Bilocal approach to the infra-red fixed point of O(N) invariant theories in 3d and its relation to higher spins

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To my mother.

The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them. In this methodological uncertainty, one might suppose that there were any number of possible systems of theoretical physics all equally well justified; and this opinion is no doubt correct, theoretically. But the development of physics has shown that at any given moment, out of all conceivable constructions, a single one has always proved itself decidedly superior to all the rest.

- Albert Einstein.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree Doctor of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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.........Day of ....2018.

Abstract

The Klebanov-Polyakov Higher-Spin Anti-de Sitter/Conformal Field Theory conjecture posits that the free $O(N)$ vector model is dual to the type A Vasiliev Higher Spin Gravity with the bulk scalar field having conformal scaling dimension $\Delta = 1$. Similarly, the critical $O(N)$ vector model in $3d$ is dual to type A Vasilev Higher Spin Gravity with bulk scalar having $\Delta = 2$. This is a weak-weak duality and accordingly allows a setting where a reconstruction of bulk physics from the boundary CFT is possible. The Jevicki-Sakita collective field theory provides an explicit realization of such a bulk reconstruction.

In this thesis, we use the collective field theory description of the large-N limit of vector models to study the $O(N)$ infra-red interacting fixed point. In particular, we compute the two-point functions for the non-linear sigma model (which is equivalent, in the infra-red, to the critical $O(N)$ vector model) and the two-time bilocal propagator. The spectrum for the $O(N)$ vector model is then obtained by looking at the poles of the connected Green's function. We then show that this same pole condition can be obtained from the homogeneous equation for the bilocal fluctuations.

We then discuss the single-time Hamiltonian formalism for the critical $O(N)$ vector model. We derive a coupled integral equation for the single-time fluctuations. This coupled integral equations allows us to write down the single-time pole condition. We show that the two-time pole condition is equivalent to the single-time pole condition. In addition, we also show that the two-time free bilocal propagator is equivalent to the single-time free bilocal propagator. A Lagrangian formulation of the single-time descrip-

tion is given and we write down the single-time propagator.

We then explain a puzzle which is that from our study of the non-linear sigma model and the the pole structure of both the two-time and the single-time propagators it would seem that both the $\Delta = 1$ and $\Delta = 2$ scalars are present. By studying the quadratic Hamiltonian determining the spectrum, we demonstrate how in the infra-red limit the state $\Delta = 1$ disappears from the spectum.

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Chapter 1

Introduction

If you can't explain it simply, you don't understand it well enough.

-Albert Einstein.

One of the most fascinating ideas in theoretical physics is the notion of Duality. In its basic incarnation, it is the simple idea that a physical system can have two completely different descriptions.

This concept is fully exploited in Superstring Theory - for more comprehensive details on Superstring Theory, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

String Theory posits that the world is made of single vibrating one dimensional entities. There are, however, certain problems with this approach. For example, the fact that superstring theory is only consistent in 9+1 dimensions or the fact that there are 10^{500} vacua - this is the so-called string Landscape or multiverse problem. Or the fact that it is difficult to verify experimentally as the Planck scale is extremely small. (For more on the debate as to why we can trust a physical theory even if it has not been verified experimentally, see [16, 17] .) [1]

[1]The second references *i.e.* [17] are there for a balanced account of superstring theory. They are largely negative and hostile..

The theory only started being taken seriously in 1984 when Green and Schwarz showed that string theory is free from the ugly infinities that plague most of attempts to find a Quantum Theory of Gravity [15].[2]

In this chapter, we will focus on one of the key dualities that was discovered during the so called Second String Revolution. (This being none other than the so called Anti-de Sitter/Conformal Field Theory Correspondence[22, 23, 24].[3])

1.1 Introduction To The AdS/CFT Correspondence

The most striking example of a Duality is the Gauge-String duality. Put simply, a gauge theory should be the same as some theory of strings. This idea has been around since 1974 [39] but it was not until 1997 when Maldacena gave a solid example of the Gauge-String Duality.

1.1.1 (Dirichlet) *Dp*-Branes

String theory is not a theory of strings.

-Robbert Dijkgraff.

To arrive at the AdS/CFT Correspondence, Maldacena looked at a stack of Dirichlet 3 branes.

To understand what is a *Dp*-brane we need to first look at the simple description of the open string action.

[2]For more on the ugly infinities (more technically, Gravity is not renormalizable) that are characteristic of Quantum Gravity, see [18, 19, 20, 21].

[3]For excellent textbooks and reviews, see [25, 26, 27, 28, 29, 30, 31, 32, 33].

As the string moves in spacetime it sweeps across an area element. The string action is then given by integrating over this area element. More formally, it is given by the Nambu-Goto action:

$$
\begin{aligned}
S_P &= -T \int dA \\
&= -T \int d^2\xi \sqrt{-h}.
\end{aligned} \tag{1.1}
$$

Here, $d^2\xi = d\sigma d\tau$ and T is the string tension[4] and h is the determinant of the induced metric *i.e.*

$$
h_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{1.3}
$$

where $\mu, \nu = 0, 1, \cdots, D$ and $a, b = 1, 2$.

This action turns out to be difficult to work with because of the appearance of the square root. A much simpler action – which is, classically, equivalent to the Nambu Goto action – is the Polyakov action which can be written as[5]

$$
\begin{aligned}
S_P &= -\frac{T}{2} \int d^2\xi \sqrt{-\gamma} \gamma^{ab} h_{ab} \\
&= -\frac{T}{2} \int d^2\xi \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu},
\end{aligned} \tag{1.4}
$$

where γ_{ab} is a dynamical auxiliary (intrinsic) metric on the worldsheet [7].

[4] The string tension can be written as

$$
T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2}, \tag{1.2}
$$

where $\alpha' = l_s^2$ is the Regge slope and l_s is the intrinsic length.

[5] The general bosonic string action is $S = -\frac{T}{2} \int d^2\xi \sqrt{-\gamma} \left(\gamma^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + i\epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \right) - \frac{1}{4\pi} \int d^2\xi \sqrt{-\gamma} \Phi(X)$. Here, $G_{\mu\nu}(X)$ is the metric tensor, $B_{\mu\nu}(X)$ is the anti-symmetrical Kalb-Ramond B field and $\Phi(X)$ is the dilaton.

To obtain the equations of motion satisfied by the string we vary the Polyakov action with respect to X^μ and find

$$\begin{aligned} 0 &= \delta S_p \\ &= -T\int d^2\xi \delta X^\mu \left(\partial_\sigma^2 - \partial_\tau^2\right) X^\mu + \frac{T}{2}\int d\tau \; \delta X^\mu \partial_\sigma X^\mu |_{\sigma=0}^{\sigma=\pi} . \end{aligned} \tag{1.5}$$

The equations of motions - namely, the wave equation - can be satisfied if we require that the boundary term (that is, the second term in (1.5)) vanishes for the open string. This means that we can either have

$$\partial_\sigma X^\mu|_{\sigma=0} = \partial_\sigma X^\mu|_{\sigma=\pi} = 0. \tag{1.6}$$

or

$$\delta X^\mu|_{\sigma=0} = \delta X^\mu|_{\sigma=\pi} = 0. \tag{1.7}$$

The boundary conditions in (1.6) are called the Neumann boundary conditions and physically mean that there is no momentum flow at the end-points. The other condition (*i.e.* (1.7)) is called the Dirichlet boundary condition.

We can now define, from a technical point of view, what a Dp-brane is. Basically, a Dp-brane is a p-dimensional hypersurface with Neumann boundary conditions for the coordinates $X^0, \ldots, X^p$ (these are the coordinates that are parallel to the brane) and Dirichlet conditions for the remaining coordinates - *i.e.* $X^{p+1}, \ldots, X^{D-1}$ - transverse to the brane [40, 41].

A more clear cut definition is that a Dp-brane is simply a hypersurface where the end points of the string are attached - this is visualized in (1.1).

Note that a $D0$-brane is none other than a point (particle) where an open string can be attached. A $D1$-brane can thought of as the usual string. And a $D2$-brane is a

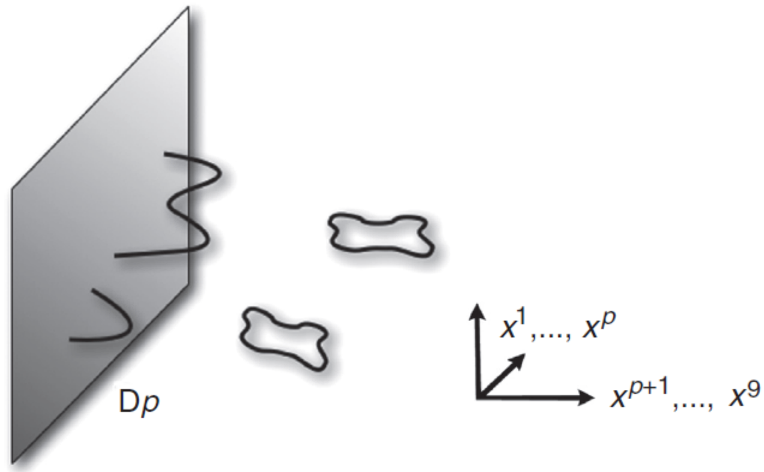


Figure 1.1: A Dp brane with open strings attached and closed strings propagating in the bulk. Picture credits [9].

membrane. (Obviously, the word p-brane is derived from membrane.)

In a manner similar to the Polyakov action for bosonic string, the action for the Dp-brane is given, partly, by what is called the (Dirac-Born-Infeld) DBI action:

$$S_{DBI} = -T_{Dp} \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det \mathcal{P}\left(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)}, \tag{1.8}$$

Here, g_s is the string coupling, $\mathcal{P}\left(g_{ab}\right)$ is the pullback of the metric onto the worldvolume of the Dp-brane (similar definitions hold for the Kalb-Ramond 2-form B field and the field strength F) *i.e.*

$$\mathcal{P}\left(g_{ab}\right) = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu} \tag{1.9}$$

and the brane tension is

$$T_{Dp} = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}}}. \tag{1.10}$$

It is useful to expand the DBI action along the flat Minkowski metric (*i.e.* $g_{\mu\nu} =$

$\eta_{\mu\nu}$,$B_{\mu\nu} = 0$) at low energies *i.e.* $\alpha' \to 0$.

In addition, we will choose our coordinates such that the first $p+1$ coordinates lie along (are parallel to) the Dp-brane and the remaining $D-p-1$ are transverse to the Dp-brane [4]. In other words, $X^a = \xi^a$ with $a = 0, 1, \cdots, p$. The pull back of the metric is $(M = p+1, \cdots, D)$

$$\begin{aligned}\mathcal{P}\left(g_{ab}\right) &= \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu} \\ &= \frac{\partial X^c}{\partial \xi^a}\frac{\partial X^d}{\partial \xi^b}\eta_{cd} + \frac{\partial X^M}{\partial \xi^a}\frac{\partial X^N}{\partial \xi^b}\eta_{MN} \\ &= \frac{\partial \xi^c}{\partial \xi^a}\frac{\partial \xi^d}{\partial \xi^b}\eta_{cd} + \frac{\partial X^M}{\partial \xi^a}\frac{\partial X^N}{\partial \xi^b}\eta_{MN} \\ &= \eta_{ab} + \partial_a X^M \partial_b X_M. \end{aligned} \tag{1.11}$$

To do such an expansion one needs the identity [7]:

$$\det M = \exp\left(\mathrm{Tr}\ln M\right), \tag{1.12}$$

where M is a general matrix.

Putting everything together, we find that the quadratic fields are

$$\begin{aligned}
2S_{DBI} &= -T_{Dp}\int d^{p+1}\xi e^{-\Phi}\sqrt{-\det\mathcal{P}\left(g_{ab}+2\pi\alpha' F_{ab}\right)} \\
&= -T_{Dp}\int d^{p+1}\xi e^{-\Phi}\left[\exp\left(\mathrm{Tr}\ln\left(\eta_{ab}+\partial_a X^M\partial_b X_M+2\pi\alpha' F_{ab}\right)\right)\right]^{1/2} \\
&= -T_{Dp}\int d^{p+1}\xi e^{-\Phi}\left[\exp\left(\mathrm{Tr}\left(\partial_a X^M\partial_b X_M-\frac{\left(2\pi\alpha'\right)^2}{2}F_{ab}F^{ab}+\cdots\right)\right)\right]^{1/2} \\
&= -T_{Dp}\int d^{p+1}\xi e^{-\Phi}\left(1+\mathrm{Tr}\left(\partial_a X^M\partial_b X_M-\frac{\left(2\pi\alpha'\right)^2}{2}F_{ab}F^{ab}\right)+\cdots\right)^{1/2} \\
&= -T_{Dp}\int d^{p+1}\xi e^{-\Phi}\left(1+\frac{1}{2}\left(\mathrm{Tr}\left(\partial_a X^M\partial_b X_M-\frac{\left(2\pi\alpha'\right)^2}{2}F_{ab}F^{ab}\right)\right)\right)+\cdots
\end{aligned} \tag{1.13}$$

The dependence on the gauge fields is

$$S_{DBI}\supset -\frac{T_p\left(2\pi\alpha'\right)^2}{4g_s}\int d^{p+1}\xi\left(F_{ab}F^{ab}\right) \tag{1.14}$$

which is a Yang-Mills term:

$$S_{YM}=-\frac{1}{2g_{YM}^2}\int d^{p+1}\xi\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{1.15}$$

provided we make the simple identification that

$$\frac{T_p\left(2\pi\alpha'\right)^2}{4g_s}=\frac{\left(2\pi\alpha'\right)^2}{4\times\left(2\pi\right)^p\alpha'^{\frac{p+2}{2}}}=\frac{1}{2g_{YM}^2} \tag{1.16}$$

which implies

$$\boxed{2g_s\left(2\pi\right)^{p-2}\alpha'^{\frac{p-3}{2}}=g_{YM}^2.} \tag{1.17}$$

The Dp-brane also carries R-R charge. To describe this phenomenon, we add to the DBI action the Wess Zumino action defined by

$$S_{WZ} = \mu_p \int \sum_q \mathcal{P}C_q \wedge e^{\hat{B}\wedge\alpha' F}. \tag{1.18}$$

Here $\mathcal{P}C_q$ is the pull-back of the RR potential.

A second description of the Dp-branes is in terms of closed strings. In this picture, the Dp-brane is identified with what is known as a p brane. In plain terms, a p-brane is a classical solution to the low energy (*i.e.* Supergravity) limit of string theory.

To explain what this means let us first consider the simple example of the Reissner-Nordström solution in $4D$.

Recall that, in the presence of charges, the Einstein-Hilbert action can be written as

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x\sqrt{-g}\,(R - G_N F_{\mu\nu}F^{\mu\nu})\,, \tag{1.19}$$

where G_N is Newton's Gravitational constant, R is the Ricci scalar. Varying the above action leads to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{1.20}$$

where the energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{1.21}$$

A solution to the resulting field equations was obtained by Reisnner and Nordström and reads:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1.22}$$

$$F = \frac{Q}{r^2} dt \wedge dr, \tag{1.23}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the two-sphere and F is the field strength.

Naively - similarly to the case when we have Schwarzchild blackholes – the Reissner-Nordström has a horizon at the singularity when g^{rr} vanishes:[6]

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{1.24}$$

The size of the horizon is real and hence $M \geq Q$ which is known as the BPS bound.

It turns out that it is interesting to look at what are called extremal blackholes. These are defined as those blackholes which saturate the BPS bound and satisfy the condition $r_+ = r_- = Q$.

In general, the Reissner-Nordström blackhole solution can be written as

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \\ &= \frac{1}{r^2}\left(r - r_-\right)\left(r - r_+\right) dt^2 + \frac{r^2}{\left(r - r_-\right)\left(r - r_+\right)} dr^2 + r^2 d\Omega_2^2. \end{aligned} \tag{1.25}$$

[6]There is a technical detail which we have ignored [4]. The two cases are not the same and we have obtained the horizon by looking at the place where the metric is singular *i.e.* $g^{rr} = 0$. We have to look at the null hypersurface condition in order to define the horizon.

For the extremal blackholes, we obtain [4]:

$$
\begin{aligned}
ds^2 &= -\frac{1}{r^2}\left(r-r_-\right)\left(r-r_+\right)dt^2 + \frac{r^2}{\left(r-r_-\right)\left(r-r_+\right)}dr^2 + r^2 d\Omega_2^2 \\
&= -\frac{\left(r-Q\right)^2}{r^2}dt^2 + \frac{r^2}{\left(r-Q\right)^2}dr^2 + r^2 d\Omega_2^2 \\
&= -\frac{\rho^2}{\left(\rho+Q\right)^2}dt^2 + \frac{\left(\rho+Q\right)^2}{\rho^2}d\rho^2 + r^2 d\Omega_2^2 \\
&= -H\left(\rho\right)^{-2}dt^2 + H\left(\rho\right)^2 d\rho^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).
\end{aligned}
\tag{1.26}
$$

with $H\left(\rho\right)$ a Harmonic function given by

$$
H\left(\rho\right) = \left(1 + \frac{Q}{\rho}\right), \qquad \rho = r - Q. \tag{1.27}
$$

In summary, the extremal Reissner-Nordström solution to the Einstein field equations reads:

$$
ds^2 = -H\left(\rho\right)^{-2}dt^2 + H\left(\rho\right)^2 dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.28}
$$

$$
F = dH \wedge dt \tag{1.29}
$$

Black p-branes are generalization to the solution given above.

Let us consider Type IIB string theory. The low energy dynamics is given by an action of the form [38]

$$
\begin{aligned}
S_{II} = \frac{1}{16\pi G_N}\Bigg[\int d^{10}x\Bigg\{e^{-2\Phi}\Bigg(R + 4\left(\partial\Phi\right)^4 - \frac{1}{12}H_3^2 - \frac{1}{2}\left(\partial\chi\right)^2 - \frac{1}{2}F_3'^2 - \frac{1}{240}F_5'^2 \\
+ \int A_4 \wedge F_3 \wedge H_3\Bigg\}\Bigg],
\end{aligned}
\tag{1.30}
$$

where

$$H_3 = dB_2, \qquad F_n = dC_{n-1} \tag{1.31}$$

$$\tilde{F}_3 = F_3 + C_0 \wedge H_3, \qquad \tilde{F}_5 = F_5 + C_2 \wedge H_3 \tag{1.32}$$

The Type IIB solution can be written as [4, 38]:

$$ds^2 = H_p(r)^{-1/2}\, \eta_{\alpha\beta} dx^\alpha dx^\beta + H_p(r)^{1/2}\, \delta_{IJ} dx^I dx^J \tag{1.33}$$

$$e^\Phi = H_p(r)^{\frac{3-p}{4}} \tag{1.34}$$

$$C_{p+1} = \left(H_p(r)^{-1} - 1\right) dx^0 \wedge dx^1 \tag{1.35}$$

with $\alpha, \beta = 1, 2, \ldots, p$ and $I, J = p+1, \ldots, D$ and

$$H_p(r) = 1 + \left(\frac{R}{r}\right)^{7-p}; \qquad R^{7-p} = \left(2\sqrt{\pi}\right)^{5-p} \Gamma\left(\frac{2-p}{2}\right) g_s N \alpha'^{\frac{7-p}{2}}. \tag{1.36}$$

The Superstring

Pure bosonic string theory lives in 26 dimensions.

However, pure bosonic string theory suffers from certain technical problems. The primary one is that it is tachyonic. In particular, the mass of the lowest energy state is

$$m^2 = -\frac{4}{\alpha'}, \tag{1.37}$$

where $\alpha' = l_s^2$ is the Regge slope.

One way to deal with the tachyon is to invoke supersymmetry.

In addition, it is obvious that we live in $(3+1)$-dimensions and that the world is fairly well described by the Standard Model of Particle Physics which has a gauge group of the form $SU(3)_C \times SU(2)_L \times U(1)_Y$.

A possible way to get realistic physics is to compactify on a Calabi-Yau 3-fold [42].

The presence of solitonic objects, *e.g.* D-branes , with masses of the form

$$m \sim \frac{1}{g_s} \tag{1.38}$$

is highly indicative that string perturbation theory is not convergent.

Let us first look at fermions in $D = 1 + 1$ dimensions.

The Dirac action reads

$$S_D = \int d^2 x \bar{\psi} \gamma^\mu \partial_\mu \psi. \tag{1.39}$$

We will choose a particular representation of the Dirac matrices which is

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.40}$$

It is straightforward to show that

$$\gamma^5 = -\gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.41}$$

The fermions are chosen as

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}. \tag{1.42}$$

The action for the fermions then reads

$$S_D = \int d^2x \psi^\dagger \begin{pmatrix} \partial_0 - \partial_1 & 0 \\ 0 & \partial_0 + \partial_1 \end{pmatrix} \psi$$
$$= \int d^2x \left\{ \chi_L \left(\partial_0 - \partial_1\right) \chi_L + \chi_R \left(\partial_0 - \partial_1\right) \chi_R \right\}. \tag{1.43}$$

After this discussion, we can now supersymmetrize the Polyakov action by writing

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\xi \eta^{ab} \left(\partial_a X^\mu \partial_b X_\mu + i\psi^\mu \gamma_a \partial_b \psi^\mu\right) g_{\mu\nu}. \tag{1.44}$$

The above action is invariant under the SUSY transformations:

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu \tag{1.45}$$
$$\delta_\epsilon \psi^\mu = \gamma^a \partial_a X^\mu. \tag{1.46}$$

Upon taking variations of the fermionic Polyakov action one finds, for the fermionic boundary term:

$$\delta S_P = \frac{1}{4\pi\alpha'} \int d\tau \left(\chi_L^\mu \delta\left(\chi_{L_\mu}\right) - \chi_R^\mu \delta\left(\chi_{L_{R\mu}}\right)\right) \tag{1.47}$$
$$\sim \int d\tau \left.\left\{\delta\left(\chi_L^\mu \chi_{L_\mu}\right) - \delta\left(\chi_R^\mu \chi_{L_{R\mu}}\right)\right\}\right|_0^\pi. \tag{1.48}$$

We note that there are two ways for the boundary term to vanish *viz.*

$$\text{Neveu} - \text{Schwarz}: \quad \chi^\mu_L(\tau,0) = -\chi^\mu_R(\tau,\pi) \tag{1.49}$$

$$\text{Ramond}, \quad \chi^\mu_L(\tau,0) = \chi^\mu_R(\tau,\pi)\,.$$

The periodic (anti periodic) bcs are called the Ramond (Neveu-Schwarz).

We can consider the mode expansion of the fermions in the Ramond (R) sector and we find

$$\chi^\mu_{R/L}(\sigma,\pi) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in\sigma_\mp} \tag{1.50}$$

with the algebra:

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m,n}. \tag{1.51}$$

In the Neveu-Schwarz (NS) sector, we have

$$\chi^\mu_{R/L}(\sigma,\pi) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-in\sigma_\mp} \tag{1.52}$$

and the oscillators satisfy the algebra:

$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r,s}. \tag{1.53}$$

Moreover, in the NS sector one has

$$b_r^\mu \left|0\right\rangle_{NS} = 0 \qquad \forall r > 0$$

This means that b_r^μ with $r < 0$ are creation operators.

For the Ramond sector, we have

$$d_m^\mu \left|0\right\rangle = 0. \tag{1.54}$$

Moreover, one can show that

$$\{d_n^\mu, d_0^\nu\} = 0, \qquad \forall n > 0. \tag{1.55}$$

What this means is that the ground state is degenerate. In particular, we note that

$$b_{-\frac{1}{2}}^\mu \left|0\right\rangle_{NS} = 0. \tag{1.56}$$

The mass can be shown to be given by

$$m^2 = \frac{1}{\alpha'}\left(\frac{1}{2} - \frac{D-2}{16}\right). \tag{1.57}$$

Accordingly, massless modes will occur only when $D = 10$.

It turns out that the NS ground state is tachyonic. More precisely, the ground state mass in the NS sector is

$$m^2 = -\frac{1}{2\alpha'}. \tag{1.58}$$

In order to get rid of this tachyon we perform a trick known as the GSO projection. This consists of introducing the operator $e^{i\pi F} = (-1)^F$ and keeping states with only an odd number of fermions. Since the ground state has zero fermions it is projected out. The NS ground state has a -1 eigenvalue.

Recall that for the massless fields in D dimensions the little group is $SO(D-2)$. Since superstrings live in $D = 10$ we have to study representations of $SO(8)$ which are given in the table below:

	$(-)^F$	$SO(8)$	m^2
NS	-1	$\mathbf{8_v}$	0
NS	-1	$\mathbf{1}$	$-\frac{1}{2\alpha'}$
R	+1	$\mathbf{8}$	0
R	+1	$\mathbf{8'}$	0

Note that we are working in what is called the RNS formalism. The reason for this is that we have superconformal invarince on the worldsheet. The problem with this formalism is that we don't have manifest spacetime supersymmetry. The alternative approach is the Green-Schwarz (GS) formalism which has manifest spacetime supersymmetry and an additional spacetime fermionic symmetry called κ symmetry.

Recall that a point-like particle, for example an electron, is charged under A_μ with field strength $F = dA$. What we mean by this is that as the point traces out a worldline in space time there will be a tangent vector to the worldline $dx^\mu(\tau)/d\tau$. This worldline carries a single index and can thus be multiplied A^μ. Thus, the action for the point particle reads [5]:

$$S = -m\int ds + q\int d\tau A_\mu(x^\mu(\tau))\frac{dx^\mu(\tau)}{d\tau} - \frac{1}{4}\int d^D x F_{\mu\nu}F^{\mu\nu}. \tag{1.59}$$

where m is the mass of the point-particle and q is what we refer to as the charge.What should be clear is that the charge is being associated with a point. We want to do something similar for the string.

Likewise the string is charged under the Kalb-Ramond $B_{\mu\nu}$ field since we can write

$$S = S_{NG} - \frac{1}{2}\int d^2\xi \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu} - \frac{1}{6}\int d^D x H_{\mu\nu\rho} H^{\mu\nu\rho}. \tag{1.60}$$

where $H = dB$.

This idea can be generalized to a Dp-brane in the sense that it will couple to the C_{p+1} potential via

$$S_{Dp} \sim -\int d^{p+1}\xi \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^{\mu_1}}{\partial \sigma}\cdots\frac{\partial X^{\mu_p}}{\partial \sigma} C_{\mu\mu_1\cdots\mu_p} \tag{1.61}$$

and the field strength being $F = dC$.

So far, we have only discussed the open string. Recall that the closed string satisfies the relation: $\sigma \sim \sigma + 2\pi$.

For the closed string the boundary conditions are

$$\chi^\mu_L(\tau, 0) = \pm\chi^\mu_L(\tau, 2\pi) \tag{1.62}$$

$$\chi^\mu_R(\tau, 0) = \pm\chi^\mu_R(\tau, 2\pi). \tag{1.63}$$

These choices can be written as

$$\chi^\mu_{R/L}(\tau, 0) = e^{2\pi i\phi}\chi^\mu_{R/L}(\tau, 2\pi). \tag{1.64}$$

The important thing is that the boundary conditions for the closed string can be chosen independently of each other. (Physically, this is suggestive of the fact that physical observably are usually given by the mod-square of the fermionic fields.)

In other words, we have four options *viz.*

$$\begin{array}{cc} (\mathrm{R},\mathrm{R}) & (\mathrm{NS},\mathrm{NS}) \\ (\mathrm{R},\mathrm{NS}) & (\mathrm{NS},\mathrm{R}) \end{array}$$

The various combinations tensor products in the various sectors are

Sector	$SO\,(8)$	m^2
(NS+,NS+)	$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$	0
(NS-,NS+)	$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$	0
(R+,R+)	$\mathbf{8'} \otimes \mathbf{8'} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$	$-\frac{1}{2\alpha'}$
(R-,R-)	$\mathbf{8} \otimes \mathbf{8'} = \mathbf{8_v} \oplus \mathbf{56_t}$	0
(NS+,R+)	$\mathbf{8_v} \otimes \mathbf{8} = \mathbf{8} \oplus \mathbf{56}$	0
(NS+,R-)	$\mathbf{8_v} \otimes \mathbf{8'} = \mathbf{8} \oplus \mathbf{56'}$	0

It turns out that we can construct two type of strings depending on whether we choose our theory to be chiral or non-chiral. The non-chiral ($\mathcal{N} = 2$ supersymmetric) is called Type IIA and can be constructed by taking the direct sum of the non-chiral (NS-,NS+) and (NS+,R-) pairs. The Type IIA direct sum is $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{8} \oplus \mathbf{56_t}$.

The theory which is chiral is called Type IIB and its' direct sum - obtained by taking the direct sum of the (NS+,NS+) , (R+,R+) and (NS+,R+) pairs - is $\mathbf{1}^2 \oplus \mathbf{28}^2 \oplus \mathbf{35} \oplus \mathbf{8}'^2 \oplus \mathbf{56}^2 \oplus \mathbf{35}_+$.

Here the **28**, **56** and $\mathbf{35}_{\pm}$ are two, three, and four-forms respectively. The **56′**s are vector spinors which we call gravitinos.

From this we can see that Type IIB string theory has the potentials: C_0, C_2, C_4, C_6 and C_8 . The corresponding field strength are F_1, F_3, F_5, F_7 and F_9. Moreover, since a Dp-brane couples to a C_{p+1}this implies that Type IIB string theory has $D1$, $D3$, $D5$, $D7$ and $D9$-branes.

Likewise, Type IIA has the potentials C_1 , C_3, C_5, C_7 and C_9 with corresponding field strengths F_2, F_4, F_6, F_8 and F_{10}. The Dp -branes in Type IIA are $D2$, $D4$, $D6$, and $D8$.

P.S. The material in this box is mostly adopted from one of Prof. Jejalla's 2016 /2017 lectures.

1.1.2 The Decoupling Limit

We now have all the necessary ingredients to give a heuristic argument for the Maldacena conjecture [22, 23, 24].

Let us consider a stack of N $D3$-branes. Then, as we mentioned in the previous section, the $D3$-branes can be regarded as black brane solutions to the Type IIB SUGRA.

For $p = 3$ in (1.33) and (1.36), we have

$$ds^2 = H_3\left(r\right)^{-1/2}\left(-dt^2 + d\vec{x}^2\right) + H_3\left(r\right)^{1/2}\left(dr^2 + r^2 d\Omega_{8-3}^2\right) \tag{1.65}$$

$$H_3\left(r\right) = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s N \alpha'^2. \tag{1.66}$$

where $\vec{x}$ denotes the coordinates along (parallel to) the brane.

Let us consider the geometry far away from the $D3$-branes *i.e.* $r \to \infty$. We see that the metric becomes

$$
\begin{aligned}
ds^2 &= H_3\left(r\right)^{-1/2}\left(-dt^2+d\vec{x}^2\right)+H_3^{1/2}\left(r\right)\left(dr^2+r^2d\Omega_5^2\right)\\
&=\left(1+\frac{R^4}{r^4}\right)^{-1/2}\left(-dt^2+d\vec{x}^2\right)+\left(1+\frac{R^4}{r^4}\right)^{1/2}\sum_{i=4}^{9}dy_i^2.\\
&=-dt^2+d\vec{x}^2+d\vec{y}^2.
\end{aligned}
\tag{1.67}
$$

That is, as expected, far away from the stack of the geometry of the branes is that $(9+1)$-dimensional Minkowski flat space.

We then consider the near horizon limit *i.e.* what happens when $r\to 0$. It is straightforward to see that as $r\to 0$, we have

$$
\begin{aligned}
ds^2 &= H_3\left(r\right)^{-1/2}\left(-dt^2+d\vec{x}^2\right)+H_3\left(r\right)^{1/2}\left(dr^2+r^2d\Omega_{8-3}^2\right)\\
&=\left(1+\frac{R^4}{r^4}\right)^{-1/2}\left(-dt^2+d\vec{x}^2\right)+\left(1+\frac{R^4}{r^4}\right)^{1/2}\left(dr^2+r^2d\Omega_5^2\right)\\
&\to\frac{r^2}{R^2}\left(-dt^2+d\vec{x}^2\right)+\frac{R^2}{r^2}dr^2+R^2d\Omega_5^2.
\end{aligned}
\tag{1.68}
$$

After a little trivial rearranging, we obtain

$$
\begin{aligned}
ds^2 &= \left[\frac{r^2}{R^2}\left(-dt^2+d\vec{x}^2\right)+\frac{R^2}{r^2}dr^2\right]+R^2d\Omega_5^2\\
&=\frac{R^2}{z^2}\left(dz^2-dt^2+d\vec{x}^2\right)+R^2d\Omega_5^2\\
&=ds^2_{AdS_5}+ds^2_{S^5},
\end{aligned}
\tag{1.69}
$$

where we have set $z=\frac{R}{r}$.

Put simply, we have found that, in the closed string picture, a stack of N $D3$ is described by $10d$ SUGRA and type IIB string theory on $AdS_5\times S^5$- this is illustrated in the cartoon in figure 1.2.

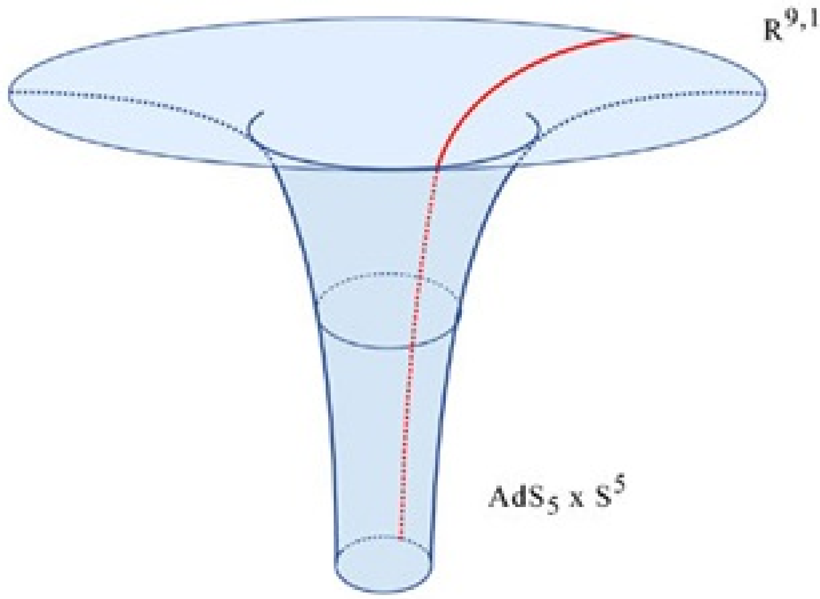


Figure 1.2: Cartoon diagram illustrating the geometry close to and far away from a stack of N $D3$-branes.

Schematically, we have the equation

$$S = S_{IIB}^{AdS_5\times S^5} + S_{IIB\,SUGRA}^{10D}. \tag{1.70}$$

We next look at the open picture description of the stack of $D3$-branes. In this picture the action can be written as [31]:

$$S = S_{bulk} + S_{brane} + S_{int}, \tag{1.71}$$

where S_{bulk} is the action of the closed string in the bulk, S_{brane} is the action along the stack of $D3$-branes and S_{int} is the action of the interactions between the modes on the brane and the one on the bulk.

In the low energy limit the higher massive string excitations will not contribute S_{bulk} and is effectively the supergravity action *i.e.*

$$S_{bulk} \sim \frac{1}{16\pi G}\int \sqrt{-g}R \sim \int (\partial h)^2 + \kappa (\partial h)\, h^2. \tag{1.72}$$

The action along the brane reduces to a super Yang-Mills theory - this is a general expansion similar to the one that led us to (1.14). In fact, detailed computations show that the theory along the stack of $D3$-branes is $\mathcal{N} = 4$ SYM [43].[7]

The action for the interactions of the bulk and the brane is of the form

$$S_{int} \to \alpha'^2 g_s \tag{1.73}$$

We see that in the low energy limit the bulk and brane dynamics are independent of one another *i.e.* they decouple and the stack is described by the action

$$S = S^{D=4}_{\mathcal{N}=4\,\text{SYM}} + S^{10D}_{IIB\,\text{SUGRA}}. \tag{1.74}$$

We now see that in the open string description the stack of $D3$-branes is described by $10D$ SUGRA and $\mathcal{N} = 4$ SYM while in the closed string picture the same stack of $D3$-branes is described by $10D$ SUGRA and Type IIB string theory on $AdS_5 \times S^5$. Given that the two descriptions are describing the same underlying physics we have to conclude that $\mathcal{N} = 4$ SYM must be equivalent to Type IIB strings on $AdS_5 \times S^5$.

Put differently, if we compare the schematic actions appearing in (1.70) and (1.74), we see that $(3 + 1)$-dimensional $\mathcal{N} = 4$ SYM is dual to Type IIB string theory on $AdS_5 \times S^5$.

What we have shown is a weak form of the AdS/CFT correspondence in the sense that we assume that the two theories are the same only at large-N.[8]

[7] As an aside, $\mathcal{N} = 4$ SYM also appears in the so called "grand unified theory" of Mathematics known as the Langlands programme [44], which brings together Number Theory (Galois groups), automorphic forms and representation theory.

[8] Since we know that $R^4 = 4\pi g_s N \alpha'^2$, it follows that working in the supergravity regime - which corresponds to a large radius of curvature - neccesarilly means that we are at large-N.

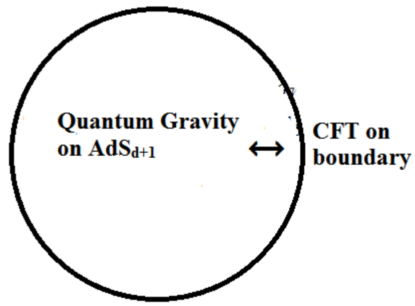


Figure 1.3: The fundamental idea behind the AdS/CFT Correspondence is that a Quantum Gravity theory on the bulk is physically equivalent to some CFT on the boundary.

The stronger form of the duality is that $\mathcal{N} = 4$ SYM is equivalent to Type II B strings at all values of the coupling at at finite N.

Black Hole Thermodynamics

It is a well-known fact that in 1915 Einstein knew of only one solution to his field equations *i.e.* flat Minkowski spacetime [67] and that within a year Karl Schwarzschild had discovered a solution to the vacuum field equations outside an object of mass M [68]. In particular, the metric that Schwarzschild found is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (1.75)$$

Note that the metric has a singularity at $r = 2M$ (the surface $r = 2M$ is called the event horizon) and $r = 0$. The singularity at $r = 2M$ (the event horizon) is actually a coordinate singularity – i.e. it can be removed by a clever choice of coordinates. In contrast, there is a physical singularity at $r = 0$. In other words, there is no choice of coordinates that can remove the singularity at $r = 0$.

The next metric to be considered was the metric for a charged body with charge Q. This metric was written down in 1916 by Reissner [69] and Nordström (1918) [70]. The Reissner-Nordström metric reads (1.22):

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.76}$$

The metric has a coordinate singularity at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{1.77}$$

Given that the position (size) of the two horizons is a real number, we have to require that

$$M \geq Q \tag{1.78}$$

RN black holes that saturate the bound - *i.e.* those for which $r_+ = r_-$ are called extremal. The next exact solutions to the field equations were only written in the 1960s. The first of these exact solutions is the Kerr metric which describes a rotating uncharged object of mass M and angular momentum J [71]. The Kerr metric is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right) dt^2 - \frac{4Mra\sin^2\theta}{r^2 + a^2\cos^2\theta} dt d\phi \\ &- \left(\frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2\cos^2\theta}\right) dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &+ \left(\frac{r^2 + a^2\cos^2\theta - 2Mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \sin^2\theta d\phi^2, \end{aligned} \tag{1.79}$$

where $a = J/M$.

The metric was quickly generalized to include the case when the object is rotating and has a charge Q [72]. The Kerr-Newman metric (which, according to the no hair theorem [73, 74], is the most general asymptotically flat black hole solution to the Einstein-Maxwell field equations) reads

$$\begin{aligned} ds^2 = &-\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + \left(dt - a\sin^2\theta d\phi\right)^2 \frac{\Delta}{\rho^2} \\ &- \left(\left(r^2 + a^2\right)d\phi - adt\right)^2 \frac{\sin^2\theta}{\rho^2}, \end{aligned} \tag{1.80}$$

where

$$a = \frac{J}{M} \tag{1.81}$$

$$\rho^2 = r^2 + a^2 \tag{1.82}$$

$$\Delta = r^2 - r_s r + a^2 + r_Q^2 \tag{1.83}$$

Here,

$$r_s = 2M, \qquad r_Q^2 = Q^2. \tag{1.84}$$

Classically nothing can escape from a black hole. This would then seem to imply that a black hole should not have any temperature whatsoever.

But thanks to Quantum Mechanics this is not the case. In fact, using semi-classical analysis, Hawking found that a black hole should emit radiation with temperature given by [76]

$$T_H = \frac{\hbar c^3}{8\pi G_N M k_B}, \tag{1.85}$$

where k_B is the Boltzmann constant, G_N is Newton's Gravitational Constant and c is the speed of light.

In addition, black holes have an entropy which is given by [63, 76]:

$$S_{BH} = \frac{A k_B c^3}{4 G_N \hbar}, \tag{1.86}$$

where A is the area of the black hole. Note that the Bekenstein-Hawking entropy scales with the area and not, as one would expect from Statistical Mechanics, with the volume. This is an extremely important notion – often called the Holographic Principle [64, 65], which states that in a theory of Quantum Gravity the degrees of freedom should be one degree less than what one would expect.[9]

It turns out that black holes obey laws which are reminiscent of the usual laws of Thermodynamics[75].

More specifically, the Zeroth Law of black hole Thermodynamics states that the surface gravity, denoted by κ, is a constant along the event horizon. This is similar to the classical Zeroth Law which states that the temperature of a system is constant at thermal equilibrium.

The First Law of Thermodynamics can be written as

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi d\Omega. \tag{1.87}$$

where M is the mass of the black hole , J is the angular momentum and Q is the charge of the black hole.

The Second Law of black hole Thermodynamics is the statement that the area of the black hole is a never decreasing function of time:

$$\frac{dA}{dt} \geq 0. \tag{1.88}$$

Recall that the Third Law of Thermodynamics posits that $S \to 0$ as $T \to 0$. Roughly speaking it is not possible to reach absolute zero in a number of finite steps. Similarly, the Third Law of black hole Thermodynamics posits that it is not possible to have $\kappa = 0$ [75, 77].

1.1.3 Symmetries And Matching Of Parameters

The first trivial check for the AdS/CFT Correspondence surely has to be the check that the symmetries on both side of the duality match.

The symmetry of the superconformal $\mathcal{N} = 4$ SYM is the Lie supergroup $PSU\,(2,2\,|4)$which has the bosonic subgroup $SU\,(2,2) \times SU\,(4) \cong SO\,(4,2) \times SO\,(6)$.

Lie super algebra

A natural generalization of a Lie algebra is what is known as a graded Lie algebra or super Lie algebra [34]. This generalization is made by deforming the standard commutator between the generators as follows [35]:

$$[t_a, t_b] = t_a t_b - (-1)^{\eta_a \eta_b}\, t_b t_a, \tag{1.89}$$

where t_a, t_b are the generators and η_a is either $+1$ or 0. Elements that have $\eta_a = 1$ are odd and are called fermionic. Similarly elements with $\eta_a = 0$ are bosonic and even [35].

Recall that a generic matrix Lie super group can be written as [36, 37]:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.90}$$

where the matrices A and D are even while B and C are odd. We will limit ourselves to the case where A, B, C and D are 4×4 complex matrices. The super Lie algebra $\mathfrak{gl}\,(4|4,\mathbb{C})$ is the vector space constructed out of these matrices. The $\mathbb{Z}_2$-graded Lie bracket $[\cdot\}$ can be constructed as

$$[X,Y\} = XY - (-1)^{XY}\, XY \tag{1.91}$$

and satisfies the super Jacobi identity:

$$(-1)^{XZ}\,[[X,Y\},Z\} + (-1)^{ZY}\,[[Z,X\},Y\} + (-1)^{XY}\,[[Y,Z\}X\} \tag{1.92}$$

Moreover, the identity supermatrix commutes with all the other elements of the super algebra and can be projected out to form the Lie super algebra $\mathfrak{pgl}\,(4|4,\mathbb{C})$ [36].

The generators of the superconformal symmetry of $\mathcal{N}=4$ SYM are P_μ, $K_\mu D$, $L_{\mu\nu}$, Q, $\bar{Q}$,S and $\bar{S}$. The Lie super algebra $\mathfrak{psu}\,(2,2\,|4)$ can then be written as the matrix

$$\begin{pmatrix} L & P & -iQ \\ K & \bar{L} & -iS \\ S & \bar{Q} & R \end{pmatrix}.$$

Here the $SO\,(4,2)$ is the conformal group in $D=4$ and the $SO\,(6)$ is the $\mathcal{R}$ symmetry of the six scalar fields of $\mathcal{N}=4$ SYM. In the bulk, the $SO\,(6)$ corresponds to the isometry group of the S^5. Likewise the $SO\,(4,2)$ is the isometry AdS_5.

Another identification is the integral five form flux of the five form flux is identified with the rank of the gauge group of $\mathcal{N} = 4$ SYM. That is,

$$\int_{S^5} F_5 = N. \tag{1.93}$$

For $p = 3$ (1.17) reduces to:

$$g_{YM}^2 = 4\pi g_s \tag{1.94}$$

From (1.66), we have

$$R^4 = 4\pi g_s \alpha'^2. \tag{1.95}$$

Therefore,

$$\boxed{\frac{R^4}{\alpha'^2} = \lambda,} \tag{1.96}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling.

This makes the AdS/CFT Correspondence a strong/weak duality. That is, when the Field theory side is strongly coupled – in other words, we are unable to make use of perturbation theory – the Gravity side is simple to deal with as we can make use of classic supergravity.

This is at the heart of the reason why the AdS/CFT Correspondence is both a powerful technique to use to study strongly coupled theories and also difficult to prove mathematically. The strategy is to try to construct a gravitational dual to the strongly coupled Quantum Field Theory. There are actually a lot of examples where this strategy actually works. For example, a very small shear to entropy ratio was obtained for the $\mathcal{N} = 4$ SYM plasma [45, 46]. More specifically, we have

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}. \tag{1.97}$$

It was then later confirmed – through experiments at Relativistic Heavy-Ion Collider (RHIC) - that the quark gluon plasma does have a value similar to the value obtained for $\mathcal{N} = 4$ SYM [47]. Note that this value cannot be obtained through perturbation theory. In addition, the Navier-Stokes equation was also derived using Holography [48]!

As amazing as it sounds that $\mathcal{N} = 4$ SYM has played a huge role in helping us understand real world physics, it still is an unphysical theory. Part of the reason is that it is superconformal. Another is that all the fields of the maximally supersymmetric $\mathcal{N} = 4$ Yang Mills theory transform in the adjoint representation.

This difficulty can be remedied by, for example, making $\mathcal{N} = 4$ SYM less symmetric *i.e.* one can break some of the supersymmetry or even add flavour/matter to the theory. The latter option has proven quite powerful in the construction of models for QCD [49, 50, 51, 52, 53].

Other examples include using holography to study strange metals [54, 55, 57, 58], scattering amplitudes [59], High T_C superconductors [60, 61, 62] *etc.*

However, this same strong/weak Duality characteristic of the AdS/CFT Correspondence makes it extremely difficult to prove the correspondence because when one side is relatively simple, the corresponding side of the duality becomes extremely strenuous to work with.

Of great consequence is that the theory – besides relating a theory with gravity to one without gravity – is that it relates a theory living on $(3 + 1)$ −dimensions to a $5d$ theory.[10]

This means that the AdS/ CFT Correspondence is holographic.

In simple terms Holography is the statement that, in any theory of quantum gravity,

[10]The S^5 is compact and does not, effectively, play any role in the discussion below.

the degrees of freedom should have one degree less than what one would have naively expected.

A simple example – actually, historically, this is the example that led us to the holographic principle in the first place– is given by looking at a black hole. Recall that the second law of black hole Thermodynamics states that the area of a blackhole never decrease.

This statement was seen to be analogous of the statement that the entropy is an ever-increasing quantity. Bekenstein showed that it was possible to associate the entropy of the black hole with its area [63]. More precisely, the Bekenstein-Hawking entropy of a blackhole is

$$S_{BH} = \frac{k_B c^3 A}{4 G_N \hbar}. \tag{1.98}$$

where k_B is the Boltzmann constant.

Note that the entropy scales with the area -and not the volume! – of the blackhole. This is suggestive that in order to describe a quantum gravitational we need only look at the boundary of the system! (In more exotic everyday terminology the world we live in is a hologram.)

Now for some terminology. The CFT_d that lives on the boundary of AdS is obviously called the theory on the boundary. On the other hand, the gravitational theory on the AdS_{d+1} space is referred to as the theory in the bulk.

1.1.4 GKPW Recipe And Correlation functions

In most physical theories one is interested in the computation correlation functions of operators. To complete the dictionary, we need to know how to relate the bulk fields to the CFT operators living in the boundary.

