Resolvability of Topological Groups

by

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Abstract

A topological group is called resolvable (ω-resolvable) if it can be partitioned into two (into ω) dense subsets and absolutely resolvable (absolutely ω-resolvable) if it can be partitioned into two (into ω) subsets dense in every nondiscrete group topology. These notions have been intensively studied over the past 20 years. In this dissertation some major results in the field are presented. In particular, it is shown that (a) every countable nondiscrete topological group containing no open Boolean subgroup is ω-resolvable, and (b) every infinite Abelian group containing no infinite Boolean subgroup is absolutely ω-resolvable.

Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Chapter 1

Introduction

Topological groups can be thought of as an abstraction of continuous groups of transformations. A topological group is a set which has two structures; one is a group and the other a topological space. These structures are linked in such a manner that the algebraic properties of the group have an effect on the topological properties of the space, and *vice versa*. This marriage of the axioms of topological spaces and groups provides an excellent foundation for an abstract theory and is a powerful concept that unifies diverse mathematical areas. The study of topological groups is thus ideal for exploring the interaction between algebraic and topological ideas.

Our main interest in this dissertation is the resolvability of topological groups. The notion of resolvability was introduced by Edwin Hewitt [17]. He defined a topological space (X, τ) to be resolvable if there exists a subset D of X for which D as well as its complement $X \setminus D$ are dense with respect to τ in X, that is, if X can be partitioned into two disjoint dense subsets. Ceder [5] then generalized this notion to higher cardinals as follows: given a cardinal number κ, the topological space X is said to be κ-resolvable if there exists a family of κ-many pairwise disjoint dense subsets of X and given the first infinite ordinal ω with cardinality $\aleph_0$, the space X is ω-resolvable if it can be partitioned into countably many disjoint dense subsets. The notion of resolvability has proved itself fertile ground for study on topological groups as evidenced by the vast literature around the subject. The study of resolvability in the context of topological groups was initiated by W. W. Comfort and Jan van Mill [9] wherein it was shown that every nondiscrete irresolvable Abelian topological group contains an infinite Boolean subgroup. It was later established in [32] and [44] that every nondiscrete ω-irresolvable Abelian topological group contains an open Boolean subgroup. This result in particular is significant because the investigation of nondicrete ω-irresolvable group topologies on Abelian groups is, in some manner, reduced to their investigation on a countable Boolean group. Under Martin's axiom there are nondicrete irresolvable topological groups [23], however, it is not possible to show that such groups exist in ZFC, i.e Zermelo-Fraenkel set theory with the axiom of choice which is the standard form of axiomatic set theory. In [31], Protasov showed that there are models of ZFC for which any nondiscrete topological group is ω-resolvable. The question of characterizing absolutely resolvable groups was raised in [9] wherein it was also mentioned that the groups $\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ for any prime p are absolutely resolvable. The purpose of this dissertation is to give self contained proofs of the main results from [43], [46] and [45].

This dissertation is arranged as follows. In Chapter 2 we give an introduction to the theory of topological groups in which we recall basic notions and results. We go on to discuss the existence of nondiscrete Hausdorff group topologies. We also show that all infinite Abelian groups admit a totally bounded group topology and we characterize when a countably infinite group admits a nondiscrete Hausdorff group topology. We conclude Chapter 2 with Illanes' theorem which states that ω-irresolvable topological spaces are finitely irresolvable.

In Chapter 3 we study the Stone-Čech compactification of a discrete space. To this end, we give basic facts about ultrafilters and show that the Stone-Čech compactification of a discrete space is the set of ultrafilters on that space. We take the point of that space to be identified with the principal ultrafilters. We give El'kin's theorem which states that a space is irresolvable if and only if there exists a converging open filter on it. Given that D is a discrete space, we show that βD is extremally disconnected and $|\beta D| = 2^{2^\kappa}$. We then extend the operation of a discrete semigroup to its Stone-Čech compactification in such a way that βS becomes a right topological semigroup. Given a semigroup endowed with a left invariant topology $\mathscr{T}$, we proceed to define the ultrafilter semigroup, $Ult(\mathscr{T})$, to be the set of all ultrafilters on that semigroup that converge to its identity in $\mathscr{T}$. We obtain that it is a closed subsemigroup of βS. We conclude Chapter 3 with a brief exposition of Martin's axiom.

In Chapter 4 we consider the resolvability of topological groups. We present two proofs for the first of our two main theorems, that every countably infinite nondiscrete topological group containing no open Boolean subgroup is ω-resolvable. In Section 4.1 we explore the notions of a local left topological group, a local homomorphism, and a local automorphism upon which the two proofs of the major theorem of the chapter are based. In Section 4.2 we present the first of the two proofs. This proof relies on the structure of a local automorphism of finite order and the fact that the inversion map is a local automorphism. In Section 4.3 we present the second of the two proofs. This proof, shorter and more transparent than the first, is based on a structure theorem for a large family of homeomorphisms of finite order on countably infinite regular spaces.

In Chapter 5 we consider the absolute resolvability of topological groups. Using the Finite Sums Theorem, we give a proof showing that every infinite Abelian group which does not contain an infinite Boolean subgroup is absolutely resolvable. We consider an Abelian group G and $C = \{y \in G : 2y \neq 0\}$ to be infinite. Using the notation of finite sums with inverses, FSI, we construct a partition $\{C_r : r < \omega\}$ of C such that whenever $(y_n)_{n<\omega}$ is a one-to-one sequence in C, $h \in G$ and $r < \omega$, we have

$$(h + FSI((y_n)_{n<\omega})) \cap C_r \neq \varnothing,$$

where

$$FSI((y_n)_{n<\omega}) = \{\sum_{n\in E} \varepsilon_n^E y_n : E \in \mathscr{P}_f(\omega) \text{ and } \varepsilon_n^E \in \{1, -1\} \text{ for all } n \in E\}$$

and $\mathscr{P}_f(\omega)$ is the set consisting of the nonempty finite subsets of ω. From this we deduce the second of our main theorems, that every infinite Abelian group which does not contain an infinite Boolean subgroup is absolutely ω-resolvable, and as a result, they can be partitioned into infinitely many subsets such that every coset modulo infinite subgroup meets each subset of the partition.

Chapter 2

Preliminaries

2.1 Basic notions: Definitions and Properties

Definition 2.1.1. *A group G endowed with a topology $\mathscr{T}$ is a* ***topological group*** *if the multiplication*

$$\mu : G \times G \ni (g, h) \mapsto gh \in G$$

and the inversion

$$\iota : G \ni g \mapsto g^{-1} \in G$$

are continuous mappings, where $G \times G$ has the product topology. If a topology makes a group into a topological group, we call it a ***group topology****.*

Topological groups have a topological structure as well as a group structure. The group structure allows us to perform algebraic operations and the topological structure allows us to speak of continuous functions. [10] and [6] have a great deal of information about topological groups.

Below we give a few examples of topological groups.

Example 2.1.2. *1. Any group G can become a topological group if we endow it with the discrete topology. We call these groups discrete.*

2. *The real line* $\mathbb{R}$ *as well as the complex plane* $\mathbb{C}$ *under addition are topological groups when taken with their usual topologies.*

3. *The topological groups considered in 1. and 2. above are all abelian. For examples of more interesting non-abelian topological groups we consider the General Linear group,* $GL_n(\mathbb{R})$*, of non-singular* $n \times n$ *real matrices as well as the Special Linear group,* $SL_n(\mathbb{R})$*, of* $n \times n$ *real matrices which have determinant* 1*. We can view these groups as topological groups with the topology defined by the subspace topology if we embed them in* $\mathbb{R}^{n \times n}$*.*

Combining the multiplication and inversion in Definition 2.1.1 gives the continuity of the function

$$\mu' : (g, h) \mapsto gh^{-1}.$$

The continuity of μ' means whenever $a, b \in G$ and U is a neighbourhood of ab, there are neighbourhoods V and W of a and b respectively, such that $VW^{-1} \subseteq U$. It follows that whenever $a_1, \ldots, a_n \in G$, $k_1, \ldots, k_n \in \mathbb{Z}$ and U is a neighbourhood of $a_1^{k_1} \cdots a_n^{k_n} \in G$, there are neighbourhoods $V_1, \ldots, V_n$ of $a_1, \ldots a_n$, respectively, such that $V_1^{k_1} \cdots V_n^{k_n} \subseteq U$.

Definition 2.1.3. *If* G_1 *and* G_2 *are topological groups and* $\psi : G_1 \to G_2$ *is a continuous homomorphism, then* ψ *is a* ***topological isomorphism*** *if it is simultaneously an isomorphism and a homeomorphism.*

Next we consider the homogeneity of topological groups. Recall that a topological space X is *homogeneous* if and only if, given distinct points $x, y \in X$, there is a homeomorphism $\varphi : X \to X$ for which $\varphi(x) = y$. To begin we note that a topological group G acts on itself by certain canonical homeomorphisms, for example, the inversion of G and the left or right translation of G by a fixed element. To verify that these are in fact homeomorphisms, let $a \in G$, then the left translation

$$\lambda_a : G \ni g \mapsto ag \in G$$

and the right translation

$$\rho_a : G \ni g \mapsto ga \in G$$

are clearly continuous mappings, for each $a \in G$, as they are merely restrictions of the multiplication in Definition 2.1.1. The inversion ι is also continuous by definition. From well-known properties of the group operation, we have that λ_a, ρ_a, and ι are one-to-one and onto in G. As ι is continuous in G and $\iota^{-1} = \iota$, we have that ι is a homeomorphism. Finally from $(\lambda_a)^{-1} = \lambda_{a^{-1}}$ and $(\rho_a)^{-1} = \rho_{a^{-1}}$, we obtain that λ_a and ρ_a are homeomorphisms. Now let $a, b \in G$, there is a homeomorphism $G \ni g \mapsto ba^{-1}g \in G$ that maps a to b. Thus we can conclude that the space of any topological group is homogeneous.

The next topic we consider is that of the separation axioms on topological groups. The separation axioms all say, in different ways, that elements that can be distinguished or separated in some weak sense must also be distinguishable or separated in some stronger sense. However, to prove the results we wish to prove about the separation axioms and many others that will follow, we must first introduce a few more important concepts.

Definition 2.1.4. *Consider a topological space X with $x \in X$. Then a **neighbourhood** of x is a subset U of X for which there exists and open set V that satisfies $x \in V \subseteq U$. Equivalently, we say U is a **neighbourhood** of x if its interior contains x.*

Definition 2.1.5. *A **filter** on a nonempty set X is a family $\mathscr{F}$ of subsets of X such that:*

1. *$\varnothing \notin \mathscr{F}$ and $X \in \mathscr{F}$,*

2. *if $C, E \in \mathscr{F}$, then $C \cap E \in \mathscr{F}$, and*

3. *if* $C \in \mathscr{F}$ *and* $C \subseteq E \subseteq X$, *then* $E \in \mathscr{F}$.

The canonical example of a filter, and one we will use frequently, is the set $\mathscr{N}_x$ of all topological neighbourhoods of $x \in X$ called the *neighbourhood filter* for the point x. The *neighbourhood system* of X is the system $\{\mathscr{N}_x : x \in X\}$ of all neighbourhood filters on X.
If $\mathscr{F} = \{A \subseteq X : E \subset C \ for \ some \ E \in \mathscr{E}\}$ is a filter, then the family $\mathscr{E}$ of subsets of X is a *base* for $\mathscr{F}$. A base for a neighbourhood filter of a point $x \in X$ is called a *neighbourhood base* at x.

A neighbourhood U of the identity element of a topological group is *symmetric* if $U = U^{-1}$. Now let U be an arbitrary neighbourhood of the identity element in a topological group G and let $V = U \cap U^{-1}$. Then plainly $V = V^{-1}$, V is a neighbourhood of the identity, and $V \subseteq U$ hence every neighbourhood of the identity contains a symmetric one.

Theorem 2.1.6 ([47])**.** *Consider the topological space* X *with the neighbourhood system* $\{\mathscr{N}_x : x \in X\}$. *Then*

1. *for each* $x \in X$ *and* $U \in \mathscr{N}_x$, $x \in U$, *and*

2. *for each* $x \in X$ *and* $U \in \mathscr{N}_x$, $\{y \in X : U \in \mathscr{N}_y\} \in \mathscr{N}_x$.

Conversely, given a set X *and a system* $\{\mathscr{N}_x : x \in X\}$ *of filters on* X *that satisfies 1-2, there exists a unique topology* $\mathscr{T}$ *on* X *such that* $\{\mathscr{N}_x : x \in X\}$ *is the neighbourhood system.*

Proof. It is clear that the neighbourhood system $\{\mathscr{N}_x : x \in X\}$ of a space X satisfies 1-2. To prove the converse we must first define the operator *int* on the subsets of X. For each $A \subseteq X$,

$$int(C) = \{x \in X : C \in \mathscr{N}_x\}$$

The existence of a unique topology $\mathscr{T}$ on X for which *int* is the interior operator for $(X, \mathscr{T})$, follows from the following conditions that the *int* operator must satisfy.

1. $int(X) = X$. For each $x \in X$, we have $X \in \mathscr{N}_x$, thus $x \in int(X)$, and so 1 holds.

2. $int(C) \subseteq C$. If $x \in int(C)$, then $C \in \mathscr{N}_x$ and by 1, we have $x \in C$.

3. $int(int(C)) = int(C)$. Let $x \in int(C)$, then $C \in \mathscr{N}_x$. By 2, $int(C) \in \mathscr{N}_x$ which gives $x \in int(int(C))$. Hence $int(C) \subseteq int(int(C))$, the converse inclusion from 2.

4. $int(C \cap E) = (int(C)) \cap (int(E))$.
   Let $x \in (int(C)) \cap (int(E))$, then $x \in int(C)$ and $int(E)$. We then have that $C \in \mathscr{N}_x$ and $E \in \mathscr{N}_x$, so $C \cap E \in \mathscr{N}_x$ It follows that $x \in (int(C)) \cap (int(E))$. Therefore $(int(C)) \cap (int(E)) \subseteq int(C \cap E)$.
   Conversely, let $x \in int(C \cap E)$, then $C \cap E \in \mathscr{N}_x$ which means $C \in \mathscr{N}_x$ and $E \in \mathscr{N}_x$. It follows that $x \in (int(C)) \cap (int(E))$. Therefore $int(C \cap E) \subseteq (int(C)) \cap (int(E))$.

We now have that a subset $U \subseteq X$ is a neighbourhood of a point $x \in X$ if and only if x is in the interior of U, and so if and only if $U \in \mathscr{N}_x$. Therefore $\{\mathscr{N}_x : x \in X\}$ is a neighbourhood for $(X, \mathscr{T})$. □

We are now ready to give a few important separation results.

Lemma 2.1.7. *Every T_0 topological group is regular and hence Hausdorff.*

Proof. Suppose the topological group G satisfies the T_0 separation property. For each $g \in G \setminus \{e\}$, there exists a neighbourhood U of the identity that does not contain g. As G is homogeneous, it is T_1. By the T_0 axiom imposed on G in the beginning, there is an open set U that contains either of the points e and g, but not the other.

Now suppose $e \in U$, then $g \notin U$. Otherwise gU^{-1} is a neighbourhood of e that does not contain g. Next, we choose a neighbourhood V of the identity such that $VV^{-1} \subseteq U$. Then for each $g \in G \setminus U$, we have $gV \cap U = \varnothing$. Otherwise $ga = b$ for some

$a, b \in V$, giving $g = ba^{-1} \in VV^{-1} \subseteq U$ which is a contradiction. Hence $cl(V) \subseteq U$. Thus for every neighbourhood U of the identity, there is a closed neighbourhood of the identity contained in U. $\square$

With significantly more work, we can show that more is true. The following result is a stronger statement than Lemma 2.1.7 and is considered the best general separation result.

Theorem 2.1.8. *Every Hausdorff topological group G is completely regular.*

Proof. See [[29], Theorem 10]. $\square$

As with Urysohn's lemma, the introduction of continuous functions in this context is rather surprising. In fact, separation by a continuous function is an extremely strong condition to impose on a space.

A topological space X is called zero-dimensional if it has a base of clopen sets, that is, the sets are both open and closed. It is important to note that if a space is T_0 and zero-dimensional, then it is completely regular. We now give a result that is an improvement over Theorem 2.1.8 for countable topological groups.

Proposition 2.1.9 ([47])**.** *Every countable regular space is normal and zero-dimensional.*

Proof. Let X be a countable regular space. First we want to show that X is normal which we will achieve by proving that every pair of disjoint closed subsets of X will have neighbourhoods that are disjoint. Let C and E be disjoint closed subsets of X and enumerate them as follows:

$$C = \{a_n : n < \omega\} \text{ and } E = \{b_n : n < \omega\}.$$

Using induction, we choose neighbourhoods U_n and V_n of a_n and b_n respectively, for each $n < \omega$ such that

1. $cl(U_n) \cap E = \varnothing$ and $C \cap cl(V_n) = \varnothing$,

2. $U_n \cap (\bigcup_{i\leq n} V_i) = \varnothing$ and $(\bigcup_{i\leq n} U_i) \cap V_n = \varnothing$, and

3. $U_n \cap V_n = \varnothing$.

We require 1 to satisfy 2 while combining 2 and 3 gives

$$(\bigcup_{i<n} U_i) \cap (\bigcup_{i<n} V_i) = \varnothing.$$

We can now see that $U = \bigcup_{n<\omega} U_n$ and $V = \bigcup_{n<\omega}$ where U and V are disjoint neighbourhoods of C and E, respectively. Thus X is normal.

All that remains is to show that the space X has dimension zero. We suppose U is an open neighbourhood of $x \in X$. We may assume, without loss of generality that U is not the whole space. By Urysohn's lemma, there is a continuous function $\varphi : X \to [0,1]$ such that $\varphi(x) = \{0\}$ and $\varphi(X \setminus U) = \{1\}$. As X is countable, there exists $s \in [0,1] \setminus \varphi(X)$. Thus $\varphi^{-1}([0,s)) = \varphi^{-1}([0,s])$ is a clopen neighbourhood of x contained in U. $\square$

From Lemma 2.1.7 and Proposition 2.1.9 above, it follows that every countable Hausdorff topological group is normal and zero-dimensional.

A neighbourhood base at $x \in X$ is *countable* if there exists a sequence of neighbourhoods U_m of x such that for every neighbourhood U of x, there exists some m for which $U_m \subseteq U$. If each point in X has a countable neighbourhood base, then X is said to be *first countable.*

We now give a topological characterization of when a topological group is metrizable.

Theorem 2.1.10. *A Hausdorff topological group G is metrizable if and only if it is first countable.*

Proof. See [[18], Theorem 8.3]. $\square$

It is a well known fact that every metric space is normal. Combining this fact with Theorem 2.1.10 immediately gives that every first countable Hausdorff topological group is also normal.

The natural question to ask at this point is; given a topological group G, is it possible to form new topological groups from G? The answer to this question is yes and it prompts us to investigate some methods to achieve this.

Given a topological group G, a subgroup H of G (with the subspace topology) is a topological group. It is easy to see that the mapping
$(g, h) \mapsto gh^{-1}$ of $H \times H$ onto H is continuous as it is merely a restriction of the corresponding mapping of $G \times G$ and G.

Before we continue in this direction we must take a small detour to address a potential concern. Consider a topological group $(G, \mathscr{T})$ and a subgroup H of G. One might be tempted to want a stronger topology $\mathscr{O}$ on G. For example, one might define H and each of its left translates to be open with respect to $\mathscr{O}$, that is, $\mathscr{O} = \{gH \cap U : g \in G, U \in \mathscr{T}\}$ hoping that $(G, \mathscr{O})$ will be a topological group. However there is no guarantee that the function

$$\mu' : (g, h) \mapsto gh^{-1}$$

will be continuous with respect to $\mathscr{O}$. It is therefore a good idea to introduce an axiomatization of the topology of a topological group. The two results that follow will serve to address this issue by characterizing the neighbourhood filter of the identity of a topological group.

Theorem 2.1.11 ([37]). *Let $(G, \mathscr{T})$ be a topological group and let $\mathscr{N}_e$ be the neighbourhood filter of the identity element e. Then*

1. *$U \in \mathscr{N}_e$ implies the existence of $V \in \mathscr{N}_e$ such that $V \cdot V \subseteq U$;*
2. *$U \in \mathscr{N}_e$ implies $U^{-1} \in \mathscr{N}_e$;*
3. *for every $U \in \mathscr{N}_e$, and $g \in G$, $gUg^{-1} \in \mathscr{N}_e$*

Furthermore, for every $g \in G$, the neighbourhood filter $\mathscr{N}_g$ is given by

$$\mathscr{N}_g = \{Ug : U \in \mathscr{N}_e\} = \{gU : U \in \mathscr{N}_e\}$$

Proof. Since $\mu(e,e) = e$ and μ is continuous, there exist, for given $U \in \mathscr{N}_e$, neighbourhoods of the identity W and W^* such that $\mu(W, W^*) \subseteq U$. Setting $V = W \cap W^*$ and noting that $VV \subseteq WW^* = \mu(W, W^*)$ proves 1. Next, we have $\iota(e) = e$ and ι is a homeomorphism. Hence, given $U \in \mathscr{N}_e$, the set $\iota(U) = U^{-1}$ is a neighbourhood of e. Therefore 2 is proved.

Finally, we let $g \in G$. Then for every $U \in \mathscr{N}_e$, we have $gUg^{-1} = \lambda_g \circ \rho_g^{-1}(U)$. We have $gUg^{-1} \in \mathscr{N}_e$. This follows from the fact that λ_g and ρ_g^{-1} are homeomorphisms and that $\lambda_g \circ \rho_g^{-1}(e) = e$. Let V be an arbitrary element of $\mathscr{N}_e$ and the set $U = g^{-1}Vg$. Then $U \in \mathscr{N}_e$ and $gUg^{-1} = V$. Hence, $\mathscr{N}_e = \{gUg^{-1} : U \in \mathscr{N}_e\}$. Since λ_g and ρ_g are homeomorphisms that map e onto g, it follows that for every $g \in G$

$$g\mathscr{N} = \{gU : U \in \mathscr{N}_e\} = \{U_g : U \in \mathscr{N}_e\} = \mathscr{N}g$$

is a neighbourhood filter of g. $\square$

Theorem 2.1.12 ([37]). *Let G be a group and the let $\mathscr{N}$ be a filter satisfying* 1, 2, *and* 3 *of Theorem 2.1.11. Then there is a unique topology $\mathscr{T}$ on G such that $\mathscr{N}_e$ is the neighbourhood filter of the identity element $e \in G$. The topology $\mathscr{T}$ is Hausdorff if and only if*

$$\bigcap \mathscr{N}_e = \{e\}$$

Proof. Given the neighbourhood system $\{g\mathscr{N} : g \in G\}$, we must show that it satisfies the conditions of Theorem 2.1.6. If we suppose that $g \in G$ and U is a neighbourhood of the identity, then from 1-2, it follows that there exists a neighbourhood of the identity V such that $VV^{-1} \subseteq U$. Then $gVV^{-1} \subseteq gU$ and 1 is satisfied. To verify 2, we let $g \in G$ and U be a neighbourhood of the identity. Then from 1, it follows that there exists a neighbourhood of the identity V such that $VV \subseteq U$. For each $h \in gV$, we have $hV \subseteq gVV \subseteq gU$, so $gV \subseteq \{h \in G : gU \in h\mathscr{N}\}$. Thus $\{h \in G : gU \in h\mathscr{N}\}$.

Theorem 2.1.6 guarantees the existence of a unique topology $\mathscr{T}$ on G such that $g\mathscr{N}$ is the neighbourhood filter of g, for every $g \in G$. This means the neighbourhoods of g are of the form gU, where $U \in \mathscr{N}_e$. All that remains is to show that $\mathscr{T}$ is a group topology. To do this, let $g, h \in G$ and $U \in \mathscr{N}_e$. Using 1-2, we choose $V \in \mathscr{N}_e$ for which $hVV^{-1}h^{-1} \subseteq U$. Then

$$gV(hV)^{-1} = gVVh^{-1} = gh^{-1}hVV^{-1}h^{-1} \subseteq gh^{-1}U.$$

We have shown that $\mathscr{T}$ is a group topology. It is thus Hausdorff if and only if it is T_1, and is T_1 if and only if $\bigcap \mathscr{N}_e = \{e\}$. □

Theorem 2.1.11 also showed that the neighbourhood system of a topological group is completely determined by the neighbourhood filter of the identity.

We can now return to our study of subgroups.

Theorem 2.1.13 ([18])**.** *Let H be a subgroup of a topological group G. Then H is open if and only if it has a nonempty interior. Every open subgroup H of G is closed.*

Proof. Let g be an interior point of H. Then there exists a neighbourhood U of the identity in G such that $gU \subseteq H$. This means that for each $h \in H$, we have $hU = hg^{-1}gU \subseteq hg^{-1}H = H$, so H is open. Since H is open, every point of H is, by definition an interior point. Now suppose that H is an open subgroup of G, then we can write the complement, H^c, of H as $H^c = \bigcup\{gH : g \notin H\}$. We know that each set gH is open, hence H^c is open. Consequently, H is closed. □

Next we show that it is possible to generate open and closed subgroups from neighbourhoods of the identity.

Theorem 2.1.14 ([18])**.** *Let U be any symmetric neighbourhood of the identity in a topological group G. Then the set $K = \bigcup_{n<\omega} U^n$ is an open and closed subgroup of G.*

Proof. Let $g \in U^r$ and $h \in U^s$. Then $gh \in U^{r+s}$ and $g^{-1} \in (U^{-1})^r$ so K is a subgroup of G. Finally, we have by Theorem 2.1.13, that K is open and closed. □

Theorem 2.1.15 ([18])**.** *Suppose that H is a subgroup of a topological group G. Then H is discrete if and only if it has an isolated point.*

Proof. Let $g \in H$ and suppose that g is isolated (in the subspace topology). In other words, there exists a neighbourhood U of the identity in G such that $(gU) \cap H = \{g\}$. Then for an arbitrary point $h \in H$, we have

$$(hU) \cap H = (hU) \cap (hg^{-1}H) = hg^{-1}((gU) \cap H) = \{h\}.$$

Thus every point of H is isolated, which confirms that H is discrete. Now if we assume that H is discrete, all of its points are, by definition isolated. $\square$

Now let H be a normal subgroup of a topological group G and consider the quotient G/H with the quotient topology $\mathscr{T}(G/H)$, in particular, the strongest topology on G/H for which the natural mapping $\pi : G \to G/H$ is continuous. Let the open sets in G/H be of the form $\{uH : u \in U\}$ where U is an open subset of G, thus $\mathscr{T}(G/H)$ consists of all sets that have the form $\{uH : u \in U\}$ where U is open.

2.2 Topologizing a Group

It was noted earlier that when considered with the discrete topology, any group can be trivially made to be a topological group. The question we must now ask ourselves is, which groups admit a nondiscrete Hausdorff group topology? It was in 1945 that A. Markov [26] posed his now famous question. He posed it as follows: Does every infinite group admit a nondiscrete Hausdorff group topology? Of particular interest to us, for the purpose of this dissertation, are countably infinite groups and infinite Abelian groups.

In this section we will give a characterization of when a countable group admits a nondiscrete Hausdorff group topology, we will prove that every countable infinite Abelian group admits a totally bounded group topology and we will briefly describe the definitive solution to Markov's problem.

First and foremost we must define exactly what we mean when we say a given group G is topologizable.

Definition 2.2.1. *A group G is topologizable if it admits a nondiscrete Hausdorff group topology.*

Definition 2.2.2. *We call a topological group G **totally bounded** if it is Hausdorff and for every nonempty open subset U of G there is a finite subset A of G such that $AU = G$.*

Theorem 2.2.3. *A topological group G is totally bounded if and only if it can be topologically and algebraically embedded into a compact Hausdorff topological group.*

Many mathematicians posed variations of Markov's question. If G is an infinite group, is it possible to endow it with a nondiscrete group topology which is metrizable? What about one which is totally bounded? In 1953 Kertész and Szele [22] showed that every infinite Abelian group admits a nondiscrete group topology which is metrizable. We tackle the problem of a totally bounded group topology on infinite Abelian groups.

Theorem 2.2.4 ([19])**.** *Consider an Abelian group G with identity e and the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For every $g \in G \setminus \{e\}$, there is a homomorphism $\varphi : G \to \mathbb{T}$ such that $\varphi(g) \neq 1$.*

Proof. Suppose that $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ is the cyclic group generated by g. Define φ on $\langle g \rangle$ by stating that $\varphi(g^n) = exp(in)$ if $\langle g \rangle$ is infinite. If $\langle g \rangle$ is of order k, we define φ on $\langle g \rangle$ by $\varphi(g^n) = exp(\frac{2n\pi i}{k})$. It will be shown that φ can be extended to G. Consider the set of all pairs (h, H), where H is a subgroup of G, $\langle g \rangle \subseteq H$, $h : H \to \mathbb{T}$ is a homeomorphism, and $\varphi \subseteq h$. This set is ordered by

$$(h_1, H_1) \leq (h_2, H_2) \text{ if and only if } H_1 \subseteq H_2 \text{ and } h_2|_{H_1} = h_1.$$

By Zorn's lemma, we obtain a maximal member (h, H). We claim that $H = G$. Suppose for contradiction that there exists some $a \in G \setminus H$. Let $H' = \{a^n b : n \in$

$\mathbb{Z}$ and $b \in H\}$. Then H' is a subgroup of G and $H \subset H'$. We show that H can be extended to a homomorphism $h' : H' \to G$, which contradicts the maximality of (h, H). Begin by assuming that there is no $n \in \mathbb{Z} \setminus \{0\}$ such that $a^n \in H$. Recall that the members of H' have a unique expression of the form $a^n b$ with $n \in \mathbb{Z}$ and $b \in H$. So we may define a homomorphism h' on H' by $h'(a^n b) = h(b)$. Otherwise, choose m to be the first positive integer for which $a^m \in H$ and choose $\theta \in \mathbb{R}$ such that $exp(i\theta) = h(a^m)$. Notice that for any $n \in \mathbb{Z}$, we have $a^n \in H$ if and only if $n = mp$ for some $p \in \mathbb{Z}$. To extend h to h', we simply state that $h'(a^n b) = exp(\frac{ni\theta}{m})h(b)$. $\square$

Theorem 2.2.5 ([47]). *Every infinite Abelian group G admits a nondiscrete totally bounded group topology.*

Proof. Suppose that G is an infinite Abelian group. By Theorem 2.2.4, for every $g \in G \setminus \{0\}$, there is a homomorphism $\varphi_g : G \to \mathbb{T}$ with $\varphi_g(g) \neq e$. Now let $H = \prod_{g \in G\setminus\{0\}} \mathbb{T}_g$ where $\mathbb{T}_g = \mathbb{T}$. Let $\varphi : G \to H$ be defined by $(\varphi(x))_g = \varphi_g(x)$. We see that φ is an injective homomorphism. From the fact that it is a subgroup of a compact group, we have that $\varphi(G)$ is totally bounded. Hence, the topology on the infinite Abelian group G which consists of the subsets $\varphi^{-1}(U)$, where U runs over open subsets of $\varphi(G)$, is as desired. $\square$

It can be shown that every infinite Abelian group G admits $2^{2^{|G|}}$ totally bounded group topologies. It is also known from a result of Taimanov [36] that all large subgroups of a permutation group and free topological groups are nontrivially topologizable. This settles the first task we have set for the section.

