Analytic and combinatorial explorations of partitions associated with the Rogers-Ramanujan identities and partitions with initial repetitions

by

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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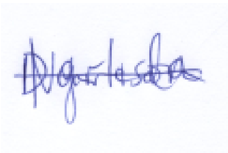

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Thanks to my family for emotional support. To the rest, I say be blessed.

Abstract

In this thesis, various partition functions with respect to Rogers-Ramanujan identities and George Andrews' partitions with initial repetitions are studied.
Agarwal and Goyal gave a three-way partition theoretic interpretation of the Rogers-Ramanujan identities. We generalise their result and establish certain connections with some work of Connor. Further combinatorial consequences and related partition identities are presented.
Furthermore, we refine one of the theorems of George Andrews on partitions with initial repetitions. In the same pursuit, we construct a non-diagram version of the Keith's bijection that not only proves the theorem, but also provides a clear proof of the refinement.
Various directions in the spirit of partitions with initial repetitions are discussed and results enumerated. In one case, an identity of the Euler-Pentagonal type is presented and its analytic proof given.

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Chapter 1

Introduction

The history of partitions dates back to the Middle Ages. Earlier discoveries with a great amount of depth stem from Leonard Euler. Euler resolved important partition problems and it can be said that he indeed laid the foundations of the subject of integer partitions [11]. A partition of n is defined to be a sequence of positive integers $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_1+\lambda_2+\ldots+\lambda_\ell = n$ and $\lambda_i \geq \lambda_{i+1}$ for all $i = 1, \ldots, \ell-1$. The summand λ_i is called a *part* of λ. The number of partitions of n is denoted by $p(n)$. For instance, there are 4 partitions of 4: (4), (3,1), (2,2), (1,1,1,1). Thus $p(4) = 4$. In order to deal with the challenge of enumerating $p(n)$, Euler introduced the idea of generating functions [21]. Using this approach, he was able to show that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

where $p(0) := 1$. The reasoning was to expand the series product in several terms, i.e.,

$$\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^4}\cdots$$

and use geometric series

$$(q^{0\cdot 1}+q^{1\cdot 1}+q^{2\cdot 1}+q^{3\cdot 1}+\ldots)(q^{0\cdot 2}+q^{1\cdot 2}+q^{2\cdot 2}+\ldots)(q^{0\cdot 3}+q^{1\cdot 3}+q^{2\cdot 3}+\ldots)\times\ldots. \quad (1.1)$$

Since a partition is a linear combination of $1, 2, 3, 4, \ldots$ with coefficients in $\mathbb{Z}_{\geq 0}$, then it is clear that such a combination occurs as an exponent once the necessary terms in the expanded version (1.1) are multiplied. Thus the generating function was realised in that manner [21].

Adapting such an approach, in general, the number of partitions of n with parts in a set $H \subseteq \mathbb{N}$ is simply the coefficient of q^n in the expansion of the series

$$\prod_{j \in H} \frac{1}{1-q^j}.$$

Various researchers have tried to come up with an explicit formula for $p(n)$. Some success has been recorded despite the fact that the formulas are quite complicated. One of them, due to Hardy and Ramanujan and perfected by Rademacher is given by

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh \left[\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right] \right)$$

where

$$A_k(n) = \sum_{0 \leq m < k : (m,k)=1} e^{\pi i [s(m,k) - \frac{1}{k} 2nm]}$$

and $s(m,k)$ is a Dedekind sum, see [11]. More recent work on formulas for $p(n)$ can be found in [19] and [16].
However, due to the nature of the formulae, efficient computation of $p(n)$ for some given n is highly compromised. The function $p(n)$ can be defined recursively. The best known recurrences include; the one based on the sum of divisors function σ and another based on pentagonal numbers. It can be shown that

$$np(n) = \sum_{j=1}^{n} p(n-j)\sigma(k),$$

where $\sigma(k)$ is the sum of divisors of k. Furthermore, from the fact that

$$\prod_{m=1}^{\infty} (1-q^m) \cdot \prod_{m=1}^{\infty} \frac{1}{1-q^m} = 1,$$

one obtains a formula

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \ldots,$$

which indeed involves pentagonal numbers $1, 2, 5, 7, \ldots$. It turns out that $p(n)$ grows exponentially, and this renders it difficult to compute $p(n)$ with large values of n, see [11]. More explicitly,

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}} \text{ as } n \to \infty.$$

Congruence relations involving $p(n)$ have also been derived and similar techniques are being used to derive results for other partition functions.
Questions on combinatorial or analytic investigation of $p(n)$ can be extended to several restricted partition functions. One may ask about exact or recurrence formulas, asymptotic estimates, generating functions, etc. for such functions. In a situation where a partition function can be expressed in terms of $p(n)$, it may be less of a burden to extend those formulas that apply to $p(n)$ to the function itself.
If $f(n), g(n)$ are partition functions, then the statement $f(n) = g(n)$ is called a partition identity given that the statement is true. Partition identities can be proved combinatorially, bijectively or by means of generating functions. For the study of the latter, see [36].
A partition identity shall be called Rogers-Ramanujan type if the generating function of one side takes the form $\prod_{n\in M} \frac{1}{1-q^n}$ for some set $M \subseteq \mathbb{N}$, and the combinatorial interpretation of the other side is stated in terms of difference conditions on parts of partitions. The underlying analytic counterpart shall also bear the same terminology: a Rogers-Ramanujan identity.
Among contributors to the subject of partition identities of Rogers-Ramanujan type are Agarwal [4], Andrews [10], Bressoud [14], Connor [20] and Hirschhorn [25].
There has been much research on the subject of partitions in areas such as combinatorial or analytic proofs for partition identities, congruence relations, statistical distributions, asymptotics, etc., see [13], [26], [31]. As pointed out in [11], Rogers-Ramanujan identities have attracted attention, ranging from their combinatorial interpretation to other applications. These are the *original* Rogers-Ramanujan identities that have set a motivation to identities of Rogers-Ramanujan type.
Besides contributing to the class of Rogers-Ramanujan identities, George Andrews introduced a study of certain partitions with early conditions [9]. In his first paper on the subject, he defined what are called *partitions with initial repetitions*, from which arise several partition identities ranging from Rogers-Ramanujan type identities to identities of Euler Pentagonal type. These partitions with initial repetitions led to a broader look at partitions with early conditions. In pursuing this work, Andrews was motivated by Sylvester's work [35].

The two major themes of this thesis are partition-theoretic interpretation of Rogers-Ramanujan identities and partitions with initial repetitions. The study of the latter took place very recently.

In Chapter One, we give a survey of important known results and tools that we require to pursue our work.
In Chapter Two, we study the three-way interpretation of Rogers-Ramanujan identities which is due to Agarwal [6]. We generalize this result, and then discuss some consequences in the subsequent chapter where also, certain applications to plane partitions are given, new partition identities and their bijections described.
In Chapter Four, we construct a non-diagram version of the Keith's bijection, a bijection given by William J. Keith. Keith [27] gave the bijection in response to an open problem posed by George Andrews. The idea was to prove one of George Andrews' theorems bijectively.
In the same chapter, we give a refinement of the same George Andrews theorem that Keith studied.
In Chapter Five, some variants of Andrews partitions with early conditions are discussed and results proved via generating functions or combinatorially. In some cases, Euler Pentagonal like identities are obtained and congruence relations presented.
The thesis ends with a conclusion in Chapter Six, where the major highlight is a statement of open problems or further related work.
Chapter Three, Section 4.1 (except a bijection for Theorem 4.2) and Section 4.2 have been written up as a research paper which has in turn been published in Utilitas Mathematica, see [29].

Chapter 2

Preliminaries

2.1 Notation and Terminology

We shall denote the set of all partitions of n with a certain property by upper case letters, e.g., $A(n)$, $B(n)$, etc, and their corresponding cardinalities by lower case letters, e.g., $a(n) = |A(n)|$, $b(n) = |B(n)|$. Unless specified otherwise, $a(0) := 1$, for any partition-enumerating function $a(n)$.

We will adopt the following notation and standard terminology:

$P(n)$: the set of partitions of n. We set $p(n) := |P(n)|$.

$\mathit{DIST}(n)$: the set of partitions of n into distinct parts. We set $\mathit{dist}(n) := |\mathit{DIST}(n)|$.

$\mathit{ODD}(n)$: the set of partitions of n into odd parts. We set $\mathit{odd}(n) = |\mathit{ODD}(n)|$.

ϕ : *Euler* map with domain $\mathit{DIST}(n)$ and range $\mathit{ODD}(n)$ for a fixed n. This map is described on page 20.

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}), \ \ (a;q)_0 := 1.$$

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \ \ (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

$$(a_1, a_2, \ldots, a_k; q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\ldots(a_k;q)_\infty.$$

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$,
$|\lambda|$: the total sum of the summands, i.e., $|\lambda| = \sum_{j=1}^{\ell} \lambda_i$.

$\#(\lambda)$ or $\#\lambda$: the length of λ which is the number of parts in λ, counting multiplicity.

$\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$: alternative notation for λ as a multiset admitting repeated parts. If emphasis is placed on multiplicity, we write $\lambda = \langle t_1^{m_{t_1}} t_2^{m_{t_2}} \ldots t_s^{m_{t_s}} \rangle$ where $t_1 > t_2 > \ldots > t_s$ and m_{t_j} is the multiplicity of t_j, $j = 1, 2, \ldots, s$. At times, instead of m_h, one writes $m_h(\lambda)$ to stand for the multiplicity of h in the partition λ.

Subpartition: A subpartition of λ is any partition β whose parts appear in λ and $m_j(\beta) \leq m_j(\lambda)$ for all j.

$\lambda \setminus \beta$: a partition obtained by deleting β from the partition λ. In this case, the assumption is that β is a subpartition of λ.

$\lambda^{j..j}$: a partition obtained from λ after deleting all parts j.

For example, $\lambda = (15, 15, 15, 6, 6, 3, 3, 2, 1)$ can be written as $\{15, 15, 15, 6, 6, 3, 3, 2, 1\}$ or $\langle 15^3 6^2 3^2 2^1 1^1 \rangle$; $\#(\lambda) = 9$, $|\lambda| = 66$, $\lambda \setminus (6, 2, 1) = (15, 15, 15, 6, 3, 3)$ and $\lambda^{15..15} = (6, 6, 3, 3, 2, 1)$.

$[\lambda]_k$: the k-modular diagram of a partition λ.

Definition 2.1. Given a partition λ, the conjugate of λ denoted by λ' is a partition whose part λ'_i is defined as

$$\lambda'_i := |\{j \in \mathbb{N} : \lambda_j \geq i\}|.$$

Definition 2.2. If λ and μ are any two partitions, we define

$$\lambda \cup \mu = \langle \ldots j^{m_j(\lambda)+m_j(\mu)} \ldots 2^{m_2(\lambda)+m_2(\mu)} 1^{m_1(\lambda)+m_1(\mu)} \rangle$$

and

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots).$$

In Definition 2.2, if one of the partitions is shorter in length than the other, append zeros to it to ensure addition. For example, let $\lambda = (5, 5, 4, 3, 1)$ and $\mu = (35, 7, 5, 5, 1, 1, 1, 1)$. Then we append zeros to λ so that

$$\lambda + \mu = (5, 5, 4, 3, 1, 0, 0, 0) + (35, 7, 5, 5, 1, 1, 1, 1) = (40, 12, 9, 8, 2, 1, 1, 1).$$

Just by using the definitions above, it can be shown that *union* and *sum* operations satisfy

$$(\lambda \cup \mu)' = \lambda' + \mu'. \tag{2.1}$$

Definition 2.3. Consider a partition $\pi = \langle \dots j^{m_j(\pi)} \dots 2^{m_2(\pi)} 1^{m_1(\pi)} \rangle$, a partition $[k, \pi]$ is constructed by replacing $j^{m_j(\pi)}$ with $(j/k)^{km_j(\pi)}$ whenever j is divisible by k, and reordering the resulting sequence into a partition. Similarly, $[k, \pi]^{-1}$ is constructed by replacing $j^{m_j(\pi)}$ with $(kj)^{m_j(\pi)/k}$ whenever $m_j(\pi)$ is divisible by k.

For example, $\pi = (24, 7, 5, 4, 2, 1)$ and $[2, \pi] = \langle 12^2 7^1 5^1 2^2 1^3 \rangle$.

Definition 2.4. A *plane partition* of n on a square is a representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$, $a \geq b \geq d \geq 0, a \geq c$ and $c \geq d$ such that $a + b + c + d = n$.

2.2 Representation of partitions

Partitions can be represented via diagrams. We discuss two ways, namely Ferrers graphs and modular diagrams.

Definition 2.5. The *Ferrers graph* of a partition λ is a left-justified array of dots with λ_i dots in the i^{th} row.

The Ferrers graph of $\lambda = (10, 10, 10, 7, 5, 5, 4)$ is shown in Figure 2.1.
There are several advantages in realising partitions as Ferrers graphs. One benefit is that the Ferrers graph of λ' can be obtained by flipping the Ferrers graph of λ along the main diagonal. Figure 2.2 shows the conjugate of $\lambda = (10, 10, 10, 7, 5, 5, 4)$.
Furthermore, Ferrers graphs can be used to prove certain partition identities such as the following partition theorem, see [11].

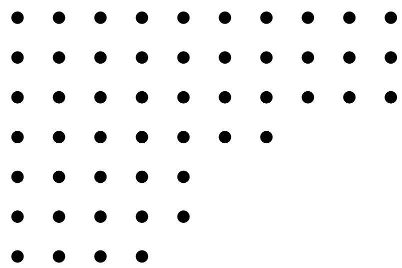

Figure 2.1: Ferrers graph of $(10, 10, 10, 7, 5, 5, 4)$

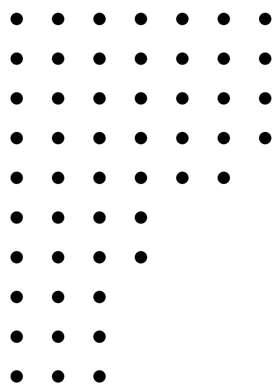

Figure 2.2: Ferrers graph of $(7, 7, 7, 7, 6, 4, 4, 3, 3, 3) = (10, 10, 10, 7, 5, 5, 4)'$

Theorem 2.1. *The number of partitions of n with at most m parts equals the number of partitions of n in which no part is greater than m.*

Ferrers graphs are used in one of the bijective proofs of the so-called Euler Pentagonal Theorem (EPT) stated below, see [11].

Theorem 2.2 (EPT)**.** *For $|q| < 1$, we have*

$$\prod_{n=1}^{\infty}(1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^m q^{\frac{m}{2}(3m-1)}.$$

The following theorem, Theorem 2.3 is a partition-theoretic interpretation of the theorem above and is due to Legendre, see [11]. A generating function proof of the theorem is given on pages 20 and 21.

Theorem 2.3. *For $i = 0, 1$, let $dist_i(n)$ denote the number of partitions λ into distinct parts such that $\#(\lambda) \equiv i \pmod 2$. Then*

$$dist_0(n) - dist_1(n) = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m \pm 1), \\ 0 & \text{otherwise.} \end{cases}$$

Besides Ferrers graphs, partitions can be represented using modular diagrams. Consider a partition λ. The k-modular diagram of λ, $[\lambda]_k$ is constructed as follows:

- Write each part $\lambda_i = b_i k + r_i$ where $0 \leq r_i < k$.
- Each part λ_i is replaced by a column in which k appears b_i times, and if $r_i \neq 0$, r_i appears once and as the first entry in the column.

Figure 2.3 shows the 5-modular diagram of $(27, 21, 21, 15, 9, 1)$. Note that each entry

$$\begin{array}{llllll} 2 & 1 & 1 & 5 & 4 & 1 \\ 5 & 5 & 5 & 5 & 5 & \\ 5 & 5 & 5 & 5 & & \\ 5 & 5 & 5 & & & \\ 5 & 5 & 5 & & & \\ 5 & & & & & \end{array}$$

Figure 2.3: The 5-modular diagram for $(27, 21, 21, 15, 9, 1)$

of a modular diagram is non-zero.

Definition 2.6. An entry i of $[\lambda]_k$ is called a *k-residue unit* if $i < k$; otherwise it is called a *k-unit*. A partition is called *k-singular* if every part is divisible by k. If each part is not divisible by k, the partition is said to be k-regular. Where the meaning of k is understood, we simply say residue unit, unit instead of k-residue unit or k-unit, respectively.

If a partition λ has parts with multiplicities less than k, then the successive parts in λ' have a difference of at most $k-1$. This motivates the following definition which was formally introduced in [34] and translated in [28].

Definition 2.7. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is said to be k-flat if $\lambda_i - \lambda_{i+1} < k$ for all $i = 1, 2, \ldots, \ell$, where we define $\lambda_{\ell+1} := 0$.

Given $[\lambda]_k$, a k-unit y_i is called a *k-edge unit* if upon removal of y_i and all the k-units under it, one obtains a k-modular diagram. Figure 2.4 illustrates this; the 5-edge units in $[(27, 21, 21, 15, 9, 1)]_5$ are those framed in boxes.

$$
\begin{array}{cccccc}
2 & 1 & 1 & 5 & 4 & 1 \\
5 & 5 & 5 & 5 & \boxed{5} & \\
5 & 5 & 5 & \boxed{5} & & \\
5 & 5 & 5 & & & \\
5 & 5 & \boxed{5} & & & \\
\boxed{5} & & & & &
\end{array}
$$

Figure 2.4: The 5-edge units of $(27, 21, 21, 15, 9, 1)$

Number the k-edge units consecutively, from top down: $y_1, y_2, \ldots, y_w$. To each y_i, build S_i as follows:

Let S_i contain the lowest k-unit in each column left of y_i not lying in $S_{i+1} \cup \ldots \cup S_w$ and y_i itself.

