EXPLORING GRADE 11 LEARNER ROUTINES ON FUNCTION FROM A COMMOGNITIVE PERSPECTIVE

A thesis submitted to the Faculty of Humanities, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy

by

Regina Miriam Essack
Student Number: 528134

Supervisor:
Prof. Jill Adler

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Declaration

I declare that this research report is my own, except as indicated in the acknowledgments, the text and the references. It is being submitted in fulfilment of the requirement for the degree Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any institution.

Regina Miriam Essack   Date

Dedication

For taking this journey with me and your unrelenting support, this is dedicated to

Omar,

Hannah,

Zara,

Marshall and Jeanne Daniels,

Cassim and Ayesha Essack

‘Hmmm... page 1 of 21, you can do it mum!’ – Zara (10)JMJ

Abstract

This study explores the mathematical discourse of Grade 11 learners on the topic function through their routines. From a commognitive perspective, it describes routines in terms of exploration and ritual. Data was collected through in-depth interviews with 18 pairs of learners, from six South African secondary schools, capturing a landscape of public schooling, where poor performance in Mathematics predominates. The questions pursued became: why does poor performance persist and what might a commognitive lens bring into view? With the discursive turn in education research, commognition provides an alternate view of learning mathematics. With the emphasis on participation and not on constraints from inherited mental ability, the study explored the nature of learner discourse on the object, function. Function was chosen as it holds significant time and weight in the secondary school curriculum. Examining learners' mathematical routines with the object was a way to look at their discourse development: what were the signifiers related to the object and what these made possible for learners to realise. Within learners' routines, I was able to characterise these realisations, which were described and categorised. This enabled a description of learner thinking over three signifiers of function in school Mathematics: the algebraic expression, table and graph.

In each school, Grade 11 learners were separated into three groups according to the levels at which they were performing, from summative scores of grade 11 assessments, so as to enable a description of discourse related to performance. Interviews were conducted in pairs, and designed to provoke discussion on aspects of function and its signifiers between learners in each pair. This communication between learners and with the interviewer provided data for description and analysis of rituals and explorations. Zooming in and out again on these routines made a characterisation of the discourse of failure possible, which is seldom done. It became apparent early in the study that learners talked of the object function, without a formal mathematical narrative, a definition in other words, of the object. The object was thus vested in its signifiers. The absence of an individualised formal narrative of the object impacts directly what is made possible for learners to realise, hence to learn.

The study makes the following contributions: first, it describes learners' discursive routines as they work with the object function. Second, it characterises the discourse of learners at different levels of performance. Third, it starts exploration of commognition as an alternate means to look

at poor performance. The strengths and limitations of the theory as it pertains to this study, are discussed later in the concluding chapter.

Keywords

commognition, discourse, communication, participation, routines, exploration, ritual, learners, learning, narratives, endorsed narratives, visual mediators.

Acknowledgements

As my study grew, she also grew me. I learned about research and more importantly about myself. My confidence grew from her advice, insight and knowledge. I saw how hard she worked on my contributions fashioning them into relevance and the precise and exacting standards she expected. I especially appreciated the experiences she made possible as I delved into commognition. For the research seminars and knowledgeable others brought in to advise, I am grateful.

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I am thankful to Professor Anna Sfard for her input in the early stages of my work.

I also thank the schools and parents of the learners I worked with, and the learners for their enthusiasm and generosity in terms of the information and time they offered to me. Frequently, discussions continued beyond five o'clock, which was the time set for the closure of the activity. They would have missed their transport home and arrived home after dark. Visiting researchers, observing on some of the episodes, also remarked on the enthusiastic participation of learners.

Mr. Prem Govender, who sadly passed away before completion of this study, is thanked for his support in providing access to data from the department of education. Mr. Cassim Seth, who also facilitated access to data from the department of education, and to this day, remains a source of support and encouragement.

Table of Contents

List of Tables

List of Figures

List of Extracts

List of Appendices

List of Acronyms/Abbreviations

FET	Further Education and Training
GET	General Education and Training
NCS	National Curriculum Statements
NSC	National Senior Certificate
SA	South Africa
SAG	Subject Assessment Guidelines
SBA	Subject Based Assessment
WMC-S	Wits Maths Connect Secondary

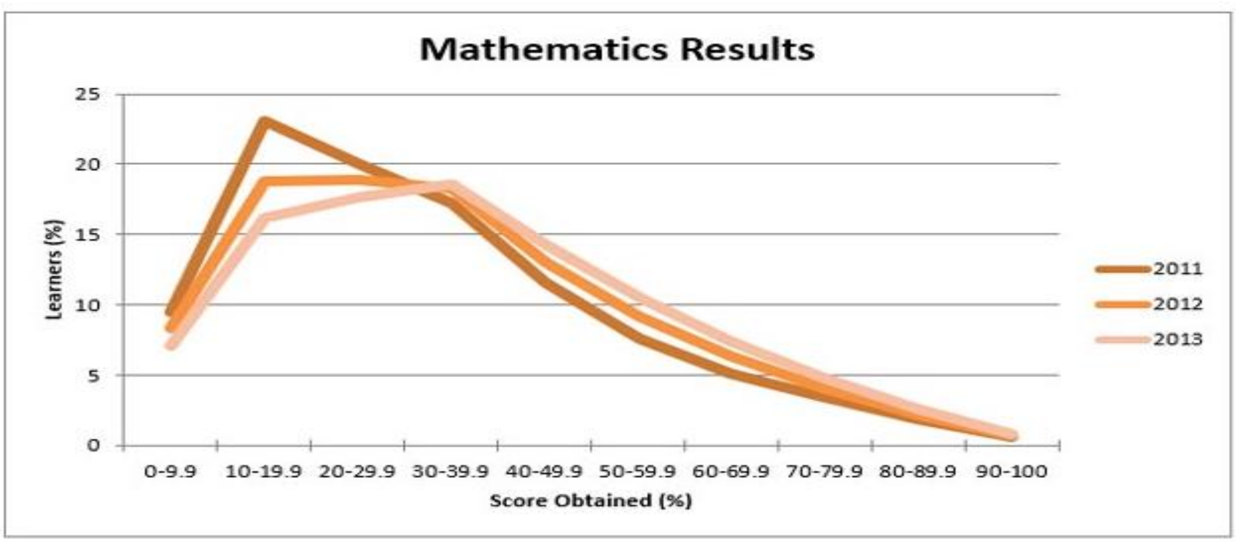
Chapter 1: Introduction

1.1 Background of the problem

South Africa offers an exciting and rich environment for educational research. Multiple social and cultural layers such as race, language, economic capability, educational access (to name but a few) interact, intersect and occasionally collide to present us (the educational research community) with ready dilemmas for enquiry. Extracted from the complex intersections of dilemmas, I note one, which locates my study within the entropy of this multi-layered system. As Taylor, Van der Berg, & Mabogoane have put it, "it is a vibrant new democracy, with great diversity in people, many of whom even twenty years post apartheid, continue to live in contexts of grave inequality, unemployment and poverty" (Taylor, van der Berg, & Mabogoane, 2013, p. 299).

Each year in January, the National Department of Education publicly presents the results of National Senior Certificate[1] (NSC) of the previous year. Each year, as the day draws near, our hopeful nation collectively holds its breath in anticipation; sadly, an inevitable disappointment follows. Focus is rapidly drawn to Mathematics performance. The skewed performance curve below shows quite starkly the high failure rate, reinforcing the ailing-system metaphor.

Figure 1 Matric results 2011 to 2013.



Source: Moneyweb

[1] The National Senior Certificate is the matriculation examination, the exit examination for South African learners after twelve years of formal schooling.

Education in South Africa is relatively well-financed compared to the situation in other developing countries (DBE, 2011c). Studies show we have high participation rates, yet poor progression and weak performance on national and international assessments when compared to other middle-income countries (Gustafsson, 2011). Our results are poor even when compared to our more disadvantaged neighbours (Taylor, Fleisch, & Shindler, 2008). Performance in mathematics appears steeped in failure, particularly at the secondary level. It does not seem to matter where in the world you are, first or third world. Venkat & Spaull (2014) has cautioned that in the abundance of reports of how bad the situation has become, we require an understanding of why they are so bad. Like many before me, for teachers and researchers, the question as to why performance in mathematics is so poor? became the compelling question which initiated and continued to motivate this study.

Being a teacher for 23 years, in both public and private schools in two of the eleven provinces in South Africa, I joined in the inevitable annual collective chorus. This resulted in constant revisions of my mathematical and teaching knowledge, though I cannot claim to have understood the dilemma of poor performance any better. To better understand this increasingly complex dilemma, I was required to acknowledge that the traditional ways we teach and learn were being changed with momentum far beyond our control. The broad transformative change has breached walls of classrooms and traditional sources of knowledge, by foregrounding communication in the age of technological advancement. Locally, August 2015 saw the rollout of electronic tablets to learners in 300 secondary schools in Gauteng, the province in South Africa where this study is located. Now, as both parent and teacher, I have to admit that the sphere of communication and information learners have access to today is far wider than I could have imagined. The classroom, therefore, has to be looked at with new eyes. I turned to the broader mathematics education literature to inform my view further.

The common thread across much of the discourse on education in South Africa (e.g. Taylor et al., op cit) is the complexity of the problem within our historical and social context. Learning mathematics, even in advantaged and well-resourced contexts, appears problematic. It was naive to expect that mathematics education research would deliver the answers I sought. I did, however, find a promising way of looking at the problem, and hopefully, of generating new and useful insights.

Within a socio-cultural framework, the present discursive turn in mathematics education research (Morgan, 2012) embraced the urgency of the communication imperative and provided tools to examine learner thinking beyond identification of errors and misconceptions and their remediation (DBE, 2013, 2014). I hope to illustrate in this study that discourse is an alternate and productive route for exploring the persistent problem of poor learner performance in mathematics, and a means to grow knowledge in an increasingly communication-driven world. It has made available routes less travelled, and revealed aspects seldom seen. But a communicational approach to explore learner thinking was not my starting point. Indeed like many others, I began by delving first into NSC mathematics learner scripts.

In 2010, I joined Wits Maths Connect-Secondary (WMC-S), a research and development project based at the University, whose goal was to improve mathematics results of ten relatively poorly performing schools in the Gauteng. In 2011, the 2009 and 2010 NSC learner scripts from ten Wits Maths Connect-Secondary (WMC-S) schools became available to my study. I had stumbled on a goldmine, and thought this sufficiently rich data on which to base my enquiry. With my focus trained on learners and learning, with data relevant to the end of formal schooling, I honed in on the topic function, which had enjoyed longevity in the curriculum and thus in learners' school mathematical experience. I felt confident to make the following critical assumptions:

- A focus on functions was justified because it claimed significant portion of the teaching time allocated to Mathematics from Grade 8 to Grade 12, as seen in curriculum documents (DBE, 2011a);
- Function related questions were allocated a significant part of the exit, high stakes NSC examination;
- The Grade 11 year marks the completion of the teaching of the topic to learners. At this stage learners have been taught all that is required by the curriculum for this topic. This point or beyond, would be the ideal stage at which to select learners, if I wanted to know what they had learned.

My initial foray into questions involving functions and related algebra on the NSC scripts presented unexpected results. Learners were not responding to questions on functions. How widespread was this across the ten project schools?

Table 1 Extract from spreadsheet of learner scores on function related questions (NSC 2010).

NSC 2010 Mathematics Paper 1

	1.1.1	1.1.2	1.1.3	1.2	1.3		3.1	3.2		4.1	4.2	4.3	4.4	4.5	
	3	4	4	7	3	21	3	4	7	2	3	2	3	4	14
A1	3	3	1	2	1	10	0	0	0	1	0	0	0	0	1
A2	1	1	1	0	0	3	0	0	0	0	0	0	0	0	0
A3	1	1	1	0	0	3	0	2	2	0	0	1	0	0	1
A4	3	0	1	0	1	5	0	0	0	0	0	1	0	0	1
A5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A6	0	0	0	0	0	0	3	0	3	0	0	0	0	0	0
A7	3	4	3	3	0	13	0	0	0	0	3	1	0	0	4
A8	1	0	1	0	0	2	2	0	2	0	0	1	0	0	1
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A11	0	2	1	0	0	3	0	0	0	0	0	0	0	0	0
A12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A13	1	4	0	0	0	5	0	0	0	0	0	0	0	0	0
A14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A15	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
A16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A18	1	0	1	0	0	2	0	0	0	0	2	1	0	0	3
A19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A20	2	1	3	3	0	9	0	0	0	0	0	0	0	0	0
A21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	5.1	5.2	5.3	5.4	5.5		6.1	6.2	6.3	6.4	6.5		10.1	10.2
	4	1	3	1	3	12	2	3	1	2	3	11	3	3
A1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A7	3	0	0	0	0	3	2	2	0	0	0	4	0	0
A8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A11	0	0	1	0	0	1	0	0	0	0	0	0	0	0
A12	1	0	0	0	0	1	0	0	0	0	0	0	0	0
A13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A14	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A15	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A18	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A19	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A20	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A21	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A23	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The table is an extract of a larger spreadsheet capturing learners' participation and performance across the ten project schools. Table 1 revealed yellow highlighted cells indicating questions which learners did not attempt on the examination; and grey cells showing a score of zero, where learners attempted an answer for which they were allocated a zero. The random selection from the larger spreadsheet shows one of the ten WMC-S schools. The other nine schools mimicked the pattern of a white page awash in yellow and grey. The following trends could be concluded from the table:

- learners were not responding to questions related to functions evident in each yellow highlighted cell containing a zero in the table. There was thus poor *participation* in questions on functions; and
- where there were responses, most learners score zero, evident in the grey cells. Thus, when learners did participate, their *performance* in questions on the topic was poor.

I called this pattern of poor participation and poor performance, marked by the dominance of zero seen across all schools, 'the presence of absence'. It signalled an inexplicably grave silence. It presented the quandary: how could learner responses in functions, after five years of secondary school Mathematics, appear as significant absence on such a high stakes examination? This result suggested to me that conventional means of assessing what learners know were not efficiently accomplishing this task. From the extract above, I could see that learners participation (or non-participation) in function related questions was indeed peculiar. Why they were participating in these ways became compelling. In asking this question, I declare that I depart from a position which sees learners as capable of participating in a mathematical discourse. My quandary thus became why they would show such poor participation and performance in questions on functions. So began my search to find out: **what** learners knew about functions and **how** they knew this. My focus was beginning to find definition.