Moreover, once we have the bulk fields we can compute – if possible – the partition function. The operator-field map bluntly says that a bulk field has a CFT operator

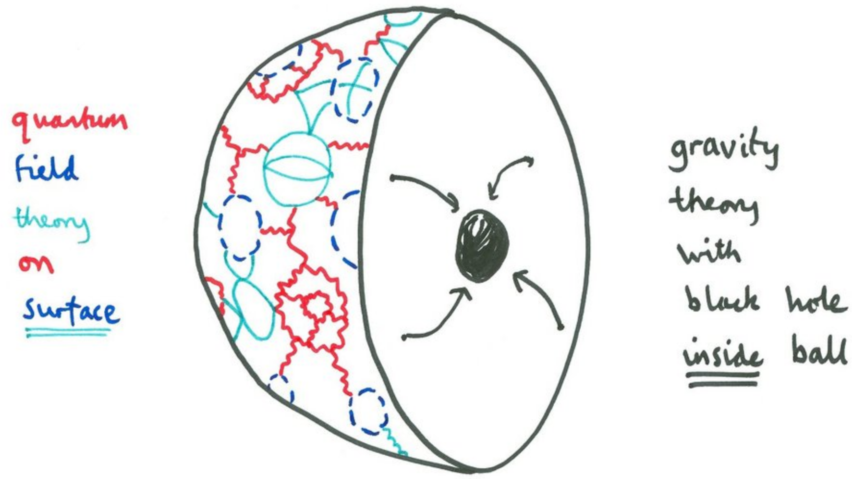


Figure 1.4: A finite temperature CFT is dual to some blackhole living in the bulk.

where the field is evaluated at the boundary. More specifically, we have [78]:

$$\mathcal{O}(x) \leftrightarrow \phi_0 = \phi(x, z_0 \to 0)\,. \tag{1.99}$$

For example, the bulk fields A_μ and $g_{\mu\nu}$ can be mapped to the CFT currents as follows:

$$A_\mu \leftrightarrow J_\mu \tag{1.100}$$

$$g_{\mu\nu} \leftrightarrow T_{\mu\nu}. \tag{1.101}$$

The operator $\mathcal{O}$ has a scaling dimension and we want to see what physical properties this will correspond to in the bulk.

The simplest way to to this is to consider the Einstein-Hilbert action with matter added. The action for such a system can be written as

$$S = \frac{1}{2\kappa^2}\int d^dx\,(R - \Lambda + \mathcal{L}_M)\,, \tag{1.102}$$

where the matter Lagrangian is

$$\mathcal{L}_M = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2. \tag{1.103}$$

We now study the dynamics of this system in AdS_{d+1}.

By varying the action, it is straightforward to show that the equations of motion are

$$\left(\partial^2 - m^2\right)\Phi = 0, \tag{1.104}$$

where

$$\partial^2 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right). \tag{1.105}$$

We will work in Poincarè coodinate and write the metric as

$$ds^2 = \frac{R^2}{z_0^2}\left(dz_0^2 + d\vec{z}^2\right). \tag{1.106}$$

Accordingly, the equations of motion read

$$\left(z_0^2\partial_{z_0}^2 + (1-d)\, z_o\partial_{z_0} + z_0^2 R^2 \partial_i\partial_i - m^2 R^2\right)\Phi = 0. \tag{1.107}$$

We can write Φ as

$$\Phi\left(z_0, \vec{z}\right) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{z}}\phi_{z_0}. \tag{1.108}$$

Inserting this into (1.107) leads to

$$z_0^2 \frac{d^2\phi_{z_0}}{dz_0^2} + (1-d)\, z_0 \frac{d\phi_{z_0}}{dz} - k^2 z_0^2 \phi_{z_0} - m^2 R^2 \phi_{z_0} = 0. \tag{1.109}$$

We are interested in the behaviour of the fields close to the boundary *i.e.* $z_0 \to 0$. Accordingly, we can ignore the term $-k^2 z_0^2 \phi_{z_0}$ and write (1.109) as[11]

$$z_0^2 \frac{d^2\phi_{z_0}}{d^2 z^2} + (1-d)\, z_0 \frac{d\phi_{z_0}}{dz} - m^2 R^2 \phi_{z_0} = 0 \tag{1.110}$$

which has solutions of the form:

$$\phi\left(z_0 \to 0, \vec{z}\right) = \phi_0 \sim z_0^{\Delta}. \tag{1.111}$$

Making use of this ansatz in (1.110) yields

$$z_0^{\Delta}\left(\Delta\left(\Delta - 1\right) + \Delta\left(1-d\right) - m^2 R^2\right) = 0 \tag{1.112}$$

which implies that

$$\Delta^2 - d\Delta - m^2 R^2 = 0. \tag{1.113}$$

The equation above is trivial to solve and we find that

$$\boxed{\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}.} \tag{1.114}$$

Moreover, since the conformal scaling dimension is real we require that

$$\frac{d^2}{4} > -m^2 R^2. \tag{1.115}$$

[11] It is not difficult to recognize that (1.109) is actually Bessel's differential equation and the general solution is $\phi_{z_0} = a_1 z_0^{(d+1)/2} K_\nu\left(k z_0\right) + a_2 z_0^{(d+1)/2} I_\nu\left(k z_0\right)$.

Let us review how to compute standard correlation functions in any standard Quantum Field Theory. The correlation function for $\mathcal{N}=4$ SYM is defined – in the path integral formalism – as

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \int \mathcal{D}\left[\text{SYM fields}\right] e^{iS_{\mathcal{N}=4SYM}}\mathcal{O}(x_1)\cdots\mathcal{O}(x_n) \tag{1.116}$$

where the $\mathcal{O}(x_i)(i=1,2,\ldots,N)$ are composite fields of $\mathcal{N}=4$ SYM.

To compute the n-point function we first introduce the generating functional:[12]

$$Z_{CFT}\left[\phi_0\right] = \int \mathcal{D}\left[\text{SYM fields}\right] e^{iS_{\mathcal{N}=4SYM}+\int\mathcal{O}\phi_0} = e^{-W[\phi_0]} \tag{1.117}$$

where $W\left[\phi_0\right]$ is the generating functional for connected diagrams.

The n-point correlation function is then obtained by taking successive derivatives of the generating functional and setting the source to zero at the end. That is,

$$\left.\frac{\delta \ln Z_{CFT}\left[\phi_0\right]}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)}\right|_{\phi_0=0} = \langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle\,.$$

The heart of the AdS/CFT correspondence lies in identifying the generating functional in the CFT with the string theory partition function:

$$\boxed{Z_{string}\left[\phi_0\right] = \left\langle \exp\left(\int \phi_0\mathcal{O}\right)\right\rangle_{CFT},} \tag{1.118}$$

with ϕ_0 being identified as the value of the bulk field at the boundary.

In addition, if we are at the strong 't Hooft coupling limit we can identify the string theory with classical SUGRA and the above relation simplifies to

[12]We will denote the source by ϕ_0 and not the customary J.

$$\left\langle \exp\left(\int \phi_0 \mathcal{O}\right)\right\rangle_{CFT} = e^{-S_{SUGRA}}. \tag{1.119}$$

To see how what we have set out above actually works in practice we will consider the simple case of the two-point correlation function.

The bulk-boundary propagator is [24, 79, 133]:

$$K_\Delta\left(z_0, \vec{z}, \vec{x}\right) = \frac{\Gamma\left(\Delta\right)}{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2}\right)}\left(\frac{z_0}{z_0^2 + \left(\vec{z} - \vec{x}\right)^2}\right)^\Delta. \tag{1.120}$$

The bulk-boundary propagator satisfies the boundary condition:

$$z_0^{\Delta - d} K_\Delta\left(z_0, \vec{z}, \vec{x}\right) \underset{z_0 \to 0}{\to} \delta\left(\vec{z} - \vec{x}\right). \tag{1.121}$$

The bulk scalar field can then be written as

$$\phi\left(z_0, \vec{z}\right) = \frac{\Gamma\left(\Delta\right)}{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2}\right)}\int d^d\vec{x}\left(\frac{z_0}{z_0^2 + \left(\vec{z} - \vec{x}\right)^2}\right)^\Delta \phi_0\left(\vec{x}\right). \tag{1.122}$$

Assuming we are in the SUGRA limit, we have

$$\ln Z_{CFT}\left[\phi_0\right] = S_{SUGRA}. \tag{1.123}$$

The gravity action is computed to be [133]:

$$\begin{aligned} S_{SUGRA} &= \frac{1}{2}\int d^{d+1}x\left(\sqrt{g}\left(\partial_\mu \phi_0\right)\left(\partial_0^\mu \phi_0\right) - m^2\phi_0^2\right) \\ &= \frac{1}{2}\int d^dx \int dz\left(\partial_\mu\left(\sqrt{g}\phi_0\partial^\mu\phi_0\right) - \phi_0\left(\sqrt{g}\partial_\mu\partial^\mu + m^2\right)\phi_0\right) \\ &= \frac{1}{2}\int d^dx\left\{\frac{R^{d+1}}{z_0^{d+1}}\left(\phi_0\frac{z_0^2}{R^2}\partial_0\phi_0\right)\right\} + \frac{1}{2}\int d^{d-1}x\int dz\left(\sqrt{g}\phi_0\partial^j\phi_0\right) \\ &= \frac{1}{2}\int d^dx\left(\frac{R^{d-1}}{z_0^{d-1}}\phi_0\partial_0\phi_0\right). \end{aligned} \tag{1.124}$$

where to move from the first to the second line we integrated by parts.

We note that [133]:

$$\begin{aligned}
\phi_0 \partial_0 \phi_0 &= \frac{\Gamma(\Delta)\,\phi_0(z_0, x)}{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2}\right)} \partial_0 \left(\int d^d \vec{y} \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{y})^2} \right)^{\Delta_+} \phi_0(\vec{y}) \right) \\
&= \frac{\Gamma(\Delta)\, z_0^{\Delta_-} \phi_0(\vec{x})}{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2}\right)} \int d^d \vec{y} \left(\frac{\Delta_+ z_0^{\Delta_+ - 1} \phi_0(\vec{y})}{\left((\vec{z} - \vec{y})^2\right)^{\Delta_+}} + \cdots \right) \\
&= \frac{\Gamma(\Delta)\, \Delta_+ z_0^{d-1} \phi_0(\vec{x})}{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2}\right)} \int d^d \vec{y} \left(\frac{\phi_0(\vec{y})}{\left((\vec{z} - \vec{y})^2\right)^{\Delta_+}} + \cdots \right),
\end{aligned} \tag{1.125}$$

where we made use of the fact that $\Delta_+ + \Delta_- = -d$.

Thus,

$$\begin{aligned}
S_{SUGRA} &= \frac{1}{2} \int d^d \vec{x} \int dz_0 \left(\frac{R^{d-1}}{z_0^{d-1}} \phi_0(\vec{x})\, \partial_0 \phi_0(\vec{y}) \right) \\
&= C_\Delta \int d^d \vec{x} \int dz_0 \left(\phi_0(\vec{x}) \int d^d \vec{y} \frac{\phi_0(\vec{y})}{\left((\vec{x} - \vec{y})^2\right)^{\Delta_+}} \right).
\end{aligned} \tag{1.126}$$

The two point-function is [133]:

$$\begin{aligned}
\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle &= \frac{\delta^2 S_{sugra}\left[\phi(\phi_0)\right]}{\delta \phi_0(x_1)\, \delta \phi_0(x_2)} \\
&= \frac{C_\Delta}{(\vec{x} - \vec{y})^{2\Delta}}.
\end{aligned} \tag{1.127}$$

which has the standard form for the two-point function of a CFT.

The three-point action can be written as [133]:

$$\begin{aligned}
S_3 &= \frac{\lambda}{3!} \int d^d \vec{z} dz_0 \phi_0^3(z_0, \vec{z}) \sqrt{g} \\
&= \frac{\lambda}{3!} \int d^d \vec{x}_1 d^d \vec{x}_2 d^d \vec{x}_3 dz_0 \sqrt{g} \phi_0(\vec{x}_1)\, \phi_0(\vec{x}_2)\, \phi_0(\vec{x}_3)\, K_\Delta(z_0, \vec{z} - \vec{x}_1)\, K_\Delta(z_0, \vec{z} - \vec{x}_2)\, K_\Delta(z_0, \vec{z} - \vec{x}_3).
\end{aligned} \tag{1.128}$$

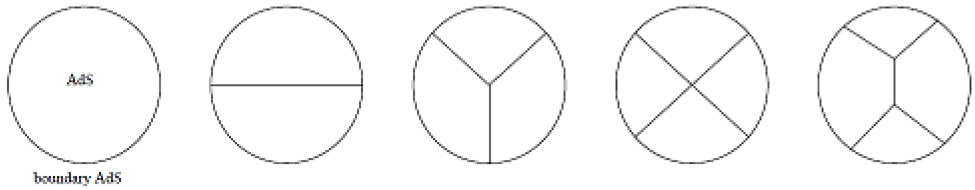


Figure 1.5: Some examples of Witten diagrams. Picture credits [30].

Likewise, the three-point function is [133]:

$$\begin{aligned}\langle \mathcal{O}(\vec{x}_1)\, \mathcal{O}(\vec{x}_2)\, \mathcal{O}(\vec{x}_3)\rangle &= \frac{\delta^3 S_{sugra}\left[\phi\left(\phi_0\right)\right]}{\delta\phi_0(x_1)\,\delta\phi_0(x_2)\,\delta\phi_0(x_3)} \\ &= \int d^d\vec{z}dz_0 K_\Delta\left(z_0, \vec{z}-\vec{x}_1\right) K_\Delta\left(z_0, \vec{z}-\vec{x}_2\right) K_\Delta\left(z_0, \vec{z}-\vec{x}_3\right).\end{aligned} \tag{1.129}$$

In fact, one can show that[13]

$$\langle \mathcal{O}(\vec{x}_1)\, \mathcal{O}(\vec{x}_2)\, \mathcal{O}(\vec{x}_3)\rangle = \frac{a_1}{\left|\vec{x}_1-\vec{x}_2\right|^{\Delta_1+\Delta_2-\Delta_3}\left|\vec{x}_2-\vec{x}_3\right|^{\Delta_2+\Delta_3-\Delta_1}\left|\vec{x}_3-\vec{x}_1\right|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{1.130}$$

Most of the correlation functions can be computed using what are known as Witten diagrams - examples of some of these diagrams are shown in Figure 1.5.

The last part of the AdS/CFT dictionary that we mention has to do with how to interpret finite temperature of the CFT on the bulk gravity side. This question was answered in one of the early canonical papers by Witten [80].[14]

Essentially, the vacuum of the CFT corresponds to pure AdS space. One tries to excite this CFT vacuum and one of the excited states will correspond to finite-temperature. There will be two candidates for the thermal dual of the finite CFT. These being a

[13]The subscript on the constant a_1 is there because there are two cubic vertices that were considered in [79]. We have only chosen the simpler vertex to illustrate how the computation works.

[14]This interpretation is depicted in (1.4).

Boundary CFT	Bulk Gravity
Operator $\mathcal{O}$	dynamical field $\phi(z_0, x)$ such that $\phi_0(x) = \lim_{z\to 0} z^{\Delta_-}\phi(z,x)$
scaling dimension Δ	mass
global symmetry, $PSU(2,2 \mid 4)$	Isometries of $AdS_5 \times S^5$
g_{YM}^2	$4\pi g_s$
Rank of gauge group, N	Five form flux through the five sphere
Finite temperature CFT	Blackhole

Table 1.1: Summary of the AdS/CFT dictionary

thermal gas or a blackhole. The thermal gas is ruled out because it leads to a tachyon and has a singularity at $z \to \infty$ [78]. This means we need to find a blackhole with the right symmetry. We can choose an ansatz of the form:

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)\,dt^2 + d\vec{x}^2\right) + \frac{R^2}{z^2} g(z)\,dz^2, \tag{1.131}$$

where

$$f(z) = \frac{1}{g(z)} = \left(1 - \frac{z^4}{z_0^4}\right). \tag{1.132}$$

One can then perform the analogous "cheap" derivation of the Hawking temperature to the metric given above and find that the black hole temperature is given by

$$T_H = \frac{1}{\pi z_0}. \tag{1.133}$$

1.2 Other Examples

M stands for Magic, Mystery, or Matrix according to taste

-Edward Witten

We saw when we looked at the closed strings in the superstring section (box) that there were two consistent ways to choose the boundary conditions. This led us to two consisted superstrings *viz.* Type IIA and Type IIB string theories. (Recall that Type IIB is chiral while Type IIA is non-chiral.)

By the beginning of 1985 another kind of superstring had been constructed by Gross, Harvey, Martinec and Rohm called the heterotic string which had either the gauge group $SO(32)$ or $E_8 \times E_8$ [81].

It turned out that there were five consistent theories. (The Heterotic strings are useful for phenomenology. That is, they are useful when one wants to reproduce things like the standard Model of Particle Physics.)

This was an embarrassment of sorts for a physical theory which purported to be a Theory of Everything.

However, it turned out that all these five theories were related by dualities to one another. For example, Type IIB and Type IIA are related by T-duality while Type I is related to heterotic $SO(32)$ by S-duality. These dualities are summarized in Figure 1.6.

What was surprising was that these superstring theories were dual to some all embodying theory whose low energy effective theory was non other than 11D SUGRA. This theory was called M-Theory [82].

A particular realization of M-Theory was given to be a matrix model in 0+1 [83] or 0+0 [84] dimensions.

M-theory also has extended solitonic objects *viz.* the $M2$-brane and the $M5$-brane. The $M2$-branes solution to $11D$ SUGRA is [85, 86]

$$ds^2 = H(r)^{-2/3}\, dx \cdot dx + H(r)^{1/3}\, dy \cdot dy \tag{1.134}$$

$$H(r) = 1 + \frac{r_2^6}{r^6}, \qquad r_2^6 = 32\pi^2 N_2 l_p^6. \tag{1.135}$$

with the electric flux given by

$$G_{012r} = -\frac{\partial}{\partial r} H(r)^{-1}, \tag{1.136}$$

while the $M5$-brane solution reads [86]:

$$ds^2 = H(r)^{-1/3}\, dx \cdot dx + H_3(r)^{2/3}\, dy \cdot dy \tag{1.137}$$

$$H(r) = 1 + \frac{r_5^6}{r^6}, \qquad r_5^3 = \pi N_5 l_p^5. \tag{1.138}$$

The natural thing was to try and obtain some AdS/CFT duality by studying a stack of $M2$ branes. This is not that trivial as most aspects of M-theory are "mysterious." In particular, the first "simple" case to be considered was the worldvolume of a stack of $N = 2$. This theory, named BLG, was formulated in terms of a three algebra [87, 88, 89, 90].

However, after some effort, Aharony, Bergman, Jafferis and Maldacena, finally found the worldvolume theory describing a stack of multiple $M2$-branes on $\mathbb{CP}^4/\mathbb{Z}_k$ orbifold [91].[15] More specifically, ABJM conjectured that M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ is dual to a $3d$ $\mathcal{N} = 6$ supersymmetric Chern-Simons matter $U(N)_k \times U(N)_{-k}$ quiver gauge theory called ABJM theory.

A few comments are in order. First, the action of the $\mathbb{Z}_k$ on the coordinates of the $\mathbb{CP}^4$

[15]Some clear expositions of the ABJM duality can be found in Chapter 20 of [26] and also the notes by Klebanov [92].

is

$$Z_i \to e^{\frac{2\pi i}{k}} Z_i, \tag{1.139}$$

and k is an integer called the level.

Second, when $k = 2$ the supersymmetry is enhanced to $\mathcal{N} = 8$. In particular, if we also set $N = 2$ we obtain the BLG theory.

ABJM consists of the following fields: 4 complex scalar fields C^I, Weyl fermions and gauge fields A^μ and the action can be written as[16]

$$\begin{aligned} S_{ABJM} = \int d^3x \Bigg[& \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho + A_\mu \partial_\nu A_\nu) \\ & - \mathrm{Tr}\left(D_\mu C_I^\dagger D^\mu C^I \right) - i \mathrm{Tr}\left(\psi^{I\dagger} \gamma^\mu D_\mu \psi_I \right) + \frac{4\pi^2}{3k^2} \mathrm{Tr}\Big(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger \\ & + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K + 4 C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6 C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger \Big) \\ & + \frac{2\pi i}{k} Tr \Big(C_I^\dagger C^I \psi^{J\dagger} \psi_J - \psi^{J\dagger} C^I C_I^\dagger \psi_J - 2 C_I^\dagger C^J \psi^{I\dagger} \psi_J \\ & + \epsilon^{IJKL} \psi_I C_J^\dagger \psi_K C_L^\dagger - \epsilon_{IJKL} \psi^{I\dagger} C^J \psi^{\dagger K} C^L \Big) \Bigg]. \end{aligned} \tag{1.140}$$

Compactifying M-theory on S^1 gives us Type IIA strings. Accordingly, when when the radius of the circle is small i.e. $k^5 \gg n$, we have that Type IIA strings on $AdS_4 \times \mathbb{CP}^3$ are dual to the ABJM theory.

Finally, the theory can be generalized to the case when one has the quiver gauge group $U(N)_k \times U(M)_{-k}$ and finds the ABJ duality [93].

To obtain other Gauge/String dualities one can try and look at "deformations" of the original $AdS_5 \times S^5$. For example, we could look at the Penrose limit of $AdS_5 \times S^5$ and

[16]This is of course only one side of the gauge group. In addition, we are being loose with the groups as they are meant to be $U(N)_k \times U(N)_{-k}$.

find that strings on the PP-background are dual to the BMN gauge theory [95].

One way that was mentioned in passing to "deform" the correspondence is to try and break some of the supersymmetries. This was done first by Klebanov and Witten by considering a stack of $D3$-branes near a conical singularity. In particular, they conjectured that Type IIB string theory on $AdS_5 \times T^{1,1}$ is dual to $(3+1)$ dimensional $\mathcal{N}=1$ SCFT [94].

Other examples of the Gauge/String duality includes lower dimensional versions:

- AdS_3/CFT_2: string theory on $AdS_3 \times S^3 \times \mathcal{M}^4$ with $\mathcal{M}^4 = \mathbb{T}^4$ (or $\mathcal{M}^4 = K3$) is dual to small $\mathcal{N} = (4,4)$ SYM [22, 96]. This duality is arrived at by replacing the $D3$-brane set up by the $D1$-$D5$ system i.e. one considers a system with N_1 $D1$ branes and N_5 $D5$ branes - for a more comprehensive review of the set up, see [97].
- Recently, the CFT dual of string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ was found to be the symmetric orbifold $\mathcal{S}_k$ [99].
- Pure gravity on AdS_3 is dual to a conformal field theory[17] with the Monster gauge group [100] - so called because it is the largest sporadic group with order of

$$\begin{aligned}\left|G\right| &= 2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^2 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71 \\ &= 808,017,424,722,512,875,886,459,904,961,710,757,005,754,368,000,000, \\ &\qquad 000 \approxeq 8 \times 10^{53}.\end{aligned}$$

More precisely, the conjecture as stated is true for an extremal CFT with central charge $c = 24$. However, Witten posits a stronger statement which holds for $c = 24k$ with $k > 1$.

[17] As a historical aside, the AdS_3case is often cited as a precursor to the AdS/CFT Correspondence. This is due to the paper by Brown and Henneaux in 1986 where they found that the asymptotic algebra of AdS_3 consists of two Virasoro algebras with central charges $c_L = c_R = 3l/2G$ [98].

- $NAdS_2/NCFT_1$: The current melonic revolution has given a candidate for the near AdS_2/ near CFT_1 . In particular, the $(0+1)$ SYK model [101, 102, 103, 104, 105, 106] (which saturates the Maldacena-Shenker-Stanford chaos bound[18] - which is a signature of blackholes and has an emergent conformal symmetry in the IR) is supposedly dual to the Jackiw-Tetelbohm dilaton gravity [108, 109, 110, 111]. [19]

What most of these dualities (except for the last one) is that they involve matrix-like fields on the CFT side and rely heavily on supersymmetry.[20]

In the next Chapter, we will consider a class of simpler AdS/CFT dualities were the dofs scale like N and not N^2 as in the matrix case.

1.3 Outline

In this thesis, we use the collective field theory approach to study the infra-red fixed point of interacting $\frac{\lambda}{N}\left(\phi^a\phi^a\right)^2$ theory in terms of the bilocals.[21] This theory is equivalent to the non-linear sigma model. We compute various correlation function in the non-linear sigma model (and the single-time bilocal approach). We identify a bound state and give an argument that the $\Delta = 1$ state is indeed not present in our spectrum. This is in agreement with the Klebanov-Polyakov HS/CFT conjecture.

This thesis is organized as follows. In Chapter 2, we give an historical review of the problems that bedazzled early attempts to find a consistent theory of higher spins in flat Minkowski spacetime. (The problem is as old as Quantum Mechanics, and was only

[18]Quantum Chaos is characterized by the out of time correlation function:

$$\langle V(t)\,W(0)\,V(t)\,W(0)\rangle \sim \frac{1}{N}e^{\lambda_L t} \tag{1.141}$$

where λ_L is the Lyapunov exponent and satisfies the bound: $\lambda_L \leq \frac{2\pi}{\beta}$ [107].

[19]The original theory was introduced in [112, 113].

[20]The Higher-Spin/CFT and AdS/SYK dualities seem to be the exception where supersymmetry is not needed. On quite general grounds, it seems that supersymmetry is essential for the AdS/CFT Correspondence [114, 115, 116].

[21]At present, most work has been done for the UV free fixed point.

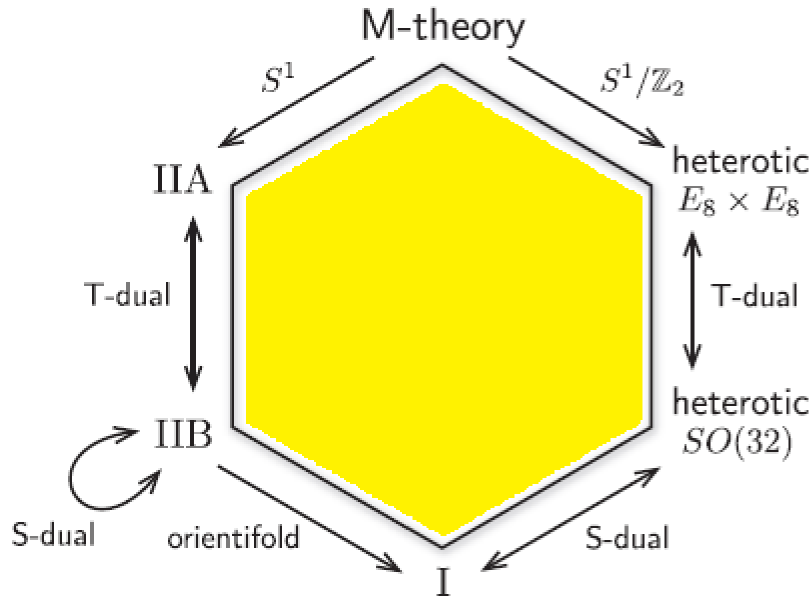


Figure 1.6: The five string theories are related to one another by dualities and to some theory called $11D$ M-theory. Picture credits [231].

finally solved in the period 1987-1992.) In particular, we will discuss various No-Go Theorems that clearly rule out higher spin interactions in flat space. We will first focus on the free massless higher spin theory of Fronsdal. This will be followed by a discussion of the fully interaction theory of Vasiliev. We then review the Klebanov-Polyakov Higher-Spin AdS_4/CFT_3 conjecture. This will be followed by a brief discussion of some of the generalizations of the conjecture and some of the main checks that have been done to test the Higher-Spin/Vector Model Duality.

In Chapter 3, we revisit the Jevick-Sakita collective field theory. In particular, we focus on rewriting the $O(N)$ vector model in terms of the $O(N)$ invariant bilocals. In section 3.1, we derive the large-N collective field theory Hamiltonian. In section 3.2, we discuss the covariant path integral two-time bilocals. More precisely, we solve for the large-N background and obtain the gap equation.

In Chapter 4, we consider the single time collective field theory Hamiltonian. In section

4.1, we discuss how to obtain the free field theory spectrum and propagator. In section 4.2, we then discuss the collective field theory map. The focus will be largely on the lightcone gauge map. (However, for completeness, we also mention the time-like map.) In section 4.3, we give a brief review of the Bethe-Salpeter equation and use it to obtain the bilocal propagator – which is really a four-point function in the original fields - for the free case theory.

In Chapter 5, we review the argument that the $O(N)$ critical vector model is equivalent, in the IR, to the non-linear sigma model.

In Chapter 6, we turn to the covariant two-time non-linear sigma model. We vary the collective field theory action of the non-linear sigma model. This leads us to a gap equation. We solve for the large-N background configurations of the bilocals and the Lagrange multiplier field. This enables us to introduce fluctuations. We shift the bilocal fluctuations and diagonalize the effective quadratic action. We then compute the position two-point correlation functions for the bilocal and Lagrange multiplier fluctuations.

In Chapter 7, we present the two-time bilocal description of the $O(N)$ critical $(\phi^2)^2$ theory. More specifically, in section 7.1, we find an expression for the bilocal two-point function which corresponds to the Bethe-Salpeter equation. Section 7.2 is devoted to finding the spectrum of the $O(N)$ critical $(\phi^2)^2$. This is done by studying the poles of the two-time collective field theory propagator.

In chapter 8, we give a Hamiltonian (single-time) description of the $O(N)$ critical$(\phi^2)^2$ theory. In section 8.1, we make use of Hamilton's equations to write down a coupled integral equation for the bilocal fluctuations. In section 8.2, we show the equivalence between the two-time and single-time bilocal descriptions. In section 8.3, we consider the single-time Lagrangian and obtain the propagator.

In Chapter 9, we begin by making some comments about the fact that the naive expectation that the spectrum of the $O(N)$ critical $(\phi^2)^2$ theory can be obtained from the free theory by replacing $\Delta = 1$ by $\Delta = 2$ looks, on the face of it, erroneous. To clarify this puzzle, we look at the simple scattering through a Dirac Delta potential. This basic example will allow us to argue that $\eta_{xx} = 0$.

In Appendix A, we expand the effective action to cubic order and extract the three-point vertices. This allows us to introduce a set of Feynman diagrams.

In Appendix B, we discuss the mode expansion for the free $O(N)$ vector model.

In Appendix C, we reformulate the quantum mechanical problem with a Dirac delta function potential in the operator language that has been used in this thesis. We invert the operator and write down the propagator for this system.

Chapter 2

Higher-Spin Gravity

What is this [higher spin(HS)/Chern Simons duality] good for? I really don't know.

-Shiraz Minwalla, Strings 2017.

The study of massless fields of spin-1 and spin-2 underlies most of 20th Century Physics. For example, the spin-1 massless $U(1)$ fields are used to describe what is said to be the most accurate theory in the history of man-kind *i.e.* Quantum Electrodynamics.[1]

The non-Abelian generalities of the spin-1 play a fundamental role in the Standard Model of Particle Physics.

The spin-2 case should describe gravity.

A natural question to ask is if there are higher spin ($s > 2$) massless cousins of the photon, gluons or gravitons.

The higher spin cousins of the photon, graviton etc. have not been ruled out (experimentally) in a phase where higher spin symmetry is spontaneously broken [117, 118].[2]

[1]More precisely, we compare the experimental value for $\left(\frac{g-2}{2}\right)_{exp} = (115965218115965230) \times 10^{-11}$, where g is the electron g factor, to the theoretical value of $\left(\frac{g-2}{2}\right)_{theory} = 0.5\left(\frac{\alpha}{\pi}\right) - 0,32848\left(\frac{\alpha}{\pi}\right)^2 + 1,19\left(\frac{\alpha}{\pi}\right)^3 = (115965230 \pm 10) \times 10^{-11}$.It turns out that the theory is correct to 16 significant figures.

[2]Note that, however, we have experimentally observed higher spin MASSIVE particles.

From a theoretical view point, there are various no go theorems that actually rule out the existence of higher spin fields [119, 120, 121, 122, 123, 124, 125, 126]. (For reviews on the No-Go theorems, see [127].)

Without going into details we can sketch a simple example of why all higher spin fields have to be trivial – *i.e.* there cannot be any interactions.

Recall that the Poincaré group consists of translations, rotations and boosts. The Poincaré algebra is

$$[P_\mu, P_\nu] = 0 \tag{2.1}$$

$$[M_{\mu\nu}, P_\rho] = i\left(g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu\right) \tag{2.2}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\varrho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}\right). \tag{2.3}$$

There are two Casimirs *viz.*[3]

$$P^2 = P_\mu P^\mu, \qquad W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \tag{2.4}$$

where W_μ is called the Pauli-Lubanski four vector.

Under very general but fairly reasonable physical assumptions – *e.g.* there are a finite number of particles *etc.* – the Coleman-Mandula theorem posits that it is impossible to mix the Poincaré symmetries with the internal symmetries in any but the most trivial manner i.e. the resulting group can at most be a direct sum of the Poincaré group and the internally symmetry group.

To see how the Coleman-Mandula theorem rules out higher spin fields one only has to remember the simple fact that massless fields are gauge fields. So, adding massless higher spin fields will introduce a bigger symmetry group than the Poincaré group and this is ruled out by the Coleman-Mandula theorem. (There are other specific theorems that

[3] A Casimir is defined as a quantity that commutes with all the elements of the Lie algebra.

also rule out, unambiguously, higher spin gauge fields *e.g.* the Weinberg soft theorems and the Weinberg-Witten theorems.)

(In fact, the study of higher spin field theories is as old as Quantum Theory and dates back to attempts by Dirac to generalize his famous spin 1/2 equation [131] and also by the likes Majorana [132] and Fierz and Pauli [133].)

No-Go Theorems

Besides the Coleman-Mandula theorem, we also have a theorem due to Weinberg that also rules out the existence of massless particles with spins $s > 2$. We consider the scattering of N particles with momenta p_i $(i = 1, 2, \ldots, N)$. In this scattering process we will assume the emission of some massless particle of spin s with momentum q. In the soft limit (i.e. $q \to 0$), the S-matrix for this scattering process takes the form [127, 128, 134]:

$$S\left(p_1, \ldots, p_N, q, \varepsilon\right) = \sum_{i=1}^{N} g_i \left[\frac{p_i^{\mu_1} \cdots p_i^{\mu_s} \varepsilon_{\mu_1 \cdots \mu_s}}{2 p_i \cdot q}\right] S\left(p_1, \ldots, p_N\right). \tag{2.5}$$

Here, $\varepsilon_{\mu_1 \cdots \mu_s}$ is the polarization tensor and the g_i are coupling constants. Under Lorentz transformations, the polarization transforms as [127]:

$$\varepsilon_{\mu_1 \cdots \mu_s}\left(q\right) \to \varepsilon_{\mu_1 \cdots \mu_s}\left(q\right) + q_{(\mu_1} \zeta_{\mu_2 \cdots \mu_s)}. \tag{2.6}$$

In order for the spurious polarizations to decouple, we have to impose the condition that

$$\boxed{\sum_{i=1}^{N} g_i^{(s)} p_i^{\mu_1} \cdots p_i^{\mu_{s-1}} = 0.} \tag{2.7}$$

When $s = 1$, (2.7) yields

$$\sum_{i=1}^{N} g_i^{(1)} = 0. \tag{2.8}$$

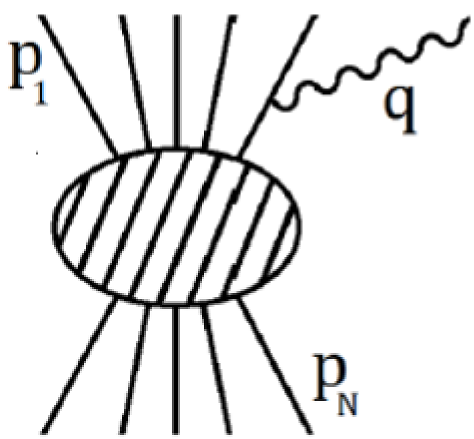
This condition expresses the intuitive fact that the total charge (*e.g.* electric charge) for the entire scattering process is conserved.

For $s = 2$, (2.7) implies that

$$\sum_{i=1}^{N} g_i^{(2)} p_i^{\mu} = 0. \tag{2.9}$$

This condition can easily be satisfied by setting $g_i = \kappa$ and $\sum_i p_i^{\mu} = 0$. Thus, (2.9) expresses nothing but the conservation of the four-momentum and the fact that all particles interact with the same strength with the graviton.

However, for $s > 2$ we cannot write down a non-trivial solution to (2.7). More precisely, the only way for (2.7) to be satisfied is by setting $g_i^{(s)} = 0$. However, this does not completely rule out higher spin interactions but only states that these interactions cannot be long-range [121, 127, 128, 134].



Instead of giving a generic argument, Aragone and Deser tried to explicitly construct an interacting massless spin-5/2 theory [123]. The action of the spin-5/2 hypergravity theory reads

$$S = \int d^4x\sqrt{-g}\Bigg[-\frac{1}{2}\bar{\psi}_{\mu\nu}\not{D}\psi^{\mu\nu} + 2\bar{\not{\psi}}_{\mu}\not{D}\not{\psi}^{\mu} - \frac{1}{2}\bar{\psi}'\not{D}\psi'$$
$$+ \left(\bar{\psi}\nabla\cdot\not{\psi} - 2\bar{\not{\psi}}_{\mu}\nabla\cdot\psi^{\mu} - \text{h.c.}\right)\Bigg], \tag{2.10}$$

where $\not{D} = \gamma^{\alpha}D_{\alpha}$, $\psi_{\mu\nu}$ is a rank two tensor spin, $\psi^{\mu} = \gamma_{\alpha}\gamma^{\mu\nu}$ and $\psi = \psi^{\alpha}_{\alpha}$ [123]. The hypergravity theory is, as expected, redundant. These redundancies can be eliminated by imposing the gauge invariance:

$$\delta\psi_{\mu\nu} = \nabla_{(\mu}\epsilon_{\nu)}, \qquad \gamma^{\mu}\epsilon_{\mu} = 0. \tag{2.11}$$

However, under these gauge transformations, the actions changes by

$$\delta S \sim \int d^4x\sqrt{-g}\left(\epsilon_{\mu}\gamma_{\nu}\psi_{\alpha\beta}R^{\mu\alpha\nu\beta}\right) \tag{2.12}$$

which is obviously non-vanishing in a general curved spacetime.

The other powerful No-Go theorems are the two Weinberg-Witten theorems [124]. These theorems states that

i) A theory with a Lorentz covariant, conserved four current J and associated conserved charge $Q = \int d^dxJ^0$ does not admit massless particles with spin $s > \frac{1}{2}$with non-zero charge under Q.

ii) A theory that allows the construction of a conserved Lorentz covariant energy-momentum tensor $T_{\mu\nu}$ for which $\int d^3xT_{0\nu}$ is the energy-momentum four-vector cannot contain massless particles of spin $s > 1$.

It is important to note that the Weinberg-Witten theorem does not cover the General

Theory Of Relativity as the energy-momentum tensor of GR is not Lorentz covariant.

The theorems are actually simple to prove. Firstly, note that it is possible to transform into a frame where we can write the null four momenta as

$$p = (|p|, \vec{p}), \qquad p' = (|p|, -\vec{p}). \tag{2.13}$$

It is straightforward to show that

$$\langle p', \pm\sigma | J^{\mu}(x) | p, \pm\sigma\rangle = \frac{g p^{\mu}}{(2\pi)^3 E} \tag{2.14}$$

$$\langle p', \pm\sigma | T^{\mu\nu}(x) | p, \pm\sigma\rangle = \frac{f p^{\mu} p^{\nu}}{(2\pi)^3 E}, \tag{2.15}$$

where s is the spin of the particle and

$$g = \int d^d x J^0 \tag{2.16}$$

$$f p^{\mu} = \int d^d x T^{0\mu}. \tag{2.17}$$

Under a rotation by an angle θ, we have

$$|U(R(\theta))| p, \pm\sigma\rangle = e^{\pm i\theta s} |p, \pm\sigma\rangle, \qquad |U(R(\theta))| p', \pm\sigma\rangle = e^{\mp i\theta s} |p', \pm\sigma\rangle. \tag{2.18}$$

Recall that under rotations the current J^{μ} and the energy-momentum tensor $T^{\mu\nu}$ transform as

$$J^{\mu} \to U^{-1}\left(R\left(\theta\right)\right) J^{\mu} U\left(R\left(\theta\right)\right) = R\left(\theta\right)^{\mu}_{\nu} J^{\nu} \tag{2.19}$$

$$T^{\mu\nu} \to U^{-1}\left(R\left(\theta\right)\right) T^{\mu\nu} U\left(R\left(\theta\right)\right) = R\left(\theta\right)^{\mu}_{\rho} R\left(\theta\right)^{\nu}_{\sigma} T^{\rho\sigma}. \tag{2.20}$$

From (2.18), (2.19) and (2.20) , we obtain

$$\left\langle p', \pm\sigma \left| J^{\mu} \right| p, \pm\sigma \right\rangle \to e^{\pm 2i\theta s} \left\langle p', \pm\sigma \left| J^{\mu} \right| p, \pm\sigma \right\rangle = R\left(\theta\right)^{\mu}_{\nu} \left\langle p', \pm\sigma \left| J^{\nu} \right| p, \pm\sigma \right\rangle \tag{2.21}$$

$$\left\langle p', \pm\sigma \left| T^{\mu\nu} \right| p, \pm\sigma \right\rangle \to e^{\pm 2i\theta s} \left\langle p', \pm\sigma \left| T^{\mu\nu} \right| p, \pm\sigma \right\rangle = R\left(\theta\right)^{\mu}_{\rho} R\left(\theta\right)^{\nu}_{\sigma} \left\langle p', \pm\sigma \left| J^{\mu} \right| p, \pm\sigma \right\rangle. \tag{2.22}$$

The rotation matrix $R\left(\theta\right)$ only has the Fourier modes 1 and $e^{\pm i\theta}$. Hence, from (2.21) we see that the only allowed values for the spin are

$$2s = 0; \pm 1. \tag{2.23}$$

or

$$s \leq \frac{1}{2}. \tag{2.24}$$

Similarly, (2.22) implies that the allowed values for the spin of a theory that has a covariant stress-energy tensor are

$$s \leq 2. \tag{2.25}$$

That is, any theory that has a Lorentz covariant stress-energy tensor cannot have massless particles with spin greater than two.

2.1 Fronsdal

Let us begin by reviewing the elementary case of the Maxwell theory. This theory is described in terms of the gauge field A_μ which satisfies the free field equation:

$$\partial_\mu F^{\mu\nu} = 0 = \partial^2 A^\nu - \partial^\nu (\partial \cdot A) \tag{2.26}$$

where the field strength is given by: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Maxwell theory is invariant under the gauge transformations of the form:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \epsilon \tag{2.27}$$

In other words, the Maxwell theory is invariant under $\delta A_\mu = \partial_\mu \epsilon$.

We next look at the linearized Einstein equations. In particular, we consider fluctuations around the Minkowski background: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. One can show that the field equations then reduce to:

$$0 = \partial^2 h_{\mu\nu} - \partial_\mu \partial^\rho h_{\nu\rho} + \partial_\nu \partial^\rho h_{\mu\rho} + \partial_{\mu\nu} h \tag{2.28}$$

where h is the determinant of the fluctuations.

The linearized field equations are invariant under transformations of the form:

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{2.29}$$

What we want is similar analogous results for free higher spin gauge fields which will be a symmetric rank s tensor which we denote by $\phi_{\mu_1\cdots\mu_s}$.[4] These equations were obtained by Fronsdal in 1978.

For simplicity, we will follow the construction in [128, 129]. We first consider the spin-s Christoffel symbols which is defined as

$$\Gamma^{(1)}_{\rho;\mu_1\cdots\mu_s} = \partial_\rho \phi_{\mu_1\cdots\mu_s} - s\partial_{(\mu_1}\phi_{\mu_2\cdots\mu_s)\rho}. \tag{2.30}$$

We recursively define

$$\Gamma^{(m)}_{\rho_1\cdots\rho_s;\mu_1\cdots\mu_s} = \partial_{\rho_1}\Gamma^{(m-1)}_{\rho_2\cdots\rho_s;\mu_1\cdots\mu_s} - \frac{s}{m}\partial_{(\mu_1|}\Gamma^{(m-1)}_{\rho_2\cdots\rho_m\rho_1|\mu_2\cdots\mu_s)}, \tag{2.31}$$

where the brackets $(\mu_1 \mid \rho_2\cdots\rho_m\rho_1 \mid \mu_2\cdots\mu_s)$ means we need to symmetrize with respect to μ_1 to μ_s while leaving indices between the vertical lines alone.

For $m = 2$ this definition reduces to

$$\begin{aligned}
\Gamma^{(2)}_{\rho_1\rho_2;\mu_1\cdots\mu_s} &= \partial_{\rho_1}\Gamma^{(1)}_{\rho_2;\mu_1\cdots\mu_s} - \frac{s}{2}\partial_{(\mu_1|}\Gamma^{(1)}_{\rho_2;\rho_1|\mu_2\cdots\mu_s)} \\
&= \partial_{\rho_1}\left(\partial_{\rho_2}\phi_{\mu_1\cdots\mu_s} - s\partial_{(\mu_1}\phi_{\mu_2\cdots\mu_s)\rho_2}\right) - \frac{s}{2}\partial_{(\mu_1|}\partial_{\rho_2}\phi_{\rho_1|\mu_2\cdots\mu_s)} + \frac{s}{2}\times s\partial_{(\mu_1|}\partial_{(\rho_1}\phi_{\mu_2\cdots\mu_s)\rho_2} \\
&= \partial_{\rho_1}\partial_{\rho_2}\phi_{\mu_1\cdots\mu_s} - s\partial_{\rho_1}\partial_{(\mu_1}\phi_{\mu_2\cdots\mu_s)\rho_2} + \frac{s\,(s-1)}{2}\partial_{(\mu_1}\partial_{\rho_1}\phi_{\mu_2\cdots\mu_s)\rho_2}.
\end{aligned} \tag{2.32}$$

The free Fronsdal equations of motion are

$$\boxed{\mathcal{F} = \Gamma^{(2)\rho}_{\rho;\mu_1\cdots\mu_s} = \partial^2\phi_{\mu_1\cdots\mu_s} - s\partial_\rho\partial_{(\mu_1}\phi^{\rho}_{\mu_2\cdots\mu_s)} + \frac{s\,(s-1)}{2}\partial_{(\mu_1}\partial_{\rho_1}\phi_{\mu_2\cdots\mu_s)\rho_2} = 0} \tag{2.33}$$

and are invariant under the gauge transformation:

[4] The rank s tensor should also be doubly traceless *i.e.* $\eta^{\mu_1\mu_2}\phi_{\mu_1\cdots\mu_s}$.

$$\delta\phi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)}. \tag{2.34}$$

In addition, the Fronsdal equations can be obtained from the action [133, 134]:

$$\begin{aligned} S = \int d^4x \Bigg(& \frac{1}{2}\partial_\alpha\phi_{\mu_1\cdots\mu_s}\partial^\alpha\phi^{\mu_1\cdots\mu_s} - \frac{s}{2}\partial_\alpha\phi^\alpha_{\mu_2\cdots\mu_s}\partial^\beta\phi_\beta^{\mu_2\cdots\mu_s} \\ & - \frac{s\,(s-1)}{2}\phi^\alpha_{\alpha\mu_3\cdots\mu_s}\partial_\beta\partial_\gamma\phi^{\beta\gamma\mu_3\cdots\mu_s} - \frac{s\,(s-1)}{4}\partial_\beta\phi^\alpha_{\alpha\mu_3\cdots\mu_s}\partial^\beta\phi_\gamma^{\gamma\mu_3\cdots\mu_s} \\ & - \frac{s\,(s-1)\,(s-2)}{8}\partial^\beta\phi^\alpha_{\alpha\beta\mu_4\cdots\mu_s}\partial_\delta\phi_\gamma^{\gamma\delta\mu_4\cdots\mu_s}\Bigg) \\ = -\frac{1}{2}\int & d^4x\phi_{\mu_1\cdots\mu_s}\left(\mathcal{F}^{\mu_1\cdots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}F_\alpha^{\alpha\mu_3\cdots\mu_s)}\right), \end{aligned} \tag{2.35}$$

Note that there are still no interactions.