S_i is called a *k-strip.*

The 5-strips of $(27, 21, 21, 15, 9, 1)$ are shown in Figure 2.5.

After removing the k-strips from $[\lambda]_k$, we are left with a flat partition called the k-flat portion of λ, which is denoted by λ_f.
Fixing k, it turns out that every λ can be written (uniquely) as a sum

$$\lambda = \lambda_f + \lambda_s \tag{2.2}$$

where λ_s is k-singular. [34]

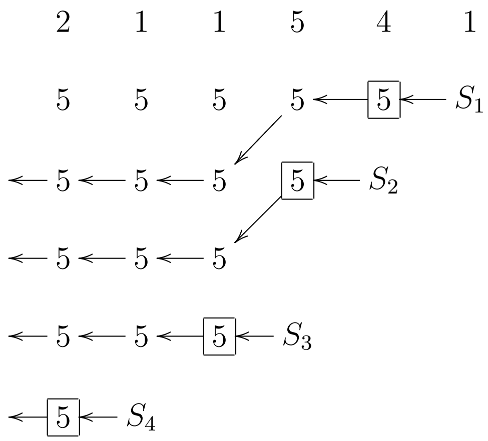

Figure 2.5: The 5-strips of $(27, 21, 21, 15, 9, 1)$

For instance, if $k = 5$, $\lambda = (27, 21, 21, 15, 9, 1)$, we have

$$\lambda_f = (7, 6, 6, 5, 4, 1) \text{ and } \lambda_s = (20, 15, 15, 10, 5).$$

2.3 q-Series and Partitions

In this section, we survey some relevant and important results in partition theory via q-series. Recall that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots .$$

Let $p(n, m)$ denote the number of partitions of n with m parts. If we require the generating function of $p(n, m)$, then a second variable z is introduced such that the coefficient of $q^i z^j$ is simply $p(i, j)$. With this reasoning, it turns out that the generating function of $p(n, m)$ is (see [13])

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n, m)q^n z^m = \prod_{n=0}^{\infty} \frac{1}{1 - zq^n}.$$

Finding the generating function of a partition function is not generally easy. In some cases, there is a need to develop special techniques for computing generating functions. A class of partitions in which parts are defined by certain inequality constraints has been studied by Carla and Wilf [18] and is a typical case where writing down the generating function is not difficult. Sometimes, one studies the combinatorial structure of partitions to derive a functional equation which may be solved. We start with an identity due to Cauchy which is paramount to our work.

Theorem 2.4. *If* $|q| < 1$, $|z| < 1$, *then*

$$1 + \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \prod_{n=0}^{\infty} \frac{(1 - azq^n)}{(1 - zq^n)}. \tag{2.3}$$

Proof. Let $A(z) = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-zq^n)}$. Since for fixed a and q, the product $A(z)$ is uniformly convergent on the open disc $|z| < 1$, we can expand it as a power series. Thus $A(z) = \sum_{n=0}^{\infty} b_n(a,q) z^n$. To simplify the notation, set $b_n := b_n(a,q)$. Now

$$\begin{aligned}
A(zq) &= \prod_{n=0}^{\infty} \frac{(1 - azq^{n+1})}{(1 - zq^{n+1})} \\
&= \frac{(1-z)(1-az)}{(1-az)(1-z)} \prod_{n=0}^{\infty} \frac{(1 - azq^{n+1})}{(1 - zq^{n+1})} \\
&= \frac{1-z}{1-az} \prod_{n=0}^{\infty} \frac{(1 - azq^{n})}{(1 - zq^{n})} \\
&= \frac{1-z}{1-az} A(z)
\end{aligned}$$

so that

$$(1 - az)A(zq) = (1 - z)A(z) \tag{2.4}$$

Substituting $\sum_{n=0}^{\infty} b_n z^n$ for $A(z)$ in (2.4) and comparing the coefficients of powers of z, one obtains the recurrence

$$b_n = \frac{1 - aq^{n-1}}{1 - q^n} b_{n-1}, \tag{2.5}$$

and it is not difficult to see that $b_0 = A(0) = 1$. Iterating (2.5) results in

$$b_n = \frac{(1 - aq^{n-1})(1 - aq^{n-2}) \dots (1 - a)}{(1 - q^n)(1 - q^{n-1}) \dots (1 - q)} b_0.$$

$\square$

For the proofs of the corollaries that follow, refer to [11].

Corollary 2.1 (Heine Transformation Formula). *For $|z| < 1$, $|b| < 1$, $|q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_n(z;q)_n}{(q;q)_n(az;q)_n} b^n. \tag{2.6}$$

Corollary 2.2 (Lebesgue's identity). *For $|q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} q^{n(n+1)/2} = \prod_{n=1}^{\infty} (1 - aq^{2n-1})(1 + q^n). \tag{2.7}$$

Corollary 2.3 (Euler). *For $|q| < 1$, $|zq| < 1$, it follows that*

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = \prod_{n=0}^{\infty} (1 + zq^n). \tag{2.8}$$

This corollary has combinatorial implications. Set $z := q$ so that we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1 + q^n). \tag{2.9}$$

The power series on the left-hand side is the generating function for $dist(n)$. To see this, observe that every partition $\lambda \in DIST(n)$ can be written in the form $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where

$$\lambda_j = \ell - j + 1 + \sum_{i=1}^{\ell-j+1} n_i$$

where $n_1, n_2, \ldots, n_\ell \geq 0$. Conversely, such a representation defines a unique partition in $DIST(n)$. Hence, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} dist(n) q^n &= 1 + \sum_{\ell \geq 1} \sum_{n_1, n_2, \ldots, n_\ell \geq 0} q^{(n_1+1)+(n_1+n_2+2)+\ldots+(n_1+n_2+n_3+\ldots+n_\ell+\ell)} \\
&= 1 + \sum_{l=1}^{\infty} q^{\ell(\ell+1)/2} \sum_{n_1=0}^{\infty} q^{\ell n_1} \cdot \sum_{n_2=0}^{\infty} q^{(\ell-1)n_2} \cdot \ldots \cdot \sum_{n_{\ell-1}=0}^{\infty} q^{2n_{\ell-1}} \cdot \sum_{n_\ell=0}^{\infty} q^{n_\ell} \\
&= 1 + \sum_{l=1}^{\infty} q^{\ell(\ell+1)/2} \frac{1}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^\ell)} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n}.
\end{aligned}$$

From another angle, consider a partition λ of n whose m parts are distinct, and let $dist(n, m)$ enumerate such partitions. We define

$$dist(0, 0) := 1. \tag{2.10}$$

Let $f(q, z)$ be the generating function for $dist(n, m)$. Then

$$f(q, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} dist(n, m) q^n z^m$$

where z, q are chosen in such a way that $|q| < 1$, $|zq| < 1$.
Now, carry out the following transformation:

- If the smallest part of λ is 1, then delete 1 and subtract 1 from each of the remaining parts. The resulting partition is in the set $DIST(n - m, m - 1)$.
- If the smallest part of λ is ≥ 2, then subtract 1 from every part. The resulting partition is in the set $DIST(n - m, m)$.

The transformation is invertible, and so

$$dist(n, m) = dist(n - m, m - 1) + dist(n - m, m). \tag{2.11}$$

Equations (2.10) and (2.11) imply that

$$f(q, 0) = 1, \tag{2.12}$$

$$f(q, z) = zqf(q, zq) + f(q, zq) = (1 + zq)f(q, zq). \tag{2.13}$$

Thus

$$f(q, z) = (1 + zq)(1 + zq^2)(1 + zq^3)(1 + zq^3) \dots$$

which implies that $f(q, z) = \prod_{n=1}^{\infty}(1 + zq^n)$. Setting $z = 1$ yields

$$\sum_{n=0}^{\infty} dist(n) q^n = \prod_{n=1}^{\infty}(1 + q^n).$$

The above technique used in which a functional equation is derived and solved is often utilised in partition theory.

Theorem 2.5 (Euler). *The number of partitions into distinct parts is equal to the number of partitions into odd parts, i.e, $dist(n) = odd(n)$.*

Proof. Recall that $\sum_{n=0}^{\infty} dist(n)q^n = \prod_{n=1}^{\infty}(1+q^n)$. Now we can write the series product as

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^n) &= \prod_{n=1}^{\infty}\frac{(1+q^n)(1-q^n)}{1-q^n}\\ &= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^n}\\ &= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}\\ &= \sum_{n=0}^{\infty} odd(n)q^n.\end{aligned}$$

□

Euler's theorem is a special case of the following theorem due to Glaisher.

Theorem 2.6 ([11, page 6])**.** *The number of partitions of n into parts not divisible by d is equal to the number of partitions of n in which parts occur at most $d-1$ times.*

The theorem has been proved bijectively. We shall describe the map ϕ, referred to as the *Euler* map (see page 10). The map $\phi : DIST(n) \to ODD(n)$ is defined as follows:
Starting with a partition whose parts are distinct, split every even part $2j$ into $j+j$. If j is even, repeat the operation until all parts are odd, and reorder the parts to form a partition. For the inverse, one looks for any repeated part j, say. Add any pair of j's to form $2j$'s, and if it turns out that the $2j$'s appear at least twice, repeat the process until you get distinct parts, and reorder the parts to form a partition.
The bijection ϕ is a special case of Glaisher's bijection [23].

Theorem 2.7 ([11, page 22])**.** *For $|q|<1$ and non-negative integers k, i, we have*

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2-in} = \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i}).$$

The above theorem has applications related to the Euler Pentagonal Theorem, Theorem 2.3. To see this, set $k=i=1$, so we have

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} = \prod_{n=0}^{\infty}(1-q^{3(n+1)})(1-q^{3n+1})(1-q^{3n+2}). \tag{2.14}$$

Note that the series product is equal to $\prod_{n=1}^{\infty}(1-q^n)$.

From the fact that

$$1-q^j = \sum_{m=0}^{1}(-1)^m q^{j\cdot m},$$

we have

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1}\cdots(-1)^{i_1+i_2+i_3+\cdots}q^{1\cdot i_1+2\cdot i_2+3\cdot i_3+\cdots}.$$

In the summation, each representation $1\cdot i_1+2\cdot i_2+3\cdot i_3+\ldots+t\cdot i_t$ for any $t<\infty$ defines a partition into distinct parts with the number of parts $i_1+i_2+i_3+\ldots$. So each partition with number of odd (or even) parts gives rise to -1 (or $+1$), respectively. Hence,

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{n=1}^{\infty}(dist_0(n)-dist_1(n))q^n.$$

The left-hand side of (2.14), can be written as

$$1+\sum_{m=1}^{\infty}(-1)^m q^{m(3m\pm1)/2}.$$

So,

$$\sum_{n=1}^{\infty}(dist_0(n)-dist_1(n))q^n = \sum_{m=1}^{\infty}(-1)^m q^{m(3m\pm1)/2}$$

thus proving Theorem 2.3.

Theorem 2.8 ([11, page 23]). *For $|q|<1$, we have*

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m-1}}.$$

Henceforth, in any theorem involving q-series, it shall always be assumed that $|q|<1$. Further representations have been derived for the sum in the left-hand side of Theorem 2.8, just like in the following theorem.

Theorem 2.9 ([15, page 22]).

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = 1+q+\sum_{n=1}^{\infty}\frac{q^{2n+1}(q^2;q^2)_n}{(1-q^3)(1-q^5)(1-q^7)\ldots(1-q^{2n+1})}.$$

Theorem 2.10 ([15, page 12]).

$$1+\sum_{n=1}^{\infty}\frac{(1+zq)(1+zq^3)(1+zq^5)\dots(1+zq^{2n-1})}{(1-q^2)(1-q^4)\dots(1-q^{2n})}q^{2n}=\prod_{n=1}^{\infty}\frac{1+zq^{2n+1}}{1-q^{2n}}.$$

2.4 The Rogers-Ramanujan identities

The celebrated Rogers-Ramanujan identities are given by the series-product representations:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}(1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}. \tag{2.15}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}(1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{2.16}$$

An analytic proof of the identities can be found in [11]. They were discovered by Rogers [30] and subsequently rediscovered by Ramanujan in 1913 [24, p. 91]. The respective identities have the following well-known partition-theoretic interpretations (see for example [11]).

Theorem 2.11. *Let $c(n)$ be the number of partitions of n with parts differing by at least 2, and $d(n)$ the number of partitions of n with parts $\equiv \pm 1 \pmod 5$. Then $c(n) = d(n)$.*

Theorem 2.12. *Let $e(n)$ denote the number of partitions of n with parts differing by at least 2 and containing no ones, and let $f(n)$ be the number of partitions of n into parts $\equiv \pm 2 \pmod 5$. Then $e(n) = f(n)$.*

It is not difficult to see that

$$\sum_{n=0}^{\infty}c(n)q^n=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}$$

and

$$\sum_{n=0}^{\infty}e(n)q^n=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}.$$

Notice that a partition λ of n whose parts differ by at least 2 can be written uniquely in the form $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ where $\lambda_j = 2j-1+\sum_{i=1}^{j} n_i$ for all $j = 1, 2, \dots, \ell$, $|\lambda| = n$ and $n_1, n_2, n_3, \dots, n_\ell \geq 0$. From this representation,

$$\sum_{n=0}^{\infty}c(n)q^n=1+\sum_{\ell=1}^{\infty}\sum_{n_1,n_2,\dots,n_\ell=0}^{\infty}q^{\lambda_1+\lambda_2+\dots+\lambda_\ell}$$

$$= 1 + \sum_{\ell=1}^{\infty} \sum_{n_1,n_2,\ldots,n_\ell=0}^{\infty} q^{\ell^2+\ell n_1+(\ell-1)n_2+\ldots+2n_{\ell-1}+n_\ell}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}$$

The procedure can be adapted to those partitions whose parts differ by at least 2 and have no ones, and the series sum derived.

Theorems 2.11 and 2.12 were further generalised by Gordon [22] as follows.

Theorem 2.13 (Gordon)**.** *Let $bb_{k,i}(n)$ denote the number of partitions of n of the form $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ where $\lambda_j - \lambda_{j+k-1} \geq 2$, and at most $i-1$ of the λ_j equal 1. Let $aa_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then for all n,*

$$aa_{k,i}(n) = bb_{k,i}(n).$$

A proof of this theorem can be found in [11].
Indeed setting $k = i = 2$ yields Theorem 2.12, and $k = i + 1 = 2$ leads to Theorem 2.11.

There are various versions of these identities, and they have witnessed several combinatorial interpretations. Andrews [12] gave several interpretations of (2.15) and (2.16) including the following related identities

$$\sum_{n=0}^{\infty} \frac{q^{3+5+\ldots+2n+1}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+2+2j}) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{2.17}$$

$$\sum_{n=0}^{\infty} \frac{q^{1+3+5+\ldots+2n-1}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+2+2j}) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{2.18}$$

Slater [33] proved the following identities of Rogers [30];

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \prod_{n=1}^{\infty}(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3}). \tag{2.19}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \prod_{n=1}^{\infty}(1-q^{5n})(1-q^{5n-1})(1-q^{5n-4}). \tag{2.20}$$

These were recently given a partition-theoretic interpretation by Argawal and Goyal in [6].

A few relevant Rogers-Ramanujan type identities due to Rogers which will be useful are listed below. They can be found in [33].

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{(q^2, q^6, q^8, q^{10}, q^{12}, q^{14}, q^{18}, q^{20}; q^{20})_\infty}{(q;q)_\infty} \tag{2.21}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \frac{(q^3, q^4, q^7, q^{10}, q^{13}, q^{16}, q^{17}, q^{20}; q^{20})_\infty}{(q;q)_\infty} \tag{2.22}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \frac{(q, q^8, q^9, q^{10}, q^{11}, q^{12}, q^{19}, q^{20}; q^{20})_\infty}{(q;q)_\infty} \tag{2.23}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \frac{(q^2, q^4, q^6, q^{10}, q^{14}, q^{16}, q^{18}, q^{20}; q^{20})_\infty}{(q;q)_\infty} \tag{2.24}$$

Connor [20] interpreted the left-hand side of the above four identities, and other partition-theoretic versions can be found in [25]. Besides the aforementioned, see [10] for more Rogers-Ramanujan identities.

Chapter 3

Partition-theoretic interpretation of Rogers-Ramanujan identities

3.1 Agarwal and his identities

Several analogous combinatorial interpretations of the Rogers-Ramanujan identities (2.15) and (2.16) and related identities have been documented in the literature (see for example [14, 17, 5, 3]).
Agarwal and Goyal [6] recently gave new partition-theoretic interpretations of (2.15) and (2.16). For a positive integer k, let $A_k(n)$ denote the set of partitions of n in which the smallest part is $\equiv k \pmod 4$ and the difference between successive parts is $\equiv 2 \pmod 4$. Thus $a_k(n) := |A_k(n)|$. They first proved that

$$\sum_{n=0}^{\infty} a_k(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}}{(q^4;q^4)_n}. \tag{3.1}$$

We maintain the notation in Theorems 2.11 and 2.12. They stated the following pair of 3-way partition identities (see Theorems 1.4 and 1.6 in [6]).

Theorem 3.1 (Agarwal-Goyal). *Let $b(n)$ denote the number of partitions of n into parts $\equiv 2 \pmod 4$. Then*

$$c(n) = d(n) = \sum_{j=0} a_1(j) b(n-j);$$

$$e(n) = f(n) = \sum_{j=0} a_3(j) b(n-j).$$

We provide a generalization of Theorem 3.1 and discuss certain combinatorial results that arise therefrom. In Section 3.2 we give generalisations of Theorems 3.2 and 3.3.

3.2 Extension of Agarwal's work

We state and prove our two main theorems in this section.