We now direct our focus to the problem of characterizing when a countably infinite group admits a nondiscrete Hausdorff group topology. First recall the notion of a filter from Definition 2.1.5. Now for every filter $\mathscr{F}$ on a group G, we denote by $\mathscr{T}(\mathscr{F})$, the largest group topology on G for which the filter $\mathscr{F}$ converges to the identity element. For every filter $\mathscr{F}$ on a group G, we can also define a filter with a base that consists of subsets having the form

$$\bigcup_{g \in G} g(F_g \cup F_g{}^{-1} \cup \{e\})g^{-1},$$

where $F_g \in \mathscr{F}$ for each $g \in G$. We denote this filter by $\tilde{\mathscr{F}}$.

It would serve us well to take a deeper look at the topology $\mathscr{T}(\mathscr{F})$. We do this with the aid of the following theorem which gives a description of $\mathscr{T}(\mathscr{F})$. Precedently, let S_n denote the group of all permutations on $\{1, \dots, n\}$ for each $n \in \mathbb{N}$.

Theorem 2.2.6 ([42]). *For every filter $\mathscr{F}$ on a group G, the neighbourhood filter of the identity element in $\mathscr{T}(\mathscr{F})$ has a base that consists of subsets having the form*

$$\bigcup_{n=1}^{\infty} \bigcup_{\pi \in S_n} \prod_{i=1}^{n} K_{\pi(i)},$$

where $(K_n)_{n=1}^{\infty}$ is a sequence of members of $\tilde{\mathscr{F}}$.

Proof. See [[47], Theorem 1.17]. □

In each of the definitions that follow, G is a countably infinite group which we enumerate as $\{g_n : n < \omega\}$ such that there are no repetitions and $g_0 = e$.

Definition 2.2.7 ([42]). *We define, for each infinite sequence in G, the set $U((a_n)_{n=1}^{\infty})$ of G as follows:*

$$U((a_n)_{n=1}^{\infty}) = \bigcup_{n=1}^{\infty} \bigcup_{\pi \in S_n} \prod_{i=1}^{n} K_{\pi(i)},$$

where

$$K_i = \bigcup_{j=0}^{\infty} g_j \{e, a_{i+j}^{\pm 1}, a_{i+j+1}^{\pm 1}, \dots\} g_j^{-1}.$$

Definition 2.2.8 ([42]). *We define, for each infinite sequence $a_1, \dots, a_n$ in G, the set $U(a_1, \dots, a_n)$ of G as follows:*

$$U(a_1, \dots, a_n) = \bigcup_{\pi \in S_n} \prod_{i=1}^{n} K_{\pi(i)}^{n},$$

where

$$K_i^n = \bigcup_{j=0}^{n-i} g_j\{e, a_{i+j}^{\pm 1}, a_{i+j+1}^{\pm 1}, \ldots\}g_j^{-1}.$$

From Definition 2.2.8 we can see that the elements of $U(a_1, \ldots, a_n)$ are all those having the form

$$g_{j_1} b_1 g_{j_1}^{-1} \cdots g_{j_n} b_n g_{j_n}^{-1},$$

where $j_i \in \{0, \ldots, n - \pi(i)\}$ and $b_i \in \{e, a_{\pi(i)+j_i}^{\pm 1}, \ldots, a_n^{\pm 1}\}$ for each $1 \leq i \leq n$ and $\pi \in S_n$. Particularly, we have $U(\varnothing) = \{e\}$ and $U(a_1) = \{e, a_1^{\pm 1}\}$.

Definition 2.2.9 ([42]). *For every finite sequence $a_1, \ldots, a_{n-1}$ in G, let $F(a_1, \ldots, a_{n-1}, x)$ be the set of group words $\phi(x)$ in the alphabet $G \cup \{x\}$ which has the variable x and which the words are of the form*

$$\phi(x) = g_{j_1} b_1 g_{j_1}^{-1} \cdots g_{j_n} b_n g_{j_n}^{-1},$$

where $j_i \in \{0, \ldots, n - \pi(i)\}$ and $b_i \in \{e, a_{\pi(i)+j_i}^{\pm 1}, \ldots, a_{n+1}^{\pm 1}, x\}$ for each $1 \leq i \leq n$ and $\pi \in S_n$. In particular, $F(x)$ consists of two group words x and x^{-1}.

It is worth noting that when the group G is Abelian, the definitions above look far less complex.

The following result is of profound significance as it gives us a way to make a countably infinite group admit a nondiscrete Hausdorff group topology.

Theorem 2.2.10 ([42]). *For every infinite sequence in a countable group G, the following statements hold:*

1. *$U((a_n)_{n=1}^{\infty})$ is a neighbourhood of the identity element in $\mathscr{T}((a_n)_{n=1}^{\infty})$,*

2. *$U((a_n)_{n=1}^{\infty}) = \bigcup_{n=1}^{\infty} U(a_1, \ldots, a_n)$,*

3. *$U(a_1, \ldots, a_n) = U(a_1, \ldots, a_{n-1}) \cup \{\phi(a_n) : \phi(x) \in F(a_1, \ldots, a_n, x)\}$ for each $n \in \mathbb{N}$, and*

4. for every $n \in \mathbb{N}$ and $\phi(x) \in F(a_1, \ldots, a_{n-1}, x)$, $\phi(e) \in U(a_1, \ldots, a_{n-1})$.

As promised earlier we illustrate Theorem 2.2.10 with a special result, called Markov's Criterion, but first we have to define the notion of an *inequality* over a group.

Definition 2.2.11 ([42]). *Let G be a group. An* ***inequality*** *over G is any expression that has the form $\varphi(x) \neq c$, where $\varphi(x)$ is a group word in the alphabet $G \cup \{x\}$ and $c \in G$.*

Theorem 2.2.12 ([42]). *(Markov's Criterion) Let G be a countable group. Then G is topologizable if and only if every finite system of inequalities over G that has a solution also has another solution.*

Proof. Necessity. Given that $\mathscr{T}$ is a nondiscrete Hausdorff group topology on G, consider any finite system of inequalities over G, say $\varphi_i(x) \neq c_i$, where $i = 1, \ldots, n$, which have a solution, say $b \in G$, that is $\varphi_i(b) \neq c_i$ for each $i = 1, \ldots, n$. Recall that $\mathscr{T}$ is a Hausdorff group topology. Then there exists a neighbourhood $U \in \mathscr{T}$ of b for which $c_i \notin \varphi_i(U)$ for each $i = 1, \ldots, n$. It follows that every element of U is a solution of the system. Furthermore, since $\mathscr{T}$ is nondiscrete, $U \setminus \{b\} \neq \varnothing$.

Sufficiency. The proof is based on Theorem 2.2.10. It suffices to construct a sequence $(b_n)_{n=1}^{\infty}$ in $G \setminus \{e\}$ for which $g_i \notin U(b_i, b_{i+1}, \ldots, b_n)$ for each $n \in \mathbb{N}$ and $i = 1, \ldots, n$. This implies that $g_i \notin U((b_n)_{n=1}^{\infty})$ for which each $i \in \mathbb{N}$ making the topology $\mathscr{T}((b_n)_{n=1}^{\infty})$ nondiscrete and Hausdorff. Begin by picking any $b_1 \in G \setminus \{e, g_1^{\pm 1}\}$, then $g_1 \notin U(b_1) = \{e, b_1^{\pm 1}\}$. Next, fix $n \in \mathbb{N}$ and assume that we have chosen the elements $b_1, \ldots, b_n \in G$ in such a way that $g_i \notin U(b_i, b_{i+1}, \ldots, b_n)$ for each $i = 1, \ldots, n$. We need to find $b_{n+1} \in G \setminus \{e\}$ such that $g_i \notin U(b_i, b_{i+1}, \ldots, b_n, b_{n+1})$ for each $i = 1, \ldots, n+1$. Since

$$U(b_i, b_{i+1}, \ldots, b_n, b_{n+1}) = U(b_i, b_{i+1}, \ldots, b_n) \cup \{\varphi(b_{n+1}) : \varphi(x) \in T(b_i, b_{i+1}, \ldots, b_n, x)\},$$

it follows that $g_i \notin U(b_i, b_{i+1}, \ldots, b_n, b_{n+1})$ for each $i = 1, \ldots, n+1$ if and only if the system of inequalities $\varphi(x) \neq g_i$ has a solution b_{n+1}, where $i = 1, \ldots, n+1$ and

$\varphi(x) \in T(b_i, b_{i+1}, \ldots, b_n, x)$. Since $\varphi(e) \in U(b_i, b_{i+1}, \ldots, b_n)$ for each $i = 1, \ldots, n+1$ and $\varphi(x) \in T(b_i, b_{i+1}, \ldots, b_n, x)$, we have that e is a solution of this system. Hence there exists a solution $b_{n+1} \neq e$. □

We conclude this section with a short discussion on the definitive solution to Markov's problem which was eventually resolved in the negative after having remained open for over three decades. The first example of a nontopologizable group was given by S. Shelah [33]. Shelah's construction relies on the use of the continuum hypothesis and is of a group G having cardinality $\aleph_1$ such that the only topologies that G admits are the two trivial ones, that is, the discrete and indiscrete topologies. This group satisfies the following conditions:

1. there is some $r \in \mathbb{N}$ such that $A^r = G$ for every $A \subseteq G$ with the cardinalities of A and G equal;

2. for every subgroup H of G with the cardinality of H being strictly less than that of G, there is some $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in G$ such that the intersection $\bigcap_{i=1}^{n} g_i^{-1} H g_i$ is finite.

In 1 above, r can be chosen to be 10000 and in 2 above, we can take $n = 2$. G. Hesse showed, in [16] that we need not assume the continuum hypothesis to construct such a nontopologizable group.

We now give a brief sketch of a nontopologizable group constructed by Ol'shanskii [28] that falls within the framework of ZFC.

The reader should note that if every element of a group G except the identity has infinite order, then G is called a *torsion-free group.* If every element of G has finite order then we say G is a *torsion-group.*

Example 2.2.13. *Let r and n be odd integers with $r \geq 2$ and $n \geq 665$, and let $A(r, n)$ be the Adian group. This group has the following properties:*

1. $A(r,n)$ is generated by r elements;

2. $A(r,n)$ is torsion-free;

3. the center of $A(r,n)$ is an infinite cyclic group $\langle c\rangle$;

4. the quotient $A(r,n)/\langle c\rangle$ is an infinite group and has period n.

The quotient $A(r,n)/\langle c\rangle$ is, in fact the Burnside group $B(r,n)$ which is the largest group on r generators that satisfy the identity $x^n = e$. By 1, the Adian group is countable. If $x \in A(r,n) \setminus \langle c\rangle$ then $x^n \in \langle c\rangle$ by virtue of $A(r,n)/\langle c\rangle$ having period n.

We want to show that $x^n \notin \langle c^n\rangle$ so we assume the contrary. Then $x^n = (c^n)^k = (c^k)^n$ for some integer k. If $z = xc^{-k}$, then $z \notin \langle c\rangle$ and $z^n = x^n c^{-kn} = e$, as $\langle c\rangle$ is the center but this contradicts that the Adian group is torsion-free.

Now suppose that $G = A(r,n)/\langle c^n\rangle$ and let $D = \langle c\rangle / \langle c^n\rangle$. We have that G is an infinite group, $D = \{e, d_1, \ldots, d_{n-1}\}$ is a proper subset of G and for each $x \in G \setminus D$, we have $x^n = \{d_1, \ldots, d_{n-1}\}$. It follows that for every T_1-topology on G in which the identity element is not an isolated point, the mapping $x \mapsto x^n$ is not continuous at e. Hence the group G does not admit a nondiscrete Hausdorff group topology.

2.3 Resolvability and Irresolvability in Topological Spaces

Definition 2.3.1. *A topological space X is said to be* ***resolvable*** *if it can be partitioned into two dense subsets. More generally, given a cardinal $\kappa \geq 2$, the space X is* ***κ-resolvable*** *if it can be partitioned into κ-many dense subsets. If X is not resolvable (or κ-resolvable) then we say it is* ***irresolvable*** *(or κ-irresolvable).*

Based on this definition, it is evident that the usage of the term “resolvable” coincides with “2-resolvable”. This implies that whether or not a space is resolvable, it must be 1-resolvable. Also observe that for every cardinal number $\kappa \geq 1$, the empty space $\varnothing$ is κ-resolvable.

Example 2.3.2. *The set of real numbers is a resolvable topological space. This is because the sets of rational and irrational numbers are disjoint dense subsets of* $\mathbb{R}$.

Definition 2.3.3 ([47])**.** *A topological space X is called **hereditarily κ-irresolvable** if every nonempty subsets of X is κ-irresolvable and **open hereditarily κ-irresolvable** if every nonempty open subset of X is κ-irresolvable.*

Here we give a short survey of some of the more general results concerning the resolvability and irresolvability of topological spaces and groups. It is useful to point out that every resolvable space X is dense-in-itself, in other words, there does not exist a point of X that is isolated in X.

Lemma 2.3.4 ([41])**.** *A space with a resolvable subspace is itself resolvable.*

Proof. Given that X is a topological space, suppose that Z_0 is a resolvable subspace of X and that $\{C_0, E_0\}$ is a partition of Z_0 into dense subsets. Let $\mathscr{P}$ be the family of all pairs $\{C, E\}$ of disjoint subsets of X such that $C_0 \subseteq C$, $E_0 \subseteq E$, $C \subseteq cl(E)$ and $E \subseteq cl(C)$. Because $\{C_0, E_0\} \in \mathscr{P}$, we have that $\mathscr{P}$ is nonempty. Now let the order on $\mathscr{P}$ be defined by

$$\{C_1, E_1\} \leq \{C_2, E_2\} \text{ if and only if } C_1 \subseteq C_2 \text{ and } E_1 \subseteq E_2.$$

Every chain $(\{C_i, E_i\})_{i\in I}$ in $\mathscr{P}$ has an upper bound $\{\bigcup_{i\in I} C_i, \bigcup_{i\in I} E_i\}$. By Zorn's lemma, there exists a maximal element $\{C, E\} \in \mathscr{P}$. It remains to show that $\{C, E\}$ is a partition of X into dense subsets. For this, it is suffient to check that $C \cup E = X$. Obviously, $Z = C \cup E$ is closed. Suppose $Z \neq X$. Then pick $x \in X \setminus Z$ and $z \in Z$. Next, suppose that $\varphi : X \to X$ is a homeomorphism with $\varphi(z) = x$ and choose an open neighbourhood U of $z \in Z$ such that $\varphi(U) \cap Z = \varnothing$. Put $C_1 = C \cup \varphi(U \cap C)$ and $E_1 = E \cup \varphi(U \cap E)$. Then $(C_1, E_1) \in \mathscr{P}$ and $\{C, E\} < \{C_1, E_1\}$ which is a contradiction. $\square$

Lemma 2.3.5 ([7])**.** *Consider a topological space X. If Y is a κ-resolvable subspace of X, then the closure of Y in X, denoted $cl_X(Y)$, is κ-resolvable.*

Proof. The relation "is dense in itself" is transitive. □

If a topological space X is of the form $X = \bigcup_{i \in I} X_i$ where $\{X_i : i \in I\}$ is a pairwise disjoint family such that each X_i is resolvable, then X is itself resolvable. To see this, notice that for each $i \in I$ there is a subset D_i of X_i such that D_i as well as its complement $X_i \setminus D_i$ are dense in X_i. It follows that $\bigcup_{i \in I} D_i$ and $X \setminus \bigcup_{i \in I} D_i$ are both dense in X. Thus X is resolvable. The following theorem generalizes this fact.

Theorem 2.3.6 ([7])**.** *The union of a family of κ-resolvable subsets of a space is κ-resolvable.*

Proof. Let X be a topological space. Given the maximal family $\mathscr{R}$ of pairwise disjoint κ-resolvable subsets of X, let $Y = \bigcup \mathscr{R}$. Now let $U := X \setminus cl_X(Y)$ be an open set. If U is nonempty then there exists some $i \in I$ for which $U \cap X_i \neq \varnothing$.

An open subspace of a κ-resolvable space is κ-resolvable, hence the family $\mathscr{R} \cup \{U \cap X_i\}$ is pairwise disjoint and has κ-resolvable members, contradicting the maximality of $\mathscr{R}$. If we choose, for $C \in \mathscr{R}$, a pairwise disjoint nonempty family $\{A_\gamma : \gamma < \kappa\}$ of dense subsets of C and then define $P_\gamma = \bigcup_{C \in \mathscr{R}} C_\gamma$ for $\gamma < \kappa$, we have that $\{P_\gamma : \gamma < \kappa\}$ is a family of κ-many pairwise disjoint dense subsets of X. We can conclude that X is κ-resolvable. □

The reader should note that the intersection of a family of κ-resolvable spaces may not be κ-resolvable. For instance, the singleton is irresolvable. Theorem 2.3.6 is of great significance as it arms one with a powerful tool to give very efficient proofs for numerous other results that would otherwise require indirect and obscure arguments to prove. We illustrate this fact with the following results whose proofs all follow from the one above.

Lemma 2.3.7 ([9])**.** *Let G be a topological group and H a subgroup of G.*

1. *If H is a proper dense subgroup of G, then G is resolvable.*

2. *If H is nonclosed, then G is resolvable.*

3. *If H is closed and nowhere dense with G/H resolvable (in the quotient topology), then G is resolvable.*

Proof. 1. Evidently H and $G \setminus H$ are dense in G.

2. It follows from part 1 that $cl(H)$ is resolvable. Now by Theorem 2.3.6, G is resolvable.

3. We first define the natural map $\phi : G \to G/H$ and then let D and $(G/H) \setminus D$ be complementary dense subsets of G/H. As ϕ is an open map, $\phi^{-1}(D)$ and $G \setminus \phi^{-1}(D)$ are dense in G. Thus G is resolvable.

□

Corollary 2.3.8 ([47]). *Let X be a topological space and $\kappa \geq 2$. Let*

$$\mathscr{R}_\kappa(X) := \bigcup\{Y : \ Y \subseteq X, \ Y \text{ is resolvable}\},$$

that is, $\mathscr{R}_\kappa(X)$ is the union of all κ-resolvable subsets of X. Then

1. *$\mathscr{R}_\kappa(X)$ is the largest κ-resolvable subset of X,*

2. *$\mathscr{R}_\kappa(X)$ is closed,*

3. *X is κ-resolvable if and only if $\mathscr{R}_\kappa(X) = X$, and*

4. *If X is κ-irresolvable, then $\mathscr{I}_\kappa(X) = X \setminus \mathscr{R}_\kappa(X)$ is hereditarily κ-irresolvable.*

It has become conventional to represent a space X as a disjoint union $X = \mathscr{R} \cup \mathscr{I}$ where $\mathscr{R} = \mathscr{R}_2(X)$ is closed and resolvable, and $\mathscr{I} = \mathscr{I}_2(X) = X \setminus \mathscr{R}$ is open hereditarily irresolvable. In this case, X is resolvable if and only if $\mathscr{I} = \varnothing$, and hereditarily irresolvable if and only if $\mathscr{R} = \varnothing$. This convention is attributed to Edwin Hewitt.

The following characterization of a homogeneous κ-irresolvable space is a consequence of the properties of $\mathscr{R}_\kappa(X)$.

Corollary 2.3.9 ([47])**.** *If X is a homogeneous space then it is κ-irresolvable if and only if it is hereditarily κ-irresolvable.*

Proof. Consider a homogeneous κ-resolvable space X and suppose that it is not hereditarily κ-irresolvable. Then $\mathscr{R}_\kappa(X)$ is a proper subset of X. If we choose $x \in \mathscr{R}_\kappa(X)$ and $y \in X \setminus \mathscr{R}_\kappa(X)$ and let $\varphi : X \to X$ be a homeomorphism with $\varphi(x) = y$, we obtain that $\varphi(\mathscr{R}_\kappa(X))$ is a subset of X which is κ-resolvable and also that $\varphi(\mathscr{R}_\kappa(X)) \setminus \mathscr{R}_\kappa(X) \neq \varnothing$. This is a contradiction so X must be hereditarily κ-irresolvable. $\square$

In what remains of this section, we will explore one of the most fundamental results of the theory of resolvability (or irresolvability) in topological spaces. This result, attributed to A. Illanes, says spaces that are ω-irresolvable are in fact finitely irresolvable. Recall that a space is ω-resolvable if it can be partitioned into countably many dense subsets.

The proof of Illanes' theorem requires the use of two results that we shall present as lemmas.

Lemma 2.3.10 ([20])**.** *Consider a topological space X and let $Q = Q(X)$ be the union of every open set in X that contains a hereditarily irresolvable dense subset. Then*

1. *Q is the largest open set in X that contains an open-hereditarily irresolvable dense subset, and*

2. *every dense subset of $X \setminus cl(Q)$ is resolvable, and thus ω-resolvable.*

Proof. Let $\mathscr{U}$ be a maximal family of pairwise disjoint open subsets of X that contain a hereditarily irresolvable dense subset. Also note that $\bigcup \mathscr{U}$ is dense in Q. Now for each $U \in \mathscr{U}$, let D_u be a dense hereditarily irresolvable subset of U and let $D_0 = \bigcup\{D_u : U \in \mathscr{U}\}$. We have that D_0 is an open-hereditarily dense subset of Q.

We now wish to show that every dense subset of $X \setminus cl(Q)$ is resolvable. To do this, consider a dense subset A of $X \setminus cl(Q)$. Now suppose, on the contrary, that A is irresolvable. It follows from Corollary 2.3.8 that there exists a nonempty, hereditarily irresolvable open subset V of A. Now let $U_1 \subseteq X$ be open with $U_1 = V \cap A$ and $V \subseteq X \setminus cl(Q)$. Then U_1 is dense in V, giving $V \in \mathscr{U}$ which contradicts the maximality of $\mathscr{U}$. Thus $X \setminus cl(Q)$ is resolvable.

All that remains is to show that $X \setminus cl(Q)$ is ω-resolvable. Since $X \setminus cl(Q)$ is resolvable, there exist disjoint dense subsets A_1 and B_1 of $X \setminus cl(Q)$. Repeating this process we obtain that there exist disjoint dense subsets A_2 and B_2 of B_1 such that $B_1 = A_2 \cup B_2$. If we proceed in this manner we can construct sequences $(A_n)_{n<\omega}$ and $(B_n)_{n<\omega}$ such that for each $n < \omega$, A_{n+1} and B_{n+1} are disjoint dense subsets of B_n with $B_n = A_{n+1} \cup B_{n+1}$. Then

$$X \setminus cl(Q) = (A_1 \cup (X \setminus cl(Q)) \setminus \bigcup\{A_n : n \geq 2\}) \cup A_2 \cup A_3 \cup \dots$$

This union is made up of pairwise disjoint dense subsets of $X \setminus cl(Q)$. Hence $X \setminus cl(Q)$ is ω-resolvable. $\square$

Lemma 2.3.11 ([20])**.** *Let X be a topological space and let D be an open-hereditarily irresolvable subset of a space X. If X is $(n+1)$-resolvable for some n then $X \setminus D$ is dense in X and n-resolvable.*

Proof. Suppose that U is a nonempty open subset of X. We now need to show that $U \setminus D$ contains a nonempty n-resolvable set. We may suppose that $D \cap U \neq \varnothing$ and let $X = A_1 \cup A_2 \cup \ldots \cup A_{n+1}$ where each A_i is dense for $i = \{1, \ldots, n+1\}$. Then we can write $D \cap U$ as follows:

$$(A_1 \cap D \cap U) \cup (A_2 \cap D \cap U) \cup \ldots \cup (A_{n+1} \cap D \cap U).$$

There exists no space that can be partitioned into finitely many nowhere dense subsets, so at least one of these sets, take $A_{n+1} \cap D \cap U$ for example, is not nowhere dense in $D \cap U$. Then there must exist an open subset U_0 of U such that

$$\varnothing \neq D \cap U_0 \subseteq cl_{D\cap U}(A_{n+1} \cap D \cap U).$$

Then $A_{n+1} \cap D \cap U_0$ is dense in U_0 but as $D \cap U_0$ is irresolvable, we must have that $(D \cap U_0) \setminus A_{n+1}$ is not dense in $D \cap U_0$. Thus there exists an open subset U_1 of U_0 such that $\varnothing \neq D \cap U_1 \subseteq A_{n+1} \cap D \cap U_0$ which means $\varnothing \neq D \cap U_1 \subseteq A_{n+1} \cap D \cap U_1$. Consequently $A_1 \cap U_1, \ldots, A_n \cap U_1$ are pairwise disjoint dense subsets of $U_1 \setminus D$.

$\square$

We now possess all the required tools to prove this fundamental result in the theory of resolvable topological spaces.

Theorem 2.3.12 ([20]). *(Illanes) If a topological space X is n-resolvable for every $n < \omega$, then it is ω-resolvable.*

Proof. By Corollary 2.3.8, all we need to show is that every open subset X_0 of X contains a nonempty ω-resolvable subset. For each n, X_0 is n-resolvable as it is an open subset of X. Let $Q(X_0)$ be a subset of X_0 as given in Lemma 2.3.10. If $Q(X_0)$ is not dense, then $X_0 \setminus cl(Q(X_0))$ is ω-resolvable. Otherwise let D_1 be a dense open-hereditarily irresolvable proper subset of $Q(X_0)$ and let $X_1 = Q(X_0) \setminus D_1$. Since $Q(X_0)$ is an open subset of X_0, it is n-resolvable for each n. Then applying Lemma 2.3.11 gives that X_1 is n-resolvable for each n.

Now let $Q(X_1)$ be a subset of X_1 as given in Lemma 2.3.10. If $Q(X_1)$ is not dense, then $X_1 \setminus cl(Q(X_1))$ is ω-resolvable. Otherwise let D_2 be a dense open-hereditarily irresolvable proper subset of $Q(X_1)$ and let $X_2 = Q(X_1) \setminus D_2$. Since $Q(X_1)$ is an open subset of X_1, it is n-resolvable for each n. Then by Lemma 2.3.11, we have that X_2 is n-resolvable for each n.

Proceeding in this way, if at some m, the subset $Q(X_m)$ of X_m is not dense, then $X_m \setminus cl(Q(X_m))$ a nonempty ω-resolvable subset of X_0 as required. Otherwise let $(D_m)_{m=1}^{\infty}$ be an infinite sequence of pairwise disjoint dense subsets of X_0, then X_0 itself is ω-resolvable. $\square$

It is worth noting that while Illanes' theorem is attributed to Alejandro Illanes, it is believed to have been proved first by Eric van Douwen but was never published. Illanes' theorem can naturally be generalized to arbitrary cardinal numbers of countable cofinality as was shown by Bhaskara Rao [3].

Chapter 3

Stone-Čech Compactification

3.1 Ultrafilters

In a nutshell, an ultrafilter is a method of convergence to infinity. The concept of an ultrafilter was first introduced by Riesz in 1909 but only gained wide use two decades later after a paper by Ulam. A great deal of information about ultrafilters can be found in [8]. [19] and [2] are also useful sources in this regard.

Recall the definition of a filter, Definition 2.1.5.

Example 3.1.1. *1. Let $\mathscr{F} = \{X\}$. Clearly $\mathscr{F}$ is a filter, known as the* ***trivial*** *filter.*

2. Given $x \in X$ and $\mathscr{F} = \{F \subseteq X : x \in F\}$. We call $\mathscr{F}$ a ***principal*** *filter generated by the element $x \in X$.*

3. Let X be an infinite set and $\mathscr{F} = \{F \subseteq X : X \setminus F \text{ is finite}\}$. The filter $\mathscr{F}$ is called the ***cofinite*** *filter.*

Filters that are not principal filters are called *nonprincipal* filters.

Definition 3.1.2. *An* ***ultrafilter*** *on a nonempty set X is a maximal filter on X, that is, a filter which is not properly contained in any other filter on X.*

Definition 3.1.3. *A family $\mathscr{A}$ of subsets of X has the* ***finite intersection property*** *if for any finite subfamily $\mathscr{B}$ of $\mathscr{A}$, $\bigcap \mathscr{B} \neq \varnothing$.*

Notice that if $\mathscr{F}$ is a filter, $\varnothing \notin \mathscr{F}$ and $\mathscr{F}$ is closed under finite intersections. This implies that $\mathscr{F}$ is a filter base so any filter $\mathscr{F}$ has the finite intersection property. If $x \in X$, we can see that $\mathscr{U} = \{U \subseteq X : x \in U\}$ is an ultrafilter and since all principal filters are of this form, any principal filter is an ultrafilter. The ultrafilter $\mathscr{U}$ is called the *principal* ultrafilter defined by x.

Principal ultrafilters are the only ultrafilters whose members we can define explicitly. The natural question to ask is: do nonprincipal ultrafilters exist? The answer to this question is yes, but not within the framework of Zermelo-Fraenkel set theory. To show the existence of nonprincipal ultrafilters, we are forced to use the axiom of choice.

Theorem 3.1.4 ([30])**.** *Every filter on a set X can be extended to an ultrafilter on X.*

Proof. Suppose that $\mathscr{F}_0$ is any filter on a set X and that P is a partially ordered set of all filters on X that contain $\mathscr{F}_0$. Now let $\gamma = \{\mathscr{F}_i : i \in I\}$ be a chain of all filters in P. From Definition 2.1.5 we have that $\bigcup_{i \in I} \mathscr{F}_i$ is an upper bound of γ in P. It follows from Zorn's lemma that P has a maximal element, say $\mathscr{F} \in P$. Then $\mathscr{F}$ is a maximal filter, consequently, an ultrafilter that contains $\mathscr{F}_0$. $\square$

Consider the cofinite filter $\mathscr{F} = \{F \subseteq X : X \setminus F \text{ is finite}\}$. Since $\mathscr{F}$ is a filter, the above application of Zorn's lemma gives that $\mathscr{F}$ is contained in an ultrafilter, say $\mathscr{U}$. For any $x \in X$, $\mathscr{U}$ cannot be the principal ultrafilter generated by x as $X \setminus \{a\} \in \mathscr{F} \subseteq \mathscr{U}$. It follows that there are indeed nonprincipal ultrafilters on any infinite set.