Finally, we mention that the free Fronsdal equations can also be written down in Anti-de Sitter and de Sitter spaces. To do this we replace the standard derivative by the covariant derivative: $\partial \to \nabla$. When this is done properly one obtains [128]:

$$0 \;=\; \left(\nabla^2 - m^2\right)\phi_{\mu_1\cdots\mu_s} \;-\; s\nabla_{(\mu_1}\nabla^\rho\phi_{\mu_2\cdots\mu_s)\rho} + \frac{s\,(s-1)}{4s}g_{(\mu_1\mu_2}\nabla^{\nu_1}\nabla^{\nu_2}\phi_{\mu_1\cdots\mu_s)\nu_1\nu_2} \tag{2.36}$$

In De Donder gauge viz.

$$\nabla^\rho\phi_{\rho\mu_1\cdots\mu_s} = 0 \tag{2.37}$$

(2.36) reduces to

$$0 = \left(\nabla^2 - m^2\right)\phi_{\mu_1\cdots\mu_s}. \tag{2.38}$$

To find the Laplacian we first rewrite, as is customary, the Poincaré metric as the Lagrangian:

$$\mathcal{L} = \frac{1}{z^2}\left(\dot{z}^2 + \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) \tag{2.39}$$

The Euler-Lagrange equations are

$$0 = \ddot{z} - \frac{1}{z}\dot{z}^2 + \frac{1}{z}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\mu \tag{2.40}$$

$$0 = \ddot{x}^\nu - \frac{2}{z}\dot{x}^\nu \dot{z}. \tag{2.41}$$

It then follows that the only non-vanishing Christoffel symbols are [133]:

$$\Gamma^z_{zz} = -\frac{1}{z}, \qquad \Gamma^\nu_{z\mu} = \frac{1}{z}\eta_{\mu\nu}, \qquad \Gamma^\nu_{z\mu} = -\frac{1}{z}\delta^\nu_\mu. \tag{2.42}$$

Let

$$F_{\alpha\mu_1\cdots\mu_s} = \nabla_\alpha \phi_{\mu_1\cdots\mu_s} \tag{2.43}$$

Then [170]:

$$\nabla^\alpha F_{\alpha\mu_1\cdots\mu_s} = \left(\partial_z - \frac{d-1}{z}\right) F_{z\mu_1\cdots\mu_s} + \partial_i F_{i\mu_1\cdots\mu_s} \tag{2.44}$$

Therefore the Laplacian is [133, 170]:

$$
\begin{aligned}
\nabla^2\phi_{\mu_1\cdots\mu_s} &= \left(\partial_z - \frac{d-1}{z}\right)\left(\partial_z\phi_{\mu_1\cdots\mu_s} - \Gamma_{z(\mu_1}\phi_{\mu_2\cdots\mu_s)}\right) \\
&+ \partial_i\left(\partial_z\phi_{\mu_1\cdots\mu_s} - \Gamma_{z(\mu_1}\phi_{\mu_2\cdots\mu_s)}\right) \\
&= \left[z^2\left(\partial_z + \frac{s-d+1}{z}\right)\left(\partial_z + \frac{s}{z}\right) - z^2\partial_i\partial_i - s\right]\phi_{\mu_1\cdots\mu_s} - 2sz\partial_{(\mu_1}\phi_{\mu_2\cdots\mu_s)z} \\
&+ s(s-1)\,\eta_{\mu_1\mu_2}\phi_{\mu_3\cdots\mu_s)zz} - s(d+2s-3)\,\delta_{z(\mu_1}\phi_{\mu_2\cdots\mu_{s)z}} + 2sz\partial_\rho\delta_{z(\mu_1}\phi_{\mu_1}\phi_{\mu_2\cdots\mu_s)}.
\end{aligned}
\tag{2.45}
$$

2.2 Vasiliev

Over the decades there have been few ways of going around what happens to be the best known No-Go theorem in Theoretical Physics.

The simplest way to get away with the Coleman-Mandula theorem is to enlarge the Poincarè algebra to the conformal algebra. There is no S-matrix in a CFT and so the theorem need not apply.

The other more familiar way is to consider a graded Lie algebra. This means that we introduce fermionic generators that satisfy the algebra:

$$
\boxed{\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} P_\mu} \tag{2.46}
$$

where $\sigma^\mu = (-\mathbb{I}, \sigma^i)$.

In 1987, Vasiliev and Fradkin found a way of circumventing the Coleman-Mandula theorem and all the other various No-Go theorems that ruled out interacting higher spin gauge theories.

The basic assumption in all the No-Go theorems was that we are working on flat Minkowski spacetime. What Vasiliev (and Fradkin during the early developmental parts of the theory) showed was that it was possible to have interacting higher spin cubic vertices in AdS [135, 136]. Later, between 1990 and 1992, Vasiliev managed to construct a fully-fledged higher spin theory with an infinite tower of massless higher spin fields [137, 138, 139]. (For reviews on Vasilievs higher spin gauge theories see [128, 133, 134, 140, 141, 142, 143, 144].)

Let us consider AdS_4 with coordinates $x^\mu(\mu = 1, \dots, 4)$. In addition, to the spacetime coordinates, we will also introduce a set of internal twistor variables $Y = \left(y^\alpha, \bar{y}^{\dot{\alpha}}\right)$ and $Z = \left(z^\alpha, \bar{z}^{\dot{\alpha}}\right)$

On this internal twistor space, we introduce the star-product [148]:

$$f(Y,Z) \star g(Y,Z) = f(Y,Z) \exp\left[\epsilon^{\alpha\beta}\left(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha}\right)\left(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}\right) + \epsilon^{\dot{\alpha}\dot{\beta}}\left(\overleftarrow{\partial}_{\bar{y}^{\dot{\alpha}}} + \overleftarrow{\partial}_{\bar{z}^{\dot{\alpha}}}\right)\left(\overrightarrow{\partial}_{\bar{y}^{\dot{\beta}}} - \overrightarrow{\partial}_{\bar{z}^{\dot{\beta}}}\right)\right] g(Y,Z). \tag{2.47}$$

which has an integral representation of the form:

$$f(Y,Z) \star g(Y,Z) = \int d^2u d^2\bar{u} d^2v d^2\bar{v} e^{-uv+\bar{u}\bar{v}} f(y+u, \bar{y}+\bar{u}, z+u, \bar{z}+\bar{u}) \times g(y+v, \bar{y}+\bar{v}, z-v, \bar{z}-\bar{v}). \tag{2.48}$$

It is not difficult to show that the star-products for the various twistor variables are

$$y^\alpha \star y^\beta = y^\alpha y^\beta + \epsilon^{\alpha\beta} \tag{2.49}$$

$$z^\alpha \star z^\beta = z^\alpha z^\beta - \epsilon^{\alpha\beta} \tag{2.50}$$

$$y^\alpha \star z^\beta = y^\alpha z^\beta - \epsilon^{\alpha\beta}. \tag{2.51}$$

The Vasiliev theory is formulated in terms of a one-form in space-time $W(x|Y;Z)$, a scalar field $B(x|Y;Z)$and also a 1-form in Z-space $S(x|Y;Z)$. It is convenient to also define

$$\mathcal{A} = W + S_\alpha dz^\alpha + S_{\dot{\alpha}} d\bar{z}^\alpha \tag{2.52}$$

and

$$\begin{aligned}\hat{\mathcal{A}} &= \mathcal{A} - \frac{1}{2} z_\alpha dz^\alpha - \frac{1}{2} \bar{z}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}} \\ &= W + \left(S_\alpha - \frac{1}{2} z_\alpha\right) dz^\alpha + \left(S_{\dot{\alpha}} - \frac{1}{2} \bar{z}_{\dot{\alpha}}\right) d\bar{z}^{\dot{\alpha}}.\end{aligned} \tag{2.53}$$

The exterior derivative is

$$d = d_x + d_Z. \tag{2.54}$$

We also introduce the Kleinian:

$$K = e^{y^\alpha z_\alpha}, \qquad \bar{K} = e^{\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}}. \tag{2.55}$$

This definition can be generalized to define the quantities

$$K\left(t\right) = e^{t y^\alpha z_\alpha}, \qquad \bar{K}\left(t\right) = e^{t \bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}}. \tag{2.56}$$

By allowing the Kleinian to act on some arbitrary function f, one can show that

$$K \star f\left(y, \bar{y}, z, \bar{z}\right) \star \bar{K} = f\left(-y, \bar{y}, -z, \bar{z}\right). \tag{2.57}$$

We now have everything we need to write down the equations of motion that for the infinite tower of massless particles. The Vasiliev equations are

$$d\hat{\mathcal{A}} + \hat{\mathcal{A}} \star \hat{\mathcal{A}} = f\,(B \star K)\, dz^2 + \bar{f}\left(B \star \bar{K}\right) d\bar{z}^2 \tag{2.58}$$

$$0 = d_x B + \hat{\mathcal{A}} \star B - B \star \pi\left(\hat{\mathcal{A}}\right) \tag{2.59}$$

where

$$\pi f\,(y, \bar{y}, z, \bar{z}, dz, d\bar{z}) = f\,(-y, \bar{y}, -z, \bar{z}, -dz, d\bar{z})\,. \tag{2.60}$$

For completeness, we also define $\bar{\pi}$ via

$$\bar{\pi} f\,(y, \bar{y}, z, \bar{z}, dz, d\bar{z}) = f\,(y, -\bar{y}, z, -\bar{z}, dz, -d\bar{z})\,. \tag{2.61}$$

The field equations involve an arbitrary function f. It turns out that - by field redefinition - the function f can be written as

$$f\,(X) = \frac{1}{4} + X e^{i\theta(X)}. \tag{2.62}$$

It also turns out that by further requiring that the Higher-Spin Gauge Theory be parity invariant, the function f takes the form:

$$f_A\,(X) = \frac{1}{4} + X, \quad \text{or} \qquad f_B\,(X) = \frac{1}{4} + iX. \tag{2.63}$$

Those theories which preserve parity with the function f given by f_A are called A-type Vasiliev. (Similarly, theories that preserve parity with the function f given by f_B are called B-type Vasiliev theories.)

The Vasiliev equations as given in are obviously written in a very compactified form. However, it is easy to rewrite the field equations in terms of the original master variables S, W and B. Indeed, one can verify that

$$d_x W + W \star W = 0 \tag{2.64}$$

$$d_Z W + d_x S + \{W, S\}_\star = 0 \tag{2.65}$$

$$d_Z S + S \star S = f\,(B \star K)\,dz^2 + \bar{f}\,(B \star K)\,dz^2 \tag{2.66}$$

$$d_x B + W \star B - B \star \pi\,(W) = 0 \tag{2.67}$$

$$d_x B + S \star B - B \star \pi\,(S) = 0 \tag{2.68}$$

The above field equations are invariant under the gauge transformation:

$$\delta W = d\epsilon + [W, \epsilon]_\star \tag{2.69}$$

$$\delta S = d_Z \epsilon + [S, \epsilon]_\star \tag{2.70}$$

$$\delta B = B \star \pi\,(\epsilon) - \epsilon \star B \tag{2.71}$$

As a possible ansatz for a solution of the Vasiliev equations, we can write

$$W = W_0\,(x\,|Y\,), \qquad S = 0, \qquad B = 0. \tag{2.72}$$

From it is easy to see that the the only non-trivial equations that we have is

$$dW_0 + W_0 * W_0 = 0. \tag{2.73}$$

Suppose that we can write the master field W_0 as [128, 139, 148]:

$$W_0\,(x \mid Y) = (e_0)_{\alpha\dot{\beta}}\, y^\alpha \bar{y}^{\dot{\beta}} + (\omega_0)_{\alpha\beta}\, y^\alpha y^\beta + (\omega_0)_{\dot{\alpha}\dot{\beta}}\, \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}, \tag{2.74}$$

where e_0 is the vierbein and ω_0 the spin-connection.

Then, (2.73) implies that

$$
\begin{aligned}
0 = &\left(d_x \left(e_0\right)_{\alpha\dot{\beta}}\right) y^{\alpha}\bar{y}^{\dot{\beta}} + \left(d_x \left(\omega_0\right)_{\alpha\beta}\right) y^{\alpha}y^{\beta} + \left(d_x \left(\omega_0\right)_{\dot{\alpha}\dot{\beta}}\right) \bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} \\
&+ \left(e_0\right)_{\alpha\dot{\beta}} \left(e_0\right)_{\gamma\dot{\delta}} \left(y^{\alpha}\bar{y}^{\dot{\beta}} * y^{\gamma}\bar{y}^{\dot{\delta}}\right) + \left(e_0\right)_{\alpha\dot{\beta}} \left(\omega_0\right)_{\gamma\delta} \left(y^{\alpha}\bar{y}^{\dot{\beta}} * y^{\gamma}y^{\delta}\right) \\
&+ \left(e_0\right)_{\alpha\dot{\beta}} \left(\omega_0\right)_{\dot{\gamma}\dot{\delta}} \left(y^{\alpha}\bar{y}^{\dot{\beta}} * \bar{y}^{\dot{\gamma}}\bar{y}^{\dot{\delta}}\right) + \left(\omega_0\right)_{\alpha\beta} \left(e_0\right)_{\gamma\dot{\delta}} \left(y^{\alpha}y^{\beta} * y^{\gamma}\bar{y}^{\dot{\delta}}\right) \\
&+ \left(\omega_0\right)_{\alpha\beta} \left(\omega_0\right)_{\gamma\delta} \left(y^{\alpha}y^{\beta} * y^{\gamma}y^{\delta}\right) + \left(\omega_0\right)_{\alpha\beta} \left(\omega_0\right)_{\dot{\gamma}\dot{\delta}} \left(y^{\alpha}y^{\beta} * y^{\dot{\gamma}}\bar{y}^{\dot{\delta}}\right) \\
&+ \left(\omega_0\right)_{\dot{\alpha}\dot{\beta}} \left(e_0\right)_{\gamma\dot{\delta}} \left(\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} * y^{\gamma}\bar{y}^{\dot{\delta}}\right) + \left(\omega_0\right)_{\dot{\alpha}\dot{\beta}} \left(\omega_0\right)_{\gamma\delta} \left(\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} * y^{\gamma}y^{\delta}\right) \\
&+ \left(\omega_0\right)_{\dot{\alpha}\dot{\beta}} \left(\omega_0\right)_{\dot{\gamma}\dot{\delta}} \left(\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} * \bar{y}^{\dot{\gamma}}\bar{y}^{\dot{\delta}}\right).
\end{aligned}
\tag{2.75}
$$

For the various star products, we obtain [128]:

$$\begin{aligned}
y^\alpha\bar{y}^{\dot\beta} * y^\gamma\bar{y}^{\dot\delta} &= y^\alpha\bar{y}^{\dot\beta}\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)y^\gamma\bar{y}^{\dot\delta} \\
&= y^\alpha\bar{y}^{\dot\beta}+\epsilon^{\alpha\delta}\bar{y}^{\dot\beta}\bar{y}^{\dot\delta}+\epsilon^{\dot\beta\dot\gamma}y^\alpha y^\gamma+\epsilon^{\alpha\delta}\epsilon^{\dot\beta\dot\gamma} \qquad (2.76)\\
y^\alpha\bar{y}^{\dot\beta} * y^\gamma y^\delta &= y^\alpha\bar{y}^{\dot\beta}\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)y^\gamma y^\delta \\
&= y^\alpha\bar{y}^{\dot\beta}y^\gamma y^\delta+\epsilon^{\alpha\gamma}\bar{y}^{\dot\beta}y^\delta+\epsilon^{\alpha\delta}\bar{y}^{\dot\beta}y^\gamma \qquad (2.77)\\
y^\alpha\bar{y}^{\dot\beta} * \bar{y}^{\dot\gamma}\bar{y}^{\dot\delta} &= y^\alpha\bar{y}^{\dot\beta}\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)\bar{y}^{\dot\gamma}\bar{y}^{\dot\delta} \\
&= y^\alpha\bar{y}^{\dot\beta}\bar{y}^{\dot\gamma}\bar{y}^{\dot\delta}+\epsilon^{\dot\beta\dot\gamma}y^\alpha\bar{y}^{\dot\delta}+\epsilon^{\dot\beta\dot\delta}y^\alpha\bar{y}^{\dot\gamma} \qquad (2.78)\\
y^\alpha y^\beta * y^\gamma\bar{y}^{\dot\delta} &= y^\alpha y^\beta\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)y^\gamma\bar{y}^{\dot\delta} \\
&= y^\alpha y^\beta y^\gamma\bar{y}^{\dot\delta}+\epsilon^{\alpha\gamma}y^\beta\bar{y}^{\dot\delta}+\epsilon^{\beta\gamma}y^\alpha\bar{y}^{\dot\delta} \qquad (2.79)\\
y^\alpha y^\beta * y^\gamma y^\delta &= y^\alpha y^\beta\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}+\frac{1}{2}\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}\epsilon^{\mu\nu}\overleftarrow{\partial_{y^\mu}}\overrightarrow{\partial_{y^\nu}}\right)y^\gamma y^\delta \\
&= y^\alpha y^\beta y^\gamma y^\delta+\epsilon^{\alpha\gamma}y^\beta y^\delta+\epsilon^{\beta\gamma}y^\alpha y^\delta+\epsilon^{\alpha\delta}y^\beta y^\gamma+\epsilon^{\beta\delta}y^\alpha y^\gamma \\
&+\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}+\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}+\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}+\epsilon^{\beta\delta}\epsilon^{\alpha\gamma} \qquad (2.80)\\
y^\alpha y^\beta * y^{\dot\gamma}\bar{y}^{\dot\delta} &= y^\alpha y^\beta\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)y^{\dot\gamma}\bar{y}^{\dot\delta} \\
&= y^\alpha y^\beta y^{\dot\gamma}\bar{y}^{\dot\delta} \qquad (2.81)\\
\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta} * y^\gamma\bar{y}^{\dot\delta} &= \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)y^\gamma\bar{y}^{\dot\delta} \\
&= \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}y^\gamma\bar{y}^{\dot\delta}+\epsilon^{\dot\alpha\dot\delta}\bar{y}^{\dot\beta}y^\gamma+\epsilon^{\dot\beta\dot\delta}\bar{y}^{\dot\alpha}y^\gamma \qquad (2.82)\\
\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta} * y^\gamma y^\delta &= \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)y^\gamma y^\delta \\
&= \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}y^\gamma y^\delta \qquad (2.83)\\
\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta} * \bar{y}^{\dot\gamma}\bar{y}^{\dot\delta} &= \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}\left(1+\epsilon^{\rho\sigma}\overleftarrow{\partial_{y^\rho}}\overrightarrow{\partial_{y^\sigma}}+\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}+\frac{1}{2}\left(\epsilon^{\dot\rho\dot\sigma}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)\left(\epsilon^{\dot\mu\dot\nu}\overleftarrow{\partial_{\bar{y}^{\dot\rho}}}\overrightarrow{\partial_{\bar{y}^{\dot\sigma}}}\right)\right)\bar{y}^{\dot\gamma}\bar{y}^{\dot\delta} \\
&= \bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}\bar{y}^{\dot\gamma}\bar{y}^{\dot\delta}+\epsilon^{\dot\alpha\dot\gamma}\bar{y}^{\dot\beta}\bar{y}^{\dot\delta}+\epsilon^{\dot\beta\dot\gamma}\bar{y}^{\dot\alpha}\bar{y}^{\dot\delta}+\epsilon^{\dot\alpha\dot\delta}\bar{y}^{\dot\beta}\bar{y}^{\dot\gamma}+\epsilon^{\dot\beta\dot\delta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\gamma}+\frac{1}{2}\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta} \\
&+\frac{1}{2}\epsilon^{\dot\beta\dot\gamma}\epsilon^{\dot\alpha\dot\delta}+\frac{1}{2}\epsilon^{\dot\alpha\dot\delta}\epsilon^{\dot\beta\dot\gamma}+\frac{1}{2}\epsilon^{\dot\beta\dot\delta}\epsilon^{\dot\alpha\dot\gamma}. \qquad (2.84)
\end{aligned}$$

Plugging (2.76), (2.77), (2.78), (2.79), (2.80), (2.81), (2.82), (2.83) and (2.84) back into

(2.75), we get

$$
\begin{aligned}
0 &= \left(d_x\,(e_0)_{\alpha\dot\beta}\right) y^\alpha \bar y^{\dot\beta} + \left(d_x\,(\omega_0)_{\alpha\beta}\right) y^\alpha y^\beta + \left(d_x\,(\omega_0)_{\dot\alpha\dot\beta}\right) \bar y^{\dot\alpha} \bar y^{\dot\beta} \\
&+ (e_0)_{\alpha\dot\beta}\,(e_0)_{\gamma\dot\delta} \left(y^\alpha \bar y^{\dot\beta} + \epsilon^{\alpha\delta} \bar y^{\dot\beta} \bar y^{\dot\delta} + \epsilon^{\dot\beta\dot\gamma} y^\alpha y^\gamma + \epsilon^{\alpha\delta}\epsilon^{\dot\beta\dot\gamma}\right) \\
&+ (e_0)_{\alpha\dot\beta}\,(\omega_0)_{\gamma\delta} \left(y^\alpha \bar y^{\dot\beta} y^\gamma y^\delta + \epsilon^{\alpha\gamma} \bar y^{\dot\beta} y^\delta + \epsilon^{\alpha\delta} \bar y^{\dot\beta} y^\gamma\right) \\
&+ (e_0)_{\alpha\dot\beta}\,(\omega_0)_{\dot\gamma\dot\delta} \left(y^\alpha \bar y^{\dot\beta} \bar y^{\dot\gamma} \bar y^{\dot\delta} + \epsilon^{\dot\beta\dot\gamma} y^\alpha \bar y^{\dot\delta} + \epsilon^{\dot\beta\dot\delta} y^\alpha \bar y^{\dot\gamma}\right) \\
&+ (\omega_0)_{\alpha\beta}\,(e_0)_{\gamma\dot\delta} \left(y^\alpha y^\beta y^\gamma \bar y^{\dot\delta} + \epsilon^{\alpha\gamma} y^\beta \bar y^{\dot\delta} + \epsilon^{\beta\gamma} y^\alpha \bar y^{\dot\delta}\right) \\
&+ (\omega_0)_{\alpha\beta}\,(\omega_0)_{\gamma\delta} \left(y^\alpha y^\beta y^\gamma y^\delta + \epsilon^{\alpha\gamma} y^\beta y^\delta + \epsilon^{\beta\gamma} y^\alpha y^\delta\right. \\
&\left.\epsilon^{\alpha\delta} y^\beta y^\gamma + \epsilon^{\beta\delta} y^\alpha y^\gamma + \epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\beta\gamma}\epsilon^{\alpha\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma} + \epsilon^{\beta\delta}\epsilon^{\alpha\gamma}\right) \\
&+ (\omega_0)_{\alpha\beta}\,(\omega_0)_{\dot\gamma\dot\delta} \left(y^\alpha y^\beta y^{\dot\gamma} \bar y^{\dot\delta}\right) \\
&+ (\omega_0)_{\dot\alpha\dot\beta}\,(e_0)_{\gamma\dot\delta} \left(\bar y^{\dot\alpha} \bar y^{\dot\beta} y^\gamma \bar y^{\dot\delta} + \epsilon^{\dot\alpha\dot\delta} \bar y^{\dot\beta} y^\gamma + \epsilon^{\dot\beta\dot\delta} \bar y^{\dot\alpha} y^\gamma\right) + (\omega_0)_{\dot\alpha\dot\beta}\,(\omega_0)_{\gamma\delta} \left(\bar y^{\dot\alpha} \bar y^{\dot\beta} y^\gamma y^\delta\right) \\
&+ (\omega_0)_{\dot\alpha\dot\beta}\,(\omega_0)_{\dot\gamma\dot\delta} \left(\bar y^{\dot\alpha} \bar y^{\dot\beta} \bar y^{\dot\gamma} \bar y^{\dot\delta} + \epsilon^{\dot\alpha\dot\gamma} \bar y^{\dot\beta} \bar y^{\dot\delta} + \epsilon^{\dot\beta\dot\gamma} \bar y^{\dot\alpha} \bar y^{\dot\delta} + \epsilon^{\dot\alpha\dot\delta} \bar y^{\dot\beta} \bar y^{\dot\gamma} + \epsilon^{\dot\beta\dot\delta} \bar y^{\dot\alpha} \bar y^{\dot\gamma}\right. \\
&= \left. + \frac{1}{2}\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta} + + \frac{1}{2}\epsilon^{\dot\beta\dot\gamma}\epsilon^{\dot\alpha\dot\delta} + \frac{1}{2}\epsilon^{\dot\alpha\dot\delta}\epsilon^{\dot\beta\dot\gamma} + \frac{1}{2}\epsilon^{\dot\beta\dot\delta}\epsilon^{\dot\alpha\dot\gamma}\right). \qquad (2.85)
\end{aligned}
$$

After some straightforward, but tedious, algebra we obtain [148]:

$$d_x\,(e_0)_{\alpha\dot\beta} + 4\,(e_0)_{\gamma\dot\beta} \wedge (\omega_0)_{\alpha\dot\delta}\,\epsilon^{\gamma\delta} - 4\,(e_0)_{\alpha\dot\gamma} \wedge (\omega_0)_{\dot\delta\dot\beta}\,\epsilon^{\dot\gamma\dot\delta} = 0 \qquad (2.86)$$

$$d_x\,(\omega_0)_{\alpha\beta} + (e_0)_{\alpha\dot\gamma} \wedge (e_0)_{\beta\dot\delta}\,\epsilon^{\dot\gamma\dot\delta} + 4\,(\omega_0)_{\alpha\gamma} \wedge (\omega_0)_{\beta\delta}\,\epsilon^{\gamma\delta} = 0 \qquad (2.87)$$

$$d_x\,(\omega_0)_{\dot\alpha\dot\beta} + (e_0)_{\gamma\dot\alpha} \wedge (e_0)_{\delta\dot\beta}\,\epsilon^{\gamma\delta} + 4\,(\omega_0)_{\dot\alpha\dot\gamma} \wedge (\omega_0)_{\dot\beta\dot\delta}\,\epsilon^{\dot\gamma\dot\delta} = 0 \qquad (2.88)$$

Since

$$(e_0)_{\alpha\dot\beta} = \frac{1}{4} e^a \sigma^a_{\alpha\dot\beta}, \qquad (\omega_0)_{\alpha\beta} = \frac{1}{16}\omega^{ab}\sigma^{ab}_{\alpha\beta}, \qquad (\omega_0)_{\dot\alpha\dot\beta} = \frac{1}{16}\omega^{ab}\sigma^{ab}_{\dot\alpha\dot\beta}, \qquad (2.89)$$

we can write (2.86), (2.87) and (2.88) as [128]:

$$d_x e_a + \omega_{ab} \wedge e_b = 0 \tag{2.90}$$

$$d_x \omega_{ab} + \omega_{ac} \wedge \omega_{cb} = 6 e_b \wedge e_a. \tag{2.91}$$

These are the equations that describe the AdS spacetime. More specifically, (2.90) is the torsion free condition for the AdS_4 spacetime and (2.91) is an expression for the AdS_4 Ricci tensor in terms the vierbeins [144].

We wish to linearize the field equations about the AdS vacuum solution. This linearization can be achieved by writing

$$W\left(x \mid Y\right) = W_0 + \lambda \tilde{W}, \qquad S = 0 + \lambda \tilde{S}, \qquad B = 0 + \lambda \tilde{B}. \tag{2.92}$$

Accordingly, (2.64) becomes

$$\begin{aligned} 0 &= d_x W + W * W \\ &\left(d_x W_0 + W_0 * W_0\right) + \lambda \left(d_x W_0 + \left\{W_0, \tilde{W}\right\}_*\right) + \mathcal{O}\left(\lambda^2\right) \\ &= D_0 \tilde{W}, \end{aligned} \tag{2.93}$$

where

$$D_0 = d + \left[W_{0,} \cdot\right]. \tag{2.94}$$

Similarly, use of (2.92) in (2.65) yields

$$\begin{aligned} 0 &= d_Z W_0 + \lambda \left(d_Z W_0 + d_x \tilde{S} + \left\{ W_0, \tilde{S} \right\}_* \right) \\ &= d_Z W_0 + \lambda \left(d_Z W_0 + D_0 \tilde{S} \right). \end{aligned} \tag{2.95}$$

From (2.66), we obtain

$$f\left(B * K\right) dz^2 + \bar{f}\left(B * \bar{K}\right) = \lambda d_Z \tilde{S} + \lambda^2 \tilde{S} * \tilde{S}. \tag{2.96}$$

Eq. (2.67) implies that

$$\begin{aligned} d_x B + W * B - B * \pi\left(W\right) = 0 \\ 0 &= \lambda \left(d_x \tilde{B} + W_0 * \tilde{B} - \tilde{B} * \pi\left(W_0\right) \right) + \mathcal{O}\left(\lambda^2\right) \\ &= \lambda \tilde{D}_0 \tilde{B} + \mathcal{O}\left(\lambda^2\right), \end{aligned} \tag{2.97}$$

where

$$\tilde{D}_0 = d_x + W_0 * \cdot - \cdot * \pi\left(W_0\right). \tag{2.98}$$

Finally, (2.68) yields

$$\begin{aligned} 0 &= d_x B + S * B - B * \pi\left(S\right) \\ &= \lambda \left(d_x \tilde{B} \right) + \mathcal{O}\left(\lambda^2\right) \end{aligned} \tag{2.99}$$

To summarize, the linearized Vasiliev's field equations read [128, 139]:

$$D_0\tilde{W} = 0 \quad (2.100)$$

$$d_Z\tilde{W} + D_0 S = 0 \quad (2.101)$$

$$d_Z\tilde{S} = (B * K)\, dz^2 + \left(B * \bar{K}\right) d\bar{z}^2 \quad (2.102)$$

$$\tilde{D}B = 0 \quad (2.103)$$

$$d_x\tilde{B} = 0 \quad (2.104)$$

2.3 Vector Models AdS_4/CFT_3

The AdS/CFT Correspondence is a weak/strong duality. What this means is that when the gravity side of the duality is computationally under control, the dual CFT is strongly coupled.

We would like to understand the gauge-string duality in more detail. What we need is a form of a weak/weak duality.

The simplest CFT is given by the $O(N)$ vector model. The action for the $O(N)$ vector model can be written as [146]:

$$S = \int d^3x \left(\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}m^2\left(\phi^i\phi^i\right) + \frac{\lambda}{4}\left(\phi^i\phi^i\right)^2\right). \quad (2.105)$$

The theory has a trivial UV fixed point at $g = 0$ and another IR fixed point for some value of the coupling constant. More precisely, the β-function for the $O(N)$ vector model is [146]:

$$\beta_\lambda = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \mathcal{O}\left(\lambda^3\right). \quad (2.106)$$

Note that the above β-function was computed in $4-\epsilon$ dimensions and hence $\epsilon = 1$ in $3d$ [146].

It is straightforward to see that the IR fixed point will occur at $\lambda = \frac{8\pi^2}{N+8}\epsilon$.

Note that the $O(N)$ vector model has an infinite amount of conserved currents.

For example, it is easy to show that [146]

$$J_\mu = \phi \overleftrightarrow{\partial_\mu} \phi \tag{2.107}$$

$$J_{\mu\nu} = T_{\mu\nu} = \partial_\mu \phi \partial_\mu \phi - \frac{1}{4(d-1)} \left[(d-2) \partial_\mu \partial_\nu + g_{\mu\nu} \partial^2 \right] \phi^2 \tag{2.108}$$

are conserved for the free case.

In general, the conserved currents can be written (schematically) as[5]

$$J_{\mu_1 \cdots \mu_s} = \phi^a \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^a + \cdots \tag{2.109}$$

Following Giombi and Yi [147], we can repackage the conserved currents as

$$\mathcal{O}_f(\vec{x}, \epsilon) = \phi^i(\vec{x}) f\left(\epsilon_\mu, \overrightarrow{\partial}_\mu, \overleftarrow{\partial_\mu}\right) \phi^i(\vec{x}), \tag{2.110}$$

where [147]:

$$f(\vec{\epsilon}, \vec{u}, \vec{v}) = e^{(u-v)\cdot\epsilon} \cosh \sqrt{2(u \cdot v)\epsilon^2 - 4(u \cdot \epsilon)(v \cdot \epsilon)}. \tag{2.111}$$

Given the identification, which is part of the AdS/CFT dictionary, that

$$J_\mu \leftrightarrow A_\mu, \qquad T_{\mu\nu} \leftrightarrow g_{\mu\nu}, \qquad J_{\mu_1 \dots \mu_s} \leftrightarrow \phi_{\mu_1 \dots \mu_s}. \tag{2.112}$$

it is understandable that this simple CFT, naively speaking, should be dual to some gravitational theory that has an infinite tower of massless higher spins. Such a theory

[5] The correct expression can be found in [146].

should live on AdS and we do indeed know of one such theory namely Vasiliev's higher spin Gauge Theory.

Another way, and this was historically the path followed, to see that the Vasiliev Higher Spin Gauge Theory could be dual to the free CFTs of matrix-valued fields was to actually start from the original AdS/CFT Correspondence. There is a particular limit in which the string tension goes to zero. (String Theory has massive higher spin excitations.)

In this tensionless limit, the massive excitations become massless and decouple from the spectra. So, it was felt that in this tensionless limit of Type IIB should be related to the free $\mathcal{N} = 4$ SYM [177, 178, 179, 180, 181].

The duality was finally conjectured in a more precise way by Klebanov and Polyakov in 2002. The Klebanov-Polyakov Higher Spin AdS/CFT Correspondence posits that the free (critical) $O\left(N\right)$ vector model is dual to the minimal type A Vasiliev with the scaling dimension of the scalar field being equal to one (two) [147]. (For an excellent review on the Higher Spin AdS/CFT Duality see [148].)

Immediately after Polyakov and Klebanov put forward their conjecture, a similar result was proposed for the fermionic Gross-Neveu model.

Recall that the Gross-Neveu vector model has an action of the form

$$S = \int d^3x \left(\bar{\psi}_i \gamma^\mu \partial_\mu \psi^i + \frac{g}{2}\left(\bar{\psi}_i \psi^i\right)^2\right). \tag{2.113}$$

According to [149, 150], it has been conjectured that the singlet sector of the Gross-Neveu is dual to type B minimal Vasiliev theory.

There is also a formulation of the duality in de-Sitter spaces . More specifically, it has been postulated that Vasiliev's Higher Spin Gauge Theory in dS_4 is holographically dual to the $SP(N)$ model [151] with action of the form:

$$S = \frac{1}{8\pi} \int d^3x \left(\Omega_{ab} \delta^{ij} \partial_i \chi^a \partial_j \chi^a + V(\chi \cdot \chi) \right), \tag{2.114}$$

where $\chi^a (a = 1, 2, \ldots, N)$ is an N component scalar field and

$$\Omega_{ab} = \begin{pmatrix} 0 & 1_{\frac{N}{2} \times \frac{N}{2}} \\ -1_{\frac{N}{2} \times \frac{N}{2}} & 0 \end{pmatrix}. \tag{2.115}$$

Recently, the Higher-Spin dS/CFT duality has been extended to the supersymmetric case [152]. More precisely, the $\mathcal{N} = 2$ supersymmetric extensions of the $SP(N)$ vector models are dual to $\mathcal{N} = 2$ Vasiliev Higher-Spin Gauge Theories in de Sitter space [152].

There is also a lower dimensional version of the duality in AdS_3 [153]. More precisely, the Gaberdieli-Gopakumar conjecture posits that the bosonic (truncated) Vasiliev-Prokushkin Higher Spin Gravity in AdS_3 [154, 155] is dual to the $\mathcal{W}_{N,k}$ minimal models [153] which can be represented in terms of the coset:

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}. \tag{2.116}$$

The central charge is of the form

$$c = (N-1)\left[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right]. \tag{2.117}$$

As is well-known the central charge is a measure of the degrees of freedom of a system. Therefore, we see that the degrees of freedom for this AdS_3/CFT_2 scale like N - this is in contrast to the standard matrix (adjoint) AdS/CFT Dualities where the degrees of freedom scale like N^2.

There are also supersymmetric extensions of the Gaberdieli-Gopakumar conjecture. More accurately, the full Vasiliev-Prokushkin Higher Spin Gravity is said to be equivalent to the $\mathcal{N}=2$ $\mathbb{CP}^N$ Kazama-Suzuki coset model [156]:

$$\frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}\left(N\right)_1}{\mathfrak{su}\left(N\right)_{k+1}\mathfrak{u}\left(N\right)_{N(N+1)(k+N+1)}} \tag{2.118}$$

Chern-Simons theories have found diverse applications in a variety of fields including in the study of knot invariants [157] and Condensed Matter Physics - *e.g.* in the fractional Hall effect [158].

Chern-Simons theories coupled to fermionic or bosonic vector models have been conjectured to be dual to the parity violating Vasiliev's Higher Spin Gauge Theories [159, 160]. One reason why this particular conjecture is interesting is due to the fact that this provides a way to embed the Vasiliev's Higher Spin Gravity into string theory [161, 162].[6] In the Gauge-Gravity Duality, a superconformal Chern-Simons-matter theory has been conjectured to be dual to Type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Moreover, Vasiliev's Higher Spin Gauge Theories allow for boundary conditions that preserve $\mathcal{N}=1,2,\ldots,6$ supersymmetry. Obviously, because of the ABJ model, the most important case to study is when $\mathcal{N}=6$ case. In such an instance, we have a triality linking vector Chern-Simons theories, Type IIA string theory and Vasiliev's Higher Spin Gravity [161]. [7] To date, this has been one of the most successful attempts to embed string theory in higher spin theories.

Recall that one the earliest forms of a duality was discovered by Coleman and Mandelstam *viz.* the bosonization duality [164, 165]. This was the duality that states that the Thirring model and the sine-Gordon model are equivalent in $(1+1)$-dimensions. Recently, as a

[6]Intuitively, the Higher Spin Gauge Theories can be thought of as the tensionless limit of some string theory. However, this notion has never been made precise.

[7]Recently, there is evidence that a deformation of this triality leads to a quadrality involving two $\mathcal{N}=5$ Vasiliev Higher-Spin theories, string/M-theory and $\mathcal{N}=5$ ABJ theory with gauge group $O\left(N\right)_{2k} \times USp\left(2N_2\right)_{-k}$ [163].

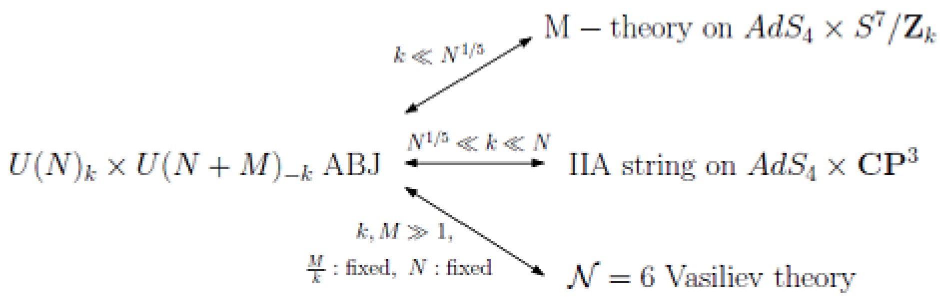


Figure 2.1: The ABJ triality. Picture credits [163].

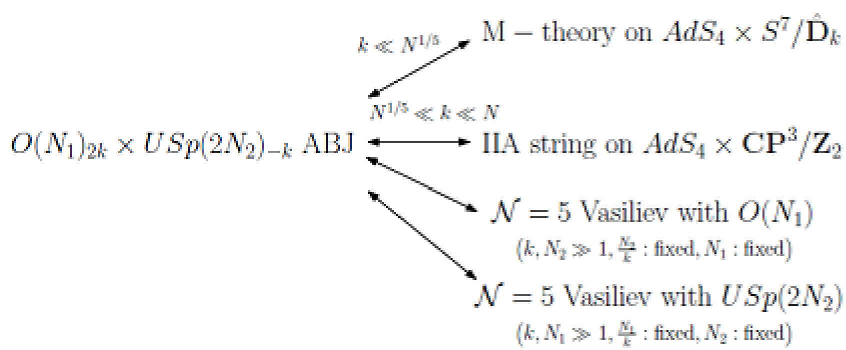


Figure 2.2: The ABJ quadrality. Picture credits [163].

result of studying the Chern-Simons vector model dualities, the bose-fermi duality has been generalized to $3d$ [160, 166]. More precisely, the free (critical) boson coupled to Chern-Simons gauge theory is dual to the critical (free) coupled to Chern Simons theory. Evidence for this $3d$ bosonization includes the matching of the thermal free energies [167] and the correlation functions [166, 168]. For example, the fermionic two point function is

$$\left\langle J_f^0\left(-q\right)J_f^0\right\rangle = -N_f\frac{\tan\left(\pi\lambda_f/2\right)}{4\pi\lambda_f}\left|q\right| \tag{2.119}$$

and the critical bosonic two point function is

$$\left\langle J_b^0\left(-q\right)J_b^0\right\rangle = N_b\frac{4\pi\lambda_b}{\tan\left(\pi\lambda_f/2\right)}\left|q\right|. \tag{2.120}$$

Here, $\lambda = \frac{N}{k}$ is the 't Hooft coupling and k is the level. What is important to note is that the two correlation functions are the same after some redefinitions [168].

The precise mapping between the parameters is given by:

$$\lambda_f = \lambda_b - \operatorname{sign}\left(\lambda_b\right) \tag{2.121}$$

$$N_f = \frac{N_b\left(1-\left|\lambda\right|_b\right)}{\left|\lambda\right|_b}. \tag{2.122}$$

At present, the evidence for the Klebanov-Polyakov $O\left(N\right)/HS$ correspondence includes the most striking and constructive derivation of the duality at the UV fixed point [209][8] and the matching of the bulk and boundary three point functions [170, 171]. Furthermore, the 1-loop correction to the (three-sphere) free-energy has also been computed on both sides and it was seen that indeed the corrections do match [172, 173, 174].

[8] There is an alternative exact renormalization constructive derivation of the higher spin bulk gravity given in by Douglas et al [169].

	Vasiliev Type A	**Vasiliev Type B**
$\Delta = 1$	Free scalar	Critical fermion
$\Delta = 2$	Critical scalar	Free fermion:

Table 2.1: The original HS/CFT dualities [146].

The value for the tree level contribution to the free $O(N)$ vector model is

$$F^{(0)} = \frac{N}{8}\left(2\ln 2 - \frac{3\zeta(3)}{\pi^2}\right) \tag{2.123}$$

while the 1-loop contribution vanishes *i.e.*

$$F^{(1)} = 0. \tag{2.124}$$

In [172], the value for the 1-loop was computed and found to be

$$F^{(1)}_{\Delta=1} = \frac{N}{8}\left(2\ln 2 - \frac{3\zeta(3)}{\pi^2}\right). \tag{2.125}$$

There is agreement on both sides of the HS AdS/CFT correspondence if we make the identification:

$$\frac{1}{G_N}F^{(0)}_{min} = \frac{(N-1)}{8}\left(2\ln 2 - \frac{3\zeta(3)}{\pi^2}\right). \tag{2.126}$$

For the critical $O(N)$ vector model such problems do not arise. Indeed, the 1-loop correction - on both sides of the duality - is

$$F^{(1)}_{critical} = F^{(1)}_{min} = -\frac{\zeta(3)}{8\pi^2}. \tag{2.127}$$

Finally, the $O(N)$ vector/Higher-spin Correspondence is in agreement with the Maldacena-Zhibodoev theorem [175, 176].

Chapter 3

Vector Models And Collective Field Theory

On your way towards becoming a bad theoretician, take your own immature theory, stop checking it for mistakes, don't listen to colleagues who do spot weaknesses, and start admiring your own infallible intelligence.

-Gerard 't Hooft.

The concept of collective fields has found many applications in diverse fields in physics. A classic example being in the Bohm-Pines theory of plasma oscillations [182, 183, 184].

In the large-N limit an analogous concept is given by the Jevicki-Sakita Collective Field Theory [185, 186].

In this Chapter, we will give a review of Collective Field Theory approach. In particular, we will obtain the large-N collective Hamiltonian for the $O(N)$ vector model and the gap-equation.

3.1 Hamiltonian And Single Time Bi-locals

Let us consider a generic theory with a kinetic term of the form

$$\hat{K} = -\frac{1}{2} \sum_{a=1}^{N} \frac{\partial}{\partial X_a} \frac{\partial}{\partial X_a}. \tag{3.1}$$

In general, the original variable X_a might well not be gauge invariant. That is, the original variable might not be written down in terms of some master field.[1]

For example, since we will be working with $O(N)$ bilocals, we wish to rewrite our theory in terms of the gauge invariant variable (the X_as are in the fundamental representation) denoted schematically by ψ_A.

It is straightforward to show that, in general

$$\begin{aligned} -2\hat{K} &= \sum_{a=1}^{N} \frac{\partial}{\partial X_a} \frac{\partial}{\partial X_a} \\ &= \left(\sum_{a=1}^{N} \frac{\partial^2 \psi_A}{\partial X_a \partial X_a} \right) \frac{\partial}{\partial \psi_A} + \left(\sum_{a=1}^{N} \frac{\partial \psi_A}{\partial X_a} \frac{\partial \psi_B}{\partial X_a} \right) \frac{\partial}{\partial \psi_A} \frac{\partial}{\partial \psi_B} \\ &= \omega_A \partial_A + \Omega_{AB} \partial_A \partial_B, \end{aligned} \tag{3.2}$$

where

$$\omega_A = \sum_{a=1}^{N} \frac{\partial^2 \psi_A}{\partial X_a \partial X_a} \tag{3.3}$$

$$\Omega_{AB} = \sum_{a=1}^{N} \frac{\partial \psi_A}{\partial X_a} \frac{\partial \psi_B}{\partial X_a}. \tag{3.4}$$

To explicitly show the Hermiticity of the kinetic term, we perform a similarity transformation:

[1] The concept of the large-N master field is due to Witten [187].

$$\partial_A \to \partial_A - \frac{1}{2}\partial_A \ln J \tag{3.5}$$

Under this similarity transformation, the kinetic term becomes

$$\begin{aligned}
-2\hat{K} &= \omega_A \partial_A + \Omega_{AB}\partial_A\partial_B \\
&\to \omega_A\left(\partial_A - \frac{1}{2}\partial_A \ln J\right) + \Omega_{AB}\left(\partial_A - \frac{1}{2}\partial_A \ln J\right)\left(\partial_B - \frac{1}{2}\partial_B \ln J\right) \\
&= \omega_A\partial_A - \frac{1}{2}\omega_A\partial_A \ln J + \Omega_{AB}\partial_A\left(\partial_B\right) - \frac{1}{2}\Omega_{AB}\partial_A\left(\partial_B \ln J\right) - \frac{1}{2}\Omega_{AB}\partial_A \ln J\partial_B + \frac{1}{4}\partial_A \ln J\partial_B \ln J \\
&= \omega_A\partial_A - \frac{1}{2}\omega_A\partial_A \ln J + \partial_A\left(\Omega_{AB}\partial_B\right) - \left(\partial_A\Omega_{AB}\right)\partial_B - \frac{1}{2}\Omega_{AB}\left(\partial_A\partial_B\right)\ln J - \frac{1}{2}\Omega_{AB}\left(\partial_B \ln J\right)\partial_A \\
&\quad - \frac{1}{2}\Omega_{AB}\partial_A \ln J\partial_B + \frac{1}{4}\partial_A \ln J\partial_B \ln J.
\end{aligned} \tag{3.6}$$

Using the fact that $\Omega_{AB} = \Omega_{BA}$ it follows that

$$\left(\partial_A\Omega_{AB}\right)\partial_B = \left(\partial_B\Omega_{BA}\right)\partial_A. \tag{3.7}$$

Accordingly, the kinetic term can then be written as

$$-2\hat{K} = \left(\omega_A - \partial_B\Omega_{BA} - \Omega_{AB}\left(\partial_B \ln J\right)\right)\partial_A - \frac{1}{2}\omega_A\partial_A \ln J + \partial_A\left(\Omega_{AB}\partial_B\right) + \frac{1}{4}\Omega_{AB}\partial_A \ln J\partial_B \ln J. \tag{3.8}$$

We require that the kinetic term be explicitly Hermitian. This means that the term multiplying the derivative ∂_A should vanish. That is,

$$\boxed{\omega_A - \partial_B\left(\Omega_{BA}\right) = \Omega_{AB}\left(\partial_B \ln J\right).} \tag{3.9}$$

In the large-N limit we have $\ln J \sim N$, $\omega_A \sim N$ and $\Omega \sim 1$ and it is clear that (3.9) simplifies to

$$\omega_A = \Omega_{AB}\left(\partial_B \ln J\right). \tag{3.10}$$

This is sufficient for obtaining the bilocal ground state and spectra. There are counterterms which we do not consider.

Inserting (3.10) into (3.8) yields

$$\begin{aligned} -2\hat{K} &= -\frac{1}{2}\omega_A \partial_A \ln J + \partial_A\left(\Omega_{AB}\partial_B\right) + \frac{1}{4}\Omega_{AB}\partial_A \ln J \partial_B \ln J \\ &= -\frac{1}{2}\Omega_{AB}\partial_B \ln J \partial_A \ln J + \partial_A\left(\Omega_{AB}\partial_B\right) + \frac{1}{4}\partial_A \ln J \partial_B \ln J \\ &= \partial_A \Omega_{AB}\partial_B - \frac{1}{4}\partial_A \ln J \varOmega_{AB}\partial_B \ln J \\ &= \partial_A \Omega_{AB}\partial_B - \frac{1}{4}\omega_A \varOmega^{-1}_{AB}\omega_B. \end{aligned} \tag{3.11}$$

Thus,

$$\begin{aligned} \hat{K} &= -\frac{1}{2}\partial_A \Omega_{AB}\partial_B + \frac{1}{8}\omega_A \varOmega^{-1}_{AB}\omega_B \\ &= -\frac{1}{2}\partial_A \Omega_{AB}\partial_B + \frac{1}{8}\left(\partial_A \ln J\right)\varOmega_{AB}\left(\partial_B \ln J\right). \end{aligned} \tag{3.12}$$

For the $O\left(N\right)$ vector model the set of gauge invariant quantities is given by the bilocals which are defined via

$$\psi_{xy} = \sum_{a=1}^{N}\phi_a\left(x\right)\phi_a\left(y\right) \tag{3.13}$$

which is only defined when $x < y$. More generally, we define

$$\varPhi_{xy} = \begin{cases} \psi_{xy} & x < y \\ \psi_{yx} & x > y \end{cases} \tag{3.14}$$

Since the bilocals Φ_{xy} are symmetric, we have

$$\frac{\partial}{\partial \psi_{xy}} \Phi_{x'y'} = \delta_{xx'}\delta_{yy'} + \delta_{xy'}\delta_{y'x}. \tag{3.15}$$

In general, the operators will act on functionals of Φ_{xy}.