I had positioned my enquiry thus in stark contrast to examination reports, which listed as a point of departure, all of that which learners were incapable. The appeal of the socio-cultural perspective lay in the way it approached learning as a social endeavour, whether learners were successful or not, whether through the interactions of a classroom or alone (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005). Failure in learning was seen as a product of a social context (ibid), the learner-in the classroom-in the learner (Lerman, 2006), in the world. This perspective provided a contrasting lens with which to explore performance in Mathematics. A learner cannot be entirely responsible for her performance in Mathematics, where it is necessary to acknowledge that all learning occurs within a larger social milieu.

My chosen research perspective situates learning in social endeavour, and with the current emphasis on communication, Sfards' (2008) communicational framework called commognition, gained resonance, anchoring my work to a theoretical frame. Commognition is a lexical combination of the words communication and cognition. It foregrounds thinking as a special form of communicating. Based on this premise, I could access learner thinking through the ways in which they communicate with themselves or with others. Knowledge, in particular, mathematical knowledge, would be a type of communication, defining a particular community, who would communicate about its objects, the mediators used, and who would follow its specific rules. Within commognition, such specialised forms of communication are known as discourses. This convergence of research perspective and theoretical framing enabled me to begin to refine the focus of my study. Instead of defining learners as capable or incapable, I was able to examine the nature of what they say and do with the mathematical object function. It would be an exploration of their *discourse* on function. Considering the definition of a discourse, this still appeared too broad. While I elaborate my research perspective connected to critical areas of the thesis in Chapter 2, and Sfard's commognitive theory in Chapter 3, it is necessary to provide a brief description of key aspects here, as these enable me to formulate the problem and research questions for this study.

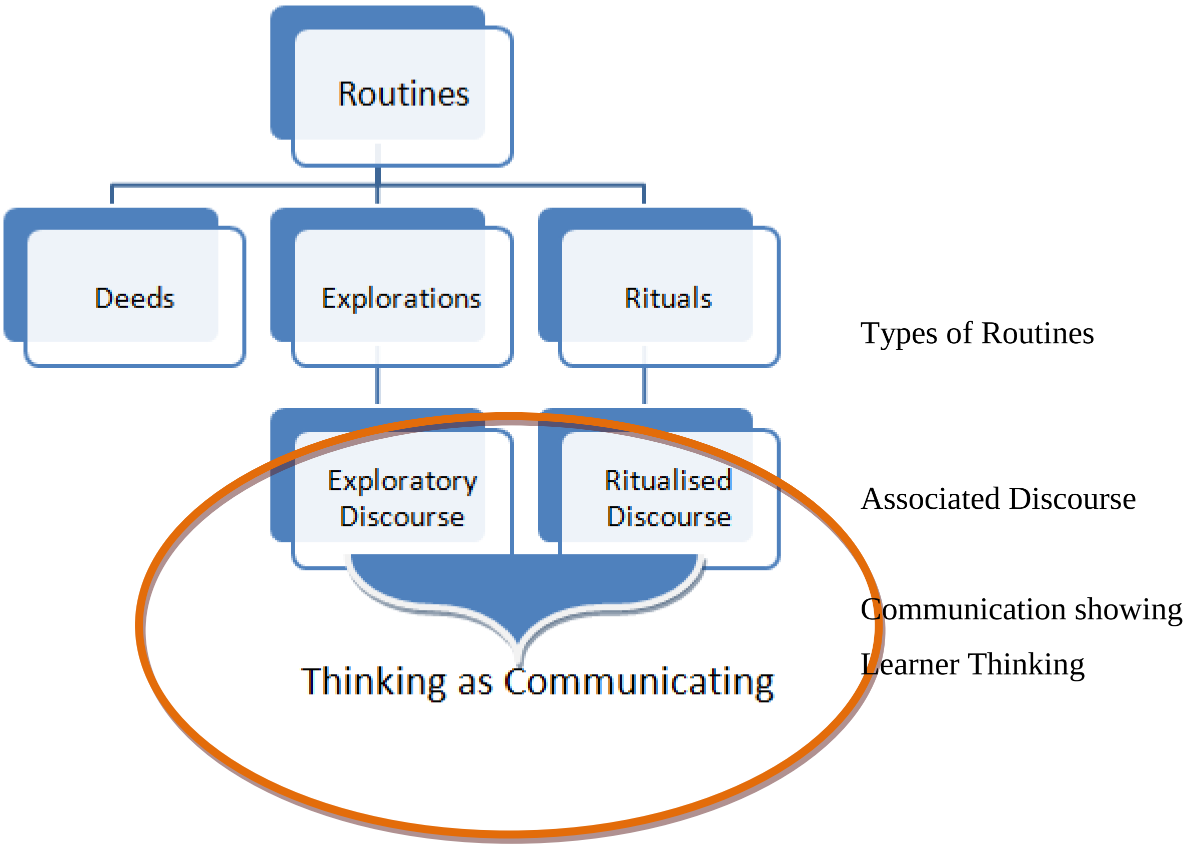
Sfard (op. cit.) defines a critical distinction between mathematics and other discourse in terms of the patterned ways we as mathematists[2] look and attend to mathematical problems, use words and mediators, create and substantiate narratives. Sfard has called these patterned ways of doing things 'routines' (Sfard, 2008, p. 9). Routines comprise two parts: the *how* and the *when* of a routine. The **how** of a mathematical routine refers to following a course of action, and matching a routine to a recognisable task. The **when** of a mathematical routine refers to the relevance of the routine in a particular instance. For emphasis, I repeat a question asked earlier, if and to what extent learners' routines include both the how and the when, and what consequences follow from this? I could speculate at this early stage that, given my own experience as a teacher, and the way in which the Mathematics curriculum has evolved in South Africa (which I elaborate on in Chapter 2), coupled with its implementation, it is likely that learners are restricted to knowledge of *how* routines in the absence of *when* routines. This is possibly a primary constraint to their full participation in Mathematics discourses. Could the presence of zeroes on the spreadsheet be an indication that learners may know *what* to do, but do not know *when* to do it?

In addition to illuminating the *how* and *when* of routines, Sfard further distinguishes three types of routines: deeds, explorations and rituals (Sfard, 2008, p. 259). The characteristics of each are detailed and exemplified in Chapter 3. Briefly and for the purposes of this introduction and statement of the problem, explorations produce endorsed narratives; deeds transform concrete objects, and rituals are socially oriented for solidarity with others. I focused on explorations and rituals, as I felt these would be most prevalent in the ways Grade 11 learners dealt with abstract objects, like functions. As a teacher, my goal would be to develop and encourage independent exploration among my learners. This initiated important questions for the study. Do learners' routines include exploration and in what ways? The focus continues to sharpen. Given my experience and the functions literature on teaching and learning discussed in Chapter 2, a dominance of ritual is expected. What does this look like? What can a focus on learner routines for ritual and exploration, reveal about learner thinking? With the focus now

[2] A "mathematist" is a participant in a mathematical discourse (Sfard, 2008, p. 128). It could be a learner in a classroom who is communicating mathematically. This contrasts with a professional mathematician, who can participate in the established formal mathematical discourse.

sharper, it was possible to represent this diagrammatically for its location in the commognitive framework.

Figure 2 The focus.



To reiterate, commognition provides a means to look at what learners know, in terms of their patterned ways of communicating mathematically (talking, reading, writing, gesturing), that is, by examining their discursive routines. As suggested above, if we regard discourse as a form of socially mediated communication, and mathematics as being a form of discourse, then routines are the repetitive, discursive patterns which characterise the discourse. This enabled me to reframe my problem as follows: What are South African learners' routines when they engage with functions? How do these routines contribute to learner participation and performance in questions related to functions? The reconstruction of questions in this form was important as it avoided characterising learners as in deficit. Learner mathematical routines on function became the focus, and these were described and characterised.

1.2 Statement of the problem

Learner performance in mathematics is poor. Traditional means of describing and investigating learner performance have provided useful initial insight into this. In this study, I have attempted to approach the problem from an alternate perspective, that of communication, to hopefully extend the ways we look at poor performance. Functions occupy significant space in the South African, secondary schooling curriculum and assessments. Significant in that questions related to functions and the allied algebra, comprise the highest weighting on the high stakes NSC. It is therefore compelling to explain poor performance on a topic that would have significant command of teaching time and curriculum presence, and significant consequences for learners who wish to proceed to tertiary study in mathematics-related fields.

The current discursive turn in education research and Sfard's commognition theory in particular, provided this alternate perspective and the set of analytical tools for this task. These enabled a look at the problem of poor performance by focusing on how learners participate in the mathematical discourse. I locate the problem and focus on persistent patterns which occur in learner discourses on function. Table 1 showed that alongside predominantly poor performance, there were learners who do participate and perform at different levels in the examination. I thus conjectured further that there might well be different levels of discourse for learners at different levels of performance. How would the discourse of a learner excelling in Mathematics compare to a learner occupying the other extreme? A description of each of these discursive spaces as well as the space between, presented an enticing challenge. When we can describe what learners consistently do, at different levels of performance, we can access possible reasons for poor performance, but also begin to look at means to improve performance from one discursive space to a higher one.

1.3 Purpose of the study

This study attempts, through a discursive lens, to explore Grade 11 learner thinking in functions, to better understand poor performance in Mathematics. A Grade 11 learner would have had a four-year engagement with the topic since Grade 8 (as specified in the curriculum and teaching work schedules). Within cognitive-based research perspectives, the term *thinking* appeared as a difficult term to operationalise or define. Commognition defines and operationalises the terms it uses rigorously. Commognition as a discursive framework, stresses the unity of the processes of thinking and communicating, and thus provides means for operationalising learner thinking through learner communication. Communication is the window into learner thinking. We can describe learner thinking through the ways learners communicate with themselves or with others. Communication is examined in this study as talking, writing and to a limited extent, gesturing. Learning mathematics or learning to think mathematically is learning to speak mathematically (Lerman, 2001) to oneself (when we think), or with others. Commognition sees learning in mathematics become evident in the way in which our discourse changes through our increased independent participation in it, which results in advancing levels of complexity and abstraction.

This study explored the discourses on function of Grade 11 learners at three different levels of performance: top, middle and low. Its purpose was to describe learner thinking at each of these performance levels, as it becomes evident through the routines learners use when working with functions. As noted, routines are persistent or repetitive discursive practices. My goal was to examine learner discourses on function, through a commognitive lens, held up to the mirror of the existing formal mathematical discourse on function. I also wanted to find the distinguishing features of mathematical discourse as they related to the three levels of performance among learners. My intention is to construct a worthwhile and useful story of learner thinking in function. Towards this end, I began with the following questions:

Research questions

1. How does learner thinking, evident in the routines they use, illuminate poor performance in functions?
    - 1.1 What are learner routines in function at different levels of performance?
    - 1.2 What are the characteristics of learners rituals and explorations at different levels of performance?
        - 1.2.1 Did learner routines on function include exploration and what did this look like?
        - 1.2.2 Where and around which aspects of the mathematical discourse do instances of exploration occur?

1.2.3 What were features of learners rituals?

2. What can a focus on routines tell us about the object, function?

Of course, a final question would prove to be: does commognition illuminate poor performance in ways that takes our understanding of the problem forward?

1.4 Importance of the study

While my study describes learner performance across three levels ranging from successful to unsuccessful, its importance is twofold. First, it offers an exploration and description of failure, through the examination of the discourses of poorly performing learners, who are frequently neglected in mathematics education research. It is a subtle, yet significant move away from deficit descriptions of learners and what they can't do, to what they actually say and do. Second, it offers the exploration of a relatively new theoretical framework, its applicability and worth in examining the quandaries of thinking and learning in general, and the function object in mathematics, in particular.

A brief note on scope and limitations. The question of scope and limitations is fully elaborated in the methodology in Chapter 4. However, it is important that I briefly discuss the most significant of these here. My study and the data that enable my story are all in English. South Africa is, however, a multilingual society, with 11 official languages. The dominant language in terms of access, power and the economy (Phakeng & Moschkovich, 2010; Setati, 2008) remains to be English as it is the language of instruction in most secondary schools. However, English is not the main language of most learners, including the learners in this study, and they take NSC examination, discussed above, in English. In Chapter 4, I outline the rationale for conducting the interviews and the gathering of data for this study in English.

1.5 Outline of chapters

This thesis intends to start conversation on and thinking around different ways of looking at learner thinking in mathematics. This chapter sketches the broad landscape of learning mathematics in South African schools. It problematises poor performance and focuses on discourse as means to examine this dilemma. In Chapter 2, I survey associated literature on the learning of functions. You will find that the language of much of the existing research is strongly embedded in the cognitive-acquisition paradigm, namely that we acquire knowledge or concepts by developing cognitive paths to this new knowledge. By proposing an alternate view, to explore learner performance, I acknowledge different and divergent paradigms as they have provided useful markers along the research trajectory for ways of looking at and judging learner performance. My approach may provide answers to some of the questions I sought, but they lead to far more, as I present a different way of looking at the problem of poor Mathematical performance. To do this and running alongside the existing literature, I elaborate a broad view of knowledge and learning functions, narrowing the focus to how these are vested in the South African mathematics curriculum. Simultaneously I draw attention to links from existing literature to my chosen discursive focus. In Chapter 3, the theoretical framing provides the definition of terms and their application in Sfards' theory of commognition, the discursive theory used as a frame for this study. It provides comprehensive description of commognition and develops the finer resolution focus on mathematical routines in learning. Chapter 4 attempts to convey the complexity of the South African school Mathematics environment, with description of the learners and schools involved. The methodology showing the passage of this study from the engagement of the role players, namely parents, teachers and learners, to the research instruments and tools, to critical adaptations that arose in connection to instruments, and ultimately to the analysis of data in the research process. Various data sources are illuminated, the way in which these were formatted for the levels of analysis that were engaged, followed by the analytical tools developed and their allied processes. Chapters 5, 6 and 7 look at the data collected across three signifiers of functions, the algebraic expression, the table, and the graph, respectively. These chapters integrate the existing theoretical constructs and introduce the new tools developed for looking at learners mathematical routines. Chapter 8 recorded the findings and conclusions of the study.

Research in socio-economic contexts of wide diversity and disparity among people, is perhaps the most exciting and challenging environment in which to work. The learners in this study were enthusiastic and generous on multiple levels and they provided impetus to work in ways that authentically captured their thinking as they participated in the mathematical discourse of function.