We now characterize when a filter on a set is an ultrafilter.

Theorem 3.1.5 ([30])**.** *Let X be set and let $\mathscr{F}$ be a filter on X. Then $\mathscr{F}$ is an ultrafilter if and only if either $F \in \mathscr{F}$ or $X \setminus F \in \mathscr{F}$, for all $F \subseteq X$.*

Proof. Suppose that $\mathscr{F}$ is a filter and $F \subseteq X$. If there exists $A \in \mathscr{F}$ such that $A \cap F = \varnothing$ then $X \setminus F \in \mathscr{F}$. If $A \cap F \neq \varnothing$ for every $A \in \mathscr{F}$ then

$$\mathscr{F}' = \{B \subseteq X : A \cap F \subseteq B \text{ for some } A \in \mathscr{F}\}$$

is a filter. We have that $F \in \mathscr{F}'$, $\mathscr{F} \subseteq \mathscr{F}'$ and as $\mathscr{F}$ is an ultrafilter, $\mathscr{F} = \mathscr{F}'$. Therefore $F \in \mathscr{F}$.

Conversely, suppose that $\mathscr{F}$ is a filter for which either $F \in \mathscr{F}$ or $X \setminus F \in \mathscr{F}$ for any subset F of X. Suppose the filter $\mathscr{F}$ is properly contained in a filter $\mathscr{F}'$. If $\mathscr{F} \neq \mathscr{F}'$, there is a $F \in \mathscr{F}$ for which $F \notin \mathscr{F}'$. However, $F \cap (X \setminus F) = \varnothing$, a contradiction of the definition of a filter. Thus $\mathscr{F} = \mathscr{F}'$ and $\mathscr{F}$ is an ultrafilter. $\square$

Let $\mathscr{U}$ be an ultrafilter on a set X. If there exists an element $x \in \bigcap \mathscr{U}$, then $\mathscr{U} \subseteq \{F \subseteq X : x \in F\}$. From the fact that $\mathscr{U}$ is a maximal filter, we have $\mathscr{U} = \{F \subseteq X : x \in F\}$, a principal ultrafilter. If $F \in \mathscr{F}$ is finite, then there exists some $x \in F$ such that $\{x\} \in \mathscr{U}$. Consequently $\mathscr{U} = \{F \subseteq X : x \in F\}$. Thus $\mathscr{U}$ is a nonprincipal ultrafilter if and only if $\bigcap \mathscr{U} = \varnothing$.

Definition 3.1.6. *A topological space X is called* ***compact*** *if and only if every open cover contains a finite subcover.*

We illustrate the fact that filter convergence is ideal when considering ultrafilters on compact Hausdorff spaces with the following theorem.

Theorem 3.1.7.

1. *A topological space X is Hausdorff if and only if every ultrafilter $\mathscr{F}$ on X converges to at most one point.*
2. *A topological X is compact if and only if every ultrafilter $\mathscr{F}$ on X converges to at least one point.*

Proof. 1. Necessity. Let $\mathscr{F}$ be a filter on a Hausdorff space X. Assume, on the contrary, that $\mathscr{F}$ converges to distinct points $a, b \in X$. Now let $U, V \in \mathscr{F}$ be disjoint neighbourhoods of a and b respectively. Then $U \cap V = \varnothing$ since X is

Hausdorff, but this contradicts the definition of a filter. So $\mathscr{F}$ must converge to at most one point.

Sufficiency. Assume for contradiction, that every filter $\mathscr{F}$ on X converges to at most one point but X is not Hausdorff. Then there are distinct points $a, b \in X$ for which any open neighboughood U of a intersects any open neighbourhood V of b. The family $\{U \in \mathscr{F} : a \in U\} \bigcup \{V \in \mathscr{F} : b \in V\}$ has the finite intersection property. Extend $\mathscr{F}$ to an ultrafilter containing this family. Then $\mathscr{F}$ converges to the distinct points a and b, contradicting the hypothesis.

2. Necessity. Assume, on the contrary, that the space X is compact but the ultrafilter $\mathscr{U}$ on X has no limit points. This means for each $x \in X$, there is some open neighbourhood U of x for which $U \notin \mathscr{U}$. So $X = \bigcup\{U \notin \mathscr{U} : x \in U\}$, and since X is compact, $X = \bigcup_{i=1}^{n} U_i$. However $X \in \mathscr{U}$, so there must be some $U_i \in \mathscr{U}$, a contradiction.

   Sufficiency. Assume that X is not compact. Then there exists an open $X = \bigcup_{i=1}^{n} U_i$ with no finite subcover. So $\bigcap_{i=1}^{n}(X \setminus U_i) = \varnothing$, but there are no empty finite subintersections so $\{X \setminus U_i\}_{i=1}^{n}$ has the finite intersection property. This allows us to pick an ultrafilter $\mathscr{U}$ that contains $\{X \setminus U_i\}_{i=1}^{n}$. Now any point $x \in X$ is contained in some U_i and since $X \setminus U_i \in \mathscr{U}$, we have that $U_i \notin \mathscr{U}$. Thus x is not a limit point of $\mathscr{U}$.

□

Theorem 3.1.8. *The product of compact topological spaces is compact.*

Proof. Suppose that $X = \prod_{i \in I} X_i$ and that each X_i is compact. We wish to show that X is compact. To this end, for each $i_0 \in I$, define the projection $\pi_{i0} : X \to X_{i0}$ of X onto X_{i0} to be $\pi_{i0}((x_i)) = x_{i0}$ and suppose that $\mathscr{U}$ is an ultrafilter on X. Then for each $i \in I$, $\pi_i(\mathscr{U}) = \{A \in X_i : \pi_i^{-1}(A) \in \mathscr{U}\}$ is an ultrafilter on X_i. It follows that since X_i is compact, $\pi_i(\mathscr{U})$ converges to some point $a_i \in X_i$. Now suppose that $W = \prod_{i \in I} U_i$ is a canonical open neighbourhood of a and $I_0 = \{i \in I : W_i \neq X_i\}$.

Then $\mathscr{U} = \bigcap_{i \in I_0} \pi_i^{-1}(W_i)$. Consequently, since I_0 is finite and $\pi_i^{-1}(W_i) \in \mathscr{U}$, we have that $W \in \mathscr{U}$. $\square$

We conclude this section with another result in the theory of resolvability on topogical spaces that is considered "fundamental". El'kin's criterion of topological irresolvability, roughly stated says a topological space $X = (X, \tau)$ is irresolvable if and only if the topology τ contains a base of some ultrafilter on X.

To arrive at El'kin's criteion, we need to first give a few results.

Proposition 3.1.9 ([11]). *If X is a topological space, it is open-hereditarily irresolvable if and only if for every (converging) maximal open filter $\mathscr{F}$ on X, $|\overline{\mathscr{F}}| < n$.*

Theorem 3.1.10 ([11]). *A topological space X is n-irresolvable if and only if there is a (converging) open filter $\mathscr{F}$ on X with $|\overline{\mathscr{F}}| < n$.*

Proof. To prove necessity, let $\mathscr{I}_n(X) = X \setminus \mathscr{R}_n(X)$ be a subspace of X. From Corollary 2.3.8, we get that $\mathscr{I}_n(X)$ is open and hereditarily n-irresolvable. Now choose a converging maximal open filter $\mathscr{F}$ on X for which $\mathscr{I}_n(X) \in \mathscr{F}$. By Proposition 3.1.9, we have $|\overline{\mathscr{F}}| < n$.

For sufficiency, suppose A_i is a partition of X into n dense sets, where $i < n$. Then there exists $j < n$ such that $\bigcup_{j \neq i < n} A_i$ belongs to the filter $\mathscr{F}$. As $\mathscr{F}$ is open we have that A_j is not dense which contradicts the density of the subsets of X. Thus X is n-irresolvable. $\square$

Corollary 3.1.11 ([11]). *(El'kin's Criterion) Let X be a topological space. Then X is irresolvable if and only if there is a (converging) open ultrafilter on X.*

3.2 The Stone-Čech Compactification

The Stone-Čech compactification was obtained independently by M. H. Stone [35] and E. Čech [4] in 1937. The approach used by M. H. Stone was to treat the relations of topology and algebra through the applications of Boolean rings. On the other hand, E. Čech demonstrated the existence of the Stone-Čech compactification of a space and used it to investigate properties of that space by embedding it in a product of lines. The method we choose to define the Stone-Čech compactification of a discrete space is close to the one used by H. Wallman [40] and the approach we adopt for discrete spaces follows the treatment in [14].

When mathematicians speak of a *compactification* they are referring to the process wherein a topological space is made to be a compact space. There are numerous methods of compactification such as the one-point compactification given by Alexandroff [1] which we can illustrate by considering the real line $\mathbb{R}$ with its usual topology. The space $\mathbb{R}$ is not compact but it can be made to be compact using the one-point compactification by adding a point ∞, which we call the *point at infinity,* not belonging to $\mathbb{R}$.

The advantage of the one-point compactification is that it is not difficult to describe, however, it is in some ways inadequate. The space of continuous functions on the one-point compactification often differs a great deal to the space of bounded continuous functions on the underlying topological space. Every continuous real function on the one-point compactification of a topological space X defines a bounded continuous real function on that space but there are some bounded continuous functions on X that do not extend to a continuous function on the one-point compactification of X.

Completely regular Hausdorff spaces have a compactification that is free of this imperfection. This special type of compactification, called the Stone-Čech compactification and denoted by βX, first appeared in the literature implicitly in 1930 in a paper by Tychonoff [38], wherein he classified completely regular spaces as those spaces that can be embedded into a product of copies of the closed unit interval I.

Definition 3.2.1. *An **embedding** of a topological space X into a topological space Y is a function $\phi : X \to Y$ which defines a homeomorphism from X onto $\phi(X)$.*

Definition 3.2.2 ([39])**.** *Let $\mathscr{F}$ be a family of functions on a space X. If for every pair of points $x, y \in X$ with $x \neq y$, there is a function $\varphi \in \mathscr{F}$ for which $\varphi(x) \neq \varphi(y)$, we say the family $\mathscr{F}$ **distinguishes points**. If for every closed set $F \in X$ and every point $x \notin F$, there is some function $\varphi \in \mathscr{F}$ for which $\varphi(x)$ misses $cl(\varphi(F))$, then we say $\mathscr{F}$ **distinguishes points and closed sets**.*

Now suppose that $\mathscr{F}$ is a family of mappings for which each $\varphi \in \mathscr{F}$ maps the space X to a space Y_φ. For each $x \in X$ define the *evaluation mapping* $\varepsilon : X \to \prod Y_\varphi$ by $\varepsilon(x)_\varphi = \varphi(x)$. Since $\pi_\varphi \circ \varepsilon = \varphi$, that is, the composition of ε with each projection is continuous, we have that ε is continuous. If the family ϵ distinguishes closed sets, ε is an open open mapping onto $\varepsilon(X)$. To see this, suppose U is an open neighbourhood of $x \in X$. Pick a function $\varphi \in \mathscr{F}$ for which $\varphi(x) \notin cl(\varphi(X \setminus U))$. Then the set of every $y \in \varepsilon(X)$ for which $y_\varphi \notin cl(\varphi(X \setminus U))$ is a neighbourhood of $\varepsilon(x)$ and is contained in $\varphi(U)$. Thus, $\varphi(U)$ is open in $\varepsilon(X)$. Obviously, if $\mathscr{F}$ distinguishes points, ε is one-to-one. It follows that if $\mathscr{F}$ distinguishes points and also distinguishes points and closed sets, ε is an embedding.

The characterization of the complete regularity given by Tychonoff shows that no larger class of spaces can be studied by means of embeddings into compact Hausdorff spaces.

Unlike the one point compactification, the Stone-Čech compactification is difficult to describe.

Consider a completely regular Hausdorff space X and define the mapping $\varepsilon : X \to C^*(X)$, where $C^*(X)$ is the space of real bounded continuous functions on X, by

$$\varepsilon(x) = e_x,$$

which associates to each x, the evaluation functional at x. We endow $C^*(X)$ with the product topology. The mapping ε is one-to-one and is an embedding. Thus if we identify the space X with $\varepsilon(X)$, we can view it as a topological subspace of $C^*(X)$.

For each bounded continuous function γ, we choose a real number $M_\gamma > 0$. This real number M_γ must satisfy $|\gamma(x)| \leq M_\gamma$ for each $x \in X$. We can see then that

$$\varepsilon(X) \subseteq \prod_\gamma [-M_\gamma, M_\gamma] := Q.$$

The set Q is a compact subset of the space of bounded continuous functions by the Tychonoff Product Theorem. Therefore the closure $cl(\varepsilon(X))$ of $\varepsilon(X)$ is also a compact subset of $C^*(X)$. In other words, $cl(\varepsilon(X))$ is a compactification of X, the Stone-Čech compactification.

3.3 Stone-Čech Compactification of a Discrete Space

We give a construction of the Stone-Čech compactification of a discrete space. It is important to note that any discrete topological space D is metrizable using the discrete metric, and as we know, this means it is Hausdorff and completely regular. Thus the space D has a Stone-Čech compactification βD.

It turns out that if D is discrete, βD can be constructed as the set of all ultrafilters on D. The topology on βD is called the Stone topology. The first step is to consider a nonempty set D and define a topology on the set of all ultrafilters on D.

Definition 3.3.1 ([19])**.** *Consider a discrete topological space D. Let βD denote the set of all ultrafilters on D. Given $A \subseteq D$, we define $\overline{A} \subseteq \beta D$ by*

$$\overline{A} = \{p \in \beta D : A \in p\}.$$

Lower case letters will denote ultrafilters on D, since we will be thinking of ultrafilters as points in a topological space. Now suppose $a \in D$. Then $\varepsilon(a) = \{A \subseteq D : a \in A\}$. Thus for each element a of D, we can think of $\varepsilon(a)$ as the principal ultrafilter corresponding to a.

For every subset A of D and $p \in \beta D$, $p \in \overline{A}$ if and only if $A \in p$. Thus $p \in \overline{A \cap B}$ if and only if $A \cap B \in p$ and this holds if and only if $A \in p$ and $B \in p$ which in turn, hold if and only if $p \in \overline{A \cap B}$. Hence $\overline{A \cap B} = \overline{A} \cap \overline{B}$ for $A, B \subseteq D$. similarly, it can be shown that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Consequently, the family $\{\overline{A} : A \subseteq D\}$ forms a base for the topology on βD, therefore we should define the topology of βD as the topology having these sets as a basis.

The reader should also make note of the following facts: $\overline{A} = \varnothing$ if and only if $A = \varnothing$. This is evident from the fact that $A \in p$ if and only if A is nonempty, by definition. So $\overline{A} = \varnothing$. Using this, we can show that $D \setminus A = \varnothing$ if and only if $\overline{D \setminus A} = \varnothing$. This holds if and only if $\beta D \setminus \overline{A} = \varnothing$ which in turn, holds if and only if $\beta D = \overline{A}$. Thus $\overline{A} = \beta D$ if and only if $A = D$.

We now give a few important properties of the space βD.

Theorem 3.3.2 ([19]). *Given any set D, βD is a compact Hausdorff space.*

Proof. Let p and q be distinct elements of βD. If $A \in p \setminus q$, then $D \setminus A \in q$ so $\overline{A}$ and $\overline{D \setminus A}$ are disjoint open subsets of βD with $p \in \overline{A}$ and $q \in \overline{D \setminus A}$. Hence, the topological space βD is Hausdorff.

It remains to show that the space βD is compact. To do this, let $\mathscr{U}$ be an open cover of the space βD. Recall that each open subset of βD is the union of sets of the form $\overline{A}$, then without loss of generality, one may assume that $\mathscr{U} = \{\overline{A} : A \in \mathscr{F}\}$, where $\mathscr{F}$ is a family of subsets of D. Now let

$$\mathscr{F}' = \{D \setminus A : A \in \mathscr{F}\}$$

and suppose that $\mathscr{F}'$ has the finite intersection property. Then $\mathscr{F}'$ must be contained in some ultrafilter p. As $\mathscr{U}$ is a cover of βD, there is some $A \in \mathscr{F}$ such that $p \in \overline{A}$. On the other hand, $D \setminus A \in p$. Since $A \in p$, it follows that $D \setminus A \in p$, however this contradicts the fact that p is a filter. Therefore, we must have that the family $\mathscr{F}'$

does not have the finite intersection property. Now pick subsets $A_1, A_2, \ldots, A_n \in \mathscr{F}$ such that

$$(D \setminus A_1) \cap (D \setminus A_2) \cap \ldots \cap (D \setminus A_n) = \varnothing.$$

Then $A_1 \cup A_2 \cup \ldots \cup A_n = D$ and $\overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n} = \beta D$. Thus $\{\overline{A_1}, \overline{A_2}, \ldots, \overline{A_n}\}$ is a finite subcover of $\mathscr{U}$. □

It is useful to point out that for every subset A of D, the set $\overline{A}$ is open and closed. To see this, let $p \in \overline{A}$. Then $A \in p$ and, consequently, the set $\overline{A}$ is a neighbourhood of each of its points, thus $\overline{A}$ is open. We claim that $\beta D \setminus \overline{A} = \overline{D \setminus A}$. This is clear as $p \in \overline{D \setminus A}$ if and only if $D \setminus A \in p$, which holds if and only if $A \notin p$, which in turn, holds if and only if $p \in \beta D \setminus \overline{A}$. Thus the set $\overline{A}$ is closed as the complement of an open set.

Theorem 3.3.3 ([19]). *Given the set D, the sets of the form $\overline{A}$ are the clopen subsets of βD.*

Proof. As we have mentioned above, each set $\overline{A}$ is open and closed. Now suppose that C is any clopen subset of βD and $\mathscr{A} = \{\overline{A} : A \subseteq D \text{ and } \overline{A} \subseteq C\}$. As C is open, we have that $\mathscr{A}$ is an open cover of C and as C is also closed, we have that it is compact by Theorem 3.3.2. So if we choose a finite family $\mathscr{F}$ of subsets of D such that $C = \bigcup\{\overline{A} : A \in \mathscr{F}\}$, the $C = \overline{\bigcup \mathscr{F}}$ as sets of the form $\overline{A}$ are closed under finite unions. □

Theorem 3.3.4 ([19]). *Given a set D,*

1. *For every subset A of D, $\overline{A} = cl_{\beta D}(\varepsilon(A))$.*

2. *For any subset A of D and any ultrafilter $p \in \beta D$, $p \in cl_{\beta D}(\varepsilon(A))$ if and only if $A \in p$.*

Proof. 1. For each $a \in A$, $\varepsilon(a) \in \overline{A}$ and therefore $cl_{\beta D}(\varepsilon(A)) \subseteq \overline{A}$. Now let $p \in \overline{A}$, if $\overline{B}$ is a neighbourhood of p, then $A \in p$ and $B \in p$ and so the intersection

$A \cap B$ is nonempty. Now pick any $a \in A \cap B$, since $\varepsilon(a) \in \varepsilon(A) \cap \overline{B}$ is nonempty and thus $p \in cl_{\beta D}(\epsilon(A))$.

2. By 1 and the definition of $\overline{A}$, we have that $p \in cl_{\beta D}(\varepsilon(A))$ if and only if $p \in \overline{A}$ which is true if and only if $A \in p$.

□

We have almost all the pieces in place to show that βD is the Stone-Čech compactification of a discrete space.

Definition 3.3.5. *Let D be a discrete space. The **Stone-Čech compactification of D** is a compact Hausdorff space Y that contains D as a dense subspace. If K is any compact Hausdorff space, then every mapping $\varphi : D \to K$ can be extended to a continuous mapping $\overline{\varphi} : Y \to K$.*

Definition 3.3.6. *Consider a filter $\mathscr{F}$ on a nonempty set X. If $A \subseteq X$ and $A \cap F \neq \varnothing$ for any $F \in \mathscr{F}$, then $\mathscr{F}|_A = \{A \cap F : F \in \mathscr{F}\}$ is a filter on A and we call it the **trace** of $\mathscr{F}$ on A. If $\varphi : X \to C$, then $\varphi(\mathscr{F}) = \{\varphi(F) : F \in \mathscr{F}\}$ is a filter base on C and we call it the **image** of $\mathscr{F}$ with respect to φ.*

We need one more result before we tackle the main problem of the section. The reader should note that if φ is a continuous mapping from a completely regular Hausdorff space X into a compact space Y, then $\overline{\varphi}$ will denote the continuous mapping from βX to Y which extends φ. Let us consider the follwing general situation.

Let X be a dense subset of a space Y, let K be a Hausdorff space and $\varphi : X \to K$ be a mapping from X into K. For every $p \in Y$, let $\mathscr{F}_p$ denote the trace of the neighbourhood filter of $p \in Y$ on X and let $\lim_{x \to p} \varphi(x)$, where $x \in X$, be the limit of the filter base $\varphi(\mathscr{F}_p)$ in K, if the limit exists. We see that if φ has a continuous extension $\overline{\varphi} : Y \to K$, then it is unique as any two extensions agree on a dense subspace and

$$\overline{\varphi}(p) = \lim_{x \to p} \varphi(x),\ x \in X$$

for every $p \in Y$.

Lemma 3.3.7 ([19]). *Let K be a regular space and suppose that for every $p \in Y$, there is $\lim_{x \to p} \varphi(x)$, $x \in X$. Define $\overline{\varphi} : Y \to K$ by*

$$\overline{\varphi}(p) = \lim_{x \to p} \varphi(x),\ x \in X.$$

Then $\overline{\varphi}$ is the continuous extension of φ.

Proof. We claim that for every subset A of X,

$$\overline{\varphi}(cl_Y(A)) \subseteq cl_Z(\varphi(A)).$$

To see this, we use the definition of $\overline{\varphi}$. Let $q \in cl_Y(A)$ and let W be a neighbourhood of $\overline{\varphi}(q) \in K$. Since $\overline{\varphi}(q) = \lim_{x \to q} \varphi(x)$, there $B \in \mathscr{F}_q$ such that $\varphi(B) \subseteq W$. We have that the intersection $B \cap A$ is nonempty and $\varphi(B \cap A) \subseteq W$. Hence, $\varphi(q) \in cl_Z(\varphi(A))$ as required.

All that remains is to show that $\overline{\varphi}$ is continuous. To do this, suppose that $p \in Y$ and U is a neighbourhood of $\overline{\varphi}(p) \in K$. Since K is a regular space, we may assume that U is closed. We can choose some $A \in \mathscr{F}_p$ such that $\varphi(A) \subseteq U$ and put $V = cl_Y(A)$. Then V is a neighbourhood of $p \in Y$ and $\overline{\varphi}(V) \subseteq U$. Thus $\overline{\varphi}$ is continuous. □

At last we have everything in place to arrive at the main result for this section.

Theorem 3.3.8 ([47]). *Given a discrete space D, βD is the Stone-Čech compactification of D.*

Proof. We have already shown that the space βD is a compact Hausdorff space. Now suppose that $\varphi : D \to K$ is any function that maps the discrete space D into K, where K is any compact Hausdorff space and let $p \in \beta D$. For every $p \in \beta D$, the trace of the neighbourhood filter of $p \in \beta D$ on D is the ultrafilter p. The ultrafilter base $\varphi(p) \in K$ is convergent because K is compact. This means $\lim_{x \to p} \varphi(x)$, where $x \in D$, exists. Hence, by Lemma 3.3.7,

$$\overline{\varphi} : \beta D \ni p \mapsto \lim_{x \to p} \varphi(x) \in K$$

is the continuous extension of φ. We can thus conclude that βD is the Stone-Čech compactification of D. $\square$

The construction we have given of βD is a special case of a more general construction. We can describe the Stone-Čech compactification of a general completely regular Hausdorff space in terms of z-ultrafilters. A *z-set* is a set of the form $\{x : \varphi(x) = 0\}$ where φ is a bounded continuous function. In other words, it is the zero set of a bounded continuous function. Every set in a discrete space is a z-set. We call a family of z-sets that satisfy the definition of a filter where only z-sets are allowed a *z-filter.* A *z-ultrafilter* is a maximal z-filter. For details about the construction of the Stone-Čech compactification of a general completely regular Hausdorff space, the reader is referred to [39].

It is important to note that whenever we deal with βD, it is customary to identify the points of D with the principal ultrafilters generated by those points. After we have identified $d \in D$ with $\varepsilon(d) \in \beta D$, we will assume $D \subseteq \beta D$ and then $D^* = \beta D \setminus \varepsilon[D]$ will become $D^* = \beta D \setminus D$.

Over the remainder of this section we will explore a few interesting facts about the space βD.

Definition 3.3.9. *A topological space X is **extremally disconnected** if the closure of every open subset is open.*

Theorem 3.3.10 ([19])**.** *The space βD is extremally disconnected.*

Proof. Suppose that U is an open subset of βD. Since $cl_{\beta D}(A)$ is an open subset of βD, the set $cl_{\beta D}(U \cap D)$ is an open subset of βD. If $x \in cl_{\beta D}(U)$ and V is an open neighbourhood of x, then the intersection $V \cap U$ is nonempty, so $V \cap U \cap D$ is nonempty, or $x \in cl_{\beta D}(U \cap D)$. Therefore $cl_{\beta D}(U) = cl_{\beta D}(U \cap D)$, which means that $cl_{\beta D}(U)$ is open. $\square$

Extremally disconnected spaces have wonderful properties. For example, any continuous function that maps a dense subspace of an extremally disconnected space into a compact space, has a continuous extension to the entire space [14]. We should show some caution here. We have shown that the space βD is extremally disconnected, however, as shown in [14], D^* is not extremally disconnected.

An exposition of the space βD would not be complete if we do not mention its cardinality. We show here that for any infinite set D, $|\beta D| = 2^{2^{|D|}}$. Given any infinite set D, denote by $U(D)$, the subset of βD that consists of uniform ultrafilters.

We define the *density* of a space X to be the smallest cardinal number of a dense subset A of X, denoted by $d(X)$.

Theorem 3.3.11 ([12]). *Hewitt-Maeczewski-Pondiczery Theorem. If $d(X_s) \leq m \geq \aleph_0$ for every $s \in S$ and $|S| \leq 2^m$,then $d(\prod_{s\in S} X_s) \leq m$.*

Proof. See [12], Theorem 2.3.15 $\square$

Theorem 3.3.12 ([47]). *Consider an infinite set D with cardinality κ. Then*

$$|U_\kappa(D)| = |\beta D| = 2^{2^\kappa}.$$

Proof. Note that every ultrafilter is a member of $\mathscr{P}(\mathscr{P}(D))$. In other words, $\beta D \subseteq \mathscr{P}(\mathscr{P}(D))$, and so $|U_\kappa(D)| \leq |\beta D| \leq 2^{2^\kappa}$.

Let $U = U_\kappa(D)$. To show that $|U| \geq 2^{2^\kappa}$, we must construct a mapping of U onto a set with cardinality 2^{2^κ}. Let I be the product of 2^κ copies of the discrete space $\{0, 1\}$. Then $|I| = 2^{2^\kappa}$. By Theorem 3.3.11, I has a dense subset A such that $|A| = \kappa$. We define A to be $\{q_\alpha : \alpha < \kappa\}$. Now we consider a partition $\{C_\alpha : \alpha < \kappa\}$ of D into subsets of cardinality κ. There exists some $p_\alpha \in U$ for each $\alpha < \kappa$ with $C_\alpha \in p$. There exists a continuous injection $\varphi : D \to A$ which we define by $\varphi(C_\alpha) = \{q_\alpha\}$. Now extend φ to $\overline{\varphi} : \beta D \to A$. Then $\overline{\varphi}(p_\alpha) = q_\alpha$. As $U \subseteq \beta D$ is closed, it follows that $\overline{\varphi}(U)$ is a compact subset of I that contains A and since βD is compact, $\overline{\varphi}$ is a continuous surjection from compact space to a Hausdorff space. Hence $\overline{\varphi}(U) = I$. Thus $|\beta D| = 2^{2^\kappa}$. $\square$

Finally, we establish a characterization of the closed subsets of βD by showing a one-to-one correspondence between the nonempty closed subsets of βD and the filters on D.

Definition 3.3.13. *Let $\mathscr{A}$ be a family of subsets of D. We define $\overline{\mathscr{A}} \subseteq \beta D$ by*

$$\overline{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} \overline{A}.$$

Theorem 3.3.14 ([47])**.** *Given a set D,*

1. *If $\mathscr{F}$ is a filter on D, then $\overline{\mathscr{F}}$ is a closed subset of βD.*

2. *If $\varnothing \neq F \subseteq \beta D$ and $\mathscr{F} = \bigcap F$, then $\mathscr{F}$ is a filter on D and $\overline{\mathscr{F}} = cl(F)$.*

Proof. 1. Let $p \in \beta D \setminus \overline{\mathscr{F}}$. If we choose $B \in \mathscr{F} \setminus p$, then $\overline{D \setminus B}$ is a neighbourhood of p which misses $\overline{\mathscr{F}}$.

2. $\mathscr{F}$ is a filter as it is the intersection of a set of filters. Further, for each $p \in F$, $\mathscr{F} \subseteq p$ giving that $F \subseteq \mathscr{F}$ and thus by 1, $cl(F) \subseteq \overline{\mathscr{F}}$.

   To prove the reverse inclusion, we suppose that $p \in \overline{\mathscr{F}}$ and $B \in p$. Next, suppose $\overline{B} \cap F = \varnothing$. Then for each $q \in F$, we have that $D \setminus B \in q$ so $D \setminus B \in \mathscr{F} \subseteq p$ which is a contradiction.

$\square$

3.4 Extending the Operation to βS

In this section we extend the operation of the discrete semigroup S to βS, its Stone-Čech compactification.

To aid us in this endeavor we introduce the notion of p-limit which first appeared in literature in [13]. We want $p-\lim_{d\in D} x_d = y$ to mean x_d is "often" "near" y. As usual in topology, the notion of nearness to a point y in a topological space is determined by neighbourhoods of that point and "often" is determined by the members of the ultrafilter p.