For the "joining" operator Ω_{AB}, we obtain:[2]

$$\begin{aligned} \Omega_{AB} &= \int dz \frac{\partial}{\partial \phi_a(z)} \sum_{b=1}^{N} \phi_b(x)\,\phi_b(y)\, \frac{\partial}{\partial \phi_a(z)} \sum_{c=1}^{N} \phi_c(x')\,\phi_c(y') = \Omega_{xy,x'y'} \\ &= \delta(x-x')\,\Phi_{yy'} + \delta(x-y')\,\Phi_{yx'} + \delta(y-x')\,\Phi_{xy'} + \delta(y-y')\,\Phi_{xx'} \end{aligned} \tag{3.16}$$

with $A = (xy)$.

Let

$$X_{AB} = X_{xy;x'y'} = \delta(x-x')\,\Phi_{yy'}. \tag{3.17}$$

Then the "joining" operator can be written as

$$\Omega_{AB} = X_{AB} + X_{A\bar{B}} + X_{\bar{A}B} + X_{\bar{A}\bar{B}}, \tag{3.18}$$

where

$$A = (x,y)\,, \qquad \bar{A} = (y'.x') \tag{3.19}$$

$$B = (x',y')\,, \qquad \bar{B} = (y',x') \tag{3.20}$$

[2] In the original formulation, in what was called loop space, the operator Ω_C used to join loops while ω_C used to split the loops. More schematically, $\Omega(C,C) = \sum \phi_{C+C'}$ and $\omega_C = \sum \phi_C \phi_{C'}$.

For the “splitting” operator, we obtain

$$\omega_A = \int dz \frac{\partial}{\partial \phi^a (z)} \frac{\partial}{\partial \phi^a (z)} \sum_{b=1}^{N} \phi^b (x) \phi^b (y)$$
$$= 2N\delta_{xy}. \tag{3.21}$$

In this notation, the equation satisfied by the Jacobian - namely, (3.10) - is

$$X_{AB}\partial_B \ln J + X_{A\bar{B}}\partial_B \ln J + X_{\bar{A}B}\partial_B \ln J + X_{\bar{A}\bar{B}}\partial_B \ln J = 2N\delta_{xy}. \tag{3.22}$$

Note that the operator Ω cannot be inverted. However, one can find a solution to (3.22) provided that:

$$X_{AB}\partial_B \ln J = \frac{1}{4}(2N\delta_{xy}) = \frac{N}{2}\delta_{xy} \tag{3.23}$$

and

$$X_{A\bar{B}}\partial_B \ln J = X_{\bar{A}B}\partial_B \ln J = X_{\bar{A}\bar{B}}\partial_B \ln J = \frac{N}{2}\delta_{xy}. \tag{3.24}$$

A solution to the above conditions is given by

$$J = [\det(\Phi_{xy})]^{N/2} \tag{3.25}$$

or

$$\ln J = \frac{N}{2} \mathrm{Tr} \ln \Phi. \tag{3.26}$$

Indeed, one can check that[3]

$$
\begin{aligned}
X_{AB}\partial_B \ln J &= \int dx' \int\limits_{x'<y'} dy' \delta\left(x-x'\right)\Phi_{yy'}\frac{\partial}{\partial\psi_{x'y'}}\left(\frac{N}{2}\mathrm{Tr}\ln\Phi\right) \\
&= \int dx' \int\limits_{x'<y'} dy' \delta\left(x-x'\right)\Phi_{yy'}\frac{N}{2}\times 2\times\left(\Phi^{-1}\right)_{y'x'} \\
&= \frac{N}{2}\int dx' \int dy' \delta\left(x-x'\right)\Phi_{yy'}\left(\Phi^{-1}\right)_{y'x'} \\
&= \frac{N}{2}\int dx' \int dy' \delta\left(x-x'\right)\delta\left(y-x'\right) = \frac{N}{2}\delta\left(x-y\right). \qquad (3.27)
\end{aligned}
$$

One can easily confirm that the other three terms yield the same result.

Now evaluating the last term appearing in (3.12) yields

$$
\begin{aligned}
-\frac{1}{4}\omega_A\Omega^{-1}_{AB}\omega_B &= -\frac{1}{4}\int dx \int\limits_{x<y} dy \int dx' \int\limits_{x'<y'} dy' \frac{\partial}{\partial\psi_{xy}}\left(\frac{N}{2}\mathrm{Tr}\log\Phi\right) \\
&\quad \times\Omega_{xy;x'y'}\frac{\partial}{\partial\psi_{x'y'}}\left(\frac{N}{2}\mathrm{Tr}\log\Phi\right) \\
&= -\frac{1}{4}\int dx \int\limits_{x<y} dy \int dx' \int\limits_{x'<y'} dy' \left(\frac{N}{2}\right)^2 2\Phi^{-1}_{yx}\Omega_{xy;x'y'}2\Phi^{-1}_{y'x'} \\
&= -\frac{1}{4}\int dx \int dy \int dx' \int dy' \frac{N^2}{4}\Phi^{-1}_{yx}\Omega_{xy;x'y'}\Phi^{-1}_{y'x'} \\
&= -\frac{1}{4}\int dx \int dy \int dx' \int dy' \frac{N^2}{4}\Phi^{-1}_{yx}\Big(\delta\left(x-x'\right)\Phi_{yy'}+\delta\left(x-y'\right)\Phi_{yx'} \\
&\quad +\delta\left(y-x'\right)\Phi_{xy'}+\delta\left(y-y'\right)\Phi_{xx'}\Big)\Phi^{-1}_{y'x'} \\
&= -\frac{1}{4}\left(\frac{N^2}{4}\right)\times 4\mathrm{Tr}\left(\Phi^{-1}\right) = -\frac{N^2}{4}\mathrm{Tr}\left(\Phi^{-1}\right). \qquad (3.28)
\end{aligned}
$$

[3]For a precise (discrete) version of the answer below, including subleading corrections to the Jacobian, see [188].

Similarly, for the first term in (3.12), we obtain

$$\begin{aligned}
\partial_A \Omega_{AB} \partial_B =& \int\limits_{x<y} dx \int dy \int\limits_{x'<y'} dx' \int dy' \frac{\partial}{\partial \psi_{xy}} \Omega_{xy;x'y'} \frac{\partial}{\partial \psi_{x'y'}} \\
=& \int dx \int dy \int dx' \int dy \frac{\partial}{\partial \Phi_{xy}} \Omega_{xy;x'y'} \frac{\partial}{\partial \Phi_{x'y'}} \\
=& 4\mathrm{Tr}\left(\frac{\partial}{\partial \Phi} \Phi \frac{\partial}{\partial \Phi} \right). \qquad (3.29)
\end{aligned}$$

By making use of (3.28) and (3.29) in (3.12), we get

$$\hat{K} = -\frac{2}{N} \mathrm{Tr}\left(\frac{\partial}{\partial \Phi} \Phi \frac{\partial}{\partial \Phi} \right) + \frac{1}{8} \mathrm{Tr}\left(\Phi^{-1} \right). \qquad (3.30)$$

To summarize the leading collective field theory Hamiltonian reads

$$\hat{H}_{Col} = -\frac{2}{N} \mathrm{Tr}\left(\frac{\partial}{\partial \Phi} \Phi \frac{\partial}{\partial \Phi} \right) + \frac{1}{8} \mathrm{Tr}\left(\Phi^{-1} \right) + V. \qquad (3.31)$$

3.2 Covariant/Path Integral And Two Time Bilocals

The Lagrangian for the $O(N)$ $(\phi^2)^2$vector model is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} m^2 \phi^a \phi^a - \frac{g}{4!} \left(\phi^a \phi^a \right)^2, \qquad (3.32)$$

where g is the coupling constant and, in this thesis, the signature is given by $(+, -, \cdots, -)$.

It is clear by now the goal of collective field theory is to rewrite a given theory in terms of (gauge) invariant quantities. For the functional/path integral description of the $O(N)$ vector model the invariants are given by the two-time bilocals:

$$
\begin{aligned}
\psi\left(x^{\mu}, y^{\mu}\right) &= \psi_{xy} \\
&= \sum_{a=1}^{N} \phi^{a}\left(t_{x}, \vec{x}\right) \phi^{a}\left(t_{y}, \vec{y}\right). \qquad (3.33)
\end{aligned}
$$

Note that we have denoted the times as t_x and t_y. This was done to emphasize the point that, in general, we work with unequal bilocals. That is, the time at $\vec{x}$ is not necessarily equal to the time at point $\vec{y}$.

By changing variables from the $O(N)$ fields ϕ^a $(a = 1, 2, \ldots, N)$ to the invariant bilocals, we introduce a non-trivial Jacobian:

$$
\int \mathcal{D}\phi = \int \mathcal{D}\psi J. \qquad (3.34)
$$

Using collective field theory, one can show that this Jacobian satisfies the same equation as that of the previous section, except that the bilocals are two-time bilocals [186, 189, 226]. The log of the Jacobian (to leading order) is given by

$$
\ln J = \frac{N}{2} \mathrm{Tr} \ln \psi. \qquad (3.35)
$$

Accordingly, the partition function becomes

$$
\begin{aligned}
Z &= \int \mathcal{D}\phi e^{iS[\phi]} \\
&= \int \mathcal{D}\psi e^{\ln J + iS} \\
&= \int \mathcal{D}\psi e^{iS_{eff}}. \qquad (3.36)
\end{aligned}
$$

where the effective action is given by

$$S_{eff} = -i\frac{N}{2}\mathrm{Tr}\ln\psi + S. \tag{3.37}$$

In order to explicitly exhibit the large-N dependence, we rescale the fields as follows:

$$\phi \to \sqrt{N}\phi, \qquad \psi \to N\psi \tag{3.38}$$

With this rescaling, the action becomes

$$\begin{aligned} S &= N\int d^dx \left(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{gN}{4!}\left(\phi^a\phi^a\right)^2\right) \\ &= N\int d^dx \left[\frac{1}{2}\left(-\int d^dy\delta\left(x-y\right)\partial_y^2\psi_{xy}\right) - \frac{1}{2}m^2\psi_{xx} - \frac{\lambda}{4!}\left(\psi_{xx}\right)^2\right], \end{aligned} \tag{3.39}$$

where $\lambda = gN$ is the 't Hooft coupling.

As $N \to \infty$, the leading large-N contribution is given by the minimum of the effective action. This shows the power of the collective field theory approach: the large-N limit emerges as a semi-classical limit of an effective action which in general is amenable to a systematic $1/\sqrt{N}$ expansion. This will also be exhibited in the Hamiltonian, where the large-N configuration is given by the minimum of an effective potential.

Using translational invariance, we write

$$\psi_{xy} = \int \frac{d^dp}{(2\pi)^d} e^{ik(x-y)}\psi_p, \tag{3.40}$$

where kx is short-hand for $k^\mu x_\mu$.

In momentum space, the effective action can be written as

$$\begin{aligned}\frac{S_{eff}}{N} &= \int d^dx \left(\left(-\frac{1}{2}\lim_{x\to y}\partial_x^2\psi_{xy}\right) + \frac{1}{2}m^2\psi_{xx} - \frac{\lambda}{4!}\left(\psi_{xx}\right)^2\right) - \frac{i}{2}\mathrm{Tr}\ln\psi \\ &= \int d^dx\left(-\frac{1}{2}\lim_{x\to y}\partial_x^2\left(\int\frac{d^dk}{\left(2\pi\right)^d}e^{ik(x-y)}\psi_k\right)\right) + \frac{1}{2}m^2\int d^dx\left(\int\frac{d^dk}{\left(2\pi\right)^d}e^{ik(x-x)}\psi_k\right) \\ &- \frac{\lambda}{4!}\int d^dx\int\frac{d^dk_1}{\left(2\pi\right)^d}e^{ik_1(x-x)}\psi_{k_1}\int\frac{d^dk_2}{\left(2\pi\right)^d}e^{ik_2(x-x)}\psi_{k_2} - \frac{i}{2}\mathrm{Tr}\ln\psi \\ &= \frac{V}{2}\int\frac{d^dk}{\left(2\pi\right)^d}k^2\psi_k + \frac{m^2V}{2}\int\frac{d^dk}{\left(2\pi\right)^d}\psi_k - \frac{\lambda V}{4!}\int\frac{d^dk_1}{\left(2\pi\right)^d}\int\frac{d^dk_2}{\left(2\pi\right)^d}\psi_{k_1}\psi_{k_2} - \frac{i}{2}\mathrm{Tr}\ln\psi.\end{aligned} \tag{3.41}$$

That is,

$$\frac{S_{eff}}{NV} = \frac{1}{2}\int\frac{d^dk}{\left(2\pi\right)^d}k^2\psi_k + \frac{m^2}{2}\int\frac{d^dk}{\left(2\pi\right)^d}\psi_k - \frac{\lambda}{4!}\int\frac{d^dk_1}{\left(2\pi\right)^d}\int\frac{d^dk_2}{\left(2\pi\right)^d}\psi_{k_1}\psi_{k_2} - \frac{i}{2}\mathrm{Tr}\ln\psi \tag{3.42}$$

The saddle point equations can be obtained by varying the effective action with respect to ψ_k:

$$\begin{aligned}0 &= \frac{\delta S_{eff}}{\delta\psi_k} \\ &= \frac{1}{2}k^2 + \frac{m^2}{2} - \frac{2\lambda}{4!}\int\frac{d^dk_1}{\left(2\pi\right)^d}\psi_{k_1} - \frac{i}{2}\psi_k^{-1},\end{aligned}$$

which implies

$$\psi_0\left(k\right) = \frac{i}{k^2 + m^2 - \frac{\lambda}{6}\int\frac{d^dl}{\left(2\pi\right)^d}\psi_0\left(l\right)}. \tag{3.43}$$

The gap-equation follows by simply integrating both sides of (3.43):

$$\boxed{s = \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2 + m^2 - s}.} \tag{3.44}$$

where

$$s = \int \frac{d^d p}{(2\pi)^d} \psi_0 \left(p\right). \tag{3.45}$$

We have assumed that these expressions are regularized, when required.

Other Examples of Gap Equations

In the next chapter, we will show that the single-time large-N background is given by (see e.g. (4.10))

$$\psi^0_{\vec{k}} = \frac{1}{2} \left(\vec{k}^2 + m^2 + \frac{\lambda}{6} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \psi_{\vec{k}} \right)^{-1/2}. \tag{3.46}$$

The gap equation in this case is readily obtained to be

$$\Delta = \frac{1}{2} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{k}^2 + m^2 + \frac{\lambda}{6} \int \frac{d^{d-1}\vec{k}'}{(2\pi)^{d-1}} \psi_{\vec{k}'}}}. \tag{3.47}$$

Recall that superconducting materials have a variety of wonderful physical phenomena including their infinite conductivity, the Meissner effect, critical field, flux quantization, isotope effect and a distinct behaviour for the specific heat [190]. In particular, close to the critical temperature the superconducting specific heat C_s is initially larger than the normal metallic specific heat C_n of a metal. It then drops below the normal specific heat and is given by [190]

$$C_s \propto \exp\left(-\frac{\Delta_0}{k_B T}\right) \tag{3.48}$$

where k_B is the Boltzmann constant and Δ_0 is the energy difference between the ground state and the excited states. This energy gap satisfies a relation of the form [190]:

$$\frac{1}{N(0)V} = \frac{1}{2}\int_0^{\hbar\omega_c} \frac{\tanh\left(\frac{1}{2}\beta\left(\xi^2+\Delta^2\right)^{1/2}\right)}{\left(\xi^2+\Delta^2\right)^{1/2}} d\xi, \tag{3.49}$$

where $\beta = \frac{1}{k_B T}$, ω_c is some critical angular frequency, V is the electron-fermion coupling potential and $N(0)$ denotes the number density at the fermi level.

A more direct field theoretic gap equation, in contrast to the gap equation in superconductivity, is the QCD gap for the dressed quark propagator $S(p)$ equation [191]:

$$S(p) = \frac{1}{Z_2\left(i\gamma^\alpha p_\alpha + m_{bare}\right) + Z_1 \int_q^\Lambda g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu S(q)\Gamma_\nu^a(q,p)} \tag{3.50}$$

Here, $D_{\mu\nu}(p-q)$ is the dressed gluon propagator, m_{bare} the bare mass $\Gamma_\nu^a(q,p)$ the quark-gluon vertex and Z_1 and Z_2 are wave-renormalization constants [191].

Chapter 4

Constructing AdS_4 At The Free Bosonic Fixed Point

We know a lot of things, but what we don't know is a lot more.

- Edward Witten.

We have already mentioned that a formal (mathematical) proof of the AdS/CFT correspondence is lacking. In particular, is the issue of the extra radial dimension. Over the years various approaches have been proposed to try and tackle these issues. Some of these approaches actually attempt to reconstruct the bulk gravity operators from the CFT data [192, 193, 194, 195] or use concepts of Multiscale Entanglement Renormalisation Ansatz (MERA) to see how the bulk spacetime emerges [196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206]. [1] The much more tractable $O(N)$vector model/HS duality could give us a glimpse of the inner workings of the AdS/CFT correspondence and the collective field theory formalism has indeed proven to be a powerful tool to elucidate these issues [209, 210, 211, 212, 213, 214, 216].[2]

[1]The second approach i.e. the "geometry from entanglement" is beyond the scope of this dissertation, but we recommend the recent book by Takayanagi and Rangamani [207].

[2]One example that illustrates the power of collective field theory is in the $N \to N-1$ shift that we

4.1 Hamiltonian/Collective Field Theory Canonical Quantization Of The Vector Model

The Lagrangian for the $O(N)$ critical vector model can be written as

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi^a\right)\left(\partial^\mu \phi^a\right) - \frac{1}{2}m^2\phi^a\phi^a - \frac{g}{4!}\left(\phi^a\phi^a\right)^2. \tag{4.1}$$

Hence, the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\pi_x^a\pi_x^a + \frac{1}{2}m^2\phi^a\phi^a + \frac{g}{4!}\left(\phi^a\phi^a\right)^2 \\ &= \frac{1}{2}\pi_x^a\pi_x^a + V.\end{aligned}$$

As is standard by now, the equal time collective field theory bilocals are

$$\psi_{\vec{x},\vec{y}}(t) = \sum_{a=1}^{N}\phi^a(t,\vec{x})\,\phi^a(t,\vec{y}). \tag{4.2}$$

and the collective field theory Hamiltonian reads

$$H \;=\; 2\mathrm{Tr}\left(\mathit{\Pi}\psi\Pi\right) + \frac{N^2}{8}\mathrm{Tr}\psi^{-1} + \int d^{d-1}x\left(\frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{1}{2}\lim_{\vec{y}\to\vec{x}} -\partial^2\psi_{\vec{y}\vec{x}} + \frac{g}{4!}\psi^2_{\vec{x}\vec{x}}\right). \tag{4.3}$$

We can rescale the bilocal fields and the conjugate momenta as

$$\psi \to N\psi, \qquad \Pi \to \frac{1}{N}\mathit{\Pi}. \tag{4.4}$$

It is then not difficult to show that the collective field theory Hamiltonian becomes

mentioned at the end of Chapter 2. In the collective field approach, the shift appears naturally once we note that the Jacobian has a term of order $\mathcal{O}\left(N^0\right)$ which after regularization is $\mu = \left(\det\psi\right)^1$.

$$H = \frac{2}{N}\mathrm{Tr}\left(\Pi\psi\Pi\right) + NV_{eff}, \tag{4.5}$$

where the effective potential is

$$V_{eff} = \frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{1}{2}\lim_{\vec{y}\to\vec{x}} -\partial^2\psi_{\vec{y}\vec{x}} + \frac{\lambda}{4!}\psi^2_{\vec{y}\vec{x}} \tag{4.6}$$

and $\lambda = gN$.

The large-N background configuration can be obtained by varying the effective potential with respect to the bilocals. The kinetic energy is subleading in N.

In momentum space one can show that the effective potential can be written as

$$V_{eff} = V\left[\frac{1}{8}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\left(\psi_{\vec{k}}\right)^{-1} + \frac{1}{2}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\left(m^2+\vec{k}^2\right) + \frac{\lambda}{4!}\left(\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\right)\right], \tag{4.7}$$

where $V = \int d^{d-1}x$ is the volume of the space we are considering and we have used the translationally invariant ansatz:

$$\psi_{\vec{x}\vec{y}} = \int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\psi_{\vec{k}}. \tag{4.8}$$

The equations of motion are

$$\begin{aligned}
0 &= \frac{\delta V_{eff}}{\delta\psi_{\vec{k}}}\\
&= -\frac{1}{8}\psi_{\vec{k}}^{-2} + \frac{1}{2}\left(\vec{k}^2+m^2\right) + \frac{\lambda}{12}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\\
\Rightarrow\psi_{\vec{k}}^{-2} &= 4\left(\vec{k}^2+m^2+\frac{\lambda}{6}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\right).
\end{aligned} \tag{4.9}$$

It follows from (4.9) that

$$\psi^0_{\vec{k}} = \frac{1}{2}\left(\vec{k}^2 + m^2 + \frac{\lambda}{6}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\right)^{-1/2}. \tag{4.10}$$

Note that the condition that the renormalized mass vanishes is the same in both the Lagrangian and Hamiltonian formulations. Imposing the condition that the renormalized mass vanishes - this is to preserve conformal invariance - leads to (we have in mind $d = 3$):

$$\psi^0_{\vec{k}} = \frac{1}{2\sqrt{\vec{k}^2}}. \tag{4.11}$$

Expanding about this large-N background, we have:

$$\psi_{\vec{x}\vec{y}} = \psi^0_{\vec{x}\vec{y}} + \frac{1}{\sqrt{N}}\eta_{\vec{x}\vec{y}}. \tag{4.12}$$

Let us now focus solely on the free $O(N)$ vector model.

By expanding (4.5), with $\Pi = \sqrt{N}p$, the quadratic collective field theory Hamiltonian can be written as [209]:

$$H_2 = 2\mathrm{Tr}\left(p\psi_0 p\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right). \tag{4.13}$$

Recall that

$$\psi^0_{\vec{x}\vec{y}} \;\; = \;\; \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}} e^{i\vec{k}_1\cdot\vec{x}_1}\psi^0_{\vec{k}_1} \tag{4.14}$$

and the fluctuations are

$$\eta_{\vec{x}\vec{y}} = \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-E_{\vec{k}_1\vec{k}_2}t} e^{i\vec{k}_1\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2}. \tag{4.15}$$

In addition, we have

$$p_{\vec{x}\vec{y}} = \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} e^{i\vec{q}\cdot(\vec{x}-\vec{y})} p_{\vec{q}}. \tag{4.16}$$

The Large-N background is (4.11):

$$\psi^0_{\vec{k}} = \frac{1}{2\left|\vec{k}\right|}. \tag{4.17}$$

Using the Hamiltonian equations of motions, we have

$$\begin{aligned} \dot{\psi}^0_{\vec{x}\vec{y}} &= \frac{\delta H_2}{\delta p_{\vec{x}\vec{y}}} \\ &= 2\left((p\psi_0)_{\vec{x}\vec{y}} + (\psi_0 p)_{\vec{x}\vec{y}}\right) = \dot{\psi}^0_{\vec{x}\vec{y}}, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} \dot{p}_{\vec{x}\vec{y}} &= -\frac{\delta H_2}{\delta \eta_{\vec{x}\vec{y}}} \\ &= -\frac{1}{8}\left(\left(\psi_0^{-1}\eta\psi_0^{-2}\right)_{\vec{x}\vec{y}} + \left(\psi_0^{-2}\eta\psi_0^{-1}\right)_{\vec{x}\vec{y}}\right) \\ &= \dot{p}_{\vec{x}\vec{y}}. \end{aligned} \tag{4.19}$$

where we have used the symmetry $\vec{x} \longleftrightarrow \vec{y}$.

Taking another time derivative of (4.18) leads to

$$
\begin{aligned}
\ddot{\psi}_{\vec{x}\vec{y}} &= 2\left(\dot{p}\psi_0 + \psi_0\dot{p}\right)_{\vec{x}\vec{y}} \\
&= 2\left(-\frac{1}{8}\right)\left(\psi_0^{-1}\eta\psi_0^{-2}\psi_0 + \psi_0^{-2}\eta\psi_0^{-1}\psi_0 + \psi_0\psi_0^{-1}\eta\psi_0^{-2} + \psi_0\psi_0^{-2}\eta\psi_0^{-1}\right) \\
&= -\frac{1}{4}\left(\psi_0^{-1}\eta\psi_0^{-1} + \psi_0^{-2}\eta + \eta\psi_0^{-2} + \psi_0^{-1}\eta\psi_0^{-1}\right) \\
&= -\frac{1}{4}\left(2\psi_0^{-1}\eta\psi_0^{-1} + \psi_0^{-2}\eta + \eta\psi_0^{-2}\right). \qquad (4.20)
\end{aligned}
$$

By using the expressions given in (4.14) and (4.15) one can easily show that in momentum space

$$
\boxed{E^2_{\vec{k}_1\vec{k}_2}A_{\vec{k}_1\vec{k}_2} = \frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\right)^2 A_{\vec{k}_1\vec{k}_2} = \omega^2_{\vec{k}_1\vec{k}_2}A_{\vec{k}_1\vec{k}_2},} \qquad (4.21)
$$

where

$$
\omega_{\vec{k}_1\vec{k}_2} = \pm\left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right). \qquad (4.22)
$$

This corresponds to the relativistic energy of two combined massless particles.

Repeating the derivation directly in momentum space, we find that the first term on the R.H.S. of (4.13) yields

$$
\begin{aligned}
2\mathrm{Tr}\left(p\psi^0 p\right) &= 2\int d^{d-1}\vec{x}_1 \int d^{d-1}\vec{x}_2 \int d^{d-1}\vec{x}_3 p_{\vec{x}_1\vec{x}_2}\psi^0_{\vec{x}_2\vec{x}_3}p_{\vec{x}_3\vec{x}_1} \\
&= 2\int d^{d-1}\vec{x}_1 \int d^{d-1}\vec{x}_2 \int d^{d-1}\vec{x}_3 \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{k}_3}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{k}_4}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \\
&\qquad e^{i\vec{k}_1\cdot\vec{x}_1 + i\vec{k}_2\cdot\vec{x}_2} e^{i\vec{p}\cdot(\vec{x}_2-\vec{x}_3)} e^{i\vec{k}_3\cdot\vec{x}_3 + i\vec{k}_4\cdot\vec{x}_1} p_{\vec{k}_1\vec{k}_2}\psi^0_{\vec{p}}p_{\vec{k}_3\vec{k}_4} \\
&= 2\int d^{d-1}\vec{k}_1 \int d^{d-1}\vec{k}_2 p_{\vec{k}_1\vec{k}_2}\psi^0_{\vec{k}_2}p_{-\vec{k}_2,-\vec{k}_1} \qquad (4.23)
\end{aligned}
$$

and for the second term, we obtain

$$\frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) =$$
$$\frac{1}{8}\int d^{d-1}\vec{x}_1\int d^{d-1}\vec{x}_2\int d^{d-1}\vec{x}_3\int d^{d-1}\vec{x}_4\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}$$
$$\times\int\frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k}_3}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k}_4}{(2\pi)^{\frac{d-1}{2}}}e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}e^{i\vec{k}_1\cdot\vec{x}_2}e^{i\vec{k}_2\cdot\vec{x}_3}e^{i\vec{q}\cdot(\vec{x}_3-\vec{x}_4)}e^{i\vec{k}_3\cdot\vec{x}_4}e^{i\vec{k}_4\cdot\vec{x}_1}\psi_{\vec{p}}^{0^{-2}}\eta_{\vec{k}_1\vec{k}_2}\psi_{\vec{p}}^{0^{-1}}\eta_{\vec{k}_3\vec{k}_4}$$
$$=\int\frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k}_3}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k}_4}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}(2\pi)^{4(d-1)}\delta\left(\vec{p}+\vec{k}_4\right)$$
$$\times\delta\left(\vec{k}_1-\vec{p}\right)\delta\left(\vec{k}_2+\vec{q}\right)\delta\left(\vec{k}_3-\vec{q}\right)\psi_{\vec{p}}^{0^{-2}}\eta_{\vec{k}_1\vec{k}_2}\psi_{\vec{p}}^{0^{-1}}\eta_{\vec{k}_3\vec{k}_4}$$
$$=\frac{1}{8}\int d^{d-1}\vec{k}_1\int d^{d-1}\vec{k}_2\psi_{\vec{k}_1}^{0^{-2}}\eta_{\vec{k}_1\vec{k}_2}\psi_{\vec{k}_1}^{0^{-1}}\eta_{-\vec{k}_2,-\vec{k}_1}. \quad (4.24)$$

The free effective quadratic collective field Hamiltonian in momentum space then reads

$$H_2 = 2\mathrm{Tr}\left(p\psi_0 p\right)+\frac{1}{8}\mathrm{Tr}\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}$$
$$=2\int d^{d-1}\vec{k}_1\int d^{d-1}\vec{k}_2 p_{\vec{k}_1\vec{k}_2}\psi_{\vec{k}_2}^0 p_{-\vec{k}_2,-\vec{k}_1}+\frac{1}{8}\int d^{d-1}\vec{k}_1\int d^{d-1}\vec{k}_2\psi_{\vec{k}_1}^{0^{-2}}\eta_{\vec{k}_1\vec{k}_2}\psi_{\vec{k}_1}^{0^{-1}}\eta_{-\vec{k}_2,-\vec{k}_1}. \quad (4.25)$$

In momentum space the equations of motion are

$$\dot{\eta} = \frac{\delta H_2}{\delta p_{\vec{k}_1\vec{k}_2}}$$
$$=2\left(\psi_{\vec{k}_2}^0 p_{-\vec{k}_2,-\vec{k}_1}+p_{-\vec{k}_2,-\vec{k}_1}\psi_{\vec{k}_1}^0\right)$$
$$=2\left(\psi_{\vec{k}_1}^0+\psi_{\vec{k}_2}^0\right)p_{-\vec{k}_2,-\vec{k}_1}, \quad (4.26)$$

and

$$\dot{p}_{\vec{k}_1\vec{k}_2} = -\frac{\delta H}{\delta \eta_{\vec{k}_1\vec{k}_2}}$$
$$= -\frac{1}{8}\left(\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1} + \psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\eta_{\vec{k}_1\vec{k}_2}. \tag{4.27}$$

Then,

$$\ddot{\eta}_{\vec{k}_1\vec{k}_2} = 2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)\dot{p}_{-\vec{k}_2,-\vec{k}_1}$$
$$= 2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)\left\{-\frac{1}{8}\left(\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1} + \psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\eta_{\vec{k}_1\vec{k}_2}\right\}$$
$$= -\frac{1}{4}\left(\psi^0_{\vec{k}_1}\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1} + \psi^0_{\vec{k}_1}\psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1} + \psi^0_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1} + \psi^0_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\eta_{\vec{k}_1\vec{k}_2}$$
$$= -\frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-2}}_{\vec{k}_2} + \psi^{0^{-2}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\eta_{\vec{k}_1\vec{k}_2}$$
$$= -\frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\right)^2\eta_{\vec{k}_1\vec{k}_2} = -\left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2\eta_{\vec{k}_1\vec{k}_2}. \tag{4.28}$$

The Lagrangian density is

$$\mathcal{L} = p\dot{x} - \mathcal{H}$$
$$= \frac{1}{2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}\dot{\eta}_{-\vec{k}_2,-\vec{k}_1}\dot{\eta}_{\vec{k}_1\vec{k}_2} - 2p_{\vec{k}_1\vec{k}_2}\psi^0_{\vec{k}_2}p_{-\vec{k}_2,-\vec{k}_1} + V$$
$$= \frac{1}{2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}\dot{\eta}_{-\vec{k}_2,-\vec{k}_1}\dot{\eta}_{\vec{k}_1\vec{k}_2} - 2\left(\frac{1}{2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}\dot{\eta}_{-\vec{k}_2,-\vec{k}_1}\right)\psi^0_{\vec{k}_2}\left(\frac{1}{2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}\dot{\eta}_{\vec{k}_1\vec{k}_2}\right)$$
$$= \frac{1}{2\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}\dot{\eta}_{-\vec{k}_2,-\vec{k}_1}\dot{\eta}_{\vec{k}_1\vec{k}_2} - \frac{1}{4\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)^2}\dot{\eta}_{-\vec{k}_2,-\vec{k}_1}\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)\dot{\eta}_{\vec{k}_1\vec{k}_2}$$
$$= \frac{1}{4\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}\dot{\eta}_{-\vec{k}_2,-\vec{k}_1}\dot{\eta}_{\vec{k}_1\vec{k}_2} - \frac{1}{16}\eta_{\vec{k}_1\vec{k}_2}\left(\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1} + \psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\eta_{\vec{k}_1\vec{k}_2}. \tag{4.29}$$

That is,

$$
\begin{aligned}
L = \frac{1}{4}\int d^{d-1}\vec{k}_1 \int d^{d-1}\vec{k}_2 \dot{\eta}_{\vec{k}_1\vec{k}_2} \frac{1}{\left(\psi^0_{\vec{k}_1}+\psi^0_{\vec{k}_2}\right)} \dot{\eta}_{-\vec{k}_2,-\vec{k}_1} - \frac{1}{16}\int d^{d-1}\vec{k}_1 \\
\times \int d^{d-1}\vec{k}_2 \eta_{\vec{k}_1\vec{k}_2} \left(\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1}+\psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\eta_{\vec{k}_1\vec{k}_2} \\
= \frac{1}{2}\int d^{d-1}\vec{k}_1 \int d^{d-1}\vec{k}_2 \eta_{\vec{k}_1\vec{k}_2} \hat{O}\eta_{-\vec{k}_2,-\vec{k}_1},
\end{aligned}
\tag{4.30}
$$

where

$$
\begin{aligned}
\hat{O}_{\vec{k}_1\vec{k}_2;\vec{k}_3\vec{k}_4} = \left(-\frac{1}{2\left(\psi^0_{\vec{k}_1}+\psi^0_{\vec{k}_2}\right)}\partial_t^2 - \frac{1}{8}\left(\psi^{0^{-1}}_{\vec{k}_2}\psi^{0^{-2}}_{\vec{k}_1}+\psi^{0^{-2}}_{\vec{k}_2}\psi^{0^{-1}}_{\vec{k}_1}\right)\right) \\
\times \delta\left(\vec{k}_1+\vec{k}_4\right)\delta\left(\vec{k}_2+\vec{k}_3\right).
\end{aligned}
\tag{4.31}
$$

For future reference, we need to find the inverse of the operator $\hat{O}_{\vec{k}_1\vec{k}_2;\vec{k}_3\vec{k}_4}$. The inverse is formally defined via

$$
\int d^{d-1}\vec{k}_3 \int d^{d-1}\vec{k}_4 \hat{O}_{\vec{k}_1\vec{k}_2;\vec{k}_3\vec{k}_4}(t)\, \hat{O}^{-1}_{\vec{k}_3\vec{k}_4;\vec{p}_1\vec{p}_2}(t-t') = \delta_{\vec{k}_1\vec{p}_1}\delta_{\vec{k}_2\vec{p}_2}\delta(t-t') \tag{4.32}
$$

with

$$
\hat{O}^{-1}_{\vec{k}_1\vec{k}_2;\vec{p}_1\vec{p}_2} = \int \frac{dE}{(2\pi)} e^{-iE(t-t')} \hat{O}^{-1}_{\vec{k}_1\vec{k}_2;\vec{p}_1\vec{p}_2;E}. \tag{4.33}
$$

It is clear to see that the inverse is given by

$$
\boxed{\hat{O}^{-1}_{\vec{k}_3\vec{k}_4;\vec{p}_1\vec{p}_2;E} = \frac{i\delta\left(\vec{k}_3+\vec{p}_2\right)\delta\left(\vec{k}_4+\vec{p}_1\right)}{\frac{E^2}{2\left(\psi^0_{\vec{k}_3}+\psi^0_{\vec{k}_4}\right)} - \frac{1}{8}\left(\psi^{0^{-1}}_{\vec{k}_3}\psi^{0^{-2}}_{\vec{k}_4}+\psi^{0^{-2}}_{\vec{k}_3}\psi^{0^{-1}}_{\vec{k}_4}\right)}}. \tag{4.34}
$$

After some manipulations, we obtain

$$
\begin{aligned}
O^{-1}_{\vec{k}_3\vec{k}_4;\vec{p}_1\vec{p}_2;E} &= \frac{i\delta\left(\vec{k}_3+\vec{p}_2\right)\delta\left(\vec{k}_4+\vec{p}_1\right)}{\frac{E^2}{2\left(\psi^0_{\vec{k}_3}+\psi^0_{\vec{k}_4}\right)}-\frac{1}{8}\left(\psi^{0^{-1}}_{\vec{k}_3}\psi^{0^{-2}}_{\vec{k}_4}+\psi^{0^{-2}}_{\vec{k}_3}\psi^{0^{-1}}_{\vec{k}_4}\right)} \\
&= \frac{2i\left(\psi^0_{\vec{k}_3}+\psi^0_{\vec{k}_4}\right)\delta\left(\vec{k}_3+\vec{p}_2\right)\delta\left(\vec{k}_4+\vec{p}_1\right)}{E^2-\frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_3}\psi^{0^{-2}}_{\vec{k}_4}+\psi^{0^{-2}}_{\vec{k}_3}\psi^{0^{-1}}_{\vec{k}_4}\right)\left(\psi^0_{\vec{k}_3}+\psi^0_{\vec{k}_4}\right)} \\
&= \frac{2i\left(\psi^0_{\vec{k}_3}+\psi^0_{\vec{k}_4}\right)}{E^2-\frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_3}+\psi^{0^{-1}}_{\vec{k}_4}\right)^2}\delta\left(\vec{k}_3+\vec{p}_2\right)\delta\left(\vec{k}_4+\vec{p}_1\right).
\end{aligned}
\tag{4.35}
$$

Since $\psi_{\vec{k}} = \frac{1}{2\left|\vec{k}\right|}$, the inverse can be written as

$$
\begin{aligned}
\hat{O}^{-1}_{\vec{k}_3\vec{k}_4;\vec{p}_1\vec{p}_2;E} &= \frac{2i\left(\psi^0_{\vec{k}_3}+\psi^0_{\vec{k}_4}\right)}{E^2-\frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_3}+\psi^{0^{-1}}_{\vec{k}_4}\right)^2}\delta\left(\vec{k}_3+\vec{p}_2\right)\delta\left(\vec{k}_4+\vec{p}_1\right) \\
&= \left(\frac{1}{\left|\vec{k}_3\right|}+\frac{1}{\left|\vec{k}_4\right|}\right)\frac{i\delta\left(\vec{k}_3+\vec{p}_2\right)\delta\left(\vec{k}_4+\vec{p}_1\right)}{E^2-\left(\left|\vec{k}_3\right|+\left|\vec{k}_4\right|\right)^2}.
\end{aligned}
\tag{4.36}
$$

As expected, the poles of the propagator correspond to the dispersion relation (4.22).

4.2 The Map

The collective field map between the bilocal coordinates and the AdS_4 space where Vasiliev's higher spin gauge theory lives was first obtained by the comparison of the generators on both sides of the correspondence in the light-cone gauge with equal x^+

quantization. The AdS generators had been computed in Metsaevs's paper [217] and are

$$\hat{p}^- = -\frac{p^x p^x + p^z p^z}{2p^+} \tag{4.37}$$

$$\hat{p}^+ = p^+ \tag{4.38}$$

$$\hat{m}^{+-} = x^+ \hat{p}^- - x^- p^+ \tag{4.39}$$

$$\hat{m}^{+x} = x^+ p^x - x p^+ \tag{4.40}$$

$$\hat{m}^{-x} = x^- p^x - x p^- + \frac{p^\theta p^z}{p^+} \tag{4.41}$$

$$\hat{d} = \left(x^+ p^- + x^- p^+ + x p^x + z p^z + d_a\right) \tag{4.42}$$

$$\hat{k}^- = -\frac{1}{2}\left(x^2 + z^2\right)\hat{p}^- + x^-\left(x^- p^+ + x p^x + z p^z + d_a\right) \tag{4.43}$$

$$\hat{k}^+ = x^+ \hat{p}^- + x^+\left(x p^x + z p^z + d_a\right) - \frac{1}{2}\left(x^2 - z^2\right) p^x + x\left(x^- p^+ + z p^z + d_a\right) + z p^\theta \tag{4.44}$$

where $\hat{p}$, $\hat{m}^{AB}(A, B = +, -, x)$, $\hat{d}$ and $\hat{k}^A$ are the generators of momentum, Lorentz rotations, dilations and the special conformal transformations respectively.

Starting from the $O(N)$ vector model, the $3d$ conformal generators were computed and found to be [209]:

$$\hat{p}^- = \hat{p}_1^- + \hat{p}_2^- \tag{4.45}$$

$$\hat{p}^+ = \hat{p}_1^+ + \hat{p}_2^+ \tag{4.46}$$

$$\hat{p}^i = \hat{p}_1^i + \hat{p}_2^i \tag{4.47}$$

$$\hat{m}^{+i} = x^+ p^i - x_1^i p_1^+ - x_2^i p_2^+ \tag{4.48}$$

$$\hat{m}^{-i} = x^+ p^i - x_1^i p_1^- - x_2^i p_2^- \tag{4.49}$$

$$\begin{aligned} \hat{d} &= x^+ p^- + x_1^- p^+ + x_2 p_2^- + x_1^i p_1^i \\ &+ x_2^i p_2^i + 2 d_\phi \end{aligned} \tag{4.50}$$

$$\begin{aligned} \hat{k}^- &= x_1^i x_1^i \frac{p_1^j p_1^j}{4p_1^+} + x_2^i x_2^i \frac{p_2^j p_2^j}{4p_2^+} \\ &+ x_1^- \left(x_1^- p_1^+ + x_1^i p_1^i + d_\phi \right) \\ &+ x_2^- \left(x_2^- p_2^+ + x_2^i p_2^i + d_\phi \right) \end{aligned} \tag{4.51}$$

$$\begin{aligned} \hat{k}^+ &= x^+ p + x^+ \left(x_1^i p_1^i + x_2^i p_2^i + 2 d_\phi \right) \\ &- \frac{1}{2} x_1^i x_1^i p_1^+ - \frac{1}{2} x_2^i x_2^i p_2^+ \end{aligned} \tag{4.52}$$

$$\begin{aligned} \hat{k}^i &= -x^+ \left(x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_1^i p_1^i + x_2^i p_2^i \right) \\ &- \frac{1}{2} x_1^j x_1^j p_1^i - \frac{1}{2} x_2^j x_2^j p_2^i \\ &+ x_1^i \left(x_1^- p_1^+ + x_1^j p_1^j + d_\phi \right) + x_2^i \left(x_2^- p_2^- + aq x_1^j p_1^j + d_\phi \right) \end{aligned} \tag{4.53}$$

where the labels $\vec{x}_1$ ($\vec{p_1}$) and $\vec{x}_2$ ($\vec{p_2}$) are the coordinates (momenta) of the bilocals. In other words, the bilocals are

$$\psi_{\vec{x}_1 \vec{x}_2} = \sum_{i=1}^{N} \phi^i \left(t, \vec{x}_1 \right) \phi^i \left(t, \vec{x}_2 \right). \tag{4.54}$$

The map can then be obtained by comparing the generators. For example, if one compares (4.38) and (4.46), one has

$$p^{+} = p_1^{+} + p_2^{+}. \tag{4.55}$$

Similarly, comparing (4.40) with (4.48) leads to

$$xp^{+} = x_1^{i} p_1^{+} - x_2^{i} p_2^{+} \tag{4.56}$$

or

$$x = \frac{x_1^{i} p_1^{+} - x_2^{i} p_2^{+}}{p_1^{+} + p_2^{+}}. \tag{4.57}$$

The light-cone map – which results from carrying all the other comparisons – can then be written as [209]:

$$x^{-} = \frac{x_1^{-} p_1^{+} + x_2^{-} p_2^{+}}{p_1^{+} + p_2^{+}} \tag{4.58}$$

$$p^{+} = p_1^{+} + p_2^{+} \tag{4.59}$$

$$x = \frac{x_1 p_1^{+} + x_2 p_2^{+}}{p_1^{+} + p_2^{+}} \tag{4.60}$$

$$p^{x} = p_1 + p_2 \tag{4.61}$$

$$z = \frac{(x_1 - x_2)\sqrt{p_1^{+} p_2^{+}}}{p_1^{+} + p_2^{+}} \tag{4.62}$$

$$p^{z} = \sqrt{\frac{p_2^{+}}{p_1^{+}}} p_1 - \sqrt{\frac{p_2^{+}}{p_2^{+}}} p_2 \tag{4.63}$$

$$\begin{aligned} p^{\theta} = {} & \sqrt{p_1^{+} p_2^{+}}\left(x_1^{-} - x_2^{-}\right) \\ & + \frac{x_1 - x_2}{2}\left(\sqrt{\frac{p_2^{+}}{p_1^{+}}} p_1 + \sqrt{\frac{p_1^{+}}{p_2^{+}}} p^{z}\right) \end{aligned} \tag{4.64}$$

$$\theta = 2 \arctan\left(\sqrt{\frac{p_2^{+}}{p_1^{+}}}\right) \tag{4.65}$$

Of great significance is the equation for z *i.e.* (4.62). As mentioned previously it is important to understand where the extra radial dimension comes from. (Traditionally, it has been interpreted as a renormalization group flow [218, 219, 220].) What the equation for z allows us to do is to give this extra dimension a physical interpretation as the distance between the poles of a dipole. A similar picture also arises in QCD [221].

To explain what θ is, we need first to consider mixed representations of the Lorentz group. This can be done effectively by considering the construction by Bengtsson et al [222]. More precisely, the mixed representations of the Lorentz group are built from creation operators. These creation operators can be repackaged into a ket $|\Phi\rangle$ in the Fock space defined as [222]:

$$|\Phi\rangle = \sum_{s=1}^{\infty} \Phi^{\mu_1 \cdots \mu_s} a^\dagger_{\mu_1} \cdots a^\dagger_{\mu_s} |0\rangle \,, \tag{4.66}$$

where $\mu = (0, 1, \ldots, z, d-1)$ and the creation and annihilation operators satisfy the commutation relations [209]:

$$\left[a^I, a^{J\dagger}\right] = \delta^{IJ}, \qquad \left[a^I, a^J\right] = \left[a^{I\dagger}, a^{J\dagger}\right] = 0. \tag{4.67}$$

Here I, J refer to the transverse directions and also the bulk AdS_{d+1} z direction.

We require all traces to vanish in order for the representations to be irreducible. This means that we need to impose the condition that

$$T\,|\Phi\rangle = 0, \qquad T = a^I a^I. \tag{4.68}$$

More intuitively, this means that we impose the constraint that only two components will survive.

On physical grounds, in four dimensions, it is convenient to complexify the oscillators and write[3]

$$\alpha = \frac{1}{\sqrt{2}}\left(a_1 + ia_2\right), \qquad \alpha^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger + ia_2^\dagger\right) \tag{4.69}$$

$$\bar{\alpha} = \frac{1}{\sqrt{2}}\left(a_1 - ia_2\right), \qquad \bar{\alpha}^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger - ia_2^\dagger\right). \tag{4.70}$$

It turns out that the ket $|\Phi\rangle$ can be expanded as [209, 222]:

$$|\Phi\rangle = \sum_{\lambda=1}^{\infty}\left(\Phi_{(\lambda)}\left(\bar{\alpha}^\dagger\right)^\lambda + \bar{\Phi}_{(\lambda)}\left(\alpha^\dagger\right)^\lambda\right)|0\rangle\,. \tag{4.71}$$

In addition, the ket $|\Phi\rangle$ satisfies the new constraint that [222]:

$$T\,|\Phi\rangle = 0, \qquad T = \bar{\alpha}\alpha. \tag{4.72}$$

In terms of the complexified oscillators, the spin matrix can be written as

$$M^{IJ} = \alpha^{I\dagger}\bar{\alpha}^J - \bar{\alpha}^{J\dagger}\alpha^I. \tag{4.73}$$

In four dimensions, the only non-vanishing component of the spin matrix is M^{xz} with the complex oscillators being given by $\alpha = e^{i\theta}$ and $\bar{\alpha} = e^{-i\theta}$.

[3]This has to do with the fact that in four dimensions any spin-s field only has two physical degrees of freedom *viz.* the two helicities$\lambda = \pm s$ [222].