Chapter 2: The landscape
Situating the study in literature

2.1 Introduction

Lerman's (2001) 'zoom lens' is a useful metaphor for engaging with educational research. It is useful because we see differently when we zoom in on or out from our research focus, learners mathematical routines. When we zoom in we can see fine detail, a finer resolution of the rituals or explorations embedded in routines, but it is circumscribed. In contrast, when we zoom out, we get a broader perspective, the 'bigger picture' of routines and their importance in learning mathematics, and their place in a mathematics education research. I have found the metaphor particularly useful in an educational context like South Africa, where there are numerous and often competing factors in the processes and outcomes ascribed to mathematics learning and performance. Zooming in and out again enables the construction of a fuller, more nuanced story, of this complex quandary. This chapter does not unfold as a traditional literature review related to my questions. Instead, I begin with a zoom out so as to briefly offer my interpretation of the orientation to knowledge and learning of the object, function. I start in this way to clear the analytic lens which I have used to survey both the literature and data, recalling that my aim is to explore and describe learner thinking on function. The learning of this abstract and complex mathematical object contributes its own features to the challenge of its learning and teaching in the South African classroom. No object develops in isolation, and learning does not occur in isolation, and this necessarily includes the ways in which we choose to conceive of it.

In the next section (2.2) I describe the view brought by my zoom out, and so the broad landscape of the object and the related literature can be reviewed. I start by focusing my chosen theoretical lens on the acquisition-participation debate, and use this to discuss the development of the function object itself within formal mathematical discourse. This zoom out provides perspective from which to 'zoom in', specifically on the way function is recontextualised first in the South African curriculum and then in the South African mathematics classroom.
Section 2.3 locates my study in the literature of the field, pertaining to the learning of the object itself and its associated routines. I begin with the wide angle focus on the work pertaining to

learning of algebra and function, and then zoom in on the finer aspects of the object relating to notation, definition, keywords; followed by the literature on multiple representations.

2.2 A Theoretical view of learning

2.2.1 Establishing the participationist view

In this study, the dynamic and complex nature of learning functions is seen through a socio-cultural lens. Sfard (1998, 2006, 2008) distills two orientations to mathematics education research: the acquisitionist and the participationist perspectives, respectively. Her clear distinction between these perspectives cleared the lens by which the data in this study was analysed. Reviews of literature over the past three decades shows most research to be in the language of the acquisitionist perspective, with theoretical frameworks originating in Piagetian constructivism (Kieran, 2006). This contrasts the participationist perspective seen in more recent work, such as that of Nachlieli & Tabach (2012). With the current discursive turn in mathematics education research (Morgan, 2006), the transition from acquisitionist to participationist perspectives spelt a change in the unit of analysis. We are required to look beyond Piagetian levels of understanding or cognition, for example, as indicators of learning, as these are difficult to operationalise and hence describe clearly and/or measure. Within a participationist view, we see learning as participation in "a patterned collective way of doing" (Sfard, 2008, p. 78). A learner's progressive and increasing participation in the historically established mathematical discourse develops their mathematical thinking. This enables a view of learner mathematical thinking in their social-cultural contexts through '*what'* and '*how*' learners communicate, as opposed to the development of their cognitive capabilities, describing 'what they *have* or *are*' (op cit, p. 75). The latter is seen frequently, for example, in literature and reports on learners' errors and misconceptions (Smith, DiSessa, & Roschelle, 1993), which frames learners in terms of some innate ability or capability. From my experience in apartheid education as a student, particularly in gateway subjects (such as mathematics), ability was often linked to race, and my thesis opposes such notions strongly.

The participationist perspective espoused by Sfard and others (Caspi & Sfard, 2012; Nachlieli & Tabach, 2012) stands in contrast to Piaget's view that learning is a product of biological make-up. I regard this Piagetian view of learning as limiting, as it suggests a ceiling to learners' capabilities. The participationist view of learning, grounded in Vygotsky, sees learning as a result of social and cultural mediation. For the learners of this study, the socio-cultural milieu is primarily the classroom, where they learn in interaction with the teacher, each other, and resources that may be provided. Learning becomes visible in the way in which a learner communicates about the mathematical objects. Communication encompasses all means a learner would use to show how they think mathematically. While I have chosen to work within a participationist perspective, I acknowledge the acquisitionist for the useful mirror and trajectories it has enabled in work such as mine. The problems are the same indicated by their persistence - it is the gaze with which we see them and explore them that differs. Another key contrast in the perspectives is seen in the way that we define and operationalise the terms we use. Terms like 'concepts' for example, are used frequently in acquisitionist frames, see Ronda (2009) discussed later in this chapter. The word was defined by Vygotsky (1987, p. 48) as 'word meaning'. Sfard (2008, p. 111) usefully extends the definition as a concept being "a symbol together with its uses". This was particularly useful in encompassing the abstraction of the object function. The particpationist will speak of objects, as opposed to concepts, as that which we talk about. Using Sfards' (2008) dialectical equivalence of '*thinking as communicating*' was foundational to this thesis, where I explore Grade 11 learner discourse on function, at the conclusion of the topic in school Mathematics. The unit of analysis, within a communicational approach, thus resides in learner utterances, and in this study, is defined as learner realisations. Realisations can be seen as that which a learner is able to communicate about the primary object function and its related objects, called secondary objects.

I now turn to give a broad overview of literature pertinent to learner thinking in mathematics and in particular in functions. Further and more detailed elaboration of discursive/commognitive framework used to analyse learner responses, my 'zoom in' is done in Chapter 3. Both chapters 2 and 3 use literature from an acquisitionist perspective, because it holds up a mirror of the prevailing work around learner thinking to the discursive turn, which arose out of the need to answer acquisitionist quandaries from an alternate perspective.

2.2.2 Learning mathematics

Within a participationist perspective, learning is commonly seen as initiation into a patterned, historically established form of activity (Ben-Zvi & Sfard, 2007; Sfard, 2008). Sfard (2008, p. 299) defines learning as "changing discourse in a lasting way". Participationists examine *what* and *how* people do in these patterned human processes (Sfard, 2007, 2008).

Learning is through collaboration or interaction with others. Sfard (2012b), in her established discursive framework, commognition, moves once again away from the traditional vision of mathematics as that which can be seen to have been ‘given to people by the world’. Rather, grounded in the socio-cultural theories of Vygotsky, meaning-making and knowledge creation result from interactions with one another. In mathematics, Vygotsky’s scientific concept translates within commognition, to a formally defined mathematical object. Vygotsky defined *scientific* concepts in an effort to regulate word use, so that concepts can be described rigourously and without ambiguity (Vygotsky, 1986). Rigour, disambiguation and regulated word use are characteristics of the formal discourse of mathematics. The counterpart of Vygotsky’s scientific concept is the *everyday*, which matches discourse to that which is learned spontaneously through everyday experience (Nachlieli & Tabach, 2012). In my study, these terms are used to describe learners’ informal discourse on functions. As learner discourse develops, the goal of learning becomes to explore the objects of the discourse. This includes their properties (Ben-Zvi & Sfard, 2007; Nachlieli & Tabach, 2012; Sfard, 2008) and secondary objects, which result from realisations of the primary object. The Vygotskian constructs of *scientific* and *everyday* (also called spontaneous) gave means to describe learners’ mathematical discourse. The everyday discourse develops prior to the scientific, the role of their mutual support in learning, appears with limited presence in mathematics research. One such study from Kim, Ferrini-Mundy & Sfard (2012) showed how English and Korean students’ spontaneous discourses facilitate discourse development that aligns to the formal mathematical discourse. This related to the meaning attributed to mathematical words in everyday talk in the different languages. While it emphasised the social aspects of learning, it sharpened the focus on the inseparability of communication and learning in mathematics.

Communication is foundational to all social behaviour. Certainly, research spanning the last two decades defines learning as, essentially, a social activity (Ben-Zvi & Sfard, 2007; Lerman, 2000, 2001, 2006; Moschkovich, 2010; Mouton, 2012; Sfard, 2008; Sfard, Forman, & Kieran, 2001). Talking and thinking are considered examples of communication, in the form of communication with others and with self (Kieran, 2001). Sfard (2008) provides the necessary development of this premise in defining thinking discursively as part of the commognitive framework, and her work can be defined as follows: *mathematics* is a discourse, a form of communication that defines a community; *learning mathematics* becomes individualising[3] the mathematical discourse; and *thinking*, an individualised form of communicating. This formulation enabled the current study to approach access to learner thinking through learner communication.

It ought to receive certain emphasis that mathematics as a discourse is defined as a specialised form of communication. Mathematics as a discourse creates its own objects, it creates the things of which it wants to communicate (Nachlieli & Tabach, 2012; Sfard, 2008). We explore these objects on our own when we think about them or with others through conversation about them it (op.cit,2012). It is through some form of communication in this way, that these objects develop in complexity and abstraction, which is an important goal of mathematics. People who participate in the mathematical discourse would belong to a community, identifiable by this specialised form of communication. School mathematics is a subset of such a community.

The school mathematical discourse is intended as an initiation into this specialised form of communication. The rigour that characterises the development of the formal mathematical discourse ensures the well-defined rules of the discourse. These rules provide the distinct, patterned ways in which we communicate mathematically. After being initiated into the discourse, a learner individualises the discourse, to become an independent participant able to solve problems (Ben-Zvi & Sfard, 2007). This thesis looks at the formation of the mathematical object, function, in school mathematics, which becomes visible through learners mathematical routines.

Mathematical routines in commognition are considered as the patterned ways in which learners communicate in mathematics. For now, this common sense definition of 'routine' is sufficient. It will be developed for its meaning, detail and importance within commognition later in Chapter 3. The focus on routines initiates important questions for this study, and illustrates the interrelationship of the theory and query in the formulation of the problem: do learners' mathematical routines advance the formation of function and its related objects? What do learner routines tell us about the ways they learn and think about the object?

Commognition provides tools to explore these questions, and as noted, these are elaborated in Chapter 3. At this point, however, it is necessary to briefly discuss an additional

[3] Individualising refers to a person's "transition from role of mere observer of practices to fully agentive participant in a discourse. Similar in meaning to Vygotsky's internalization [sic]". (Sfard, 2008, p. 79).

element of the commognitive view of learning in some more detail. The framework labels two kinds of learning: meta-level learning, defined as learning that leads to a change in the meta-rules of the discourse; this is opposed to object-level learning, which expands the existing assortment of routines and endorsed narratives. Sfards' work has been further developed in various studies likewise significant for this study (Nachlieli & Tabach, 2012).

Learning, as discussed already within the participationist view, results in the development or growth in discourse, through participation in the discourse, by oneself or with others. The discourse on functions poses challenge to individualising the discourse as it subsumes the discourses on algebraic expressions, graphs and tables (Nachlieli & Tabach, 2012; Sfard, 2008) the representations of function, which are focused on in this study. This development of discourse over multiple representations is called horizontal, and can be contrasted with vertical development, which describes growth or learning within each of the separate representations (Nachlieli & Tabach, 2012). The word *function* when used, thus comes to signify an encapsulated whole in which these separate representations reside. In the language of commognition, the word function is then a *signifier* for *realisations*, or procedures, that pair a signifier with a specific object or the products that result from these procedures. These realisations were the unit of analysis in this study. Sfard has noted, "*a mathematical object thus becomes a signifier with its realisation tree* ... a realisation tree is a hierarchically organised set of all the realisations of a given signifier together with realisations of these realisations" (Sfard, 2012a, p. 4). This implies discursive expansion for the learner, as opposed to cognitive growth.

One such study, that looks at learning of functions in this hierarchical way, is through the notion of "growth points" by Ronda (2009). Growth points mark specific points in learning function, which are progressive and necessary to understand the concept. With its cognitivist underpinning, it marks the primary contrast in the distinction between the acquisitionist and partcipationist views alluded to earlier. In growth points, we acquire knowledge and competency in prescribed processes to ascend the hierarchy of levels. Growth points thus entail a description of learners as 'having acquired knowledge or concepts and competency in' processes as they traverse the points describing associated cognitive jumps. Commognition, in contrast, marks the connection or link between realisations and objects. Growth of the mathematical discourse becomes evident in the concrete and visible ways in learner talk. Objectification is one such way, where talk of processes is replaced with talk of objects. Learning is evident in progress through discursive levels which grow mathematical objects in complexity and abstraction. The descriptions centre on *what* and *how* a learner communicates mathematically. The critical distinction is thereby once again drawn between the participationist and acquisitionist views and frames informing the exploration of data in this study.

With the broad landscape of how learning and knowledge description is understood within the current research frame, I draw focus on the formal, historically developed, endorsed discourse of the object, function. The summary provided of the development of this object is a very good illustration of how "discourse develops as a product of human actions" (Sfard, 2012a, p. 2). Discourse thus grows, contingent on human needs and from the basic human drive, to think for increasing complexity. In addition, it serves to hold up the mathematical discourse that learners are required to participate in before turning to functions in the school curriculum. The illumination of the formal endorsed narrative of function at this point, serves as a reference for the learner discourses on function as they appear in Chapters 5 to 7.

2. 3 The Development of the object function

2.3.1 Historical development

The historical development of function speaks to the dynamic growth of discourse in a basic human quest for complexity and abstraction (Sfard, 2008). Sfard has noted that "function was born as a result of a long search after a model for physical phenomena involving variable quantities" (Sfard, 1991, p. 14). The development of the discourse on function can be summarised as follows:

1707-1783: Euler defines function as a dependence relation;

1805-1859: Dirichlet function as an arbitrary correspondence between real numbers;

1932- : Bourbaki, generalises Dirichlets' definition where function is defined as a correspondence between two sets. Direchlet-Bourbaki allows function to be conceived as a mathematical object (Sfard, 1991).

We can easily see how narratives make necessary changes that are contingent on their usefulness. The definition of function that coheres to the versions offered in South African school textbooks which suggests a specific relationship is noted below:

> "A function is a special relationship between input and output values where every input value has only one output value. Note that different input values can have the same output values" (Laridon, et al., 2011, p. 139).

The development of this definition is illustrative of the social construction of knowledge, and certainly the evolution of the discourse on function to speaking of a specific relationship between variables. While the notion of relationship is stressed at the beginning of the definition, it cannot be assumed that learners will develop realisations that speak to the notion of relationship. In particular, in an environment which emphasises an orientation to process, such as ours in South Africa, learner focus can easily revert to the process of input values for output values in a function relationship. Realisations made possible by this endorsed narrative should signify to learners the properties and features of all similar objects accumulated in the discourse of function. It may be recalled that commognition defines an object as a signifier together with its realisations. The endorsed narrative, the definition of function, is a means that connects and encapsulates the discourses of the various representations for learners. Thus, instead of learning a collection of disparate, unconnected rules and concepts, objectification becomes a means to connect and subsume all related discourses to an encapsulating object (Sfard, 2008). The formal narrative above has its discursive place, where it does not exist for learners to reconstruct or recite passively. In teaching or guidance by a knowledgeable other, it is filled with meaning and is to be used as a trigger for objectification (Nachlieli & Tabach, 2012). In teaching, the equivalence of representations and the relationship between variables serve as the mentioned 'triggers', and then need to be made explicit. The examination of this formal narrative, for what it is and signifies, serves as point of reference in this study for how learners have formed the object function.