Definition 3.4.1 ([19]). *Let D be a discrete space, let p be an ultrafilter in βD, let $\langle x_d\rangle_{d\in D}$ be an indexed family in a topological space X, and let y be a point in X. Then $p-\lim_{d\in D} x_d = y$ if and only if for each neighbourhood V of y, $\{d\in D : x_d\in V\}\in p$.*

Let us consider the topological spaces X and Y, with a subset A of X and a function $\varphi : A\to Y$. Also let x be in the closure of A in X and y be a point of the space Y. We write $\lim_{a\to x}\varphi(a) = y$ if and only if, for every neighbourhood V of y, there exists a neeighbourhood U of x such that $\varphi(A\cap U)\subseteq V$. Notice that if $\lim_{a\to x}\varphi(a)$ exists, it is unique.

We shall now show that for functions defined on βD, the notion of limit corresponds to the notion of p-limit.

Theorem 3.4.2 ([19]). *Let Y be a topological space, let D be a discrete space, let p be an ultrafilter in βD and let y be a point of the space Y. If A is a member of p and $\varphi : A\to Y$, then $p-\lim_{a\in A}\varphi(a)$ if and only if $\lim_{a\to p}\varphi(a) = y$.*

Proof. Necessity. Suppose that $p-\lim_{a\in A}\varphi(a) = y$. Then, if V is a neighbourhood of y, $\varphi^{-1}(V)\in p$. Let $B=\varphi^{-1}(V)$, then $\overline{B}$ is a neighbourhood of p by Theorem 3.3.4 and $\varphi(\overline{B}\cap A)=\varphi(B)\subseteq V$. Thus $\lim_{a\to p}\varphi(a) = y$.

Sufficiency. Suppose that $\lim_{a\to p}\varphi(a) = y$. Then, if V is a neighbourhood of y, there is a neighbourhood U of p in βD such that $\varphi(U\cap A)\subseteq V$. Now $U\cap A\in p$ and since $U\cap A\subseteq\varphi^{-1}(V)$, then it follows that $\varphi^{-1}(V)\in p$. Thus $p-\lim_{a\in A}\varphi(a) = y$. $\square$

Theorem 3.4.3 ([19]). *Let D be a discrete space, let p be an ultrafilter in βD, and let*

$\langle x_d \rangle_{d\in D}$ *be an indexed family in a space* X. *If* X *is a compact space, then* $p-\lim_{d\in D} x_d$ *exists and is unique.*

Proof. Let us assume that $p - \lim \langle x_d \rangle_{d\in D}$ does not exist and for each $y \in X$, we pick an open neighbourhood U_y of y such that $\{d \in D : x_d \in U_y\} \notin p$. Then $\{U_y : y \in X\}$ is an open cover of X. Now we choose a finite subset F of X such that $X = \bigcup_{y\in F} U_y$. Then $D = \{d \in D : x_d \in U_y\}$ so we choose some $y \in F$ for which $\{d \in D : x_d \in U_y\} \in p$ which is a contradiction. Hence, $p - \lim \langle x_d \rangle_{d\in D}$ exists whenever X is compact. Furthermore, it is obvious that this p-limit is unique. □

Theorem 3.4.4 ([19])**.** *Let* D *be a discrete space, let* p *be an ultrafilter in* βD, *let* X *and* Y *be topological spaces, let* $\langle x_d \rangle_{d\in D}$ *be an indexed family in* X, *and let* $\varphi : X \to Y$. *If* φ *is continuous and* $p - \lim_{d\in D} x_d$ *exists, then*

$$p - \lim_{d\in D} \varphi(x_d) = \varphi(p - \lim_{d\in D} x_d).$$

Proof. Suppose that U is a neighbourhood of $\varphi(p - \lim_{d\in D} x_d)$ and then choose a neighbourhood V of $p-\lim_{d\in D} x_d$ such that $\varphi(V) \subseteq U$. Now let $A = \{d \in D : x_d \in V\}$. Then A is a member of p and

$$A \subseteq \{d \in D : \varphi(x_d) \in U\}.$$

□

If we view $\langle d \rangle_{d\in D}$ as an indexed family in βD, where D is a discrete space, then we have $p - \lim_{d\in D} d = d$, where $p \in \beta D$. If we consider this fact along with the above theorem and let φ be a function that maps D into a compact space X, with $\overline{\varphi} : \beta D \to X$ as the continuous extension of φ, then it follows that

$$\overline{\varphi}(p) = p - \lim_{d\in D} \varphi(d) \text{ for all } p \in \beta D.$$

We can now consider a semigroup S with the discrete topology and extend the binary operation of this discrete semigroup to its Stone-Čech compactification. We follow the convention of supposing that $S \subseteq \beta S$.

Theorem 3.4.5 ([19]). *Consider the discrete space S and the binary operation, $\cdot$, defined on S. There is a unique binary operation $* : \beta S \times \beta S \to \beta S$ for which the follwing conditions hold:*

1. *For every $s, t \in S$, $s * t = s \cdot t$,*

2. *For each ultrafilter $q \in \beta S$, the mapping $\rho_q : \beta S \to \beta S$ is continuous, $\rho_q(p) = p * q$,*

3. *For each point $s \in S$, the mapping $\lambda_s : \beta S \to \beta S$ is continuous, where $\lambda_s(q) = s * q$.*

Proof. We will establish the uniqueness and the existence of the binary operation $*$ simultaneously. The first step is to define $*$ as we are forced to define it, first by defining it on $S \times \beta S$ and then by extending $*$ to the rest of $\beta S \times \beta S$.

To define $*$ on $S \times \beta S$, we consider any $s \in S$ and define $\eta_s : S \to S \subseteq \beta S$ by $\eta_s(t) = s \cdot t$. Then there exists a continuous function $\lambda_s : \beta S \to \beta S$ for which $\lambda_s|_S = \eta_s$. For $s \in S$ and $q \in \beta S$. we define $s * q = \lambda_s(q)$. From this we have that 3 is satisfied and so is 1 since λ_s extends η_s. We also have that the extension λ_s is unique as continuous functions that agree on a dense subspace are equal. So there can be no other possible definition of $*$ that satisfies 1 and 3.

Now we extend $*$ to the rest of $\beta S \times \beta S$. Given an ultrafilter q in βS, we define $\xi_q : S \to \beta S$ by $\xi_q(s) = s * q$. Then there is a continuous function $\rho_q : \beta S \to \beta S$ such that $\rho_q|_S = \xi_q$. For $p \in \beta S \setminus S$, we define $p * q = \rho_q(p)$. It should be noted that if $s \in \beta S$, then $\rho_q(s) = \xi_q(s) = s * q$ so for every $p \in \beta S$, $\rho_q(p) = p * q$. Therefore 2 is satisfied. Once more, by the uniqueness of the continuous extensions, there is no other possible definition that would satisfy all three conditions. $\square$

We wish to verify the associativity of the extended operation, however, it would be prudent to better understand the operations. The opeartion on βS can be characterized in terms of limits. We should keep in mind ρ_q is continuous for every $q \in \beta S$ and that λ_s is continuous for every member s of S. Now let $\cdot$ be a binary operation on a discrete space S and let $p, q \in \beta S$. Whenever $s \in S$ and $q \in \beta S$, $s \cdot q = \lim_{t \to q} s \cdot t$ and whenever $p, q \in \beta S$,

$$p \cdot q = \lim_{s \to p}(\lim_{t \to q} s \cdot t), \text{ where } t \in S.$$

Furthermore, if $P \in p$ and $Q \in q$, then

$$p \cdot q = p - \lim_{s \in P}(q - \lim_{t \in Q} s \cdot t).$$

Theorem 3.4.6 ([19])**.** *Suppose that S is a semigroup. Then the extended operation on βS is associative.*

Proof. Let $p, q, r \in \beta S$ and let $x, y, z \in S$.

$$\begin{aligned}
(p \cdot q) \cdot r &= (\lim_{x \to p} x \cdot q) \cdot r && \text{by continuity of } \rho_q \\
&= (\lim_{x \to p} \lim_{y \to q} x \cdot y) \cdot r && \text{by continuity of } \lambda_x \\
&= \lim_{x \to p} \lim_{y \to q} (x \cdot y) \cdot r && \text{by continuity of } \rho_r \\
&= \lim_{x \to p} \lim_{y \to q} \lim_{z \to r} (x \cdot y) \cdot z && \text{by continuity of } \rho_{x \cdot y} \\
&= \lim_{x \to p} \lim_{y \to q} \lim_{z \to r} x \cdot y \cdot z
\end{aligned}$$

and

$$\begin{aligned}
p \cdot (q \cdot r) &= \lim_{x \to p} (x \cdot (q \cdot r)) && \text{by continuity of } \rho_{q \cdot r} \\
&= \lim_{x \to p} (x \cdot \lim_{y \to q} y \cdot r) && \text{by continuity of } \rho_r \\
&= \lim_{x \to p} (x \cdot \lim_{y \to q} \lim_{z \to r} y \cdot z) && \text{by continuity of } \lambda_y \\
&= \lim_{x \to p} \lim_{y \to q} \lim_{z \to r} x \cdot (y \cdot z) && \text{by continuity of } \lambda_x \\
&= \lim_{x \to p} \lim_{y \to q} \lim_{z \to r} x \cdot y \cdot z
\end{aligned}$$

so $(p \cdot q) \cdot r = p \cdot (q \cdot r)$. □

As we have verified the associativity of the extended operation, we now know that βS is a semigroup if S is a semigroup.

A semigroup S endowed with a topology is called a *right topological semigroup* if for each $a \in S$, the right translation $\rho_a : S \ni x \mapsto xa \in S$ is continuous. The *topological center* of a right topological semigroup S, which we denote $\Lambda(S)$, consists of all $b \in S$ such that the left translation $\lambda_b : S \ni x \mapsto bx \in S$ is continuous. So it follows from Theorem 3.4.5 and Theorem 3.4.6 that βS is a compact right topological semigroup with $S \subseteq \Lambda(\beta S)$.

Definition 3.4.7. *Consider a semigroup S endowed with a topology and a compact right topological semigroup T. A* ***semigroup compactification*** *of S is a pair (ϕ, T), where $\phi : S \to T$ is a continuous homomorphism such that $\phi(S)$ is dense in T and $\phi(S) \subseteq \Lambda(T)$.*

Theorem 3.4.8 ([19])**.** *Consider the Hausdorff right topological semigroups T and R. Suppose that S is a dense subsemigroup of T such that $S \subseteq \Lambda(T)$ and $\phi : T \to R$ is a continuous mapping such that $\phi(S) \subseteq \Lambda(R)$. If $\phi|_S$ is a homomorphism, then ϕ is also a homomorphism.*

Proof. Let $a, b \in S$ and let $q \in T$. Then

$$\begin{aligned}
\phi(aq) &= \phi(\lim_{b\to q} ab) && \text{by continuity of } \lambda_a \\
&= \lim_{b\to q} \phi(ab) && \text{by continuity of } \phi \\
&= \lim_{b\to q} \phi(a)\phi(b) && \text{since } \phi|_S \text{ is a homomorphism} \\
&= \phi(a) \lim_{b\to q} \phi(b) && \text{by continuity of } \lambda_{\phi(a)} \\
&= \phi(a)\phi(q).
\end{aligned}$$

Now let $p, q \in T$. Then

$$\begin{aligned}
\phi(pq) &= \phi(\lim_{a\to p} aq) && \text{by continuity of } \rho_q \\
&= \lim_{a\to p} \phi(aq) && \text{by continuity of } \phi \\
&= \lim_{a\to p} \phi(a)\phi(q) \\
&= (\lim_{a\to q} \phi(a))\phi(q) && \text{by continuity of } \rho_{\phi(q)} \\
&= \phi(p)\phi(q).
\end{aligned}$$

□

Consider a semigroup S endowed with the discrete topology and any homomorphism ϕ of S into a compact Hausdorff right topological semigroup T such that $\phi(S) \subseteq \Lambda(T)$. If $\overline{\phi} : \beta S \to T$ is the continuous extension of ϕ, then applying Theorem 3.4.8 gives that $\overline{\phi}$ is a homomorphism. Thus the Stone-Čech compactification βS of a discrete semigroup is the largest semigroup compactification of S. From this point onwards, whenever we speak of βS, we will be referring to the **Stone-Čech compactification of of the discrete semigroup** S.

It is possible to describe the operation of βS in terms of ultrafilters. This means for ultrafilters $p, q \in \beta S$, we can characterize the subsets of S which are members of the ultrafilters pq.

Note that given a given a semigroup S, $B \subseteq S$ and $s \in S$,

$$s^{-1}B = \{a \in S : sa \in B\} = \lambda_s^{-1}(B).$$

The notation $s^{-1}B$ is an alternative notation for $\lambda_s^{-1}(B)$, and using it does not imply that s has an inverse in B.

Theorem 3.4.9 ([19]). *Consider the semigroup S, $B \subseteq S$, $s \in S$ and the ultrafilters $p, q \in \beta S$. Then*

1. *$B \in s \cdot q$ if and only if $s^{-1}B \in q$, and*

2. *$B \in p \cdot q$ if and only if $\{a \in S : a^{-1}B \in q\} \in p$.*

Proof. 1. Necessity. Suppose $B \in s \cdot q$. Then $\overline{B}$ is a neighbourhood of $s \cdot q$. Since λ_s is continuous, there is some $Q \in q$ such that $s \cdot \overline{Q} \subseteq \overline{B}$. It follows that $sQ \subseteq B$ and so $s^{-1}B \in q$.

Sufficiency. Let $s^{-1}B \in q$ and assume on the contrary that $B \notin s \cdot q$. Then $S \setminus B \in s \cdot q$ so ,by the necessity we have just established, $s^{-1}(S \setminus B) \in q$. But this is a contradiction since

$$(s^{-1}B) \cap (s^{-1}(S \setminus B)) = \varnothing.$$

2. Necessity. Suppose that $B \in p \cdot q$. Since ρ_q is continuous, there is some $P \in p$ such that $\overline{P} \cdot q \subseteq \overline{B}$. Then for every $z \in P$, $B \in z \cdot q$ and so by 1, $z^{-1}B \in q$. Hence, $\{z \in S : z^{-1}B \in q\} \in p$.

Sufficiency. Let $\{z \in S : z^{-1}B \in q\} \in p$ and assume on the contrary, that $B \notin p \cdot q$. Then, $S \setminus B \in p \cdot q$ so, by the necessity we have just established, $\{z \in S : z^{-1}(S \setminus B) \in q\} \in p$. However

$$(z^{-1}B) \cap (z^{-1}(S \setminus B)) = \varnothing$$

for each $z \in S$. It follows that

$$\{z \in S : z^{-1}B \in q\} \cap \{z \in S : z^{-1}(S \setminus B) \in q\} = \varnothing,$$

which is a contradiction.

□

3.5 Ultrafilter Semigroups

Definition 3.5.1. *A **left topological semigroup** is a semigroup S endowed with a topology $\mathscr{T}$ such that for every $c \in S$, the left translation*

$$\lambda_c : S \ni c \mapsto cs \in S$$

is a continuous mapping of the space S to itself. The topology $\mathscr{T}$ on S is called a ***left invariant topology.***

Given a semigroup S, a topology $\mathscr{T}$ on S is left invariant if and only if for all $c \in S$ and $U \in \mathscr{T}$,

$$c^{-1}U = \lambda_c^{-1}(U) = \{s \in S : cs \in U\} \in \mathscr{T}.$$

It is important to note that the left translations in a left topological group are homeomorphisms. So if S is a group, we can determine a left invariant topology on S completely by the neighbourhood filter of the identity. We can characterize topologies having the property that for every $c \in S$, the neighbourhood filter of c is

$$c\mathscr{N} = \{cU : U \in \mathscr{N}_e\}$$

on semigroups as follows:

Lemma 3.5.2 ([47])**.** *Given a semigroup S with identity, let $\mathscr{T}$ be a topology on S and $\mathscr{N}_e$ the neighbourhood filter of the identity in $\mathscr{T}$. Then the follwing statements are equivalent.*

1. *For every $c \in S$, $c\mathscr{N}$ is a neighbourhood base at c,*
2. *For every $c \in S$, the left translation λ_c is continuous and open, and*
3. *For every $c \in S$ and $U \in \mathscr{T}$, both $c^{-1}U \in \mathscr{T}$ and $cU \in \mathscr{T}$.*

Proof. $(1 \Rightarrow 2)$. We have to show that λ_c is open and continuous. To show continuity, suppose that $c \in S$ and U is a neighbourhood of $\lambda_c(b) = cb$. Now choose a set $V \in \mathscr{N}_e$ such that $cbV \subseteq U$. Then bV is a neighbouhood of b and $\lambda_c(bV) = cbV \subseteq U$. To show that λ_c is open, suppose that $b \in S$ and U is a neighbourhood of b. Now choose a set $V \in \mathscr{N}_e$ such that $bV \subseteq U$. Then abV is a neighbourhood of $\lambda_c(b)$ and $\lambda_c(U) \supseteq \lambda_c(bV) = cbV$.

$(2 \Rightarrow 3)$. We know that λ_c is continuous, so $c^{-1}U = \lambda_c^{-1}(U) \in \mathscr{T}$. We also know that λ_c is open, so $cU = \lambda_c(U) \in \mathscr{T}$.

$(3 \Rightarrow 1)$. If U is an open neighbourhood of the identity, then clearly $c \in cU \in \mathscr{T}$, so cU is an open neighbourhood of c. Conversely, suppose that V is an open neighbourhood of c and $U = c^{-1}V$. Then U is an open neighbourhood of the identity and $cU \subseteq V$. $\square$

Much like we did with the topological group G, we can also characterize the neighbourhood filter of the identity of a left topological semigroup. The following theorem achieves this.

Theorem 3.5.3 ([47]). *Suppose that S is a left topological semigroup with identity and $\mathscr{N}_e$ is the neighbourhood filter of the identity. Then*

1. *For every $U \in \mathscr{N}_e$, $e \in U$, and*

2. *For every $U \in \mathscr{N}_e$, $\{s \in S : s^{-1}U \in \mathscr{N}_e\} \in \mathscr{N}_e$.*

Conversely, if S is a semigroup with identity and a filter $\mathscr{N}_e$ on S that satisfies 1-2, then there exists a left invariant topology on S in which for each $c \in S$, $c\mathscr{N}$ is a neighbourhood base at c.

Proof. Let S be a left topological semigroup with identity and let $U \in \mathscr{N}_e$. It is obvious that $e \in U$ and if we set $V = int(U)$, we obtain that $V \in \mathscr{N}_e$ and for every $s \in V$, $s^{-1}U \in \mathscr{N}_e$. So 1-2 are satisfied.

Conversely, let $\mathscr{N}_e$ be a filter on S that satisfies 1-2. For every $s \in S$, suppose that $\mathscr{N}_s$ is a neighbourhood filter with a base $s\mathscr{N}$. We now show that whenever $s \in S$ and $U \in \mathscr{N}_s$, we have

(i) $s \in U$, and

(ii) $\{t \in S : U \in \mathscr{N}_t\} \in \mathscr{N}_s$.

We obtain (i) from 1 and the definition of $\mathscr{N}_s$. In order to arrive at (ii), we let $U_0 = s^{-1}U$, $V_0 = \{r \in S : U_0 \in \mathscr{N}_r\}$, and $V = sV_0$. Notice that $U_0 \in \mathscr{N}_e$, so by 2, we have $V_0 \in \mathscr{N}_e$, and as a result $V \in \mathscr{N}_s$. We claim that for every $t \in V$, $U \in \mathscr{N}_t$. To verify this, we write $t = sr$ for some $r \in V_0$. Then $s^{-1}U = U_0 \in \mathscr{N}_r = r\mathscr{N}$, and so $U \in sr\mathscr{N} = \mathscr{N}_t$. If we apply Theorem 2.1.6, we obtain that there is a topology $\mathscr{T}$ on S such that $\{\mathscr{N}_s : s \in S\}$ is the neighbourhood system. Lemma 3.5.2 gives that $\mathscr{T}$ is left invariant. □

The following simple general fact will be useful down the line.

Lemma 3.5.4 ([47]). *Let $\mathscr{F}$ be a filter on a semigroup S. If $\{s \in S : s^{-1}F \in \mathscr{F}\} \in \mathscr{F}$ for every $F \in \mathscr{F}$, then $\bar{\mathscr{F}}$ is a closed subsemigroup of βS.*

Proof. As $\overline{\mathscr{F}}$ is a closed subset of βS, all we need to do is show that $\overline{\mathscr{F}}$ is a subsemigroup. Suppose that $p, q \in \overline{\mathscr{F}}$ and $F \in \mathscr{F}$. We have to show that $F \in pq$. To do this, begin by setting $B = \{s \in S : s^{-1}F \in \mathscr{F}\}$, and for every $s \in B$, let $C_s = s^{-1}F$. Then $B \in \mathscr{F} \subseteq p$, $C_s \in \mathscr{F} \subseteq q$, and $\bigcup_{s \in B} sB_s \subseteq F$. Hence, $F \in pq$. □

Lemma 3.5.5 ([47]). *Consider a left topological semigroup with identity $(S, \mathscr{T})$ and let $\mathscr{N}_e$ be the neighbourhood filter of the identity. Then*

1. *$\overline{\mathscr{N}_e}$ is a closed subsemigroup of βS,*

2. *For every open subset U, $\overline{U} \cdot \overline{\mathscr{N}_e} \subseteq \overline{U}$, and*

3. *If $\mathscr{T}$ is a T_1-topology, then*

$$\overline{\mathscr{N}_e} \setminus \{e\} = \{p \in S^* : p \text{ converges to } e \text{ in } \mathscr{T}\}.$$

Proof. 1. This is a consequence of Theorem 3.5.3 and Lemma 3.5.4.

2. Suppose that $p \in \overline{U}$ and $q \in \overline{\mathscr{N}_e}$. The set U is open, so for every $s \in U$, there is some $V_s \in \mathscr{N}_s$ such that $sV_s \subseteq U$. Then $\bigcup_{s \in U} sV_s \subseteq U$. Since $U \in p$ and $V_s \in q$, we have that $U \in pq$. Thus $pq \in \overline{U}$.

3. Clearly, $\overline{\mathscr{N}_e} \setminus \{e\}$ is a closed subset of βS. Since $\mathscr{T}$ satisfies the T_1 separation axiom, $\bigcap \mathscr{N}_e = \{e\}$. Then $\overline{N}_e \setminus \{e\} \subseteq S^*$, hence

$$\overline{\mathscr{N}_e} \setminus \{e\} = \{p \in S^* : p \text{ converges to } e \text{ in } \mathscr{T}\}.$$

All that remains is to show that $\overline{\mathscr{N}_e} \setminus \{e\}$ is a subsemigroup. To do this, suppose that $p, q \in \overline{\mathscr{N}_e} \setminus \{e\}$. We have to show that for every $U \in \mathscr{N}_e$, $p, q \in \overline{U} \setminus \{e\}$. Obviously, we may assume that U is open. Then $U \setminus \{e\}$ is also open, as $\mathscr{T}$ is a T_1-topology. Since $p \in \overline{U \setminus \{e\}}$, we obtain by 2 that $pq \in \overline{U \setminus \{e\}} = \overline{U} \setminus \{e\}$.

□

Definition 3.5.6 ([47])**.** *Given a T_1 left topological semigroup with identity $(S, \mathscr{T})$, define $Ult(\mathscr{T}) \subseteq \beta S$ by*

$$Ult(\mathscr{T}) = \{p \in S^* : p \textit{ converges to } e \textit{ in } \mathscr{T}\}.$$

We call $Ult(\mathscr{T})$ the ***ultrafilter semigroup*** *of $\mathscr{T}$.*

If we combine Definition 3.5.6 and Lemma 3.5.5, we see that $Ult(\mathscr{T})$ is a closed subsemigroup of βS if it is nonempty. Consequently, a general problem of investigation arises, that is, the interaction between the properties of the left invariant topology $\mathscr{T}$ on a group or semigroup and the algebraic structure of the corresponding ultrafilter semigroup $Ult(\mathscr{T})$. This has been a particularly fruitful endeavour with numerous striking results having been obtained in this manner. Notably, that (i) there exists a maximal regular left topological group within the framework of ZFC, (ii) every nondiscrete left topological group is ω-resolvable, (iii) the existence of a maximal topological group is impossible to establish in ZFC, etc.

Lemma 3.5.7 ([47])**.** *Let $\mathscr{F}$ be a filter on a semigroup S and let $X \in \mathscr{F}$. Suppose that R is a compact Hausdorff right topological semigroup and $\varphi : X \to R$. Assume*

1. *For every $x \in X$, there is some $U_x \in \mathscr{F}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $y \in U_x$, and*

2. $\varphi(X) \subseteq \Lambda(R)$.

Then for every $p \in \overline{X}$ *and* $q \in \overline{\mathscr{F}}$*, we have* $\overline{\varphi}(pq) = \overline{\varphi}(p)\overline{\varphi}(q)$*, where* $\overline{\varphi} : \overline{X} \to R$ *is the continuous extension of* φ*.*

Proof. For every $x \in X$, we have

$$\begin{aligned}
\overline{\varphi}(xq) &= \overline{\varphi}(\lim_{y\to q} xy), \quad y \in U_x \\
&= \lim_{y\to q} \varphi(xy) \\
&= \lim_{y\to q} \varphi(x)\varphi(y) \quad \text{by 1} \\
&= \varphi(x) \lim_{y\to q} \varphi(y) \quad \text{by 2} \\
&= \varphi(x)\overline{\varphi}(q).
\end{aligned}$$

Then

$$\begin{aligned}
\overline{\varphi}(pq) &= \overline{\varphi}(\lim_{x\to p} xq), \quad x \in X \\
&= \lim_{x\to p} \overline{\varphi}(xq) \\
&= \lim_{x\to p} \varphi(x)\overline{\varphi}(q) \\
&= (\lim_{x\to p} \varphi(x))\overline{\varphi}(q) \\
&= \overline{\varphi}(p)\overline{\varphi}(q).
\end{aligned}$$

□

3.6 Martin's Axiom

In the final section of this chapter, we give an elementary introduction to Martin's axiom. We must first recall a few basic notions from set theory.

If P is a nonempty set and the relation $\leq$ is a partial order, then the pair $(P, \leq)$ is a *partially ordered set.* A nonempty subset G of P is a *filter* in P if; (i) for every

$a, b \in G$, there exists some $c \in G$ such that $c \leq a$ and $c \leq b$, and (ii) for every $a \in G$ and $b \in P$, $a \leq b$ implies $b \in G$.

Let $a, b \in P$. The elements a and b are *compatible* if and only if there exists some $c \in P$ such that $c \leq a$ and $c \leq b$. If there is no such $c \in P$, then a and b are *incompatible.* Finally, a subset D of P is called *dense* if for every $a \in P$ there is some $b \in D$ such that $b \leq a$.

A subset of pairwise incompatible elements is called an *antichain* in P and if every antichain in P is countable then it is said that P has the *countable chain condition.*

Martin's axiom is a statement, introduced by D. A. Martin and R. M. Soloway in 1970, that is independent of the usual axioms of ZFC. Stated informally, Martin's axiom tells us that all cardinals $\kappa < \mathfrak{c}$ behave more or less like $\aleph_0$.

Definition 3.6.1. *Let κ be an infinite cardinal. Martin's Axiom, denoted by $MA(\kappa)$ is the assertion that if P is a partial order that satisfies the countable chain condition and $\mathscr{D}$ is a family of dense sets in P with $|\mathscr{D}| \leq \kappa$, then there is a filter G on P such that $G \cap D \neq \varnothing$, for every $D \in \mathscr{D}$.*

It is a theorem of ZFC that $MA(\mathfrak{c})$ does not hold, where $\mathfrak{c}$ is the cardinality of the continuum. Martin's axiom can thus be restated as follows:

Definition 3.6.2. *For every cardinal κ, if $\omega \leq \kappa < \mathfrak{c}$, then $MA(\kappa)$ is true.*

We will conclude this short exposition of Martin's axiom by showing that if follows from the continuum hypothesis.

Theorem 3.6.3. *The Continuum Hypothesis implies Martin's Axiom.*

Proof. Suppose the partially ordered set P has the countable chain condition. Now let $\mathscr{D} = \{D_n : n < \omega\}$ be a family of dense subsets of P. It follows from the continuum hypothesis that $\mathscr{D}$ is countable. Pick $(a_1) \in D_1$. Now inductively construct a sequence $(a_n)_{n<\omega}$ in P such that $a_n \in D_n$ and $a_{n+1} \leq a_n$. The filter $G = \{b \in P : a_n \leq b \text{ for some } n < \omega\}$ is generated by $(a_n)_{n<\omega}$ as required. $\square$

Chapter 4

Resolvability of Topological Groups

In this chapter, we relax our assumption that all topological spaces are Hausdorff.

4.1 Local Left Groups and Local Homomorphisms

Given a set X and some subset Y of $X \times X$, a function that maps Y to X is called a *partial multiplication* on X. A *partial semigroup* is a set with a partial multiplication such that $(xy)z = x(yz)$ whenever $x, y, z \in X$ and $(xy)z$, $x(yz)$ are defined.

Definition 4.1.1 ([47]). *Consider a topological space X satisfying the T_1 separation axiom and having a distinguished element e such that X is also a partial semigroup. The T_1-space X is a **local left topological group** if there exists some left topological group G wherein X can be topologically embedded in such a way that the following conditions hold:*

1. *e is the identity element of G,*
2. *the partial multiplication defined on X is the partial operation induced by the multiplication of G,*
3. *for every $a \in X$, there exists a neighbourhood U_a of the identity e in X such that $aU_a \subseteq X$, and*

4. for every $a \in X$, $aX \cap X$ is a neighbourhood of a in X.

Concerning local left topological groups, whenever we write $a \cdot b$ it will be assumed that $b \in U_a$ and whenever we write $a \cdot U$, where U is a neighbourhood of the identity, it will be assumed that $U \subseteq U_a$. If the T_1-space X can be embedded in a Hausdorff zero-dimensional left topological group G such that the above conditions are satisfied, then we call X a *regular local left topological group.*

A basic example of a local left topological group is an open neighbourhood of the identity e of a left topological group satisfying the T_1 separation axiom. We can see this by taking an open subset V of G such that $a \in V \subseteq X$ for every $a \in X$ and then take $U_a = a^{-1}V \cap X$. Furthermore, in any local left topological group, we may suppose that $U_e = X$.