The light-cone map allows us to map the bilocals to the higher-spin fields via

$$\hat{\mathcal{H}}\left(p^+, p, p^z, \theta\right) = \int dp_1^+ dp_1 dp_2^+ dp_2 \mathcal{J}\left(p_1^+, p_1, p_2^+, p_2\right)$$
$$\delta\left(p_1^+ + p_2^+ - p^+\right)\delta\left(p_1 + p_2 - p\right)\delta\left(\sqrt{\frac{p_2^+}{p_1^+}}p_1 - \sqrt{\frac{p_2^+}{p_2^+}}p_2 - p^z\right)$$
$$\delta\left(2\arctan\left(\sqrt{\frac{p_2^+}{p_1^+}}\right) - \theta\right)\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right). \tag{4.74}$$

where all the transformations are point-like in momentum space and the Jacobian is

$$\mathcal{J}\left(p_1^+, p_1, p_2^+, p_2\right) = \frac{1}{p_1^+} + \frac{1}{p_2^+}. \tag{4.75}$$

The inverse map is given by

$$\tilde{\Psi}\left(p_1^+, p_1, p_1^+, p_1\right) = \int dp^+ dp dp^z d\theta \mathcal{J}^{-1}\left(p^+, p, p^z, \theta\right)\delta\left(p_1^+ - p^+ \cos^2\frac{\theta}{2}\right)$$
$$\times\,\delta\left(p_2^+ - p^+ \sin^2\frac{\theta}{2}\right)\delta\left(p_1 - \frac{(1+\cos\theta) + p\sin\theta}{2}\right)$$
$$\times\,\delta\left(p_2 - \frac{(1+\cos\theta) + p\sin\theta}{2}\right)\hat{\mathcal{H}}\left(p^+, p, p^z, \theta\right). \tag{4.76}$$

The above light-cone map has been generalized to the case when we have a time-like gauge and reads ($E = E_1 + E_2$) [215]:

$$\vec{p} = \vec{p_1} + \vec{p_2} \tag{4.77}$$
$$p^z = 2\sqrt{|\vec{p_1}|\,|\vec{p_2}|}\sin\left(\frac{\varphi_2 - \varphi_1}{2}\right) \tag{4.78}$$
$$\theta = \arctan\left(\frac{2\vec{p_2}\times\vec{p_1}}{\left(|\vec{p_1}| - |\vec{p_2}|\right)p}\right). \tag{4.79}$$

The angles φ_1 and φ_2 are defined via [215]:

$$\vec{p_1} = \left(|\vec{p_1}|\cos\varphi_1, |\vec{p_1}|\sin\varphi_1, \right), \qquad \vec{p_2} = \left(|\vec{p_2}|\cos\varphi_2, |\vec{p_2}|\sin\varphi_2\right). \tag{4.80}$$

In time-like gauge, the map between the bilocals and the higher spin field can be written as

$$\begin{aligned}\hat{\mathcal{H}}\left(\vec{p_1},\vec{p_2},\theta\right) = &\int d\vec{p_1}d\vec{p_2}\mathcal{J}\left(\vec{p_1},\vec{p_2}\right)\delta\left(\vec{p_1}+\vec{p_2}-\vec{p}\right)\delta\left(\arctan\left(\frac{2\vec{p}_2\times\vec{p}_1}{\left(|\vec{p_1}|-|\vec{p_2}|\right)p}\right)-\theta\right)\\ &\times\delta\left(\sqrt{2\,|\vec{p_1}|\,|\vec{p_1}|-2\vec{p_1}\cdot\vec{p_2}}-\theta\right)\tilde{\Psi}\left(\vec{p_1},\vec{p_2}\right),\end{aligned} \tag{4.81}$$

where the Jacobian is given by

$$\mathcal{J}\left(\vec{p_1},\vec{p_2}\right) = \frac{1}{|\vec{p_1}|} + \frac{1}{|\vec{p_2}|}. \tag{4.82}$$

4.3 The Two Time Free Bilocal Collective Field Propagator/ Bethe Salpeter

The Bethe-Salpeter equation [223] is an important result that plays a huge role in the analysis of the four-point function for bound systems.[4] It arises naturally when the four-point function has a rung added to it at each level in perturbation. The resulting ladder diagrams can be added together to give an effective vertex.

The four point function is defined via [225]:

$$G\left(x_1,x_2,x_3,x_4\right) = \left\langle 0\left|T\left[\phi\left(x_1\right)\phi\left(x_2\right)\phi^\dagger\left(x_3\right)\phi^\dagger\left(x_4\right)\right]\right|0\right\rangle. \tag{4.83}$$

where T denotes the usual time ordering from QFT and we consider complex scalar fields.

[4] In fact, the Bethe-Salpeter equation was first written down by Nambu in 1950 [224].

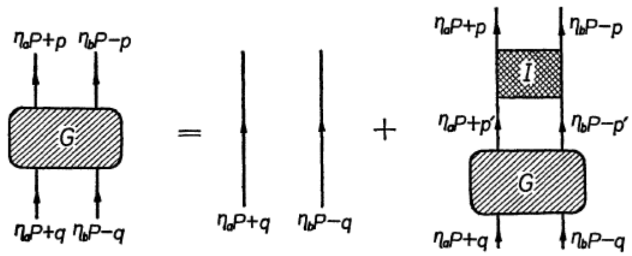


Figure 4.1: Schematic representation of the Bethe-Saltpeter equation in momentum space. Picture credits [225].

The framework for solving the Bethe-Salpeter equation can be set up for a theory with quartic interactions.

Generally, the Bethe-Saltpeter equation for the four-point, in coordinate space, can be written as [225]:

$$G\left(x_1,x_2,y_1,y_2\right)=G\left(x_1-y_1\right)G\left(x_2-y_2\right)+\int dz_1\int dz_2\int dz_1^{'}\int dz_2^{'}$$
$$\times\, G\left(x_1-z_1\right)G\left(x_2-z_2\right)I\left(z_1,z_2;z_1',z_2'\right)G\left(z_1',z_2';y_1,y_2\right), \quad (4.84)$$

where $I\left(z_1,z_2;z_1',z_2'\right)$ represents the sum over all the two-particle irreducible Feynman diagrams.

In momentum space this reads

$$\left[G\left(\eta_a P+p\right)G\left(\eta_b P-p\right)\right]^{-1}G\left(p,q;P\right)=\delta\left(p-q\right)$$
$$+\int dq' I\left(p,q';P\right)G\left(q',q;P\right), \quad (4.85)$$

where η_a, η_b are arbitrary parameters such that $\eta_a+\eta_b=1$. A diagrammatic representation of the above equation is given in Figure 4.1.

For the free $O\left(N\right)$ vector model the action, in terms of bilocals, reads

$$S=N\int d^dx\left(-\int d^dy\delta\left(x-y\right)\partial_y^2\psi_{xy}\right)-i\frac{N}{2}\mathrm{Tr}\ln\psi. \tag{4.86}$$

By expanding the action – or the log of the Jacobian - to second order one can extract the quadratic part of the action:

$$S_2^{eff}=\frac{i}{4}\mathrm{Tr}\psi_0^{-1}\eta\psi_0^{-1}\eta. \tag{4.87}$$

In momentum space, the effective quadratic action is

$$\begin{aligned}
S_2^{eff} &= \frac{i}{4}\mathrm{Tr}\eta\psi_0^{-1}\eta\psi_0^{-1}\\
&= \frac{i}{4}\int d^dx_1d^dx_2d^dx_3d^dx_4\left(\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x_3}\eta_{x_3x_4}\psi^{0^{-1}}_{x_3x_4}\right)\\
&= \frac{i}{4}\int d^dx_1d^dx_2d^dx_3d^dx_4\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\eta_{k_1k_2}e^{-ik_1x_1-ik_2x_2}\\
&\times\int\frac{d^dp_1}{(2\pi)^d}\psi_{0p_1}^{-1}e^{ip_1(x_1-x_2)}\int\frac{d^dk_3}{(2\pi)^d}\int\frac{d^dk_4}{(2\pi)^d}\eta_{k_3k_4}e^{-ik_3x_3-ik_4x_4}\int\frac{d^dp_1}{(2\pi)^d}\psi_{0p_1}^{-1}e^{ip_1(x_1-x_2)}\\
&= \frac{i}{2}\int\frac{d^dk_1d^dk_2d^dk_3d^4k_4}{(2\pi)^d}\eta_{k_1k_2}\hat{O}_{k_1k_2;k_3k_4}\eta_{k_3k_4},
\end{aligned} \tag{4.88}$$

where

$$\hat{O}_{k_1k_2;k_3k_4}=\frac{i}{2}\psi_{k_1}^{-1}\psi_{k_2}^{-1}\delta\left(-k_2-k_3\right)\delta\left(-k_1-k_3\right). \tag{4.89}$$

The propagator - which is the inverse of $\hat{O}$ - is defined through

$$\int d^dk_3\int d^dk_4\hat{O}_{k_1k_2;k_3k_4}\hat{O}^{-1}_{k_3k_4;p_1p_2}=i\delta\left(k_1+p_1\right)\delta\left(k_2+p_2\right). \tag{4.90}$$

We will assume that the inverse is of the form [226]:

$$\hat{O}^{-1}_{k_3k_4;p_1p_2}=A\left(k_3,k_4\right)\delta\left(k_3-p_2\right)\delta\left(k_4-p_1\right). \tag{4.91}$$

We plug this ansatz into (4.90) and find

$$\int d^d k_3 \int d^d k_4 \hat{O}_{k_1 k_2; k_3 k_4} \hat{O}^{-1}_{k_3 k_4; p_1 p_2} = \int d^d k_3 \int d^d k_4 \Bigg[\frac{1}{2} \psi^{-1}_{k_3} \psi^{-1}_{k_4}$$
$$\delta(-k_1 - k_3)\, \delta(-k_2 - k_4) \Bigg] \times A(k_3, k_4)\, \delta(k_3 - p_2)\, \delta(k_4 - p_1)$$
$$= \frac{1}{2} \psi^{-1}_{-k_1} \psi^{-1}_{-k_2} A(-k_1, -k_2)\, \delta(k_1 + p_2)\, \delta(k_2 + p_2)\,. \tag{4.92}$$

Comparing of (4.90) and (4.92) implies that

$$1 = \frac{1}{2} \psi^{-1}_{-k_1} \psi^{-1}_{-k_2} A(-k_1, -k_2) \tag{4.93}$$

or

$$A(k_1, k_2) = 2\psi_{k_1} \psi_{k_2}. \tag{4.94}$$

Thus, the free propagator is given by

$$\boxed{\hat{O}^{-1}_{k_3 k_4; p_1 p_2} = 2\psi_{k_3} \psi_{k_4} \delta(k_3 - p_2)\, \delta(k_4 - p_1)\,.} \tag{4.95}$$

which is illustrated in Figure 4.2.

In coordinate space, we have

$$\partial_x^2 \partial_y^2 \eta_{xy} = 0. \tag{4.96}$$

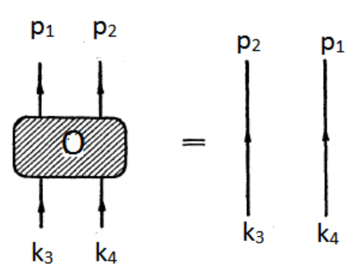


Figure 4.2: The bilocal propagator for the free $O(N)$ vector model.

Chapter 5

$\left(\phi^2\right)^2$ and the Zinn-Justin Argument

Science is beautiful when it makes simple explanations of phenomena or connections between different observations.

-Stephen Hawking.

The Lagrangian for the $O\left(N\right)\ \left(\phi^2\right)^2$ model can be written as

$$\mathcal{L} =\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{1}{2}m^2\phi^a\phi^a + \frac{\lambda}{4!N}\left(\phi^a\phi^a\right)^2 . \tag{5.1}$$

Note that this is the same Lagrangian that is used in [227, 228, 229] once we have made the trivial identification that $m^2 = r$ and $\frac{\lambda}{N} = u$.

Moreover, recall that the Hubbard-Stratonovich transformation is useful when we wish to decouple a quartic interaction through the introduction of an auxialiary field. More precisely, it is the simple identity - which follows from trivial manipulation of a Gaussian integral - that states:

$$\exp\left(-\hat{\rho}_m K_{mn}\hat{\rho}_n\right) = \int\mathcal{D}\sigma\exp\left(-\frac{1}{4}\sigma_m K^{-1}_{mn}\sigma_n - \sigma_m\hat{\rho}_n\right), \tag{5.2}$$

where, as is traditional, we work in Euclidean space and $\hat{\rho} = \phi^a \phi^a$ for bosons.

By using the Hubbard-Stratonovich transformation, we obtain[1]

$$\exp\left(-\left(\frac{1}{2}m^2\phi^a\phi^a + \frac{\lambda}{4!N}\left(\phi^a\phi^a\right)^2\right)\right) = \int \mathcal{D}\sigma \exp\left(-\frac{3}{2\lambda}\sigma^2 - \frac{3m^2}{\lambda}\sigma - \frac{1}{2}\sigma\left(\phi^a\phi^a\right)\right). \tag{5.3}$$

This approach is also called the auxiliary field method.

There is, however a more elementary way to see that is correct. Suppose the Lagrangian of the $O(N)$ vector model can be written as

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi^a\partial^\mu\phi^a\right) + a\sigma^2 + b\sigma + \frac{1}{2}\sigma\left(\phi^a\phi^a\right). \tag{5.4}$$

The coefficients a and b are determined by considering the classical equations of motion. It is simple to see that the equations of motion for the Lagrangian in (5.4) are

$$0 = \frac{\delta\mathcal{L}}{\delta\sigma} = 2a\sigma + b + \frac{1}{2}\left(\phi^a\phi^a\right). \tag{5.5}$$

Therefore, the auxiliary field σ is explicitly given by

$$\sigma = -\frac{1}{2a}\left(b + \frac{1}{2}\left(\phi^a\phi^a\right)\right). \tag{5.6}$$

Substituting the expression that we have found for the auxiliary field back into (5.4), we

[1]The same result can be obtained by making use of the equations of motion. We will illustrate how this is done when we introduce the collective field theory approach below.

get

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi^a\partial^\mu\phi^a\right) + a\left(-\frac{1}{2a}\left(b+\frac{1}{2}\left(\phi^a\phi^a\right)\right)\right)^2 \\
&+ b\left(-\frac{1}{2a}\left(b+\frac{1}{2}\left(\phi^a\phi^a\right)\right)\right) + \frac{1}{2}\left(\phi^a\phi^a\right)\left(-\frac{1}{2a}\left(b+\frac{1}{2}\left(\phi^a\phi^a\right)\right)\right) \\
&= \frac{1}{2}\left(\partial_\mu\phi^a\partial^\mu\phi^a\right) + \frac{b^2}{4a} + \frac{1}{16a}\left(\phi^a\phi^a\right)^2 + \frac{b}{4a}\left(\phi^a\phi^a\right) \\
&- \frac{b^2}{2a} - \frac{b}{4a}\left(\phi^a\phi^a\right) - \frac{b}{4a}\left(\phi^a\phi^a\right) - \frac{1}{8a}\left(\phi^a\phi^a\right)^2 \\
&= \frac{1}{2}\left(\partial_\mu\phi^a\partial^\mu\phi^a\right) - \frac{b}{4a}\left(\phi^a\phi^a\right) - \frac{1}{16a}\left(\phi^a\phi^a\right)^2 + Const.
\end{aligned} \tag{5.7}
$$

Now, by simply comparing (5.1) with (5.7), we obtain

$$
\frac{b}{4a} = -\frac{1}{2}m^2, \qquad \frac{1}{16a} = -\frac{\lambda}{4!N}. \tag{5.8}
$$

Therefore,

$$
a = -\frac{3N}{2\lambda}, \qquad b = \frac{3m^2N}{\lambda}. \tag{5.9}
$$

Thus, the path integral for the $O(N)\left(\phi^2\right)^2$ vector model can be written as [227, 228, 229]:

$$
\begin{aligned}
Z &= \int \mathcal{D}\phi e^{-S[\phi]} \\
&= \int \mathcal{D}\phi\mathcal{D}\sigma \exp\left(-N\int d^dx\left(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{3}{2\lambda}\sigma^2 + \frac{3m^2}{\lambda}\sigma + \frac{1}{2}\sigma\left(\phi^a\phi^a\right)\right)\right).
\end{aligned} \tag{5.10}
$$

We now have to determine the (mass) dimension of the auxiliary field σ. This can be done easily by noting that

$$\left[\sigma^2\right] = \left[m^2\sigma\right]. \tag{5.11}$$

That is,

$$2\left[\sigma\right] = 2 + \left[\sigma\right] \tag{5.12}$$

or $[\sigma] = 2$. (This result holds in all dimensions.)

In general, an operator $\mathcal{O}$ with classical (mass) dimension Δ will be irrelevant if $\Delta - d > 0$ [230]. This can be seen by writing down the perturbative expansion of the effective action [231]:

$$\mathcal{L}_{eff}\left[\phi\right] = \mathcal{L}_l\left[\phi\right] + \sum_i c_i \frac{\mathcal{O}_{\Delta_i}}{M^{\Delta_i - d}}. \tag{5.13}$$

Here $\mathcal{L}_l\left[\phi\right]$ is the renormalizable part of the effective action, and the c_is are dimensionless constants [231].[2] It is thus clear that operators with dimensions $\Delta_i > d$ will dominate in the UV and become negligible as we flow to the the IR *i.e.* operators with dimension larger than the space-time dimension are irrelevant.

In particular, the operator $\mathcal{O} = \sigma^2$ has (mass) dimension four and will be irrelevant for $d < 4$. Thus, we can drop the term that is quadratic in the auxiliary field in (5.10). Therefore, in the critical domain, the path integral for the $\left(\phi^2\right)^2$ takes the form:

$$\begin{aligned} Z &= \int \mathcal{D}\phi\mathcal{D}\sigma \exp\left(-N\int d^dx \left(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \sigma\left(x\right)\left(\frac{3m^2}{\lambda} + \frac{1}{2}\left(\phi^a\phi^a\right)\right)\right)\right) \\ &= Const. \int \mathcal{D}\phi\delta\left(\frac{3m^2}{\lambda} + \frac{1}{2}\left(\phi^a\phi^a\right)\right)\exp\left(-N\int d^dx\left(\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a\right)\right), \end{aligned} \tag{5.14}$$

[2]We consider heavy particles with mass $M > \Lambda$ with Λ being the cutoff. At low energies, $E < \Lambda$, the contribution of the operator $\mathcal{O}_\Delta$ is proportional to $\left(E/M\right)^{\Delta-d}$ [231].

where in the last line we have made use of the functional integral representation of the Dirac delta function:

$$\delta\left(x-y\right)=Const.\int\mathcal{D}\lambda e^{i\lambda(x-y)}. \tag{5.15}$$

Note that (5.14) is the path integral for the $O\left(N\right)$ non-linear σ-model.

A few comments are in order. First, the argument given in this Chapter does not constitute a proof that the critical $O(N)$ vector model is equivalent to the non-linear σ-model. One of the reason for this is that we could add an operator of the form $\left(\phi^a\phi^a\right)^3$ in (5.1).[3] This operator is marginal in $d=3$ and relevant in $d<3$. One would then have to show that the β-function, for the $\left(\phi^a\phi^a\right)^3$ operator, is positive and the operator is thus marginally irrelevant. Second, away from the free fixed point not much is known about how the dimensions actually flow. Accordingly, we might have other relevant operators appearing in our Lagrangian. Finally, when $d=2$ there are many fixed points and the motivation may be found wanting.

In summary, we have given a motivation (not a rigorous mathematical proof) that, in the large-N , the non-linear σ-model and the $\left(\phi^2\right)^2$ vector model are equivalent.

[3]The authors in [159, 166] include the marginal operator $\left(\phi^a\phi^a\right)^3$ in the bosonic part of their action.

Chapter 6

Non-linear Sigma Model And The Two Time Bilocal Description

God does not care about our mathematical difficulties. He integrates empirically.

Albert Einstein.

In the previous chapter, we showed that the critical $(\phi^2)^2$ vector model is equivalent to the non-linear sigma model. This provides us with another scheme with which to compute observables on the CFT side. We will use the non-linear sigma model, in the collective field theory language, to compute the bilocal two-point functions.

Let us consider the $O(N)$ sigma model with action given by

$$Z = \int \mathcal{D}\vec{S} \int \mathcal{D}\alpha \exp\left[-\frac{N}{g}\int d^d x \left(\frac{1}{2}\partial_\mu \vec{S}\partial^\mu \vec{S} + \frac{1}{2}\alpha(x)\left(\vec{S}^2 - 1\right)\right)\right]. \tag{6.1}$$

where $\alpha(x)$ is a Lagrange multiplier which is there to make sure that the constraint $\vec{S}^2 = 1$ is satisfied.

We rescale the action and the coupling as follows:

$$\vec{S} \to g\vec{S}, \qquad g \to \lambda. \tag{6.2}$$

The action can then be written as

$$S = N \int d^dx \left(\frac{1}{2}\partial_\mu \vec{S}\partial^\mu \vec{S} + \frac{\alpha(x)}{2}\left(\vec{S}^2 - \frac{1}{\lambda}\right)\right). \tag{6.3}$$

Using the Jacobian obtained from previous chapters, we have

$$\begin{aligned} Z &= \int \mathcal{D}\vec{S} \int \mathcal{D}\alpha e^{-N\int d^dx\left(\frac{1}{2}\partial_\mu \vec{S}\partial^\mu \vec{S} + \frac{\alpha(x)}{2}\left(\vec{S}^2 - \frac{1}{\lambda}\right)\right)} \\ &= \int \mathcal{D}\psi \int \mathcal{D}\alpha e^{-S_{eff}}. \end{aligned} \tag{6.4}$$

where $\psi_{xy} = \vec{S}(x) \cdot \vec{S}(y)$ and the effective action is

$$S_{eff} = N\left\{ -\frac{1}{2}\mathrm{Tr}\ln\psi + \int d^dx \left(-\frac{1}{2}\lim_{y\to x}\partial^2\psi_{xy} + \frac{1}{2}\alpha\psi_{xx} - \frac{1}{2\lambda}\alpha(x)\right)\right\}. \tag{6.5}$$

The large-N background follows from varying the effective action with respect to the bilocals. It follows that the saddle point equation, obtained by performing this variation, is

$$\psi_{xy} = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2+\alpha}. \tag{6.6}$$

Similarly, when we vary the effective action with respect to the Lagrange multiplier α, we obtain

$$\psi_{xx} = \frac{1}{\lambda} \tag{6.7}$$

or

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+\alpha} = \frac{1}{\lambda}. \tag{6.8}$$

Using simple cut-off regularization we see that, for $d = 3$, the integral above is linearly divergent.

However, we could opt for dimensional interpolation and analytic continuation methods to make sense of the linear divergence.

More precisely, one can show that for arbitrary d [170, 232]:

$$\boxed{\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+\alpha} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) \alpha^{\frac{d-2}{2}}.} \tag{6.9}$$

For $d = 3$, this leads to

$$\begin{aligned} \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) \alpha^{\frac{d-2}{2}} &= \frac{1}{(4\pi)^{3/2}} \Gamma\left(1-\frac{3}{2}\right) \alpha^{\frac{3-2}{2}} \\ &= \frac{1}{8\pi^{3/2}} \left(-2\sqrt{\pi}\right) \alpha^{1/2} \\ &= -\frac{1}{4\pi^2} \alpha^{1/2}. \end{aligned} \tag{6.10}$$

Using (6.9) in the gap equation – *i.e.* (6.8) – yields

$$\frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) \alpha^{\frac{d-2}{2}} = \frac{1}{\lambda} \tag{6.11}$$

which implies that

$$\alpha = \left[\frac{(4\pi)^{d/2}}{\Gamma\left(1-\frac{d}{2}\right)}\right]^{\frac{2}{d-2}} \frac{1}{\lambda^{\frac{2}{d-2}}}. \tag{6.12}$$

This means that when $d > 2$ and $\lambda \to \infty$, the Lagrange multiplier α, which plays the role of a mass, goes to zero.

For $d = 3$ (6.12) becomes

$$\alpha = \left[\frac{(4\pi)^{3/2}}{\Gamma\left(1-\frac{3}{2}\right)}\right]^{\frac{2}{3-2}} \frac{1}{\lambda^{\frac{2}{3-2}}} \\ = \frac{64\pi^2}{\lambda^2}. \tag{6.13}$$

Consequently, in the strong coupling limit we can write

$$\psi_0\left(k\right) = \frac{1}{k^2}. \tag{6.14}$$

The fluctuations for the two fields are

$$\alpha = 0 + \frac{i}{\sqrt{N}}\tilde{\alpha}\left(x\right) \tag{6.15}$$

$$\psi_{xy} = \psi^0_{xy} + \frac{1}{\sqrt{N}}\eta_{xy}. \tag{6.16}$$

The conventions for the α field are the same as those used by Ruhl but differ from Giombi and Yi. The convention used here has the advantage that it will turn α into a real Lagrange multiplier.

Since

$$\mathrm{Tr}\ln\psi = \ldots - \frac{1}{2}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right). \tag{6.17}$$

We can then write the quadratic effective action as

$$S_2^{eff} = -\frac{1}{2}\left(-\frac{1}{2}\right)\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{\alpha}\eta\right). \tag{6.18}$$

Furthermore, we will assume that we have written the Lagrange multiplier in the form of a diagonal matrix. That is,

$$\tilde{\alpha}_{xy} = \tilde{\alpha}_x \delta\left(x - y\right). \tag{6.19}$$

The effective quadratic action then takes the form

$$S_2^{eff} = \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{\alpha}\eta\right). \tag{6.20}$$

We vary the effective quadratic action with respect to $\tilde{\alpha}_x$ and find that $\eta_{xx} = 0$. To remedy this problem, we will have to "shift"" the η fluctuations or colloquially speaking complete the square. More specifically, when we vary with respect to η_{xy}, we find

$$\begin{aligned} 0 &= \frac{\delta S_2^{eff}}{\delta \eta_{xy}} \\ &= \frac{1}{2}\left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{yx} + \frac{i}{2}\tilde{\alpha}\delta\left(x-y\right). \end{aligned} \tag{6.21}$$

This suggests that we write

$$\eta = -i\left(\psi_0\tilde{\alpha}\psi_0\right) + \tilde{\eta}. \tag{6.22}$$

The effective quadratic action then becomes

$$\begin{aligned} S_2^{eff} &= \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) + \frac{i}{2}\mathrm{Tr}\left(\tilde{\alpha}\eta\right) \\ &= \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\left(-i\left(\psi_0\tilde{\alpha}\psi_0\right) + \tilde{\eta}\right)\psi_0^{-1}\left(-i\left(\psi_0\tilde{\alpha}\psi_0\right) + \tilde{\eta}\right)\right) \\ &+ \frac{i}{2}\mathrm{Tr}\left(\tilde{\alpha}\left(-i\left(\psi_0\tilde{\alpha}\psi_0\right) + \tilde{\eta}\right)\right) \\ &= \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right) + \frac{1}{4}\mathrm{Tr}\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right), \end{aligned} \tag{6.23}$$

which after a trivial relabeling reads

$$\boxed{S_2^{eff} = \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) + \frac{1}{4}\mathrm{Tr}\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right).} \tag{6.24}$$

Let us now move to momentum space and write

$$\tilde{\eta}_{xy} = \int \frac{d^dk_1}{(2\pi)^d}\int \frac{d^dk_2}{(2\pi)^d} e^{ik_1x}e^{ik_2y}\tilde{\eta}_{k_1k_2} \tag{6.25}$$

$$\tilde{\alpha}_x = \int \frac{d^dk}{(2\pi)^{d/2}} e^{ikx}\tilde{\alpha}_k. \tag{6.26}$$

Accordingly,

$$\begin{aligned}
&\frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) = \frac{1}{4}\int d^dx_1\int d^dx_2\int d^dx_3\int d^dx_4\left(\psi^{-1}_{0x_1x_2}\eta_{x_2x_3}\psi^{-1}_{0x_3x_4}\eta_{x_4x_1}\right)\\
&= \frac{1}{4}\int d^dx_1\int d^dx_2\int d^dx_3\int d^dx_4\int\frac{d^dp_1}{(2\pi)^d}\psi^{-1}_{0x_1x_2}e^{ip_1(x_1-x_2)}\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}e^{ik_1x_2}e^{ik_2x_3}\tilde{\eta}_{k_1k_2}\\
&\int\frac{d^dp_2}{(2\pi)^d}\psi^{-1}_{0x_3x_4}e^{ip_2(x_3-x_4)}\int\frac{d^dk_3}{(2\pi)^d}\int\frac{d^dk_4}{(2\pi)^d}e^{ik_3x_4}e^{ik_4x_1}\tilde{\eta}_{k_3k_4} = \frac{1}{4}\int d^dk_1\int d^dk_2\\
&\times\tilde{\eta}_{k_1k_2}\left(\psi_0^{-1}\right)_{k_1}\left(\psi_0^{-1}\right)_{k_2}\tilde{\eta}_{-k_2,-k_1}.
\end{aligned} \tag{6.27}$$

and

$$\begin{aligned}
&\frac{1}{4}\mathrm{Tr}\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right) = \frac{1}{4}\int d^dx_1\int d^dx_2\tilde{\alpha}_{x_1}\psi^0_{x_1x_2}\tilde{\alpha}_{x_2}\psi^0_{x_2x_1}\\
&= \frac{1}{4}\int d^dx_1\int d^dx_2\int\frac{d^dk_1}{(2\pi)^{d/2}}e^{ik_1x_1}\tilde{\alpha}_{k_1}\int\frac{d^dp_1}{(2\pi)^d}e^{ip_1(x_1-x_2)}\psi^0_{p_1}\int\frac{d^dk_2}{(2\pi)^{d/2}}e^{ik_1x_2}\tilde{\alpha}_{k_2}\\
&\times\int\frac{d^dp_2}{(2\pi)^d}e^{ip_2(x_2-x_1)}\psi^0_{p_1} = \frac{1}{4}\int d^dk_1\tilde{\alpha}_{k_1}\left(\int\frac{d^dp_1}{(2\pi)^d}\psi^0_{p_1}\psi^0_{k_1+p_1}\right)\tilde{\alpha}_{-k_1}.
\end{aligned} \tag{6.28}$$

Thus, the effective quadratic action – in momentum space – is

$$S_2^{eff} = \frac{1}{4}\int d^dk_1 \int d^dk_2 \tilde{\eta}_{k_1k_2}\left(\psi_0^{-1}\right)_{k_1}\left(\psi_0^{-1}\right)_{k_2}\tilde{\eta}_{-k_2,-k_1}$$
$$+\frac{1}{4}\int d^dk_1 \tilde{\alpha}_{k_1}\left(\int \frac{d^dp}{(2\pi)^d}\psi_p^0\psi_{k_1+p}^0\right)\tilde{\alpha}_{-k_1}.$$

Moreover, we have

$$\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2}\frac{1}{(k-p)^2} = -\frac{(k^2)^{\frac{d}{2}-2}\pi\Gamma\left(\frac{d}{2}-1\right)}{(4\pi)^{d/2}\sin\left(\frac{\pi d}{2}\right)\Gamma(d-2)} \tag{6.29}$$

$$= \frac{1}{8|k|}, \qquad d=3. \tag{6.30}$$

The $3d$ effective quadratic action can then be written as

$$S_2^{eff} = \frac{1}{4}\int d^dk_1 \int d^dk_2 \tilde{\eta}_{k_1k_2}k_1^2k_2^2\tilde{\eta}_{-k_2,-k_1} + \frac{1}{4}\int d^dk_1\tilde{\alpha}_{k_1}\left(\frac{1}{8|k|}\right)\tilde{\alpha}_{-k_1}. \tag{6.31}$$

The propagators can be read off from the above effective quadratic action. Indeed, we have

$$\langle \tilde{\eta}_{k_1k_2}\tilde{\eta}_{p_1p_2}\rangle = \frac{2}{k_1^2k_2^2}\delta(k_2+p_2)\delta(k_1+p_1)$$
$$= \frac{1}{k_1^2k_2^2}\left(\delta(k_2+p_2)\delta(k_1+p_1)+\delta(k_2+p_2)\delta(k_1+p_1)\right). \tag{6.32}$$

and

$$
\begin{aligned}
\langle\tilde{\alpha}_{k_1}\tilde{\alpha}_{k_2}\rangle &= 2\left[-\frac{\left(k_1^2\right)^{\frac{d}{2}-2}\pi\Gamma\left(\frac{d}{2}-1\right)\delta\left(k_1+k_2\right)}{\left(4\pi\right)^{d/2}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(d-2\right)}\right] \\
&= 16\left|k_1\right|\delta\left(k_1+k_2\right), \qquad d=3.
\end{aligned} \tag{6.33}
$$

This is in agreement with the results obtained in [232].

Let us now consider the space time correlators. In particular, in position space we have

$$
\begin{aligned}
\langle\eta_{x_1x_2}\eta_{x_3x_4}\rangle &= 2\int\frac{d^dk_1}{\left(2\pi\right)^d}\frac{e^{ik_1(x_1-x_4)}}{k_1^2}\int\frac{d^dk_2}{\left(2\pi\right)^d}\frac{e^{ik_2(x_2-x_3)}}{k_2^2} \\
&= \int\frac{d^dk_1}{\left(2\pi\right)^d}\frac{e^{ik_1(x_1-x_4)}}{k_1^2}\int\frac{d^dk_2}{\left(2\pi\right)^d}\frac{e^{ik_2(x_2-x_3)}}{k_2^2}+\left(x_3\leftrightarrow x_4\right).
\end{aligned} \tag{6.34}
$$

We make use of the result that

$$
\int d^dp\left(p^2\right)^{\alpha}e^{ipx} = \gamma_d\alpha\left(x^2\right)^{-\frac{d}{2}-\alpha}, \tag{6.35}
$$

where

$$
\gamma_d = \pi^{d/2}\alpha^{2\alpha+d}\frac{\Gamma\left(\alpha+\frac{d}{2}\right)}{\Gamma\left(-\alpha\right)}. \tag{6.36}
$$

For the special case $\alpha=-1$, we obtain

$$
\int\frac{d^dp}{\left(2\pi\right)^d}\frac{e^{ipx}}{p^2} = \frac{2^{d-2}}{\left(4\pi\right)^{d/2}}\Gamma\left(\frac{d}{2}-1\right)\left(x^2\right)^{1-\frac{d}{2}}. \tag{6.37}
$$

$$
= \frac{1}{4\pi}\frac{1}{|x|}, \qquad d=3. \tag{6.38}
$$

This enables us to write

$$\langle \eta_{x_1x_2}\eta_{x_3x_4}\rangle = \left(\frac{2^{d-2}}{(4\pi)^{d/2}}\Gamma\left(\frac{d}{2}-1\right)\right)^2 \left(\left(x_{13}^2\right)^{1-\frac{d}{2}}\left(x_{24}^2\right)^{1-\frac{d}{2}} + \left(x_{14}^2\right)^{1-\frac{d}{2}}\left(x_{23}^2\right)^{1-\frac{d}{2}}\right)$$
$$\to \left(\frac{1}{4\pi}\right)^2 \left(\frac{1}{\left(x_{13}^2\right)^{1/2}}\frac{1}{\left(x_{24}^2\right)^{1/2}} + \frac{1}{\left(x_{14}^2\right)^{1/2}}\frac{1}{\left(x_{23}^2\right)^{1/2}}\right). \qquad (6.39)$$

Next, using the fact that

$$\int \frac{d^dp}{(2\pi)^d} e^{ipx}\left(p^2\right)^{2-\frac{d}{2}} = \frac{2^{4-d}}{(\pi)^{d/2}}\frac{\left(x^2\right)^{-2}}{\Gamma\left(\frac{d}{2}-2\right)}$$
$$\to -\frac{1}{\pi^2}\frac{1}{\left(x^2\right)^2}, \qquad d=3. \qquad (6.40)$$

we find

$$\langle \tilde{\alpha}_{x_1}\tilde{\alpha}_{x_2}\rangle = 2\left[-\frac{(4\pi)^{d/2}\sin\left(\frac{\pi d}{2}\right)\Gamma(d-2)}{\pi\Gamma\left(\frac{d}{2}-1\right)}\right]\int \frac{d^dk}{(2\pi)^d} e^{ik(x_1-x_2)}\left(k^2\right)^{2-\frac{d}{2}}$$
$$= -2^5\left(\frac{\sin\left(\frac{\pi d}{2}\right)}{\pi}\right)\frac{\Gamma(d-2)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-2\right)}\frac{1}{\left(x_{12}^2\right)^2}$$
$$\to -\frac{16}{\pi^2}\frac{1}{\left(x_{12}^2\right)^2} \qquad d=3. \qquad (6.41)$$

Note that as expected $\Delta = 2$.

A few comments are in order. First, the correlation function appearing in (6.39) is the same for both the free and critical $\left(\phi^2\right)^2$ $O(N)$ vector model. In addition, if we set $x_1 = x_2$ and $x_3 = x_4$ in (6.39), we obtain

$$\langle \eta_{x_1x_1}\eta_{x_3x_3}\rangle = \left(\frac{1}{4\pi}\right)^2\left(\frac{1}{\left(x_{13}^2\right)^{1/2}}\frac{1}{\left(x_{13}^2\right)^{1/2}} + \frac{1}{\left(x_{13}^2\right)^{1/2}}\frac{1}{\left(x_{13}^2\right)^{1/2}}\right)$$
$$= \frac{1}{8\pi^2}\frac{1}{\left(x_{13}^2\right)}. \qquad (6.42)$$

From this expression we can easily read off the conformal dimension is $\Delta = 1$. Second, we have learnt that the critical $O(N)$ vector model, which we are describing here with the non-linear sigma model, is not simply described by the scaling dimension being two. What we are seeing is that the critical theory consists of a state which is identical to the free theory plus an additional state. We will return to these issues in later chapters.

Chapter 7

$\left(\phi^2\right)^2$ Two-time Bilocal Description

The best way to understand something is to first start out confused.

-Ahmed Almheiri.

We begin by making a simple observation. Let us consider the free scalar field theory with a Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2}\phi^2. \tag{7.1}$$

The Euler-Lagrange equations are

$$\begin{aligned} 0 &= -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) + \frac{\partial \mathcal{L}}{\partial \phi} \\ &= \left(-\partial^2 + m^2 \right) \phi. \end{aligned} \tag{7.2}$$

which is the Klein-Gordon equation.

The action for the free scalar field can be written as

$$S = \frac{1}{2}\int d^d x \phi \left(-\partial^2 + m^2\right)\phi$$
$$= \frac{1}{2}\int d^d x \phi \hat{O}\phi, \tag{7.3}$$

where $\hat{O} = -\partial^2 + m^2$.

In addition, the propagator for the free scalar field is

$$G(p) = \frac{1}{p^2 + m^2}. \tag{7.4}$$

The observation we make is that the classical equations of motion can be obtained by requiring that

$$\hat{O}\phi = \left(-\partial^2 + m^2\right)\phi = 0. \tag{7.5}$$

Moreover, note that the on-shell condition - *i.e.* $p^2 = m^2$ - can be obtained by looking at the poles of the propagator.

In this chapter, we will make analogous statements for the $O(N)$ critical vector model.

7.1 Collective Field Propagator

In this section, we obtain the bilocal propagator for the critical $O(N)$ vector model.

The effective action for the critical $O(N)$ vector model reads

$$S_{eff} = N\int d^d x\left[\frac{1}{2}\left(-\int d^d y \delta(x-y)\partial_y^2\psi_{xy}\right) - \frac{1}{2}m^2\psi_{xx} - \frac{\lambda}{4!}\left(\psi_{xx}\right)^2\right] - \frac{Ni}{2}\mathrm{Tr}\ln\psi. \tag{7.6}$$

The saddle-point equation gave us an expression for the background field (3.43):

$$\psi_k^0 = \frac{i}{k^2}. \tag{7.7}$$

Our intention now is to expand the bilocals about this large-N background configuration, and extract the effective quadratic action. More specifically, we write

$$\psi_{xy} = \psi_{xy}^0 + \frac{1}{\sqrt{N}}\eta_{xy}. \tag{7.8}$$

The log term in the effective action can be expanded as[1]

$$\begin{aligned} \operatorname{Tr}\ln\left(\psi^0 + \frac{1}{\sqrt{N}}\eta\right) &= \operatorname{Tr}\ln\left[\psi^0\left(1 + \frac{1}{\sqrt{N}}\left(\psi^0\right)^{-1}\eta\right)\right] \\ &= \operatorname{Tr}\ln\psi^0 + \operatorname{Tr}\left(\frac{1}{\sqrt{N}}\left(\psi^0\right)^{-1}\eta\right) + \operatorname{Tr}\left[-\frac{1}{2N}\left(\psi^0\right)^{-1}\eta\left(\psi^0\right)^{-1}\eta\right] + \cdots \end{aligned} \tag{7.9}$$

Thus, the quadratic effective action reads

$$\begin{aligned} S_2^{eff} &= \frac{i}{4}\operatorname{Tr}\left(\left(\psi^0\right)^{-1}\eta\left(\psi^0\right)^{-1}\eta\right) - \frac{\lambda}{4!}\int d^dx\eta_{xx}^2 \\ &= \frac{i}{4}\int d^dx_1d^dx_2d^dx_3d^dx_4\left(\psi^0\right)^{-1}_{x_1x_2}\eta_{x_2x_3}\left(\psi^0\right)^{-1}_{x_3x_4}\eta_{x_4x_1} - \frac{\lambda}{4!}\int d^dx\eta_{xx}^2. \end{aligned} \tag{7.10}$$

We move into momentum space and write the fluctuations as (recall that our signature is $(+,-,\cdots,-)$):

$$\eta_{xy} = \int\frac{d^dk_1}{(2\pi)^{d/2}}\int\frac{d^dk_2}{(2\pi)^{d/2}}e^{-ik_1x-ik_2y}\eta_{k_1k_2}. \tag{7.11}$$

and the translationally invariant background bilocal as

$$\psi_{0xy}^{-1} = \int\frac{d^dk_1}{(2\pi)^d}e^{ik_1(x-y)}\psi_{0k_1}^{-1}. \tag{7.12}$$

[1]We use the trivial fact that $\ln(1+x) = x - \frac{x^2}{2} + \ldots$.

Plugging (7.11) and (7.12) back into (7.10), we get

$$
\begin{aligned}
S_2^{eff} &= \frac{i}{4}\int d^dx_1 d^dx_2 d^dx_3 d^dx_4 \left(\psi^0\right)^{-1}_{x_1x_2}\eta_{x_2x_3}\left(\psi^0\right)^{-1}_{x_3x_4}\eta_{x_4x_1} - \frac{\lambda}{4!}\int d^dx\eta^2_{xx} \\
&= \frac{i}{4}\int d^dx_1 d^dx_2 d^dx_3 d^dx_4 \int \frac{d^dk_1}{(2\pi)^d} e^{ik_1(x_1-x_2)}\psi^{0^{-1}}_{k_1}\int \frac{d^dk_2}{(2\pi)^{d/2}}\int \frac{d^dk_3}{(2\pi)^{d/2}} e^{-ik_2x_2-ik_3x_3}\eta_{k_2k_3} \\
&\quad\times\int \frac{d^dk_4}{(2\pi)^d} e^{ik_4(x_3-x_4)}\psi^{0^{-1}}_{k_4}\int \frac{d^dk_5}{(2\pi)^{d/2}}\int \frac{d^dk_6}{(2\pi)^{d/2}} e^{-ik_5x_4-ik_6x_1}\eta_{k_5k_6} - \frac{\lambda}{4!}\int d^dx \\
&\quad\times\int \frac{d^dk_1}{(2\pi)^{d/2}}\int \frac{d^dk_2}{(2\pi)^{d/2}} e^{-ik_1x-ik_2x}\eta_{k_1k_2}\int \frac{d^dk_3}{(2\pi)^{d/2}}\int \frac{d^dk_4}{(2\pi)^{d/2}} e^{-ik_3x-ik_4x}\eta_{k_3k_4} \qquad (7.13)
\end{aligned}
$$

After some simple manipulations, the quadratic action simplifies to

$$
\begin{aligned}
S_2^{eff} &= \frac{i}{4}\int \frac{d^dk_1}{(2\pi)^{2d}}\int d^dk_2\int d^dk_3\int d^dk_4\eta_{k_2k_3}\psi^{0^{-1}}_{k_1}\psi^{0^{-1}}_{k_4} \\
&\quad\times\delta\left(-k_1-k_2\right)\delta\left(-k_3-k_4\right)\eta_{k_4k_1} - \frac{\lambda}{4!}\int \frac{d^dk_1}{(2\pi)^d}\int d^dk_2\int d^dk_3\int d^dk_4\eta_{k_2k_3}\eta_{k_1k_2} \\
&\quad\times\delta\left(k_1+k_2+k_3+k_4\right)\eta_{k_3k_4} \\
&= \frac{1}{2}\int d^dk_1\int d^dk_2\int d^dk_3\int d^dk_4\eta_{k_1k_2}\hat{O}_{k_1k_2;k_3k_4}\eta_{k_3k_4}. \qquad (7.14)
\end{aligned}
$$

where in order to arrive at the last line we have interchanged $k_1 \leftrightarrow k_3$ in the first term on the RHS of the quadratic action and the operator $\hat{O}_{k_1k_2;k_3k_4}$ is defined as

$$
\hat{O}_{k_1k_2;k_3k_4} = \frac{i}{2}\psi^{0^{-1}}_{k_3}\psi^{0^{-1}}_{k_4}\delta\left(k_2+k_3\right)\delta\left(k_1+k_4\right) - \frac{2\lambda}{4!}\frac{1}{(2\pi)^d}\delta\left(k_1+k_2+k_3+k_4\right). \qquad (7.15)
$$

Our goal is to invert the operator $\hat{O}_{k_1k_2;k_3k_4}$. More specifically, we need to find some operator $\hat{O}^{-1}_{k_3k_4;p_1p_2}$ such that

$$
\int d^dk_3\int d^dk_4\hat{O}_{k_1k_2;k_3k_4}\hat{O}^{-1}_{k_3k_4;p_1p_2} = i\delta\left(k_1-p_1\right)\delta\left(k_2-p_2\right). \qquad (7.16)
$$

For our ansatz, we will take:[2]

$$\hat{O}^{-1}_{p_1p_2;p_3p_4} = A\left(p_1,p_2\right)\delta\left(p_1+p_4\right)\delta\left(p_2+p_3\right) + G\left(p_1,p_2,p_3,p_4\right)\delta\left(p_1+p_2+p_3+p_4\right). \tag{7.17}$$

When we plug this ansatz into (7.16), we get

$$\begin{aligned}\int d^dk_3\int d^dk_4\hat{O}_{k_1k_2;k_3k_4}\hat{O}^{-1}_{k_3k_4;p_1p_2} &= \int d^dk_3\int d^dk_4\Bigg[\frac{i}{2}\psi^{0^{-1}}_{k_3}\psi^{0^{-1}}_{k_4}\times\delta\left(k_3+k_2\right)\delta\left(k_1+k_4\right)\\ &-\frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^d}\delta\left(k_1+k_2+k_3+k_4\right)\Bigg]\times\Bigg[A\left(k_3,k_4\right)\delta\left(k_3+p_2\right)\delta\left(k_4+p_1\right)\\ &+G\left(k_3,k_4,p_1,p_2\right)\delta\left(k_3+k_4+p_1+p_2\right)\Bigg] = -\frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^d}\delta\left(k_1+k_2-p_2-p_1\right)A\left(-p_2,-p_1\right)\\ &-\frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^d}\delta\left(k_1+k_2-p_2-p_1\right)\int d^dk_3G\left(k_3,-k_1-k_2-k_3,p_1,p_2\right)\\ &+\frac{i}{2}\psi^{0^{-1}}_{-k_1}\psi^{0^{-1}}_{-k_2}A\left(-k_2,-k_1\right)\delta\left(p_1-k_1\right)\delta\left(p_2-k_2\right)\\ &+\frac{i}{2}\psi^{0^{-1}}_{-k_1}\psi^{0^{-1}}_{-k_2}G\left(-k_2,-k_1,p_1,p_2\right)\delta\left(-k_1-k_2+p_2+p_1\right).\end{aligned} \tag{7.18}$$

By comparing (7.16) and (7.18), one can conclude that

$$A\left(-k_2,-k_1\right) = 2\psi^0_{-k_2}\psi^0_{-k_1}, \tag{7.19}$$

and

$$\begin{aligned}G\left(-k_2,-k_1,p_1,p_2\right) &= -\frac{8\lambda i}{4!}\frac{1}{\left(2\pi\right)^d}\psi^0_{-k_1}\psi^0_{-k_2}\psi^0_{-p_1}\psi^0_{-p_2}\\ &-\frac{4\lambda i}{4!}\frac{1}{\left(2\pi\right)^d}\psi^0_{-k_1}\psi^0_{-k_2}\int d^dk_3G\left(k_3,-k_1-k_2-k_3,p_1,p_2\right)\end{aligned} \tag{7.20}$$

where the whole equation sits inside the delta function: $\delta\left(-k_1-k_2+p_2+p_1\right)$.

[2]This has been explained in [226].

We can make use of the standard technique of iteration, which underlies the Bethe-Salpeter equation, in order to solve the integral equation given in (7.20). However, we will opt for the simpler method where we first attempt to turn (7.20) into an algebraic equation.