Early theoretical frameworks analysing learners understanding of function focused on multiple representations and translation between them (Janvier, 1987; Kaput, 1989). The purpose of working across multiple representations of a function is to connect the different representations to the object they signify and to "establish a sense of invariance" (Slavit, 1997, p.

264). Invariance is best described by an example of a straight line. With the exception of univariate relations (like $x=5$), we can generally describe the gradient and intercepts of straight lines. These properties are invariant across this class of function. Other frameworks combine process-object theory and the different representations of function (Moschkovich, Schoenfeld, & Arcavi, 1993). One route, grounded in Piaget's (cognitively based) theory of reflective abstraction, leads the learner in computational processes to a process conception, that later becomes encapsulated as a mental object (Dubinsky & Harel, 1992). This earlier work uses the keywords: concept, understanding, mental objects, and schema. Even Sfard's earlier work (1991), in genesis, straddles the boundary between these cognitive descriptions, using the keywords listed, before it developed into a fully discursive framework. She has described *concepts* as conceived as a process before they are conceived as a mathematical *mental object* (Sfard, 1991, 1992). In the process-object theory thus, objectification is a result of experience at performing actions on objects (Sfard, 1991).

A second route to objectification of the process-object conception of a mathematical concept can be through understanding its properties. Working with various function classes, noticing their properties, learners conceive of functions as objects either possessing or not possessing the class properties (Monk, 1988; Ronda, 2009; Slavit, 1997). Borrowing from both perspectives and in summary there appears discursively, two routes to the objectification of function for learners: the first is through the reification[4] of processes, and the second is through the developing a discourse of the functions features and properties.

Mathematical objects grow as a consequence of our participation in a conversation about them. This was the purpose of the development of the object function illustrated above, where we see different mathematicians at different times, engaging 'conversations' which change the ways that the object is defined. The extent of the discourse of function expected of South African learners is defined in the National Curriculum Statement (NCS), imbued with its South African identity, where our approach to the object impacts it teaching and learning, and the resources which become available to learners. The curriculum, since the advent of formal democracy, has seen transitions from an outcome focus (in the NCS) to a greater content focus at present (in the

[4] Reification in commognitive theory, replaces talk of processes with talk of objects (Sfard, 2008).

NCS- CAPS[5]) (DBE, 2007, 2011b). In South Africa traditionally, we initiate functions through an algebraic pointwise orientation (DBE, 2007), which backgrounds the defining relationship narrative. The implications of this approach are interesting, and how they unfold in terms of how learners work with functions is discussed in the analysis chapters. The NCS is currently prescriptive in developing the trajectory of the topic for teachers in terms of a list of content topics to be followed and the assessments specified.

2.3.2 The South African curriculum

Chisholm (2005) notes the dominance of procedural approaches to mathematics teaching in South Africa. I will argue that a procedural approach is not problematic in itself if processes are reified, as a necessary step to objectification. The curriculum, which drives teaching and related resources, prescribes to teacher's structure and direction to progress through its listed topics. What learners are to learn is stated in a content format (du Plessis, 2013). The presentation as a list of topics, possibly suggests to teachers a procedural orientation, as they do not describe the strategies and thinking learners are required to use (Ronda, 2009). To understand functions (implying their acquisition), learners have to objectify functions on multiple levels, to relate old knowledge to new, and to consolidate what is learned (Watson & Harel, 2013). The work from an acquisitionist frame above provides an area of common purpose to the discursive-participationist, that is, the focus on objectification. The commognitivist/discursive researcher will seek first to operationalise the terms mentioned like, understand, objectify, coordinate, consolidate, before exploring a particular quandary. So, as stated earlier, thinking, to the commognitivist, can be accessed through communication. How learners think, as well as, attention to the participation in and development of a mathematical discourse, appears not to be emphasised in the structuring of the South African curriculum. As long as a curriculum is shrouded in being highly prescriptive, in terms of content and assessment, it compromises the attention that is necessary to aid and develop learners' mathematical thinking. This reduces mathematics to a check list of executionable steps attached to specific content that can be learned without agency (Nachlieli & Tabach, 2012). As a consequence, a learner could engage a ritualised practice, an execution of procedures, which do not serve to develop a complex

[5] CAPS is the Curriculum &Assessment Policy Statements. CAPS is not a new curriculum. It is an amendment to the NCS. It specifies the movement from the previous outcome format of the Curriculum to a content format (Pinnock, 2011).

mathematical reasoning. The description of the development of the topic on functions in the school mathematics curriculum follows to highlight its leaning towards a ritualised practice. The South African National Curriculum Statements, NCS (CAPS) (DBE, 2007, 2011b) specifies, for Grades 10 to 12, what learners are expected to know by Grade 12. A summary, extracted from the NCS-CAPS is presented below. I have italicised for emphasis on what I believe to be the broad intent of the topics related to functions.

1. *Working flexibly* across all representations: numerical, graphical, verbal and symbolic, is stressed.
2. Graphs begin with point-by-point plotting and progress to the *generalisation* of the effects of parameters on the linear, quadratic, hyperbolic, parabolic graphs and the graphs of the fundamental trigonometric ratios.
3. In Grade 10, vertical transformations of these graphs are specified.
4. The development in Grade 11 specifies the horizontal transformation of the graphs and includes the exponential graph.
5. The work in Grade 12 *investigates* the inverses of these graphs.

The key point to emphasise is that the initiation into function is through a pointwise orientation (Even, 1998). This is the point-by-point plotting of ordered pairs, which satisfy the algebraic expression of the function, progressing to recognition of discernible features of the particular graph and possible transformations of the function. Functions are taught and learned in compartments of the various function types: linear, parabolic, hyperbolic, exponential and basic trigonometric graphs. The curriculum is silent on specifying what 'working flexibly', 'generalisation' and 'investigates' (indicated in italics above) mean and how these can be achieved in practice. Without elaboration and emphasis on these important ways of thinking, the curriculum is left open to interpretation, not with regards to what has to be learned, but how it should be learned. Foundational to any ritualised practice is the need for social acceptance (Sfard, 2008). Ritual appears to gain in presence and prevalence in the contexts of grave social inequality and pervasive poor performance, where success in mathematics seems to guarantee access to further study and better employment opportunities (Sfard, 2012b). Behaving in ways that are predictable and to which everyone conforms, provides a sense of security to those participating in the mathematical discourse, but could easily constitute a mirage of mathematical thinking. Mathematical procedures or processes which are not objectified are a good indication of ritual.

The content in the curriculum builds and specifies the processes required for the graphs studied in school Mathematics. It also lists the characteristics or properties of the functions in terms of its domain and range, intercepts, turning points, asymptotes, shape, symmetry, periodicity and amplitude. The final three content areas then turn to average gradient, intervals of increase and decrease, discrete and continuous graphs. The discourse of function as a specific relationship between two variables, called the input and output, is absent or assumed up to this point, which we need to remember is the termination of the topic in the school curriculum. The object function is meant to be filled with meaning as a self-standing mathematical object in its own right. It is intended to subsume the discourses of the separate representations as representations of a specific function relationship. The subsuming discourse thus offered a useful marker for which to check. In school Mathematics, function appears presented as several separate function types: the linear, quadratic, exponential, logarithmic, cubic and various trigonometric graphs.. This is despite the explicit curriculum statement that learners are expected to work flexibly across representations. How we are to work flexibly, and the gains from this flexibility, remain opaque to learners and teachers. This is observed in literature, which found that while learners may be able to transform one representation into another, this offers no means to assess if learners see equivalence between representations or not (Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013).

This section thus far has reflected the affordances and constraints of the South African curriculum as it pertains to the teaching and learning of functions. I acknowledge that no curriculum document can specify all detail to fine resolution. What is noted for purposes of this study is that which the curriculum leaves open for interpretation. This becomes important in the exploration of learner discourse for how learners engage the object. To develop a discourse means to build longer realisation strands or fuller realisation trees, since the curriculum does not provide guidance on how teachers are to develop these realisations. In addition, deliberate and focused attention to developing the connecting discourse, which links the individual functions to each other and the subsuming object, needs to be developed alongside an understanding of why it is so important to learning the discourse of the object. This is suggested as a means to break the ritualised loop and initiate exploration into function.

Relating this to commognitive theory, the curriculum document shows emphasis on the *how* of a mathematical routine, where learners use prompts for a mathematical routine in order to execute it. The extract below is taken from a report from the department of education on the NSC examination 2011 (DBE, 2011d, p. 101), and refers to learner responses to questions on functions and graphs.. The extract was chosen for its direct link to supporting the curriculum as a report on a national examination, and its specificity in the advice offered in teaching the function topic. It upholds the acquisitionist lens, with reference to use of words such as errors and misconceptions, and understanding. In addition, in the first paragraph, it draws attention to spontaneous responses by learners to find the intercepts-decisions not made on interpretation of a question, but merely from perceived prompts. A learner recognises familiar cues and initiates a familiar routine, regardless of what is sought by the question asked. Such learner narratives based on prompts, was indicative of ritualised discourse, especially when the question is not answered. The document, while progressive in discursive terms by the keywords and phrases it uses, does not make explicit the means teachers could use to implement the recommendations.

For instance, in the second paragraph, it is suggested that teachers 'discuss' the information obtained from the graph with learners. This points to developing narratives on the properties of graphs. It would seem that the document is alluding to forming an objectified notion of function, through objectifying the features of the graphical representation. This was considered progressive, as teachers were to engage learners in the discourse of function. The second progressive stance noted is that teaching does not end at the sketching of the graph. However, the document is silent on connecting the graphical representation to the primary object, function and developing meaning of keywords such as 'asymptotes'. It appears sufficient in the document for learners to curve their sketch of hyperbolas to approaching 'asymptotes' as opposed to 'being shaped away from the asymptotes'. This backgrounds the meaning of the keyword 'asymptote', significant in learning about functions, and is crucial in linking the graph to algebraic reasoning around the significance of asymptotes. This can have significant consequence for the development of a function discourse. Further review of the report shows the focus on common errors which learners make on the topic relating to calculations involving fractions, particularly and the assigning of coordinates to points on univariate functions (literature on such functions are discussed later in this chapter).

Figure 3 Department of Basic Education Report on NSC 2011.

QUESTION 5: FUNCTIONS AND GRAPHS

(i) COMMON ERRORS AND MISCONCEPTIONS

In previous papers, the calculations for *x*- and *y*- intercepts of the hyperbola were asked as a single question. Learners need to read for understanding and not just assume what the question is asking. Many learners calculated the *x*- and *y*-intercepts as a unit rather than specifically answering the individual questions. Learners generally knew how to find the intercepts with the axes.

Question 5.1.4 was poorly answered. This suggests that learners and teachers do not spend enough time discussing information that can be obtained from graphs. It is important when studying functions that teaching does not stop at how to sketch the curve, but rather that questions which require using the graph to deduce properties of the function also be considered. In this case, $f(x) > 0$ when the curve is above the *x*-axis, and so the answer to the question was $-3 < x < 3$. Learners who attempted this question failed to notice that x should also be less than 3. The implication is that the concept of an asymptote is not well understood and should be explained clearly by all teachers.

Question 5.1.5 required finding the average gradient between 2 points on the curve. Many learners knew the correct formula to use. Errors were, however, often made, in that arbitrary *y* – coordinates were used for the points where x is – 2 or 0. f (–2) needs to be calculated by substitution in the equation, while the *y*-intercept is already known.

Some learners found Question 5.2 unfamiliar and consequently omitted it. Learners who attempted it generally scored at least some marks.

(ii) SUGGESTIONS FOR IMPROVEMENT

Since calculation errors were common, it is important for teachers to emphasise in Grades 9 and 10 the importance of working with algebraic fractions. Teachers should also take care to show learners how to shape their curves, as many curved the arms of the hyperbola away from the asymptotes.

Teachers should note that the need for learners to know the effect of the various parameters in the equation of the parabola is clearly stipulated in the SAG document. In teaching the function, learners should first know the basic curve and then be given the opportunity via worksheets to investigate the effects of changing the values of the various parameters. This could also be the topic of one of the SBA investigations in grade 11.

The document suggests the approach of working with parameters of the equation and linking these to changes in the graph. The teaching approach is suggested through worksheets to investigate the changes. This is, again, a critical silence, as the discourse to be developed with learners is not specified. On the whole though, while the report has several limitations and oversights, it does offer a wider interpretation of the curriculum. Developing a discourse however, is highly specialised, and unlike the curriculum which lists topics, the pedagogical meta-discourse of the keywords, narratives, visual mediators and routines is critical, yet remains unspecified for teachers. The last sentence in the extract urges teachers to place similar tasks to their worksheets into assessments (called a SBA or subject-based assessment). The emphasis on assessment as a primary goal for learning mathematics, the strong prescription of frequency and content of assessments, can have unfortunate implications for learning and teaching. No literature could be found which indicates that in contexts of poor performance these lead to improvements. They do however encourage a ritualised practice on both these fronts, teaching and learning, and compromise discourse development to levels of abstraction and complexity. Commognitively, working with or for complexity and abstraction becomes a marker for successful participation in mathematics. The relevance of the routine or *when* it becomes applicable is not explicit in documents provided to teachers, compared to the emphasis on the execution of the routine itself, prevalent in textbooks and teaching resources. As a result, research notes that the object function itself is never fully formed, but becomes a disparate collection of distinct, individual relationships represented by the linear, quadratic, hyperbolic functions (Even, 1998; Gagatsis & Shiakalli, 2004; Nachlieli & Tabach, 2012; Vinner & Dreyfus, 1989).

In short, the South African curriculum provides a structured and organised conceptual list of topics for teachers to follow in the learning of function. Strategies for thinking or means to develop complexity are not explicit in the curriculum. With our history of inequality and a prevailing socio-economic context of grave disadvantage, as well as our complex cultural and language diversity, the curriculum has inserted necessary imperatives for redress and equality. It does not appear to favour a particular teaching/learning paradigm, and thus, as described earlier provides a list of topics for teachers to interpret. The way in which learners develop thinking and advance their participation in the levels of objectified discourse possible for the object function remains unspecified for teachers, in our context of poor performance and complex social dynamics.

In South Africa, the curriculum is the driver of classroom practice around the object. How then does it unfold in the classroom? What are the implications for learner participation and performance? The next section looks at how the curriculum manifests itself in the classroom as an experience of the object for learners in school Mathematics. Here, discussion is not restricted to South Africa, but rather zooms further out to research on school Mathematics more generally.