Theorem 3.2. *For positive integers k, γ and t, let $A_k(n, \gamma, t)$ denote the set of partitions of n in which the smallest part is $\equiv k \pmod{t}$ and the difference between successive parts $\equiv \gamma \pmod{t}$. Thus $a_k(n, \gamma, t) := |A_k(n, \gamma, t)|$. Then*

$$\sum_{n=0}^{\infty} a_k(n, \gamma, t) q^n = \sum_{n=0}^{\infty} \frac{q^{kn+\gamma n(n-1)/2}}{(q^t; q^t)_n}$$

where we define $a_k(0, \gamma, t)$ to be 1.

Note that (3.1) follows from Theorem 3.2 on setting $\gamma = 2$ and $t = 4$.

For a positive number s, set
$b'(n)$: the number of partitions of n whose parts are $\equiv 2s \pmod{4s}$ and $\not\equiv 0, s \pmod{2s}$.

$d'(n)$: the number of partitions of n whose parts are $\not\equiv 0, 2s, 3s \pmod{5s}$.

$F'(n)$: the set of partitions of n whose parts are $\not\equiv 0, s, 4s \pmod{5s}$, and set $f'(n) := |F'(n)|$.

$c'(n)$: the set of partitions of n in which either

a. there exists a part j which is distinct such that

   i. all parts greater than j are distinct and the part difference is $\geq 2s$, all parts less than j have the part difference of at most $s-1$ and the smallest part is at most $s-1$ in size.

   ii. the difference between j and the largest part less than j is at least s.

or

b. all parts are distinct with part difference of at least $2s$ and the smallest part is at least s.
or

c. the difference between parts is at most $s-1$ and the smallest part is at most $s-1$ in size.

We set $c'(n) := |C'(n)|$.

$E'(n)$: the set of partitions of n in which either

a. there exists a part j which is distinct such that

   i. all parts greater than j are distinct and the part difference is $\geq 2s$, all parts less than j have the part difference of at most $s-1$ and the smallest part is at most $s-1$ in size.

   ii. the difference between j and the largest part less than j is at least $2s$.

or

b. all parts are distinct with part difference of at least $2s$ and the smallest part is at least $2s$.
or

c. the difference between parts is at most $s-1$ and the smallest part is at most $s-1$ in size.

We set $e'(n) := |E'(n)|$.

Theorem 3.3. *Let s be a positive integer. Then*

$$c'(n) = d'(n) = \sum_{j=0}^{n} a_s(j, 2s, 4s) b'(n-j); \tag{3.2}$$

$$e'(n) = f'(n) = \sum_{j=0}^{n} a_{3s}(j, 2s, 4s) b'(n-j), \tag{3.3}$$

where $a_k(n, \gamma, t)$ is defined in Theorem 3.2.

Note that Theorem 3.1 may be obtained by setting $s = 1$ in Theorem 3.3.
The following identities due to Rogers from Slater's list in [33] will be used in the proofs.

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3}); \tag{3.4}$$

$$\sum_{n\geq 0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-1})(1-q^{5n-4}). \tag{3.5}$$

3.2.1 Proof of Theorem 3.2

Proof. Let $A_k(m,n)$ be the set of partitions in $A_k(n,t,\gamma)$ which have exactly m parts, $m \geq 1$, and set $a_k(m,n) := |A_k(m,n)|$.
We split these partitions into two groups: Those with smallest part equal to k, and those with smallest part greater than k.
For those whose smallest part is equal to k, deleting k and then subtracting γ from each part leaves us with partitions in $A_k(m-1, n-k-\gamma(m-1))$.
For those partitions with the smallest part greater than k, subtracting t from each part leaves us with partitions in $A_k(m, n-tm)$. The transformations are invertible, and thus we have

$$a_k(m,n) = a_k(m-1, n-k-\gamma(m-1)) + a_k(m, n-tm).$$

For $|q| < 1$ and $|zq| < 1$, let

$$g(z,q,t,\gamma,k) = \sum_{n\geq 0}\sum_{m\geq 0} a_k(m,n) z^m q^n.$$

Clearly $g(0,q,t,\gamma,k) = 1$, and we have

$$\sum_{m\geq 1}\sum_{n\geq k+\gamma(m-1)} a_k(m-1, n-k-\gamma(m-1)) z^m q^n = zq^k g(zq^\gamma, q, t, \gamma, k)$$

and

$$\sum_{m\geq 0}\sum_{n\geq tm} a_k(m, n-tm) z^m q^n = g(zq^t, q, t, \gamma, k)$$

so that

$$g(z,q,t,\gamma,k) = zq^k g(zq^\gamma, q, t, \gamma, k) + g(zq^t, q, t, \gamma, k).$$

Thus we seek a function $g(z,q,t,\gamma,k)$ that satisfies (3.6) and (3.7)

$$g(0,q,t,\gamma,k)=1 \tag{3.6}$$

$$g(z,q,t,\gamma,k)=zq^k g(zq^\gamma,q,t,\gamma,k)+g(zq^t,q,t,\gamma,k). \tag{3.7}$$

We claim that

$$g(z,q,t,\gamma,k)=\sum_{n\geq 0}\frac{q^{kn+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n.$$

For the sake of notation, set $g(z,q):=g(z,q,t,\gamma,k)$. We have

$$\begin{aligned}
zq^k g(zq^\gamma,q)+g(zq^t,q) &= zq^k\sum_{n\geq 0}\frac{q^{kn+\gamma n+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n+\sum_{n\geq 0}\frac{q^{tn+kn+\gamma n((n-1)/2}}{(q^t;q^t)_n}z^n\\
&= zq^k+\sum_{n\geq 1}\frac{q^{k+kn+\gamma n+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^{n+1}+1+\sum_{n\geq 1}\frac{q^{(t+k)n+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n+1\\
&\quad zq^k+\sum_{n\geq 2}\frac{q^{kn+\gamma(n-1)+\gamma(n-1)(n-2)/2}}{(q^t;q^t)_{n-1}}z^n+\sum_{n\geq 1}\frac{q^{(t+k)n+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n\\
&= 1+\sum_{n\geq 1}\frac{q^{nk+\gamma(n-1)+\gamma(n-1)(n-2)/2}}{(q^t;q^t)_{n-1}}+\sum_{n\geq 1}\frac{q^{(t+k)n+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n\\
&= 1+\sum_{n\geq 1}\left(\frac{q^{nk+\gamma n(n-1)/2}}{(q^t;q^t)_{n-1}}+\frac{q^{tn+kn+\gamma n(n-1)/2}}{(q^t;q^t)_n}\right)z^n\\
&= 1+\sum_{n\geq 1}\frac{q^{nk+\gamma n(n-1)/2}(1-q^{tn})+q^{tn+kn+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n\\
&= \sum_{n\geq 0}\frac{q^{kn+\gamma n(n-1)/2}}{(q^t;q^t)_n}z^n=g(z,q).
\end{aligned}$$

Indeed (3.7) is satisfied. Furthermore, (3.6) is satisfied too. Thus

$$\sum_{n\geq 0}a_k(n,\gamma,t)q^n=g(1,q,t,\gamma,k)=\sum_{n\geq 0}\frac{q^{kn+\gamma n(n-1)/2}}{(q^t;q^t)_n}.$$

□

3.2.2 Proof of Theorem 3.3

Lemma 3.1. *The following equations hold*

$$\sum_{n=0}^{\infty}e'(n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{5sn})(1-q^{5sn+s})(1-q^{5sn+4s})}{1-q^n}. \tag{3.8}$$

$$\sum_{n=0}^{\infty} c'(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{5sn})(1-q^{5sn+2s})(1-q^{5sn+3s})}{1-q^n}. \tag{3.9}$$

Proof. For (3.8), let $\lambda \in E'(n)$. It is not difficult to see that the conjugate λ' is a partition of n in which

a. if i appears at least $2s$ times, then all integers less than i appear at least $2s$ times and those parts greater than i appear less than s times,

b. if there is no part appearing at least $2s$ times, then all parts appear less than s times.

Since conjugation is an invertible transformation, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} e'(n)q^n &= \sum_{n=0}^{\infty} \frac{q^{2s(1+2+3+\ldots+n)}}{(q;q)_n} \prod_{j=n+1}^{\infty} (1+q^j+q^{2j}+\ldots+q^{(s-1)j}) \\
&= \sum_{n=0}^{\infty} \frac{q^{2s(1+2+3+\ldots+n)}}{(q;q)_n} \prod_{j=n+1}^{\infty} \frac{1-q^{sj}}{1-q^j} \\
&= \sum_{n=0}^{\infty} \frac{q^{sn(n+1)}}{(q;q)_\infty} \frac{(q^s;q^s)_\infty}{(q^s;q^s)_n} \\
&= \prod_{n=1}^{\infty} \frac{1-q^{ns}}{1-q^n} \cdot \sum_{n=0}^{\infty} \frac{q^{sn(n+1)}}{(q^s;q^s)_n} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{5sn})(1-q^{5sn+s})(1-q^{5sn+4s})}{1-q^n}
\end{aligned}$$

Let $\lambda \in C'(n)$. We observe that λ' is characterised by the property that a) if a part i appears at least s times, then all integers less than i appear at least $2s$ times. We point out that Keith [27] found the generating function for the number of such conjugate partitions, which is equal to the right-hand side of (3.9). □

Let $k = s, \gamma = 2s$ and $t = 4s$ in Theorem 3.2. Then

$$\begin{aligned}
\sum_{n\geq 0} a_s(n, 2s, 4s)q^n &= \sum_{n\geq 0} \frac{q^{sn^2}}{(q^{4s};q^{4s})_n} \\
&= \frac{(-q^s;q^{2s})_\infty}{(q^{2s};q^{2s})_\infty} \prod_{n=1}^{\infty} (1-q^{5sn})(1-q^{5sn-2s})(1-q^{5sn-3s}).
\end{aligned}$$

The last equality follows from (3.4).

$$\begin{aligned}\sum_{n\geq 0} b'(n)q^n &= \prod_{n=0}^{\infty}(1-q^{4sn+2s})^{-1}\frac{(1-q^{2sn+s})(1-q^{2sn+2s})}{1-q^{n+1}}\\ &= \prod_{n=0}^{\infty}\frac{(1-q^{2sn+s})(1-q^{2sn+2s})}{(1-q^{2sn+s})(1+q^{2sn+s})(1-q^{n+1})}\\ &= \prod_{n=0}^{\infty}\frac{1-q^{2sn+2s}}{1+q^{2sn+s}}\cdot\frac{1}{1-q^{n+1}}\\ &= \frac{(q^{2s};q^{2s})_\infty}{(-q^s;q^{2s})_\infty}\cdot\prod_{n=0}^{\infty}(1-q^{n+1})^{-1}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\prod_{n=0}^{\infty}(1-q^{n+1})\cdot\sum_{n\geq 0}b'(n)q^n\cdot\sum_{n\geq 0}a_s(n,2s,4s)q^n\\ &\qquad= \prod_{n=1}^{\infty}(1-q^{5sn})(1-q^{5sn-2s})(1-q^{5sn-3s})\end{aligned}$$

so that

$$\sum_{n\geq 0}b'(n)q^n\cdot\sum_{n\geq 0}a_s(n,2s,4s)q^n = \sum_{n\geq 0}d'(n)q^n,$$

and (3.2) follows after taking Lemma 3.1 into account.

Let $k=3s, \gamma=2s$ and $t=4s$ in Theorem 3.2. Then we have

$$\sum_{n\geq 0}a_{3s}(n,2s,4s)q^n = \sum_{n\geq 0}\frac{q^{sn^2+2sn}}{(q^{4s};q^{4s})_n}.$$

Use (3.5) with $q := q^s$, and Lemma 3.1, the rest follows. For example, consider $n=8$ and $s=2$. We have $d'(8)=15$ and observe from Table 3.1 that

$$\sum_{j=0}^{8}a_2(j,4,8)b'(8-j) = 9\times 1+5\times 1+1\times 1 = 15.$$

Partitions in $C'(8)$ are:
$3+1+1+1+1+1$, $4+1+1+1+1$, $4+2+1+1$,
$5+1+1+1$, $6+1+1$, $6+2$, 8, $7+1$,

Table 3.1: Enumerating Partitions

j	$b'(j)$	$a_2(8-j,4,8)$	Partitions in the set $A_2(8-j,4,8)$
0	1	1	$6+2$
1	1	0	$--$
2	1	0	$--$
3	2	0	$--$
4	3	0	$--$
5	4	0	$--$
6	5	1	2
7	7	0	$--$
8	9	1	$\emptyset$

$5+2+1, \quad 1+1+1+1+1+1+1+1, \quad 2+1+1+1+1+1+1,$
$2+2+1+1+1+1, \quad 2+2+2+1+1, \quad 3+2+1+1+1,$
$3+2+2+1.$
Clearly,

$$c'(8) = d'(8) = \sum_{j=0}^{8} a_2(j,4,8)b'(8-j).$$

Furthermore, we note that partitions in $F'(8)$ are:
$1+1+3+3, \quad 1+1+1+1+1+1+1+1, \quad 1+1+1+1+4,$
$1+1+1+1+1+3, \quad 1+3+4, \quad 4+4, \quad 1+1+1+5$, $3+5$,
$1+7, \quad 1+1+6$. There are 10 partitions.

Those enumerated by $E'(8)$ are:
$5+1+1+1, \quad 6+1+1, \quad 8, \quad 7+1,$
$1+1+1+1+1+1+1+1, \quad 2+1+1+1+1+1+1, \quad 2+2+1+1$
$+1+1, \quad 2+2+2+1+1, \quad 3+2+1+1+1$, $3+2+2+1$.

We also observe that $a_6(j,4,8) = 0$ for $j = 1,2,3,4,5,7,8$, and $a_6(0,4,8) = a_6(6,4,6) = 1$. Hence

$$\sum_{j=0}^{8} a_6(j,4,8)b'(8-j) = a_6(0,4,8)\cdot b'(8) + a_6(6,4,6)\times b'(2) = 10.$$

$$\text{Indeed} \quad e'(8) = f'(8) = \sum_{j=0}^{8} a_6(j,4,8)b'(8-j).$$

Chapter 4

Further combinatorial consequences

In this chapter, we look at some consequences of the results in the previous chapter. More explicitly, we give combinatorial proofs of certain partition theorems which are inspired partly by (3.1) and the work of Connor [20]. We also relate partitions with parity-alternating parts to plane partitions on a square. In Section 4.3, an explicit bijection is given involving one of Andrews' interpretations of Rogers-Ramanujan identities and the other one due to Agarwal and Goyal that has been previously visited. In the same section, we turn to coefficients of a fifth order mock theta function given by

$$\zeta(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1+q)(1+q^2)\dots(1+q^n)}.$$

It turns out that the coefficients of powers of q in $\zeta(q)$ are associated with the partition function $e(n)$ in Theorem 2.12, see [32]. We give an explicit bijection for an inequality involving the coefficients.

4.1 Bijections for related partition identities

For positive integers k, γ and t, let $A_k(n, \gamma, t, l)$ denote the set of partitions of n of length l in which the smallest part is $\equiv k \pmod{t}$ and the difference between successive parts $\equiv \gamma \pmod{t}$. Set $a_k(n, \gamma, t, l) = |A_k(n, \gamma, t, l)|$. Observe that a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ in $A_k(n, \gamma, t, l)$ is such that

$$\lambda_l = tn_l + k,$$

$$\lambda_{l-1} = t(n_{l-1} + n_l) + \gamma + k,$$

and in general,

$$\lambda_i = t(n_i + n_{i+1} + n_{i+2} + \ldots + n_l) + \gamma(l-i) + k, \ \ 1 \leq i \leq l. \tag{4.1}$$

From (4.1), note that the generating function for $a_k(n,\gamma,t,l)$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} a_k(n,\gamma,t,l)q^n &= \sum_{n_1,n_2,\ldots,n_l=0}^{\infty} q^{\sum_{i=1}^{l}(t(n_i+n_2+\ldots+n_l)+\gamma(l-i)+k)} \text{ by (4.1)} \\
&= q^{kl+\gamma l(l-1)/2} \sum_{n_1=0}^{\infty} q^{tn_1} \cdot \sum_{n_2=0}^{\infty} q^{2tn_2} \sum_{n_{l-1}=0}^{\infty} q^{t(l-1)n_{l-1}} \ldots \sum_{n_l=0}^{\infty} q^{tln_l} \\
&= \frac{q^{kl+\gamma l(l-1)/2}}{(1-q^t)(1-q^{2t})(1-q^{3t})\ldots(1-q^{lt})} \\
&= \frac{q^{kl+\gamma l(l-1)/2}}{(q^t;q^t)_l}.
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} a_k(n,\gamma,t,l)q^n = \frac{q^{kl+\gamma l(l-1)/2}}{(q^t;q^t)_l} \tag{4.2}$$

For $j \geq 0$, $s \geq 1$, let

$$A_{k,\gamma,t}^{j,s}(n) = \cup_{l=1}^{\infty}\{\lambda \in A_k(n,\gamma,t,l) : l \equiv j \pmod{2s}\}. \tag{4.3}$$

Set

$$a_{k,\gamma,t}^{j,s}(n) := |A_{k,\gamma,t}^{j,s}(n)|.$$

Let $U_s(n,h)$ denote the set of partitions of n of the form $c_1 + c_2 + \ldots + c_m$ where $m \equiv h \pmod{s}$, $c_i > c_{i+1}$ for $i \equiv h \pmod{s}$ and $c_i \geq c_{i+1}$ for $i \not\equiv h \pmod{s}$. Set $u_s(n,h) = |U_s(n,h)|$.

Set

$R_1(n)$: the set of partitions of n of the form $c_1 + c_2 + \ldots + c_m$ where $c_1 > c_2 \geq c_3 > c_4 \geq c_5 > c_6 \geq \ldots$, $c_m > 0$. Set $r_1(n) := |R_1(n)|$.

$R_2(n)$: the set of partitions of n of the form $c_1 + c_2 + \ldots + c_m$ where $c_1 \geq c_2 > c_3 \geq c_4 > \ldots$, $c_m > 0$. Set $r_2(n) := |R_2(n)|$.