Let us suppose that Y is a local left topological group and denote by Y_d, the partial semigroup Y reendowed with the discrete topology. As we have shown in the case of βS, we can extend the partial operation of Y_d in a natural way, to βY_d. This extension is given by

$$\lim_{y \to p} \lim_{z \to q} yz,$$

where $y, z \in Y$, making βY_d a right topological partial semigroup.

The product pq is defined if and only if

$$\{y \in Y : \{z \in Y : yz \text{ is defined}\} \in q\} \in p.$$

This can be seen by first supposing that pq is defined. As pq is a filter on Y, we have $Y \in pq$. The definition of pq gives that $Y \in pq$ if and only if $\{y \in Y : y^{-1}Y \in q\} \in p$ which holds if and only if $\{y \in Y : \{z \in Y : yz \text{ is defined}\} \in q\} \in p$. Conversely, we can suppose $\{y \in Y : \{z \in Y : yz \text{ is defined}\} \in q\} \in p$. Define the family

$$\mathscr{F} = \{A \subseteq Y : \{y \in Y : y^{-1}A \in q\} \in p\}.$$

Obviously $\varnothing \in \mathscr{F}$. The assumption that $\{y \in Y : \{z \in Y : yz \text{ is defined}\} \in q\} \in p$, along with the equality $y^{-1}Y = \{z \in Y : yz \text{ is defined}\}$ implies that $Y \in \mathscr{F}$. Now let $A, B \in \mathscr{F}$. Since $y^{-1}(A \cap B) = y^{-1}A \cap y^{-1}B$ for any $y \in Y$, we have $A \cap B \in \mathscr{F}$. Finally, for any $a \in \mathscr{F}$ and any subset B of Y with $A \subseteq B$, we see that $y^{-1}A \subseteq y^{-1}B$ and as p is a filter, we have that $B \in \mathscr{F}$. Thus the family $\mathscr{F}$ is a filter. Hence, pq is defined.

Definition 4.1.2 ([47])**.** *Let Y be a local left topological group,*

$$Ult(Y) = \{p \in Y_d^* : p \text{ converges to } e \text{ in } Y\}.$$

In other words, $Ult(Y)$ is the set of all nonprincipal ultrafilters on Y converging to the identity.

Suppose that $p \in \beta Y_d$ and $q \in Ult(Y)$. For every $a \in Y$, we have $U_a \in q$ and as ab is defined for $b \in U_a$, we have $U_a \subseteq \{b \in Y : ab \text{ is defined}\}$. Therefore $\{b \in Y : ab \text{ is defined}\} \in q$. Consequently, $\{a \in Y : \{b \in Y : ab \text{ is defined}\} \in q\} = Y \in p$. Hence pq is defined. We can conclude that $Ult(Y)$ is a closed subsemigroup of Y_d^*.

Given a set Y and a nonempty subset Z of Y, consider the bijection

$$\mathscr{F} \mapsto \mathscr{F} \cap Z := \{F \cap Z : F \in \mathscr{F}\}$$

and

$$\mathscr{F}' \mapsto \mathscr{F}'' := \{F \subset Y : E \subseteq F \text{ for some } E \in \mathscr{F}'\},$$

where $\mathscr{F}$ ranges over filters on Y that contain Z and $\mathscr{F}'$ ranges over filters on Z. Notice that ultrafilters and principal ultrafilters are sent to ultrafilters and principal ultrafilters respectively. Additionally, we let G be a local left topological group with topology $\mathscr{T}$ and suppose that Y is an open neighbourhood of the identity in G. We also suppose that $p \in Ult(\mathscr{T})$, then $Y \in p$. Consequently, for every neighbourhood W of the identity in G, $Y \cap W \in p$. We see that the set $Y \cap W$, where W is a neighbourhood of the identity in Y. It follows that $p \cap Y := \{B \cap Y : B \in p\} \in Ult(Y)$, so we can identify $Ult(Y)$ with $Ult(\mathscr{T})$.

Definition 4.1.3 ([47]). *Consider the local left topological groups Y and Z. A mapping $\varphi : Y \to Z$ is a **topological local homomorphism** if $\varphi(e_Y) = e_Z$ and for all $y \in Y \setminus \{e\}$, there exists a neighbourhood U of the identity e in Y such that $\varphi(yz) = \varphi(y)\varphi(z)$ for every $z \in U$. If $\varphi : Y \to Z$ is simultaneously a local homomorphism and homeomorphism then we call φ a **local isomorphism.***

It is important to note that if $\varphi : Y \to Z$ is a local isomorphism, then $\varphi^{-1} : Z \to Y$ is also a local isomorphism.

Theorem 4.1.4 ([34]). *Suppose that Y and Z are local left topological groups and that $\varphi : Y \to Z$ is a topological local isomorphism. Then, $\overline{\varphi}(Ult(Y)) = Ult(Z)$ and*

$$\varphi^* := \overline{\varphi}|_{Ult(Y)} : Ult(Y) \to Ult(Z)$$

is a topological isomorphism.

Proof. Suppose $p \in Ult(Y)$ and that W is a neighbourhood of e_Z. The continuity of φ and the fact that $\varphi(e_Y) = e_Z$ gives that $\varphi^{-1}(W)$ is a neighbourhood of e_Y and, therefore, $\varphi^{-1}(W) \in p$. Hence $W \in \overline{\varphi}(p)$. Then $\overline{\varphi}(p) \in Ult(Z)$ and, therefore, $\overline{\varphi}(Ult(Y)) \subseteq Ult(Z)$.

Conversely, suppose that $q \in Ult(Z)$. As $\overline{\varphi}$ is surjective, $q = \overline{\varphi}(p)$ for some $p \in \beta Y_d$. Now suppose that V is a neighbourhood of e_Y. Since $\varphi(V)$ is a neighbourhood of e_Z, $\varphi(V) \in q$, giving $U = \varphi^{-1}(\varphi(V)) \in q$. Hence $p \in Ult(Y)$. Therefore, $Ult(Z) \subseteq \overline{\varphi}(Ult(Y))$. Thus $\overline{\varphi}(Ult(Y)) = Ult(Z)$.

It remains to show that φ^* is a homeomorphism. Since $\overline{\varphi}$ is a bijection, φ^* is also a bijection. Now suppose that $A \subseteq \beta Y_d$ and $B \subseteq \beta Z_d$ are open. Then,

$$\varphi^*(Ult(Y) \cap A) = \overline{\varphi}(Ult(Y) \cap A) = \overline{\varphi}(Ult(Y)) \cap \overline{\varphi}(A) = Ult(Z) \cap \overline{\varphi}(A)$$

is an open subset of $Ult(Z)$ and

$$\varphi^{*-1}(Ult(Z) \cap B) = \overline{\varphi}^{-1}(Ult(Z)) \cap \overline{\varphi}(B) = Ult(Y) \cap \overline{\varphi}^{-1}(B)$$

is an open subset of $Ult(Y)$. Therefore, φ^* is a homeomorphism. $\square$

Definition 4.1.5 ([47])**.** *Given a local left topological group Y and a semigroup S, a mapping $\varphi : Y \to S$ is a* ***local homomorphism*** *if for every $y \in Y \setminus \{e\}$, there exists a neighbourhood U of the identity such that $\varphi(yz) = \varphi(y)\varphi(z)$ for all $z \in U \setminus \{e\}$.*

Theorem 4.1.6 ([47])**.** *Suppose that $\varphi : Y \to R$ is a local homomorphism from a local left topological group Y into a compact right right topological semigroup R such that $\varphi(Y) \subseteq \Lambda(R)$ and that $\overline{\varphi} : \beta Y_d \to R$ is the continuous extension of φ. Now define $\varphi : Ult(Y) \to R$ by $\varphi^* = \overline{\varphi}|_{Ult(Y)}$. Then φ^* is a continuous homomorphism and if, in addition, $\varphi(U \setminus \{e\})$ is dense in R for every neighbourhood U of the identity e in Y, then φ^* is surjective.*

Proof. It follows from Lemma 3.5.7 that φ^* is a homomorphism. Now suppose for every neighbourhood U of the identity $e$in Y, $\varphi(U \setminus \{e\})$ is dense in R and let $r \in R$. Then for each neighbourhood V of $r \in R$, there exists $a \in U \setminus \{e\}$ such that $\varphi(a) \in V$. This implies that there exists $p \in Ult(\mathscr{T})$ such that $\overline{\varphi}(p) = r$. Hence φ^* is surjective. $\square$

Since a finite discrete semigroup is a compact right topological semigroup which is equal to its topological center, then $\varphi^* = \overline{\varphi}|_{Ult(Y)} : Ult(Y) \to R$ is a homomorphism where Y is a local left topological group, R is a finite discrete semigroup, $\varphi : Y \to R$ is a local homomorphism and $\overline{\varphi} : \beta Y_d \to R$ is the continuous extension of φ.

If a homomorphism of an ultrafilter semigroup can be induced by a local homomorphism, then it is called *proper.* Therefore the induced homomorphisms $\varphi^* : Ult(Y) \to R$ and $\varphi^* : Ult(Y) \to Ult(Z)$ are proper.

Definition 4.1.7 ([47])**.** *Suppose that $\mathbb{B} = \bigoplus_\omega \mathbb{Z}_2$ is the countable Boolean group. Equip $\mathbb{B} = \bigoplus_\omega \mathbb{Z}_2$ with the topology obtained by taking as a neighbourhood base at zero the subgroups $H_\alpha = \{y \in \mathbb{B} : supp(y) \cap \alpha = \varnothing\}$, where $\alpha < \omega$. Note that for* *any $c = (c_n) \in \mathbb{B}$, $supp(c) = \{n < \omega : a_n \neq 0\}$. An element $c \in \mathbb{B}$ is called **basic** if $supp(c)$ is a nonempty interval in ω. Each nonzero element $c \in \mathbb{B}$ can be uniquely represented in the form $c = c_0 + c_1 + \cdots + c_l$, where*

1. *for every $i \leq l$, c_i is basic, and*

2. *for every $i \leq l-1$, $max\, supp(c_i) + 2 \leq min\, supp(c_{i+1})$.*

*This decomposition is called **canonical.***

We can extend a mapping φ_0 from the set of basic elements of $\mathbb{B}$ into a semigroup T to the mapping $\varphi : \mathbb{B} \to T$ by $\varphi(c) = \varphi(c_0)\varphi(c_1)\cdots\varphi(c_l)$, where $c = c_0 + c_1 + \cdots + c_l$ is the canonical decomposition of c. Now suppose that $y, z \in \mathbb{B}$ such that $max\, supp(y) + 1 < min\, supp(z)$ and that $y = y_0 + y_1 + \cdots + y_l$ is the canonical decomposition of y and $z = z_0 + z_1 + \cdots + z_m$ is the canonical decomposition of z. It follows that $y + z = y_0 + y_1 + \cdots + y_l + z_0 + z_1 + \cdots + z_m$ is the canonical decomposition of $y + z$. Therefore

$$\begin{aligned}
\varphi(y+z) &= \varphi(y_0 + y_1 + \cdots + y_l + z_0 + z_1 + \cdots + z_m) \\
&= \varphi(y_0)\varphi(y_1)\cdots\varphi(y_l)\varphi(z_0)\varphi(z_1)\cdots\varphi(z_m) \\
&= [\varphi(y_0)\varphi(y_1)\cdots\varphi(y_l)][\varphi(z_0)\varphi(z_1)\cdots\varphi(z_m)] \\
&= \varphi(y)\varphi(z)
\end{aligned}$$

Thus φ is a local homomorphism.

Definition 4.1.8 ([47]). *Consider the set L of all words w on the alphabet $\mathbb{Z}_m$, where $m \geq 2$. Also suppose that L includes the empty word $\varnothing$. Denote by $|w|$ the length of a word w for each $w \in L$. If all nonzero letters in w form a final subword we say the nonempty word $w \in L$ is basic. Particularly, all nonempty zero words, that is, words in which every letter is zero, are basic. For each $u, w \in L$ for which $|u| + 2 \leq |w|$ and the first $|u| + 1$ letters are zero, let $u + w \in L$ be defined as the resulting of substituting u for the initial subword having length $|u|$ in w. Each nonempty word $w \in L$ can be uniquely represented as $w = w_0 + w_1 + \cdots + w_l$ where*

1. *for every $i \leq l$, w_i is basic, and*

2. *for every $i < l$, w_i is nonzero.*

This decomposition is called ***canonical.***

We now give a theorem describing the structure of a local homomorphism. In addition to the main operation $^\frown$, we will also utilise the partial operation $+$ on L.

Theorem 4.1.9 ([47]). *Suppose that Y is a countably infinite regular local left topological group and that $\{U_y : y \in Y \setminus \{e\}\}$ is a family of neighbourhoods of the identity e in Y. Then there exists a continuous bijection $\psi : Y \to \mathbb{B}$ with $\psi(e) = 0$ such that*

1. *$\psi^{-1}(H_\alpha) \subseteq U_y$ whenever max supp$(\psi(y)) + 2 \leq n$, and*

2. *$\psi(yz) = \psi(y) + \psi(z)$ whenever max supp$(\psi(y)) + 2 \leq$ min supp$(\psi(z))$.*

Proof. Suppose that $L = L(\mathbb{Z}_2)$ is the set of all words on the alphabet $\mathbb{Z}_2$ including the empty word $\varnothing$ and since Y is countably infinite, enumerate Y without repetitions as $\{e, y_1, y_2, \ldots\}$. To each word w in L, we shall assign a point $y(w)$ in Y and a clopen neighbourhood $Y(w)$ of $y(w)$ such that

(a) $y(0^n) = e$, $Y(\varnothing) = Y$, and $Y(0^n) \subseteq U_{y(u)}$ for all $u \in L$ with $|u| \leq n - 2$,

(b) $Y(w^\frown 0) \cap Y(w^\frown 1) = \varnothing$ and $Y(w^\frown 0) \cup Y(w^\frown 1) = Y(w)$,

(c) $y(w) = y(w_0)y(w_1)\cdots y(w_l)$ and $Y(w) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l)$ where $w = w_0 + w_1 + \cdots + w_l$ is the canonical decomposition of w, and

(d) $y_n \in \{y(u) : u \in L \text{ and } |u| = n\}$.

We choose $Y(0)$ to be a clopen neighbourhood of the identity e such that $y_1 \notin Y(0)$. Next, we put $Y(1) = Y \setminus Y(0)$, $y(0) = e$ and $y(1) = y_1$. Now fix $n \geq 2$ and assume that $Y(w)$ and $y(w)$ have been constructed for each word w with $|w| < n$ such that conditions $(a) - (d)$ are satisfied. We point out that the subsets $Y(w)$, where

$|w| = n-1$, form a partition of Y, so one of these subsets, say $Y(v)$, contains y_n. Now suppose that $v = v_0 + v_1 + \cdots + v_k$ is the canonical decomposition. Then we can write $Y(v) = y(v_0)y(v_1)\cdots y(v_{k-1})Y(v_k)$ and $y_n = y(v_0)y(v_1)\cdots y(v_{k-1})z_n$ for some $z_n \in Y(v_k)$.

In the case that $z_n = y(v_k)$, we choose $Y(0^n)$ to be a clopen neighbourhood of the identity e such that $Y(0^n) \subseteq U_{y(w)}$ for every word w in L with $|w| = n-1$, we have $Y(w) \setminus y(w)Y(0^n) \neq \varnothing$. Now for every basic word w with $|w| = n-1$, we put $Y(w^\frown 1) = Y(w) \setminus Y(w^\frown 0)$ where $y(w^\frown 0) = y(w)$ and $Y(w^\frown 0) = y(w)Y(0^n)$. We also choose $y(w^\frown 1)$ to be any element of $Y(w^\frown 1)$. We also put $y(0^n) = e$ and in the case that $z_n \neq y(v_k)$, we pick $Y(0^n)$ in addition so that $z_n \notin y(v_k)Y(0^n)$, and put $y(v_k^\frown 1) = z_n$. For every nonbasic nonzero word w with $|w| = n$, we let $Y(w)$ and $y(w)$ be defined by condition (c). Then

$$y(w) = y(w_1)y(w_2)\cdots y(w_l) \in y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l) = Y(w)$$

and if $y_n \notin \{y(w) : |w| = n-1\}$, we have

$$y_n = y(v_0)y(v_1)\cdots y(v_{k-1})y(v_k^\frown 1) = y(v^\frown 1) \in \{y(w) : |w| = n\}.$$

In order to check condition (b), we let $|w| = n-1$. In the case that the word w is basic, we have $Y(w^\frown 0) = y(w)Y(0^n) \subset Y(w)$ and $Y(w^\frown 1) = Y(w) \setminus y(w)Y(0^n)$. Now assume that the word w is nonbasic and let the canonical representation of w be $w = w_0 + w_1 + \cdots + w_l$. If w_l is zero, then $w^\frown 0 = w_0 + w_1 + \cdots + w_{l-1} + 0^n$ is the canonical decomposition of $w^\frown 0$. It follows that

$$Y(w^\frown 0) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(0^n) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l^\frown 0).$$

Otherwise the canonical decomposition is $w^\frown 0 = w_0 + w_1 + \cdots w_{l-1} + w_l + 0^n$, and then

$$Y(w^\frown 0) = y(w_0)y(w_1)\cdots y(w_{l-1})y(w_l)Y(0^n) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l^\frown 0).$$

In any event,

$$Y(w^\frown 0) = y(w_0)y(w_1)\cdots y(w_{l-1})y(w_l)Y(0^n) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l^\frown 0).$$

Now, since $w^\frown 1 = W_0 + w_1 + \cdots + w_{l-1} + w_l^\frown 1$ is the canonical decomposition of $w^\frown 1$, we have

$$Y(w^\frown 1) = y(w_0)y(w_1)\cdots y(w_{l-1})y(w_l)Y(0^n) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l^\frown 1).$$

Consequently,

$$\begin{aligned} Y(w^\frown 0) \cup Y(w^\frown 1) &= y(w_0)y(w_1)\cdots y(w_{l-1})[Y(w_l^\frown 0) \cup Y(w_{l-1}^\frown 1)] \\ &= y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l) \\ &= Y(w) \end{aligned}$$

and clearly $Y(w^\frown 0) \cap Y(w^\frown 1) = \varnothing$.

Finally, for each $y \in Y$, there exists a word w in L having a nonzero last letter such that $y = y(w)$, so $\{u \in L : y = y(u)\} = \{w^\frown 0^n : n < \omega\}$. Therefore $\psi : Y \to \mathbb{B}$ can be defined by setting for each $w = b_0 b_1 \cdots b_n \in L$,

$$\psi(y(w)) = \overline{w} = (b_0, b_1, \ldots, b_n, 0, 0, \ldots) \qquad (*)$$

It is obvious that ψ is a bijection and that $\psi(0) = e$ so ψ is continuous since for each $w = b_0 b_1 \cdots b_n \in L$, $Y(w)$ consists of all elements $y \in Y$ such that $\psi(y) = (b_0, b_1 \ldots, b_n, \ldots)$. And as $\psi^{-1}(H_\alpha) = Y(0^n) \subseteq U_{y(w)}$ for every $w \in L$ with $|w| \leq n-2$, and so 1. is satisfied.

All that remains is to check 2. To do this, we suppose $max\, supp(\psi(y)) + 2 \leq min\, supp(\psi(z))$. Now choose the words $w, u \in L$ having nonzero last letters such that

$$y = y(w) \text{ and } z = y(u). \qquad (**)$$

Additionally, consider $w = w_0 + w_1 + \cdots + w_l$ and $u = u_0 + u_1 + \cdots + u_r$ to be the canonical decompositions of w and u rspectively. We have $z \in Y(0^{|w|+1})$, so the canonical decomposition of $w + u$ is $w + u = w_0 + w_1 + \cdots + w_l + u_0 + u_1 + \cdots + u_r$. and by $(**)$

$$yz = y(w_0)y(w_1)\cdots y(w_l)y(u_0)y(u_1)\cdots y(u_r) = y(w+u). \qquad (***)$$

By $(*)$ and $(***)$, it follows that

$$\psi(yz) = \psi(y(w+u)) = \overline{w+u} = \overline{w} + \overline{u} = \psi(y(w)) + \psi(y(u)) = \psi(y) + \psi(z),$$

so 2. is satisfied. □

Given the countable nondiscrete regular local left topological groups Y and Z, consider the bijections $\psi : Y \to \mathbb{B}$ and $\xi : Z \to \mathbb{B}$ that are guaranteed by Theorem 4.1.9. Then there is a bijection $\varphi := \xi^{-1} \circ \psi : Y \to Z$ such that both φ and φ^{-1} are local homomorphisms.

4.2 Resolving by Local Automorphisms of Finite Order

Definition 4.2.1. *Given a local left topological group Y, a* ***local automorphism*** *of Y is a local isomorphism of Y onto itself. If a local automorphism is the identity map of some neighbourhood of the identity, it is called* ***trivial****.*

Definition 4.2.2. *A group G is* ***Boolean*** *if every element of G, other than the identity, has order 2. A* ***Boolean part*** *of a group G is the subset*

$$B(G) = \{g \in G : g^2 = e\}.$$

If a group G is Boolean, then for each $a, b \in G$, we have $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$ since for each $a \in G$ with $a \neq e$, we have $a^2 = e$ if and only if $a \cdot a = e$ which is true if and only if $a = a^{-1}$. Then every Boolean group G is Abelian.

Consider a topology $\mathscr{T}$ on a group G such that all shifts are continuous. For every element $h \in G$, h-conjugation $g \mapsto h^{-1}gh$ on G is an example of a topological automorphism. It is a trivial local automorphism if and only if the centralizer of h is open. If $h^n = e$, then h-conjugation is of order $\leq n$. Now let G be Abelian and suppose that the topology $\mathscr{T}$ on G is such that all shifts and inversions are continuous. An inversion on G is a topological automorphism and has order ≤ 2. It is a trivial local automorphism if and only if there exists an open Boolean subgroup in G.

In the general case, when G is not Abelian, inversion on G is not a local automorphism. Now let $h \in G$. For each h, there exists a neighbourhood of $e \in G$ such that $hg = gh$ for all $g \in U$. Consequently, $g^{-1}h^{-1} = h^{-1}h^{-1}$ and then $(hg)^{-1} = g^{-1}h^{-1} = h^{-1}h^{-1}$. Therefore if each conjugation on G is trivial, inversion is a local automorphism. An inversion on G is trivial if and only if there exists an open Boolean subgroup in G. This can be seen by picking a neighbourhood U of e such that $U^2 \subseteq B(G)$. This means for any $g, k \in U$, $(gk)^2 = gkkg = e$. Consequently, $gk = kg$. Hence, $\langle U \rangle$, a subgroup generated by U is contained in $B(G)$.

Definition 4.2.3. *Consider a bijection φ on a set Y. If $\varphi^n = id_Y$ and n is the smallest integer that satisfies this property then we say φ has **finite order n**.*

Definition 4.2.4. *Let φ be a mapping of a set Y into itself. A subset $Z \subseteq Y$ is said to be **invariant** with respect to φ if $\varphi(Z) \subseteq Y$. A family $\mathscr{A}$ of subsets of Y is said to be **invariant** if for each $Z \in \mathscr{A}$, $\varphi(Z) \in \mathscr{A}$. For each $y \in Y$, define the **orbit** with respect to φ of the element y by $O(\varphi, y) = \{\varphi^n(y) : n < \omega\}$.*

Notice that if $|O(\varphi, y)| = \omega$, then for any pair of distinct $n, m \in \mathbb{N}$ such that $n, m < \omega$, $\varphi^n(y) \neq \varphi^m(y)$ and if $|O(\varphi, y)| = n$, then $\varphi^n(y) = y$.

Lemma 4.2.5 ([43])**.** *Given a space Y, suppose that $\varphi : Y \to Y$ is a homeomorphism. Let $y \in Y$ with $|O(\varphi, y)| = k$ for some $k \in \mathbb{N}$ and let U be a neighbourhood of y for which the family $\{\varphi^j(U) : j < k\}$ is disjoint. If there exists $n \in \mathbb{N}$ such that $\varphi^n|_U = id_U$, then there exists an open neighbourhood $W \subseteq U$ of y for which the family $\varphi^j(W) : j < k\}$ is invariant. If Y is zero dimensional, then W can be chosen to be clopen.*

Proof. Notice that $n = kl$ for some $l \in \mathbb{N}$. Now pick an open neighbourhood V of y for which $\varphi^{j+ik}(V) \subseteq \varphi^j(U)$ for every $j < k$ and $i < l$. Particularly, $\varphi^{ik}(V) \subseteq U$ for each $i < l$. We can do this because $\varphi^k(y) = y$ which is true because $|O(\varphi, y)| = k$. We can put $W = \bigcup_{i<l} \varphi^{ik}(V)$. Then $W \subseteq U$ is an open neighbourhood of y and $\varphi^k(W) = \bigcup_{i<l} \varphi^{(i+l)k}(V)$. Since $\varphi^{kl} = \varphi^0$, it follows that

$$\varphi^k(W) = \bigcup_{i<l} \varphi^{ik}(V) = W.$$

Consequently, the family $\{\varphi^j(W) : j < k\}$ is invariant. □

Given a space Y with a distinguished point $e \in Y$ and a homeomorphism $\varphi : Y \to Y$ with $\varphi(e) = e$, it follows from lemma 4.2.5 that one of the following two possibilities must hold: Either (1) for every neighbourhood U of $e \in Y$ and every $n \in \mathbb{N}$, there exists some $y \in U$ with $|O(\varphi, y)| > n$, or (2) there exists an open invariant(with respect to φ) neighbourhood U of $e \in Y$ for which $\varphi|_U$ is of finite order, is true.

Definition 4.2.6 ([43])**.** *Suppose that Y is a space with a distinguished point $e \in Y$ and that $\varphi : Y \to Y$ is a homeomorphism of finite order with $\varphi(e) = e$. A **spectrum** of φ is the set $spec(\varphi) = \{|O(\varphi, y)| : y \in Y, y \neq e\}$, and more generally, for any subset $Z \subseteq Y$, a **spectrum** of φ on Z is the set $spec(\varphi, Z) = \{|O(\varphi, y)| : y \in Z, y \neq e\}$. If for each neighbourhood U of $e \in Y$, $spec(\varphi, U) = spec(\varphi)$, then φ is called **spectrally irreducible**. Finally, a neighbourhood U of a point $y \in Y$ is called **spectrally minimal** if for any neighbourhood $W \subseteq U$ of y, $spec(\varphi, W) = spec(\varphi, U)$.*

Given a space Y with a distinguished point $e \in Y$ and a homeomorphism of finite order $\varphi : Y \to Y$ with $\varphi(e) = e$, it follows from Lemma 4.2.5 that there exists an open invariant(with repect to φ) neighbourhood U of $e \in Y$ for which $\varphi|_U$ is spectrally irreducible. This means that whenever we study homeomorphisms of finite order in a neighbourhood of a fixed point, our investigation can be restricted to considering only the homeomorphisms that are spectrally irreducible.

Lemma 4.2.7 ([43])**.** *Suppose that Y is a local left topological group, φ is a spectrally irreducible local automorphism of finite order on Y, $e \neq y_0 \in Y$ with $|O(\varphi, y)| = t$,* *and U is a spectrally minimal neighbourhood of y_0. Then*

$$spec(\varphi, U) = \{lcm(k, r) : r \in \{e\} \cup spec(\varphi)\},$$

where $lcm(t, r)$ is the least common multiple of t and r.

Proof. For every $y \in O(\varphi, y_0)$, suppose that W_y is a neighbourhood of the identity e for which

(a) $\varphi(yz) = \varphi(y)\varphi(z)$ for each $z \in W_y$, and

(b) $z \mapsto yz$ is a homeomorphism, where $z \in W_y$ and $yz \in yW_y$.

Now pick a neighbourhood W of e for which $W \subseteq \bigcap_{y \in O(\varphi, y_0)} W_y$, $y_0 W \subseteq U$, and the family of subsets yW with $y \in O(\varphi, y_0)$ is disjoint. Suppose that φ has order n and pick a neighbourhood V of e for which $\varphi^i(V) \subseteq W$ for $0 \leq i \leq n-1$. This inclusion holds for $i < \omega$. We now show that for all $z \in V$, $\varphi^i(y_0 z) = \varphi^i(y_0)\varphi^i(z)$. The case $i = 0$ is trivial. We obtain, inductively, that

$$\varphi^i(y_0 z) = \varphi\varphi^{i-1}(y_0 z) = \varphi[\varphi^{i-1}(y_0)\varphi^{i-1}(z)] = \varphi^i(y_0)\varphi^i(z).$$

We now suppose $z \in V$, $|O(\varphi, z)| = r$, $s = lcm(t, r)$ and claim that $|O(\varphi, y_0 z)| = s$. Indeed,

$$\varphi^s(y_0 z) = \varphi^s(y_0)\varphi^s(z) = y_0 z.$$

Next, if $\varphi^i(y_0 z) = y_0 z$ for some i, then $\varphi^i(y_0)\varphi^i(z) = y_0 z$. It follows from the fact that the family yW with $y \in O(\varphi, y_0)$ is disjoint, that $\varphi^i(y_0) = y_0$. It is also true that $\varphi^i(z) = z$ so $r|i$. Hence $s|i$. □

Given that φ is a spectrally irreducible local automorphism of finite order, Lemma 4.2.7 tells us that the spectrum of φ is a finite subset of $\mathbb{N}$ that contains the least common multiple of any pair of elements of the spectrum of φ, that is, closed with respect to taking the least common multiple. Next, consider any finite subset of $\mathbb{N}$ that is closed under the least common multiple. We now endevour to give a spectrally irreducible local automorphism of the corresponding spectrum.

Example 4.2.8. *Suppose that σ is any finite subset of $\mathbb{N}$ closed under the least common multiple, $\sigma = \{t_1, t_2, \ldots, t_k\}$, $t_1 < t_2 < \cdots < t_k$, $m = 1 + \sum_{t\in\sigma} t$ and $\bigoplus_\omega \mathbb{Z}_m$ is the direct sum of ω copies of the group $\mathbb{Z}_m$. Let $\bigoplus_\omega \mathbb{Z}_m$ be endowed with the topology obtained by taking as a neighbourhood base at zero the subgroups*

$$H_\alpha = \{y \in \bigoplus_\omega \mathbb{Z}_m : y(i) = 0 \textit{ for all } i < \alpha\},$$

where $\alpha < \omega$. We also let π be the coordinatewise permutation on the local left topological group $\bigoplus_\omega \mathbb{Z}_m$ induced by the product of disjoint cycles

$$\pi_0 = (1, \ldots, t_1)(t_1 + 1, \ldots, t_1 + t_2) \cdots (t_1 + \cdots + t_{k-1} + 1, \ldots, t_1 + \cdots + t_k).$$

Then π is a homeomorphism with $\pi(0) = 0$, $spec(\pi, H_\alpha) = \sigma$ for all $\alpha < \omega$, and $\pi(y+z) = \pi(y)+\pi(z)$ whenever $max\, supp(y) < min\, supp(z)$. Consequently, π is a spectrally irreducible local automorphism on $\bigoplus_\omega \mathbb{Z}_m$ of spectrum σ. It is called ***standard****.*

We now wish to give a theorem describing the structure of a local automorphism. However, to prove this theorem, we will require the following result.