2.3.3 School Mathematics

School Mathematics is a confluence of several convergent tensions brought through disparate influences. This section examines learning in terms of:

1. the curriculum and its influence on school Mathematics;
2. school Mathematics and its role in the development of a mathematical discourse;
3. the significant contribution of a knowledgeable other; and
4. learner autonomy.

Learning depends not only on the curriculum or its related policy statements, it also depends on school and classroom context, the teaching learners have access to, expectations related to assessments (Watson & Harel, 2013) and most importantly, it depends on the learner. This study shows consideration of the encompassing social context of learning, the learner in a classroom in a learner (Lerman, 2006) and indeed its contributing complexity. Learner mathematical discourse thus becomes an important tool to view the complexity of learning in the context of the classroom, with the learner realisations as a focus. Discursively, the realisations that are made possible in the classroom for learners are largely a product of what the curriculum prescribes. Due to the poor performance rates in Mathematics as a school subject, witnessed annually, and discussed in Chapter 1, the curriculum has become increasingly prescriptive in attempt to address the persistent problem. This can be seen in the evolution of policy statements from the initial Curriculum 2005 to the current Curriculum Assessment Policy Statement, an amendment to the National Curriculum Statement.

The discourse of school Mathematics derives directly from the National Curriculum Statements. Learning mathematics, as developing a discourse, presents an inherent paradox, namely that “familiarity with objects of the discourse is a precondition for participation, but at the same time participation in the discourse is a precondition for gaining familiarity” (Sfard, 2008, p. 161). The building of successive discursive layers entails following rules of the historically established narratives of the discourse (op cit). The high stakes associated with success in Mathematics results in an emphasis on rules, and this with the inherent circularity of learning, account for why an objectified discourse is found less frequently in school Mathematics. Learners who are successful are able to follow the rules when they pick up cues to do so. Much research from acquisitionist and participationist frames, attempts to explain the difficulties learners experience in this regard. De Lima & Tall (2007) talk of an encapsulation of mathematical actions that contribute to the formation of a mathematical concept. Other theorists (Dubinsky & Harel, 1992; Sfard, 1991, 2008) note that the shift from process to object is a difficult one for learners. To contrast the two sources above, we see the parallels in describing learning: actions to concept and then process to object. This thesis will search the space between the paralleled theoretical labels described. What is required is research that describes the way in which learners are to navigate or transition the difficulty they experience. This thesis characterises the transitioning space as a discursive one. A learner, in developing an increasingly objectified way of communicating, through participating in the school mathematical discourse, will most likely have the mathematical means to transition the difficulty described. With an emphasis on rules in the school mathematical discourse, I drew focus to learners' mathematical routines in the way in which they objectify.

Within a content focused curriculum, with low to no guidance on object formation, it is expected that few learners will talk of mathematical objects. Discursively, the main goal for a learner who engages the school mathematical discourse, is to be able to tell stories by means of which to talk about the mathematical objects they explore (Sfard, 2013b). School-type learning is an activity in which the student modifies and extends her discursive repertoire (Ben-Zvi & Sfard, 2007). The discursive repertoire of school Mathematics, algebra in particular, should incorporate both process and object ways of communicating, where it can currently be observed to be skewed in favour of the process. With regards to mathematical routines in commognitive theory, learners are meant to know *how* to complete a mathematical routine, but also *when* that routine becomes applicable. Acquisitionist literature confirms this link between knowing a procedure and knowing the effect of a procedure (De Lima & Tall, 2007). The discursive distinction between 'how' and 'when', helps focus the aspects of learning evident in routines which is the focus of this study. The literature that follows explores objectification, routes to achieving it, and its role in learning.

While more objectified ways of communicating may encourage the growth of complexity and abstraction in discourse, participation in the discourse with a knowledgeable other is recognised as a means to achieve this objectification (De Lima & Tall, 2007; Sfard, 2008). Commognitive theory stresses the importance of the *knowledgeable other* (Sfard, 2008), who is usually a teacher, but could be a learner, who would play a critical role in both object level and meta-level learning. This suggests that the path to exploration in mathematics is hardly automatic, or achievable through independent endeavour by a learner, but rather, through participation in a scaffolded conversation with a knowledgeable other. To build the notion of this transitioning space and what it could hold for learning, it would seem that a knowledgeable other and a more objectified discourse reside here as catalysts to exploration. Commognition acknowledges that we cannot assume learners will transition from one level of objectification to a higher level, or transition discursive challenges they face, as mathematicians do in a breakthrough. In fact, learners are predicted to reach a dead-end (Ben-Zvi & Sfard, 2007). School Mathematics, as with all levels of study in mathematics, does not hold the expectation that learners will redevelop and rediscover the mathematical discourse that already exists. The intention is for learners to become fully-fledged participants in already established formal, endorsed mathematical discourse (Sfard, 2008) through exploration. Exploration compels a fuller encompassing interpretation of our content focused curriculum. Does school mathematics adequately prepare learners for exploration?[6]

The knowledgeable other comes with valuable experience in mathematical routines. They give learners a view into how to work these routines in automated and embodied ways, as with riding a bicycle. Commognition places significant importance in the imitation of an expert in any learning process. Imitation of the mathematical routine and the ways of working with fluency within the routine is acknowledged as an important first step in learning (Sfard, 2008). If learners begin to individualise a mathematical routine initially through the imitation of an expert, this raises important questions about the competence of the teacher. As a result, much research linked to teacher knowledge in mathematics, and linking teaching to learner performance, can be found (Bryne, 1983; Monk, 1994; Sorto & Sapire, 2011). These have resulted in the development of useful constructs like pedagogical content knowledge (PCK) (Shulman, 1986) and Adlers' (2006) *mathematics for teaching*. The teacher, in a context of lack of resources, as in the majority of South African schools, becomes the primary resource for learners and plays an important role in developing the mathematical discourse (Adler & Pillay, forthcoming).

Research pertaining to the South African context specifically, shows the problems which exist in connection with the knowledgeable other, pervasive on many levels of schooling. At primary school level, teachers have been shown to know only what is expected of them from the

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[6] Exploration is a type of mathematical routine whose goal is primarily to produce endorsable narratives. This is developed in detail in a commognitive sense in Chapter 3.

curriculum and that some may know less (Taylor, 2009). Four years later, Taylor (2013) lists the following obstacles to learning on a longitudinal, multi-level study of primary school teachers:

1. South African Grade 6 Mathematics teachers have inadequate subject matter knowledge for learners to be prepared for subjects continued in secondary school; and
2. teachers' subject matter knowledge is a barrier to learning.

At the secondary school level, Bansilal et al. (2014) showed that in a sample of 253 teachers from a single province in South Africa, when assessed on a NSC examination paper, could pass with an average of 57 percent. The matriculation pass rate in the province was 53.6% in the same year (op cit). This team describes teachers performance as decreasing as the cognitive demand of the paper increased. The NSC is structured according to Blooms Taxonomy of cognitive levels. On the level of problem solving, teachers averaged 26 percent. These shocking observations imply that South African teachers, at both the primary and secondary levels of schooling, do not possess the mathematics knowledge that is needed, and therefore cannot promote a disciplined development of the mathematical discourse among learners. Such research implies our learners will have cumulative gaps in their knowledge, through all phases of school Mathematics. Primary school Mathematics teaching appears inadequate to support the abstractedness which is to develop through the further education and training (FET) phase. The knowledge learners are exposed to at secondary level is thus a product of the accumulated gaps in teachers' knowledge. The teacher is the learners' initiation into the specialised mathematical discourse and literature surveyed above, imply a disconnection between learning and the mathematical discourse used in instruction. The Taylor study (2013) attributes what appears as 'dead ends' in specialised knowledge, across schools, to the knowledgeable other. This in combination with the lack of agency that learners feel in highly processual learning environments (discussed below) means that mathematical discourse development is punctured at several points along a learners' school life.

Examination of the role of the knowledgeable other is also related to learner autonomy. We build learner autonomy through watching how experts work (Ben-Zvi & Sfard, 2007). The experts' role initiates learners into the discourse. The learners' sense of autonomy drives the individualising of the discourse and learning to higher levels of abstraction. The processual patterns of learning, seen in learner responses on the NSC, show an emphasis on the 'how' of mathematical routines. This contributes to diminished agency among learners, where recall or reliance on memory as opposed to mathematical reasoning is prioritised. When the object is obscured by the emphasis on process, it results in a discourse of pseudo-objects, which resemble the genuine concept, but are not fully formed objects (Berger, 2005). Learners develop autonomy and agency when they begin to participate in object-level discourses independently (Ben-Zvi & Sfard, 2007). Independent object level participation, is exploring mathematical objects, and is an indication that learners have individualised the formal mathematical discourse. Learners develop and extend realisations of these objects. New levels of discourse are often presented by the knowledgeable other. These are meant to initiate further exploration, rather than to become the object of learning by remaining in a ritualised practice. The consequences of the orientation of the curriculum (discussed in the previous section), combined with access to a knowledgeable other, the specialised knowledge of the teacher; all appear to exert critical influences on school mathematics. How do learners in this study pursue exploration of mathematical objects? Is it thoughtless and ritualised or an exercise in rationalisation and logical deduction? What is evident of the nature of the school mathematical discourse in relation to the formal discourse which emerges as learners work with the object function? These questions underpin this study, which characterises learner discourse as both exploratory and ritualised.

This section described the trajectory and tensions of the object-function being investigated, from formal mathematics to its eventual presence in school mathematics. The next section can now locate relevant aspects of the object in the field of teaching and learning functions in mathematics education research.

2.4 Describing learning of function

The literature in this section relates to learning in algebra, which supports the development of the object function. It is presented at a wider focus before the detailed literature of function is more closely discussed. The broader algebraic discourse is foundational to the development of the discourse on function. Commognitively, we can say that the discourse on functions subsumes algebraic discourse. While there is much valuable work already done on algebra, I attempt to establish a discursive position informed by Sfard (2008), Caspi & Sfard (2012), and Watson (2009). Watson in particular provides a useful review of literature on algebra, and I select from this as I develop the focus on working with symbols and the connection between learners' formal and informal discourses. These provide a basis to begin my discussion on functions.

2.4.1 Algebra

In South African school mathematics, the importance with which algebra is regarded is evident in the presence it shows in the national curriculum statements and national assessments like the high-stakes National Senior Certificate. It receives the largest portion of time allocated to teaching Mathematics in the classroom and the highest mark allocation on the NSC (DBE, 2011b). Like many other countries, much research centres on algebra, where we focus on improving results (Knuth, Stephens, McNeil, & Alibali, 2006). A large portion of existing literature looks at algebra developing spontaneously from the need to formalise and generalise arithmetic, where school algebra is seen as a meta-discourse of arithmetic (Sfard, 2008); informal algebraic discourse appears to emerge early from the need to generalise arithmetic (Caspi & Sfard, 2012). Algebra is the way we express generalisations. To explore learners discourse on functions in school mathematics, it is assumed that the layers of discourse development would follow the progression from arithmetic to algebra to function.

Spontaneous meta-arithmetic reveals patterns and features which may not be a part of learners everyday discourse (Caspi & Sfard, 2012). It has been shown that when learners are allowed to use their own methods of calculation, they find algebraic structures for themselves (Watson, 2009). This does not preclude the involvement of the knowledgeable other in the selection of the activities supporting this learning, as well as for the induction of the learner into the formal mathematical discourse, the routines and syntax. Despite the suggested natural tendency towards algebra, Ben-Zvi (2007) notes that the broader algebraic discourse, and the object function in particular, are new to students, and require meta-level learning before learners can independently participate in the new commognitive activity. The knowledgeable other, notably the teacher, plays an important role in the formalisation of the arithmetic to algebra. An essential part of formalisation here entails explaining how one generalises (Watson, 2009). This is a necessary aspect of the meta-learning involved. It points to the transitioning space alluded to in the previous section. While the Further Education and Training (FET) curriculum for Grades 10 to 12 as discussed in 2.3 (ii) mentions generalising as an important skill for learners, it is silent on specifying how this generalising ought to be carried out. How can teachers provide effective opportunities for learners to generalise and to communicate how they do this? This marks an important transition from informal algebraic talk to formal algebraic discourse.

The decision to grow formal algebra from informal gets support from acquisitionist and participationist theories of learning, both frames sharing the emphasis on connections. Acquisitionists describe the change as connections of phenomena into rich concepts, where the process of making links leads to the compression of knowledge from complicated phenomena to rich concepts, with usable properties and coherent links to other ideas (De Lima & Tall, 2007); if the two forms, the formal and informal, do not connect, then true concept development does not take place (Daniels, 2007). Participationist discursive frames discuss the transition using notions of connections between the formal and informal (H. Venkat & Adler, 2012), thus characterising the transitioning space. One such means, encapsulation, involves learners informal ways of generalising being formalised into working algebraically. Algebra allows compression of a learner's formal and informal discourse and disambiguation of the informal. For me, encapsulation and compression enable connections, which are made discursively in the transitioning space to a higher level of discourse. Formal discourse, in comparison to informal, is compact (Caspi & Sfard, 2012) and as a result of the algebraic apparatus, you can say more with less.

In addition, algebra enables us to see similarities in objects and their features. Objects may be classed according to their common properties or behaviour (Sfard, 2008). A survey of research shows that these connections seem spontaneous for some learners, in the way that learners attempt to generalise arithmetic (Arzarello, 1992; Booth, 1984; Caspi & Sfard, 2012; Dekker & Dolk, 2011) and problematic for others (Herscovics & Linchevski, 1994). The emphasis on the connection between formal and informal can be seen in the works of Vygotsky, who observed the connection between the everyday (informal spontaneous) and scientific (formal) concepts (Vygotsky, 1986). Algebraic ability developing closely alongside learners' own discourse ensures continuity between discourses, establishing the important connection between the formal and informal, between what they are familiar with and what they are to learn (Caspi & Sfard, 2012; Daniels, 2007). Algebra also visibilises properties, features and behaviours of objects difficult to discern if communicated in everyday talk. Drawing learners attention to features of the algebraic representation minimises the danger of purely ritualised learning, where the algebraic discourse is used as a discourse for others (Sfard, 2008), in the execution of recognisable processes in isolation of the objects to which they are connected.

