_i z_i^1 + \alpha_i \triangle z')_{i=1}^N \\ (\gamma_j z_j^2 + \beta_j \triangle z)_{j=1}^M \\ -z'(0^-) - \zeta \triangle z + \sum_{j=1}^M \bar{\beta}_j z_j^2 \end{bmatrix} \right\rangle$$

$$= \int_{-a}^{0} (-y''z + yz'') + \int_{0}^{b} (-y''z + yz'') + \sum_{i=1}^{N} \left\{ (\delta_i y_i^1 + \alpha_i \triangle y')\overline{z_i^1} - y_i^1 \overline{(\delta_i z_i^1 + \alpha_i \triangle z')} \right\}$$
$$+ \sum_{j=1}^{M} \left\{ (\gamma_j y_j^2 + \beta_j \triangle y)\overline{z_j^2} - y_j^2 \overline{(\gamma_j z_j^2 + \beta_j \triangle z)} \right\} + \frac{(-y'(0^-) - \zeta \triangle y + \sum_{j=1}^M \beta_j y_j^2)\overline{z_0}}{\eta}$$

$$- \frac{y_0 \overline{(-z'(0^-) - \zeta \triangle z + \sum_{j=1}^M \beta_j z_j^2)}}{\eta}.$$

Using the domain of the operator, we get

$$\begin{aligned}
\langle L_2 Y, Z\rangle - \langle Y, L_2 Z\rangle = & -y'(0^-)z(0^-) + y(0^-)z'(0^-) + y'(0^+)z(0^+) - y(0^+)z'(0^+) \\
& - z(0^+)(y'(0^+) - y'(0^-)) + y(0^+)(z'(0^+) - z'(0^-)) + \sum_{j=1}^M \beta_j \triangle y\overline{z_j^2} \\
& - \sum_{j=1}^M y_j^2 \overline{\beta_j \triangle z} - y'(0^-) \triangle z - \zeta \triangle y \triangle z + \sum_{j=1}^M y_j^2 \overline{\beta_j \triangle z} \\
& + z'(0^-) \triangle y + \zeta \triangle y \triangle z - \sum_{j=1}^M \beta_j \triangle y\overline{z_j^2} \\
= & \ 0
\end{aligned}$$

Thus L_2 is symmetric. □

Chapter 6

Conclusion

In this dissertation we have computed the forward Darboux-Crum transformation on the eigenparameter dependent transmission conditions of the Sturm-Liouville eigenvalue problem. We have as well illustrated its effect on the potential, q, of the problem. It was inductively shown that the forward Darboux-Crum transformation increases the eigenparameter dependence of the transmission condition in each step of induction, thus yielding a hierarchy of Sturm-Liouville problems with transmission conditions of the aforementioned type.

We then tackled the problem of moving down the hierarchy back to our original, simplified transmission condition using the inverse Darboux-Crum transformation. Backwards induction using this transformation allowed us to decrease the eigenparameter dependence of our transmission condition in each step of induction eventually resulting in an eigenparameter independent transmission condition. It is shown that both the forward and inverse transformation each increases the eigenparameter dependence of the transmission condition and it is in fact the choice of the Nevanlinna function coefficients that determines which one of the transformations will decrease the eigenparameter dependence of the transmission condition. Finally, we posed the Sturm-Liouville eigenvalue problem with transmission conditions depending on the eigenparameter as Nevanlinna functions in suitable Pontryagin and Hilbert spaces. We then defined a symmetric operator for each class of our transmission conditions. This is the first step of posing the boundary value problem in a functional analytic framework.

This dissertation concentrates on the behaviour of the transmission condition, however, one can also transform the boundary conditions resulting in a transformation of the entire problem. We can then compare the eigenvalues of the original problem with that of the transformed problem i.e. determine if eigenvalues are lost, gained or remain the same. Binding et al. study the effect of the Darboux transformation on eigenparameter dependent boundary conditions in [3].

Further work in this topic would involve a rigorous functional analytic approach using operator theory. One would pose the above work in suitable Pontryagin and Hilbert spaces as done in this dissertation. However, instead of just showing symmetry of the resulting operators we will extend this to show that the operators are self-adjoint with compact resolvent. Formulating the problem in such a manner will allow one to illustrate the transformations given in this dissertation using operator factorization as Binding et al. have done in [2] for Sturm-Liouville problems with eigenvalue-dependent boundary conditions. The resulting hierarchy of Sturm-Liouville problems developed in this dissertation can be applied to the study of inverse spectral problems for Sturm-Liouville eigenvalue problems with transmission conditions having Nevanlinna dependence on the eigenparameter. The inverse spectral problem is one in which, given spectral data, one would recover the potential, boundary conditions and transmission conditions of the Sturm-Liouville problem.

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