Lemma 4.2.9 ([43])**.** *Suppose that Y is a countable nondiscrete regular space having a distinguished point $e \in Y$, $\varphi : Y \to Y$ is a homeomorphism of finite order with $\varphi(e) = e$, $V \subseteq Y$ is clopen invariant, $H \subseteq V$ is finite invariant, and $\mathscr{P}$ is a clopen invariant partition of V such that for each $I \in \mathscr{P}$, $spec(\varphi, H \cap I) = spec(\varphi, I)$. Then there exists a clopen invariant partition $\{V(y) : y \in H\}$ of V inscribed into $\mathscr{P}$ such that for each $y \in H$, $V(y)$ is a spectrally minimal neighbourhood of y.*

Proof. Let $V = \{y_n : n < \omega\}$ with $y_0 \in H$. For each $y \in H$, we construct an increasing sequence $(V_n(y))_{n<\omega}$ of clopen spectrally minimal neighbourhoods of y such that for every $n < \omega$, the family $\{V_n(y) : y \in H\}$ is disjoint, inscribed into $\mathscr{P}$ and invariant, and $y_n \in V_n = \bigcup_{y\in H} V_n(y)$. The subsets $V(y) = \bigcup_{n<\omega} V_n(y)$, $y \in H$, will then be as desired. Now we proceed inductively on n. Let z_i, for $i < l$, represent the orbits in H and let $|O(\varphi, z_i)| = k_i$. For each $i < l$, we pick a neighbourhood U_i of z_i for which $\varphi^j(U_i)$ is a spectrally minimal neighbourhood of $\varphi^j(z_i)$ for every $j < k_i$, and the family $\{\varphi^j(U_i) : i < l, j < k_i\}$ is disjoint and inscribed into $\mathscr{P}$. By Lemma 4.2.5, There exists a clopen neighbourhood $W_i \subseteq U_i$ of z_i for which the family $\{\varphi^j(W_i) : j < k_i\}$ is invariant. Put $V_0(\varphi^j(z_i)) = \varphi^j(W_i)$. Now fix $n > 0$ and we assume that $V_{n-1}(y)$, $y \in H$ has been constructed as required. Without loss of generality we may assume that $y_n \notin V_{n-1}$. Let $|O(\varphi, y_n)| = k$ and let $y_n \in I_n \in \mathscr{P}$. Using lemma 4.2.5 again, we pick a clopen neighbourhood W_n of y_n such that for each $j < k$, $\varphi^j(W_n)$ is a spectrally minimal neighbourhood of $\varphi^j(y_n)$, and the family

$$\{\varphi^j(W_i) : j < k\} \cup \{V_{n-1}(y) : y \in H\}$$

is disjoint, inscribed into $\mathscr{P}$ and invariant. Pick $c_n \in H \cap I_n$ with $|O(\varphi, c_n)| = k$ and for each $j < k$, we put $V_n(\varphi^j(x_n)) = V_{n-1}(\varphi^j(x_n)) \cup \varphi^j(W_n)$. For each $y \in H \backslash O(\varphi, x_n)$, put $V_n(y) = V_{n-1}(y)$. $\square$

Theorem 4.2.10 ([43]). *Given a countable nondiscrete regular local left topological group Y, suppose that $\varphi : Y \to Y$ is a spectrally irreducible local automorphism of finite order, $\sigma = spec(\varphi)$, and $m = 1 + \sum_{t \in \sigma} t$. Let π be the standard permutation on the local left topological group $\bigoplus_\omega \mathbb{Z}_m$ of spectrum σ. Then there exists a continuous bijection $\psi : Y \to \bigoplus_\omega \mathbb{Z}_m$ with $\psi(e) = 0$ such that*

1. *$\varphi : \psi^{-1} \circ \pi \circ \psi$, and*
2. *$\psi(yz) = \psi(y) + \psi(z)$ whenever $max\, supp(\psi(y)) + 2 \leq min\, supp(\psi(z))$.*

If Y is first countable, then ψ can be chosen to be a homeomorphism.

Proof. Suppose that $L = L(\mathbb{Z}_m)$ is the set of all words on the alphabet $\mathbb{Z}_m$ including the empty word $\varnothing$ and since Y is countably infinite, enumerate Y without repetitions as $\{e, y_1, y_2, \ldots\}$. The permutation π_0 on $\mathbb{Z}_m$, which induces the standard permutation π on $\bigoplus_\omega \mathbb{Z}_m$, also induces the permutation π_1 on L. If $w = \gamma_0\gamma_1 \cdots \gamma_n$, then $\pi_1(w) = \pi_0(\gamma_0)\pi_0(\gamma_1) \cdots \pi_0(\gamma_n)$. Instead of writing π_0 and π_1, it is convenient to just write π.

To each word $w \in L$, we assign a point $y(w) \in Y$ and a clopen spectrally minimal neighbourhood $Y(w)$ of $y(w)$ such that

(a) $y(0^n) = e$ and $Y(\varnothing) = Y$,

(b) $\{Y(w^\frown\gamma) : \gamma \in \mathbb{Z}_m\}$ is a partition of $Y(w)$,

(c) $y(w) = y(w_0)y(w_1)\cdots y(w_{l-1})y(w_l)$ and $Y(w) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l)$ where $w = w_0 + w_1 + \cdots + w_l$ is the canonical decomposition of w,

(d) $\varphi(y(w)) = y(\pi(w))$ and $\varphi(Y(w)) = Y(\pi(w))$, and

(e) $y_n \in \{y(u) : u \in L \text{ and } |u| = n\}$.

Now let $\sigma = \{t_1, t_2, \ldots, t_k\}$, $t_1 < t_2 < \cdots < t_k$. We choose, for each $i = 1, \ldots, k$, a representative λ_i of the orbit in $\mathbb{Z}_m \setminus \{0\}$ of lengths t_i. Now using Lemma 4.2.9, we pick a clopen invariant neighbourhood V_1 of $e \in Y$ for which $y_1 \notin V_1$ and $spec(\varphi, Y \setminus V_1) = spec(\varphi)$ and put $y(0) = e$, $Y(0) = V_1$. Then pick the points $b_i \in Y \setminus V_1$, for $1 \le i \le k$, with pairwise disjoint orbits of lengths t_i for which $y_1 \in \bigcup_{i=1}^k O(\varphi, b_i)$. Now we put, for each $1 \le i \le k$ and $j < t_i$, $y(\pi^j(\lambda_i)) = \varphi^j(b_i)$.

Next, using Lemma 4.2.9, there exists an invariant partition $\{Y(\gamma) : \gamma \in \mathbb{Z}_m \setminus \{0\}\}$ of $Y \setminus V_1$ for which $Y(\gamma)$ is a clopen spectrally minimal neighbourhood of $y(\gamma)$. Fix $n > 1$ and assume that we have constructed $Y(w)$ and $y(w)$ for every $w \in L$ with $|w| < n$ in such a way that conditions (a)-(e) are satisfied. Notice that the subsets $Y(w)$ with $|w| = n - 1$, form a partition of Y so one of these subsets, say $Y(v)$, contains y_n. Now let v have the canonical decomposition $v = v_1 + v_2 + \cdots + v_q$. Then $Y(v) = y(v_0)y(v_1)\cdots y(v_{q-1})Y(v_q)$ and $y_n = y(v_0)y(v_1)\cdots y(v_{q-1})z_n$ for some $z_n \in Y(v_q)$. Next, we pick a clopen invariant neighbourhood V_n of $e \in Y$ such that for all basic words w with $|w| = n - 1$,

(i) $y(w)V_n \subset Y(w)$,

(ii) $\varphi(yz) = \varphi(y)\varphi(z)$ for all $z \in V_n$, and

(iii) $spec(\varphi, Y(w) \setminus y(w)V_n) = spec(Y(w))$.

If $z_n \neq y(v_q)$, we choose V_n in addition so that (iv) $z_n \notin y(v_q)V_n$. Put $y(0^n) = e$ and $Y(0^n) = V_n$.

Let $w \in L$ be an arbitrary nonzero basic word with $|w| = n-1$ and let $O(\varphi, w) = \{w_j : j < t\}$, where $w_{j+1} = \pi(w_j)$ for $j < t-1$ and $\pi(w_{t-1}) = w_0$. Put $Z_j = Y(w_j) \setminus y(w_j)V_n$. Using Lemma 4.2.7, we pick points $c_i \in Z_0$ for $1 \leq i \leq k$, with pairwise disjoint orbits of lengths $lcm(t_i, t)$. If $v_q \in O(\varphi, w)$, we choose c_i in addition so that $z_n \in \bigcup_{n=1}^{k} O(\varphi, c_i)$. For each $1 \leq i \leq k$ and $j < t_i$, we have $y(\pi^j(w^\frown \gamma_i)) = \varphi^j(c_i)$. Then by Lemma 4.2.9, we inscribe an invariant partition

$$\{Y(u^\frown \gamma) : u \in O(\varphi, w), \gamma \in \mathbb{Z}_m \setminus \{0\}\}$$

into the partition $\{Z_i : j < t\}$ such that $Y(u^\frown \gamma)$ is a clopen spectrally minimal neighbourhood of $y(u^\frown \gamma)$. If w is a nonbasic word in L with $|w| = n$, we define $y(w)$ and $Y(w)$ by condition (c).

We now wish to check conditions (b) and (d). To do this, we let $|w| = n-1$ and w have the canonical decomposition $w = w_0 + w_1 + \cdots + w_l$. Then

$$Y(w^\frown 0) = y(w_0)y(w_1)\cdots y(w_l)Y(0^n) = y(w_0)y(w_1)\cdots y(w_{l-1})y(w_l)Y(0^n)$$

and

$$Y(w^\frown \gamma) = y(w_0)y(w_1)\cdots y(w_{l-1})Y(w_l^\frown \gamma),$$

so (b) is satisfied. Next,

$$\begin{aligned}
\varphi(y(w)) &= \varphi(y(w_0)y(w_1)\cdots y(w_{l-1})y(w_l)) \\
&= \varphi(y(w_0))\varphi(y(w_1)y(w_2)\cdots y(w_{l-1})y(w_l)) \\
&= \varphi(y(w_0))\varphi(y(w_1))\varphi(y(w_2)y(w_3)\cdots y(w_{l-1})y(w_l))
\end{aligned}$$

Proceeding in this manner, we have that

$$\begin{aligned}
\varphi(y(w)) &= \varphi(y(w_0))\varphi(y(w_1))\cdots\varphi(y(w_{l-1}))\varphi(y(w_l)) \\
&= y(\pi(w_0))y(\pi(w_1))\cdots y(\pi(w_{l-1}))y(\pi(w_l)) \\
&= y(\pi(w_0)\pi(w_1)\cdots\pi(w_{l-1})\pi(w_l)) \\
&= y(\pi(w)),
\end{aligned}$$

so (d) is satisfied.

To check (e), we let $y_n \notin \{y(w) : |w| = n-1\}$. Then

$$\begin{aligned}
y_n &= y(v_0)y(v_1)\cdots y(v_{q-1})z_n \\
&= y(v_0)y(v_1)\cdots y(v_{q-1})y(v_q^\frown\gamma) \\
&= y(v^\frown\gamma).
\end{aligned}$$

Now, for each $y \in Y$, there exists a word $w \in L$ with a nonzero last letter such that $y = y(w)$, so $\{u \in L : y = (u)\} = \{w^\frown 0^n : n < \omega\}$. Hence, we can define $\psi : Y \to \bigoplus_\omega \mathbb{Z}_m$ by setting for all $w = \gamma_0\gamma_1\cdots\gamma_n \in L$,

$$\psi(y(w)) = \overline{w} = (\gamma_0, \gamma_1, \ldots, \gamma_n, 0, 0, \ldots).$$

Notice that ψ is a bijection, $\psi(e) = 0$ and since for every $x = (\gamma_i)_{i<\omega} \in \bigoplus_\omega \mathbb{Z}_m$, $\psi^{-1}(x + H_\alpha) = Y(\gamma_0\gamma_1\cdots\gamma_{n-1})$, we have that ψ is continuous. Now let $y = y(w)$. Then

$$\begin{aligned}
\psi(\varphi(y(w))) &= \psi(y(\pi(w))) \\
&= \overline{\pi(w)} \\
&= \pi(\overline{w}) \\
&= \pi(\psi(y(w))).
\end{aligned}$$

Therefore part 1 is satisfied.

Next, let $y = y(w)$, $w = w_0 + w_1 + \cdots + w_l$ and $n = max\, supp(\psi(y)) + 2$. Let $z \in \psi^{-1}(H_\alpha)$, $z = y(u)$ and $u = u_0 + u_1 + \cdots + u_p$. Then

$$
\begin{aligned}
\psi(yz) &= \psi(y(w_0)y(w_1)\cdots y(w_l)y(u_0)y(u_1)\cdots y(u_p)) \\
&= \psi(y(w+u)) \\
&= \overline{w+u} \\
&= \overline{w}+\overline{u} \\
&= \psi(y(w))+\psi(y(u)) \\
&= \psi(y)+\psi(z).
\end{aligned}
$$

Finally, if Y is first countable, $\{Y(0^n) : n < \omega\}$ can be chosen as a neighbourhood base at e. In that case ψ is a homeomorphism. □

Theorem 4.2.11 ([43]). *Suppose that $\mathscr{T}_0$ is a topology on a countable nondiscrete regular local left topological group Y, φ is a nontrivial spectrally irreducible local automorphism on $(Y, \mathscr{T}_0)$ of finite order and t is the least number of $spec(\varphi)\backslash\{e\}$. Then we can can partition Y into countably many subsets which are dense in any nondiscrete topology $\mathscr{T}$ on Y such that*

1. *$(Y, \mathscr{T})$ is a local left topological group,*
2. *φ is a homeomorphism on $(Y, \mathscr{T})$, and*
3. *For any neighbourhoods U, W of the identities in the topologies $\mathscr{T}, \mathscr{T}_0$ respectively, we have $t \in spec(\varphi, U \cap W)$.*

Proof. Suppose that $\psi : (Y, \mathscr{T}_0) \to \bigoplus_\omega \mathbb{Z}_m$ is a continuous local isomorphism obtained from Theorem 4.2.10 and that T is a t-element orbit with respect to π in $\mathbb{Z}_m$. Given an arbitrary point $y \in Y$, we let $\psi(y)$ be a sequence of coordinates that belong to T. Next, we let the first element of the sequence be denoted by $L(y)$ and the last element of the sequence be denoted by $R(y)$. Also suppose that $\upsilon(y)$ is the number of pairs of neighbouring distinct elements in $\psi(y)$. Notice that if $|O(\varphi, y)| = t$, then $\psi(y)$

has coordinates from T and $\upsilon(y) = \upsilon(\varphi(y))$. If $y, z \in Y$ and $max\, supp(\psi(y)) + 2 < min\, supp(\psi(z))$, then

$$\upsilon(yz) = \begin{cases} \upsilon(y) + \upsilon(z) & \text{if } R(y) = L(y), \\ \upsilon(y) + \upsilon(z) + 2 & \text{if } R(y) \neq L(z). \end{cases}$$

Furthermore, if $|O(\varphi, y)| = |O(\varphi, z)| = t$, then there exists some $i < t$ such that

$$\upsilon(y \cdot \varphi^i(z)) = \upsilon(y) + \upsilon(z), \upsilon(y \cdot \varphi^{i+1}(z)) = \upsilon(y) + \upsilon(z) + 2.$$

Now enumerate $Y_n = \{y \in Y : min\, supp(\upsilon(y)) = n\}$ for $n < \omega$. That the family Y_n for $n < \omega$ is disjoint, is trivial. It remains to show that Y_n is dense in $(Y, \mathscr{T})$. To do this, we let $y \in Y$ and U be a neighbourhood of the identity of $(Y, \mathscr{T})$. We can put $k = 2^{n+1}$ and choose elements $y_1, y_2, \ldots, y_k$ in U such that

(a) $|O(\varphi, y_j)| = k$,

(b) $max\, supp(\psi(y)) + 2 < min\, supp(\psi(y_1)), max\, supp(\psi(y_j)) + 2 < min\, supp(\psi(y_{j+1}))$, and

(c) $z_1 \cdots z_k \in U$ for any $z_i \in O(\varphi, y_j)$.

Consequently, there exists an element of Y_n among the elements $y \cdot z_1 \cdots z_k \in yU$ where $z_j \in O(\varphi, y_j)$. □

From Theorem 4.2.11, we see that a countably infinite regular local left topological group with a nontrivial local automorphism of finite order is ω-resolvable. Also, if we endow a countably infinite group with a regular ω-irresolvable group topology wherein all shifts are continuous, then the centralizer of any element of finite order is open.

Theorem 4.2.12 ([43]). *Suppose that $\mathscr{T}_0$ is a nondiscrete regular topology on a countable group G with continuous shifts and inversion and that the Boolean part $B(G)$ is not a neighbourhood of the identity of $(G, \mathscr{T}_0)$. Now suppose that $B = B(G) \setminus \{e\}$ and Y is an open symmetric neighbourhood of the identity of $(G, \mathscr{T}_0)$ that satisfies one of the following conditions:*

1. $Y \cap B = \varnothing$,

2. $e \in cl(B)$ *and the centralizer of each element of* $B \cap Y$ *is open.*

Then there is a partial operation $+$ *on* Y *such that for every topology* $\mathscr{T}$ *containing* $\mathscr{T}_0$ *on* G *with continuous shifts and inversion,* $(Y, \mathscr{T}|_Y, +)$ *is a local left topological group and the inversion on* $(Y, \mathscr{T}|_Y, +)$ *is a local automorphism.*

Proof. We look at Case 2. We will apply the concepts of "spectrally minimal neighbourhood" and "invariant partition" to the inversion.

Suppose that Y is a semigroup of words on the alphabet $\mathbb{Z}_4$. Since Y is countable, we enumerate it as $\{y_n : n < \omega\}$ with $y_0 = e$ and $Y(\varnothing) = Y$.

For each $n \in \mathbb{N}$, we define $\{Y(w) : w \in L \text{ and } |w| = n\}$ to be a clopen partition of Y, $\{y(w) \in Y(w) : w \in L \text{ and } |w| = n\}$ to be a subset and $y(w) + z$ with $w \in L$, $|w| = n-1$, $z \in Y(0^n)$ to be a partial operation on L such that the following conditions are satisfied:

(a) $\{Y(w^\frown\gamma) : \gamma \in \mathbb{Z}_4\}$ is a partition of $Y(w)$, $y(w^\frown 0) = y(w)$, $w \in L$ and $|w| = n-1$,

(b) $Y(w)$ is a spectrally minimal neighbourhood of $y(w)$, for basic words w with $|w| = n$, $Y(0^n)$ is contained in the centralizer of each element of order 2 from the set $\{y(w) : w \text{ is basic and } |w| = n-1\}$,

(c) for all basic words with $|w| = n$, $(Y(w))^{-1} = Y(w^{-1})$ and $(y(w))^{-1} = y(w^{-1})$,

(d) for any basic word w with $|w| = n-1$ and $z \in Y(0^n)$, we have $y(w) + z = z \cdot y(w)$ if the last letter from the set $\{1, 3\}$ in the word w is 3, $y(w) + z = y(w) \cdot z$, otherwise,

(e) if $w \in L$, $|w| = n-1$, and $z \in Y(0^n)$, $y(w) + z = y(w_1 + \cdots + w_{l-1}) + (y(w_l) + z)$,

(f) $Y(w) = y(w_1 + \cdots + w_{l-1}) + Y(w_l)$ and $y(w) = y(w_1 + \cdots + w_{l-1}) + y(w_l)$ where $w \in L$ and $|w| = n$,

(g) $y_n \in \{y(w) : w \in L$ and $|w| = n$.

We choose as $Y(0)$, a clopen symmetric neighbourhood of e such that the set $U = Y \setminus Y(0)$ contains elements of order 2 and $y_1 \notin Y(0)$. Put $y(0) = e$ and pick the elements $y(1), y(2), y(3)$ in U such that $(y(1))^{-1} = y(3)$, $(y(2))^{-1} = y(2)$, and $y_1 \in \{y(\gamma) : \gamma = 1, 2, 3\}$. Then we choose a clopen invariant parition $\{Y(\gamma) : 1, 2, 3\}$ of U such $Y(\gamma)$ is a spectrally minimal neighbourhood of $y(\gamma)$.

Assume that the partition $\{Y(w) : w \in L$ and $|w| = n\}$, the subset $\{y(w) \in Y(w) : w \in L$ and $|w| = n\}$ and the partial operation $y(w) + z$, with $w \in L$, $|w| = n$ and $z \in Y(0^n)$, have already been defined. One of these partitions, say $Y(v)$, contains $y_{n+1} \in Y(v_q)$. Since $Y(v) = y(v_1 + \cdots + v_{q-1}) + Y(v_q)$, we have $y_{n+1} = y(v_1 + \cdots + v_{q-1}) + z_{n+1}$ for some $z_{n+1} \in Y(v_q)$. We choose as $Y(0^{n+1})$, a clopen symmetric neighbourhood of e such that after we have defined (d) for the case $n+1$, $Y(w^\frown 0) = y(w) + Y(0^{n+1}) \subset Y(w)$, $\sigma(Y(w) \setminus Y(w^\frown 0)) = \sigma(Y(w))$, for basic words w with $|w| = n$, $Y(0^n)$ is contained in the centralizer of each element of order 2 from the set $\{y(w) : w$ is basic and $|w| = n$ and $z_{n+1} \notin Y(v_q^\frown 0)$, if, of course, $z_{n+1} \neq y(v_q)$. For each basic w with $|w| = n$, put $y(w^\frown 0) = y(w)$.

Let w be a nonzero basic word in L with $|w| = n$ and $w^{-1} = w$. Pick the elements $y(w^\frown \gamma)$, where $\gamma = 1, 2, 3$ in the set $U = Y(w) \setminus Y(w^\frown 0)$ such that $(y(w^\frown 1))^{-1} = y(w^\frown 3)$, $(y(w^\frown 2))^{-1} = y(w^\frown 2)$ and $z_{n+1} \in \{y(w^\frown \gamma) : \gamma \in \mathbb{Z}_4\}$, if $w = v_q$. Then we choose a clopen invariant partition $\{Y(w^\frown \gamma) : \gamma = 1, 2, 3\}$ of the set U such that $Y(w^\frown \gamma)$ is a spectrally minimal neighbourhood of $y(w^\frown \gamma)$.

Let w be a nonzero basic word in L with $|w| = n$ and $w^{-1} \neq w$. Put $U_1 = Y(w) \setminus Y(w^\frown 0)$, $U_2 = Y(w^{-1}) \setminus Y(w^{-1\frown} 0)$, $U = U_1 \cup U_2$. Since $Y(0^{n+1})$ is symmetric, $U^{-1} = U_2$. Now pick the distinct elements $y(w^\frown \gamma) \in U_1$, $y(w^{-1\frown} \gamma) \in U_2$, $\gamma = 1, 2, 3$ such that $(y(w^\frown \gamma))^{-1} = y(w^{-1\frown} \gamma)$ and $z_{n+1} \in \{y(w^\frown \gamma), y(w^{-1\gamma}) : \gamma \in \mathbb{Z}_4\}$, if $v_q \in \{w, w^{-1}\}$. Then we inscribe the clopen invariant partition $\{Y(u^\frown \gamma) : u = w, w^{-1}, \gamma = 1, 2, 3\}$ into the partition $\{U_1, U_2\}$ of U such that $Y(u^\frown \gamma)$ is a spectrally minimal neighbourhood of $y(u^\frown \gamma)$.

Once we have defined $Y(w)$, $y(w)$ and $y(w)+z$ fr all basic words w with $|w| = n+1$, we shall define them for nonbasic words w with $|w| = n+1$ by conditions (e) and (f) for the case $n+1$. Once this process is complete, we shall obtain the bijection $L' \ni w \mapsto y(w) \in Y$ and the partial operation $y(w)+z$ where $w \in L'$ and $z \in Y(0^{|w|+1})$. Trivially, the operation $y(w)+z$, where $z \in Y(0^{|w|+1})$ maps homeomorphically the neighbourhood $Y(0^{|w|+1})$ of e onto the neighbourhood $Y(w^\frown 0)$ of $y(w)$. Furthermore, this is valid for any topology $\mathscr{T} \supseteq \mathscr{T}_0$ with continuous shifts. To see that $y(v+u) = y(v)+y(u)$, if $max\, supp(v)+2 < min\, supp(u)$, we use the induction in the length of the canonical decomposition of u.

We have

$$\begin{aligned}
y(v+u) &= y(v+u_1+u_2+\cdots+u_p) \\
&= y(v+u_1+u_2+\cdots u_{p-1})+y(u_p) \\
&= y(v+u_1+u_2+\cdots+u_{p-2})+(y(u_{p-1})+y(u_p)) \\
&= y(v+u_1+u_2+\cdots+u_{p-2})+y(u_{p-1}+u_p) \\
&= y(v)+y(u).
\end{aligned}$$

Consequently, the partial operation $+$ is associative. If $max\, supp(v)+2 < min\, supp(u)$ and $max\, supp(u)+2 < min\, supp(w)$, then

$$\begin{aligned}
(y(v)+y(u))+y(w) &= y(v+u)+y(w) \\
&= y(v+u+w) \\
&= y(v)+y(u+w) \\
&= y(v)+(y(u)+y(w)).
\end{aligned}$$

Therefore $Y_{\mathscr{T}}$ is a local left topological group. We check that the inversion on $Y_{\mathscr{T}}$ is a local automorphism. Let w be a word in a subsemigroup L' of L and $z \in Y(0^{|w|+1})$.

If w is a nonzero basic word and $w = w^{-1}$ then

$$\begin{aligned}
(y(w)+y)^{-1} &= (y(w)\cdot z)^{-1} \\
&= (z\cdot y(w))^{-1} \\
&= (y(w))^{-1}\cdot z^{-1} \\
&= (y(w))^{-1}+z^{-1}.
\end{aligned}$$

If w is a nonzero basic word, $w = w^{-1}$ and the last letter of w from the set $\{1,3\}$ is, say 3, then the last letter of w^{-1} from the set $\{1,3\}$ is 1 and we have

$$\begin{aligned}
(y(w)+z)^{-1} &= (z\cdot y(w))^{-1} \\
&= (y(w))^{-1}\cdot z^{-1} \\
&= y(w^{-1})\cdot z^{-1} \\
&= y(w^{-1})+z^{-1} \\
&= (y(w))^{-1}+z^{-1}.
\end{aligned}$$

In the general case, we use the induction in the length of the canonical decomposition of w :

$$\begin{aligned}
(y(w)+z)^{-1} &= (y(w_1+\cdots+w_{l-1})+(y(w_l))+z)^{-1} \\
&= (y(w_1+\cdots+w_{l-1}))^{-1}+(y(w_l)+z)^{-1} \\
&= (y(w_1+\cdots+w_{l-1}))^{-1}+((y(w_l))^{-1}+z^{-1}) \\
&= y(w_1^{-1}+\cdots+w_{l-1}^{-1})+(y(w_l^{-1})+z^{-1}) \\
&= y(w^{-1})+z^{-1} \\
&= (y(w))^{-1}+z^{-1}.
\end{aligned}$$

We can consider Case 1 analogously by taking a semigroup L to be a semigroup of words on the alphabet $\mathbb{Z}_3$. $\square$

We now give the main result of this section.

Theorem 4.2.13 ([43])**.** *Let $\mathscr{T}$ be a regular ω-resolvable topology on a countable group G with continuous shifts and inversion. Then a Boolean part is a neighbourhood of the identity.*

Proof. Consider a countable group G. Endow G with a regular topology with continuous shifts and inversion. Suppose the Boolean part $B(G)$ is not a neighbourhood of the identity. We have to show that G is ω-resolvable. To do this, suppose $B = B(G) \setminus \{e\}$. Since we know that the centralizer of any element of finite order in a countable group with a regular ω-irresolvable topology with continuous shifts is open, it follows that if there exists some element in B such that the centralizer of this element is not open in G, then G is ω-resolvable. Hence, suppose the centralizer of each element of B is open in G. If $e \in cl(B)$, we put $Y = G$, otherwise Y becomes an open symmetric neighbourhood of the identity of G disjoint with B. We obtain from Theorem 4.2.12, that there exists a partial operation $+$ on Y for which $(Y, +)$ is a local left topological group and the inversion on $(Y, +)$ is a local automorphism. Since $B(G)$ is not a neighbourhood of the identity of G, the inversion on Y is not a trivial local automorphism. It follows from the fact that a countable regular local left topological group with a nontrivial local automorphism of finite order is ω-resolvable, that Y is ω-resolvable. Consequently, G is also ω-resolvable. □

To conclude this section we give the following result.

Theorem 4.2.14 ([43])**.** *Suppose that a countably infinite group having a finite Boolean part that can be embedded into a compact topological group. Then it can be partitioned into countably many subsets dense in any nondiscrete topology with continuous shifts and inversion.*

Proof. Suppose that G is a countably infinite group with a finite Boolean part that can be embedded into a compact topological group, $\mathscr{T}_0$, is a totally bounded group topology on G, Y is a local left topological group obtained from $(G, \mathscr{T}_0)$ by removing elements of order 2, $+$ is a partial operation on Y guaranteed by Theorem 4.2.12, and

$\{Y_n : n < \omega\}$ is a partition of a local left topological group $(Y, +)$ by the inversion guaranteed by Theorem 4.2.11. We show that every Y_n is dense in many nondiscrete topology $\mathscr{T}$ on G with continuous shifts and inversion. Now suppose that $\mathscr{T}_1 = \mathscr{T}_0 \vee \mathscr{T}$ is the supremum of $\mathscr{T}_0$ and $\mathscr{T}$. The topology $\mathscr{T}_1$ has as neighbourhoods of the identity, subsets of the form $U \cap W$, where U, W are neighbourhoods of the identity in the topologies $\mathscr{T}_0$ and $\mathscr{T}$. The shifts and inversion in $\mathscr{T}_1$ are continuous. Because $\mathscr{T}_0$ is totally bounded, we have that $\mathscr{T}_1$ is nondiscrete. It follows $\mathscr{T}_1 \supseteq \mathscr{T}_0$, Y with the topology $\mathscr{T}_1|_Y$ and operation $+$ is a nondiscrete local left topological group and the inversion is a nontrivial local automorphism on Y. Therefore, Y_n is dense in $\mathscr{T}_1$ and, consequently, Y_n is dense in $\mathscr{T}$. □

4.3 Resolving by Regular Homeomorphisms of Finite Order

In this section we assume that all spaces are Hausdorff.