$R_3(n)$: the set of partitions of n of the form $c_1 + c_2 + \ldots + c_{2m}$ where $c_1 \geq c_2 > c_3 \geq c_4 > c_5 \geq c_6 > \ldots$, $c_{2m} > 0$. Set $r_3(n) := |R_3(n)|$.

$R_4(n)$: the set of partitions of n of the form $c_1 + c_2 + \ldots + c_{2m+1}$ where $c_1 \geq c_2 \geq c_3 > c_4 \geq c_5 > c_6 \geq \ldots$, $c_{2m+1} > 0$. Set $r_4(n) := |R_4(n)|$. It turns out that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} r_1(n) q^n, \tag{4.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} r_2(n) q^n, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} r_3(n) q^n, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} r_4(n) q^n. \tag{4.7}$$

For the above equations, see [20], [25].

Now consider (4.2), and set $l = 2j+1$, $k = 1$. Then

$$\sum_{n=0}^{\infty} a_{1,2,4}^{1,1}(n) q^n = \sum_{j=0}^{\infty} \frac{q^{(2j+1)^2}}{(q^4;q^4)_{2j+1}}$$

which implies that

$$\sum_{n=0}^{\infty} a_{1,2,4}^{1,1}(4n+1) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} r_2(n) q^n \text{ by (4.5).}$$

Similarly, we obtain

$$\sum_{n=0}^{\infty} a_{1,2,4}^{0,1}(4n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} r_1(n) q^n \text{ by (4.4).}$$

$$\sum_{n=0}^{\infty} a_{3,2,4}^{1,1}(4n+3) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} r_4(n) q^n \text{ by (4.12).}$$

$$\sum_{n=0}^{\infty} a_{3,2,4}^{0,1}(4n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} r_3(n)q^n \text{ by (4.6).}$$

Hence Theorem 4.1 follows. However, we also provide a bijective proof.

Theorem 4.1. *Let n be a nonnegative integer. Then*

$$|A_{1,2,4}^{0,1}(4n)| = |R_1(n)|. \tag{4.8}$$

$$|A_{1,2,4}^{1,1}(4n+1)| = |R_2(n)|. \tag{4.9}$$

$$|A_{3,2,4}^{0,1}(4n)| = |R_3(n)|. \tag{4.10}$$

$$|A_{3,2,4}^{1,1}(4n+3)| = |R_4(n)|. \tag{4.11}$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $c = (c_1, c_2, \ldots, c_m)$ be partitions in the underlying sets of the left- and right-hand sides of each identity. We define a bijection $\lambda \mapsto c$ in each case.

In (4.8), we have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in A_{1,2,4}^{0,1}(4n)$. Set

$$c_i = \frac{\lambda_i - 2(l-i) - 1}{4} + \frac{l}{2} - \begin{cases} 0, & i = 1 \\ \frac{i}{2}, & i \geq 2 \text{ and } i \text{ even} \\ \frac{i-1}{2}, & i \geq 3 \text{ and } i \text{ odd.} \end{cases}$$

where

$$c_m = \begin{cases} c_{l-1}, & c_l = 0 \\ c_l, & \text{otherwise.} \end{cases}$$

On the other hand, assuming that $c = (c_1, c_2, \ldots, c_m) \in R_1(n)$. When inverting the above transformation, we set

$$l = \begin{cases} m+1, \text{and } \lambda_l := 1, & \text{if } m \text{ is odd} \\ m, & \text{if } m \text{ is even.} \end{cases}$$

Thus (4.8) is proved.

For (4.9), we have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in A_{1,2,4}^{1,1}(4n+1)$. Set

$$c_i = \frac{\lambda_i - 2(l-i) - 1}{4} + \frac{l-1}{2} - \begin{cases} 0, & i = 1, 2 \\ \frac{i-2}{2}, & i \geq 4 \text{ and } i \text{ even} \\ \frac{i-1}{2}, & i \geq 3 \text{ and } i \text{ odd.} \end{cases}$$

where

$$c_m = \begin{cases} c_{l-1}, & c_l = 0 \\ c_l, & \text{otherwise.} \end{cases}$$

To invert the map, we set

$$l = \begin{cases} m+1, \text{and } \lambda_l := 1, & \text{if } m \text{ is even} \\ m, & \text{if } m \text{ is odd.} \end{cases}$$

By similar reasoning we obtain, respectively, for (4.10) and (4.11):

$$c_i = \frac{\lambda_i - 2(l-i) - 3}{4} + \frac{l}{2} - \begin{cases} 0, & i = 1, 2 \\ \frac{i-1}{2}, & i \geq 3 \text{ and } i \text{ odd} \\ \frac{i-2}{2}, & i \geq 4 \text{ and } i \text{ even.} \end{cases}$$

and

$$c_i = \frac{\lambda_i - 2(l-i) - 3}{4} + \frac{l-1}{2} - \begin{cases} 0, & i = 1, 2, 3 \\ \frac{i-2}{2}, & i \geq 4 \text{ and } i \text{ even} \\ \frac{i-3}{2}, & i \geq 5 \text{ and } i \text{ odd.} \end{cases}$$

□

Lastly, we remark that the relations (4.10) and (4.11) may be generalized as follows.

Theorem 4.2. *Let $v, p \geq 1$ and $w \geq 0$ be integers. Then for $n \geq 0$,*

$$|A^{w,p}_{v(1+2p),2v,4vp}(4vpn + 2pwv + vw^2)| = |U_{2p}(n, w)|.$$

Note that $r_4(n) = u_2(n, 1)$ and $r_3(n) = u_2(n, 0)$.
Consider $n = 7, p = 2, w = 1, v = 1$. Now $U_4(n, 1)$ has partitions of n of the form $c_1 + c_2 + c_3 + \ldots + c_{4m+1}$ such that

$$c_1 \geq c_2 \geq c_3 \geq c_4 \geq c_5 > c_6 \geq c_7 \geq c_8 \geq c_9 > c_{10} \geq \ldots, c_{4m+1} > 0.$$

Partitions in $U_4(7, 1)$ are:

$$7,\ 3+1+1+1+1,\ 2+2+1+1+1$$

and those in $A^{1,2}_{5,2,8}(61)$ are:

$$61,\ 29+11+9+7+5,\ 21+19+9+7+5.$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in A^{w,p}_{v(1+2p),2v,4vp}(4vpn + 2pwv + vw^2)$. Note that $l \equiv w \bmod 2p$. Set

$$x_i = \frac{\lambda_i - 2v(l-i) - v(1+2p)}{4vp} + \frac{l-w}{2p}.$$

Let $c = (c_1, c_2, \ldots, c_l)$ be defined as

$$c_i = x_i - \begin{cases} 0, & i = 1, 2, \ldots, w+2p \\ \frac{i-w-1}{2p}, & i \geq w+1+2p,\ i \equiv\ w+1 \bmod 2p \\ \frac{i-w-2}{2p}, & i \geq w+2+2p,\ i \equiv\ w+2 \bmod 2p \\ \frac{i-w-3}{2p}, & i \geq w+3+2p,\ i \equiv\ w+3 \bmod 2p \\ \vdots & \vdots \\ \frac{i-w-2p}{2p}, & i \geq w+4p,\ i \equiv\ w+2p \bmod 2p. \end{cases}$$

By (4.1), $\lambda_i - 2v(l-i) - v(1+2p)$ is divisible by $4vp$, where $t = 4vp$, $k = v(1+2p)$ and $\gamma = 2v$. So $\frac{\lambda_i - 2v(l-i) - v(1+2p)}{4pv} \geq 0$. Now for $i = 1, 2, \ldots, w+2p-1$,

$$c_i - c_{i+1} = \frac{\lambda_i - \lambda_{i+1} - 2v}{4vp} \geq 0 \ \Rightarrow\ c_i \geq c_{i+1}.$$

For $i \equiv w \bmod 2p$, we have

$$c_i - c_{i+1} = \frac{\lambda_i - \lambda_{i+1} - 2v}{4vp} + 1 > 0,$$

i.e. $c_i > c_{i+1}$.
For $i \not\equiv w \bmod 2p$ and $i > w+2p$, we have $c_i - c_{i+1} = \frac{\lambda_i - \lambda_{i+1} - 2v}{4vp} \geq 0$. Thus $c_i \geq c_{i+1}$. Furthermore, note that

$$\begin{aligned} |c| &= \sum_{i=1}^{l} \frac{\lambda_i - 2v(l-i) - v(1+2p)}{4vp} + 2p \sum_{i=1}^{\frac{l-w}{2p}-1} i + 2p\frac{l-w}{2p} + w\frac{l-w}{2p} \\ &= \frac{1}{4vp}\left(\sum_{i=1}^{l} \lambda_i - 2v\sum_{i=1}^{l-1} i - lv(1+2p)\right) + p\left(\frac{l-w}{2p}\right)^2 + p\frac{l-w}{2p} + w\frac{l-w}{2p} \end{aligned}$$

$$= \frac{1}{4vp}(|\lambda| - vl^2 - 2plv) + \frac{(l-w)^2}{4p} + \frac{l-w}{2} + w\frac{l-w}{2p}$$

so that

$$4vp|c| = |\lambda| - 2vpw - vw^2$$

which implies

$$4vp|c| + 2pwv + vw^2 = |\lambda|.$$

Since $|\lambda| = 4vp|c| + 2pwv + vw^2$, it follows that $|c| = n$. Thus $c \in U_{2p}(n, w)$ and the transformation $\lambda \mapsto c$ is invertible.

□

Several combinatorial interpretations of the left-hand sides of (4.4), (4.5), (4.6) and (4.7) are found in the literature (see for instance [1], [4]).

We now explore the link with plane partitions on a square.

4.2 Applications to plane partitions

Let $AB(n, l)$ denote the set of partitions of n into exactly l parts, where first part is even, and all parts alternate in parity. Set $ab(n, l) := |AB(n, l)|$.

Let $ce(n)$ denote the number of partitions of n into exactly 4 parts where parts alternate in parity.

By (4.2), the generating function for $a_1(n, 1, 2, l)$ is

$$\sum_{n=0}^{\infty} a_1(n, 1, 2, l)q^n = \frac{q^{l+l(l-1)/2}}{(q^2; q^2)_l}. \tag{4.12}$$

Observe that $a_1(n, 1, 2, l)$ is the number of partitions of n of exact length l, the first part is odd and parts alternate in parity.
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in A_1(n, 1, 2, l)$. Then $(\lambda_1+1, \lambda_2+1, \ldots, \lambda_l+1) \in AB(n+l, l)$.
On the other hand, if $(\pi_1, \pi_2, \ldots, \pi_l) \in AB(n, l)$, then
$(\pi_1 - 1, \pi_2 - 1, \ldots, \pi_l - 1) \in A_1(n - l, 1, 2, l)$. Hence,

$$a_1(n - \ell, 1, 2, \ell) = ab(n, l).$$

So we have

$$\begin{aligned}
\sum_{n=0}^{\infty} ab(n,l)q^n &= \sum_{n=0}^{\infty} a_1(n-l,1,2,l)q^n \\
&= q^l \sum_{n=0}^{\infty} a_1(n,1,2,l)q^n \\
&= q^l \cdot \frac{q^{l+l(l-1)/2}}{(q^2;q^2)_l} \\
&= \frac{q^{l(l+3)/2}}{(q^2;q^2)_l}.
\end{aligned}$$

Therefore, $ab(n,4)$ is generated by

$$\sum_{n=0}^{\infty} ab(n,4)q^n = \frac{q^{14}}{(q^2;q^2)_4}.$$

But $ce(n) = ab(n,4) + a_1(n,1,2,4)$ so that

$$\begin{aligned}
\sum_{n=0}^{\infty} ce(n)q^n &= \sum_{n=0}^{\infty} ab(n,4)q^n + \sum_{n=0}^{\infty} a_1(n,1,2,4)q^n \\
&= \frac{q^{14}}{(q^2;q^2)_4} + \frac{q^{10}}{(q^2;q^2)_4} \\
&= \frac{q^{14}+q^{10}}{(q^2;q^2)_4},
\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} ce(2n)q^n = \frac{q^5(1+q^2)}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} ce(2n+10)q^n &= \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\
&= \frac{1}{(1-q)(1-q^2)^2(1-q^3)}.
\end{aligned}$$

But $\frac{1}{(1-q)(1-q^2)^2(1-q^3)}$ is the generating function of plane partitions on a square [7]. Refer to Definition 2.4 for plane partition a square. Thus we have the following theorem.

Theorem 4.3. *Let $ap(n)$ be the number of plane partitions of n on a square. Then for $n \geq 0$, we have*

$$ap(n) = ce(2n+10)$$

where $ap(0) := 1$.

We construct an explicit bijection as follows;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} (2a+4, 2b+3, 2c+2, 2d+1), & b \geq c \\ (2a+3, 2c+2, 2b+3, 2d+2), & b < c. \end{cases}$$

If we impose a restriction on $ap(n)$, counting only those plane partitions that have positive parts, we get a stronger version as follows.

Theorem 4.4. *Let $\bar{ap}(n)$ be the number of plane partitions on a square such that the parts are strictly positive. Then for all $n \geq 4$,*

$$\bar{ap}(n) = ce(2n+2).$$

The corresponding bijection for Theorem 4.4, which we construct, is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} (2a+2, 2b+1, 2c, 2d-1), & b \geq c \\ (2a+1, 2c, 2b+1, 2d) & b < c. \end{cases}$$

4.3 Agarwal and Andrews case

Andrews [12] gave an interpretation of (2.17) and (2.18). The left-hand sides of (2.17) and (2.18) are the generating functions of $w_1(n)$ and $w_3(n)$, respectively. Andrews defined these functions as $w_i(n) = |W_i(n)|$ where $i = 1, 3$ and

$W_1(n)$: the number of partitions of n into distinct parts in which the smallest even part is larger than twice the number of odd parts.
$W_3(n)$: the number of partitions of n into distinct parts, each larger than 1 in which each even part is larger than twice the number of odd parts.

Denote by $\tilde{B}(n)$ the set of partitions of n into distinct even parts, and $\tilde{b}(n) := |\tilde{B}(n)|$. Maintaining the notation introduced in Theorem 3.1, the following proposition is clear and we give an explicit bijection.

Proposition 4.1. *For $k = 1, 3$,*

$$w_k(n) = \sum_{j=0}^{n} a_k(j)\tilde{b}(n-j).$$

Bijection

Define a map $W_k(n) \to \cup_{j=0}^{n}(\tilde{B}(n-j) \times A_k(j))$ by $\lambda \mapsto (\tilde{u}, \tilde{v})$ via the algorithm

1. For $\lambda \in W_k(n)$, let u be the subpartition of λ comprising of all even parts of λ, and v the subpartition comprising of all odd parts.

2. Recall the notation $v = (v_1, v_2, \ldots, v_{\#v})$. If $v_i - v_{i+1} \equiv 0 \pmod 4$, then delete 2 from v_1 to v_i, and add the part $2i$ to u. Here, we define $v_{\#v+1} = -k$ for convenience.

3. The resulting partitions $\tilde{u}$ and $\tilde{v}$ are such that $\tilde{u} \in \tilde{B}(n-j)$ and $\tilde{v} \in A_k(j)$ for some $0 \leq j \leq n$.

Example

$\lambda = (18, 15, 14, 12, 11, 5) \in W_3(75)$. Then
Step 1: $u = (18, 14, 12)$, $v = (15, 11, 5)$.
Step 2: $v_1 - v_2 = 15 - 11 \equiv 0 \pmod 4$, so we delete 2 from v_1 and add 2 to u. So we update u and v. At this level, $u = (18, 14, 12, 2)$ and $v = (13, 11, 5)$.
Now $v_2 - v_3 = 11 - 5 = 6 \not\equiv 0 \pmod 4$. So we do nothing, and thus $u = (18, 14, 12)$, $v = (13, 11, 5)$.
Set $v_4 = -k = -3$, $v_3 - v_4 = 8 \equiv 0 \pmod 4$. We delete 2 from v_1 to v_3, add $2(3) = 6$ to u. So $v = (11, 9, 3)$ and $u = (18, 14, 12, 6, 2)$.
Step 3: $\tilde{u} = (18, 14, 12, 6, 2)$ and $\tilde{v} = (11, 9, 3)$.

The inverse of the bijection

Given $(\tilde{u}, \tilde{v})$ such that $\tilde{u} \in \tilde{B}(n-j)$ and $\tilde{v} \in A_k(j)$ for some $0 \leq j \leq n$. To find $\lambda \in W_k(n)$, we perform the following operation.

1. For every $\tilde{u}_j$, $j \geq 1$, such that $\tilde{u}_j \leq 2 \times$ (the number of odd parts in $\tilde{v}$), add the partition $\langle 2^{\tilde{u}_j/2} \rangle$ to $\tilde{v}$ and delete the part $\tilde{u}_j$ from $\tilde{u}$.

2. Then λ is the union of the two resulting partitions in (1). Note that $\lambda \in W_k(n)$.

For instance, in the previous example $\tilde{u} = (18, 14, 12, 6, 2)$ and $\tilde{v} = (11, 9, 3)$. Note that 6,2 are the parts in $\tilde{u}$ satisfying the condition in (1). So for 2, $\tilde{v}$ updates to $\tilde{v} = (11, 9, 3) + (2) = (13, 9, 3)$. For 6, the partition updates to $(13, 9, 3) + (2, 2, 2) = (15, 11, 5)$. Similarly, $\tilde{u}$ updates to $(18, 14, 12)$
Applying step 2: $\lambda = (15, 11, 3) \cup (18, 14, 12) = (18, 15, 14, 12, 11, 5)$.

4.4 Related new partition identities

We continue with related combinatorial consequences.