Within commognitive theory, objects are defined as those things being talked about. The discursive nature of objects attributes to them the dynamism of definition and redefinition (Sfard, 2008). This was illustrated in the development of the formal definition of a mathematical function in 2.3(i). Algebra has its own objects and own rules (Caspi & Sfard, 2012; Sfard, 2008) and these evolve and grow through what they make possible mathematically. The discursive growth essentially advances in layers of complexity. The highest level of informal discourse comprises formalising all layers of discourse that are developmentally prior (Caspi & Sfard, 2012). This is seen in reflecting on the operations and processes of arithmetic as facilitating the transition to a level of algebraic reasoning. The reflection on the informal appears as foundational to the transition from the informal to formal. While arithmetic seems to facilitate the transition into algebra, the way in which to effectively cross this divide in discourse development remains under-examined in literature. There is sufficient work which locates various transitions in algebraic development and describes the difficulties learners encounter related to these. De Lima &Tall (2007), for example, show that students do not encapsulate algebraic expressions from process to object. Commognitively, this objectification, where learners move from talk of process to talk of objects in an alienated way (Sfard, 2008), is signalled in this thesis as an important step to transition to developing an exploratory discourse. It will be discussed in Chapter 3.

The symbolisation characteristic of algebra is an important marker of the transitioning space. Symbolisation is visible in transition from informal to formal, and it is significant in the development of algebraic thinking (Caspi & Sfard, 2012). The need to generalise arithmetic introduces the specialised symbols and notation, which in part, characterise the formal algebraic discourse. It is useful at this point to summarise the characteristics of the formal discourse to establish contrast with the informal:

1. Disambiguation refers to the explicit rules of the discourse ensures that differing interpretations do not arise.
2. Standardisation ensures that everyone uses the same rules when communicating.
3. Compression is made possible by the formal discourse enables us to say more with less. Lengthy statements can be represented by concise expressions which can be manipulated. This occurs through:

    - *Reification*, which turns talk of process to talk of objects.

- *Symbolisation*, which introduces ideographs, nouns and verbs are replaced by symbols. Involves change in visual mediation from nouns and verbs to ideographs. The symbols exist independent of the language and they convey meaning.

(op cit, p. 46)

Discursive literature further makes the distinction between colloquial and informal discourse; colloquial discourse being visually mediated by concrete material objects independent of the discourse (Sfard, 2008), where "colloquial discourses are also known as spontaneous or everyday" (op cit, p. 132) as they develop from patterns which occur in our real life. They contrast the formal, which is symbolically laden, and involves algorithms and processes that manipulate these symbols. These characteristics were useful markers in examining learner mathematical discourse. They allow for distinction between the formal and informal discourses, as well as for examination of the transitional discursive space in which the informal links to and becomes the formal. The introduction of mathematical symbols entails meta-level learning for learners and their use exists in the problematic transitional space described. As stated earlier, literature needs to examine how the informal algebraic discourse can effectively be used in teaching, to support the development of the formal.

Thus far, I have examined the characteristics of a formal mathematical discourse. Literature makes it possible to colsely examine, or 'zoom in' on, the components of both the formal and informal discourse. Caspi & Sfard (2012) decompose algebraic discourse, both the formal and informal, into a hierarchy of layers, ordered according to what the layer is about, "by parsing the canonic given discourse into a sequence of discourses – every element is a meta-discourse of the previous one" (op cit, p. 46). This is an extension of Sfard's (2008) work which similarly describes a mathematical discourse as developing by formalising and annexing its own meta-discourses. I interpret these elements as abstract objects and talk of these arises from prior discursive levels. Layers increase in complexity and what is made possible by them as we traverse them from the informal. A mathematics classroom is thus distinct from other classrooms – you would easily discern a History classroom from a Mathematics one, even at the highest levels of school Mathematics. This is due to the characteristics of the mathematical discourse. Sfard (2008) characterises the distinction as a result of the specialised keywords, visual mediators, narratives and routines that are used when we communicate mathematically. An intuitive understanding of these characteristics suffices as this point as I focus on discursive layers. They will be elaborated in Chapter 3 for their role in commognition and the focus of this study.

Caspi & Sfard (2012) propose five discursive levels, and the traversing of these levels, as marking developmental discursive milestones. The extension of a learners' discourse through these levels entails the change in words or how they are used, the rules that are to be followed and even in the way in which objects are mediated visually. As the layers arise from ones prior, they exert influence on each other. Learners move from constant-value algebra, in the first three layers, to variable value algebra, the next two layers. Constant-value algebra suggests that its objects are specific values, which learners work with or seek. This is evident in the pointwise means learners use in function, for example. Variable-value algebra deals with numerical variation, as becomes evident in working from a relationship orientation to functions. Informal algebra, because of the limitations on the use of symbols, and the meanings that are attached to them, is restricted in the levels it can reach in the hierarchy.

Literature from a constructivist perspective also reflects the distinction between working informally and formally. Learners will have to decide whether and how to bring their informal knowledge into a task, or, if they approach it formally, how to represent relationships algebraically and how to operate on them (Watson, 2009). Generally, theoretical perspectives cohere on a common point of departure, the symbolic nature of mathematics and the need for a mathematist[7] to be able to operate or manipulate these symbols according to rules and conventions that are formally established and endorsed. This idea of operation on and manipulation of symbols is extended by Sfard & Caspi (2012) with the opportunity for learners to unpack complex algebraic statements and restructure them. The discursive implication is that it establishes a connecting discourse between symbols and their meaning, with the objects to which they are related. Processes in such a hierarchy then become connected to the objects with which they are concerned. The implication for learners is that once they have mastered communicating the endorsed narratives, they are expected to unpack and repack through the discursive layers. The 'zoom in' and 'zoom out' metaphor refers not only to looking with a research-orientated eye, but also for learning, as it describes the mobility backwards and

[7] Recall a "mathematist" is a word used by (Sfard, 2008) to describe a person participating in the mathematical discourse. This is to allow for the distinction between participants in the discourse and professional mathematicians.

forwards through discursive layers. Caspi & Sfard (2012) describe their five developmental levels as Level 1 - Processual ; Level 2 - Granular; Level 3 to 5- Objectified. Characteristics of each of these levels are summarised below:

Figure 4 Description of discursive levels in learning algebra.

Level 1 –Processual	Level 2 - Granular	Levels 3, 4, 5- Objectified
• Focus is on numerical calculations. • Calculations follow a linear order. • Equations are solved by simple undoing. • Algebraic objects are the *unknown* (the number being sought) and the *given* (the number that is used in operations).	• Auxiliary calculations appear as granules on the chain of operation. • Auxiliary calculations are interpreted as objects. • These are communicated as complex clauses e.g. square the difference between the first term and common difference. • Verbs e.g. *add* are replaced with nouns e.g. *sum*. • Marked by a shift from process to result.	• Complex algebraic expressions describe objects and relations between objects. • Communication is alienated, without the human element (when compared to granular).

In this study, these distinct levels, processual, granular, objectified, have provided a means to characterise learner realisations, as they become evident in learner mathematical routines. Granular discourse was seen as occupying the overlapping discursive space between the processual and objectified utterances. The symbiotic link between informal and formal knowledge, hence discourse, is noted in all education research perspectives. Commognition/discursive literature provides a means to describe the transition from one to the desired other. Discursive literature, by means of notions like granular, allows examination of the space between the formal and informal discourse, and enables a view of how learners discursively navigate the space of overlap. In perhaps its most distilled form, this space is representative of learning as a space in which a change or development in discourse can occur. The space described above in terms of learning is also seen as learners' transition between the various objects of algebra towards its purposive objectification in mathematical functions. Having developed the detail used to view algebra as discourse, the literature relating to the functions as they relate to the methodology of this study, is discussed in the next section.

2.4.2 Function

Early literature attributes the development of the object function to the need to show variation (Ayalon, Lerman, & Watson, 2011; Bakar & Tall, 1991; Bloedy-Vinner, 2001; Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The early work of Sfard saw function born as a result of a long search after a mathematical description for physical phenomena involving variable quantities (Sfard, 1991). This seemed to be the common discourse of the time, e.g. where function was seen as being the first tool for dealing with changing, rather than with constant magnitudes (Dubinsky & Harel, 1992). More recently, discursive literature marks the complexity of the transition from algebraic discourse, with its own objects and rules, to the notion of function that "is new to learners and for which much meta-level learning has to occur before learners gain reasonable command of the commognitive activity" (Ben-Zvi & Sfard, 2007, p. 135).

Unlike algebra with its links to arithmetic, in function, learners have little informal preparation on which to build a new discourse, no preceding discursive layer. Possible previous experience could have involved predicting terms of a sequence, where the relationship between variables is backgrounded. Learners essentially work with a single set of numbers. Thus I would agree that, the new discursive object, function, has little or no spontaneous mathematical experience from which to be developed (Nachlieli & Tabach, 2012). In addition, because of the circularity of discourse development and object construction, effort from both the teacher (knowledgeable other) and the learner, must be invested in coming to grips with an unfamiliar, abstract mathematical object (Ben-Zvi & Sfard, 2007) called a function. My teaching experience shows that learners come to the mathematical classroom with ‘everyday’ spontaneous, amalgam meanings of the word derived from English and Natural Science classrooms. The extension of these meanings to a formal mathematical narrative is not a direct translation from a purely concrete meaning to an abstract mathematical one. This is especially difficult in multilingual contexts, where English as the language of instruction is not the first or second language of many our mathematical learners. Learners are expected to engage a conversation on functions as a precondition for objectification, before they build realisations of the object or are able to relate the object to subsuming discourses (Ben-Zvi & Sfard, 2007). The discourse on functions subsumes the discourse on algebra.

To reiterate, this study examines how the discourse on function has developed for a group of Grade 11 learners in terms of their discursive routines. Research into teaching and learning in function reveal two main themes: theory to analyse how learners understand function as a concept, and explanations for the difficulties learners have in learning function. There is much research rooted in an acquisitionist perspective. Many learners believe that all functions should be definable by a single algebraic formula (Carlson, Oehrtman, & Thompson, 2008). Learners are found to have difficulty distinguishing between an algebraically-defined function, and an equation (Carlson, 1998). This is attributed to the many uses for the equal sign and that teachers and the curriculum refer to a formula as an equation (Carlson, et al., 2008). Students are shown to benefit from explicit efforts to help them distinguish between functions and equations (Carlson, et al., 2008). Even (1990, p. 530) looked at the definition of a function as “every element in the domain corresponding to exactly one element in the range”, and has observed that learners do not understand the meaning of this definition. They can recite the definition correctly, but they could not explain what it means, nor why function is defined the way it is. Ronda (2009, p. 34) has called definitions an ‘abstraction of a concept’. She has further noted that “knowing the definition of a concept does not necessarily translate into an object conception” (op cit, p. 34). Reciting a formal definition, therefore does not guarantee that the learner has objectified. This is only a partial development of the discursive object. The current study examines learner discourse on function and asks, if through related processes on the object or through the recognition of its features, have learners objectified function?

Earlier constructivist research shows that a primarily procedural orientation to using functions to solve specific problems lacks meaning and coherence for learners (Carlson, 1998). In discursive terms, this relates to learners knowing the *how* of a mathematical routine, and not the *when*. Learners who think about functions only in terms of symbolic manipulations and procedural techniques are unable to comprehend a general mapping of a set of input values to a set output values; they also lack the conceptual structures for modelling function relationships in which the function value (output variable) changes simultaneously with continuous changes in the input variable (Carlson, 1998; Carlson, et al., 2008; Monk & Nemirovsky, 1994; Thompson, 1994a). This research showed that the strong emphasis on procedures without accompanying tasks to develop deep understanding of the concept has not been effective for building foundational function conceptions. Tasks should allow for meaningful interpretation and use of function in various contexts. Carlson et al. (2008) showed that learners, when probed to explain their thinking, typically providing some memorised rule or procedure to support their ideas. They appear to resort to memorised facts to guide their explanations (op.cit). Such literature raised questions as to how learners work with variables in the current study.

To describe this object or function that learners find problematic, other cognitive-based research attempts to develop a conceptual map for functions (Ayalon, et al., 2011) and emphasises the function concept as foundational in advanced study of mathematics. Notions of variation and the particular relationship between variables theme such research. The importance of understanding function through concept maps, was an attempt to understand how learners' see function at various stages of its development. To this end, Ronda (2009), developed growth points, which are 'big ideas' of learners understanding of function. These provide a way to monitor and analyse learner understanding of functions in equation form. Growth points describe learners' big ideas in terms of strategies, knowledge and procedures that learners apply in working with tasks and problem situations (Ronda, 2009). While cognitively based as the development of schema, growth points (GP) deviate from the order of knowledge and skills learners are to acquire as specified in the NCS. Primarily, this occurs as a result of the process orientation of the South African curriculum. Growth points are helpful, as they flag transitions in learning about functions. Growth points are evident when learners' progress:

1. from generating values from the equation (GP1);
2. to finding an equation from the relationship between values (GP2);

3. to equations describing properties of relationships (GP3); and
4. to functions as objects that can be manipulated and transformed (GP4).

Growth points 1 to 4 listed above are restricted to working with algebraic expressions for functions only. I have restricted myself to growth points of this representation as it was the representation where learners in the current study had the most to say. It has been interpreted to mark transitions in learning that the broader literature had already uncovered. It made it possible to see if the learners in my study progressed through these growth points in the order suggested, and how these transitions showed in an objectified discourse.

Certainly, growth points have an advantage in a curriculum with an orientation to process and assessments such as the NCS. It is the context of continued of poor performance and poorly prepared teachers that mathematics is recontextualised in our classrooms. Under these (and other) sinister constraints, we have to question whether it offers significant difference from what we have already been doing. Growth points offer transitions for objectification through process or properties. How is it possible to tell that a learner has objectified or shown a mastery of process at each of these levels? This challenge to operationalisation brought on by the many acquisitionist studies reviewed, became a research imperative of my study. Would learners in my study cohere to the order specified by growth points in their knowledge of functions? If not, what did this show about our transitions in developing discourse on function? Would an examination of discourse for objectification show that learners could ‘unpack and repack’ through the levels flexibly?

Broader conceptual approaches examined function across multiple representations and developed constructs to describe and view learning. The earliest acquisitionist based work I could find that described learning in ‘layers’, to contrast the discursive approach used here in describing layers of formal and informal discourse, was that of Freudenthal (1978). Coming from a process-object conception of learning, learning through engaging process at one level produces what learners will observe at the next level (op cit). Similarly, other acquisitionist approaches classified mathematical concepts as operational and structural. ‘Operational’ suggests approaching a concept as a process, whereas ‘structural’ suggests approaching a concept as an object (Dubinsky & Harel, 1992; Sfard, 1991). Computation leads to a process conception which can later be encapsulated as a mental object (Sfard, 1991). By process-object theory, an object conception is attained, generally after experience has been gained performing actions on the object (Ronda, 2009). Another route to objectification, lists understanding the properties of functions (Ronda, 2009; Slavit, 1997). Here, learners conceive of a function through the properties functions have or do not have. This characteristic was described in the section on algebra. It recurs under function as a result of the subsuming nature of the discourses. Slavit (1997) zooms further in on properties to classify them as global or local properties. Global properties pertain to class of functions, for example, in the way that symmetry can similarly be described for all quadratic functions; local properties result from properties of individual or a selection of ordered pairs of the function (op cit). Perceiving a function as an object, in school Mathematics, brings function closer to learners' experience, when we consider the abstract definitions of functions, like the set-theory definition, which is removed from learners' experiences (Sfard, 1992).