Definition 4.3.1 ([46]). *Suppose that Y is a topological space with a distinguished point $e \in Y$, $\varphi : Y \to Y$ is a homeomorphism with $\varphi(e) = e$. The homeomorphism φ is* ***regular*** *if for every $y \in Y \setminus \{e\}$, there exists a homeomorphism ϑ_y of a neighbourhood of the identity e onto a neighbourhood of y such that $\varphi\vartheta_y|_U = \vartheta_{\varphi(y)}\varphi|_U$ for some neighbourhood U of e.*

Suppose that the topological space Y admits a regular homeomorphism. This means that for any pair of points $y, z \in Y$, there exists a homeomorphism ϑ of a neighbourhood of y onto a neighbourhood of z with $\vartheta(y) = z$. Furthermore, if we take the space Y to be zero-dimensional and Hausdorff, then we can choose ϑ to be a homeomorphism of Y onto itself. Therefore, a countably infinite zero-dimensional space that admits a regular homeomorphism is homogeneous.

Now we claim that a local automorphism is regular. To see this, we suppose that

Y is a local left topological group and $\varphi : Y \to Y$ is a local automorphism. For each $y \in Y \setminus \{e\}$, we pick a neighbourhood U_y of e for which yz is defined for every $z \in U_y$, yU_y is a neighbourhood of y and $\lambda_y : y \mapsto yz$ is a homeomorphism where $y \in U_y$ and $yz \in yU_y$. Now we put $\vartheta_y = \lambda_y$. We have that $\vartheta_y(e) = y$. Next, we pick a neighbourhood W_y of e such that $W_y \subseteq U_y$, $\varphi(W_y) \subseteq U_{\varphi(y)}$ and $\varphi(yz) = \varphi(y)\varphi(z)$ for every $y \in W_y$. Then for every $z \in W_y$, $\varphi\vartheta_y(z) = \varphi(yz) = \varphi(y)\varphi(z) = \vartheta_{\varphi(y)}\varphi(z)$. It follows that the concept of a regular homeomorphism is a generalization of the concept of a local automorphism on a local left topological group.

Lemma 4.3.2 ([46])**.** *Suppose that Y is a Hausdorff space with a distinguished point $e \in Y$, $\varphi : Y \to Y$ is a spectrally irreducible regular homeomorphism of finite order, $e \neq y_0 \in Y$ with $|O(\varphi, U)| = t$ and U is a spectrally minimal neighbourhood of y_0. Then*

$$spec(\varphi, U) = \{lcm(t, r) : r \in \{e\} \cup spec(\varphi)\}$$

Proof. For every $y \in O(\varphi, y_0)$, suppose that ϑ_y is a homeomorphism of a neighbourhood U_y of e onto a neighbourhood of y such that $\varphi\vartheta_y|_{W_y} = \vartheta_{\varphi(y)}\varphi|_{W_y}$ for some neighbourhood $W_y \subseteq U_y$ of e. Now pick a neighbourhood W of e for which $W \subseteq \bigcap_{y \in O(\varphi, y_0)} W_y$, $\vartheta_{y_0}(W) \subseteq U$, and the subsets $\vartheta_y(W)$, where $y \in O(\varphi, y_0)$, are pairwise disjoint. Suppose that φ has order n and pick a neighbourhood V of e for which $\varphi^i(V) \subseteq W$ for $0 \leq i \leq n-1$. This inclusion holds for $i < \omega$. We now show that for all $z \in V$, $\varphi^i\vartheta_{y_0}(z) = \vartheta_{\varphi^i(y_0)}\varphi^i(z)$. The case $i = 0$ is trivial. We obtain, inductively, that

$$\varphi^i\vartheta_{y_0}(z) = \varphi\varphi^{i-1}\vartheta_{y_0}(z) = \varphi\vartheta_{\varphi^{i-1}(y_0)}\varphi^{i-1}(z) = \vartheta_{\varphi^i(y_0)}\varphi^i(z).$$

We now suppose $z \in V$, $|O(\varphi, z)| = r$ and $s = lcm(t, r)$ and claim that $|O(\varphi, \vartheta_{y_0}(z))| = s$. Indeed,

$$\varphi^s(\vartheta_{y_0}(z)) = \vartheta_{\varphi^s(y_0)}(\varphi^s(z)) = \vartheta_{y_0}(z).$$

Next, if $\varphi^i(\vartheta_{y_0}(z)) = \vartheta_{y_0}(z)$ for some i, then $\vartheta_{\varphi^i(y_0)}(\varphi^i(z)) = \vartheta_{y_0}(z)$. It follows from the fact that the subsets $\vartheta_y(W)$ with $y \in O(\varphi, y_0)$ are pairwise disjoint, that $\varphi^i(y_0) = y_0$, so $t|i$. It is also true that $\varphi^i(z) = z$ since ϑ_{y_0} is an injection, and so $r|i$. Hence $s|i$. $\square$

Similar to what we did with local automorphisms, Lemma 4.3.2 shows that the spectrum of a spectrally irreducible regular homeomorphism of finite order is a finite subset of $\mathbb{N}$ closed with respect to taking the least common multiple.

We now give a theorem describing the structure of a large family of homeomorphisms of finite order on countable regular spaces. As an application of this structure theorem, we will prove that every countable nondiscrete topological group which contains no open Boolean subgroup is ω-resolvable. Using this structure theorem, the desired result will be more apparent than in Section 4.3. In truth we will have proved a more general theorem.

The following notation regarding the group $\bigoplus_\omega \mathbb{Z}_m$ will be useful. For every $y \in \bigoplus_\omega \mathbb{Z}_m$, let μ_y denote the shift in $\bigoplus_\omega \mathbb{Z}_m$ by y, that is, we define $\mu_y : \bigoplus_\omega \mathbb{Z}_m \to \bigoplus_\omega \mathbb{Z}_m$ by $\mu_y(z) = y + z$.

Theorem 4.3.3 ([46]). *Suppose that Y is a countably infinite nondiscrete regular space with a distinguished point $e \in Y$, $\varphi : Y \to Y$ is a spectrally irreducible regular homeomorphism of finite order, $\sigma = spec(\varphi)$, and $m = 1 + \sum_{t\in\sigma} t$. Suppose that π is the standard permutation on $\bigoplus_\omega \mathbb{Z}_m$ of spectrum σ, and for all $b \in \bigoplus_\omega \mathbb{Z}_m$, let $\mu_b : \bigoplus_\omega \mathbb{Z}_m \to \bigoplus_\omega \mathbb{Z}_m$ be defined by $\mu_b(y) = b + y$. Then there exists a continuous bijection $\psi : Y \to \bigoplus_\omega \mathbb{Z}_m$ with $\psi(e) = 0$ such that*

1. *$\varphi = \psi^{-1}\pi\psi$, and*

2. *for every $y \in Y$, $\lambda_y = \psi^{-1}\mu_{\psi(y)}\psi$ is a homeomorphism of Y onto itself.*

Moreover, if Y is a local left topological group and φ is a local automorphism, then we can choose ψ so that

3. *$\lambda_y(z) = yz$, whenever $\max supp(\psi(y)) + 1 < \min supp(\psi(z))$.*

Proof. Suppose that $L = L(\mathbb{Z}_m)$ is the set of all words on the alphabet $\mathbb{Z}_m$ including the empty word $\varnothing$. The permutation π_0, which induces the standard permutation π

on $\bigoplus_\omega \mathbb{Z}_m$, also induces the permutation π_1 on L. If $w = \gamma_0\gamma_1\cdots\gamma_n$, then $\pi_1(w) = \pi_0(\gamma_0)\pi_0(\gamma_1)\cdots\pi_0(\gamma_n)$. Instead of writing π_0 and π_1, it is convenient to just write π. For each $y \in Y\setminus\{e\}$, we choose a homeomorphism ϑ_y of a neighbourhood of e onto a neighbourhood of y with $\vartheta_y(e) = y$ such that $\varphi\vartheta_y = \vartheta_{\varphi(y)}\varphi|_U$ for some neighbourhood U of e. We also put $\vartheta_1 = id_Y$. If Y is a local left topological group and φ is a local automorphism, we choose ϑ_y in such a way that $\vartheta_y(z) = yz$ and since Y is countably infinite, we enumerate Y as $\{y_n : n < \omega\}$ with $y_0 = e$.

To each word $w \in L$, we assign a point $y(w) \in Y$ and a clopen spectrally minimal neighbourhood $Y(w)$ of $y(w)$ such that

(a) $y(0^n) = e$ and $Y(\varnothing) = Y$,

(b) $\{Y(w^\frown\gamma) : \gamma \in \mathbb{Z}_m\}$ is a partition of $Y(w)$,

(c) $y(w) = \vartheta_y(w_0)\vartheta_y(w_1)\cdots\vartheta_y(w_{l-1})(y(w_l))$ and $Y(w) = \vartheta_y(w_0)\vartheta_y(w_1)\cdots\vartheta_y(w_{l-1})(Y(w_l))$, where w has the canonical decomposition $w = w_0 + w_1 + \cdots + w_l$,

(d) $\varphi(y(w)) = y(\pi(w))$ and $\varphi(Y(w)) = Y(\pi(w))$,

(e) $y_n \in \{y(u) : u \in L \text{ and } |u| = n\}$.

Now let $\sigma = \{t_1, t_2, \ldots, t_k\}$, $t_1 < t_2 < \cdots < t_k$. We choose, for each $i = 1, \ldots, k$, a representative λ_i of the orbit in $\mathbb{Z}_m \setminus \{0\}$ of length t_i. We pick a clopen invariant neighbourhood V_1 of e such that $y_1 \notin V_1$ and $spec(\varphi, Y \setminus V_1) = spec(\varphi)$ and put $y(0) = e$ and $Y(0) = V_1$. Then pick the points $b_i \in Y \setminus V_1$, for $1 \le i \le k$, with pairwise disjoint orbits of lengths t_i for which $y_1 \in \bigcup_{i=1}^k O(\varphi, b_i)$. Now we put, for each $1 \le i \le k$ and $j < t_i$, $y(\pi^j(\lambda_i)) = \varphi^j(b_i)$.

Next, using Lemma 4.2.9, there exists an invariant partition $\{Y(\gamma) : \gamma \in \mathbb{Z}_m \setminus \{0\}\}$ of $Y \setminus V_1$ for which $Y(\gamma)$ is a clopen spectrally minimal neighbourhood of $y(\gamma)$. Fix $n > 1$ and assume that we have constructed $Y(w)$ and $y(w)$ for every $w \in L$ with $|w| < n$ in such a way that conditions (a)-(e) are satisfied. Notice that the subsets $Y(w)$ with $|w|n - 1$, form a partition of Y so one of these subsets, say $Y(v)$, contains y_n. Now let v have the canonical decomposition $v = v_0 + v_1 + \cdots + v_q$. Then $Y(w) = \vartheta_y(v_0)\vartheta_y(v_1)\cdots\vartheta_y(v_{q-1})(Y(v_q))$ and $y_n = \vartheta_y(v_0)\vartheta_y(v_1)\cdots\vartheta_y(v_{q-1})(z_n)$ for some $z_n \in Y(v_q)$.

Next, we pick a clopen invariant neighbourhood V_n of $e \in Y$ such that for all basic words w with $|w| = n-1$,

(i) $\vartheta_{y(w)}(V_n) \subset Y(w)$,

(ii) $\varphi\vartheta_{y(w)}|_{V_n} = \vartheta_{\varphi(y(w))}\varphi|_{V_n}$, and

(iii) $spec(\varphi, Y(w) \setminus \vartheta_{y(w)}(V_n)) = spec(Y(w))$.

If $z_n \neq y(v_q)$, we choose V_n in addition so that (iv) $z_n \notin \vartheta_{y(v_q)}(V_n)$. Put $y(0) = e$ and $Y(0^n) = V_n$.

Let $w \in L$ be an arbitrary nonzero basic word with $|w| = n-1$ and let $O(\varphi, w) = \{w_j : j < t\}$, where $w_{j+1} = \pi(w_j)$ for $j < t-1$ and $\pi(w_{t-1}) = w_0$. Put $Z_j = Y(w_j) \setminus \vartheta_{y(w_j)}(V_n)$. Using Lemma 4.3.2, we pick points $c_i \in Z_0$ for $1 \leq i \leq k$, with pairwise disjoint orbits of lengths $lcm(t_i, t)$. If $v_q \in O(\varphi, w)$, we choose c_i in addition so that $z_n \in \bigcup_{i=1}^{k} O(\varphi, c_i)$. For each $1 \leq i \leq k$ and $j < t_i$, we put $y(\pi^j(w^\frown\gamma_i)) = \varphi^j(c_i)$. Then by Lemma 4.2.9, we inscribe an invariant partition

$$\{Y(u^\frown\gamma) : u \in O(\varphi, w), \gamma \in \mathbb{Z}_m \setminus \{0\}\}$$

into the partition $\{Z_j : j < t\}$ such that $Y(u^\frown\gamma)$ is a clopen spectrally minimal neighbourhood of $y(u^\frown\gamma)$. If w is a nonbasic word in L with $|w| = n$, we define $y(w)$ and $Y(w)$ by condition (c).

We now wish to check conditions (b) and (d). To do this, we let $|w| = n-1$ and let $w = w_0 + w_1 + \cdots + w_l$. Then

$$Y(w^\frown 0) = \vartheta_{y(w_0)}\vartheta_{y(w_1)}\cdots\vartheta_{y(w_l)}(Y(0^n)) = \vartheta_{y(w_0)}\vartheta_{y(w_1)}\cdots\vartheta_{w_{l-1}}(\vartheta_{y(w_l)}(Y(0^n)))$$

and

$$Y(w^\frown\gamma) = \vartheta_{y(w_0)}\vartheta_{y(w_1)}\cdots\vartheta_{y(w_{l-1})}(Y(w_l^\frown\gamma)),$$

so (b) is satisfied. Next,

$$\begin{aligned}
\varphi(y(w)) &= \varphi\vartheta_{y(w_0)}\vartheta_{y(w_1)}\cdots\vartheta_{y(w_{l-1})}(y(w_l)) \\
&= \vartheta_{\varphi(y(w_0))}\varphi\vartheta_{y(w_1)}\vartheta_{y(w_2)}\cdots\vartheta_{y(w_{l-1})}(y(w_l)) \\
&= \vartheta_{\varphi(y(w_0))}\vartheta_{\varphi(y(w_1))}\varphi\vartheta_{y(w_2)}\vartheta_{y(w_3)}\cdots\vartheta_{y(w_{l-1})}(y(w_l))
\end{aligned}$$

Proceeding in this manner, we obtain

$$\begin{aligned}
\varphi(y(w)) &= \vartheta_{\varphi(y(w_0))}\vartheta_{\varphi(y(w_1))}\cdots\vartheta_{\varphi(y(w_{l-1}))}\varphi(y(w_l)) \\
&= \vartheta_{y(\pi(w_0))}\vartheta_{y(\pi(w_1))}\cdots\vartheta_{y(\pi(w_{l-1}))}(y(\pi(w_l))) \\
&= y(\pi(w_0)\pi(w_1)\cdots\pi(w_{l-1})\pi(w_l)) \\
&= y(\pi(w)),
\end{aligned}$$

so (d) is also satisfied.

To check (e), we let $y_n \notin \{y(w) : |w| = n-1\}$. Then

$$\begin{aligned}
y_n &= \vartheta_{y(v_0)}\vartheta_{y(v_1)}\cdots\vartheta_{y(v_{q-1})}(z_n) \\
&= \vartheta_{y(v_0)}\vartheta_{y(v_1)}\cdots\vartheta_{y(v_{q-1})}(v_q^\frown\gamma) \\
&= y(v^\frown\gamma).
\end{aligned}$$

Now, for every $y \in Y$, there exists a word $w \in L$ with a nonzero last letter such that $y = y(w)$, so

$$\{u \in L : y = y(u)\} = \{w^\frown 0^n : n < \omega\}.$$

Hence, we can define $\psi : Y \to \bigoplus_\omega \mathbb{Z}_m$ by setting for all $w = \gamma_0\gamma_1\cdots\gamma_n \in L$,

$$\psi(y(w)) = \overline{w} = (\gamma_0, \gamma_1, \ldots, \gamma_n, 0, 0, \ldots).$$

Notice that ψ is a bijection, $\psi(e) = 0$ and since for every $x = (\gamma_i)_{i<\omega} \in \bigoplus_\omega \mathbb{Z}_m$, $\psi^{-1}(x + H_\alpha) = Y(\gamma_0\gamma_1\cdots\gamma_n)$, we have that ψ is continuous. Now let $y = y(w)$. Then

$$\begin{aligned}
\psi(\varphi(y(w))) &= \psi(y(\pi(w))) \\
&= \overline{\pi(w)} \\
&= \pi(\overline{w}) \\
&= \pi(\psi(y(w))).
\end{aligned}$$

Therefore part 1 is satisfied.

Next, let $y = y(w)$, $w = w_0 + w_1 + \cdots + w_l$ and $n = max\, supp(\psi(y)) + 1$. We begin by showing that $\lambda_y|_{\psi^{-1}(H_\alpha)} = \vartheta_{y(w_0)}\vartheta_{y(w_1)} \cdots \vartheta_{y(w_l)}|_{\psi^{-1}(H_\alpha)}$. Let $z \in \psi^{-1}(H_\alpha)$, $z = y(u)$ and $u = u_0 + u_1 + \cdots + u_k$. Then

$$\begin{aligned}
\varphi\vartheta_{y(w_0)}\vartheta_{y(w_1)} \cdots \vartheta_{y(w_l)}(z) &= \varphi\vartheta_{y(w_0)}\vartheta_{y(w_1)} \cdots \vartheta_{y(w_l)}\vartheta_{y(u_0)}\vartheta_{y(u_1)} \cdots \vartheta_{y(u_{k-1})}(y(u_k)) \\
&= \psi(y(w + u)) \\
&= \overline{w + u} \\
&= \overline{w} + \overline{u} \\
&= \psi(y(w)) + \psi(y(u)) \\
&= \mu_{\psi(y)}\psi(z).
\end{aligned}$$

It follows from (c) that $\vartheta_{y(w_0)}\vartheta_{y(w_1)} \cdots \vartheta_{y(w_l)}$ homeomorphically maps $Y(0^n)$, a neighbourhood of e, onto $Y(y^\frown 0)$, a neighbourhood of y, and so does λ_y. In order to see that λ_y homeomorphically maps a neighbourhood of an arbitrary point $z \in Y$ onto a neighbourhood $x = \lambda_y(z)$, it will be sufficient to check that $\lambda_y = \lambda_x(\lambda_z)^{-1}$. Indeed, $x = \psi^{-1}\mu_{\psi(y)}\psi(z) = \psi^{-1}(\psi(y) + \psi(z))$ and then

$$\begin{aligned}
\lambda_x(\lambda_z)^{-1} &= \psi^{-1}\mu_{\psi(y)+\psi(z)}\psi(\psi^{-1}\mu_{\psi(z)}\psi)^{-1} \\
&= \psi^{-1}\mu_{\psi(y)+\psi(z)}\psi\psi^{-1}(\mu_{\psi(z)})^{-1}\psi \\
&= \psi^{-1}\mu_{\psi(y)+\psi(z)}\mu_{-\psi(z)}\psi \\
&= \psi^{-1}\mu_{\psi(y)}\psi \\
&= \lambda_y
\end{aligned}$$

Therefore part 2 is satisfied.

Finally, let $y = y(w)$ and $w = w_0 + w_1 + \cdots + w_l$. If $l = 0$, then $\lambda_y(z) = \psi_y(z) = yz$.

We proceed, by induction on l, to obtain

$$\begin{aligned}
\lambda_y(z) &= \psi_{y(w_0)}\psi_{y(w_1)}\cdots\psi_{y(w_l)}(z) \\
&= \psi_{y(w_0)}\psi_{y(w_1)}\cdots\psi_{y(w_{l-1})}(y(w_l)\cdot z) \\
&= y(w_0+w_1+\cdots+w_{l-1})\cdot(y(w_l)\cdot z) \\
&= [y(w_0+w_1+\cdots+w_{l-1})\cdot y(w_l)]\cdot z \\
&= [\psi_{y(w_0)}\psi_{y(w_1)}\cdots\psi_{y(w_{l-1})}(y(w_l))]\cdot z \\
&= y(w)\cdot z.
\end{aligned}$$

□

Condition 3 in Theorem 4.3.3, where Y is a local left topological group and φ is a local automorphism, is essentially Theorem 4.2.10. If we put $\varphi = id_Y$, the first part of Theorem 4.3.3 tells us that every countable homogeneous regular space admits a Boolean group operation with continuous translations.

The purpose of Theorem 4.3.3 is to characterize spectrally irreducible regular homeomorphisms of finite order on countable regular spaces. Recall that to generate the topology of $\bigoplus_\omega \mathbb{Z}_m$, we take as a base at zero the subgroups $H_\alpha = \{y \in \bigoplus_\omega \mathbb{Z}_m : y(i) = 0 \text{ for all } i < \alpha\}$ where $\alpha < \omega$. Let $\varphi : Y \to Y$ be a spectrally irreducible homeomorphism of finite order. If there exists, for some m, a continuous bijection $\psi : Y \to \bigoplus_\omega \mathbb{Z}_m$ with $\psi(e) = 0$ such that

1. $\psi\varphi\psi^{-1}$ is a coordinatewise permutation on $\bigoplus_\omega \mathbb{Z}_m$, and

2. for every $y \in Y$, $\psi^{-1}\mu_{\psi(y)}\psi : Y \to Y$ is a homeomorphism, then φ is regular.

This can be seen as follows. Suppose $\pi = \psi\varphi\psi^{-1}$. Additionally, for every $y \in Y \setminus \{e\}$, suppose $\alpha(y) = max\, supp(\psi(y)) + 1$, $U_y = \psi^{-1}(H_{\alpha(y)})$ and $\vartheta_y = \psi^{-1}\mu_{\psi(y)}\psi|_{U_y}$. Then for every $z \in U_y$, $\varphi(z) \in U_y = U_{\varphi(y)}$ and

$$\begin{aligned}
\varphi\vartheta_y(z) &= \psi^{-1}\pi\psi\psi^{-1}\mu_{\psi(y)}\psi(z) \\
&= \psi^{-1}\pi\mu_{\psi(y)}\psi(z) \\
&= \psi^{-1}\pi(\psi(y) + \psi(z)) \\
&= \psi^{-1}(\pi(\psi(y)) + \pi(\psi(z))) \\
&= \psi^{-1}(\psi(\varphi(y)) + \psi(\varphi(z))) \\
&= \psi^{-1}\mu_{\psi(\varphi(y))}\psi(\varphi(z)) \\
&= \vartheta_{\varphi(y)}\varphi(z).
\end{aligned}$$

Definition 4.3.4. *Given a topological space Y with a distinguished point $e \in Y$, let $\varphi : Y \to Y$ be a homeomorphism with $\varphi(e) = e$. It is said that φ is* ***nontrivial*** *if every neighbourhood of e contains a nonfixed point.*

Theorem 4.3.5 ([46]). *A countably infinite regular space which admits a nontrivial regular homeomorphism of finite order is ω-resolvable.*

Proof. Suppose that Y is a countable regular space with a distinguished point $e \in Y$, $\varphi : Y \to Y$ is a nontrivial regular homeomorphism of finite order. We know from Lemma 2.3.4 that a homogeneous space which contains an ω-resolvable subspace is itself also ω-resolvable. Therefore, we may assume that φ is spectrally irreducible. Consider a bijection $\psi : Y \to \bigoplus_\omega \mathbb{Z}_m$ guaranteed by Theorem 4.3.3 and let C denote an orbit in $\mathbb{Z}_m$(with respect to π_0) of the least possible length $t > 1$. Now let

$$Z = \{y \in Y : \text{ there exists a coordinate of } \psi(y) \text{ belonging to } C\}.$$

Notice that every $y \in Y$ with $|O(\varphi, y)| = t$ belongs to Y. For every $y \in Y$, consider the sequence of coordinates of $\psi(y)$ which belong to C and let $\eta(x)$ be defined to be the number of pairs of distict neighbouring elements in the sequence. Additionally, if the sequence is nonempty, denote the first and last elements in the sequence by $\beta(y)$ and

$\xi(y)$ respectively. Then whenever $y, z \in Z$ and $max\, supp(\psi(y))+1 < min\, supp(\psi(z))$,

$$\eta(\lambda_y(z)) = \begin{cases} \eta(y)+\eta(z) & \text{if } \xi(y) = \beta(z), \\ \eta(y)+\eta(z)+1 & otherwise. \end{cases}$$

Let the partition $\{Z_n : n < \omega\}$ of Z be defined by

$$Z_n = \{y \in Z : \eta(y) \equiv 2^n (mod\, 2^{n+1})\},$$

that is, Z_n consists of all $y \in Z$ such that the index of the nonzero digit that is furthest to the left in the binary expansion of $\eta(y)$ is n.

We now wish to show that every Z_n is dense in Y. To do this, suppose that $y \in Y$ and U is an open neighbourhood of e. We have to show that $\lambda_y(U) \cap Z_n \neq \varnothing$. We put $k = 2^{n+1}$ and then we choose by induction $y_1, \ldots, y_k \in U$ such that

(a) $|O(\varphi, y)| = t$,

(b) $max\, supp(\psi(y_j)) + 1 < min\, supp(\psi(y_{j+1}))$, and if $y \neq 0$, then
$max\, supp(\psi(y)) + 1 < min\, supp(\psi(y_1))$,

(c) $\lambda_{z_1}\lambda_{z_2}\cdots\lambda_{z_k}(e) \in U$ whenever $z_j \in O(\varphi, y_j)$.

Without loss of generality, we may assume that $\xi(y_j) = \beta(y_{j+1})$, and that if $y \in \mathbb{Z}$, then $\xi(y) = \beta(y_1)$. Foe every $l = 0, 1, \ldots, k-1$, let $x_l \in U$ be defined by

$$z_l = \lambda_{y_1}\lambda_{\varphi(y_2)}\cdots\lambda_{\varphi^l(y_{l+1})}\lambda_{\varphi^l(y_{l+2})}\cdots\lambda_{\varphi^l(y_k)}(e)$$

(Particularly, $x_0 = \lambda_{y_1}\lambda_{y_2}\cdots\lambda_{y_k}(e)$.) Then

$$\psi(\lambda_y(x_l)) = \psi(y) + \psi(y_1) + \pi\psi(y_2) + \cdots + \pi^l\psi(y_{l+1})\pi^l\psi(y_{l+2}) + \cdots + \pi^l\psi(y_k).$$

Consequently, $\eta(\lambda_y(x_0)) = \eta(y)+\eta(y_1)+\cdots+\eta(y_k)$ and $\eta(\lambda_y(x_0)) = \eta(x_0)+l$. Hence, for some l, $\eta(\lambda_y(z_l)) \equiv 2^n (mod\, 2^{n+1})$, so $\lambda_y(x_l) \in Z_n$. $\square$

Lemma 4.3.6 ([46]). *Suppose that Y is a homogeneous topological space with a distinguished point $e \in Y$ and $\varphi : Y \to Y$ is a local homeomorphism of finite order n with $\varphi(e) = e$. If for every $y \in Y \setminus \{e\}$ with $|O(\varphi, y)| = t < n$, there exists a homeomorphism ϑ_y of a neighbourhood U of e onto a neighbourhood of y with $\vartheta_y(e) = y$ such that $\varphi^t \vartheta_y(z) = \vartheta_y \varphi^t(z)$ for all $z \in U$, then φ is regular. Particularly, if for every $y \in Y \setminus \{e\}$, $|O(\varphi, y)| = n$, then φ is regular.*

Proof. Consider an arbitrary orbit in Y distinct from $\{e\}$ and enumerate this orbit as $\{y_i : i < t\}$, where $y_{i+1} = \varphi(y_i)$ for $i = 0, \dots, t-2$ and $\varphi(y_{t-1}) = y_0$. If $t = n$, we choose as ϑ_{y_0} any homeomorphism of a neighbourhood U of e onto a neighbourhood y_0 with $\vartheta_{y_0}(e) = y_0$. If $t < n$, we choose ϑ_{y_0} with the additional condition that $\varphi^t \vartheta_{y_0}(z) = \vartheta_{y_0} \varphi^t(z)$ for all $z \in U$. For every $i = 1, \dots, t-1$, we put $\vartheta_{y_i} = \varphi^i \vartheta_{y_0} \varphi^{-i}|_U$. Then for every $i = 1, \dots, t-1$ and $z \in U$,

$$\varphi \vartheta_{y_i}(z) = \varphi \varphi^i \vartheta_{y_0} \varphi^{-i}(z) = \varphi^{i+1} \vartheta_{y_0} \varphi^{-(i+1)} \varphi(z).$$

If $i < t-1$, then $\varphi^{i+1} \vartheta_{y_0} \varphi^{-(i+1)} \varphi(z) = \vartheta_{y_{i+1}} \varphi(z)$, so $\varphi \vartheta_{y_i}(z) = \vartheta_{y_{i+1}} \varphi(z)$. Hence, all that remains is to check that $\varphi \vartheta_{y_{t-1}}(z) = \vartheta_{y_0} \varphi(z)$. If $t = n$, then

$$\varphi \vartheta_{y_{t-1}}(z) = \varphi^t \vartheta_{y_0} \varphi^{-t} \varphi(z) = id_Y \varphi(z) = \vartheta_{y_0} \varphi(z).$$

If $t < n$, then

$$\varphi \vartheta_{y_{t-1}}(z) = \varphi^t \vartheta_{y_0} \varphi^{-t} \varphi(z) = \vartheta_{y_0} \varphi^t \varphi^{-t} \varphi(z) = \vartheta_{y_0} \varphi(z).$$

□

We now give a proposition which tells us that every nondiscrete topological group which contains no open Boolean subgroup admits a nontrivial regular homeomorphism of order 2.