Let $t(n)$ be the number of partitions of n in which

(i) odd parts are distinct and greater than or equal to 3, and each is $\geq$ largest even part + 3

(ii) the largest even part is repeated, and

(iii) all positive even integers less than the largest even part appear as parts.

Then observe that

$$\sum_{n=0}^{\infty} t(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2+4+\ldots+(2n-2)+2n+2n}}{(1-q^2)(1-q^4)\ldots(1-q^{2n})} \prod_{j=0}^{\infty}(1+q^{2n+3+2j}),$$

and also note that

$$\sum_{n=0}^{\infty} w_1(n)q^n = \sum_{n=0}^{\infty} \frac{q^{1+3+\ldots+(2n-1)}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+2+2j}),$$

$$\sum_{n=0}^{\infty} w_3(n)q^n = \sum_{n=0}^{\infty} \frac{q^{3+5+\ldots+(2n+1)}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+2+2j}).$$

Let $w_4(n)$ be the number of partitions in $W_1(n)$ and having 1 as a part and $RR(n)$ be the set of partitions of n in which parts differ by at least 2 and the smallest part is greater than 2. Set $rr(n) := |RR(n)|$. Then we have the following theorem.

Theorem 4.5. *For all n, we have*

$$w_4(n+1) = t(n) = rr(n).$$

Proof. Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} w_4(n)q^n &= \sum_{n=0}^{\infty} w_1(n)q^n - \sum_{n=0}^{\infty} w_3(n)q^n \\
&= \sum_{n=0}^{\infty} \frac{q^{1+3+\ldots+2n-1}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+2+2j}) - \sum_{n=0}^{\infty} \frac{q^{3+5+\ldots+2n+1}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+2+2j}) \\
&= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}(-q^{2n+2};q^2)_\infty - \sum_{n=1}^{\infty} \frac{q^{(n+1)^2-1}}{(q^2;q^2)_n}(-q^{2n+2};q^2)_\infty \\
&= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}(1-q^{2n})(-q^{2n+2};q^2)_\infty \\
&= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_{n-1}}(-q^{2n+2};q^2)_\infty \\
&= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^2;q^2)_n}(-q^{2n+4};q^2)_\infty \\
&= \sum_{n=0}^{\infty} \frac{q^{(n^2+2n+1)}}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{q^{m(m+1)+2mn+2m}}{(q^2;q^2)_m} \quad \text{by (2.8)} \\
&= q\sum_{m=0}^{\infty} \frac{q^{m(m+1)+2m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{n^2+2n+2mn}}{(q^2;q^2)_n} \\
&= q\sum_{m=0}^{\infty} \frac{q^{m(m+1)+2m}}{(q^2;q^2)_m} \prod_{j=0}^{\infty}(1+q^{2m+3+2j}) \quad \text{by (2.8).}
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} w_4(n+1)q^n = \sum_{n=0}^{\infty} t(n)q^n.$$

Furthermore, note that $w_1(n+1) - w_3(n+1) = c(n+1) - e(n+1)$. However, $c(n+1) - e(n+1)$ enumerates partitions λ of $n+1$, which have 1 as a part and the difference between parts is at least 2. Deleting the part 1 from λ yields a partition in $RR(n)$, and the transformation is invertible. Hence the result follows. $\square$

For a partition with even length, i.e., $\pi = (\pi_1, \pi_2, \ldots, \pi_{2m})$, by 'the central parts' we shall mean the parts π_m and π_{m+1} .

Theorem 4.6. *For all n even,*

$$w_4(n) = \sum_{j=0}^{\lfloor \frac{n}{4} \rfloor} \kappa(4j)\tilde{b}(n-4j)$$

where $\kappa(n)$ denotes the number of partitions in $A_1(n)$ with the restriction that the central parts differ by at most 4.

Proof. For $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_l$ in $A_k(n)\,(k = 1, 3)$, the parts can be constructed in the following manner:

$$\lambda_l = 4n_l + k,$$

$$\lambda_{l-1} = \lambda_l + 4n_{l-1} + 2 = 4(n_l + n_{l-1}) + k + 2,$$

$$\lambda_{l-2} = 4(n_l + n_{l-1} + n_{l-2}) + k + 4,$$

and generally for any $j = 1, 2, \ldots, l$,

$$\lambda_j = 4(n_j + n_{j+1} + \ldots + n_l) + k + 2(l - j)$$

for some non-negative integers $n_1, n_2, \ldots, n_l$.
Assume n is even. Note that $n = \sum_{i=1}^{l} \lambda_i \equiv l^2 - l + kl \pmod 4$. The length l cannot be odd, otherwise it will contradict the parity of n. So l must be even implying that $n \equiv 0 \pmod 4$. Hence

$$a_k(4j + 2) = 0 \text{ for all}\, j \geq 0. \tag{4.13}$$

Assume n is odd. If $k = 1$, then $n \equiv l^2 \pmod 4$ which implies that l cannot be even. However, for l odd, we have $n \equiv 1 \pmod 4$. On the other hand, if $k = 3$, we observe that $n \equiv l^2 + 2l \equiv 3 \pmod 4$. Hence we have

$$a_k(j) = 0 \text{ for all}\, j \equiv 4 - k \pmod 4. \tag{4.14}$$

Let $S_i = \{j | j \equiv i \pmod 4\} \subseteq \{0, 1, 2, \ldots, n\}$. For n even, we have

$$\begin{aligned}
w_4(n) &= w_1(n) - w_3(n) \\
&= \sum_{r=0}^{n} (a_1(r) - a_3(r))\tilde{b}(n - r) \text{ by by Proposition 4.1} \\
&= \sum_{r \in S_0} (a_1(r) - a_3(r))\tilde{b}(n - r) + \sum_{r \in S_1} (a_1(r) - a_3(r))\tilde{b}(n - r) \\
&\qquad + \sum_{r \in S_2} (a_1(r) - a_3(r))\tilde{b}(n - r) + \sum_{r \in S_3} (a_1(r) - a_3(r))\tilde{b}(n - r) \\
&= \sum_{r \in S_0} (a_1(r) - a_3(r))\tilde{b}(n - r) + \sum_{r \in S_1} a_1(r)\tilde{b}(n - r) - \sum_{r \in S_3} a_3(r)\tilde{b}(n - r) \\
&\qquad \text{(by (4.13) and (4.14))}
\end{aligned}$$

$$= \sum_{r\in S_0} (a_1(r) - a_3(r))\tilde{b}(n-r) \text{ since } \tilde{b}(n-r) = 0 \text{ for all } r \in S_1 \cup S_3.$$

It remains to show that $\kappa(r) = a_1(r) - a_3(r)$.
Note that all parts of partitions in $A_1(n)$ have even length. We have the set partitioning

$$A_1(n) = A_{1,\le 4}(n) \cup A_{1,>4}(n)$$

where $A_{1,\le 4}$ is the set of all those partitions in $A_1(n)$ with the central parts differing by at most 4 and $A_{1,>4}$ is the set of those partitions in $A_1(n)$ with central parts differing by more than 4.

Now, the map $A_3(n) \to A_{1,>4}(n)$ defined by

$$(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_l) \mapsto (\lambda_1 + 2, \dots, \lambda_{\frac{l}{2}} + 2, \lambda_{\frac{l}{2}+1} - 2, \lambda_{\frac{l}{2}+2} - 2, \dots, \lambda_l - 2)$$

is a bijection. So $a_3(n) = a_{1,>4}(n)$ implies that $\kappa(r) = a_1(r) - a_3(r)$. Hence

$$w_4(n) = \sum_{r\in S_0} \kappa(r)\tilde{b}(n-r),$$

and the identity follows. $\square$

4.5 A bijection for an inequality on partitions related to $\zeta(q)$

Consider partitions in $E(n)$, let $e_i(n)$ denote the number of partitions in $E(n)$ whose largest part is $\equiv i \pmod 2$. Then it turns out that

$$\zeta(q) = \sum_{n=0}^{\infty} (e_0(n) - e_1(n))q^n = 1 + q^2 - q^3 + q^4 - q^5 + 2q^6 + \dots,$$

see [32]. The following theorem follows, and we give a bijective proof.

Theorem 4.7. *For $n \ge 2$,*

$$e_0(n) - e_1(n) > 0 \textit{ for every } n \equiv 0 \pmod 2, \tag{4.15}$$

$$e_0(n) - e_1(n) < 0 \textit{ for every } n \equiv 1 \pmod 2. \tag{4.16}$$

Proof. Let $\mathcal{E}_i(n)$ denote the set of partitions in $E(n)$ with the largest part is $\equiv i \pmod 2$. Thus $e_i(n) = |\mathcal{E}_i(n)|$.
Assume n is even. Define the following sets:

$$\begin{aligned}
K_1 &= \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathcal{E}_1(n) : \exists\, j \geq 2\,, \lambda_j \text{ is odd}, \lambda_{j+1} \text{ is even } \lambda_i \text{ is odd} \\
&\quad \forall i = 2, \ldots, j\} \\
K_2 &= \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathcal{E}_1(n) : \exists\, j \geq 2\,, \lambda_j \text{ is even}, \\
&\quad \lambda_{j+1} \text{ is odd } \lambda_i \text{ is even } \forall i = 2, \ldots, j\} \\
K_3 &= \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathcal{E}_1(n) : \lambda_j \text{ is odd} \,\forall j = 2, \ldots, l\}
\end{aligned}$$

Note that

$$K_i \cap K_j = \emptyset \text{ for } i \neq j, \text{ and } \mathcal{E}_1(n) = \bigcup_{i=1}^{3} K_i.$$

$$J_0 = \{(\lambda_1, \lambda_2) \in \mathcal{E}_0(n) : \lambda_1 - \lambda_2 \geq 4, \lambda_2 \text{ is even}\},$$

$$\begin{aligned}
J_1 &= \{(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_l) \in \mathcal{E}_0(n) : \lambda_2, \lambda_3 \text{ are both even}, \lambda_1 - \lambda_2 \geq 4\}, \\
J_2 &= \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathcal{E}_0(n) : \exists\, j \geq 3\,, \lambda_j, \lambda_{j+1} \text{ are both even}, \lambda_i \text{ is odd} \\
&\quad \forall i = 2, \ldots, j-1, \lambda_{j-1} - \lambda_j \geq 3, \lambda_1 - \lambda_2 \geq 3\}, \\
J_3 &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{E}_0(n) : \lambda_2, \lambda_3 \text{ are both odd}, \lambda_1 - \lambda_2 \geq 4\}, \\
J_4 &= \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathcal{E}_0(n) : \exists\, j \geq 3\,, \lambda_j, \lambda_{j+1} \text{ are both odd}, \lambda_i \text{ is even} \\
&\quad \forall i = 2, \ldots, j-1\,, \lambda_1 - \lambda_2 \geq 4, \lambda_{j-1} - \lambda_j \geq 3\}, \\
J_5 &= \{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathcal{E}_0(n) : l \geq 4, \lambda_l \text{ is even}, \lambda_j \text{ is odd} \,\forall j = 2, \ldots, l-1\}.
\end{aligned}$$

Observe that

$$J_i \cap J_k = \emptyset \text{ for } i \neq k, \text{ and } \bigcup_{i=0}^{5} J_i \subset \mathcal{E}_0(n).$$

The proper containment is justified since the 1-length partition (n) is in $\mathcal{E}_0(n)$ but not in any of the J_i's.

Define $\beta(\lambda)$ as follows:
If $\lambda \in K_1$, $\beta(\lambda)$ is the smallest integer $j \geq 2$ such that λ_{j+1} is even.
If $\lambda \in K_2$, $\beta(\lambda)$ is the smallest integer $j \geq 2$ such that λ_{j+1} is odd.

Then the map $\mathcal{E}_1(n) \to \bigcup_{i=0}^{5} J_i$ defined by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \mapsto \begin{cases} (\lambda_1 + 1, \lambda_2, \ldots, \lambda_{l-1}, \lambda_l - 1), & \lambda \in K_3 \\ (\lambda_1 + 1, \lambda_2, \ldots, \lambda_{\beta(\lambda)} - 1, \lambda_{\beta(\lambda)+1}, \ldots, \lambda_l), & \lambda \in K_1 \cup K_2, \end{cases}$$

is a bijection; and under this map, K_i's have their images as follows:

$$K_1 \to J_1 \cup J_2,$$

$$K_2 \to J_3 \cup J_4,$$

$$K_3 \to J_0 \cup J_5.$$

The inverse is defined by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \mapsto \begin{cases} (\lambda_1 - 1, \lambda_2, \ldots, \lambda_{l-1}, \lambda_l + 1), & \lambda \in J_0 \cup J_5 \\ (\lambda_1 - 1, \lambda_2, \ldots, \lambda_{t(\lambda)} + 1, \lambda_{t(\lambda)+1}, \ldots, \lambda_\ell), & \lambda \in J_1 \cup J_2 \cup J_3 \cup J_4, \end{cases}$$

where $t(\lambda)$ is computed as follows:
If $\lambda \in J_2$, $t(\lambda)$ is the smallest integer $j \geq 3$ such that λ_j and λ_{j+1} are both even.
If $\lambda \in J_4$, $t(\lambda)$ is the smallest integer $j \geq 3$ such that λ_j and λ_{j+1} are both odd.
If $\lambda \in J_1 \cup J_3$, then $t(\lambda) = 2$.

Since $|\bigcup_{i=0}^{5} J_i| < e_0(n)$, (4.15) follows.

Assume n is odd. For $i = 1, 2, 3$, define K_i' to be exactly K_i, except that the largest part is even. Similarly, for $i = 0, 1, 2, \ldots, 5$, let J_i' be exactly J_i, except that the largest part is odd. Observe that $|\bigcup_{i=0}^{5} J_i'| < e_1(n)$ since the 1-length partition $(n) \in \mathcal{E}_1(n)$ and
$(n) \notin \bigcup_{i=0}^{5} J_i'$.
Then the above map, where we replace $\mathcal{E}_1(n)$ with $\mathcal{E}_0(n)$, J_i with J_i' and K_i with K_i', i.e., $\mathcal{E}_0(n) \to \bigcup_{i=0}^{5} J_i'$ defines a bijection that proves (4.16). $\square$

<u>Example</u>

Consider $n = 24$. Then we have $\mathcal{E}_1(n)$-partitions:
(21,3), (19,5), (17,5,2), (15,9), (15,7,2), (15,6,3), (13,11), (13,9,2), (13,8,2), (13,7,4), (11,9,4), (11,8,5), (11,7,4,2), (9,7,5,3) and (17,7).
Thus $e_1(n) = 15$. Observe that

$$K_1 = \{(17, 5, 2),\ (15, 7, 2),\ (13, 9, 2),\ (13, 7, 4),\ (11, 9, 4),\ (11, 7, 4, 2)\}.$$

$$K_2 = \{(15, 6, 3),\ (13, 8, 3),\ (11, 8, 5)\}.$$

$$K_3 = \{(13,11),\ (19,5),\ (17,7),\ (21,3),\ (15,9),\ (9,7,5,3)\}.$$

Indeed $\bigcup_{i=1}^{3} K_i = \mathcal{E}_1(n)$ and the K_i's are mutually disjoint.
Furthermore, note that $\mathcal{E}_0(n)$-partitions are:
(24), (22,2), (20,4), (18,6), (18,4,2), (16,8), (16,6,2), (16,5,3), (14,10), (14,8,2), (14,7,3), (14,6,4), (12,10,2), (12,9,3), (12,8,4), (12,7,5), (12,6,4,2), (10,8,6), (10,8,4,2) and (10,7,5,2).
Indeed $e_0(n) = 20 > e_1(n)$.

$$J_0 = \{(22,2),\ (20,4),\ (18,6)\ (16,8),\ (14,10)\}.$$

$$J_1 = \{(18,4,2),(16,6,2)\ (14,8,2),\ (14,6,4),(12,8,4),(12,6,4,2)\}.$$

$$J_2 = \emptyset, \quad J_3 = \{(16,5,3),\ (14,7,3),\ (12,7,5)\}.$$

$$J_4 = \emptyset, \quad J_5 = \{(10,7,5,2)\}.$$

It is clear that the J_i's are mutually disjoint.
Applying the map on the K_i's, we have the following images. We shall write $\mathrm{Im}(D)$ to denote the image of the set D under the map. We note that
$\mathrm{Im}(K_1) = \{(18,4,2),\ (16,6,2),\ (14,8,2),\ (14,6,4),\ (12,8,4),\ (12,6,4,2)\} = J_1 \cup J_2$.
$\mathrm{Im}(K_2) = \{(16,5,3),\ (14,7,3),\ (12,7,5)\} = J_3 \cup J_4$.
$\mathrm{Im}(K_3) = \{(22,2),\ (20,4),\ (18,6),\ (16,8),\ (14,10),\ (10,7,5,2)\} = J_0 \cup J_5$.
The map is verified.

Chapter 5

Partitions with initial repetitions

5.1 Notation

$A_0(n,k)$: the set of partitions of n with initial k-repetitions.
$a_0(n,k) := |A_0(n,k)|$.
$C_0(n,k)$: the set of partitions of n in which no part appears more than $2k-1$ times.
$c_0(n,k) := |C_0(n,k)|$.
$E_0(n,k) = P_0(n,k)$: the set of partitions of n with all parts appearing at most $k-1$ times.
$E_j(n,k)$: the set of partitions with initial k-repetitions in which the largest part that is repeated at least k times actually occurs with multiplicity in $\{kj, kj+1, kj+2, \ldots, kj+k-1\}$, $j \geq 1$.
$P_j(n,k)$: the set of partitions in which parts $< j$ occur at most $k-1$ times, j occurs at least k but at most $2k-1$ times, and parts $> j$ occur at most $2k-1$ times.
$e_j(n,k) = |E_j(n,k)|$ for all $j \geq 0$.
$p_j(n,k) = |P_j(n,k)|$ for all $j \geq 0$.