Another dichotomy noted in literature is that between a global approach on the one hand, and a pointwise approach on the other, when it comes to working with functions (Even, 1998; Monk, 1988). Pointwise means to plot, read or deal with discrete points, as distinct from the process, operational and local conceptions described above. A global approach looks at the behaviour of a function. It seems, similarly, to mirror the structural, object, and global approach to function described in literature. Both Slavit (1997) and Even (1998) use the term 'global' in similar ways. This entails examining a function as an object, with properties or features, and which behaves in ways that are describable and predictable. While both these approaches have their strengths, the global approach appears to give a better and more powerful understanding of the relationship between the graphical and symbolic representations (Even, 1998). It is also a stated goal in the curriculum. Learners would need to reverse algebraic processes and have flexibility across all representations (Ronda, 2009). This is synonymous with the packing and repacking, or working forwards and backwards, as previously described. Ronda's growth points 3 and 4, mentioned above, seem to develop from a global understanding of a particular representation. In doing this, learners begin to explore dynamic function relationships, with regards to how one variable changes while imagining change in the other (Carlson, et al., 2008). Individualising such changes means to be able to discuss and interpret the changes to important features of the graph and its shape (op cit). The South African Mathematics curriculum includes the global conception of functions as part of its topics in Grade 11, relating to the transformations of functions, where changes of critical features occur under transformation. The questions that lead from this discussion are: from learners discourse on function, how can I describe their notions of function? Are learners oriented to process or object? Do they have a global or pointwise view of functions from school Mathematics? And, what implications do these orientations have for learning function?

The discussion of the South African curriculum earlier, showed an orientation to process, through emphasis on symbolic manipulation and procedural techniques. Research below shows that this can account for learner difficulties with constant functions (e.g., y=5), also known as special or univariate functions. Research over two decades reports that these are not considered as functions because they do not vary (Bakar & Tall, 1991; Carlson, et al., 2008; Dubinsky & Wilson, 2013). Carlson et al. (2008, p. 12) have shown that ‘learners carry out rote procedures’ when asked to solve these special functions. Earlier Carlson (1998) found 7% of A-students could produce a correct example of a function where all output values were equal to one another. This may indicate learners still stuck in a pointwise approach. Venkat & Adler (2012) showed a ‘block’ in flexibility, when learners have access to a particular method, which did not provide ways of dealing with the special cases of horizontal and vertical lines, as it does not allow learners to focus on the features of the input or other representations. Explanations supporting this idea of reliance on one route or lack of connection between routes is supported (Ayalon, et al., 2011). Within a rigid procedural orientation, Dubinsky & Wilson (2013) explain that learners expect a one-to-one correspondence between variables before they are able to see a functional relationship. The criterion of a change in one variable resulting in a change of another is a necessary condition for recognition of a functional relationship for learners. This notion appears as problematic for learners to individualise across all the research surveyed.

It is clear from the review of literature that common problems persist in learning function despite great theoretical strides in pedagogy. Given the work, describing the difficulties learners have with the idea of variation, it became important for me to locate my enquiry in learners’ mathematical routines with functions required in school mathematics. The connections between the function as an object and the processes involved appear to reside in a transitioning gap. Literature suggests that learners could objectify through two routes, namely through reifying processes, or through the features and properties of functions. When we understand as teachers and researchers what learners know through the way in which they communicate what they know, we can work towards developing an approach to function discourse that makes objectification, hence abstraction and complexity, possible. It is worthwhile asking whether a focus on learner routines in function would illuminate their thinking around the transitioning space from informal to formal function discourse; from algebra to function; from the various representations of function to the object itself; and from the individual features and properties of various functions to a class of function.

2.4.3 Notation, definitions, keywords

In this study, I focus on learners' mathematical routines. The empirical data shows learners can express their routines verbally. A challenge to communication however, arose for learners in the ways that they communicated the meaning and their use of specialised mathematical symbols and keywords. Formal notation which characterises work in function, such as $f(x)$, is a visual mediator. These are visible objects, symbolic artefacts, created and operated on as part of the process of communication (Sfard, 2008). They hold definition attached to them by formal mathematics, and they can symbolise or represent a mathematical object, but they also hold the meanings that learners attribute to them. The broader literature points to how learners conceive of function and what they do within the context of the formal, established structures on function, of which definitions and notation are a part. The formality that learners encounter in school mathematics focuses on literature about notation, keywords and definitions.

One of the purposes of formal education is to prepare learners for the formal thinking that is available to others (Ayalon, et al., 2011). Learners can benefit from explicit effort to promote their understanding of function notation (Carlson, et al., 2008). Translating this to discursive contexts means to prepare learners to participate in the school mathematical discourse. Literate or formal mathematical discourses have among their salient characteristics, a heavy reliance on written symbols and an arsenal of algorithms for making use of the special notation (Sfard, 2008). Different notations can grow different conceptualisations, ranging from the process conception to the object conception (Watson & Harel, 2013). For example, in school mathematical discourses, $f(x)$ can be seen as representing a process, where, for an input x into a function f, there is an output $f(x)$. This is the meaning attached to the symbols of the visual mediator, $f(x)$. By contrast, f can also indicate an object; no process being suggested by its symbol. Notation is used interchangeably in texts and curriculum materials, and contributes to the sense of confusion learners communicate about the function object. This duality of meaning was discussed earlier in the way that the words ‘expression’ and ‘equation’ are conflated. Research shows that function notation is particularly problematic for learners whose previous experience has led to limited sense of what letters symbolise (Watson & Harel, 2013). In such cases, learners can interpret literal symbols as shorthand labels for objects (Kucheman, 1981; McNeil & Weinberg, 2010). The letter f could be taken to stand for ‘function’. This raised a flag for this study: how do learners interpret f or $f(x)$?

A weak understanding of functions has been observed in learners’ inability to express function relationships using function notation, where weaknesses showed in not knowing what each symbol in an algebraically defined function means (Carlson, et al., 2008). In highly procedural orientations, similar to classrooms in this study, Carlson (1998) found that learners’ weak understanding of functions could be linked to poor ability to express a function using algebraic symbols and function notation from function values. A discourse that connects symbols with meaning is essential for thinking and learning. Dubinsky & Wilson (2013) mark a transition between iconic and abstract symbols. They say that learners are to begin symbolising iconically, using pictures or diagrams from their real-life experiences. So symbolism links from learners’ informal discourse. These express features which learners see, or are familiar with. These can then be replaced by the abstract symbols used according to convention in formal mathematics. This work resonates with commognition, as it talks of mathematising as moving from ordinary talk (people talk) to the regimented (feature talk) of mathematics. Slavit (1997) has argued that learners should have a proceptual understanding of function notation. Proceptual implies a deeper understanding of function than that which results from just an action or a process orientation. This means that learners should be able to understand notation as cueing both an action and an object, and should work with them flexibly. How do learners interpret and use function notation required in school Mathematics?

Another important component of formal discourse is the mathematical definition or endorsed narrative. Vygotsky states that “the development of the scientific concept begins with a verbal definition. As part of an organised system, the verbal definition descends to the concrete; it descends to the phenomena which the word represents” (Vygotsky, 1987, p. 168). In South Africa, like in most countries, our ‘scientific concepts’ or formal mathematical knowledge, is specified in the NCS. The requirements, order, emphasis and approaches to the formal knowledge on function differ from country to country. The list below is intended to contrast two other approaches with our own.

1. The UK has an informal approach to function. Formal narratives are reserved for the final year of school (year 12) to prepare those intending advanced study in mathematics (Ayalon, et al., 2011).
2. In the US, textbooks reflect the curriculum and begin almost immediately with the definition of function related concepts, such as relation, function, domain and range (Watson & Harel, 2013).
3. In South Africa, learners encounter the formal notation, as part of pedagogical strategy involving ‘function machines’, where, given a domain, learners substitute into the algebraic expression given, to find the values of the range. Learners’ transition from expressions that initially read as *y*= ... to *f(x)* =... (DBE, 2011b). The transition, the significance of different forms of representation, are not emphasised in curriculum, its associated documents, classroom texts and learning programmes in schools.

Not frequently found in research is the importance and the role of objectification in learner communication, as this connects to the formal narrative and explicit approaches to teach formal narratives and notation in learning. From Vygotsky (earlier quote), formal notation and narrative are as important as the automation and embodiment of process for learners to develop a more objectified discourse. Formal mathematical discourse catalyses or bridges the space between the informal and formal, serving as the link between them. It is noted already that this is particularly difficult in building a discourse on function, as learners have no prior informal experience with the object. As Nachlieli & Tabach have noted, “function has no spontaneously developed precursor-no mathematical predecessor” (Nachlieli & Tabach, 2012, p. 11). More and more research appears to be filling and at once characterising the transitioning space. In South African secondary education, in practice and curriculum documents, it would appear that the learning of a formal narrative coupled with the emphasis on the relationship between variables, is not described explicitly, leaving the teaching of formal narratives at the discretion of the teacher. The context of rampant poor performance could contribute to the formal narrative being deemed too difficult to learn, and hence, will not command attention in teaching.

Typical studies over time regarding the learning of formal definitions, show that to describe learners' understanding of a mathematical concept, especially to describe initial understanding, focus should be placed on the actions on the concept, its properties and representations and not so much on the definition (Bloedy-Vinner, 2001; Ronda, 2009; Vinner & Dreyfus, 1989). Surprisingly, within the discursive realm, research shows negligible influence of definitions on learners use of words (Nachlieli & Tabach, 2012). This is probably due to classroom language not being just a list of technical terms or a recital of definitions, but which involves the use of these terms in relation to each other, across a wide variety of contexts (Lemke, 1990). Venkat & Adler (2012) have suggested the important connection of the initial representation with subsequent transformations and resulting representations. This can be interpreted as the formal definition and the meaning of symbols in the symbolic representation, as having place in what the study calls 'resulting representations'. The formal definition provides a means for encapsulating the various disparate and compartmentalised discourses of the different representations of a function. Extending the role of the formal definition as facilitating abstraction is seen in studies which regard defining as responsible for learner beginning to appreciate abstractedness and to stop learners from relating abstract mathematical objects to a specific concrete things (Nachlieli & Tabach, 2012). They suggest combining formal narratives with symbols and examples as an optimal approach. Such research suggests that formal definitions transition learners from the everyday discourse to the formal mathematical. The same research details this process as the learner participating in the new discourse of the formal definition, by relying on previously informal use of words. In 'recycling' their old uses in the developing formal discourse, learners are not generally aware of inconsistencies with the formal definition (Nachlieli & Tabach, 2012). Consistent 'recycling', it appears, produces changes that bring learners closer to the formal. Does this perhaps suggest a means of navigating the transitioning space between informal and formal?

Work connected to the notion of recycling and reworking words, refers to inconsistencies which arise particularly out of learners' colloquial use of words to describe what they saw in graphs and how they regarded the object function as well. A keyword, *discontinuity*, for example, has been found to be frequently used to describe a 'gap' or a 'hole', or as a graph that 'jumped'. Conceptualisations based colloquially in 'holes', 'jumps' and 'poles' have been noted in literature to lead to misconceptions in more complex mathematics, such as the defining of the derivative (Carlson, et al., 2008). Without the formal definition of the object function, and developed discourse of its meaning, learners regarded discontinuous functions as ‘weird’ or strange. Non-calculable functions are seen as aberrations (Watson & Harel, 2013). The absence of a formal, mathematical definition for the object function showed that learners tend to identify function with one of its representations or realisations, either the graph or algebraic formula (Even, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Sfard, 1992), seldom a table (Dubinsky & Wilson, 2013).

In support of the work above it was found that the word ‘function’ is taken to refer to the algebraic formula in one context and the graph in another, seldom related to or represented by both of them at the same time (Nachlieli & Tabach, 2012). Rather than creating a unified discourse of these representations, learners develop a collection of disparate unrelated discourses, involving a key word which may be used in different ways on different occasions (op.cit). This study was able to examine the role explicit definitions functions have in connecting and unifying discourses. This is the context of literature in which learners who were able to reproduce the definitions were found to act in ways that contradict the definition (Nachlieli & Tabach, 2012). Could this be a result of ritualised learning of a formal definition, without the reasoning required, or a connection to be established between the object and the definition? The literature cited above suggests a lack of means to connect representations of function to each other, and to the formal defining narrative. Based on this, the question arises as to whether this is a pedagogic imperative. Along with this, could learners without a formal definition of function or formal means to deal with the object be able to make these connections on their own?

A consideration of pedagogic approaches is likewise not so straight forward. The presentation of an objectified discourse, the idea of function as an object, a ‘thing’, if introduced too early remains beyond the comprehension of many students (Sfard, 1992). I argue that there are ways of thinking in formal mathematics that must be learned together with the mathematics. Paying attention to that transitioning space where learners build connections between objects already existing and new, and between the informal and the formal means of communicating about these, is a highly complex task for both teaching and learning. Commognition calls this meta-level learning. Mostly it develops by participation in the discourse with a knowledgeable other. It is unlikely that learners will develop the narratives that exist in formal mathematics on their own or stumble on these ways of working. Formal mathematics need not be the singular, driving goal of learning mathematics, but rather it can be used to transition from informal to formal discourse, encapsulate related objects by their similarities, and condense and simplify ways of working mathematically, thus providing means for learners to explore mathematics and escape the hold of ritualised practice.

The learning of function is indeed complex. The span of literature covered here and the perspectives examined and synthesised were aimed to develop my understanding of how learners develop a discourse on function, particularly in the back grounding of a formal definition of function as in the SA curriculum. Research in the field has helped focus my investigation into learner discourses, bringing into focus the need to establish discursive connections on the multiple levels established here. Despite this wealth of research, none has yet engaged with explicit pedagogic approaches to do so. Commognition however, is based on learners building onto preceding discursive layers to learn; confirming it as a prudent choice of framework. With certain literature as discussed above having drawn my eye to the necessity of connection, it becomes important to examine further literature on multiple representations as they are relevant to function.