First, define

$$\int d^dk_3 G\left(k_3, -k_1 - k_2 - k_3, p_1, p_2\right) = \int d^dk_3 \int d^dk_4 G\left(k_3, k_4, p_1, p_2\right) \\ \times \delta\left(k_3 + k_4 + p_2 + p_1\right) \stackrel{\text{def}}{=} \alpha\left(p_2, p_1\right). \tag{7.21}$$

It then becomes clear that we first need to multiply (7.20) through by $\delta\left(k_1 + k_2 + p_1 + p_2\right)$ and then integrate over k_1 and k_2. In fact, when we do this we find that

$$\alpha\left(p_2, p_1\right) = -\frac{8\lambda i}{4!}\frac{1}{\left(2\pi\right)^d}\alpha\left(p_2, p_1\right)\int d^dk_1 \int d^dk_2 \delta\left(k_1 + k_2 - p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2} \\ - \frac{4\lambda i}{4!}\frac{1}{\left(2\pi\right)^d}\alpha\left(p_2, p_1\right)\int d^dk_1 \int d^dk_2 \delta\left(k_1 + k_2 - p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2}. \tag{7.22}$$

The equation above is trivial to solve and we obtain:

$$\alpha\left(p_2, p_1\right) = \frac{-\frac{8\lambda i}{4!}\frac{1}{(2\pi)^d}\psi^0_{-p_1}\psi^0_{-p_2}\int d^dk_1 \int d^dk_2 \delta\left(k_1 + k_2 - p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2}}{1 + \frac{4\lambda i}{4!}\frac{1}{(2\pi)^d}\int d^dk_1 \int d^dk_2 \delta\left(k_1 + k_2 - p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2}}. \tag{7.23}$$

Inserting this back into (7.20) yields:

$$G\left(-k_2, -k_1, p_1, p_2\right) = -\frac{8\lambda i}{4!}\frac{1}{\left(2\pi\right)^d}\psi^0_{-k_1}\psi^0_{-k_2}\psi^0_{-p_1}\psi^0_{-p_2} - \frac{4\lambda i}{4!}\frac{1}{\left(2\pi\right)^d}\psi^0_{-k_1}\psi^0_{-k_2}\alpha\left(p_2, p_1\right) \\ \times \frac{\frac{-8\lambda i}{4!}\frac{1}{(2\pi)^d}\psi^0_{-p_1}\psi^0_{-p_2}\int d^dk_1 \int d^dk_2 \delta\left(k_1 - k_2 + p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2}}{1 + \frac{4\lambda i}{4!}\frac{1}{(2\pi)^d}\int d^dk_1 \int d^dk_2 \delta\left(k_1 + k_2 - p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2}} \\ = \frac{-\frac{8\lambda i}{4!}\left(2\pi\right)^{-d}\psi^0_{-k_1}\psi^0_{-k_2}\psi^0_{-p_1}\psi^0_{-p_2}}{1 + \frac{4\lambda i}{4!}\frac{1}{(2\pi)^d}\int d^dk_1 \int d^dk_2 \delta\left(k_1 + k_2 - p_1 - p_2\right)\psi^0_{-k_1}\psi^0_{-k_2}}. \tag{7.24}$$

Therefore, the effective propagator is given by

$$\boxed{\begin{aligned}\hat{O}^{-1}_{k_1k_2;p_1p_2} &= 2\psi^0_{-p_1}\psi^0_{-p_2}\delta\left(k_1+p_2\right)\delta\left(k_2+p_1\right)\\ &+\frac{-\frac{8\lambda i}{4!}\frac{1}{(2\pi)^d}\psi^0_{-k_1}\psi^0_{-k_2}\psi^0_{-p_1}\psi^0_{-p_2}}{1+\frac{4\lambda i}{4!}\frac{1}{(2\pi)^d}\int d^dk_1\int d^dk_2\delta\left(k_1+k_2-p_1-p_2\right)\psi^0_{k_1}\psi^0_{k_2}}\delta\left(k_1+k_2+p_1+p_2\right).\end{aligned}} \quad (7.25)$$

The integral appearing in the denominator, when $d=3$, yields [208]:

$$\begin{aligned} iI &= \frac{i}{(2\pi)^d}\int d^dk_1\psi^0_{k_1}\psi^0_{k_1-p_1-p_2}\\ &= \frac{1}{(2\pi)^3}\int d^3k_{1E}\frac{1}{k^2_{1E}}\frac{1}{(k_1-p_1-p_2)^2_E}\\ &= \frac{1}{8}\frac{1}{|p_1+p_2|}. \end{aligned} \quad (7.26)$$

Hence,

$$\boxed{\begin{aligned}\hat{O}^{-1}_{k_1k_2;p_1p_2} &= 2\psi^0_{-p_1}\psi^0_{-p_2}\delta\left(k_1+p_2\right)\delta\left(k_2+p_1\right)\\ &+\frac{-\frac{8\lambda i}{4!}\frac{1}{(2\pi)^3}\psi^0_{-k_1}\psi^0_{-k_2}\psi^0_{-p_1}\psi^0_{-p_2}}{1+\frac{4\lambda}{4!}\frac{1}{8|p_1+p_2|}}\delta\left(k_1+k_2+p_1+p_2\right).\end{aligned}} \quad (7.27)$$

Therefore, at the critical point - *i.e.* when we take the limit $\lambda\to\infty$ - the connected part of (7.27) becomes:

$$G_{conn}\left(k_2k_1;p_1p_2\right)\sim\frac{|p_1+p_2|}{(2\pi)^3}, \quad (7.28)$$

and

$$
\hat{O}^{-1}_{k_1k_2;p_1p_2} = -2\frac{1}{p_1^2}\frac{1}{p_2^2}\delta\left(k_1+p_2\right)\delta\left(k_2+p_1\right)
$$
$$
-\frac{16i\left|p_1+p_2\right|}{\left(2\pi\right)^3}\frac{1}{k_1^2}\frac{1}{k_2^2}\frac{1}{p_1^2}\frac{1}{p_2^2}\delta\left(k_1+k_2+p_1+p_2\right). \tag{7.29}
$$

This agrees with (6.32) and (6.33) up to leg factors in the $\Delta = 2$ channel.

Accordingly, in $3d$ coordinate space:

$$
\begin{aligned}
G\left(x-y\right) &= -\int \frac{d^3p}{\left(2\pi\right)^3} e^{ip(x-y)}\left(p^2\right)^{1/2} \\
&= -\frac{2^{3+1}\pi^{3/2}\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}\left(x-y\right)^{-2\times\frac{1}{2}-3} \\
&= \frac{C}{\left(x-y\right)^{2\cdot 2}}.
\end{aligned} \tag{7.30}
$$

i.e. $\Delta = 2$ which is the same result as the one obtained for the non-linear sigma model.

Note that the two-time bilocal propagator is made up of two parts *viz.* the part that we obtain in the free case and a bound state with scaling dimension of two.

For finite λ, the pole condition is:

$$
1 = -\frac{4i\lambda}{4!}\frac{1}{\left(2\pi\right)^d}\int d^dk_1\psi^0_{k_1}\psi^0_{k_1-p_1-p_2}. \tag{7.31}
$$

Note that the integral appearing in (7.31) is written in Minkowski space. We need to express the integral in terms of an integral in Euclidean space. This is easily done by putting $E_M = iE_E$. Then the integral above yields

$$
\begin{aligned}
\int \frac{d^d k}{(2\pi)^d} \psi^0_{k_1} \psi^0_{k_1-p_1-p_2} &= \int \frac{d^d k}{(2\pi)^d} \frac{i}{k_M^2} \frac{i}{(k-p_1-p_2)_M^2} \\
&= -i \int dE_E \int d^{d-1} k_E \frac{1}{(-k_E^2)} \frac{1}{\left(-(k-p_1-p_2)_E^2\right)} \\
&= -i \int dE_E \int d^{d-1} k_E \frac{1}{k_E^2} \frac{1}{(k-p_1-p_2)_E^2},
\end{aligned}
\tag{7.32}
$$

where we have made use of the fact that $k_M^2 = -k_E^2$.

Since

$$
\begin{aligned}
\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2} \frac{1}{(k-p_1-p_2)_E^2} &= \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(2-d/2)}{\Gamma(d-2)} (p_1+p_2)^{d/2-1} \\
&\to \frac{1}{8} \frac{1}{|p_1+p_2|}, \qquad d=3.
\end{aligned}
\tag{7.33}
$$

Then it follows that in $3d$ (7.31) becomes

$$
\begin{aligned}
1 &= -\frac{4i\lambda}{4!} \left(-i \int \frac{d^d k}{(2\pi)^d} \frac{1}{k_E^2} \frac{1}{(k-p_1-p_2)_E^2} \right) \\
&= -\frac{\lambda}{48} \frac{1}{|p_1+p_2|}.
\end{aligned}
\tag{7.34}
$$

That is,

$$
\boxed{|p_1+p_2| = -\frac{\lambda}{48}.}
\tag{7.35}
$$

or

$$E^2 - (\vec{p}_1 + \vec{p}_2)^2 = -\frac{\lambda^2}{48^2}. \tag{7.36}$$

This signals the appearance of a tachyon on the CFT side of the duality.

Note that on the bulk gravity side masses can be negative due to the fact that this does not lead to any instabilities. This effect arises because the gravitational contribution will stabilize the negative mass squared particle in AdS. In fact, the Breitenlohner-Freedman bound [233, 234] states that in Anti-de Sitter spaces

$$m^2 R^2 \geq -\frac{d^2}{4}. \tag{7.37}$$

The tachyon can be seen as a hint that our large-N configuration is not stable.

7.2 Equations Of Motion And Pole Condition

In the previous subsection, we found that the connected Green's function is given by

$$G(-k_2, -k_1, p_1, p_2) = \frac{-\frac{8\lambda i}{4!}\frac{1}{(2\pi)^d}\psi^0_{-k_1}\psi^0_{-k_2}\psi^0_{-p_1}\psi^0_{-p_2}}{1 + \frac{4\lambda i}{4!}\frac{1}{(2\pi)^d}\int d^dk_1 \int d^dk_2 \delta(k_1 + k_2 - p_1 - p_2)\psi^0_{k_1}\psi^0_{k_2}}, \tag{7.38}$$

where the large-N background configuration is given by

$$\psi^0_k = \frac{i}{k^2}. \tag{7.39}$$

We obtained the spectrum by looking at the poles of the two-point function. In other words, the spectrum for the $O(N)$ vector model follows from

$$0 = 1 + \frac{4\lambda i}{4!}\frac{1}{(2\pi)^d}\int d^dk_1 \int d^dk_2 \delta(k_1 + k_2 - p_1 - p_2)\psi^0_{k_1}\psi^0_{k_2} \tag{7.40}$$

or

$$1 = \frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\int d^d k_1 \frac{1}{k^2}\frac{1}{(k-p_1-p_2)^2}. \tag{7.41}$$

We now confirm that this pole/mass condition for the bound state follows also follows from the homogeneous condition:

$$0 = \int d^d k_3 \int d^d k_4 \hat{O}_{k_1 k_2; k_3 k_4} \eta_{k_3 k_4}, \tag{7.42}$$

where

$$\hat{O}_{k_1 k_2; k_3 k_4} = \frac{i}{2}\psi^{0^{-1}}_{k_3}\psi^{0^{-1}}_{k_4}\delta(k_3+k_2)\,\delta(k_1+k_4) - \frac{2\lambda}{4!}\frac{1}{(2\pi)^d}\delta(k_1+k_2+k_3+k_4) \tag{7.43}$$

As usual we write

$$\eta_{xy} = \int \frac{d^d k_1}{(2\pi)^{d/2}}\int \frac{d^d k_2}{(2\pi)^{d/2}} e^{-ik_1 x} e^{-ik_2 y}\eta_{k_1 k_2}. \tag{7.44}$$

Hence,

$$\begin{aligned} 0 &= \frac{i}{2}\int d^d k_3 \int d^d k_4 \psi^{0^{-1}}_{k_3}\psi^{0^{-1}}_{k_4}\delta(k_3+k_2)\,\delta(k_1+k_4)\,\eta_{k_3 k_4} - \frac{2\lambda}{4!}\frac{1}{(2\pi)^d} \\ &\qquad \times \int d^d k_3 \int d^d k_4 \delta(k_1+k_2+k_3+k_4)\,\eta_{k_3 k_4} \\ &= \frac{i}{2}\psi^{0^{-1}}_{-k_2}\psi^{0^{-1}}_{-k_1}\eta_{-k_2,-k_1} - \frac{2\lambda}{4!}\frac{1}{(2\pi)^d}\int d^d k_3 \int d^d k_4 \delta(k_1+k_2+k_3+k_4)\,\eta_{k_3 k_4}, \end{aligned} \tag{7.45}$$

which implies that

$$\eta_{-k_2,-k_1} = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\psi^0_{-k_2}\psi^0_{-k_1}\int d^dk_3\int d^dk_4\delta\left(k_1+k_2+k_3+k_4\right)\eta_{k_3k_4}.$$
$$= -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\psi^0_{-k_2}\psi^0_{-k_1}\int d^dk\eta_{k,-k_1-k_2-k} \tag{7.46}$$

or equivalently

$$\boxed{\eta_{k_1k_2} = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\psi^0_{k_1}\psi^0_{k_2}\int d^dk\eta_{k,k_1+k_2-k}.} \tag{7.47}$$

Multiplying 7.47 by $\delta\left(k_2+k_1-p_2-p_1\right)$ and integrating over k_1 and k_2 yields

$$\int d^dk_1\int d^dk_2\delta\left(k_2+k_1-p_2-p_1\right)\eta_{k_1k_2} = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}$$
$$\times\int d^dk_1\int d^dk_2\delta\left(k_2+k_1-p_2-p_1\right)\psi^0_{k_2}\psi^0_{k_1}\int d^dk_3\int d^dk_4\delta\left(k_1+k_2-k_3-k_4\right)\eta_{k_3k_4}$$
$$= -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\int d^dk_1\psi^0_{-k_1}\psi^0_{k_1-p_1-p_2}\int d^dk_3\int d^dk_4\delta\left(k_3+k_4-p_2-p_1\right)\eta_{k_3k_4}. \tag{7.48}$$

Define

$$A\left(p_1,p_2\right) = \int d^dk_1\int d^dk_2\delta\left(k_2+k_1-p_2-p_1\right)\eta_{k_1k_2} \tag{7.49}$$

Then (7.48) becomes

$$A\left(p_1,p_2\right) = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\int d^dk_1\psi^0_{k_1}\psi^0_{k_1-p_1-p_2}A\left(p_1,p_2\right) \tag{7.50}$$

or

$$1 = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\int d^d k_1 \psi^0_{k_1}\psi^0_{k_1-p_1-p_2}. \tag{7.51}$$

This is the pole condition for the bound state that we found previously in (7.34).

The solution to (7.47) is given by

$$\eta_{k_1 k_2} = \frac{b\,(k_1+k_2)}{k_1^2 k_2^2}, \tag{7.52}$$

where b is an arbitrary constant. Indeed, when we plug this into the R.H.S. of (7.47) we obtain:

$$\begin{aligned}
-\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\psi^0_{k_2}\psi^0_{k_1}\int d^d k_3 \int d^d k_4 \delta\,(k_1+k_2-k_3-k_4)\,\eta_{k_3 k_4} = -\frac{4i\lambda}{4!} \\
\times \frac{1}{(2\pi)^d}\psi^0_{k_2}\psi^0_{k_1}\int d^d k_3 \int d^d k_4 \delta\,(k_1+k_2-k_3-k_4)\left(\frac{b\,(k_3+k_4)}{k_3^2 k_4^2}\right) \\
= -\frac{4i\lambda}{4!}\frac{i}{k_2^2}\frac{i}{k_1^2}\frac{b\,(k_1+k_2)}{1} \times \frac{i}{8}\frac{1}{\left((k_1+k_2)^2\right)^{1/2}} \\
= -\frac{\lambda}{48\left((k_1+k_2)^2\right)^{1/2}}\frac{b\,|k_1+k_2|}{k_1^2 k_2^2} \\
= \frac{b\,|k_1+k_2|}{k_1^2 k_2^2} = \eta_{k_1,k_2}.
\end{aligned} \tag{7.53}$$

where in order to arrive at the last line we had to make use of the pole condition *i.e.* (7.34).

In summary, we have obtained the two-time bilocal propagator. The propagator consists of a term identical to that of the free theory plus a bound state term that has a scaling dimension of two in the IR limit. In this limit, it is similar to that of the non-linear sigma model. Moreover, we showed that the bound state spectrum can be obtained by looking at the poles of the connected propagator. This same pole condition was then shown to arise from the homogeneous equation.

Chapter 8

$\left(\phi^2\right)^2$ Single-Time Bilocal Description

If you really want to contribute to our theoretical understanding of physical laws - and it is an exciting experience if you succeed! - there are many things you need to know. First of all, be serious about it!

-Gerard 't Hooft.

8.1 Hamiltonian Equations Of Motion

The collective field theory Hamiltonian can be written as (4.5):

$$H = 2\mathrm{Tr}\left(\Pi\psi\Pi\right) + \frac{N^2}{8}\mathrm{Tr}\psi^{-1} + \int d^{d-1}x \left(\left(-\frac{1}{2}\lim_{x\to y}\partial^2\psi_{xy}\right) - \frac{1}{2}m^2\psi_{xx} - \frac{g}{4!}\left(\psi_{xx}\right)^2\right). \tag{8.1}$$

We then introduce fluctuations via

$$\psi = \psi^0 + \frac{1}{\sqrt{N}}\eta. \tag{8.2}$$

In addition, it is clear that

$$\psi^{-1} = \psi_0^{-1}\left(1 + \frac{1}{\sqrt{N}}\psi_0^{-1}\eta\right)^{-1}$$
$$= \psi_0^{-1}\left(1 - \frac{1}{\sqrt{N}}\psi_0^{-1}\eta + \frac{1}{N}\psi_0^{-1}\eta\psi_0^{-1}\eta + \cdots\right). \tag{8.3}$$

The quadratic Hamiltonian then reads:

$$H^{(2)} = 2\mathrm{Tr}\left(p\psi^0 p\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) + \frac{\lambda}{4!}\int d^{d-1}x\eta_{xx}^2. \tag{8.4}$$

The trace appearing in the Hamiltonian is defined in the functional sense. More precisely,

$$\mathrm{Tr}\left(A\right) = \int d^{d-1}xA\left(x,x\right). \tag{8.5}$$

We can then write the quadratic Hamiltonian as

$$\begin{aligned} H^{(2)} &= 2\mathrm{Tr}\left(p\psi^0 p\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) + \frac{\lambda}{4!}\int d^{d-1}x\eta_{xx}^2 \\ &= 2\int d^{d-1}x_1\int d^{d-1}x_2\int d^{d-1}x_3\int d^{d-1}x_4 p_{x_1x_2}\psi^0_{x_2x_3}p_{x_3x_1} \\ &+ + \frac{1}{8}\int d^{d-1}x_1\int d^{d-1}x_2\int d^{d-1}x_3\int d^{d-1}x_4 \\ &\times \int d^{d-1}x_5\psi^{0^{-1}}_{x_1x_2}\eta_{x_2x_3}\psi^{0^{-1}}_{x_3x_4}\eta_{x_4x_5}\psi^{0^{-1}}_{x_5x_1} + \frac{\lambda}{4!}\int d^{d-1}x\eta_{xx}^2. \end{aligned} \tag{8.6}$$

Recall that Hamilton's equations are given by

$$\dot{p}_i = -\frac{\delta H}{\delta q_i} \tag{8.7}$$

$$\dot{q}_i = \frac{\delta H}{\delta p_i}. \tag{8.8}$$

Accordingly, the equations of motion for our system are

$$\begin{aligned}
\dot{\eta}_{xy} = \frac{\delta H}{\delta p_{xy}} \\
&= 2\int d^{d-1}x_1 \int d^{d-1}x_2 \int d^{d-1}x_3 \psi^0_{x_2x_3} p_{x_3x_1} \delta\left(x - x_1\right) \\
&\times \delta\left(y - x_2\right) + 2\int d^{d-1}x_1 \int d^{d-1}x_2 \int d^{d-1}x_3 p_{x_1x_2} \psi^0_{x_2x_3} \\
&\times \delta\left(x - x_3\right)\delta\left(x - x_1\right) \\
&= 2\int d^{d-1}x' \left(\psi^0_{yx'} p_{x'x} + p_{yx'}\psi^0_{x'x}\right),
\end{aligned} \tag{8.9}$$

and

$$\begin{aligned}
\dot{p}_{xy} &= -\frac{\delta H}{\delta \eta_{xy}} \\
&= -\frac{1}{8}\int d^{d-1}x_1 \int d^{d-1}x_2 \int d^{d-1}x_3 \int d^{d-1}x_4 \int d^{d-1}x_5 \psi^{0-1}_{x_1x_2} \psi^{0-1}_{x_3x_4} \eta_{x_4x_5} \psi^{0-1}_{x_5x_1} \delta\left(x - x_2\right)\delta\left(y - x_3\right) \\
&-\frac{1}{8}\int d^{d-1}x_1 \int d^{d-1}x_2 \int d^{d-1}x_3 \int d^{d-1}x_4 \int d^{d-1}x_5 \psi^{0-1}_{x_1x_2} \eta_{x_2x_3} \psi^{0-1}_{x_3x_4} \psi^{0-1}_{x_5x_1} \delta\left(x - x_4\right)\delta\left(y - x_5\right) \\
&+ \frac{\lambda}{12}\eta_{xx}\delta\left(x - y\right) \\
&= -\frac{1}{8}\int d^{d-1}x_1 \int d^{d-1}x_2 \int d^{d-1}x_3 \left[\psi^{0-1}_{yx_1} \eta_{x_1x_2} \psi^{0-1}_{x_2x_3} \psi^{0-1}_{x_3x} + \psi^{0-1}_{yx_1} \psi^{0-1}_{x_1x_2} \eta_{x_2x_3} \psi^{0-1}_{x_3x}\right] \\
&+ \frac{\lambda}{12}\eta_{xx}\delta\left(x - y\right)
\end{aligned} \tag{8.10}$$

where we have used the $x \leftrightarrow y$ symmetry.

To summarize, the equations of motion are:

$$\dot{\eta}_{xy} = 2\int d^{d-1}x'\left(\psi^0_{yx'}p_{x'x} + p_{yx'}\psi^0_{x'x}\right)$$

$$\dot{p}_{xy} = -\frac{1}{8}\int d^{d-1}x_1\int d^{d-1}x_2\int d^{d-1}x_3\left[\psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x_3}\psi^{0^{-1}}_{x_3x} + \psi^{0^{-1}}_{yx_1}\psi^{0^{-1}}_{x_1x_2}\eta_{x_2x_3}\psi^{0^{-1}}_{x_3x}\right] + \frac{\lambda}{12}\eta_{xx}\delta\left(x-y\right). \quad (8.11)$$

Schematically, the equations of motion can be written as:

$$\dot{\eta} = 2\left(\psi^0 p + p\psi^0\right) \quad (8.12)$$

$$\dot{p} = -\frac{1}{8}\left(\psi^{0^{-1}}\eta\psi^{0^{-1}}\psi^{0^{-1}} + \psi^{0^{-1}}\psi^{0^{-1}}\eta\psi^{0^{-1}}\right) + \frac{\lambda}{12}\eta\delta. \quad (8.13)$$

Once the equations have been written in this form, it becomes clear what the strategy to decouple the set of differential equations is. More specifically, we will need to differentiate (8.12) with respect to time. This will then introduce terms involving $\dot{p}$ into (8.12). These terms can then be eliminated by making use of (8.13). The final expression will be an equation involving only η. That is,

$$\ddot{\eta} = 2\left(\psi^0\dot{p} + \dot{p}\psi^0\right) = -\frac{1}{4}\left[\psi^0\psi^{0^{-1}}\eta\psi^{0^{-1}}\psi^{0^{-1}} + \psi^0\psi^{0^{-1}}\psi^{0^{-1}}\eta\psi^{0^{-1}}\right] + \frac{\lambda}{6}\left(\psi^0\eta\delta\right) - \frac{1}{4}\left[\psi^{0^{-1}}\eta\psi^{0^{-1}}\psi^{0^{-1}}\psi^0 + \psi^{0^{-1}}\psi^{0^{-1}}\eta\psi^{0^{-1}}\psi^0\right] + \frac{\lambda}{6}\left(\eta\delta\psi^0\right). \quad (8.14)$$

which can be simplified to:

$$\ddot{\eta} = -\frac{1}{4}\left[\eta\psi^{0^{-1}}\psi^{0^{-1}} + \psi^{0^{-1}}\eta\psi^{0^{-1}} + \psi^{0^{-1}}\eta\psi^{0^{-1}} + \psi^{0^{-1}}\psi^{0^{-1}}\eta\right] + \frac{\lambda}{6}\left(\eta\delta\psi^0\right) + \frac{\lambda}{6}\left(\psi^0\eta\delta\right). \quad (8.15)$$

The fluctuations, in terms of the Fourier transform, can be written as

$$\eta_{xy} = \int \frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{ik_1x_1 - ik_2x_2} \eta_{k_1k_2} \quad (8.16)$$

Therefore, the second order homogeneous equation for the fluctuations reads:

$$\begin{aligned}
-\int \frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{ik_1x_1 - ik_2x_2} \eta_{k_1k_2} E^2 &= \ddot{\eta}_{xy} \\
&= -\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \Big[\eta_{yx_1}\psi^{0^{-1}}_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x} \\
&\qquad + \psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\psi^{0^{-1}}_{x_1x_2}\eta_{x_2x}\Big] \\
&+ \frac{\lambda}{6}\int d^{d-1}x_1 \left(\eta_{x_1x_1}\delta\left(y-x\right)\psi^0_{x_1x} + \psi^0_{yx_1}\eta_{x_1x_1}\delta\left(x_1-x\right)\right) \\
&= -\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \Big[\eta_{yx_1}\psi^{0^{-1}}_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x} \\
&\qquad + \psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\psi^{0^{-1}}_{x_1x_2}\eta_{x_2x}\Big] \\
&\qquad + \frac{\lambda}{6}\eta_{yy}\psi^0_{yx} + \frac{\lambda}{6}\psi^0_{yx}\eta_{xx}. \quad (8.17)
\end{aligned}$$

Recall that the translationally invariant transform of the background bilocal is given by

$$\psi^0_{xy} = \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}} e^{ik(x-y)}\psi^0_k. \quad (8.18)$$

and for the fluctuations:

$$\eta_{xy} = \int \frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{ik_1x - ik_2y} \eta_{k_1k_2}. \tag{8.19}$$

First, let us consider the term that is quadratic in $\psi^{0^{-1}}$. In momentum space, this term yields:

$$\begin{aligned}
&-\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \left[\eta_{yx_1}\psi^{0^{-1}}_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\eta_{x_1x_2}\psi^{0^{-1}}_{x_2x} + \psi^{0^{-1}}_{yx_1}\psi^{0^{-1}}_{x_1x_2}\eta_{x_2x}\right] \\
&= -\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \int \frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} e^{ik_1y - ik_2x_1} e^{-iEt}\eta_{k_1k_2} \int \frac{d^{d-1}k_3}{(2\pi)^{d-1}} e^{ik_3(x_1-x_2)}\psi^{0^{-1}}_{k_3} \\
&\qquad \times \int \frac{d^{d-1}k_4}{(2\pi)^{d-1}} e^{ik_4(x_2-x)}\psi^{0^{-1}}_{k_4} \\
&-\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}} e^{ik_1(y-x_1)}\psi^{0^{-1}}_{k_1} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_3}{(2\pi)^{\frac{d-1}{2}}} e^{ik_2x_1 - ik_3x_2} e^{-iEt}\eta_{k_2k_3} \\
&\qquad \times \int \frac{d^{d-1}k_4}{(2\pi)^{d-1}} e^{ik_4(x_2-x)}\psi^{0^{-1}}_{k_4} \int \frac{d^{d-1}k_4}{(2\pi)^{d-1}} e^{ik_4(x_2-x)}\psi^{0^{-1}}_{k_4} \\
&-\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}} e^{ik_1(y-x_1)}\psi^{0^{-1}}_{k_1} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_3}{(2\pi)^{\frac{d-1}{2}}} e^{ik_2x_1 - ik_3x_2}\eta_{k_2k_3} \\
&\qquad \times e^{-iEt} \int \frac{d^{d-1}k_4}{(2\pi)^{d-1}} e^{ik_4(x_2-x)}\psi^{0^{-1}}_{k_4} \\
&-\frac{1}{4}\int d^{d-1}x_1 \int d^{d-1}x_2 \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}} e^{ik_1(y-x_1)}\psi^{0^{-1}}_{k_1} \int \frac{d^{d-1}k_2}{(2\pi)^{d-1}} e^{ik_2(x_1-x_2)}\psi^{0^{-1}}_{k_2} \int \frac{d^{d-1}k_3}{(2\pi)^{\frac{d-1}{2}}} \\
&\qquad \times \int \frac{d^{d-1}k_4}{(2\pi)^{\frac{d-1}{2}}} e^{ik_3x_2 - ik_4x_1} e^{-iEt}\eta_{k_3k_4} \\
&= -\frac{1}{4}\int \frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{ik_1y - ik_2x} \left(\psi^{0^{-1}}_{k_1}\psi^{0^{-1}}_{k_1} + \psi^{0^{-1}}_{k_2}\psi^{0^{-1}}_{k_2} + 2\psi^{0^{-1}}_{k_1}\psi^{0^{-1}}_{k_2}\right) \\
&\qquad \times \eta_{k_1k_2}.
\end{aligned} \tag{8.20}$$

Similarly, the term linear in the bilocals yields:

$$\begin{aligned}\frac{\lambda}{6}\eta_{yy}\psi^0_{yx} &= \frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{i(k_1-k_2)y}e^{ik_3(y-x)}e^{-iEt}\eta_{k_1k_2}\psi^{0^{-1}}_{k_3}\\ &+\frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{i(k_2-k_3)y}e^{ik_1(y-x)}e^{-iEt}\eta_{k_2k_3}\psi^{0^{-1}}_{k_1}\\ &=\frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{i(k_1-k_2+k_3)y}e^{-ik_3x}e^{-iEt}\eta_{k_1k_2}\psi^{0^{-1}}_{k_3}.\end{aligned}\tag{8.21}$$

Let

$$q=\frac{1}{2}\left(k_1-k_2+k_3\right);\qquad p=\frac{1}{2}\left(k_1+k_2+k_3\right).\tag{8.22}$$

Then (8.21) becomes

$$\begin{aligned}&\frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{i(k_1-k_2+k_3)y}e^{-ik_3x}e^{-iEt}\eta_{k_1k_2}\psi^{0^{-1}}_{k_3}\\ &\quad=\frac{\lambda}{6}\int\frac{d^{d-1}k_3}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}q}{(2\pi)^{\frac{d-1}{2}}}\psi^{0^{-1}}_{k_3}e^{-ik_3x}e^{iqy}e^{-iEt}\int\frac{d^dp}{(2\pi)^{d-1}}\eta_{p+q-k_3,p-q-k_3}\\ &=\frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\psi^{0^{-1}}_{k_2}e^{-ik_2x}e^{ik_1y}e^{-iEt}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}\eta_{k_3+k_1-k_2,k_3-k_1-k_2}.\end{aligned}\tag{8.23}$$

Likewise,

$$
\begin{aligned}
\frac{\lambda}{6}\psi^0_{yx}\eta_{xx} &= \frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{ik_3(y-x)}e^{ix(k_1-k_2)}e^{-iEt}\eta_{k_1k_2}\psi_{k_3}\\
&= \frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{iyk_3}e^{-ix(k_2-k_1+k_3)}e^{-iEt}\psi_{k_3}\eta_{k_1k_2}\\
&= \frac{\lambda}{6}\int\frac{d^{d-1}l}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}p}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}e^{iyk_3}e^{-ixl}e^{-iEt}\psi_{k_3}\eta_{p-l-k_3,p+l-k_3}\\
&= \frac{\lambda}{6}\int\frac{d^{d-1}k_3}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}l}{(2\pi)^{\frac{d-1}{2}}}e^{iyk_3}e^{-ixl}e^{-iEt}\psi_{k_3}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\eta_{p-l-k_3,p+l-k_3}\\
&= \frac{\lambda}{6}\int\frac{d^{d-1}k_1}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}k_2}{(2\pi)^{\frac{d-1}{2}}}e^{iyk_1}e^{-ixk_2}e^{-iEt}\psi_{k_3}\int\frac{d^{d-1}k_3}{(2\pi)^{d-1}}\eta_{k_3-k_2-k_1,k_3+k_2-k_1}.
\end{aligned}
\tag{8.24}
$$

Plugging (8.20), (8.23)and (8.24) back into (8.17), we obtain

$$
\boxed{E^2_{k_1k_2}\eta_{k_1k_2} = \frac{1}{4}\left(\psi^{0^{-1}}_{k_1}+\psi^{0^{-1}}_{k_2}\right)^2\eta_{k_1k_2}+\frac{\lambda}{6}\left(\psi^0_{k_1}+\psi^0_{k_2}\right)\int\frac{d^{d-1}l}{(2\pi)^{d-1}}\eta_{k_1+k_2-l,l},}
\tag{8.25}
$$

or equivalently

$$
\boxed{\eta_{k_1k_2} = \frac{\frac{\lambda}{6}\left(\psi^0_{k_1}+\psi^0_{k_2}\right)}{E^2_{k_1k_2}-\frac{1}{4}\left(\psi^{0^{-1}}_{k_1}+\psi^{0^{-1}}_{k_2}\right)^2}\int\frac{d^{d-1}l}{(2\pi)^{d-1}}\eta_{k_1+k_2-l,l}.}
\tag{8.26}
$$

Multiplying (8.26) by $\delta(k_1+k_2-p_1-p_2)$ and then integrating over k_1 and k_2 leads us to the single-time pole condition:

$$
1 = \frac{\lambda}{12}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{E^2_p-\left(\left|\vec{k}\right|+\left|\vec{p}-\vec{k}\right|\right)^2}\left(\frac{1}{\left|\vec{k}\right|}+\frac{1}{\left|\vec{p}-\vec{k}\right|}\right).
\tag{8.27}
$$

The integral equation for the fluctuations given in (8.26) is the single-time analogue version of (7.47). However, (8.26) is much more difficult to solve. In the two-time description, the resulting equation was solved by making use of the pole condition.

We can rewrite (8.25) as

$$E^2_{k_1k_2}\eta_{k_1k_2} = \int dk_1^{'} \int dk_2^{'} K\left(k_1, k_2; k_1^{'}, k_2^{'}\right)\eta_{k_1^{'}k_2^{'}} \tag{8.28}$$

where the kernel is given by

$$K\left(k_1, k_2; k_1^{'}, k_2^{'}\right) = \frac{1}{4}\delta\left(k_1 - k_1^{'}\right)\delta\left(k_2 - k_2^{'}\right) + \frac{\lambda}{6}\delta\left(k_1 + k_2 - k_1^{'} - k_2^{'}\right)\left(\psi^0_{k_1} + \psi^0_{k_2}\right). \tag{8.29}$$

8.2 From The Two-Time To The Single Time Equations Of Motion And The Spectrum

In this section, we will show that the various quantities of interest that were related in the single time picture are equivalent to the same quantities that were obtained in the other alternative description.

We begin with the equations of motion or colloquially speaking the pole conditions.

Recall that the two-time pole condition is

$$\begin{aligned} 1 &= \frac{i\lambda}{6}\frac{1}{(2\pi)^d}\int d^dk \frac{1}{k^2}\frac{1}{(k - p_1 - p_2)^2} \\ &= \frac{i\lambda}{6}\int\frac{d^dk}{(2\pi)^{d-1}}\int\frac{dE}{2\pi}\frac{1}{E^2 - \vec{k}^2 + i\epsilon}\frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon}, \end{aligned} \tag{8.30}$$

where we have introduced the usual Feynman $i\epsilon$ prescription which has the effect of shifting the poles that lie along the real axis upwards or downwards from the axis.

Moreover, note that the pole in the integrand occur at $E = \pm\left(\left|\vec{k}\right| - i\epsilon\right)$ and $E = E_p \pm \left(\left|\vec{k} - \vec{p}\right| - i\epsilon\right)$. We choose to close the contour on the LHP. Evaluating the residues, we obtain

$$\int \frac{dE}{2\pi} \frac{1}{E^2 - \vec{k}^2 + i\epsilon} \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon} = \int \frac{dE}{(2\pi)} f_1(E), \tag{8.31}$$

where

$$f_1(E) = \frac{1}{E^2 - \vec{k}^2 + i\epsilon} \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon}. \tag{8.32}$$

The residue at $E = \left|\vec{k}\right|$ is

$$\begin{aligned} \text{Res}\left[f_1\left(\left|\vec{k}\right|\right)\right] &= \lim_{E \to \left|\vec{k}\right|} \left(E - \left|\vec{k}\right|\right) f_1(E) \\ &= \lim_{E \to \left|\vec{k}\right|} \left(E - \left|\vec{k}\right|\right) \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2} \frac{1}{\left(E - \left|\vec{k}\right|\right)} \frac{1}{\left(E + \left|\vec{k}\right|\right)} \\ = \frac{1}{2\left|\vec{k}\right|} \frac{1}{\left(\left|\vec{k}\right| - E_p\right)^2 - \left(\vec{k} - \vec{p}\right)^2} &= \frac{1}{2\left|\vec{k}\right|} \frac{1}{\left|\vec{k}\right| - E_p - \left|\vec{k} - \vec{p}\right|} \frac{1}{\left|\vec{k}\right| - E_p + \vec{k} - \vec{p}}. \end{aligned} \tag{8.33}$$

Likewise, the other residue gives us

$$\begin{aligned} \text{Res}\left[f_1\left(E_p + \left|\vec{k} - \vec{p}\right|\right)\right] &= \lim_{E \to E_p + \left|\vec{k} - \vec{p}\right|} \left(E - E_p - \left|\vec{k} - \vec{p}\right|\right) f_1(E) \\ = \lim_{E \to E_p + \left|\vec{k} - \vec{p}\right|} &\left(E - E_p - \left|\vec{k} - \vec{p}\right|\right) \frac{1}{E^2 - \vec{k}^2} \frac{1}{E - E_p + \left|\vec{k} - \vec{p}\right|} \frac{1}{E - E_p - \left|\vec{k} - \vec{p}\right|} \\ = \frac{1}{2\left|\vec{k} - \vec{p}\right|} \frac{1}{\left(E_p + \left|\vec{k} - \vec{p}\right|\right)^2 - \vec{k}^2} &= \frac{1}{2\left|\vec{k} - \vec{p}\right|} \frac{1}{E_p + \left|\vec{k} - \vec{p}\right| - \left|\vec{k}\right|} \frac{1}{E_p + \left|\vec{k} - \vec{p}\right| + \left|\vec{k}\right|}. \end{aligned} \tag{8.34}$$

The residue theorem then allows us to write the integral in (8.31) as

$$\int \frac{dE}{(2\pi)} f_1(E) = -i \Bigg[\frac{1}{2\left|\vec{k}\right|} \frac{1}{\left|\vec{k}\right| - E_p - \left|\vec{k}-\vec{p}\right|} \frac{1}{\left|\vec{k}\right| - E_p + \vec{k}-\vec{p}} + \frac{1}{2\left|\vec{k}-\vec{p}\right|} \frac{1}{E_p + \left|\vec{k}-\vec{p}\right| - \left|\vec{k}\right|} \frac{1}{E_p + \left|\vec{k}-\vec{p}\right| + \left|\vec{k}\right|} \Bigg]. \tag{8.35}$$

Let $\vec{k} \to \vec{k_1}$ and $\vec{p} - \vec{k} = \vec{k_2}$. In terms of the new variables, we have

$$\int \frac{dE}{(2\pi)} f_1(E) = -\frac{i}{2} \Bigg[\frac{1}{\left|\vec{k_1}\right|} \frac{1}{\left|\vec{k_1}\right| - E_p - \left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_1}\right| - E_p + \left|\vec{k_2}\right|} + \frac{1}{\left|\vec{k_1}\right|} \frac{1}{\left|\vec{k_1}\right| - E_p - \left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_1}\right| - E_p + \left|\vec{k_2}\right|} \Bigg]. \tag{8.36}$$

It is more convenient to symmetrize and write

$$\int \frac{dE}{(2\pi)} f_1(E) = -\frac{i}{4} \Bigg[\frac{1}{\left|\vec{k_1}\right|} \frac{1}{\left|\vec{k_1}\right| - E_p - \left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_1}\right| - E_p + \left|\vec{k_2}\right|} + \frac{1}{\left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_2}\right| + E_p - \left|\vec{k_1}\right|} \frac{1}{\left|\vec{k_2}\right| + E_p + \left|\vec{k_1}\right|} + \frac{1}{\left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_1}\right| + E_p - \left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_1}\right| + E_p + \left|\vec{k_2}\right|} + \frac{1}{\left|\vec{k_1}\right|} \frac{1}{\left|\vec{k_1}\right| + E_p - \left|\vec{k_2}\right|} \frac{1}{\left|\vec{k_1}\right| + E_p + \left|\vec{k_2}\right|} \Bigg]. \tag{8.37}$$

For the sum of the two terms with a prefactor of $\left|\vec{k}_1\right|^{-1}$, we obtain

$$
\begin{aligned}
&\frac{1}{\left|\vec{k}_1\right|}\frac{1}{\left|\vec{k}_1\right|-E_p-\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right|-E_p+\left|\vec{k}_2\right|}+\frac{1}{\left|\vec{k}_1\right|}\frac{1}{\left|\vec{k}_1\right|+E_p-\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right|+E_p+\left|\vec{k}_2\right|}\\
&=\frac{1}{\left|\vec{k}_1\right|}\frac{\left(E_p+\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)E_p+\left(\left|\vec{k}_1\right|+\left|\vec{k}_1\right|\right)+\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)E_p-\left(\left|\vec{k}_1\right|+\left|\vec{k}_1\right|\right)}{\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p-\left(\left|\vec{k}_1\right|+\left|\vec{k}_1\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|+\left|\vec{k}_1\right|\right)\right)}\\
&+\frac{1}{\left|\vec{k}_1\right|}\frac{\left(E_p+\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right)+\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)E_p-\left(\left|\vec{k}_1\right|+\left|\vec{k}_1\right|\right)}{\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p-\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right)}.
\end{aligned}
\tag{8.38}
$$

Likewise, the two terms with the $\left|\vec{k}_2\right|^{-1}$ yield

$$
\begin{aligned}
&\frac{1}{\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right|+E_p-\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right|+E_p+\left|\vec{k}_2\right|}+\frac{1}{\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right|+E_p-\left|\vec{k}_2\right|}\frac{1}{E_p-\left|\vec{k}_2\right|-\left|\vec{k}_1\right|}\\
&=\frac{1}{\left|\vec{k}_2\right|}\frac{\left(E_p+\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right)+\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)}{\left(E_p-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right)\left(E_p+\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\right)\left(E_p-\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right)}.
\end{aligned}
\tag{8.39}
$$

Hence,

$$
\begin{aligned}
&\int \frac{dE}{(2\pi)} f_1(E) = \frac{1}{\left|\vec{k}_1\right|} \\
&\times \frac{\Big(E_p + \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p + \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)\Big) + \Big(E_p - \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big) E_p - \Big(\left|\vec{k}_1\right| + \left|\vec{k}_1\right|\Big)}{\Big(E_p - \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p + \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)\Big)\Big(E_p + \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p - \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)\Big)} \\
&+ \frac{1}{\left|\vec{k}_2\right|} \frac{\Big(E_p + \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p + \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)\Big) + \Big(E_p - \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p - \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)}{\Big(E_p - \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p + \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)\Big)\Big(E_p + \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)\Big)\Big(E_p - \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)\Big)} \\
&= -\frac{i}{4} \frac{1}{\Big(E_p^2 - \Big(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\Big)^2\Big)\Big(E_p^2 + \Big(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\Big)^2\Big)} \Bigg\{ \frac{1}{\left|\vec{k}_1\right|} \Bigg[\Big(E_p^2 + \left|\vec{k}_1\right|^2 - \left|\vec{k}_2\right|^2 + 2E_p\left|\vec{k}_2\right|\Big) \\
&\quad + \Big(\Big(E_p^2 - \left|\vec{k}_1\right|^2 + \left|\vec{k}_2\right|^2 - 2E_p\left|\vec{k}_1\right|\Big)\Big)\Bigg] + \frac{1}{\left|\vec{k}_2\right|}\Bigg[\Big(E_p^2 + \left|\vec{k}_2\right|^2 - \left|\vec{k}_1\right|^2 + 2E_p\left|\vec{k}_2\right|\Big) \\
&\qquad\qquad + \Big(\Big(E_p^2 + \left|\vec{k}_2\right|^2 - \left|\vec{k}_2\right|^2 - 2E_p\left|\vec{k}_2\right|\Big)\Big)\Bigg]\Bigg\}.
\end{aligned}
\tag{8.40}
$$

Some of the terms cancel out and we are then left with

$$\begin{aligned}
\int \frac{dE}{(2\pi)} f_1(E) = & -\frac{i}{4} \frac{1}{\left(E_p^2 - \left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)^2\right)\left(E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2\right)} \\
& \times \left[\frac{2}{\left|\vec{k}_1\right|}\left(E_p^2 + \left|\vec{k}_1\right|^2 - \left|\vec{k}_2\right|^2\right) + \frac{2}{\left|\vec{k}_2\right|}\left(E_p^2 + \left|\vec{k}_2\right|^2 - \left|\vec{k}_1\right|^2\right)\right] \\
& = -\frac{i}{2} \frac{1}{\left(E_p^2 - \left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)^2\right)\left(E_p^2 + \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2\right)} \\
& \times \left[E_p^2\left(\frac{1}{\left|\vec{k}_1\right|} + \frac{1}{\left|\vec{k}_2\right|}\right) - \left(\left|\vec{k}_2\right|^2 - \left|\vec{k}_1\right|^2\right)\left(\frac{1}{\left|\vec{k}_1\right|} - \frac{1}{\left|\vec{k}_2\right|}\right)\right] \\
& = -\frac{i}{2} \frac{1}{\left(E_p^2 - \left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)^2\right)\left(E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2\right)} \\
& \times \left[\frac{E_p^2}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right) - \frac{\left(\left|\vec{k}_2\right|^2 - \left|\vec{k}_1\right|^2\right)\left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\right] \\
& = -\frac{i}{2} \frac{1}{\left(E_p^2 - \left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)^2\right)\left(E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2\right)} \\
& \times \frac{\left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\left[E_p^2 - \left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)^2\right] \\
& = -\frac{i}{2} \frac{1}{E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2}\left(\frac{1}{\left|\vec{k}_1\right|} + \frac{1}{\left|\vec{k}_2\right|}\right). \quad (8.41)
\end{aligned}$$

The above result is of great significance, as we discuss in the following.

To summarize, we have then

$$\int \frac{dE}{2\pi} \frac{1}{E^2 - \vec{k}^2 + i\epsilon} \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon} = -\frac{i}{2}$$

$$\times \frac{1}{E_p^2 - \left(\left|\vec{k}\right| + \left|\vec{k} - \vec{p}\right|\right)^2} \left(\frac{1}{\left|\vec{k}\right|} + \frac{1}{\left|\vec{k} - \vec{p}\right|}\right). \quad (8.42)$$

Therefore,

$$1 = \frac{i\lambda}{6} \frac{1}{(2\pi)^d} \int d^d k_1 \frac{1}{k^2} \frac{1}{(k - p_1 - p_2)^2}$$

$$= \frac{i\lambda}{6} \left(-\frac{i}{2}\right) \int \frac{d^{d-1}k_1 d^{d-1}k_2}{(2\pi)^{d-1}} \frac{\delta\left(\vec{p} - \vec{k}_1 + \vec{k}_2\right)}{E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2} \left(\frac{1}{\left|\vec{k}_1\right|} + \frac{1}{\left|\vec{k}_2\right|}\right), \quad (8.43)$$

or

$$\boxed{1 = \frac{\lambda}{12} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{E_p^2 - \left(\left|\vec{k}\right| + \left|\vec{p} - \vec{k}\right|\right)^2} \left(\frac{1}{\left|\vec{k}\right|} + \frac{1}{\left|\vec{p} - \vec{k}\right|}\right).} \quad (8.44)$$

This is the pole condition that we found in the Hamiltonian approach (8.27).

We have thus demonstrated that the equations of motions in the two descriptions are equivalent.

For completeness, we carry out the same analysis for the free part of the propagator.

The free $O(N)$ vector model propagator is

$$\hat{O^{-1}}_{k_3 k_4; p_1 p_2} = 2\psi^0_{k_3}\psi^0_{k_4}\delta(k_4 + p_1)\,\delta(k_3 + p_2), \quad (8.45)$$

where all the momenta are 4 vectors.