5.2 Andrews' theorem and Keith's bijection

In the paper [9], George Andrews introduced partitions with initial repetitions. These partitions are defined as follows.

Definition 5.1. Let k be a positive integer. A partition with initial k-repetitions is a partition in which if j appears at least k times, then all positive integers less than j appear as parts at least k times.

The definition implies that partitions in which no part appears at least k times are trivially considered as partitions with initial k-repetitions.

The following theorem, which relates such partitions to certain partitions with restricted multiplicities of parts, was proved.

Theorem 5.1 ([9, page 2])**.** *For all n,*

$$a_0(n,k) = c_0(n,k).$$

The proof, which uses elementary techniques of q-series, arises from the identity

$$\sum_{n=0}^{\infty} \frac{q^{1+2+3+\ldots+n}}{(q;q)_n} \prod_{j=n+1}^{\infty} (1+q^j+q^{2j}+\ldots+q^{(k-1)j}) = \prod_{n=1}^{\infty} \frac{1-q^{2kn}}{1-q^n}$$

since the left-hand side of the series is the generating function for $a_0(n,k)$, and the right-hand side generates $c_0(n,k)$. In the same paper, several partition theorems were proved including the following theorem.

Theorem 5.2 ([9, page 3])**.** *Let $f_e(n)$ denote the number of partitions of n in which no odd parts are repeated and if an even part $2j$ is repeated then each even positive integer smaller then $2j$ appears in the partition as a repeated part and no odd integers smaller than $2j$ appear. Then $f_e(n)$ is equal to the number of partitions of n into parts $\not\equiv 0, \pm 2 \pmod 7$.*

The proof depends on the identity

$$(-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n}} = \prod_{n \not\equiv 0,\pm 2 \pmod 7}^{\infty} \frac{1}{1-q^n}$$

which is one of the so called Rogers-Selberg identities.

Motivated by Sylvester, who had studied flushed and unflushed partitions [35], Andrews broadened the study of partitions with initial repetitions to partitions with early conditions [8].

As Andrews proved Theorem 5.1 via generating functions, the problem of finding a bijective proof was settled by William Keith [27]. It must be stated that Keith's bijection makes use of modular diagrams and the concept of flat partitions. His bijection is recalled below:

Keith's Algorithm

Let $\lambda \in C_0(n, k)$.

I. Conjugate λ, obtaining λ'.

II. Remove all k-strips from λ', obtaining (π, δ). (δ has distinct parts).

III. Construct $\alpha = \pi + \delta$.

IV. Conjugate α. Thus $\alpha' \in A_0(n, k)$.

In the following section, we present a non-diagram version of Keith's bijection, and refine Theorem 5.1. The original Rogers-Ramanujan identities are treated in the spirit of Theorem 5.2 and some results presented.

5.3 A non-diagram model of Keith's bijection

New version

Let $\lambda \in C_0(n, k)$.

I. Let $v_0 = \langle \lambda_j^k : m_{\lambda_j}(\lambda) \geq k \rangle$. Write $\lambda = v_0 \cup v$, where v is the subpartition of λ obtained by removing v_0 from λ.

II. Conjugate v_0.

III. Compute $u = [k, v_0']$.

IV. Compute $u \cup v$. Thus $u \cup v \in A_0(n, k)$.

Lemma 5.1. *If λ is k-singular and its parts are distinct, then $\lambda = [k, [k, \lambda]']'$.*

Proof. Clearly $\lambda = (kr_1, kr_2, \ldots, kr_s)$ for some $r_1 > r_2 > \ldots > r_s > 0$. Thus we have

$$\begin{aligned}
[k,[k,\lambda]']' &= [k,[k,(kr_1,kr_2,\ldots,kr_s)]']' \\
&= [k,(r_1^k,r_2^k,\ldots,r_s^k)']' \\
&= [k,\big((sk)^{r_s},((s-1)k)^{r_{s-1}-r_s},((s-2)k)^{r_{s-2}-r_{s-1}},\ldots,k^{r_1-r_2}\big)]' \\
&= \big(s^{kr_s},(s-1)^{k(r_{s-1}-r_s)},(s-2)^{k(r_{s-2}-r_{s-1})},\ldots,1^{k(r_1-r_2)}\big)' \\
&= (kr_1,kr_2,\ldots,kr_s) = \lambda.
\end{aligned}$$

□

From Keith's algorithm, note that $\delta = (\delta_1, \delta_2, \ldots, \delta_t)$ where t is the number of k-strips and δ_i is the sum of k-units in the i^{th} strip. So

$$\delta = (kr_1, kr_2, kr_3, \ldots, kr_t) = k(r_1, r_2, r_3, \ldots, r_t)$$

for some $r_1 > r_2 > r_3 > \ldots r_t > 0$. By modular diagram manipulation, one observes that

$$\lambda' = k(r_1, r_2, \ldots, r_t)' + \pi. \tag{5.1}$$

From the new version, we have $\lambda = v_0 \cup v$ which implies that

$$\lambda' = v_0' + v'. \tag{5.2}$$

Equations (5.1) and (5.2) give

$$v_0' + v' = k(r_1, r_2, \ldots, r_t)' + \pi.$$

Note that v' and π are k-flat, and from the uniqueness property (2.2), we must have

$$v' = \pi, \text{ and } v_0' = k(r_1, r_2, \ldots, r_t)'.$$

So $v_0' = k(r_1, r_2, \ldots, r_t)' = \langle (kt)^{r_t}, (k(t-1))^{r_{t-1}-r_t}, \ldots, k^{r_1-r_2} \rangle$ so that $v_0 = \langle r_1^k, r_2^k, \ldots, r_t^k \rangle$. Thus

$$[k, \delta] = v_0 \quad \text{and} \quad v = \pi'. \tag{5.3}$$

Theorem 5.3. *The two algorithms are equivalent.*

Proof. The image of Keith's algorithm α' is such that

$$\begin{aligned}\alpha' &= (\pi+\delta)' \\ &= \pi' \cup \delta' \\ &= \pi' \cup ([k,[k,\delta]']')' \text{ by Lemma 5.1} \\ &= \pi' \cup [k,[k,\delta]'] \\ &= v \cup [k, v_0'] \text{ by (5.3)} \\ &= v \cup u,\end{aligned}$$

which is the image of the new algorithm. □

The inverse of our non-diagram model is given as follows

The inverse

Let $\pi \in A_0(n,k)$ and h be the largest part of π that is repeated at least k times. Let $\psi(h,\pi)$ be the subpartition of π whose parts are all those greater than h. Clearly

$$\pi = \langle j^{m_j(\pi)} : j = 1,2,\ldots,h\rangle \cup \psi(h,\pi).$$

1. For $j = 1,2,\ldots,h$, write $m_j(\pi) = k\alpha_j + r_j$ where $0 \le r_j \le k-1$ and $\alpha_j \ge 1$. Define $\underline{w}$ and $\underline{v}$ as follows

$$\underline{w} = \langle (k\cdot j)^{\alpha_j} : j = 1,2,3,\ldots,h\rangle$$

$$\underline{v} = \langle j^{r_j} : j = 1,2,3,\ldots,h\rangle.$$

2. The image of π is given by

$$\lambda := \underline{w}' \cup \underline{v} \cup \psi(h,\pi) \in C_0(n,k).$$

5.4 Refining Andrews' theorem

In this section, we refine Theorem 5.1. Recall the theorem statement;

$$a_0(n,k) = c_0(n,k) \text{ for all } n \ge 1.$$

Proposition 5.1. *For a non-negative integer j, let $a_j(n,k)$ be the number of partitions of n with initial k-repetitions in which all parts occur at most $k-1$ times or the largest part repeated at least k times actually occurs at least $kj+k$ times. Then $a_j(n,k)$ is equal to the number of partitions of n in which all parts $\leq j$ have multiplicity $\leq k-1$ and all parts $> j$ have multiplicity $\leq 2k-1$.*

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_j(n,k)q^n &= \sum_{n=0}^{\infty} \frac{q^{k(1+2+3+\ldots+n)+kjn}}{(1-q)(1-q^2)\ldots(1-q^n)} \prod_{i=n+1}^{\infty}(1+q^i+q^{2i}+\ldots+q^{(k-1)i}) \\
&= \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(n^2-n)/2+(kj+k)n}}{(q^k;q^k)_n} \\
&= \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \prod_{n=0}^{\infty}(1+q^{kn+kj+k}) \text{ by } (2.8), z = q^{kj+k}, q := q^k \\
&= \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \frac{(-q^k;q^k)_\infty}{(-q^k;q^k)_j} \\
&= \prod_{n=1}^{j} \frac{1}{(1+q^{kn})} \prod_{n=1}^{\infty} \frac{1-q^{2kn}}{1-q^n} \\
&= \prod_{n=1}^{j} \frac{1-q^{kn}}{1-q^{2kn}} \prod_{n=1}^{\infty} \frac{1-q^{2kn}}{1-q^n} \\
&= \prod_{n=1}^{j} \frac{1-q^{kn}}{1-q^n} \prod_{n=j+1}^{\infty} \frac{1-q^{2kn}}{1-q^n} \\
&= \prod_{n=1}^{j}(1+q^n+q^{2n}+\ldots+q^{(k-1)n}) \prod_{n=j+1}^{\infty}(1+q^n+q^{2n}+\ldots+q^{(2k-1)n}).
\end{aligned}$$

□

Note that $a_0(n,k) = |A_0(n,k)|$ enumerates Andrews' partitions with initial k-repetitions. With the proposition above, we are ready to refine Theorem 5.1.

Theorem 5.4. *For all $j \geq 0$, we have $e_j(n,k) = p_j(n,k)$, and this is a refinement of Theorem 5.1.*

Proof. For $j \geq 1$, $e_j(n,k) = a_{j-1}(n,k) - a_j(n,k)$ so that the generating function for

$e_j(n,k)$ is

$$\begin{aligned}\sum_{n=0}^{\infty}(a_{j-1}(n,k)-a_j(n,k))q^n &= \left(\prod_{n=1}^{j-1}\frac{1}{1+q^{kn}}-\prod_{n=1}^{j}\frac{1}{1+q^{kn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{2kn}}{1-q^n}\\ &= \left(1-\frac{1}{1+q^{kj}}\right)\prod_{n=1}^{j-1}\frac{1}{1+q^{kn}}\prod_{n=1}^{\infty}\frac{1-q^{2kn}}{1-q^n}\\ &= q^{kj}\prod_{n=1}^{j}\frac{1}{1+q^{kn}}\prod_{n=1}^{\infty}\frac{1-q^{2kn}}{1-q^n}\end{aligned}$$

which is equal to

$$q^{\overbrace{j+j+\ldots+j}^{k}}\prod_{n=1}^{j}(1+q^n+q^{2n}+\ldots+q^{(k-1)n})\prod_{n=j+1}^{\infty}(1+q^n+q^{2n}+\ldots+q^{(2k-1)n}),$$

the generating function for $p_j(n,k)$.
Furthermore, observe that for fixed k,

$$\bigcup_{j=0} E_j(n,k)=A_0(n,k) \text{ and } E_j(n,k)\cap E_i(n,k)=\emptyset \text{ for } i\neq j \tag{5.4}$$

and

$$\bigcup_{j=0} P_j(n,k)=C_0(n,k) \text{ and } P_j(n,k)\cap P_i(n,k)=\emptyset \text{ for } i\neq j. \tag{5.5}$$

Thus the theorem is indeed a refinement of Theorem 5.1. □

<u>Example</u>

We consider the set $A_0(10,3)$. Elements of this set are as follows.

$(10),(9,1),(8,2),(8,1,1),(7,3),(7,2,1),(6,4),(6,3,1),(6,2,2),(6,2,1,1),(5,5),$

$(5,4,1),(5,3,2),(5,3,1,1),(5,2,2,1),(4,4,2),(4,4,1,1),(4,3,3),(4,3,2,1),$

$(4,2,2,1,1),(3,3,2,2),(3,3,2,1,1),(3,3,1,1,1,1),(6,1,1,1,1),(7,1,1,1),$

$(2,1,1,1,1,1,1,1,1),(2,2,1,1,1,1,1,1),(2,2,2,1,1,1,1),(3,1,1,1,1,1,1,1),$

$(3,2,1,1,1,1,1),(3,2,2,1,1,1),(4,1,1,1,1,1,1),(4,2,1,1,1,1),(4,3,1,1,1),$

$$(5,1,1,1,1,1),(5,2,1,1,1),(1,1,1,1,1,1,1,1,1,1).$$

Similary, the set $C_0(10,3)$ has the elements:

$$(10),(9,1),(8,2),(8,1,1),(7,3),(7,2,1),(7,1,1,1),(6,4),(6,3,1),(6,2,2),$$

$$(6,2,1,1),(6,1,1,1,1),(5,5),(5,4,1),(5,3,2),(5,3,1,1),(5,2,2,1),$$

$$(5,2,1,1,1),(5,1,1,1,1,1),(4,4,2),(4,4,1,1),(4,3,3),(4,3,2,1),(4,3,1,1,1),$$

$$(4,2,2,2),(4,2,2,1,1),(4,2,1,1,1,1),(3,3,3,1),(3,3,2,2),(3,3,2,1,1),$$

$$(3,3,1,1,1,1),(3,2,2,2,1),(3,2,2,1,1,1),(3,2,1,1,1,1,1),(2,2,2,2,2),$$

$$(2,2,2,2,1,1),(2,2,2,1,1,1,1).$$

We now demonstrate the refinement on $|A_0(10,3)| = |C_0(10,3)| = 37$.

$$\underline{j=0}$$

$E_0(10,3) = P_0(10,3)$-partitions:

$$(10),(9,1),(8,2),(8,1,1),(7,3),(7,2,1),(6,4),(6,3,1),(6,2,2),(6,2,1,1),(5,5),$$

$$(5,4,1),(5,3,2),(5,3,1,1),(5,2,2,1),(4,4,2),(4,4,1,1),(4,3,3),(4,3,2,1),$$

$$(4,2,2,1,1),(3,3,2,2),(3,3,2,1,1)$$

Thus $e_0(10,3) = p_0(10,3) = 22$.

$$\underline{j=1}$$

$E_1(10,3)$-partitions:

$$(6,1,1,1,1),(7,1,1,1),(3,3,1,1,1,1),(2,2,2,1,1,1,1),(3,2,2,1,1,1),$$

$$(4,2,1,1,1,1),(4,3,1,1,1),(5,1,1,1,1,1),(5,2,1,1,1),(3,2,1,1,1,1,1).$$

$P_1(10,3)$-partitions:

$$(6,1,1,1,1),(7,1,1,1),(3,3,1,1,1,1),(2,2,2,1,1,1,1),(3,2,2,1,1,1),$$

$$(4,2,1,1,1,1),(4,3,1,1,1),(5,1,1,1,1,1),(5,2,1,1,1),(3,2,1,1,1,1,1).$$

We have $e_1(10,3) = |E_1(10,3)| = |P_1(10,3)| = p_1(10,3) = 10$.

$\underline{j = 2}$

$E_2(10,3)$-partitions:

$$(2,1,1,1,1,1,1,1,1),(2,2,1,1,1,1,1,1),(3,1,1,1,1,1,1,1),(4,1,1,1,1,1,1).$$

$P_2(10,3)$-partitions:

$$(4,2,2,2),(3,2,2,2,1),(2,2,2,2,2),(2,2,2,2,1,1).$$

We have $e_2(10,3) = p_2(10,3) = 4$.

$\underline{j = 3}$

$E_3(10,3)$-partitions:

$$(1,1,1,1,1,1,1,1,1,1).$$

$P_3(10,3)$-partitions:

$$(3,3,3,1).$$

We have $e_3(10,3) = p_3(10,3) = 1$.

Clearly, (5.4) and (5.5) are satisfied.

Combinatorial Proof of Theorem 5.4

The new (non-diagram) version of Keith's bijection given in previous the section makes the situation more explicit. We now illustrate how this is done with the theorem.

Proof. Let $\lambda \in P_j(n,k)$. Then after applying step I, we have

$$v_0 = \lambda_1^k \lambda_2^k \ldots \lambda_m^k j^k$$

$$v = j^r \cup w$$

for some $\lambda_1 > \lambda_2 > \ldots > \lambda_m$ with $\lambda_m > j$, $0 \le r \le k-1$, and some partition w whose parts have multiplicity $\le k-1$.

Step II: $v_0' = [(m+1)k]^j (mk)^{\lambda_m - j}[(m-1)k]^{\lambda_{m-1}-\lambda_m} \ldots (2k)^{\lambda_2-\lambda_3} k^{\lambda_1-\lambda_2}$.

Step III: $u = [k, v_0'] = (m+1)^{kj} m^{k(\lambda_m - j)} \ldots 2^{k(\lambda_2-\lambda_3)} 1^{k(\lambda_1-\lambda_2)}$.

Step IV: $u \cup v = (m+1)^{kj} m^{k(\lambda_m - j)} \ldots 2^{k(\lambda_2-\lambda_3)} 1^{k(\lambda_1-\lambda_2)} \cup j^r \cup w$.

Since $j^r \cup w$ has all its parts with multiplicity $\leq k-1$, note that the largest part repeated at least k times is $m+1$ and has multiplicity $kj+q$ with $0 \leq q \leq k-1$. So $u \cup v \in E_j(n,k)$.
Conversely, let $\mu \in E_j(n,k)$. After applying the inverse algorithm and careful analysis, it is not difficult to see that the resulting partition is in $P_j(n,k)$. $\square$

5.5 On the original Rogers-Ramanujan identities

Andrews [8] pointed out that the original Rogers-Ramanujan identities can be treated using his approach. Indeed notice this in the equations (2.17) and (2.18). In any case, in light of Andrews' approach [8], we observe that odd parts are gap free. It is possible to switch to a setting where smaller parts are even and gap free. This may be more fruitful.