Proposition 4.3.7 ([46]). *Given a nondiscrete topological group G which does not contain an open Boolean subgroup, suppose that for every $g \in G$ of order 2, the conjugation $G \ni h \mapsto ghg^{-1} \in G$ is a trivial local automorphism. Then the inversion $G \ni h \mapsto h^{-1} \in G$ is a nontrivial regular homeomorphism.*

Proof. Denote the inversion by φ and let $B = B(G)$. We have that φ is a homeomorphism of order ≤ 2. We also know that B is the set of fixed points of φ, in particular, $\varphi(e) = e$. Since a group contains an open Boolean subgroup whenever the Boolean part of that group is a neighbourhood of e, we have that B is not a neighbourhood of e as G does not contain an open Boolean subgroup. So φ is nontrivial.

We now show that φ is regular. Let $g \in G \setminus \{e\}$ and $|O(\varphi, g)| < 2$, then $g \in B$. But then there exists a neighbourhood U of e such that $ghg^{-1} = h$ for all $h \in U$, that is, $gh = hg$. Now we define $\vartheta_g : U \to gU$ by $\vartheta_g(h) = gh$. We have that

$$\varphi\vartheta_g(h) = (gh)^{-1} = (hg)^{-1} = g^{-1}h^{-1} = gh^{-1} = \vartheta_g\varphi(h).$$

Hence, by Lemma 4.3.6, φ is regular. $\square$

Finally, by combining Theorem 4.3.6 and Proposition 4.3.7, we arrive at our coveted result.

Theorem 4.3.8 ([46]). *Let G be a countable nondiscrete topological group not containing an open Boolean subgroup. Then G is ω-resolvable.*

It is worth noting to the reader that if we consider G in Theorem 4.3.8 to be Abelian, we could simplify the proof of Theorem 4.3.8 a great deal. Notice that if a topological group is Abelian, then the inversion is a local automorphism. This means we do not need the added heavy machinery of Theorem 4.3.8. It is sufficient to just use Theorem 4.2.10. The case where G is Abelian also makes it unnecessary to restict G to being countably infinite.

Finally, if there exists a countably infinite nondiscrete ω-irresolvable topological group then there exists a P-point in ω^*. This means that it is impossible to establish the existence of a countably infinite nondiscrete ω-irresolvable topological group in ZFC. For a proof of this see [[47], Theorem 12.13].

Chapter 5

Absolute Resolvability

5.1 The Finite Sums Theorem

Definition 5.1.1. *Given a set* Y, $\mathscr{P}_f(Y) = \{E : \varnothing \neq E \subseteq Y$ *and* E *is finite*$\}$.

Definition 5.1.2 ([19])**.** 1. *Given an infinite sequence* $(y_n)_{n=1}^{\infty}$ *in a semigroup* $(S, \cdot)$, *the set of* ***finite products*** *of the sequence is defined by*

$$FP((y_n)_{n=1}^{\infty}) = \{\prod_{n \in E} y_n : E \in \mathscr{P}_f(\mathbb{N})\}.$$

Given a finite sequence $(y_n)_{n=1}^{m}$ *in a semigroup* $(S, \cdot)$, *the set of finite products of the sequence is given by*

$$FP((y_n)_{n=1}^{m}) = \{\prod_{n \in E} y_n : E \in \mathscr{P}_f(\{1, 2, 3, \ldots, m\})\}.$$

2. *Given an infinite sequence* $(y_n)_{n=1}^{\infty}$ *in a semigroup* $(S, +)$, *the set of* ***finite sums*** *of the sequence is given by*

$$FS((y_n)_{n=1}^{\infty}) = \{\sum_{n \in E} y_n : E \in \mathscr{P}_f(\mathbb{N})\}.$$

Given a finite sequence $(y_n)_{n=1}^{m}$ *in a semigroup* $(S, +)$, *the set of finite sums of the sequence is given by*

$$FS((y_n)_{n=1}^{m}) = \{\sum_{n \in E} y_n : E \in \mathscr{P}_f(\{1, 2, 3, \ldots, m\})\}.$$

Theorem 5.1.3. *(Finite Sums) Let G be an infinite group and suppose it is partitioned into a finite number of cells. Then there exists a one-to-one sequence $(y_n)_{n<\omega}$ in G with $FS((y_n)_{n<\omega})$ contained in one cell.*

5.2 Partitions and Sums with inverses in Abelian groups

Definition 5.2.1. *Given a group G, a subset A of G is* ***absolutely dense*** *if it is dense in every nondiscrete group topology on G. Given a cardinal $\kappa \geq 2$, if G can be partitioned into κ-many absolutely dense subsets, we say G is* ***absolutely*** ***κ-resolvable****.*

Recall Theorem 4.2.14 from Section 4.2. From Theorem 4.2.14, we obtain

Corollary 5.2.2 ([43])**.** *Let G be a countably infinite group with finitely many elements of order 2 that can be embedded in a compact topological group. Then G is absolutely ω-resolvable.*

Theorem 5.2.3 ([45])**.** *Every infinite Abelian group G which contains no infinite Boolean subgroup is absolutely resolvable.*

Proof. Suppose that $\{C_0, C_1\}$ is an absolutely dense subset of G. Now suppose for contradiction that G contains an infinite Boolean subgroup B. Put $I_i = C_i \cap B$ for each $i < 2$. By Theorem 5.1.3, there exists a one-to-one sequence $(y_n)_{n<\omega}$ in B with $FS((y_n)_{n<\omega})$ contained in one cell, suppose I_0. Now let $B_0 = FS((y_n)_{1\leq<\omega}) \cup \{0\}$. Since B is Boolean, we have that B_0 is an infnite subgroup, and $y_0 + B_0 \subseteq FS((y_n)_{n<\omega}) \subseteq I_0$. Next, suppose that $\mathscr{T}$ is any nondiscrete group topology on G in which B is open. Then $y_0 + B$ is open. Since $y_0 + B \subseteq I_0$, we have that $(y_0 + B) \cap C_1 = \varnothing$. Hence, C_1 is not dense in $\mathscr{T}$, contradicting the hypothesis. $\square$

Definition 5.2.4 ([19]). *Consider an additive group G. Given a sequence $(y_n)_{n<\omega}$ in G, the set of* ***finite sums with inverses*** *of the sequence is defined by*

$$FSI((y_n)_{n<\omega}) = \{\sum_{n\in E} \varepsilon_n^E y_n : E \in \mathscr{P}_f(\omega) \text{ and } \varepsilon_n^E \in \{1,-1\} \text{ for all } n \in E\}.$$

Consider an Abelian group G. Notice that G can only be absolutely κ-resolvable whenever $\kappa \leq \omega$. We have seen, from Theorem 5.2.3, the answer of the absolute resolvability question for all Abelian groups and $\kappa = 2$. The solution for countable Abelian groups can be found in [44]. The following theorem settles the matter for all Abelian groups.

Theorem 5.2.5 ([45]). *Suppose that G is an Abelian group and $C = G \setminus B(G)$ is infinite. Then there exists a disjoint partition $\{C_r : r < \omega\}$ of C such that when $(y_n)_{n<\omega}$ is a one-to-one sequence in C, $h \in G$ and $r < \omega$, we have*

$$(h + FSI((y_n)_{n<\omega})) \cap C_r \neq \varnothing.$$

The proof of Theorem 5.2.5 involves a great deal of work. We will first have to prove the case where the group G is not necessarily Abelian, but is a direct sum of finite groups and then extend the proof to the general case. The proof of Theorem 5.2.5 for direct sums of finite groups relies on the following facts:

Lemma 5.2.6 ([45]). *Consider an increasing sequence $(a_n)_{n<\omega}$ in $\mathbb{N}$ such that $a_0 = 1$ and $a_{n+1} - a_n$ tends to infinity. Then there exists a mapping $\nu : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ having the following properties:*

1. *if $b \in [a_n, a_{n+1})$, then $\nu(b) = \{b_0, b_1, \ldots, b_{n-1}\}$, where $b_l \in [a_l, a_{l+1})$ for all $l \neq n-1$ (in particular, if $b \in [a_0, a_1)$, then $\nu(b) = \varnothing$,) and*
2. *for every $c \in \mathbb{N}$, there exists some $a \in \mathbb{N}$ such that whenever $b, d \in \mathbb{N}$, $0 < |b - d| \leq c$, $u \in \nu(b)$, $v \in \nu(c)$ and $u, v \geq a$, one has $|u - v| > c$.*

$[\mathbb{N}]^{<\omega}$ is the set of finite subsets of $\mathbb{N}$.

Proof. We can assume without loss of generality that $a_{n+1} - a_n \geq 5$ for $n < \omega$. For each $n < \omega$, we choose $c_n \geq 2$ such that $c_n^2 + c_n - 1 \leq a_{n+1} - a_n$ and $c_n \to \infty$. We now wish to define ν. To do this, suppose $l < \omega$, $b \geq a_{l+1}$ and choose the largest integer $k \geq 0$ such that $a_{l+1} + kc_l^2 \leq b$. Then choose the largest integer $i \geq 0$ such that $a_{l+1} + kc_l^2 + ic_l \leq b$, and then put $j = b - a_{l+1} - kc_l^2 - ic_l$. By these means we can represent a in the form $b = a_{l+1} + kc_l^2 + ic_l + j$, where k is a nonnegative integer and $i, j \in \{0, 1, \ldots, c_l - 1\}$. Put $b_l = a_l + jc_l + i$. Since $b_l \in [a_l, a_l + c_l^2)$, the mapping ν as we have defined it satisfies condition 1.

We now check condition 2. Given $c \in \mathbb{N}$, choose $n_0 < \omega$ such that $c_n \geq c + 2$ for all $n \geq n_0$ and put $a = a_{n_0}$. Now let $b, d \in \mathbb{N}$, $0 < |b - d| \leq c$, $u \in \nu(b)$, $v \in \nu(d)$ and $u, v \geq a$. Since

$$a_{l+1} - b_l = a_{l+1} - a_l - jc_l - i \geq c_l^2 + c_l - 1 - jc_l - i \geq c_l,$$

one may assume that $u = b_l$ and $v = d_l$ for some $l \geq n_0$. Let

$$b = a_{l+1} + kc_l^2 + ic_l + j, \; d = a_{l+1} + k'c_l^2 + i'c_l + j'.$$

Then $b_l = jc_l + i$, $d_l = j'c_l + i'$. Thus we have that

(a) $b - d = [(k - k')c_l + (i - i')]c_l + (j - j')$,

(b) $b_l - d_l = (j - j')c_l + (i - i')$.

Notice that $|i - i'| < c_l$ and $|j - j'| < c_l$. It follows from (b) that if $|j - j'| > 1$, $|b_l - d_l| > c_l$ so we may assume that $|j - j'| \leq 1$. But then (a) gives that $b - d$ is different from a multiple of c_l by $1, -1$ or 0. Since $|b - d| \leq c \leq c_l - 2$, this multiple of c_l has to be 0, so $(k - k')c_l + (i - i') = 0$, giving that $k = k'$ and $i = i'$. Hence, $|j - j'| = 1$, and we obtain that $|b_l - d_l| = c_l$. □

Definition 5.2.7 ([45])**.** *Given a sequence $(y_n)_{n<\omega}$ in a group G, the sequence $(z_n)_{n<\omega}$ in G is a* ***sum subsystem*** *of $(y_n)_{n<\omega}$ if there exists a sequence $(H_n)_{n<\omega}$ in $\mathscr{P}_f(\omega)$ for which max $H_n <$ min H_{n+1} and $z_n = \sum_{i \in H_n} y_i$ for every $n < \omega$.*

We can restate the Finite Sums Theorem in terms of the notion of a sum subsystem [19].

Proposition 5.2.8 ([45]). *Consider a sequence $(y_n)_{n<\omega}$ in a group G. Then when G is partitioned into finitely many cells, there exists a sum subsystem $(z_n)_{n<\omega}$ of $(y_n)_{n<\omega}$ with $FS((z_n)_{n<\omega})$ contained in one cell.*

Proposition 5.2.9 ([45]). *Let $(y_n)_{n<\omega}$ be a sequence in a totally bounded topological group G. Then for every neighbourhood V of $0 \in G$,*

$$FS((y_n)_{n<\omega}) \cap V \neq \varnothing.$$

Proof. Suppose the group G has the completion $\overline{G}$. Now pick an open neighbourhood W_0 of $0 \in \overline{G}$ for which $W_0 \cap G \subseteq V$ and for every $y \in \overline{G} \setminus W_0$, pick an open neighbourhood W_y of $y \in \overline{G}$ for which $(W_y + W_y) \cap W_y = \varnothing$. The sets W_y, where $y \in (\overline{G} \setminus W_0) \cup \{0\}$, make an open cover of $\overline{G}$ and because $\overline{G}$ is compact, there is a finite subcover $\{W_y : y \in E\}$, where E is a finite subset of $(\overline{G} \setminus W) \cup \{0\}$. By Proposition 5.2.8, there exist $y \in E$ and a sum subsystem $(z_n)_{n<\omega}$ of $(y_n)_{n<\omega}$ such that $FS((z_n)_{n<\omega}) \subseteq V_y$. Consequently, $z_0, z_1, z_0 + z_1 \in W_y$ gives that $y = 0$, and so $z_0 \in W_0 \cap G \subseteq V$. Thus $FS((y_n)_{n<\omega}) \cap V \neq \varnothing$. □

Corollary 5.2.10 ([45]). *Let $(y_n)_{n<\omega}$ be a sequence in a totally bounded topological group G. Then when V and V_y, $y \in V$, are neighbourhoods of $0 \in G$, there exists a sum subsystem $(z_n)_{n<\omega}$ of $(y_n)_{n<\omega}$ such that $z_0 \in V$ and for every $n < \omega$, $z_{n+1} \in \bigcap_{i \leq n} V_{z_i}$.*

Proof. By Proposition 5.2.9, there exists $z_0 \in FS((y_n)_{n<\omega}) \cap V$. Fix $l < \omega$ and take for granted that we have constructed a sum subsystem $(z_n)_{n\leq l}$ such that $z_0 \in V$ and for every $n < l$, $z_{n+1} \in \bigcap_{i \leq n} V_{z_i}$. Let $z_l = \sum_{n \in H_l} y_n$ and let $n_{l+1} = max\, H_l + 1$. By Proposition 5.2.9, there exists

$$z_{l+1} \in FS((y_n)_{n_{l+1} \leq n < \omega}) \cap \bigcap_{i \leq l} V_{z_i}.$$

The constructed sequence is as required. □

The following is a restatement of Theorem 5.2.5 with G being a direct sum of finite groups, not necessarily Abelian.

Theorem 5.2.11 ([45])**.** *Given $\kappa \geq \omega$, suppose, for each $\gamma < \kappa$, that G_γ is a finite group under addition. Additionally, suppose that $G = \bigoplus_{\gamma<\kappa} G_\gamma$ and $C = G \setminus B(G)$ is infinite. Then there exists a disjoint partition $\{C_r : r < \omega\}$ of C such that when $(y_n)_{n<\omega}$ is a one-to-one sequence in C, $h \in G$ and $r < \omega$, we have*

$$(h + FSI((y_n)_{n<\omega})) \cap C_r \neq \varnothing.$$

Proof. Given an infinite cardinal number κ, consider the group $G = \bigoplus_{\gamma<\kappa}$. For each $\gamma < \kappa$, G_γ is a finite group, not necessarily Abelian. For every $y \in G$, put

$$supp(y) = \{\gamma < \kappa : y(\gamma) \neq 0_\gamma\} \text{ and } supp_0(y) = \{\gamma < \kappa : y(\gamma) \notin B(G_\gamma)\}.$$

Since $B(G) = \bigoplus_{\gamma<\kappa} B(G_\gamma)$, we have that $y \in C$ if and only if $supp_0(y) \neq \varnothing$. For every $\gamma < \kappa$, $G_\gamma \setminus B(G_\gamma)$ is a disjoint union of subsets with two elements having the form $\{z, -z\}$. Then we set $sgn(z) = 1$ and $sgn(-z) = -1$. As a result, every $y \in C$ has a finite sequence $sgn(y(\gamma))$, where $\gamma \in supp_0(y)$, assigned to it.

The pivotal idea of the construction can be seen in Lemma 5.2.6.

Now suppose that $\nu : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ is a function assured by Lemma 5.2.6. For every $y \in C$, let the subset $supp_\nu(y) \subseteq supp_0(y)$ and the nonnegative integer $\eta(y)$ be defined as follows. Let $|supp(y)| = k$ and let $k \in [a_n, a_{n+1})$. Then $|supp(y)| = \{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$ for some $\gamma_0 < \gamma_1 < \cdots < \gamma_{k-1} < \kappa$ and $\nu(k) = \{k_0, k_1, \ldots, k_{n-1}\}$ for some $k_l \in [a_n, a_{l+1})$, $l \leq n-1$. Put $supp_\nu(y) = \{\gamma_{k_0}, \gamma_{k_1}, \ldots, \gamma_{k_{n-1}}\}$ and let $\eta(y)$ be defined as the number of pairs of distinct neighbouring elements in the sequence

$$sgn(y(\gamma_{k_0})), sgn(y(\gamma_{k_1})), \ldots, sgn(y(\gamma_{k_{n-1}})).$$

Notice that if $k \in [a_0, a_1)$, then $\nu(k) = \varnothing$, and then $supp_\nu(y) = \varnothing$ and $\eta(y) = 0$. Next, let the partition $\{C_r : r < \omega\}$ of C by

$$C_r = \{y \in C : \eta(y) \equiv 2^r (mod\, 2^{r+1})\}.$$

Equivalently, C_r is made up of all $y \in C$ for which the nonzero digit that is farthest to the left in the binary expansion of $\eta(y)$ has the index r. Now suppose that $(y_n)_{n<\omega}$ is a one-to-one sequence in C, $h \in G$ and $r < \omega$. It shall be shown that $(h + FSI((y_n)_{n<\omega})) \cap C_r \neq \varnothing$. Note that if $(z_n)_{n<\omega}$ is a sum subsystem of $(y_n)_{n<\omega}$, then

$$FSI((z_n)_{n<\omega}) \subseteq FSI((y_n)_{n<\omega}).$$

By Corollary 5.2.10, we may assume that

$$supp(h) \cap supp(y_i) = \varnothing \text{ and } supp(y_i) \cap supp(y_j) = \varnothing$$

for all $i < j < \omega$. Furthermore, we may assume that $(min\, supp_0(y_n))_{n<\omega}$ is an increasing sequence. Let

$$\xi = sup\{min\, supp_0(y_n) : n < \omega\}.$$

We can write every $y \in G$ uniquely in the form $y = y' + y''$, where $y', y'' \in G$,

$$supp(y') = supp(y) \cap \xi \text{ and } supp(y'') = supp(y) \setminus supp(y').$$

Put $l = 2^{r+1} - 1$. We pick a sum subsystem $(z_n)_{n<l}$ of $(y_n)_{n<\omega}$ by induction in such a way that

(a) $max\, supp_0(h') < min\, supp_0(z'_0)$ and $max\, supp_0(z'_i) < min\, supp_0(z'_{i+1})$ for $i < l - 1$, and

(b) each of the intervals $(|supp(h')|, |supp(h' + z'_0)|]$ and

$$(|supp(h' + z'_0 + z'_1 + \cdots + z'_i)|, |supp(h' + z'_0 + z'_1 + \cdots + z'_i + z'_{i+1})|],$$

where $i < l - 1$, contains some interval $[a_n, a_{n+1})$.

Let $g = h + z_0 + z_1 + \cdots + z_{l-1}$, $z_{l-1} = \sum_{n \in H} y_n$ and $n_0 = max\, H + 1$. We claim that there is $z_l \in FS((y_n)_{n_0 \leq n<\omega})$ such that

(i) $max\, supp_0(g') < min\, supp_0(z_l)$,

(ii) the interval $(|supp(g')|, |supp(g+z_l)|]$ contains some $[a_n, a_{n+1})$, and

(iii) $supp_\nu(g+z_l) \cap supp_0(g'') = \varnothing$.

We assume the contrary. By Proposition 5.2.8, there exist $\delta \in supp_0(g'')$ and a sum subsystem $(x_n)_{n<\omega}$ of $(y_n)_{n_0 \le n<\omega}$ such that $max\, supp_0(g') < min\, supp_0(x_n)$ and $\delta \in supp_\nu(g+x)$ for every $z \in FS((x_n)_{n<\omega})$. Put $c = |supp(g+x_0)|$ and suppose that a is a constant assured by Lemma 5.2.6. Next, we choose $x \in FS((y_n)_{1\le n<\omega})$ such that $max\, supp_0(g'+x_0') < min\, supp_0(x)$ and $|supp(g'+x')| \ge a$. Now let $b = |supp_0(g+x_0+x)|$ and $d = |supp_0(g+x_0)|$. Then

$$0 < b - d = |supp_0(x_0)| \le |supp_0(g+x_0)| = c.$$

Let $u = |supp_0(g+x_0+x) \cap \delta|$, $v = |supp_0(g+x) \cap \delta|$, $w = |supp_0(g+x_0) \cap \delta|$ and $s = |supp_0(g) \cap \delta|$. Then $u = v + w - s$. It follows that $u - v = w - s$. This is a contradiction since $u \in \nu(b)$, $v \in \nu(d)$, $u, v \ge a$ and $w - s \le w \le c$. Let us consider the element $h_0 = h + z_0 + z_1 + \cdots + z_l$. By the construction, $supp_\nu(h_0) \cap supp_0(z_i) \ne \varnothing$ and for each $i \le l-1$, $\varnothing \ne supp_\nu(h_0) \cap supp_0(z_i) \subseteq supp_0(z_i')$. Therefore we have that

$$\beta_0 < \gamma_1 \le \beta_1 < \gamma_2 \le \beta_2 < \gamma_3 \le \beta_3 < \cdots < \gamma_{l-1} \le \beta_{l-1} < \gamma_l,$$

where $\gamma_i = min(supp_\nu(h_0) \cap supp_0(z_i))$, $\beta_i = max(supp_\nu(h_0) \cap supp_0(z_i))$. Put $\varepsilon_0 = 1$. By induction on $i = 1, 2, \ldots, l$, we pick $\varepsilon_i \in \{1, -1\}$ so that

$$\varepsilon_{i-1} z_{i-1}(\beta_{i-1}) = \varepsilon_i z_i(\gamma_i).$$

One may assume, without loss of generality, that $\varepsilon_i = 1$, so that $h_0 = h + z_0 + z_1 + \cdots + z_l$. For each $i \le l$, put

$$h_i = h + z_0 - z_1 + \cdots + (-1)^i z_i + (-1)^i z_{i+1} + \cdots + (-1)^i z_l.$$

Then $\eta(h_i) = \eta(h_0) + i$. Hence, there exists $j \le l$ such that $\eta(h_i) \equiv 2^r (mod\, 2^{r+1})$. It follows that $h_j \in C_r$. $\square$

The following proposition is a consequence of Theorem 4.2.10.

Proposition 5.2.12 ([45]). *Given a countably infinite Abelian topological group G, let $B(G)$ be neither open nor discrete. Then there exists a continuous bijection $\varphi : G \to \bigoplus_\omega \mathbb{Z}_4$ such that*

(a) $\varphi(-y) = -\varphi(y)$ *for every* $y \in G$,

(b) $\varphi(y+z) = \varphi(y) + \varphi(z)$ *for* $y, z \in G \setminus \{0\}$ *with* $\max supp(\varphi(y)) + 2 \leq \min supp(\varphi(z))$.

We now deduce from Proposition 5.2.12, the result we need to extend Theorem 5.2.11 to Theorem 5.2.5.

Theorem 5.2.13 ([45]). *Consider an infinite Abelian group G. Endow G with the largest totally bounded group topology and let $|G| = \kappa$. Then there exists a continuous injection $\varphi : G \to \bigoplus_\kappa \mathbb{Z}_4$ such that*

1. $\varphi(-y) = -\varphi(y)$ *for all* $y \in G$,
2. $\varphi(y+z) = \varphi(y) + \varphi(z)$ *for all* $y, z \in G$ *with* $S(\varphi(y)) \cap S(\varphi(z)) = \varnothing$, *where for each* $b \in \bigoplus_\kappa \mathbb{Z}_4$, $S(b) = supp(b) \cup (supp(b) + 1)$.

The topology on $\bigoplus_\kappa \mathbb{Z}_4$ is the one induced by the product topology on $\prod_\kappa \mathbb{Z}_4$. What Theorem 5.2.13 actually does is allow us to identify an arbitrary Abelian group G of cardinality κ with a subset of $\bigoplus_\kappa \mathbb{Z}_4$ so that

(i) for every $y \in G$, $-y \in \bigoplus_\kappa \mathbb{Z}_4$ is the inverse of y in G,

(ii) for every $y, z \in G$ with $S(y) \cap S(z) = \varnothing$, $y + z \in \bigoplus_\kappa \mathbb{Z}_4$ is the sum of y and z in G, and

(iii) the largest totally bounded group topology on G is stronger than that induced from $\bigoplus_\kappa \mathbb{Z}_4$.

Proof. It is well known in group theory that every Abelian group can be embedded into a direct sum of groups isomorphic to $\mathbb{Z}(p^\infty)$ or $\mathbb{Q}$. Therefore, without loss of generality we may assume that

$$G = \bigoplus_{\gamma<\kappa} G_\gamma,$$

where for each $\gamma < \kappa$,

$$|G_\gamma| = |B(G_\gamma)| = |G_\gamma : B(G_\gamma)| = \omega.$$

For each $\gamma < \kappa$, endow G_γ with a totally bounded group topology and suppose that

$$\varphi_\gamma : G_\gamma \to \bigoplus_\omega \mathbb{Z}_4$$

is a continuous bijection guaranteed by Proposition 5.2.12. Now define

$$\varphi : G \to \bigoplus_{\gamma<\kappa} \bigoplus_{[\gamma,\gamma+\omega)} \mathbb{Z}_4.$$

by

$$\varphi = \bigoplus_{\gamma<\kappa} \varphi_\gamma.$$

□

Having proved Theorem 5.2.5 in the general case, we obtain from it, the main result of this chapter.

Corollary 5.2.14 ([45])**.** *Suppose that G is an infinite Abelian group which contains no infinite Boolean subgroup, then G is absolutely ω-resolvable.*

Proof. Consider a partition $\{C_r : r < \omega\}$ of $C = G \setminus B(G)$ assured by Theorem 5.2.5 and a nondiscrete group topology $\mathscr{T}$ on G. We claim that each C_r is absolutely dense in $\mathscr{T}$. Suppose that $h \in G$ and V is an open neighbourhood of $0 \in G$. By the assumption, $B(G)$ is finite, so we pick $y_0 \in V \setminus B(G)$ with $-y_0 \in V$. Fix $l < \omega$ and assume that we have constructed a one-to-one sequence $(y_n)_{n\le l}$ in $V \setminus B(G)$ with $FSI((y_n)_{n\le l}) \subseteq U$. Next, let $H_l = FSI((y_n)_{n\le l}) \cup \{0\}$ and pick $y_{l+1} \in V \setminus$

$(B(G) \cup \{y_n : n \leq l\})$ such that $H_l + \varepsilon y_{l+1} \subseteq V$ for each $\varepsilon \in \{1, -1\}$. Consequently, we get a one-to-one sequence $(y_n)_{n<\omega}$ in $V \setminus B(G)$ with $FSI((y_n)_{n<\omega}) \subseteq V$. Since $(h + FSI((y_n)_{n<\omega})) \cap C_r \neq \varnothing$, we have that $(h + V) \cap C_r \neq \varnothing$ as well. $\square$

We obtain another corollary from Theorem 5.2.5. This result is related to the Graham-Rothschild Theorem [15] which we can state as follows: If an infinite Abelian group having finite exponent is partitioned into finitely many subsets, then there exist arbitrarily large finite cosets contained in one subset of the partition.

Corollary 5.2.15 ([45])**.** *For every infinite Abelian group G which contains no infinite Boolean subgroup, there exists a partition $\{C_r : r < \omega\}$ of G such that for every infinite subgroup H of G, $h \in G$ and $r < \omega$, we have $(h + H) \cap C_r \neq \varnothing$.*

It is important to mention that the cardinal number ω in Theorem 5.2.5, Corollary 5.2.14 and Corollary 5.2.15 is maximally possible.

Chapter 6

Conclusion

We have given a theorem describing the structure of a local automorphism of finite order and used it to show that a countably infinite nondiscrete regular local left topological group Y can be partitioned into countably many dense subsets in any nondiscrete topology $\mathscr{T}$ on Y such that (i) $(Y, \mathscr{T})$ is a local left topological group; (ii) the nontrivial spectrally irreducible local automorphism of finite order, φ, on $(Y, \mathscr{T}_0)$ is a homeomorphism on $(Y, \mathscr{T})$; and (iii) given that t is the least number of $spec(\varphi) \setminus \{e\}$, $t \in spec(\varphi, U \cap W)$ for any neighbourhoods U, W of the identity in the topologies $\mathscr{T}, \mathscr{T}_0$. From this, it was obtained that (i) a countable regular local left topological group having a nontrivial local automorphism of finite order is ω-resolvable, and (ii) given a countable group endowed with a regular ω-irresolvable topology with continuous shifts and inversion, the centralizer of any element of finite order is open. It was also shown that there is a partial operation $+$ on an open symmetric neighbourhood, Y, of the identity of a countable group endowed with a nondiscrete regular topology having continuous shifts and inversion such that $(Y, +)$ is a local left topological group and inversion on it is a local automorphism. Using all of this, it was shown that every countably infinite nondiscrete topological group which contains no open Boolean subgroup is ω-resolvable. The same result was then proved with the aid of a theorem describing the structure of a large family of homeomorphisms of finite order on countably infinite regular spaces. Therefore it is true that every countably infinite nondiscrete ω-irresolvable topological group contains a countably infinite open Boolean subgroup, however, this problem in the case of uncountable topological groups remains open.

Finally, the problem of absolute resolvability for all Abelian groups was resolved. It was shown that there is a partititon $\{C_r : r < \omega\}$ of $C = G \setminus B(G)$ where G is an Abelian group and C is infinite such that when $(y_n)_{n<\omega}$ is one-to-one sequence in C, $h \in G$ and $r < \omega$, we have $(h + FSI((y_n)_{n<\omega})) \cap C_r \neq \varnothing$. This was shown first by considering G as a direct sum of finite groups and then extending it to the general case. We then deduced from this that every infinite Abelian group which contains no infinite Boolean subgroup is absolutely ω-resolvable.

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