The large-N background has already been obtained. In Minkowski space it is given by

$$\psi_k^0 = \frac{i}{k^2}. \tag{8.46}$$

Therefore, the free $O(N)$ propagator can be written as

$$\begin{aligned} \hat{O^{-1}}_{k_1 k_2; p_1 p_2} &= 2\psi_{k_3}^0 \psi_{k_4}^0 \delta(k_4 + p_1)\,\delta(k_4 + p_1) \\ &= -2\frac{1}{E_1^2 - \vec{k}_1^{\,2} + i\epsilon} \frac{\delta(k_4 + p_1)\,\delta(k_3 + p_2)}{E_2^2 - \vec{k}_2^2 + i\epsilon}. \end{aligned} \tag{8.47}$$

It turns out that its more convenient to change variables and work with

$$E = E_1 + E_2, \qquad \omega = E_1 - E_2 \tag{8.48}$$

or

$$E_1 = \frac{1}{2}(E + \omega), \qquad E_2 = \frac{1}{2}(E - \omega). \tag{8.49}$$

We can make use of this to write

$$\begin{aligned} \frac{-2}{E_1^2 - \vec{k}_1^{\,2} + i\epsilon} \frac{1}{E_2^2 - \vec{k}_2^2 + i\epsilon} &= \frac{-2}{\frac{1}{4}(E+\omega)^2 - \vec{k}_1^{\,2} + i\epsilon} \\ \frac{1}{\frac{1}{4}(E-\omega)^2 - \vec{k}_2^2 + i\epsilon} &= \frac{32}{(E+\omega)^2 - 4\vec{k}_1^{\,2} + i\epsilon} \frac{1}{(E-\omega)^2 - 4\vec{k}_2^{\,2} + i\epsilon}. \end{aligned} \tag{8.50}$$

It is clear that we need to perform the integral

$$\begin{aligned} &-32\int \frac{d\omega}{2\pi} \frac{32}{(E+\omega)^2 - 4\vec{k}_1^{\,2} + i\epsilon} \frac{1}{(E-\omega)^2 - 4\vec{k}_2^{\,2} + i\epsilon} \\ &\qquad = -\int \frac{d\omega}{2\pi} f(\omega), \end{aligned} \tag{8.51}$$

where

$$\frac{32}{(\omega+E)^2-4\vec{k}_1^{\,2}+i\epsilon}\frac{1}{(\omega-E)^2-4\vec{k}_2^{\,2}+i\epsilon}=\frac{i\lambda}{6}\int\frac{d^dk_1}{(2\pi)^{d-1}}$$
$$\times\int\frac{dE}{2\pi}\frac{1}{E^2-\vec{k}^2+i\epsilon}\frac{1}{(E-E_p)^2-\left(\vec{k}-\vec{p}\right)^2+i\epsilon}. \quad (8.52)$$

The poles of the function $f(\omega)$ are at $\omega=-E\pm\left(2\left|\vec{k_1}\right|\mp i\epsilon\right)$ and $\omega=E\pm\left(2\left|\vec{k_2}\right|\mp i\epsilon\right)$. We will choose to close our counter on the UHP - see Figure 8.1. The residue at $\omega=E-2\left|\vec{k_2}\right|$ is

$$\text{Res}\left[f\left(E-2\left|\vec{k_1}\right|\right)\right]=\lim_{\omega\to E-2\left|\vec{k_2}\right|}\left(\omega-E+2\left|\vec{k_2}\right|\right)f(\omega)$$
$$=\lim_{\omega\to E-2\left|\vec{k_2}\right|}\left(\omega-E+2\left|\vec{k_2}\right|\right)\frac{32}{(\omega+E)^2-4\vec{k}_1^{\,2}+i\epsilon}\frac{1}{\left(\omega-E-2\left|\vec{k_2}\right|\right)\left(\omega-E+2\left|\vec{k_2}\right|\right)}$$
$$=\frac{32}{\left(2E-2\left|\vec{k_2}\right|\right)^2-4\vec{k}_1^{\,2}}\frac{1}{\left(-4\left|\vec{k_2}\right|\right)}. \quad (8.53)$$

Likewise, the residue at $\omega=-E-2\left|\vec{k_1}\right|$ yields

$$\text{Res}\left[f\left(-E-2\left|\vec{k_1}\right|\right)\right]=\lim_{\omega\to -E-2\left|\vec{k_1}\right|}\left(\omega+E+2\left|\vec{k_2}\right|\right)f(\omega)$$
$$\lim_{\omega\to -E-2\left|\vec{k_1}\right|}\left(\omega+E+2\left|\vec{k_1}\right|\right)\frac{1}{\left(\omega+E+2\left|\vec{k_1}\right|\right)\left(\omega+E-2\left|\vec{k_1}\right|\right)}\frac{1}{(E-\omega)^2-4\vec{k}_2^{\,2}+i}$$
$$=\frac{1}{\left(-4\left|\vec{k_1}\right|\right)}\frac{1}{\left(2E+2\left|\vec{k_1}\right|\right)^2-4\vec{k}_2^{\,2}}. \quad (8.54)$$

By making use of (8.53) and (8.54) the integral in (8.47) becomes

$$\begin{aligned}
\int \frac{d\omega}{2\pi} f(\omega) &= 2\pi i \sum_k \mathrm{Res} f(z_k) \\
&= \frac{2\pi i}{2\pi}\left(\frac{32}{\left(2E-2\left|\vec{k}_2\right|\right)^2-4\vec{k}_1^{\,2}}\frac{1}{\left(-4\left|\vec{k}_2\right|\right)}+\frac{1}{\left(-4\left|\vec{k}_1\right|\right)}\frac{1}{\left(2E+2\left|\vec{k}_1\right|\right)^2-4\vec{k}_2^{\,2}}\right) \\
&= -\frac{i}{2}\left(\frac{1}{\left(E-\left|\vec{k}_2\right|\right)^2-\vec{k}_1^{\,2}}\frac{1}{\left(\left|\vec{k}_2\right|\right)}+\frac{1}{\left(E+\left|\vec{k}_1\right|\right)^2-\vec{k}_2^{\,2}}\frac{1}{\left(\left|\vec{k}_1\right|\right)}\right). \qquad (8.55)
\end{aligned}$$

The term inside the brackets yields

$$\begin{aligned}
&\frac{1}{\left(E-\left|\vec{k}_2\right|\right)^2-\vec{k}_1^{\,2}}\frac{1}{\left(-\left|\vec{k}_2\right|\right)}+\frac{1}{\left(E+\left|\vec{k}_1\right|\right)^2-\vec{k}_2^{\,2}}\frac{1}{\left(-\left|\vec{k}_1\right|\right)} \\
&\quad= \frac{\left(\left(E+\left|\vec{k}_1\right|\right)^2-\vec{k}_2^{\,2}\right)\left|\vec{k}_1\right|+\left(\left(E-\left|\vec{k}_2\right|\right)^2-\vec{k}_1^{\,2}\right)\left|\vec{k}_2\right|}{\left(\left(E-\left|\vec{k}_2\right|\right)^2-\vec{k}_1^{\,2}\right)\left(\left(E+\left|\vec{k}_1\right|\right)^2-\vec{k}_2^{\,2}\right)\left|\vec{k}_1\right|\left|\vec{k}_2\right|} \\
&= \frac{\left(E^2+\vec{k}_1^{\,2}+2E\left|\vec{k}_1\right|-\vec{k}_2^{\,2}\right)\left|\vec{k}_1\right|+\left(E^2+\vec{k}_2^{\,2}-2E\left|\vec{k}_2\right|-\vec{k}_1^{\,2}\right)\left|\vec{k}_2\right|}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left(E-\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\left(E-\left|\vec{k}_2\right|+\left|\vec{k}_1\right|\right)\left(E+\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\left(E+\left|\vec{k}_2\right|+\left|\vec{k}_1\right|\right)}.
\end{aligned} \qquad (8.56)$$

Note that

$$\begin{aligned}
&\left(E-\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\left(E-\left|\vec{k}_2\right|+\left|\vec{k}_1\right|\right)\left(E+\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\left(E+\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right) \\
&\quad= \left(E-\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)\left(E+\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\left(E-\left|\vec{k}_2\right|+\left|\vec{k}_1\right|\right)\left(E+\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right) \\
&\qquad= \left(E^2-\left(\left|\vec{k}_2\right|+\left|\vec{k}_1\right|\right)^2\right)\left(E^2-\left(\left|\vec{k}_2\right|-\left|\vec{k}_1\right|\right)^2\right). \qquad (8.57)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\left(E^2+\vec{k_1}^2+2E\left|\vec{k_1}\right|-\vec{k_2}^2\right)\left|\vec{k_1}\right|+\left(E^2+\vec{k_2}^2-2E\left|\vec{k_2}\right|-\vec{k_1}^2\right)\left|\vec{k_2}\right|}{\left|\vec{k_1}\right|\left|\vec{k_2}\right|\left(E-\left|\vec{k_1}\right|-\left|\vec{k_2}\right|\right)\left(E-\left|\vec{k_1}\right|+\left|\vec{k_2}\right|\right)\left(E+\left|\vec{k_2}\right|-\left|\vec{k_1}\right|\right)\left(E+\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)}\\
&=\frac{E^2\left|\vec{k_1}\right|+\vec{k_1}^2\left|\vec{k_1}\right|+2E\left|\vec{k_1}\right|\left|\vec{k_1}\right|-\vec{k_2}^2\left|\vec{k_1}\right|+E^2\left|\vec{k_2}\right|+\vec{k_2}^2\left|\vec{k_2}\right|-2E\vec{k}_2^2-\vec{k_1}^2\left|\vec{k_2}\right|}{\left|\vec{k_1}\right|\left|\vec{k_2}\right|\left(E^2-\left(\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)^2\right)\left(E^2-\left(\left|\vec{k_2}\right|-\left|\vec{k_1}\right|\right)^2\right)}\\
&=\frac{E^2\left(\left|\vec{k_1}\right|+\left|\vec{k_2}\right|\right)+2E\left(\left|\vec{k_1}\right|+\left|\vec{k_2}\right|\right)\left(\left|\vec{k_1}\right|-\left|\vec{k_2}\right|\right)+\vec{k}_1^3+\vec{k}_1^3-\vec{k_2}^2\left|\vec{k_1}\right|-\vec{k_2}^2\left|\vec{k_2}\right|}{\left|\vec{k_1}\right|\left|\vec{k_2}\right|\left(E^2-\left(\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)^2\right)\left(E^2-\left(\left|\vec{k_2}\right|-\left|\vec{k_1}\right|\right)^2\right)}\\
&=\frac{E^2\left(\left|\vec{k_1}\right|+\left|\vec{k_2}\right|\right)+2E\left(\left|\vec{k_1}\right|+\left|\vec{k_2}\right|\right)\left(\left|\vec{k_1}\right|-\left|\vec{k_2}\right|\right)+\left(\left|\vec{k_1}\right|^2-\left|\vec{k_2}\right|^2\right)\left(\left|\vec{k_1}\right|-\left|\vec{k_2}\right|\right)}{\left|\vec{k_1}\right|\left|\vec{k_2}\right|\left(E^2-\left(\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)^2\right)\left(E^2-\left(\left|\vec{k_2}\right|-\left|\vec{k_1}\right|\right)^2\right)}\\
&=\frac{\left|\vec{k_1}\right|+\left|\vec{k_2}\right|}{\left|\vec{k_1}\right|\left|\vec{k_2}\right|}\frac{E^2-2E\left(\left|\vec{k_1}\right|-\left|\vec{k_2}\right|\right)+\left(\left|\vec{k_1}\right|-\left|\vec{k_2}\right|\right)^2}{\left(E^2-\left(\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)^2\right)\left(E^2-\left(\left|\vec{k_2}\right|-\left|\vec{k_1}\right|\right)^2\right)}\\
&=\left(\frac{1}{\left|\vec{k_1}\right|}+\frac{1}{\left|\vec{k_2}\right|}\right)\frac{1}{\left(E^2-\left(\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)^2\right)}. \qquad (8.58)
\end{aligned}$$

Consequently, we have

$$-\int\frac{d\omega}{2\pi}f\left(\omega\right)=\frac{i}{2}\left(\frac{1}{\left|\vec{k_1}\right|}+\frac{1}{\left|\vec{k_2}\right|}\right)\frac{1}{\left(E^2-\left(\left|\vec{k_2}\right|+\left|\vec{k_1}\right|\right)^2\right)}. \qquad (8.59)$$

which is - up to some constant factors - the single-time equation of motion that we found in (4.36).

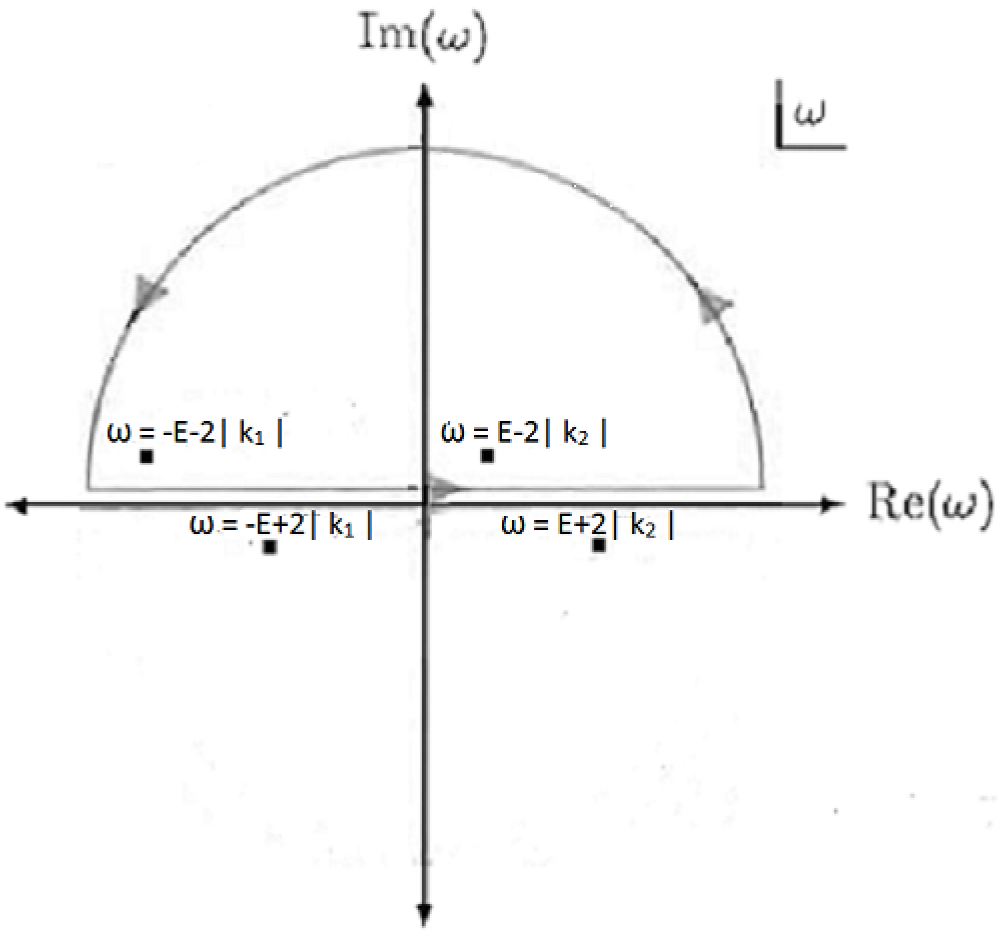


Figure 8.1: The contour chosen for the evaluation of the integral in (8.52).

8.3 Single-Time Lagrangian

But each new paper from him [Witten] gave me the joy of reading, and the question, "why am I needed?"

-Joseph Polchinski, *Memories of a Theoretical Physicist.*

For completeness, in this section we will discuss the single-time Lagrangian formalism. The large-N collective field theory Hamiltonian can be written as (8.4):

$$H_2 = 2\mathrm{Tr}\left(p\psi_0 p\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) + \frac{\lambda}{4!}\int d^{d-1}x\eta_{xx}^2. \tag{8.60}$$

The fluctuations are

$$\eta_{xy} = \int \frac{d^2k_1}{(2\pi)}\int \frac{d^2k_2}{(2\pi)} e^{i\vec{k_1}\cdot\vec{x}_1} e^{i\vec{k_2}\cdot\vec{x}_2}\eta_{\vec{k_1}\vec{k_2}}. \tag{8.61}$$

while the momentum can be written in terms of the Fourier transform as

$$p_{xy} = \int \frac{d^2k_1}{(2\pi)}\int \frac{d^2k_2}{(2\pi)} e^{i\vec{k_1}\cdot\vec{x}_1} e^{i\vec{k_2}\cdot\vec{x}_2} p_{\vec{k_1}\vec{k_2}} \tag{8.62}$$

In momentum space, one can show that the quadratic Hamiltonian becomes

$$H_2 = 2\int dk_1\int dk_2 p_{k_1k_2}\psi^0_{k_2}p_{-k_2,-k_1} + \frac{1}{8}\int dk_1\int dk_2 \psi_{k_1}^{0^{-2}}\eta_{k_1k_2}\psi^{0^{-1}}_{-k_2,-k_1} \tag{8.63}$$

$$+\frac{\lambda}{4!}\int \frac{dk_1dk_2dp_1dp_2}{(2\pi)^2}\delta\left(\vec{k_1}+\vec{k_2}+\vec{p_1}+\vec{p_2}\right)\eta_{\vec{k}_1\vec{k}_2}\eta_{\vec{p_1}\vec{p_2}}. \tag{8.64}$$

The equations of motion (*i.e.* (8.12)) lead us to conclude that

$$p_{k_1k_2} = \frac{1}{2\left(\psi^0_{k_1}+\psi^0_{k_2}\right)}\dot{\eta}_{-k_2,-k_1}. \tag{8.65}$$

We now perform the standard Legendre transformation and write the single time Lagrangian as

$$L_2 = \frac{1}{4}\int dk_1 \int dk_2 \dot{\eta}_{k_1k_2} \frac{1}{\left(\psi^0_{k_1} + \psi^0_{k_2}\right)} \dot{\eta}_{k_1k_2} - \frac{1}{16}\int dk_1 \int dk_2 \eta_{k_1k_2}$$
$$\times \left(\psi^{-2}_{k_1}\psi^{-1}_{k_2} + \psi^{-2}_{k_2}\psi^{-1}_{k_1}\right) \dot{\eta}_{-k_2,-k_1} - \frac{\lambda}{4!}\int \frac{dk_1 dk_2 dp_1 dp_2}{(2\pi)^2} \delta\left(\vec{k_1} + \vec{k_2} + \vec{p_1} + \vec{p_2}\right) \eta_{\vec{k_1}\vec{k_2}} \eta_{\vec{p_1}\vec{p_2}}. \tag{8.66}$$

We then Fourier transform to the energy description. The action in this language reads

$$S_2 = \int dk_1 \int dk_2 \int dE \eta_{k_1k_2E} \left(\frac{E^2}{4\left(\psi^0_{k_1} + \psi^0_{k_2}\right)} - \frac{1}{16}\left(\psi^{-2}_{k_1}\psi^{-1}_{k_2} + \psi^{-2}_{k_2}\psi^{-1}_{k_1}\right) \right)$$
$$\eta_{-k_2,-k_1} - \frac{\lambda}{4!}\int dE dk_1 dk_2 dp_1 dp_2 \eta_{\vec{k_1}\vec{k_2},E} \frac{\delta\left(\vec{k_1} + \vec{k_2} + \vec{p_1} + \vec{p_2}\right)}{(2\pi)^2} \eta_{\vec{p_1}\vec{p_2},E} \tag{8.67}$$

The partition function, schematically, is

$$Z \sim e^{iS_2} \sim e^{-\frac{1}{2}\eta O \eta}. \tag{8.68}$$

More formally, we have

$$e^{iS_2} = \exp -\frac{1}{2}\int dk_1 \int dE_1 \int dk_2 \int dE_2 \eta_{k_1k_2,E_1} \hat{O}_{k_1k_2,E;p_1p_2,E_2}. \tag{8.69}$$

with

$$\hat{O}_{k_1k_2,E;p_1p_2,E_2} = -i\Bigg[\frac{E^2}{2\left(\psi^0_{k_1}+\psi^0_{k_2}\right)} - \frac{1}{8}\left(\psi^{-2}_{k_1}\psi^{-1}_{k_2}+\psi^{-2}_{k_2}\psi^{-1}_{k_1}\right)$$
$$\times\delta\left(\vec{k_1}+\vec{p_2}\right)\delta\left(\vec{k_2}+\vec{p_1}\right)\delta\left(E_1+E_2\right) + \frac{2i\lambda}{4!}\frac{1}{(2\pi)^2}\delta\left(\vec{k_1}+\vec{k_2}+\vec{p_1}+\vec{p_2}\right)\delta\left(E_1+E_2\right)\Bigg]. \tag{8.70}$$

We want to find the inverse of the operator O. More precisely, we must find O^{-1} such that

$$\int dk_3\int dk_4\int dE\hat{O}_{k_1k_2,E_1;k_3k_4,E_2}\hat{O}^{-1}_{k_3k_4,E_2;p_1p_2,E_3} = \delta\left(k_1-p_1\right)$$
$$\times\delta\left(k_2-p_1\right)\delta\left(k_2-p_1\right)\delta\left(E_1-E_2\right). \tag{8.71}$$

Our ansatz for the inverse reads

$$\hat{O}_{k_3k_4,E_2;p_1p_2,E_3} = iG^{0^{-1}}\left(\vec{k_3},\vec{k_4}\right)\delta\left(\vec{k_3}+\vec{p_2}\right)\delta\left(\vec{k_4}+\vec{p_1}\right)\delta\left(E_2+E_3\right)$$
$$+\,\delta\left(\vec{k_3}+\vec{k_4}+\vec{p_1}+\vec{p_2}\right)G_{k_3k_4;p_1p_2}\delta\left(E_2+E_3\right). \tag{8.72}$$

We insert this ansatz into (8.71) and obtain

$$\int dk_3\int dk_4\int dE_2\hat{O}_{k_1k_2,E_1;k_3k_4,E_2}\hat{O}^{-1}_{k_3k_4,E_2;p_1p_2,E_3} = \int dk_3\int dk_4\int dE$$
$$\Bigg[\frac{E^2}{2\left(\psi^0_{k_1}+\psi^0_{k_2}\right)} - \frac{1}{8}\left(\psi^{-2}_{k_1}\psi^{-1}_{k_2}+\psi^{-2}_{k_2}\psi^{-1}_{k_1}\right)\delta\left(\vec{k_1}+\vec{p_2}\right)$$
$$\times\delta\left(\vec{k_2}+\vec{p_1}\right)\delta\left(E_1+E_2\right) + \frac{2i\lambda}{4!}\frac{1}{(2\pi)^2}\delta\left(\vec{k_1}+\vec{k_2}+\vec{p_1}+\vec{p_2}\right)\delta\left(E_1+E_2\right)\Bigg]$$
$$\times\left[iG^{0^{-1}}\left(\vec{k_3},\vec{k_4}\right)\delta\left(\vec{k_3}+\vec{p_2}\right)\delta\left(\vec{k_4}+\vec{p_1}\right)\delta\left(E_2+E_3\right)+\delta\left(\vec{k_3}+\vec{k_4}+\vec{p_1}+\vec{p_2}\right)G_{k_3k_4;p_1p_2}\delta\left(E_2+E_3\right)\right]$$
$$= A+B+C+D. \tag{8.73}$$

The first term in (8.73) yields

$$A = -\int dk_3 \int dk_4 \int dE G^0 (k_1, k_2) G^{-1} (k_3, k_4) \delta (k_1 + k_4) \delta (k_3 + k_2)$$
$$\times \delta (k_3 + p_2) \delta (E_1 + E_2) \delta (E_2 + E_1) = G^0 (k_1, k_2) G^{0^{-1}} (-k_2, -k_1) \delta (-k_2 + p_2)$$
$$\times \delta (-k_1 + p_1) \delta (-E_1 + E_3) . \tag{8.74}$$

For the second term in (8.73), we obtain

$$B = -i \int dk_3 \int dk_4 \int dE_2 G^0 (k_1, k_2) \delta (k_1 + k_4) \delta (k_3 + k_3) \delta (E_1 + E_2)$$
$$\times \delta (k_3 + k_4 + p_1 + p_2) \delta (E_2 + E_3) G_{k_3 k_4; p_1 p_2} = -i G^0 (k_1, k_2) \delta (-k_2 - k_1 + p_1 + p_2)$$
$$\delta (-E_1 + E_3) G_{-k_2, -k_1; p_1 p_2} = -i G^0 (k_1, k_2) \delta (p_1 + p_2 - k_1 - k_2) \delta (E_3 - E_1) G_{-k_2, -k_1; p_1 p_2} . \tag{8.75}$$

Likewise, the third term in (8.73) leads to

$$C = \frac{2i\lambda}{4!} \frac{1}{(2\pi)^2} \int dk_3 \int dk_4 \int dE_2 \delta (k_1 + k_2 + k_3 + k_4) \delta (E_1 + E_2)$$
$$\times i G^{0^{-1}} (k_3, k_4) \delta (k_3 + p_2) \delta (k_4 + p_1) \delta (E_2 + E_3) = \frac{2i\lambda}{4!} \frac{1}{(2\pi)^2}$$
$$\times \delta (k_1 + k_2 + k_3 + k_4) \delta (E_1 - E_3) i G^{0^{-1}} (-p_2, -p_1) . \tag{8.76}$$

Finally, the last term in (8.73) gives

$$D = \frac{2i\lambda}{4!} \frac{1}{(2\pi)^2} \int dk_3 \int dk_4 \int dE_2 \delta (k_1 + k_2 + p_1 + p_2) G_{k_3 k_4; p_1 p_2}$$
$$\times \delta (E_1 + E_2) \delta (k_3 + k_4 + p_1 + p_2) = \frac{2i\lambda}{4!} \frac{1}{(2\pi)^2} \delta (p_1 + p_2 - k_1 - k_2)$$
$$\delta (E_3 - E_1) \int dk_3 \int dk_4 \delta (k_3 + k_4 + p_1 + p_2) G_{k_3 k_4; p_1 p_2} . \tag{8.77}$$

Substituting (8.74), (8.75), (8.76) and (8.77) in (8.73), we have

$$0 = -iG^0\left(k_1, k_2\right) G_{-k_2,-k_1;p_1p_2} + \frac{2i\lambda}{4!}\frac{1}{\left(2\pi\right)^2} iG_0^{-1}\left(-p_2, -p_1\right)$$
$$+ \frac{2i\lambda}{4!}\frac{1}{\left(2\pi\right)^2}\int dk_3 \int dk_4 \delta\left(k_3 + k_4 + p_1 + p_2\right) G_{k_3k_4;p_1p_2} \tag{8.78}$$

or

$$\frac{2i\lambda}{4!}\frac{1}{\left(2\pi\right)^2} iG_0^{-1}\left(-p_2, -p_1\right) = iG^0\left(k_1, k_2\right) G_{-k_2-k_1;p_1p_2} - \frac{2i\lambda}{4!}\frac{1}{\left(2\pi\right)^2}\int dk_3 \int dk_4$$
$$\times \delta\left(k_3 + k_4 + p_1 + p_2\right) G_{k_3k_4;p_1p_2}. \tag{8.79}$$

Since $G_{k_1k_2;p_1p_2}$ is symmetric under the interchange $\left(k_1, k_2\right) \rightarrow \left(-k_2, -k_1\right)$, we can rewrite (8.79) as

$$\frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^2} iG_0^{-1}\left(p_1, p_2\right) = iG^0\left(k_1, k_2\right) G_{k_1k_2;p_1p_2} - \frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^2}\int dk_3 \int dk_4$$
$$\delta\left(k_3 + k_4 + p_1 + p_2\right) G_{k_3k_4;p_1p_2}. \tag{8.80}$$

Solving for $G_{k_1k_2;p_1p_2}$, we obtain

$$G_{k_1k_2;p_1p_2} = \frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^2} G_0^{-1}\left(k_1, k_2\right) G_0^{-1}\left(p_1, p_2\right) + \frac{2\lambda}{4!}\frac{1}{\left(2\pi\right)^2} G_0^{-1}\left(k_1, k_2\right)$$
$$\int dk_3 \int dk_4 \delta\left(k_3 + k_1 + p_1 + p_2\right) G_{k_3k_4;p_1p_2}. \tag{8.81}$$

We define

$$\alpha_{p_1p_2} = \int dk_1 dk_2 \delta\left(k_1 + k_2 + p_1 + p_2\right) G_{k_1k_2;p_1p_2}. \tag{8.82}$$

We can rewrite (8.81) as

$$G_{k_1k_2;p_1p_2} = \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,G_0^{-1}(p_1,p_2) + \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,\alpha_{p_1p_2}. \tag{8.83}$$

In addition, if we multiplier (8.81) by $\delta(k_1 + k_2 + p_1 + p_2)$ and integrate over k_1 and k_2, (8.81) becomes:

$$\begin{aligned}
\alpha_{p_1p_2} = & \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\int dk_1 dk_2 \delta(k_1 + k_2 + p_1 + p_2)\,G_0^{-1}(p_1p_2) \\
& + \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\int dk_1 dk_2 \delta(k_1 + k_2 + p_1 + p_2)\,G_0^{-1}(k_1k_2) \\
& \times \int dk_3 \int dk_4 \delta(k_3 + k_4 + p_1 + p_2)\,G_{k_3k_4;p_1p_2} \\
& = \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}G_0^{-1}(p_1,p_2) + \frac{2\lambda}{4!}\beta_{p_1p_2}\frac{1}{(2\pi)^2}\alpha_{p_1p_2},
\end{aligned} \tag{8.84}$$

where

$$\beta_{p_1p_2} = \int dk_3 dk_4 \delta(k_3 + k_4 + p_1 + p_2)\,G_{k_3k_4;p_1p_2}. \tag{8.85}$$

Solving (8.84) for $\alpha_{p_1p_2}$, we obtain

$$\boxed{\alpha_{p_1p_2} = \frac{\frac{2\lambda}{4!}\beta_{p_1p_2}G_0^{-1}(p_1,p_2)}{1 - \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}.} \tag{8.86}$$

Plugging this solution for $\alpha_{p_1p_2}$ into (8.84) yields

$$\begin{aligned}
G_{k_1k_2;p_1p_2} &= \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,G_0^{-1}(p_1,p_2) + \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,\alpha_{p_1p_2} \\
&= \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\left[G_0^{-1}(p_1,p_2) + \frac{\frac{2\lambda}{4!}\beta_{p_1p_2}\frac{1}{(2\pi)^2}G_0^{-1}(p_1,p_2)}{1-\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}\right] \\
&= \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\left[\frac{1-\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}+\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}{1-\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}\right]G_0^{-1}(p_1,p_2) \\
&= \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,G_0^{-1}(p_1,p_2)\left[\frac{1}{1-\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}\right]. \qquad (8.87)
\end{aligned}$$

That is,

$$\boxed{G_{k_1k_2;p_1p_2} = \frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,G_0^{-1}(p_1,p_2)\left[\frac{1}{1-\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}\right].} \qquad (8.88)$$

Putting everything together, the full Green's function is

$$\boxed{\begin{aligned}
\hat{O}^{-1}_{k_3k_4,E_2;p_1p_2,E_3} &= iG^{0^{-1}}\left(\vec{k_3},\vec{k_4}\right)\delta\left(\vec{k_3}+\vec{p_2}\right)\delta\left(\vec{k_4}+\vec{p_1}\right)\delta\left(E_2+E_3\right) \\
&\quad+\delta\left(\vec{k_3}+\vec{k_4}+\vec{p_1}+\vec{p_2}\right)\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}G_0^{-1}(k_1,k_2)\,G_0^{-1}(p_1,p_2)\left[\frac{1}{1-\frac{2\lambda}{4!}\frac{1}{(2\pi)^2}\beta_{p_1p_2}}\right].
\end{aligned}} \qquad (8.89)$$

In the strong 't Hooft limit ($\lambda \to \infty$), this simplifies to:

$$\boxed{\begin{aligned}
\hat{O}^{-1}_{k_3k_4,E_2;p_1p_2,E_3} &= iG^{0^{-1}}\left(\vec{k_3},\vec{k_4}\right)\delta\left(\vec{k_3}+\vec{p_2}\right)\delta\left(\vec{k_4}+\vec{p_1}\right)\delta\left(E_2+E_3\right) \\
&\quad+\delta\left(\vec{k_3}+\vec{k_4}+\vec{p_1}+\vec{p_2}\right)G_0^{-1}(k_1,k_2)\,G_0^{-1}(p_1,p_2)\,\frac{1}{\beta_{p_1p_2}}.
\end{aligned}} \qquad (8.90)$$

We find, once more, that the propagator consists of a part which is identical to the free theory and a bound state.

Chapter 9

Collective Field Canonical Quantization, Spectrum And Map

Three Rules of Work: Out of clutter find simplicity. From discord find harmony. In the middle of difficulty lies opportunity.

-Albert Einstein.

According to the Klebanov-Polyakov conjecture [147], adding the "double-trace" operator

$$\delta S_{CFT} = \frac{\lambda}{4!} \int d^3x \left(\phi^i \phi^i\right)^2 \tag{9.1}$$

to the free theory on the CFT side corresponds to merely changing the boundary conditions of the bulk scalar field. In other words, one expects that the results for the critical $(\phi^2)^2$ theory can, in principle, be obtained from the free theory when one replaces $\Delta = 1$ by $\Delta = 2$.

We have, however, stumbled upon a puzzle. The puzzle is that our simple expectation seems to be invalid. This puzzle was already clear when we considered the non-linear sigma model and the pole structure of both the two-time and the single-time propagator. The results that we obtain seem to indicate or suggest that one has the previous degrees of freedom of the free case plus the degrees of freedom associated with some bound state.

A similar (related) puzzle is that the homogeneous equation (or equations of motion) only give us a mass condition for the bound state. Naturally, one can ask where are the "free modes" in this instance?

To understand this better, we return to the two-time equations of motion in coordinate space.

Recall that the quadratic effective action is

$$S_2^{eff} = \frac{i}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) - \frac{\lambda}{4!}\int d^d x \eta_{xx}^2. \tag{9.2}$$

and the equations of motion are

$$\left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{xy} = -\frac{i\lambda}{6}\delta\left(x-y\right)\eta_{xx}. \tag{9.3}$$

Since $\hat{p} = -i\partial$, it is straightforward to see that $\langle x\left|\partial\right|y\rangle = i\langle x\left|\hat{p}\right|y\rangle$. For the matrix element, we obtain

$$\begin{aligned} i\langle x\left|\hat{p}\right|y\rangle &= i\int dp_1\int dp_2 \langle x\left|p_1\right.\rangle\langle p_1\left|\hat{p}\right|p_2\rangle\langle p_2\left|y\right.\rangle \\ &= i\int dp_1\int dp_2\frac{e^{ip_1x}}{\sqrt{2\pi}}\delta\left(p_1-p_2\right)p_1\frac{e^{-ip_2y}}{\sqrt{2\pi}} \\ &= i\int\frac{dp}{2\pi}e^{ip(x-y)}p \\ &= i\left(-i\frac{\partial}{\partial x}\right)\delta\left(x-y\right) = \frac{\partial}{\partial x}\delta\left(x-y\right). \end{aligned} \tag{9.4}$$

That is,

$$\boxed{\langle x\left|\partial\right|y\rangle = \frac{\partial}{\partial x}\delta\left(x-y\right).} \tag{9.5}$$

Similarly, one can show that

$$\boxed{\left\langle x\left|\partial^2\right|y\right\rangle = \frac{\partial^2}{\partial x^2}\delta\left(x-y\right).} \tag{9.6}$$

The large-N background is

$$\psi_k^0 = \frac{i}{k^2}. \tag{9.7}$$

Therefore,

$$\left\langle x\left|\psi_k^{0^{-1}}\right|y\right\rangle = -i\partial_x^2\delta\left(x-y\right). \tag{9.8}$$

The L.H.S. of (9.3) becomes

$$\begin{aligned}\left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{xy} &= \int dx_1\int dx_2\left(\psi_0^{-1}\right)_{xx_1}\eta_{x_1x_2}\left(\psi_0^{-1}\right)_{x_2y}\\ &= \int dx_1\int dx_2\left(-i\partial_x^2\delta\left(x-x_1\right)\right)\eta_{x_1x_2}\left(-i\partial_{x_2}^2\delta\left(x_2-y\right)\right)\\ &= -\partial_x^2\partial_y^2\eta_{xy}.\end{aligned} \tag{9.9}$$

Accordingly, the equations of motion - *i.e.* (9.3) - can be written as

$$\boxed{\partial_x^2\partial_y^2\eta_{xy} = -\frac{i\lambda}{6}\delta\left(x-y\right)\eta_{xx}.} \tag{9.10}$$

We observe the presence of a Dirac delta function in terms of the relative coordinates $x - y$.

In order to understand the problem at hand we will consider a very simple example that will give us a hint as to how to proceed.

Let us focus on an extremely simple $1d$ problem that involves scattering from a Dirac delta-function potential. This is described by the equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + v_0\delta\left(x\right)\right)\psi\left(x\right) = E\psi\left(x\right). \tag{9.11}$$

This equation is easy to solve using potential scattering methods. One finds - when one moves to momentum space - that the most general solution is given by [235]:

$$\psi\left(x\right) = \psi_{\text{inc}}\left(x\right) + \int \frac{dk}{(2\pi)}\frac{e^{ikx}}{E - \frac{\hbar^2 k^2}{2m}} v_0\psi\left(0\right) \tag{9.12}$$

where $\psi_{\text{inc}}\left(x\right) = e^{ik_0 x}$ is the incident wave and satisfies the free Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_{\text{inc}} = E\psi_{inc} \tag{9.13}$$

For $x > 0$, we close the contour in the LHP and perform the integral using the residue theorem. Since the integrand only has a single pole at $k_0 = \sqrt{2mE/\hbar^2}$ in the LHP, we get

$$\int \frac{dk}{(2\pi)}\frac{e^{ikx}}{E - \frac{\hbar^2 k^2}{2m}} = \frac{me^{i\sqrt{2mE/\hbar^2}x}}{i\hbar\sqrt{2mE}}. \tag{9.14}$$

Taking $x < 0$ yields

$$\int \frac{dk}{(2\pi)}\frac{e^{ikx}}{E - \frac{\hbar^2 k^2}{2m}} = \frac{me^{-i\sqrt{2mE/\hbar}x}}{i\hbar\sqrt{2mE}}. \tag{9.15}$$

Therefore, the solution can be written as

$$\psi\left(x\right)=\psi_{\mathrm{inc}}\left(x\right)+\frac{me^{ik_0|x|}}{i\hbar^2k_0}v_0\psi\left(0\right). \tag{9.16}$$

When $x=0$ the solution simplifies to

$$\psi\left(0\right)=\psi_{\mathrm{inc}}\left(0\right)+\frac{m}{i\hbar^2k_0}v_0\psi\left(0\right) \tag{9.17}$$

or

$$\psi\left(0\right)=\frac{i\hbar^2k_0}{i\hbar^2k_0-mv_0}\psi_{inc}\left(0\right). \tag{9.18}$$

Accordingly, the wavefunction can be written as

$$\psi\left(x\right)=\psi_{inc}\left(x\right)+\frac{me^{ik_0|x|}}{i\hbar^2k_0-mv_0}v_0\psi_{inc}\left(0\right). \tag{9.19}$$

What we have done may seem elementary but there are few physical things that one can learn from this. Namely, that the total wave will consist of a superposition of the “free” incident wave and the scattered wave. This heuristic argument, if formalized properly, can assist us in resolving the above puzzle with the free modes still being present even when we add the “double trace” operator to the free $O(N)$ theory.

Recall that the IR theory corresponds to taking $\lambda\to\infty$ [208]. What this translates to in our simple toy model is looking at how the wavefunction behaves as we take $v_0\to\infty$. It is straightforward to see that

$$\lim_{v_0\to\infty}(\psi(0)) = \lim_{v_0\to\infty}\left(\psi_{inc}(0) + \frac{m}{i\hbar^2 k_0 - m v_0} v_0 \psi_{inc}(0)\right)$$
$$= \psi_{inc}(0) - \psi_{inc}(0) = 0. \tag{9.20}$$

This reasoning is already suggestive that a similar argument may lead to the conclusion that $\eta_{xx} = 0$.

To obtain the bound state we can look at the poles of (9.19). It is clear that (9.19) has a pole when $i\hbar^2 k_0 = m v_0$. Thus,

$$k_0 = -i\frac{m v_0}{\hbar^2} \tag{9.21}$$

which is purely imaginary. Accordingly, $2mE = \hbar^2 k_0^2$ is negative. That is, $E < 0$ and we have a bound state. Since $E < 0$ and $k_0 = i\sqrt{2m|E|}$, it follows from (9.21) that v_0 is negative [235].

We can arrive at the same conclusion by looking at the homogeneous equation *i.e.* (9.16) without the incident wave. The homogeneous equation is

$$\psi(x) = \frac{m e^{ik_0|x|}}{i\hbar^2 k_0} v_0 \psi(0). \tag{9.22}$$

The consistency condition at $x = 0$ leads to

$$1 = \frac{m}{i\hbar^2 k_0} v_0 \tag{9.23}$$

which is exactly what we had previously in (9.21).

To make the analogue with our problem much more clearer, let us move to momentum space. The wavefunction can be written as

$$\psi(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} A_k \tag{9.24}$$

and the incident wavefunction reads

$$\psi_{inc}(x) \doteq \varphi(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} \varphi_k. \tag{9.25}$$

Since $\psi_{\text{inc}}(x) = e^{ik_0 x}$, we have

$$\begin{aligned}
\varphi_k &= \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} \varphi(x), \\
&= \sqrt{2\pi} \int \frac{dk}{2\pi} e^{i(k_0 - k)x} \\
&= \sqrt{2\pi} \delta(k - k_0).
\end{aligned} \tag{9.26}$$

where k_0 is the wavenumber of the incident wave.

In momentum space (9.12) becomes

$$A_k = \varphi_k + \frac{v_0}{E - \frac{\hbar^2 k^2}{2m}} \int dk' A_{k'}. \tag{9.27}$$

Integrating the equation above yields:

$$\int dk A_k = \int dk \varphi_k + v_0 \int dk \frac{1}{E - \frac{\hbar^2 k^2}{2m}} \int dk' A_{k'} \tag{9.28}$$

which implies that

$$\int dk A_k \left(1 - v_0 \int dk' \frac{1}{E - \frac{\hbar^2 k'^2}{2m}} \right) = \int dk \varphi_k. \tag{9.29}$$

That is,

$$\int dk A_k = \frac{1}{1 - v_0 \int dk' \frac{1}{E - \frac{\hbar^2 k'^2}{2m}}} \int dk \varphi_k \tag{9.30}$$

Substituting this back into (9.27), we obtain

$$\begin{aligned} A_k &= \varphi_k + \frac{v_0}{E - \frac{\hbar^2 k^2}{2m}} \int dk' A_{k'} \\ &= \varphi_k + \frac{v_0}{E - \frac{\hbar^2 k^2}{2m}} \frac{1}{1 - v_0 \int dk' \frac{1}{E - \frac{\hbar^2 k'^2}{2m}}} \int dk \varphi_k. \end{aligned} \tag{9.31}$$

Integrating the final result leads to

$$\int dk A_k = \int dk \varphi_k + \int d\bar{k} \frac{1}{E - \frac{\hbar^2 \bar{k}^2}{2m}} v_0 \frac{1}{1 - v_0 \int dk' \frac{1}{E - \frac{\hbar^2 k'^2}{2m}}} \int dk \varphi_k \tag{9.32}$$

Taking the limit $v_0 \to \infty$, we obtain:

$$\begin{aligned} \int dk A_k &= \int dk \varphi_k + \int d\bar{k} \frac{1}{E - \frac{\hbar^2 \bar{k}^2}{2m}} v_0 \frac{1}{-v_0 \int dk' \frac{1}{E - \frac{\hbar^2 k'^2}{2m}}} \int dk \varphi_k \\ &= \int dk \varphi_k - \int dk \varphi_k = 0. \end{aligned} \tag{9.33}$$

The above intimations provide us with a way of tackling our original problem. In direct analogue to (9.27), the full solution to the scattering potential problem *i.e.* (8.25):

$$E^2_{k_1 k_2} \eta_{k_1 k_2} = \frac{1}{4} \left(\psi^{0^{-1}}_{k_1} + \psi^{0^{-1}}_{k_2} \right)^2 \eta_{k_1 k_2} + \frac{\lambda}{6} \left(\psi^0_{k_1} + \psi^0_{k_2} \right) \int \frac{d^{d-1} l}{(2\pi)^{d-1}} \eta_{k_1 + k_2 - l, l}, \tag{9.34}$$

can be written as

$$\eta_{k_1 k_2} = \varphi_{k_1 k_2} + \frac{\lambda \left(\frac{1}{|k_1|} + \frac{1}{|k_2|} \right)}{E^2 - (|k_1| + |k_2|)^2} \int \frac{d^3 k}{(2\pi)^2} \eta_{k.k_1 + k_2 - k}, \tag{9.35}$$

where[1]

$$E^2 = \left(|p_1| + |p_2|\right)^2, \qquad \varphi_{k_1 k_2} \sim \delta\left(k_1 - p_1\right)\delta\left(k_2 - p_2\right) \tag{9.36}$$

and $\varphi_{k_1 k_2}$ solves the free equation of motion.

Integrating both sides of (9.35), and using the results (7.33) and (8.42), yields[2]

$$\int d^2 k \eta_{k,k_1+k_2-k} = \int d^2 k \varphi_{k,k_1+k_2-k} + \lambda \left(\frac{1}{4\left(-p_\mu p^\mu\right)^{1/2}} \right) \int d^2 k \eta_{k,k_1+k_2-k}. \tag{9.37}$$

Hence,

$$\left(1 - \frac{\lambda}{4\left(-p_\mu p^\mu\right)^{1/2}}\right) \int d^2 k \eta_{k,k_1+k_2-k} = \int d^2 k \varphi_{k,k_1+k_2-k} \tag{9.38}$$

or

$$\int d^2 k \eta_{k,k_1+k_2-k} = \frac{1}{1 - \frac{\lambda}{4\left(-p_\mu p^\mu\right)^{1/2}}} \int d^2 k \varphi_{k,k_1+k_2-k}. \tag{9.39}$$

Plugging (9.39) into (9.35), we get

$$\begin{aligned}
\eta_{k,k_1+k_2-k} &= \varphi_{k,k_1+k_2-k} + \frac{\lambda\left(\frac{1}{|k|} + \frac{1}{|k_1+k_2-k|}\right)}{E^2 - \left(|k|^2 + |k_1+k_2-k|^2\right)^2} \int \frac{d^2 k}{(2\pi)^2} \eta_{k.k_1+k_2-k} \\
&= \varphi_{k,k_1+k_2-k} + \frac{\left(\frac{1}{|k|} + \frac{1}{|k_1+k_2-k|}\right)\lambda}{E^2 - \left(|k|^2 + |k_1+k_2-k|^2\right)^2} \frac{1}{1 - \frac{\lambda}{4\left(-p_\mu p^\mu\right)^{1/2}}} \int d^2 k \varphi_{k,k_1+k_2-k}.
\end{aligned} \tag{9.40}$$

[1] The energy E , in analogue to the $1d$ quantum mechanical problem involving a Dirac delta potential, only depends on the momenta of the incident wave and is thus fixed.

[2] Here, $p^\mu = \left(E, \vec{k_1} + \vec{k_2}\right)$.

We integrate the above equation and find that

$$\int d^2k\eta_{k,k_1+k_2-k} = \int d^2k\varphi_{k,k_1+k_2-k} + \int d^2k' \frac{\left(\frac{1}{|k'|} + \frac{1}{|k_1+k_2-k'|}\right)\lambda}{E^2 - \left(|k'|^2 + |k_1+k_2-k'|^2\right)^2} \times \frac{1}{1 - \frac{\lambda}{4(-p_\mu p^\mu)^{1/2}}} \int d^2k\varphi_{k,k_1+k_2-k}. \quad (9.41)$$

In IR $\lambda \to \infty$ and accordingly (9.41) simplifies to:

$$\begin{aligned}\int d^2k\eta_{k,k_1+k_2-k} &= \int d^2k\varphi_{k,k_1+k_2-k} + \int d^2k' \frac{\left(\frac{1}{|k'|} + \frac{1}{|k_1+k_2-k'|}\right)\lambda}{E^2 - \left(|k'|^2 + |k_1+k_2-k'|^2\right)^2} \\ &\quad \times \frac{1}{-\frac{\lambda}{4(-p_\mu p^\mu)^{1/2}}} \int d^2k\varphi_{k,k_1+k_2-k}. \\ &= \int d^2k\varphi_{k,k_1+k_2-k} + \frac{\lambda}{4\left(-p_\mu p^\mu\right)^{1/2}} \times \frac{1}{-\frac{\lambda}{4(-p_\mu p^\mu)^{1/2}}} \int d^2k\varphi_{k,k_1+k_2-k} \\ &= \int d^2k\varphi_{k,k_1+k_2-k} - \int d^2k\varphi_{k,k_1+k_2-k} = 0. \quad (9.42)\end{aligned}$$

That is,

$$\eta_{xx} \sim \int d^2k\eta_{k,k_1+k_2-k} = 0. \quad (9.43)$$

To summarize, we have demonstrated that $\eta_{xx} = 0$.

Chapter 10

Conclusions And Outlook

Those who will finish the course will do so only because they did not, as fatigue sets in, convince themselves that the road ahead is still too long, that the inclines are too steep, that the loneliness is impossible to bear and that the prize itself is of doubtful value.

- Thabo Mvuyelwa Mbeki, 1999.

The conjecture by Klebanov and Polyakov, which states that the Vasiliev higher spin gauge theory in AdS_4 is dual to the $O(N)$ vector model, provides us with a tractable form of the AdS/CFT correspondence. (The fact that the duality is tractable, however, is a qualified statement. In particular, the CFT side of the duality is relatively not too complicated. However, the gravity side suffers from the fact that, at present, the Vasiliev field equations cannot be obtained from an action.[1] This is reminiscent, for example, of the $6d$ $(2,0)$ SCFT which also doesn't have an action. More boldly, it has been suggested that these examples, and others, show that we need to have other formulations of Quantum Field Theories that are independent of a Lagrangian formalism [238]. For the Vasiliev higher spin gravity, however, there is the so-called unfolded formalism [239].

[1]For attempts at finding a Lagrangian for Vasiliev higher spin gravity, see [236, 237].

Nonetheless, much progress in higher spin holography is made to be challenging because of a lack of a Lagrangian.)

In this thesis, we generalized the standard results, obtained for the free critical point, to the infra-red interacting fixed point. In particular, the collective field theory provides a way to reconstruct the bulk gravity side from the $O(N)$ vector model.

The first two chapters of the dissertation were largely didactic or of an introductory nature. In particular, in the first chapter we reviewed the original Maldacena conjecture *i.e.* the AdS/CFT correspondence. In Chapter 2, we discussed the problem of constructing a consistent interacting higher spin theory in flat space. We discussed how the No-Go theorems were evaded by Vasiliev. We then discussed the Klebanov-Polyakov conjecture.