Theorem 5.5. *Let $r = 1, 3$ and $cc_1(n,r)$ denote the number of partitions of n in which even parts appear without gaps, odd parts are distinct and the smallest odd part is at least r + largest even part. Then $cc_1(n,r)$ is equal to the number of partitions of n with parts $\equiv \pm r \pmod 5$.*

Proof.

$$\begin{aligned}
\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} &= \sum_{m=0}^{\infty} \frac{q^{3+5+\ldots+2m+1}}{(q^2;q^2)_m} \prod_{j=0}^{\infty}(1+q^{2m+2+2j}) \text{ by (2.17)} \\
&= \sum_{m=0}^{\infty} \frac{q^{m^2+2m}}{(q^2;q^2)_m} \prod_{j=0}^{\infty}(1+q^{2m+2+2j}) \\
&= \sum_{m=0}^{\infty} \frac{q^{m(m+1)+m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{n(n+1)+2nm}}{(q^2;q^2)_n}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{q^{(2n+1)m+m(m+1)}}{(q^2;q^2)_m} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+3+2j}).
\end{aligned}
$$

By a similar manipulation and using (2.18), we have

$$
\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \prod_{j=0}^{\infty}(1+q^{2n+1+2j}).
$$

□

If we disregard the condition for distinct odd parts in partitions in Theorem 5.5, then we end up with the following

Theorem 5.6. *Let $r = 1, 3$ and $cc_2(n, r)$ denote the number of partitions of n in which even parts appear without gaps and the smallest odd part is at least r + the largest even part. Then $cc_2(n, r)$ is equal to the number of partitions of n with parts $\equiv r, 2 \pmod 4$.*

Proof.

$$
\begin{aligned}
\sum_{n=0}^{\infty} cc_2(n,r)q^n &= \frac{1}{\prod_{j=0}^{\infty}(1-q^{2j+r})} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{1}{\prod_{j=n}^{\infty}(1-q^{2j+r})} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{1}{\prod_{j=n}^{\infty}(1-q^{2j+r})} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{1}{(q^{2n+r};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{(q^r;q^2)_n}{(q;q^2)_\infty} \\
&= \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} (q^r;q^2)_n \\
&= \frac{1}{(q;q^2)_\infty} \prod_{n=1}^{\infty}(1-q^{4n-r})(1+q^{2n}) \text{ by (2.7)} \\
&= \frac{\prod_{n=1}^{\infty}(1-q^{4n-r})(1+q^{2n})(1-q^{2n})}{(q;q^2)_\infty (q^2;q^2)_\infty}
\end{aligned}
$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{4n-r})(1-q^{4n})}{1-q^n}$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{4n+r})(1-q^{4n+2})}.$$

$\square$

Bijection

Let μ be a partition of n in which even parts appear without gaps and the smallest odd part is at least r + the largest even part.

1. Conjugate μ, obtaining μ'.

2. If μ' has no part with odd multiplicy, set $\bar{\alpha} := \mu'$ and go to step 4. Otherwise, decompose $\mu' = (\alpha, \beta)$ where

   β is the subpartition of μ' whose largest part has odd multiplicity and α is the subpartition of μ' whose parts are all the parts (counting multiplicities) in μ' that are greater than the largest part of β. Observe that $\alpha = \mu' \backslash \beta$, and it may be empty. Recall that β can be written as $\beta = \langle \beta_1^{m_{\beta_1}(\beta)} \beta_2^{m_{\beta_2}(\beta)} \ldots \beta_m^{m_{\beta_m}(\beta)} \rangle$, $\beta_1 > \beta_2 > \ldots > \beta_m$. We use this notation of β in the next step.

3. a. If the multiplicity of β_1 (the largest part of β) is not congruent to r modulo 4, then update α and β by removing the subpartition $\langle \beta_1^2 \rangle$ from β and 'adding' it to α by taking the union. In other words, α and β become
$$\beta := \beta \setminus \langle \beta_1^2 \rangle \quad , \alpha := \alpha \cup \langle \beta_1^2 \rangle.$$

   b. If the multiplicity of β_j for $j = 2, 3, \ldots, m$ is not congruent to 0 modulo 4, then update α and β by removing the subpartition $\langle \beta_j^2 \rangle$ from β and 'adding' it to α by taking the union. In other words, α and β become
$$\beta := \beta \setminus \langle \beta_j^2 \rangle \ , \alpha := \alpha \cup \langle \beta_j^2 \rangle.$$

   Now call the new updated α and β, $\bar{\alpha}$ and $\bar{\beta}$, respectively. Observe that $\mu' = \bar{\alpha} \cup \bar{\beta}$.

4. Work out $[2, \bar{\alpha}]^{-1}$.

5. Compute $c = \phi[2, \bar{\alpha}]^{-1}$ and then $[2, c]^{-1}$. For the map ϕ, see page 20.

6. Find $\lambda = \bar{\beta}' \cup [2, c]^{-1}$.

Then λ is a partition into parts $\equiv r, 2 \pmod 4$.

Before giving the inverse mapping, let us look at an example.

Example

Let $r = 1$, $\mu = \langle 23^2 17^1 11^3 8^1 6^3 4^1 2^4 \rangle$. Then $\mu' = \langle 15^2 11^2 10^2 7^2 6^3 3^6 2^6 \rangle$. Thus $\alpha = 15^2 11^2 10^2 7^2$ and $\beta = 6^3 3^6 2^6$. Updating α and β yields: $\bar{\alpha} = \langle 15^2 11^2 10^2 7^2 6^2 3^2 2^2 \rangle$ and $\bar{\beta} = \langle 6^1 3^4 2^4 \rangle$.
Now we have $[2, \bar{\alpha}]^{-1} = (30, 22, 20, 14, 12, 6, 4)$ so that $c = \phi[2, \bar{\alpha}]^{-1} = \langle 15^2 11^2 7^2 5^4 3^6 1^4 \rangle$, and so $[2, c]^{-1} = \langle 30^1 22^1 14^1 10^2 6^3 2^2 \rangle$.
But $\bar{\beta}' = \langle 9^2 5^1 1^3 \rangle$ so that the image is

$$\lambda = \bar{\beta}' \cup [2, c]^{-1} = \langle 30^1 22^1 14^1 10^2 9^2 6^3 5^1 2^2 1^3 \rangle.$$

The inverse

Let λ be a partition of n into parts $\equiv r, 2 \pmod 4$. Write $\lambda = (\lambda_1, \lambda_2)$ where λ_1 is the subpartition of λ with parts $\equiv 2 \pmod 4$. On the other hand, λ_2 has all parts $\equiv r \pmod 4$. Define

$$\mu = \lambda_2 + h'$$

where $h = [2, \phi^{-1}[2, \lambda_1]]$. Then μ is enumerated by $cc_2(n, r)$.

E.g., consider $\lambda = \langle 30^1 22^1 14^1 10^2 9^2 6^3 5^1 2^2 1^3 \rangle$ in the example above ($r = 1$). Then $\lambda_1 = \langle 30^1 22^1 14^1 10^2 6^3 2^2 \rangle$ and $\lambda_2 = \langle 9^2 5^1 1^3 \rangle$.
Now $[2, \lambda_1] = \langle 15^2 11^2 7^2 5^4 3^6 1^4 \rangle$ so that $\phi^{-1}[2, \lambda_1] = (30, 22, 20, 14, 12, 6, 4)$.
So $h = [2, (30, 22, 20, 14, 12, 6, 4)] = \langle 15^2 11^2 10^2 7^2 6^2 3^2 2^2 \rangle$ and that $h' = \langle 14^2 12^1 10^3 8^1 6^3 4^1 2^4 \rangle$.
Thus

$$\mu = \langle 9^2 5^1 1^3 \rangle + \langle 14^2 12^1 10^3 8^1 6^3 4^1 2^4 \rangle = \langle 23^2 17^1 11^3 8^1 6^3 4^1 2^4 \rangle.$$

Chapter 6

Variations and Consequences

We look at certain partition functions that are related to those encountered in the previous chapter.

6.1 Notation

$H(n)$: the set of partitions of n in which if j is even then all parts less than j are even. Note that partitions of n into odd parts are included in $H(n)$. We set $h(n) := |H(n)|$.
$\bar{H}(n)$: the set of partitions of n in which if j is odd then all parts less than j are odd. Note that partitions of n into even parts are included in $\bar{H}(n)$. We set $\bar{h}(n) := |\bar{H}(n)|$.
$DD(n)$: the set of partitions in $\bar{H}(n)$ where the smallest part is odd and greater than 1 or the only odd part allowed is 1 and even parts are greater than 2. We set $dd(n) := |DD(n)|$.
$C(n, i)$: the set of partitions of n in which the first i smallest parts are distinct, $i \geq 1$.
$TAU(n)$: the set of partitions of n in which if an even part is repeated, it is the smallest and only repeats twice, all even parts greater than it are distinct. Note that all partitions of n with distinct even parts are automatically in $TAU(n)$.
We also recall the notation $ODD(n)$, $odd(n)$, $P(n)$, $p(n)$ introduced already on page 10.

6.2 Related partition theorems and bijective proofs

We start by investigating the case where smaller parts are even and larger parts odd.

Proposition 6.1. *Define* $h(0) := 1$*. For all* $n \geq 1$*,*

$$h(2n) = h(2n+1), \tag{6.1}$$

$$h(n) = p(0) + p(1) + p(2) + \ldots + p(\lfloor n/2 \rfloor). \tag{6.2}$$

Combinatorially, (6.1) is clear. We demonstrate the generating function proof for (6.2).
Using the 2-dissection of the generating function for $h(n)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} h(n)q^n &= \sum_{j=0}^{\infty} h(2j)q^{2j} + \sum_{j=0}^{\infty} h(2j+1)q^{2j+1} \\
&= \sum_{j=0}^{\infty} h(2j)q^{2j} + \sum_{j=0}^{\infty} h(2j)q^{2j+1} \text{ by (6.1)} \\
&= (1+q)\sum_{j=0}^{\infty} h(2j)q^{2j}.
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} h(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n}}{(1-q^2)(1-q^4)\ldots(1-q^{2n})(1-q^{2n+1})(1-q^{2n+3})\ldots},$$

we thus have

$$\begin{aligned}
(1+q)\sum_{j=0}^{\infty} h(2j)q^{2j} &= \frac{1}{(q;q^2)_\infty} + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^2)(1-q^4)\ldots(1-q^{2n})(1-q^{2n+1})(1-q^{2n+3})\ldots} \\
&= \sum_{n=0}^{\infty} \frac{q^{2n}}{(1-q^2)(1-q^4)\ldots(1-q^{2n})(1-q^{2n+1})(1-q^{2n+3})\ldots} \\
&= \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n(q^{2n+1};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{2n}(q;q^2)_n}{(q^2;q^2)_n(q;q^2)_\infty}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}(q;q^2)_n}{(q^2;q^2)_n} \\
&= \frac{1}{(q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{1-q^{2n+1}}{1-q^{2n}} \text{ by Theorem 2.10} \\
&= \frac{(q;q^2)_\infty}{(q;q^2)_\infty (1-q)(q^2;q^2)_\infty} \\
&= \frac{1}{(1-q)} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})}
\end{aligned}
$$

so that

$$\sum_{j=0}^{\infty} h(2j) q^{2j} = \frac{1}{(1-q^2)} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})}.$$

Setting $q := q^{1/2}$, we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} h(2n) q^n &= \frac{1}{(1-q)} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \\
&= \sum_{n=0}^{\infty} q^n \cdot \sum_{n=0}^{\infty} p(n) q^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} p(r) \right) q^n,
\end{aligned}
$$

which, together with (6.1) implies (6.2).
Alternatively, the identity $h(2n) = \sum_{j=0}^{n} p(j)$ can be proved bijectively. To demonstrate this, fix n and let $\lambda \in p(j)$ for some $j = 0, 1, \ldots, n$. Then the map

$$\lambda \mapsto \lambda + \lambda + \langle 1^{2n-2|\lambda|} \rangle$$

is a bijection from the set of partitions in $P(0) \cup P(1) \cup \ldots \cup P(n)$ onto the set of partitions in $H(2n)$.

The inverse is given by

$$\mu \mapsto \frac{1}{2}(\mu - \langle 1^{oddp(\mu)} \rangle)$$

where $oddp(\mu)$ is the number of odd parts in μ.

Example

Let $n = 20$.

$$(4,4,3,2,1,1,1) \mapsto (8,8,6,4,2,2,2) + (1,1,1,1,1,1,1,1) = (9,9,7,5,3,3,3,1),$$

$$(5,3,3,2,2,2,1,1) \mapsto (10,6,6,4,4,4,2,2) + (1,1) = (11,7,6,4,4,4,2,2).$$

Now let us 'reverse' the even-odd condition in partitions in $H(n)$. Then we end up with partitions in $\bar{H}(n)$.

Proposition 6.2. *For $n \geq 0$,*

$$dd(n) = odd(n).$$

$$\begin{aligned}
\sum_{n=0}^{\infty} dd(n)q^n &= \frac{1}{(1-q)(1-q^4)(1-q^6)\ldots} + \sum_{n=1}^{\infty} \frac{q^{2n+1}}{(1-q^3)(1-q^5)\ldots(1-q^{2n+1})(q^{2n+2};q^2)_\infty} \\
&= \frac{1+q}{(q^2;q^2)_\infty} + \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n+1}(q^2;q^2)_n}{(1-q^3)(1-q^5)\ldots(1-q^{2n+1})} \\
&= \frac{1}{(q^2;q^2)_\infty} \left(1 + q + \sum_{n=1}^{\infty} \frac{q^{2n+1}(q^2;q^2)_n}{(1-q^3)(1-q^5)\ldots(1-q^{2n+1})} \right) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \text{ by Theorem 2.9} \\
&= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})} \sum_{n=0}^{\infty} q^{n(n+1)/2} \\
&= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})} \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^{2n-1}} \text{ by Theorem 2.8} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}
\end{aligned}$$

and the result follows.

Bijective proof

Let λ be a partition of n in $DD(n)$. Subtract 1 from every even part, and take the union of the result with the partition $\langle 1^{\#e}\rangle$, where $\#e$ is the number of even parts in λ. This resulting partition is in $ODD(n)$. This transformation is invertible and we give its inverse below.

Suppose $\mu \in ODD(n)$. Define λ as

$$\lambda = \begin{cases} \mu^{1..1} + \langle 1^{m_1(\mu)}\rangle, & \text{if } m_1(\mu) < \#\mu^{1..1} \\ \left(\mu^{1..1} + \langle 1^{\#\mu^{1..1}}\rangle\right) \cup \langle 1^{m_1(\mu)-\#\mu^{1..1}}\rangle, & \text{otherwise.} \end{cases}$$

In either case, it is not difficult to see that λ is in $DD(n)$.

Recall that in the third from the last step of the generating function proof,

$$\sum_{n=0}^{\infty} dd(n)q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Thus $dd(n)$ can be expressed in terms of the function $p(n)$ over triangular numbers. Denote by $\Delta(n)$ the set of triangular numbers $0 \le r \le n$ such that $n - r$ is even. Then

$$dd(n) = \begin{cases} \sum_{r\in\Delta(n)} p((n-r)/2), & \text{if } n \text{ is even} \\ \sum_{r\in\Delta(n)\setminus\{0\}} p((n-r)/2), & \text{otherwise.} \end{cases}$$

Observe that if λ is a partition in $DD(n)$ in which the only odd part allowed is 1 and even parts > 2, then

$$\mu = \begin{cases} \frac{1}{2}\left(\left(\lambda \setminus \langle 1^{m_1(\lambda)}\rangle\right) \cup \langle 2^{m_1(\lambda)/2}\rangle\right), & \text{if } n \text{ is even} \\ \frac{1}{2}\left(\left(\lambda \setminus \langle 1^{m_1(\lambda)}\rangle\right) \cup \langle 2^{(m_1(\lambda)-1)/2}\rangle\right), & \text{if } n \text{ is odd} \end{cases}$$

is in $P(\frac{n}{2}) \cup P(\frac{n-1}{2})$. Conversely, if $\mu \in P(\frac{n}{2}) \cup P(\frac{n-1}{2})$, then

$$\lambda = \begin{cases} \langle 1^{2m_1(\mu)}\rangle \cup (2\mu \setminus \langle 2^{m_1(\mu)}\rangle), & \text{if } \mu \in P(\frac{n}{2}) \\ \langle 1^{2m_1(\mu)+1}\rangle \cup (2\mu \setminus \langle 2^{m_1(\mu)}\rangle), & \text{if } \mu \in P(\frac{n-1}{2}) \end{cases}$$

is a partition in $DD(n)$ in which the only odd part allowed is 1 and even parts > 2. The mapping $\lambda \mapsto \mu$ is a one-to-one correspondence, and so the following corollary follows.

Corollary 6.1. *The number of partitions in $\bar{H}(n)$ such that the smallest part is odd and larger than 1 is equal to*

$$\sum_{r\in\Delta(n)\setminus\{0,1\}} p((n-r)/2).$$

We now turn our attention to some restrictions on distinct parts in the following section.