In Chapter 3, we reviewed the Jevicki-Sakita collective field theory. In particular, we wrote down the Large-N collective field theory for the $O(N)$ vector model. We then introduced the two-time bilocals. The collective field theory approach allowed us to write down an effective action. By varying the effective action, we obtained the large-N background field. Moreover, the form of the saddle-point equations enabled us to write down the gap equation.

In Chapter 4, we reviewed the collective theory bulk reconstruction of AdS_4. More precisely, we gave the map from $3d$ to $AdS_4 \times S^1$ in both the lightcone and timelike gauges. We then discussed the canonical or path integral formulation. We extracted the quadratic Hamiltonian and obtain the spectrum for the free $O(N)$ vector model. We then considered the single time Lagrangian. At the quadratic level, this could be written in terms of an operator. We inverted the operator and found the single-time propagator. After a brief discussion of the Bethe-Salpeter equation, we obtained the two-time free bilocal propagator.

In Chapter 5, we reviewed the argument that the critical $O(N)$ vector model – in the IR – is equivalent to the non-linear sigma model.

In Chapter 6, we looked at the non-linear sigma model. The non-linear sigma model was written in terms of the bilocals and a Lagrange multiplier. Using the collective field Jacobian, we found an effective action. We varied the effective action with respect to the bilocals. The resulting gap equation allowed us to solve for the Lagrange multiplier field. In the IR we require the Lagrange multiplier field to vanish. This requirement means that we have to take the large ‘t Hooft limit. The other equation of motion follows when we vary the effective action with respect to the Lagrange multiplier field. The resulting equation of motion forces us to shift the fluctuations. After we shift the fluctuations the action decouples. That is, the effective actions consist of the free part that we had previously in Chapter 5 and an extra state involving the Lagrange multiplier field. We commented on how this was slightly puzzling as the naïve guess would be that we simply replace the scaling dimension of one by two. We moved into momentum space and obtained the two-point functions for the bilocal fluctuations and the Lagrange multiplier field. The correlation functions were then written out in coordinate space.

In Chaper 7, we began by giving a simple discussion of how one can obtain the spectrum, of the scalar field theory, by looking at the poles of the propagator. In particular, we consider the critical $O(N)$ vector model. We write down the effective quadratic action in terms of an operator. We inverted this operator to obtain the two-time bilocal propagator for the critical $O(N)$ vector model. We then checked that the scaling dimension for the propagator was indeed two.We then looked at the poles of the propagator. We then showed how the pole condition can also be obtained by looking at the homogeneous equation. The pole condition was in terms of the standard loop integral over the momenta. We found that the spectrum was tachyonic unless we made the ’t Hooft coupling to be negative. We proceeded to write the solution to the equation that the fluctuations satisfied.

In Chapter 8, we revisited the Hamiltonian single-time formalism. This time we added the quartic interaction (quartic in the original $O(N)$ fields). We obtained a coupled integral equation for the spectra. This coupled integral equation was then written in terms of a kernel. We then showed that the pole condition in the two-time description is equivalent to the single-time pole condition. Furthermore, starting from the two-time bilocal propagator, we managed to demonstrate that we can obtain the single-time propagator. We then considered the single-time Lagrangian formalism for the critical $O(N)$ vector model. As is standard, we obtained the Green's function.

In Chapter 9, we revisited the puzzle of the fact that we seem to obtain the free part plus an additional state. We considered the simple case of scattering from a Dirac potential. This allowed us to argue that when the parameter in front of the Dirac delta function goes to infinity, the wavefunction vanishes. Using this observation, we demonstrated that $\eta_{xx} = 0$. For completeness, we also rewrote the quantum mechanical problem in a formalism closer to the one we have been using in this thesis.[2]

In this thesis, we mainly focused on pure $O(N)$ vector models. Of immediate value would be to obtain the general mode expansion, which is beyond the scope of this thesis. It is likely that the map to AdS_4 will provide the coordinates in which this can be implemented in a natural way.

Future works will include finding a stable large-N configuration, or elucidate on the meaning of the bound state. In particular, it might be the case that the fluctuations about another background yields the spectrum of the dual higher spin theory.

As mentioned in Chapter 2, vector models coupled to Chern-Simons gauge theory are dual to parity violating Vasiliev theory. The action for these models can be written as [160]:

[2] This will be contained in an appendix.

$$
\begin{aligned}
S &= S_{fermion} + S_{CS} \\
&= \int d^3x \bar{\psi} \gamma^\mu D_\mu \psi + \frac{ik}{4\pi} \int d^3x \mathrm{Tr} \left(\epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \right).
\end{aligned} \tag{10.1}
$$

where the covariant derivative is

$$
\left(D_\mu\right)_{ij} = \partial_\mu \delta_{ij} + A_\mu^a T_{ij} \tag{10.2}
$$

Here T_{ij} are the usual $U(N)$ generators.

It is convenient to work in lightcone coordinates defined via

$$
A^{\pm} = \frac{1}{\sqrt{2}} \left(A^1 + iA^3 \right) \tag{10.3}
$$

The action then reads [160]:

$$
S = \int d^3x \bar{\psi} \left(\gamma^\mu D_\mu \right) \psi + \frac{k}{8\pi} \int d^3x \left(A_3^a \partial_- A_+^a - A_+^a \partial_- A_3^a \right). \tag{10.4}
$$

The collective bilocals are

$$
\sigma_{\alpha\beta}(x, y) = \bar{\psi}_\alpha^i (x) \psi_\beta^i (y) \tag{10.5}
$$

At present, using collective field theory, we have managed to reproduce the free energy and the gap equation found in the literature *i.e.* [160]:

$$
\Sigma(q) = -2\pi i \lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k-q)_-} \gamma^{[3|} \frac{1}{\left(\gamma^\mu (ik_\mu) + \Sigma(k) \right)} \gamma^{|+]}, \tag{10.6}
$$

where

$$\gamma^{[3|}A\gamma^{|+]} = \gamma^3 A\gamma^+ - \gamma^+ A\gamma^3. \tag{10.7}$$

In addition, we can extract the effective quadratic action. The operator that we still need to invert is

$$\begin{aligned} \hat{O}_{\alpha_1\alpha_2}\left(k_1k_2;k_3k_4\right)_{\alpha_3\alpha_4} &= -\delta\left(k_1-k_4\right)\delta\left(k_2-k_3\right)\sigma^{-1}_{0,\alpha_4\alpha_1}\left(k_1\right)\sigma^{-1}_{0,\alpha_2\alpha_3}\left(k_3\right) \\ &+ \frac{4\pi i\lambda}{\left(2\pi\right)^3}\frac{\delta\left(k_1+k_3-k_2-k_4\right)}{\left(k_2-k_3\right)_-}\left(\gamma^{3^T}\right)_{\alpha_4\alpha_1}\left(\gamma^{+^T}\right)_{\alpha_2\alpha_3}. \end{aligned} \tag{10.8}$$

In the ABJ triality, an important part of the dictionary is the relation between the Vasiliev parity breaking phase and the ‘t Hooft coupling *viz.*

$$\theta_0\left(\lambda\right) = \frac{\pi}{2}\lambda. \tag{10.9}$$

It would be useful to see how such a relation would arise in the collective field theory approach.

Another objective would be to try and see how the bulk (at least at the linearized level) can be derived using collective fields.

Finally, recall that the extra AdS_{d+1} radial direction, in the context of stochastic quantization, is interpreted as the extra fictitious time [240]. In [241, 242], the Wilson-Polchinski exact renormalization description of higher spin holography was revisited.[3] In particular, starting from $2+1$ free Majorana fermions, the authors of [241] were able to derive a set of non-linear equations that resemble the standard Vasiliev higher spin equations. A natural question to ask is how this renormalization picture related to the stochastic collective field description [189].

[3]This is a continuation of the bulk reconstruction using the renormalization group that was mentioned in Chapter 2.

Appendix A

Three-Point Function Vertices

If you don't make mistakes, you're not working on hard enough problems. And that's a big mistake.

-Frank Wilczek.

In this appendix, we will continue the analysis that we started in Chapter 6. In particular, we will consider cubic vertices.

Since

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots \tag{A.1}$$

we can write the effective cubic action as

$$S_3^{eff} = -\frac{N}{2}\left(\frac{1}{3}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\eta\right)\right)\frac{1}{\left(\sqrt{N}\right)^3} \tag{A.2}$$

or

$$\boxed{S_3^{eff} = -\frac{1}{6\sqrt{N}}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\eta\right).} \tag{A.3}$$

We then shift the fluctuations as follows

$$\eta = \tilde{\eta} - i\left(\psi_0 \tilde{\alpha} \psi_0\right). \tag{A.4}$$

Thus,

$$\begin{aligned} S_3^{eff} &= -\frac{1}{6\sqrt{N}} \mathrm{Tr}\left(\psi_0^{-1} \eta \psi_0^{-1} \eta \psi_0^{-1} \eta\right) \\ &= -\frac{1}{6\sqrt{N}} \mathrm{Tr}\left\{\psi_0^{-1}\left[\tilde{\eta} - i\left(\psi_0 \tilde{\alpha} \psi_0\right)\right] \psi_0^{-1}\left[\tilde{\eta} - i\left(\psi_0 \tilde{\alpha} \psi_0\right)\right] \psi_0^{-1}\left[\tilde{\eta} - i\left(\psi_0 \tilde{\alpha} \psi_0\right)\right]\right\}. \end{aligned} \tag{A.5}$$

Multiplying everything out leads to a term cubic in $\tilde{\eta}$, *i.e.*,

$$S_3^{eff} \supset -\frac{1}{6\sqrt{N}} \mathrm{Tr}\left(\psi_0^{-1} \tilde{\eta} \psi_0^{-1} \tilde{\eta} \psi_0^{-1} \tilde{\eta}\right). \tag{A.6}$$

We also find a term cubic in the $\tilde{\alpha}$:

$$\begin{aligned} S_3^{eff} &\supset -\frac{1}{6\sqrt{N}} \left(-i\right)^3 \mathrm{Tr}\left[\psi_0^{-1}\left(\psi_0 \tilde{\alpha} \psi_0\right) \psi_0^{-1}\left(\psi_0 \tilde{\alpha} \psi_0\right) \psi_0^{-1}\left(\psi_0 \tilde{\alpha} \psi_0\right)\right] \\ &= -\frac{i}{6\sqrt{N}} \mathrm{Tr}\left(\tilde{\alpha} \psi_0 \tilde{\alpha} \psi_0 \tilde{\alpha} \psi_0\right), \end{aligned} \tag{A.7}$$

a term quadratic in $\tilde{\alpha}$ an linear in $\tilde{\eta}$:

$$\begin{aligned} S_3^{eff} &\supset -\frac{1}{6\sqrt{N}} \times 3\left(-i\right)^2 \mathrm{Tr}\left(\tilde{\eta} \tilde{\alpha} \psi_0 \tilde{\alpha}\right) \\ &= \frac{1}{2\sqrt{N}} \mathrm{Tr}\left(\tilde{\eta} \tilde{\alpha} \psi_0 \tilde{\alpha}\right). \end{aligned} \tag{A.8}$$

and a term linear in $\tilde{\alpha}$ and quadratic in $\tilde{\eta}$:

$$\begin{aligned} S_3^{eff} &\supset -\frac{1}{6\sqrt{N}}\text{Tr}\left(\left(\tilde{\alpha}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right)\right) \\ &= \frac{i}{2\sqrt{N}}\text{Tr}\left(\tilde{\alpha}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right). \end{aligned} \tag{A.9}$$

Thus, the effective cubic action is

$$\begin{aligned} S_3^{eff} &= -\frac{1}{6\sqrt{N}}\text{Tr}\left(\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right) + \frac{i}{2\sqrt{N}}\text{Tr}\left(\tilde{\alpha}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right) \\ &+ \frac{1}{2\sqrt{N}}\text{Tr}\left(\tilde{\eta}\tilde{\alpha}\psi_0\tilde{\alpha}\right) - \frac{i}{6\sqrt{N}}\text{Tr}\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right). \end{aligned} \tag{A.10}$$

In momentum space, the cubic vertex which is cubic in the $\tilde{\eta}$ is

$$\begin{aligned} -\frac{1}{6\sqrt{N}}\text{Tr}\left(\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right) &= -\frac{1}{6\sqrt{N}}\int d^dk_1d^dk_2d^dk_3 \\ \eta_{k_1k_2}\eta_{-k_2,k_3}\eta_{-k_3,-k_1}\psi_{k_1}^0\psi_{k_2}^0\psi_{k_3}^0 &= \frac{1}{6\sqrt{N}}\int d^dk_1d^dk_2d^dk_3k_1^2k_2^2k_3^2\eta_{k_1k_2}\eta_{-k_2,k_3}\eta_{-k_3,-k_1}. \end{aligned} \tag{A.11}$$

Similarly, the cubic vertex which is cubic in $\tilde{\alpha}$ is

$$\begin{aligned} -\frac{i}{6\sqrt{N}}\text{Tr}\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right) &= -\frac{i}{6\sqrt{N}}\int d^dp_1d^dp_2\tilde{\alpha}_{p_1}\tilde{\alpha}_{p_2}\tilde{\alpha}_{-p_1-p_2} \\ \times\int\frac{d^dk}{(2\pi)^{\frac{3d}{2}}}\psi_k^0\psi_{k-p}^0\psi_{k-p_1-p_2}^0 &= -\frac{i}{6\sqrt{N}}\int\frac{d^dp_1d^dp_2}{(2\pi)^{d/2}}\tilde{\alpha}_{p_1}\tilde{\alpha}_{p_2}\tilde{\alpha}_{-p_1-p_2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k-p_1)^2} \\ \times\frac{1}{(k-p_1-p_2)^2} &= -\frac{i}{6\sqrt{N}}\int\frac{d^dp_1d^dp_2}{(2\pi)^{d/2}}\tilde{\alpha}_{p_1}\tilde{\alpha}_{p_2}\tilde{\alpha}_{-p_1-p_2}\int\frac{d^dk}{(2\pi)^d} \\ &\times\frac{1}{(k+p_1)^2}\frac{1}{k^2}\frac{1}{(k-p_2)^2}. \end{aligned} \tag{A.12}$$

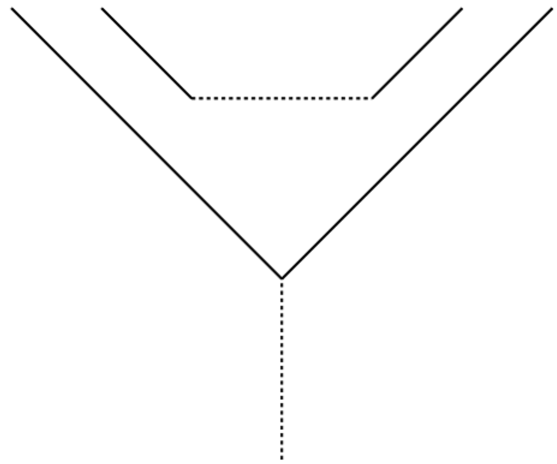


Figure A.1: The $\mathrm{Tr}\left(\tilde{\alpha}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right)$ vertex.

For the vertex that is quadratic in $\tilde{\alpha}$ and linear in $\tilde{\eta}$, we obtain

$$\begin{aligned}\frac{1}{2\sqrt{N}}\mathrm{Tr}\left(\tilde{\eta}\tilde{\alpha}\psi_0\tilde{\alpha}\right) &= \frac{1}{2\sqrt{N}}\int\frac{d^dk_1d^dk_2d^dk_3}{(2\pi)^d}\eta_{k_1k_2}\tilde{\alpha}_{-k_2-k_1}\psi^0_{k_3}\tilde{\alpha}_{-k_3-k_1} \\ &= \frac{1}{2\sqrt{N}}\int\frac{d^dk_1d^dk_2d^dk_3}{(2\pi)^d}\eta_{k_1k_2}\frac{\tilde{\alpha}_{-k_2-k_3}\tilde{\alpha}_{k_3-k_1}}{k_3^2}.\end{aligned} \tag{A.13}$$

Finally, the vertex which linear in $\tilde{\alpha}$ and quadratic in $\tilde{\eta}$ is

$$\begin{aligned}\frac{i}{2\sqrt{N}}\mathrm{Tr}\left(\tilde{\alpha}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right) &= \frac{i}{2\sqrt{N}}\int\frac{d^dk_1d^dk_2d^dk_3}{(2\pi)^{d/2}}\eta_{k_1k_2}\psi^{-1}_{k_2}\eta_{-k_2,k_3}\tilde{\alpha}_{-k_1-k_3} \\ &= \frac{i}{2\sqrt{N}}\int\frac{d^dk_1d^dk_2d^dk_3}{(2\pi)^{d/2}}\eta_{k_1k_2}k_1^2\eta_{-k_2,k_3}\tilde{\alpha}_{-k_1-k_3}.\end{aligned} \tag{A.14}$$

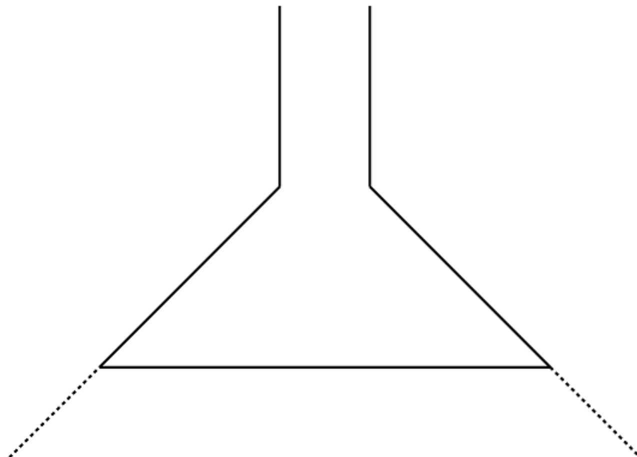


Figure A.2: The $\mathrm{Tr}\,(\tilde{\eta}\tilde{\alpha}\psi_0\tilde{\alpha})$ vertex.

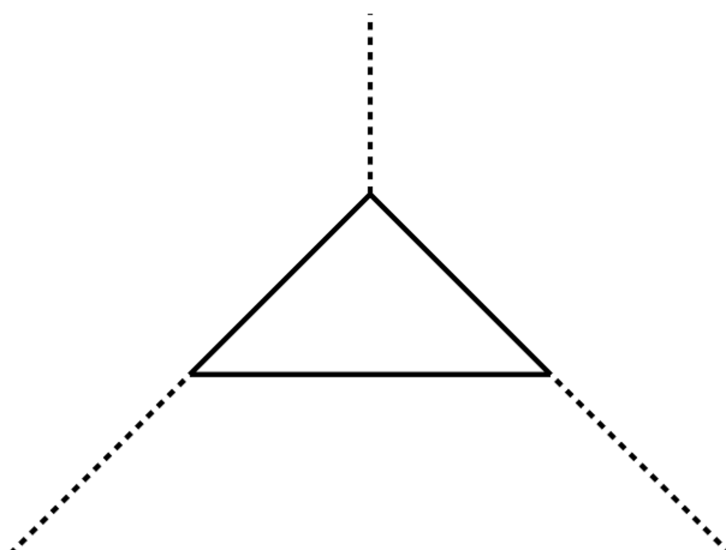


Figure A.3: The $\mathrm{Tr}\,(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0)$ vertex.

Appendix B

Mode Expansion

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

-Sidney Coleman.

Recall that the quadratic effective action is

$$H_2 = 2\mathrm{Tr} p\psi_0 p + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right), \tag{B.1}$$

and the equations of motion are

$$\begin{aligned}\dot{\eta}_{xy} &= \frac{\delta H_2}{\delta p_{xy}} \\ &= 2\left(p\psi_0\right)_{yx} + 2\left(\psi_0 p\right)_{yx}\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}\dot{p}_{xy} &= -\frac{\delta H_2}{\delta \eta_{xy}} \\ &= -\frac{1}{8}\left(\left(\psi_0^{-1}\eta\psi_0^{-2}\right)_{yx} + \left(\psi_0^{-2}\eta\psi_0^{-1}\right)_{yx}\right).\end{aligned} \tag{B.3}$$

Suppose we have oscillator expansions of the form

$$\eta_{xy} = e^{-i\omega t} e^{ik_1 x} e^{ik_2 y} A_{k_1 k_2} \tag{B.4}$$

$$p_{xy} = e^{-i\omega t} e^{ik_1 x} e^{ik_2 y} B_{k_1 k_2}. \tag{B.5}$$

Then using this in (B.2), we obtain

$$2\left(\psi_{k_1} + \psi_{k_2}\right) B_{k_1 k_2} = -i\omega A_{k_1 k_2}. \tag{B.6}$$

Or

$$B_{k_1 k_2} = -\frac{i\omega}{2\left(\psi_{k_1} + \psi_{k_2}\right)} A_{k_1 k_2}, \tag{B.7}$$

where

$$\omega = \left(\psi_{k_1}^{-1} + \psi_{k_2}^{-1}\right). \tag{B.8}$$

Then

$$\eta_{xy} = \int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}} \Bigg(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{ik_1 x} e^{ik_2 y} \alpha_{k_1 k_2} A_{k_1 k_2} + e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{-ik_1 x} e^{-ik_2 y} \alpha^*_{k_1 k_2} A^\dagger_{k_1 k_2} \Bigg), \tag{B.9}$$

and

$$p_{xy} = \int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}} \Bigg(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{ik_1 x} e^{ik_2 y} \left(-\frac{i\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha_{k_1k_2} A_{k_1k_2}$$
$$+ e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{-ik_1 x} e^{-ik_2 y} \left(\frac{i\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha^*_{k_1k_2} A^\dagger_{k_1k_2} \Bigg). \quad \text{(B.10)}$$

We will fix $\alpha_{k_1k_2}$ such that η_{xy} and p_{xy} are conjugates of each other. That is, we require that

$$[p_{xy}, \eta_{x'y'}] = -i\delta\left(x - x'\right)\delta\left(y - y'\right). \quad \text{(B.11)}$$

The commutator of η_{xy} and p_{xy} is

$$\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk'_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk'_2}{\left(\sqrt{2\pi}\right)^{d-1}} \Bigg\{ e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{ik_1 x} e^{ik_2 y} \left(-\frac{i\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha_{k_1k_2}$$
$$\times e^{i\left(\omega_{k'_1}+\omega_{k'_2}\right)t} e^{-ik'_1 x} e^{-ik'_2 y} \alpha^*_{k'_1k'_2} \left[A_{k_1k_2}, A^\dagger_{k'_1k'_2}\right] + e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{-ik_1 x} e^{-ik_2 y} \left(\frac{i\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha^*_{k_1k_2}$$
$$\times e^{-i\left(\omega_{k'_1}+\omega_{k'_2}\right)t} e^{ik'_1 x} e^{ik'_2 y} \alpha_{k'_1k'_2} \left[A^\dagger_{k_1k_2}, A_{k'_1k'_2}\right]$$
$$= \int \frac{dk_1}{(2\pi)^{d-1}} \int \frac{dk_2}{(2\pi)^{d-1}} \Bigg(e^{ik_1(x-x')} e^{ik_2(y-y')} \left(-\frac{i\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha_{k_1k_2}\alpha^*_{k_1k_2}$$
$$+ e^{-ik_1(x-x')} e^{-ik_2(y-y')} \left(-\frac{i\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha_{k_1k_2}\alpha^*_{k_1k_2} \Bigg). \quad \text{(B.12)}$$

In order for (B.11) to hold we require that

$$\left(\frac{1\left(\omega_{k_1}+\omega_{k_2}\right)}{2\left(\psi_{k_1}+\psi_{k_2}\right)}\right) \alpha_{k_1k_2}\alpha^*_{k_1k_2} = \frac{1}{2} \quad \text{(B.13)}$$

or

$$\alpha_{k_1k_2} = \sqrt{\frac{\psi_{k_1} + \psi_{k_2}}{\omega_{k_1} + \omega_{k_2}}}. \tag{B.14}$$

Recall that

$$\omega_k = \sqrt{k^2} = \frac{1}{2\psi_k}. \tag{B.15}$$

Hence,

$$\begin{aligned}
\alpha_{k_1k_2} &= \sqrt{\frac{\psi_{k_1} + \psi_{k_2}}{\omega_{k_1} + \omega_{k_2}}} \\
&= \sqrt{\frac{\psi_{k_1} + \psi_{k_2}}{\frac{1}{2\psi_{k_1}} + \frac{1}{2\psi_{k_2}}}} \\
&= \sqrt{2\psi_{k_1}\psi_{k_2}} = \frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}
\end{aligned} \tag{B.16}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{(\omega_{k_1} + \omega_{k_2})}{(\psi_{k_1} + \psi_{k_2})}\alpha_{k_1k_2} &= \frac{1}{2}\frac{(\omega_{k_1} + \omega_{k_2})}{(\psi_{k_1} + \psi_{k_2})}\sqrt{\frac{\psi_{k_1} + \psi_{k_2}}{\omega_{k_1} + \omega_{k_2}}} \\
&= \frac{1}{2}\sqrt{\frac{\omega_{k_1} + \omega_{k_2}}{\psi_{k_1} + \psi_{k_2}}} \\
&= \frac{1}{2}\alpha^{-1}_{k_1k_2} = \sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}.
\end{aligned} \tag{B.17}$$

Therefore, the mode expansion for the fluctuations becomes

$$\begin{aligned}
\eta_{xy} = \int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}} \frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}} \Bigg(& e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{ik_1x} e^{ik_2y} A_{k_1k_2} \\
& + e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{-ik_1x} e^{-ik_2y} A^{\dagger}_{k_1k_2} \Bigg).
\end{aligned} \tag{B.18}$$

Likewise, the mode expansion of the momentum

$$p_{xy} = \int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}} \left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right) \Bigg(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{ik_1 x} e^{ik_2 y} A_{k_1 k_2} + e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{-ik_1 x} e^{-ik_2 y} A^{\dagger}_{k_1 k_2} \Bigg). \quad \text{(B.19)}$$

Now that we have the mode expansions, the next thing one can do is to look at the expansion for the effective quadratic action.

Recall that

$$H_2 = 2\mathrm{Tr}p\psi_0 p + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right). \quad \text{(B.20)}$$

Using the mode expansions, we can write the first term of the effective quadratic action as

$$\begin{aligned} 2\mathrm{Tr}p\psi_0 p = 2 &\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk}{(2\pi)^{d-1}} \int \frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}} \int \frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}} \int dx \\ \times \int dy \int dz &\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right) \left(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{ik_1 x} e^{ik_2 y} A_{k_1 k_2} + e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t} e^{-ik_1 x} e^{-ik_2 y} A^{\dagger}_{k_1 k_2}\right) e^{ik(y-z)} \\ \times \psi_k &\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right) \left(e^{-i\left(\omega_{k_1'}+\omega_{k_2'}\right)t} e^{ik_1' z} e^{ik_2' x} A_{k_1' k_2'} + e^{i\left(\omega_{k_1'}+\omega_{k_2'}\right)t} e^{-ik_1' z} e^{-ik_2' x} A^{\dagger}_{k_1' k_2'}\right). \end{aligned} \quad \text{(B.21)}$$

The $AA^\dagger$ term yields

$$
\begin{aligned}
&2\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk}{(2\pi)^{d-1}}\int \frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}\int dx\int dy\int dz \\
&\times\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)e^{ik(y-z)}e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik_1x}e^{ik_2y}A_{k_1k_2}e^{i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}e^{-ik_1'z}e^{-ik_2'x}A^\dagger_{k_1'k_2'}\psi_k \\
&\quad= 2\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk}{(2\pi)^{d-1}}\int \frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}(2\pi)^{3(d-1)} \\
&\qquad\qquad\times\delta\left(k_1-k_2'\right)\delta\left(k_2+k\right)\delta\left(k_1'+k\right) \\
&\qquad\times\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)A_{k_1k_2}A^\dagger_{k_1'k_2'}\psi_k e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}\psi_k \\
&\qquad= 2\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)A_{k_1k_2}A^\dagger_{k_2k_1} \\
&\qquad\qquad\qquad\qquad= 2\int dk_1\int dk_2\left(\frac{\omega_{k_1}\omega_{k_2}}{2}\right)\psi_{k_2}A_{k_1k_2}A^\dagger_{k_2k_1}. \quad \text{(B.22)}
\end{aligned}
$$

For the $A^\dagger A$ term, we obtain

$$
\begin{aligned}
&2\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk}{(2\pi)^{d-1}}\int \frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}\int dx\int dy\int dz \\
&\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik(y-z)}e^{-ik_1x}e^{-ik_2y}A^\dagger_{k_1k_2}e^{-i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}e^{ik_1'z}e^{ik_2'x}A_{k_1'k_2'}\psi_k \\
&\quad= 2\int \frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk}{(2\pi)^{d-1}}\int \frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int \frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}(2\pi)^{3(d-1)} \\
&\qquad\qquad\times\delta\left(k_2'-k_1\right)\delta\left(k-k_2\right)\delta\left(k_1'-k\right) \\
&\qquad\times e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)A^\dagger_{k_1k_2}A_{k_1'k_2'}\psi_k \\
&\qquad\qquad\qquad\qquad= 2\int dk_1\int dk_2\left(\frac{\omega_{k_1}\omega_{k_2}}{2}\right)\psi_{k_2}A^\dagger_{k_1k_2}A_{k_2k_1}. \quad \text{(B.23)}
\end{aligned}
$$

The AA term is

$$
\begin{aligned}
&2\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}\int dx\int dy\int dz\\
&\times\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik(y-z)}e^{ik_1x}e^{ik_2y}A_{k_1k_2}e^{-i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}e^{ik_1'z}e^{ik_2'x}A_{k_1'k_2'}\\
&=2\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}(2\pi)^{3(d-1)}\\
&\qquad\times\delta\left(k_1+k_2'\right)\delta\left(k_2+k\right)\delta\left(k_1'-k\right)\\
&\quad\times\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}A_{k_1k_2}A_{k_1'k_2'}\\
&=-2\int dk_1\int dk_2\left(\frac{\omega_{k_1}\omega_{k_2}}{2}\right)e^{-2i\left(\omega_{k_1}+\omega_{k_2}\right)t}A_{k_1k_2}A_{-k_2,-k_1}\psi_{k_2}. \qquad \text{(B.24)}
\end{aligned}
$$

Finally, the $A^\dagger A^\dagger$ yields

$$
\begin{aligned}
&2\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}\int dx\int dy\int dz\\
&\times\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-ik_1x}e^{-ik_2y}A^\dagger_{k_1k_2}e^{ik(y-z)}e^{i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}e^{-ik_1'z}e^{-ik_2'x}A^\dagger_{k_1'k_2'}\\
&=2\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dk_1'}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk_2'}{\left(\sqrt{2\pi}\right)^{d-1}}(2\pi)^{3(d-1)}\\
&\qquad\times\delta\left(-k_1-k_2\right)\delta\left(-k_2+k\right)\delta\left(-k-k_1'\right)\\
&\quad\times\left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right)e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}A^\dagger_{k_1k_2}A^\dagger_{k_1'k_2'}\\
&=-2\int dk_1\int dk_2\left(\frac{\omega_{k_1}\omega_{k_2}}{2}\right)e^{2i\left(\omega_{k_1}+\omega_{k_2}\right)t}\psi_{k_2}A^\dagger_{k_1k_2}A_{-k_2,-k_1}. \qquad \text{(B.25)}
\end{aligned}
$$

Next, we consider the second term on the RHS of B.1. Writing out the trace and using the mode expansion for the fluctuations, we obtain

$$
\begin{aligned}
&\frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) = \frac{1}{8}\int dx_1\cdots\int dx_4 \eta_{x_1x_2}\psi^{-1}_{0x_2x_3}\eta_{x_3x_4}\psi^{-2}_{x_4x_1} \\
&= \int \frac{dk_1}{(2\pi)^{d-1}}\cdots\int\frac{dk_4}{(2\pi)^{d-1}}\int\frac{dk}{(2\pi)}\int\frac{dl}{(2\pi)}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}e^{ik(x_2-x_3)}e^{il(x_4-x_1)} \\
&\quad\times\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\left(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik_1x_1}e^{ik_2x_2}A_{k_1k_2}+e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-ik_1x_1}e^{-ik_2x_2}A^\dagger_{k_1k_2}\right) \\
&\quad\times\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}\left(e^{-i\left(\omega_{k_3}+\omega_{k_4}\right)t}e^{ik_3x_3}e^{ik_4x_4}A_{k_3k_4}+e^{i\left(\omega_{k_3}+\omega_{k_4}\right)t}e^{-ik_3x_3}e^{-ik_4x_4}A^\dagger_{k_3k_4}\right). \qquad \text{(B.26)}
\end{aligned}
$$

We multiply everything out and find the $AA^\dagger$ term is

$$
\begin{aligned}
&\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}e^{ik(x_2-x_3)}e^{il(x_4-x_1)} \\
&\quad\times\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik_1x_1}e^{ik_2x_2}A_{k_1k_2}e^{i\left(\omega_{k_3}+\omega_{k_4}\right)t}e^{-ik_3x_3}e^{-ik_4x_4}A^\dagger_{k_3k_4} \\
&\quad= \frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}(2\pi)^{4(d-1)}\delta(k_1-l) \\
&\qquad\times\delta(k_2+k)\,\delta(k_3-k)\,\delta(k_4-l) \\
&\qquad\times\frac{\psi_k^{-1}\psi_l^{-2}}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}A_{k_1k_2}e^{i\left(\omega_{k_3}+\omega_{k_4}\right)t}A^\dagger_{k_3k_4} \\
&\qquad= \frac{1}{8}\int dk_1\int dk_2\frac{1}{2\omega_{k_1}\omega_{k_2}}\psi_k^{-1}\psi_l^{-2}A^\dagger_{k_1k_2}A_{k_2k_1}. \qquad \text{(B.27)}
\end{aligned}
$$

For the $A^\dagger A$ term, we have

$$\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}e^{ik(x_2-x_3)}e^{il(x_4-x_1)}$$
$$\times\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-ik_1x_1}e^{-ik_2x_2}A^\dagger_{k_1k_2}e^{-i\left(\omega_{k_3}+\omega_{k_4}\right)t}e^{ik_3x_3}e^{ik_4x_4}A_{k_3k_4}$$
$$=\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}\left(2\pi\right)^{4(d-1)}\delta\left(l+k_1\right)$$
$$\times\delta\left(k-k_2\right)\delta\left(k_3-k\right)\delta\left(k_4+l\right)\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}A^\dagger_{k_1k_2}e^{-i\left(\omega_{k_3}+\omega_{k_4}\right)t}A_{k_3k_4}$$
$$=\frac{1}{8}\int dk_1\int dk_2\psi_{k_2}^{-2}\psi_{k_1}^{-1}\frac{1}{2\omega_{k_1}\omega_{k_2}}\psi_k^{-1}\psi_l^{-2}A^\dagger_{k_1k_2}A_{k_2k_1}. \quad \text{(B.28)}$$

The AA term is found to be

$$\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}e^{ik(x_2-x_3)}e^{il(x_4-x_1)}$$
$$\times\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik_1x_1}e^{ik_2x_2}A_{k_1k_2}e^{-i\left(\omega_{k_3}+\omega_{k_4}\right)t}e^{ik_3x_3}e^{ik_4x_4}A_{k_3k_4}$$
$$=\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}\left(2\pi\right)^{4(d-1)}\delta\left(k_1-l\right)$$
$$\times\delta\left(k+k_2\right)\delta\left(k_3-k\right)\delta\left(k_4+l\right)\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}A_{k_1k_2}A_{k_3k_4}$$
$$=\frac{1}{8}\int dk_1\int dk_2\frac{1}{2\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}e^{-2i\left(\omega_{k_1}+\omega_{k_2}\right)t}A_{k_1k_2}A_{-k_2,-k_1}. \quad \text{(B.29)}$$

The final term that we will evaluate is the $A^\dagger A^\dagger$ term and we find

$$\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\psi_k^{-1}\psi_l^{-2}e^{ik(x_2-x_3)}e^{il(x_4-x_1)}$$
$$\times\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-ik_1x_1}e^{-ik_2x_2}A^\dagger_{k_1k_2}e^{i\left(\omega_{k_3}+\omega_{k_4}\right)t}e^{-ik_3x_3}e^{-ik_4x_4}A^\dagger_{k_3k_4}$$
$$=\frac{1}{8}\int\frac{dk_1}{\left(\sqrt{2\pi}\right)^{d-1}}\cdots\int\frac{dk_4}{\left(\sqrt{2\pi}\right)^{d-1}}\int\frac{dk}{(2\pi)^{d-1}}\int\frac{dl}{(2\pi)^{d-1}}\int dx_1\cdots\int dx_4\,(2\pi)^{4(d-1)}\,\delta\left(-l-k_1\right)$$
$$\times\,\delta\left(k-k_2\right)\delta\left(-k-k_3\right)\delta\left(-k_4+l\right)\frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}\omega_{k_4}}}e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}A^\dagger_{k_1k_2}e^{i\left(\omega_{k_3}+\omega_{k_4}\right)t}A^\dagger_{k_3k_4}$$
$$=\frac{1}{8}\int dk_1\int dk_2\frac{1}{2\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-2}\psi_{k_1}^{-1}e^{2i\left(\omega_{k_1}+\omega_{k_2}\right)t}A^\dagger_{k_1k_2}A^\dagger_{k_3k_4}. \quad \text{(B.30)}$$

Putting everything together, we have

$$H_2=\int dk_1\int dk_2\left(\omega_{k_1}\omega_{k_2}\psi_{k_2}+\frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}\right)A_{k_1k_2}A^\dagger_{k_2k_1}$$
$$+\int dk_1\int dk_2\left(\omega_{k_1}\omega_{k_2}\psi_{k_2}+\frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}\right)A^\dagger_{k_1k_2}A_{k_2k_1}$$
$$+\int dk_1\int dk_2\left(-\omega_{k_1}\omega_{k_2}\psi_{k_2}+\frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}\right)e^{2i\left(\omega_{k_1}+\omega_{k_2}\right)t}A_{k_1k_2}A_{-k_2,,-k_1}$$
$$+\int dk_1\int dk_2\left(-\omega_{k_1}\omega_{k_2}\psi_{k_2}+\frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}\right)e^{-2i\left(\omega_{k_1}+\omega_{k_2}\right)t}A^\dagger_{k_1k_2}A^\dagger_{-k_2,,-k_1}. \quad \text{(B.31)}$$

We symmetrize the term appearing in the brackets *i.e.* we write

$$-\omega_{k_1}\omega_{k_2}\psi_{k_2}+\frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}=-\frac{1}{2}\omega_{k_1}\omega_{k_2}\left(\psi_{k_1}+\psi_{k_2}\right)+\frac{1}{32\omega_{k_1}\omega_{k_2}}\left(\psi_{k_2}^{-1}\psi_{k_1}^{-2}+\psi_{k_1}^{-1}\psi_{k_2}^{-2}\right)$$
$$=-\frac{1}{2}\omega_{k_1}\omega_{k_2}\left(\frac{1}{2\omega_{k_1}}+\frac{1}{2\omega_{k_2}}\right)+\frac{1}{32\omega_{k_1}\omega_{k_2}}\left(2\omega_{k_2}\times 4\omega_{k_1}^2+2\omega_{k_1}\times 4\omega_{k_2}^1\right)$$
$$=-\frac{1}{4}\left(\omega_{k_2}+\omega_{k_1}\right)+\frac{1}{4}\left(\omega_{k_1}+\omega_{k_2}\right)$$
$$=0 \quad \text{(B.32)}$$

and

$$\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2} = \frac{1}{4}\left(\omega_{k_2}+\omega_{k_1}\right) + \frac{1}{4}\left(\omega_{k_1}+\omega_{k_2}\right)$$
$$= \frac{1}{2}\left(\omega_{k_1}+\omega_{k_2}\right). \tag{B.33}$$

Therefore,

$$\begin{aligned}
H_2 &= \int dk_1 \int dk_2 \left(\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}\right) A_{k_1k_2}A^{\dagger}_{k_2k_1} \\
&+ \int dk_1 \int dk_2 \left(\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_1}^{-2}\right) A^{\dagger}_{k_1k_2}A_{k_2k_1} \\
&= \int dk_1 \int dk_2 \frac{1}{2}\left(\omega_{k_1}+\omega_{k_2}\right)\left(A_{k_1k_2}A^{\dagger}_{k_2k_1} + A^{\dagger}_{k_1k_2}A_{k_2k_1}\right) \\
&= \int dk_1 \int dk_2 \frac{1}{2}\left(\omega_{k_1}+\omega_{k_2}\right)\left(A_{k_1k_2}A^{\dagger}_{k_1k_2} + A^{\dagger}_{k_1k_2}A_{k_1k_2}\right).
\end{aligned} \tag{B.34}$$

The cubic Hamiltonian is

$$\begin{aligned}
H_3 &= 2\mathrm{Tr}\left(p\eta p\right) - \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) \\
&= -\frac{\sqrt{2}}{3}\int dk_1 \int dk_2 \int dk_3 \left(\omega_{k_1}+\omega_{k_2}+\omega_{k_3}\right) A_{k_1k_2}A_{-k_2,k_3}A_{-k_3,-k_1} \\
&- \frac{1}{\sqrt{2}}\int dk_1 \int dk_2 \int dk_3 \omega_{k_2} A_{k_1k_2}A_{-k_2,k_3}A^{\dagger}_{k_3k_1}.
\end{aligned} \tag{B.35}$$

Appendix C

The Propagator For Scattering from a Dirac Delta Potential

In this appendix, we compute the propagator for the quantum mechanical problem with the Dirac delta function potential.

Recall that the quantum mechanical system that we wish to consider is (9.11):

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + v_0\delta\left(x\right)\right)\psi\left(x\right) = E\psi\left(x\right) \tag{C.1}$$

By finding the propagator for this system we mean that we wish to rewrite the above problem as

$$\int dy\hat{O}\left(x,y\right)G\left(y,x'\right) = \delta\left(x-x'\right), \tag{C.2}$$

where

$$\hat{O}(x,y) = \delta(x-y)\left[E - \frac{\hbar^2}{2m}\frac{d^2}{dy^2} - v_0\delta(y)\right]. \tag{C.3}$$

and $G(y,x')$ in momentum space can be written as

$$G(y,x') = \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 y} e^{ik_3 x'} G_{k_2 k_3}. \tag{C.4}$$

Inserting this into the L.H.S. (C.2), we obtain

$$\begin{aligned}
\int dy \hat{O}(x,y) G(y,x') &= \int dy \delta(x-y)\left[E - \frac{\hbar^2}{2m}\frac{d^2}{dy^2} - v_0\delta(y)\right] \\
\times \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 y} e^{ik_3 x'} G_{k_2 k_3} &= \int dy \delta(x-y)\left(E - \frac{\hbar^2}{2m}\frac{d^2}{dy^2}\right) \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 y} e^{ik_3 x'} G_{k_2 k_3} \\
&\quad - v_0 \int dy \delta(x-y)\delta(y) \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 y} e^{ik_3 x'} G_{k_2 k_3}. \qquad \text{(C.5)}
\end{aligned}$$

For the last term, we have

$$\begin{aligned}
-v_0 \int dy \delta(x-y)\delta(y) \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 y} e^{ik_3 x'} G_{k_2 k_3} &= -v_0 \delta(x) \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 x} e^{ik_3 x'} G_{k_2 k_3} \\
&= -v_0 \int \frac{dp}{2\pi} e^{ipx} \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 x} e^{ik_3 x'} G_{k_2 k_3} \\
&= -v_0 \int \frac{dp}{2\pi} \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{i(p+k_2)x} e^{ik_3 x'} G_{k_2 k_3} \\
&= -v_0 \int \frac{dk_2}{\sqrt{2\pi}} \int \frac{dk_3}{\sqrt{2\pi}} e^{ik_2 y} e^{ik_3 x'} \int \frac{dp}{2\pi} G_{p,k_3}. \qquad \text{(C.6)}
\end{aligned}$$

The term involving the kinetic (second derivative) term yields

$$\int dy \delta\left(x-y\right)\left(E-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\right)\int\frac{dk_2}{\sqrt{2\pi}}\int\frac{dk_3}{\sqrt{2\pi}}e^{ik_2y}e^{ik_3x'}G_{k_2k_3} = \int\frac{dk_2}{\sqrt{2\pi}}\int\frac{dk_3}{\sqrt{2\pi}}e^{ik_2y}e^{ik_3x'} \times\left(E-\frac{\hbar^2}{2m}k_2^2\right)G_{k_2k_3}. \tag{C.7}$$

Therefore,

$$\left(E-\frac{\hbar^2k_1^2}{2m}\right)G_{k_1k_2}-v_0\int\frac{dk_2}{2\pi}G_{k_2k_1}=\delta\left(k_1+k_3\right), \tag{C.8}$$

which implies that

$$\boxed{\hat{O}_{k_1k_2}=\left(E-\frac{\hbar^2k_1^2}{2m}\right)\delta\left(k_1-k_2\right)-\frac{v_0}{2\pi}.} \tag{C.9}$$

We will take our propagator to be of the form

$$G_{k_1k_2}=\frac{1}{E-\frac{\hbar^2k_1^2}{2m}}\delta\left(k_1+k_2\right)+B_{k_2k_3}. \tag{C.10}$$

To determine $B_{k_2k_3}$ we plug our ansatz for the propagator into

$$\int dk_2\hat{O}_{k_1k_2}G_{k_2k_3}=\delta\left(k_2+k_3\right). \tag{C.11}$$

We note that

$$\int dk_2 \hat{O}_{k_1k_2} G_{k_2k_3} = \int dk_2 \left[\left(E - \frac{\hbar^2 k_1^2}{2m}\right)\delta\left(k_1 - k_2\right) - \frac{v_0}{2\pi}\right]$$

$$\left(\frac{1}{E - \frac{\hbar^2 k_2^2}{2m}}\delta\left(k_2 + k_3\right) + B_{k_2k_3}\right) = \int dk_2 \left(E - \frac{\hbar^2 k_1^2}{2m}\right)\delta\left(k_1 - k_2\right)\frac{1}{E - \frac{\hbar^2 k_2^2}{2m}}\delta\left(k_2 + k_3\right)$$

$$+ \int dk_2 \left(E - \frac{\hbar^2 k_1^2}{2m}\right)\delta\left(k_1 - k_2\right) B_{k_2k_3} - \frac{v_0}{2\pi}\int dk_2 \frac{1}{E - \frac{\hbar^2 k_2^2}{2m}}\delta\left(k_2 + k_3\right)$$

$$- \frac{v_0}{2\pi}\int dk_2 B_{k_2k_3}$$

$$= \delta\left(k_1 + k_3\right) + \left(E - \frac{\hbar^2 k_1^2}{2m}\right) B_{k_1k_3} - \frac{v_0}{2\pi}\frac{1}{E - \frac{\hbar^2 k_3^2}{2m}}\frac{1}{E - \frac{\hbar^2 k_2^2}{2m}} - \frac{v_0}{2\pi}\int dk_2 B_{k_2k_3}. \quad \text{(C.12)}$$

Thus,

$$0 = \left(E - \frac{\hbar^2 k_1^2}{2m}\right) B_{k_1k_3} - \frac{v_0}{2\pi}\frac{1}{E - \frac{\hbar^2 k_3^2}{2m}} - \frac{v_0}{2\pi}\int dk_2 B_{k_2k_3}. \quad \text{(C.13)}$$

or upon trivial rearranging:

$$B_{k_1k_3} = \frac{v_0}{2\pi}\frac{1}{E - \frac{\hbar^2 k_3^2}{2m}}\frac{1}{E - \frac{\hbar^2 k_1^2}{2m}} + \frac{v_0}{2\pi}\frac{1}{E - \frac{\hbar^2 k_1^2}{2m}}\int dp B_{p_1k_3}. \quad \text{(C.14)}$$

We integrate the equation above and find

$$\beta_{k_3}\left(1 - \frac{v_0}{2\pi}\alpha\right) = v_0 \frac{1}{E - \frac{\hbar^2 k_3^2}{2m}}\alpha, \quad \text{(C.15)}$$

where

$$\alpha = \frac{1}{2\pi}\int \frac{dp}{E - \frac{\hbar^2 p^2}{2m}}, \qquad \beta_{k_3} = \int dp \beta_{pk_3}. \quad \text{(C.16)}$$

Solving for β_{k_3} in (C.15), we obtain

$$\beta_{k_3} = v_0 \frac{1}{E - \frac{\hbar^2 k_3^2}{2m}} \frac{\alpha}{1 - v_0 \alpha}. \tag{C.17}$$

Substituting this back on the R.H.S of (C.14) leads to

$$B_{k_1 k_2} = \frac{1}{E - \frac{\hbar^2 k_1^2}{2m}} \frac{1}{E - \frac{\hbar^2 k_2^2}{2m}} \frac{v_0}{2\pi} \frac{1}{1 - v_0 \int \frac{dp}{2\pi} \frac{1}{E - \frac{\hbar^2 p^2}{2m}}}. \tag{C.18}$$

Therefore,

$$\boxed{G_{k_1 k_2} = \frac{1}{E - \frac{\hbar^2 k_1^2}{2m}} \delta\left(k_1 + k_2\right) + \frac{1}{E - \frac{\hbar^2 k_1^2}{2m}} \frac{1}{E - \frac{\hbar^2 k_2^2}{2m}} \frac{v_0}{2\pi} \frac{1}{1 - v_0 \int \frac{dp}{2\pi} \frac{1}{E - \frac{\hbar^2 p^2}{2m}}}.} \tag{C.19}$$

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