6.3 Further restrictions on repetition of parts

Theorem 6.1. *For $j \geq 1$, let $w(n,j)$ denote the number of partitions of n in which the largest repeated part is j. Then $w(n,j)$ is equal to the number of partitions in which the largest even part is $2j$.*

Proof. Note that

$$\prod_{n=1}^{j-1}\frac{1}{1-q^n}\prod_{m=j}^{\infty}(1+q^m)$$

is the generating function for the number of partitions in which all parts greater than $j-1$ are distinct. Thus

$$\begin{aligned}\prod_{n=1}^{j-1}\frac{1}{1-q^n}\prod_{m=j}^{\infty}(1+q^m) &= \prod_{n=1}^{j-1}\frac{1}{1-q^n}\prod_{m=1}^{j-1}\frac{1}{(1+q^m)}\prod_{m=1}^{\infty}(1+q^m)\\ &= \prod_{n=1}^{j-1}\frac{1}{1-q^{2n}}\prod_{m=1}^{\infty}(1+q^m)\\ &= \frac{1}{(q^2;q^2)_{j-1}(q;q^2)_\infty},\ \text{using Theorem 2.5.}\end{aligned}$$

Clearly

$$\prod_{n=1}^{j}\frac{1}{1-q^n}\prod_{m=j+1}^{\infty}(1+q^m)-\prod_{n=1}^{j-1}\frac{1}{1-q^n}\prod_{m=j}^{\infty}(1+q^m)$$

is the generating function for $w(n,j)$. Thus

$$\begin{aligned}\sum_{n=0}^{\infty}w(n,j)q^n &= \frac{1}{(q^2;q^2)_j(q;q^2)_\infty}-\frac{1}{(q^2;q^2)_{j-1}(q;q^2)_\infty}\\ &= \frac{1}{(q;q^2)_\infty}\frac{q^{2j}}{(q^2;q^2)_j}.\end{aligned}$$

□

Corollary 6.2. *The number of partitions of n in which there is at least one repeated part is equal to the number of partitions of n with at least one part even.*

Corollary 6.2 from that the fact that

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} w(n,j)q^n = \frac{1}{(q;q)_\infty} - \frac{1}{(q;q^2)}.$$

In the above corollary, we note that such a part j is characterised by being the largest repeated part. We impose further restrictions on the multiplicity of the largest repeated part. Interestingly, this leads to some restriction on the smallest even part in some cases.
Let $k \geq 2$ and define $\tilde{g}(0,k) := 1$. For $n \geq 1$, let $\tilde{g}(n,k)$ denote the number of partitions in which all parts are distinct or the largest repeated part occurs at least k times. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \tilde{g}(n,k)q^k &= \prod_{n=0}^{\infty}(1+q^n) + \sum_{n=1}^{\infty} \frac{q^{kn}}{(q;q)_n} \prod_{m=n+1}^{\infty}(1+q^m)\\ &= \sum_{n=0}^{\infty} \frac{q^{kn}}{(q;q)_n} \prod_{m=n+1}^{\infty}(1+q^m)\\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{kn}}{(q^2;q^2)_n}\\ &= \frac{1}{(q;q^2)_\infty} \prod_{m=0}^{\infty} \frac{1}{(1-q^{k+2m})} \text{ by Theorem 2.4.}\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} \tilde{g}(n,k)q^n = \frac{1}{(q;q^2)_\infty} \prod_{m=0}^{\infty} \frac{1}{(1-q^{k+2m})}. \tag{6.3}$$

Theorem 6.2. *If* $n \equiv 1 \pmod 2$, *then*

$$\sum_{m=1}^{n} \tilde{g}(m,3) \equiv 1 \pmod 2.$$

Observe that

$$\sum_{n=0}^{\infty} \tilde{g}(n,3)q^n = \frac{1}{(q;q^2)_\infty} \prod_{m=0}^{\infty} \frac{1}{(1-q^{3+2m})}$$

$$= \frac{1-q}{(q;q^2)_\infty} \prod_{m=0}^{\infty} \frac{1}{(1-q^{1+2m})}$$

which implies that

$$\frac{1}{1-q} \sum_{n=0}^{\infty} \tilde{g}(n,3) q^n = \frac{1}{(q;q^2)_\infty^2}$$

Expanding the right-hand side as a power series, notice that there is no power of q which has odd exponent, and the result follows after expanding $\frac{1}{1-q}$ on the left as a power series.

It is not difficult to see that

$$\tilde{g}(n,3) + \tilde{g}(n-1,3) \equiv 0 \pmod{2} \text{ for every } n \text{ odd.} \tag{6.4}$$

This comes when we multiply $1+q$ to the generating function for $\tilde{g}(n,3)$, i.e.

$$(1+q) \sum_{n=0}^{\infty} \tilde{g}(n,3) q^n = \frac{1-q^2}{(q;q^2)_\infty^2}.$$

Note that for k even, (6.3) implies that $\tilde{g}(n,k)$ is equal to the number of partitions in which the smallest even part is greater than or equal to k. Setting $k=2$ yields the unrestricted partition function $p(n)$.

Finally, for $i \geq 1$, let $b(n,i)$ be the number of partitions of n with the restriction:

a. the number of parts $= i$ and parts are distinct or

b. the number of parts is greater than i, the first i smallest parts are distinct and the $(i+1)^{\text{th}}$ part is repeated.

Proposition 6.3. *The generating function for $b(n,i)$ is*

$$B(i) = q^{i(i+1)/2} \left(\frac{1}{(q;q)_i} + \sum_{n=i+1}^{\infty} \frac{q^{(n-i)(i+1)} - q^{n(i+2)-(i+1)^2}}{(q;q)_n} \right).$$

Proof

Observe that
$$b(n,i) = c(n,i) - c(n,i+1) \tag{6.5}$$
where $c(n,i) := |C(n,i)|$. A partition $(\ldots, \lambda_i, \ldots, \lambda_2, \lambda_1) \in C(n,i)$ is such that
$$\lambda_j = \begin{cases} j + n_1 + n_2 + \ldots + n_j, & 1 \leq j \leq i \\ \lambda_i + 1 + n_{i+1} + n_{i+2} + \ldots + n_j, & j > i. \end{cases}$$
for some non-negative integers $n_1, n_2, \ldots, n_i, n_{i+1}, \ldots$.
So we have
$$\begin{aligned}
\sum_{n=0}^{\infty} c(n,i) q^n &= \sum_{l=i}^{\infty} \sum_{(\lambda_1, \lambda_2, \ldots, \lambda_l) \in C(n,i)} q^{\lambda_1 + \ldots + \lambda_l} \\
&= \sum_{l=i}^{\infty} q^{i(i+1)/2 + l - i + ln_1 + (l-1)n_2 + \ldots + (l-i+1)n_i + (l-i)n_{i+1} + (l-i-1)n_{i+2} + \ldots + 2n_{l-1} + n_l} \\
&= \sum_{l=i}^{\infty} q^{i(i+1)/2 + l - i} \sum_{n_1, n_2, \ldots, n_i = 0}^{\infty} q^{ln_1 + (l-1)n_2 + \ldots + (l-i+1)n_i} \\
&\qquad \times \sum_{n_{i+1}, n_{i+2}, \ldots, n_l = 0}^{\infty} q^{(l-i)n_{i+1} + (l-i-1)n_{i+2} + \ldots + 2n_{l-1} + n_l} \\
&= \sum_{l=i}^{\infty} q^{i(i+1)/2 + l - i} \cdot \frac{1}{(1-q^l)(1-q^{l-1}) \ldots (1-q^{l-i+1})} \\
&\qquad \times \frac{1}{(1-q^{l-i})(1-q^{l-i-1})(1-q^{l-i-2}) \ldots (1-q^2)(1-q)} \\
&= \sum_{l=i}^{\infty} \frac{q^{i(i+1)/2 + l - i}}{(q;q)_l}.
\end{aligned}$$
We have
$$\begin{aligned}
B(i) &= \sum_{n=0}^{\infty} b(n,i) q^n \\
&= \sum_{n=0}^{\infty} (c(n,i) - c(n,i+1)) q^n \text{ by (6.5)} \\
&= \sum_{n=i}^{\infty} \frac{q^{i(i+1)/2 + (n-i)(i+1)}}{(q;q)_n} - \sum_{n=i+1}^{\infty} \frac{q^{(i+1)(i+2)/2 + (n-i-1)(i+2)}}{(q;q)_n}
\end{aligned}$$

$$= \frac{q^{i(i+1)/2}}{(q;q)_i} + \sum_{n=i+1}^{\infty} \left(\frac{q^{i(i+1)/2+(n-i)(i+1)}}{(q;q)_n} - \frac{q^{(i+1)(i+2)/2+(n-i-1)(i+2)}}{(q;q)_n} \right),$$

and the result follows.

On the other hand, denote by $gg(n,i)$ the number of partitions of n in which the largest part is repeated at least $i+1$ times. It is not difficult to see that

$$\sum_{n=0}^{\infty} gg(n,i)q^n = \sum_{n=0}^{\infty} \frac{q^{(i+1)n}}{(q;q)_n}.$$

A parity-relationship between the above partition functions is reflected in the following conjectures

Conjecture 6.1. *For $n \geq 1$,*

$$b(n,1) - gg(n+3,1) \equiv 1 \pmod 2. \tag{6.6}$$

Conjecture 6.2. *For $n \geq 1$,*

$$b(n,2) - gg(n+6,2) \equiv \begin{cases} 0 \pmod 2, & n \equiv 0,1 \pmod 4 \\ 1 \pmod 2, & \textit{otherwise.} \end{cases} \tag{6.7}$$

Conjecture 6.3. *For $n \geq 1$,*

$$b(n,3) - gg(n+10,3) \equiv \begin{cases} 0 \pmod 2, & n \equiv 5,6,8,10,11 \pmod{12} \\ 1 \pmod 2, & \textit{otherwise.} \end{cases} \tag{6.8}$$

Conjecture 6.4. *For $n \geq 1$,*

$$b(n,4) - gg(n+15,4) \equiv \begin{cases} 0 \pmod 2, & n \equiv 1,4,7 \pmod 8 \\ 1 \pmod 2, & \textit{otherwise.} \end{cases} \tag{6.9}$$

Experimental verification of conjectures has been done using the Sage code developed in Appendix B, see the tables in that section. Alternatively, Maple can be used.

In the following section, we investigate the situation in which even parts are distinct except the smallest with restricted multiplicity.

6.4 An identity of the Euler-Pentagonal type

Recall the description of partitions in $TAU(n)$. Let $\tau(n) := |TAU(n)|$. For example, $TAU(8)$-partitions are:

$$(8),(7,1),(6,2),(5,3),(5,1,1,1),(4,4),(4,2,1,1),(3,3,1,1),(3,1,1,1,1,1),$$
$$(6,1,1),(5,2,1),(4,3,1),(4,2,2),(4,1,1,1,1),(3,3,2),(3,2,1,1,1),$$
$$(2,1,1,1,1,1,1),(1,1,1,1,1,1,1,1).$$

Furthermore, we partition the set $TAU(n)$ via parity of the number of distinct even parts. If parity is even, denote the number of partitions by $\tau_e(n)$ and similarly, by $\tau_o(n)$ if the parity is odd. Then we have an identity of the Euler-Pentagonal type as follows.

Theorem 6.3. *For all non-negative integers n, we have*

$$\tau_e(n) - \tau_o(n) = \begin{cases} 1, & \textit{if } n = 3m, 3m+1, m \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

where $\tau_e(0) - \tau_o(0) := 1$.

<u>Proof</u>

Note that

$$\begin{aligned}
\sum_{n=0}^{\infty}(\tau_e(n) - \tau_o(n))q^n &= \sum_{n=0}^{\infty} q^{2n+2n}(q^{2n+2};q^2)_\infty(q^{2n+1};q^2)_\infty^{-1} \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{4n}}{(q^2;q^2)_n}(q;q^2)_n \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{(q;q^2)_\infty}{(q^4;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(q^2;q^2)_n}(q^4;q^2)_n \\
&\quad \big(\text{by } (2.6), a = c = 0, b = q, t = q^4\big). \\
&= (1-q^2)\sum_{n=0}^{\infty}\frac{q^n}{(q^2;q^2)_n}(q^4;q^2)_n \\
&= (1-q^2)\sum_{n=0}^{\infty}\frac{(1-q^{2n+2})q^n}{1-q^2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(1-q^{2n+2})q^n \\
&= \sum_{n=0}^{\infty} q^n - \sum_{n=0}^{\infty} q^{3n+2} \\
&= \sum_{n=0}^{\infty} q^{3n} + \sum_{n=0}^{\infty} q^{3n+1}.
\end{aligned}
$$

Table 6.1 demonstrates the theorem.

Table 6.1: Identity of the EPT type

n	$n \pmod 3$	$\tau_e(n)$	$\tau_o(n)$	$\tau_e(n)$ - $\tau_o(n)$
1	1	1	0	1
2	2	1	1	0
3	0	2	1	1
4	1	3	2	1
5	2	3	3	0
6	0	5	4	1
7	1	7	6	1
8	2	9	9	0
9	0	12	11	1
10	1	16	15	1
11	2	20	20	0
12	0	27	26	1
13	1	35	34	1
14	2	44	44	0
15	0	56	55	1
16	1	71	70	1
17	2	88	88	0

Chapter 7

Conclusion

We have focused on the subject of Rogers-Ramanujan identities and partitions with initial repetitions.

The Rogers-Ramanujan identities have had a lot of impact in the world of partition theory and beyond. With regard to their combinatorial interpretation, much research has been done. We looked at one partition-theoretic interpretation of the identities due to Agarwal and Goyal, and generalised their result. Some connections with the work of Connor were established as part of combinatorial consequences. Some results on partition functions related to those of Rogers-Ramanujan type were determined as well. In particular, a bijection for the inequality involving the coefficients of a fifth order mock theta function was exhibited.

We conducted a deeper look into one of George Andrews' theorems on partitions with initial repetitions. We refined the theorem. Not only was the refinement presented but also, its combinatorial proof that represents a new version (non-diagram version) of Keith's bijection was given.

Furthermore, various partition functions in the spirit of partitions with initial repetitions were examined. Congruence relations were exhibited, including conjectures in (6.6), (6.7), (6.8) and (6.9). However, despite strong experimental evidence of the conjectures, their proofs have not been found, and this remains to be an open problem. In one case, an identity of the Euler-Pentagonal type was presented. It must be stated that the generating function proof used utilizes the theory of q-series. It would be interesting if an explicit bijection proof was found.

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7.1 Appendix A

Sage code for Theorem 6.3

In the following codes: TAU(n) outputs partitions enumerated by $\tau(n)$ and TAUDIFF(n) represents $\tau_e(n) - \tau_o(n)$.

```
def TAU(n):
    L = []
    for h in Partitions(n).list():
        c0 = [p for p in h if p%2 == 0]
        c1 = [p for p in h if p%2 ==1]
        if len(c0)==0:
            L.append(h)
        if len(c0)>0 and len([p for p in c0 if list(h).count(p)==1])== len(c0):
                L.append(h)
        if len(c0)> 1 and list(h).count(min(c0))== 2:
            cdiff = [c0[i] - c0[i+1] for i in range(len(c0)-1)]
            if len(c1)==0 and cdiff.count(0)==1:
                L.append(h)
            if len(c1) > 0 and cdiff.count(0) == 1 and min(c1)>min(c0):
                L.append(h)

    return L

def TAUDIFF(n):
    L0 = []
    L1 =[]
    for h in TAU(n):
        cc = [p for p in h if p%2 == 0 and list(h).count(p)==1]
        if len(cc)==0:
            L0.append(h)
        if len(cc) > 0 and len(cc)%2 == 0:
            L0.append(h)
        if len(cc)> 0 and len(cc)%2 == 1:
```

```
            L1.append(h)

    return len(L0)- len(L1)
```

Now one types TAU(8); to find $\tau(8)$ and TAUDIFF(8); for $\tau_e(8) - \tau_o(8)$.

7.2 Appendix B

Sage code for conjectures in (6.6) through (6.9)

The functions $B(n,i)$ and $GG(n,i)$ return sets of partitions enumerated by $b(n,i)$ and $gg(n,i)$, respectively

```
def B(n,i):
    L= []
    for h in Partitions(n).list():
        if len(h) == i:
            dist = [p for p in h if list(h).count(p)== 1]
            if len(dist) == len(h):
                L.append(h)
        if len(h) > i and list(h).count(h[len(h)-(i+1)])>1:
            c = 0
            for j in [1..i]:
                if list(h).count(h[len(h)-j])==1:
                    c = c + 1
            if c == i:
                L.append(h)
    return L

def GG(n,i):
    L = []
    for h in Partitions(n).list():
        if list(h).count(h[0])>= i + 1:
            L.append(h)
    return L
```

Now $b(n,i)$ = len(B(n,i)) and $gg(n,i)$ = len(GG(n,i)).
For conjectures in (6.6), (6.7), see Tables 7.1 and 7.2, respectively.

Table 7.1: Example on (6.6)

n	$b(n,1)$	$gg(n+3,1)$	$b(n,1) - gg(n+3,1) \pmod 2$
1	1	2	1
2	1	2	1
3	1	4	1
4	1	4	1
5	2	7	1
6	1	8	1
7	3	12	1
8	3	14	1
9	4	21	1
10	5	24	1
11	9	34	1
12	8	41	1
13	14	55	1
14	17	66	1
15	23	88	1
16	28	105	1
17	40	137	1
18	46	165	1
19	65	210	1
20	78	253	1
21	101	320	1
22	124	383	1
23	163	478	1
24	193	574	1
25	249	708	1
26	302	847	1
27	378	1039	1
28	457	1238	1
29	572	1507	1
30	683	1794	1

Table 7.2: Example on (6.7)

n	$n \pmod 4$	$b(n,2)$	$gg(n+6,2)$	$b(n,2) - gg(n+6,2) \pmod 2$
1	1	0	2	0
2	2	0	3	1
3	3	1	4	1
4	0	1	5	0
5	1	2	6	0
6	2	2	9	1
7	3	3	10	1
8	0	3	13	0
9	1	5	17	0
10	2	4	21	1
11	3	6	25	1
12	0	7	33	0
13	1	9	39	0
14	2	8	49	1
15	3	13	60	1
16	0	13	73	0
17	1	18	88	0
18	2	19	110	1
19	3	25	130	1
20	0	28	158	0
21	1	39	191	0
22	2	41	230	1
23	3	54	273	1
24	0	63	331	0
25	1	81	391	0
26	2	91	468	1
27	3	119	556	1
28	0	136	660	0
29	1	171	779	0
30	2	200	927	1