2.4.4 Multiple representations

Much of what is written emphasises the separateness of the discourses around the representations studied in secondary school, namely: the algebraic representation, the table and the graph. It has been found that learners have difficulty making transitions from one meditational mode to another or from one representation to another. Tables, graphs and expressions might be multiple representations to us, but there is no evidence they are multiple representations of anything to learners (Thompson, 1994b). As emphasised in prior sections, there are two necessary discursive connections that have to be sought: first, the connection and transformation across the different representations themselves, and second, the connectedness of the representations to the new subsuming discursive object. The dominance of literature from an acquisitionist/Piagetian paradigm mostly examines the flexibility of moving between representations by transforming from one representation to another. It flags, for the participationist/discursive researcher, useful points of contention in learning, where dominant research paradigms are unable to adequately account for learners' poor performance. A discourse connecting representations to subsuming discourse of function is scarce in extant literature.

Regarding flexibility in working with the different representations, Ronda (2009) found that a full understanding of the concept function necessitates the understanding of, and the ability to work with each of the representations. This suggests conceptually and discursively, seeing the equivalence of the representations, and hints at the larger object. Other similar work suggests that the ability to identify and represent the same thing in different ways, and flexibility in moving from one representation to another, allows learners to see rich relationships and develop a better conceptual understanding, which broadens and strengthens one's ability to solve problems (Even, 1998; Slavit, 1997). This extends and links flexibility to problem-solving. The ability to solve problems and develop in abstraction is a goal of mathematics. However, the discrete packaging of the different representations, which best describes the pedagogy of school mathematics, does not necessarily enable the notion of equivalence among representations to develop. Couple this with the absence of the relationship notion, could account for function being seen in a single representation only. This severely curtails the opportunity for learners to solve problems as complexity increases and may account for the poor performance in function described in Chapter 1.

Learning functions is not simple; there are multiple layers for learners to connect when dealing with this abstract mathematical object. Sierpinska (1992) found learners have difficulty making sense of covariation, that is, seeing function as the rate at which one quantity changes with respect to another. Multiple meaning attached to symbols, in an expression or equation, adds to the complexity. For example, learners' early experience with equations involve equations not as a function, but as a statement where one quantity equals another involving a single variable. The equals sign is interpreted as a signal to 'do something', or 'perform an operation', rather than denoting a relationship of equality between the expressions on either side of the equal sign (Kieran, 2007). When a function is represented by an equation, it shows a relationship between two quantities or an arrangement of algebraic symbols, which can be manipulated and transformed. The dichotomy of process-object at this point becomes critical, depending on which receives emphasis. The arrangement of symbols in an equation conveys conceptual knowledge and possibly an object conception (Ronda, 2009). The parameters of the equation themselves signify entities which can be used to reason (Kaput, 1989). However it had been shown that learners are not acquainted with the roles of parameters in different representations (Even, 1998). Understanding function from an equation is considered a major conceptual node for learners (Ronda, 2009). An equation would require that a learner develops a discourse of the object in its entirety, but also a discourse of its individual component symbolic parts. Literature on how to achieve flexibility and develop an objectified notion of function was difficult to find. To focus on flexibility, Venkat & Adler (2012) talk of producing transformation sequences that connect across representations. Now this goes beyond the emphasis of transformation sequences within a particular representation. It is wider-arching, in seeking to connect sequences between representations. With transformation in emphasis, particularly in school Mathematics, learners concentrate on moving symbols around, as opposed to connecting the symbols and process to the object represented by those symbols. For this reason, pure process orientations were found to be absent of meaning for learners, who appear unable to offer interpretations or use function in and across varied representational instances (Carlson, et al., 2008). A process orientation is defined as an understanding of the transformational activity performed on a function (Slavit, 1997).

The literature of multiple representations was scoured for notions of the ways that learners see equivalence. From any view, acquisitionist or participationist, the research largely seems to suggest that working with two representations, particularly the transformation of one to the other, is taken as a sign that learners see equivalence. It is thus important to ask learners to substantiate their thinking, and thus gain access to whether and how the representations remain separate or are related.

2.5 Conclusion

The broad sweep of literature in the learning of functions discussed above provides a basis for deeper discussion of the theoretical frame in the next chapter. It must be noted at this point, that the discursive turn in educational research is relatively new. School textbooks, related curriculum documents, and the spread of literature reviewed in this study do not depart from a discursive, commognitive view. As a result, the graph, table, and algebraic expression examined in this study, are referred to as multiple representations of function in these documents. In commognition the representations are regarded as signifiers of the object function. A commognitive framework is motivated for as a means to fill the spaces that are silent in literature. Communication is that which links the separate discursive spaces of the (representations) signifiers; and communication the tool that will allow learners to encapsulate the different signifiers into the single discursive object, function.

Chapter 3:
Locating the Study in Commognition

This chapter elaborates detail of Sfards' (2008) commognitive theory, with particular focus on those aspects of the theory that are most germane to this study. I begin with a reminder of key tenets of this discursive approach to learning and thinking, and to mathematics as a discourse, such that I might zoom in from there, when locating the study within the theory. The goal of this chapter is twofold: the first is the clarification of the terms and language of commognition as they are relevant and used to describe learner discourses; the second, is perhaps the more important, and occurs when I reflect in the final chapter on what commognition has enabled in the study, and expand on the potential for taking the theory forward in work such as mine. I build towards my focus on learners' mathematical routines using commognition and the characterisation of these routines as exploratory or ritualised shown in the analysis chapters which follow.

3.1 Why commognition?

Commognition is a discursive theory used here to describe learning. Commognition regards learning as individualisation of a 'patterned collective activity' (Sfard, 2008, p. 570). This held the initial attraction to the theory. If we reflect on mathematics, we see patterned ways in which think, do, see, and communicate. Even in an intuitive or everyday sense, we can see how these can be called routines. Mathematics has distinctive and characteristic routines, which constitute that which makes reading a text or entering a classroom instantly identifiable as mathematical. The notion of 'collective activity' was identified for its efficacy to the study for two reasons. First, it described learning (or not learning) as the result of collective participation, a social endeavour. This, I argue, stood in contrast to reports which locate failure and poor performance in the learner. Second, collective activity was coherent with my view of knowledge and learning as, rather than something thrust onto learners, as something external to them. Commognition is a term that encompasses thinking (individual cognition) and interpersonal communication. The word itself is a combination of the words communication and cognition. It stresses the fact that these two processes are different (interpersonal and intrapersonal) manifestations of the same phenomenon. We can see thinking as communicating (Sfard, 2008). From a commognitive perspective, "thinking is an individualised form of interpersonal communication" and "school learning is a process of modifying and extending ones discourse" (Ben-Zvi & Sfard, 2007, p. 81; Sfard, 2007, 2008). Learning is communication with others and ourselves, it is dialogical (Sfard, 2008), and as such, is socially and culturally produced. We modify and change what we know as we learn (Sfard, 2008).

There is very little examination or elaboration of failure as a concept, including to what or to whom to ascribe blame for this in a classroom context. As a society, we handle failure with a measure of shame and speak infrequently about it. The research on errors and misconceptions attempts to open up this conversation, but was of only partial help in this thesis, where the question of failure sprung initially from an overwhelming absence of responses from learners on a high-stakes national assessment. It's time that we confront the phenomenon of failure as a collective doing.I would argue that failure needs to be confronted collectively. As discussed in earlier chapters, the rationale is in order "to replace discussion of what people *are* and *have*, with *what* and *how* they do" (Sfard, 2008, p. 75). To this end, commognition provided tools here to examine learner discourses across different performance levels, so as to better understand how to improve at each. Communication, within the commognitive framework, is considered as a patterned, collective activity, defined by the discourse in which we participate (op.cit). Success or poor performance is thus seen as a product of collective doing, as learning is intrinsically social (Ben-Yehuda, et al., 2005). Learners in Mathematics gradually increase their participation in the discourse, a specialised form of communication, as they individualise actions which are permitted. Learners enjoy increasing autonomy from the decisions they make as they come to know a mathematical object, from a state of being an initial passive participant, to becoming a fully fledged participant in the discourse (Nachlieli & Tabach, 2012; Sfard, 2008). For learners who fail, it seemed valuable to examine the results of the disconnection and to characterise these with reference to learners who are better-performing.

The disconnection in this process has been a quandary in research for decades. Early research as well, traced a similar trajectory of developing autonomy, where it describes learners as moving from 'legitimate peripheral participants'(Lave & Wenger, 1991) to independent performers, who can undertake a task on their own. Literature, irrespective of perspective, sees learners in school Mathematics, begin participation in a mathematical discourse with the assistance of a knowledgeable other, usually a teacher. Learning implies they become more and more independent in their participation, as the teacher gradually removes support and scaffolding in order to help them grow discursively. How do learners who are failing in mathematics communicate this independence? If it does exist for them, what forms does it take? Advancing the notion of a trajectory, commognition places thinking as developmentally secondary to communicating (Sfard, 2008). I interpret this as a learner being able to communicate mathematically in order to be able to think mathematically. This justified for me the focus on how learners communicate mathematically. In addition, it allows us to explore the communication of failure.

In earlier work, Sfard (2007) specifies different types of commognition, distinctive due to their patterns, objects and the types of mediators used. These distinctions are now elaborated. Different types of communication bring some people together, while excluding others. These are called discourses. Diverse domains of knowledge (mathematics, physics etc.) learned at school, are special types of communication. Simply put, the discourse of a mathematics classroom makes it distinct from other classrooms. Recontextualisation for the classroom can create further deviations from the formal mathematical discourse. A discourse is called mathematical if it deals with mathematical objects (Ben-Yehuda, et al., 2005). From a learning perspective, we can consider objects as those things being spoken of or referred to in discourse. From a teaching perspective, they are those things that you want your learners to know. Objects can be abstract mathematical things or processes. Discursively, they are defined as a mathematical signifier, with its realisation tree, which Sfard has defined as “a hierarchically organised set of all realisations of a given signifier”, together with realisations based on those realisations (Sfard, 2012a, p. 4). School learning is an activity where learners modify and extend their discursive repertoire of mathematical objects.

Discourses are made distinct by their vocabularies (key words and their use), visual mediators, routines and the narratives (Sfard, 2008). Mathematical knowledge, more than any other discourse, overlaps several domains of knowledge (physics and chemistry are examples). Learning mathematics is defined as individualising the mathematical discourse, to communicate with others and oneself. Developing a discourse or learning is a dynamic process of constantly reworking the old with the new. Sfard (2008) defines learning as a commognitive activity, which entails reasoning, abstracting, objectifying, and subjectifying. Reasoning is the systematic derivation of utterances from other utterances. Abstracting is creating and communicating about objects that are not tangible or concrete. Objectifying, substitutes nouns for processes leaving out the human performer. Subjectifying focuses on the human performer of the action. These activities stand for inspection when we examine what learners can or cannot do. We see a developing discourse as dynamic, in cycles of contraction and expansion, building successive discursive layers that grow in abstraction and complexity. As growth in abstraction and complexity increase, the need to formalise or objectify the discourse increases. This is necessary for the effectiveness of the communication between participants.

From the above commognitive activities described, reasoning and abstracting can rest on the simplified description offered. Objectification and subjectification need to be clarified for their importance and for how they were used in my study. My assumption is that the distinction between successful and unsuccessful learners in mathematics is the ways they have and means they use to explore mathematical objects like function.

3.1.1 Objectification

In the current study, it was useful to make the distinction between learner communication, of processes or about objects, and to examine the way in which learners talked about mathematical objects. Commognition defines objectification as having two related parts: reification and alienation. Reification showed learners transforming the talk of process into the talk of object. Reification allowed a learner to capture the lengthy description of all processes relating to an object into an entity which defined the object itself. In being concise we increase the flexibility and applicability of what we communicate, where alienation involves communication of mathematics in an impersonal way.

There are several advantages to objectification. The literature showed that the elimination of talk of human action in mathematical discourse contributed discursive changes that were linked to improved performance (Ben-Yehuda, et al., 2005), making the ways we communicate mathematically more effective, while providing an anchor for the various processes we execute by attaching them to that which we have objectified. Once we objectify, that is, create an object, we establish a ‘thing’ which has permanence in our discourse, even if this ‘thing’ is an abstract entity. Onto this object, we build and accumulate knowledge, through generating successive layers of discourse, increasing in complexity and abstraction. Our processes are no longer independent or random. They link to layers of discourse or across them. Reified processes are in direct relation to the object. This emphasises the connection of process to the object, but also relates objects to one another. Objectification thus underlies the patterned ways in which we work mathematically. The efficiency afforded by objectification enables us to explore mathematical objects.

While objectification makes a powerful contribution to building a mathematical discourse, the literature shows it can contribute to the lack of autonomy a learner feels in the formal, established mathematical discourse. As Sfard has noted, "objectified descriptions deprive a person of a sense of agency, restrict her sense of responsibility, and, in effect, exclude and disable just as much as they enable and create" (Sfard, 2008, p. 56). This is possibly because the objectified nature of the formal discourse could seem removed from the learners' experience as well as from the ways in which learners might communicate in everyday speech. This serves to reinforce a point already made, that mathematics not only amounts to *what* is said, but is likewise constituted in *how* it is said. Coming to grasp the abstract object function itself is a challenge for learners. To poorly performing learners, the objectified discourse, must serve to alienate them further, judging from their non-responses on the NSC examination.

The initiation into formal mathematical discourse is gradual, guided by a knowledgeable other, where it is a question as to how much of an objectified discourse was the learner exposed to in school mathematics? The prospect of an answer to this question is compelling and presented as an extension to this study. Despite the challenges to objectification described, in both teaching and learning, an objectified discourse is an essential part of communicating mathematically. While it may be a challenge to communication, it is an essential part of independent participation and a necessary skill in the exploration of mathematical objects.

3.1.2 Subjectification

In contrast to objectification, where the emphasis is on the process and its object, subjectification focuses on the person engaging the mathematical discourse. Learners show a discursive focus on what they did, which shifts attention away from the mathematics itself. Subjectification therefore does not show the alienation characteristic of objectified discourse. Subjectification can be useful if a learner is able to talk about what she did with the object, accompanied by reflection on these actions. This reflection is essential to connecting a learners' informal discourse with the formal. In relation to developing a discourse of independent exploration, a subjectified discourse, with a focus on the person and her actions, will obscure the object and the potential for connections with other similar objects.

Research shows that in addition to the person-in-the-process emphasis, learner talk will also show instances of asking for help, or offering help to one another when working with peers (Webb & Mastergeorge, 2003). Such research puts the focus on the social aspects of learning. Learners can ask for help on problems, offer mathematical ideas to a peer, or talk about the course they have taken to solve a problem. Such instances can be helpful to learners as they mathematise. I personally have found it useful when a peer corrects my mathematics, suggests a more economical and efficient way of reasoning, or even encourages me to persist in finding solutions. A disadvantage of subjectifying is found in instances where it halts reasoning, such as ‘take it to the other side and change the sign’. Such subjectifying utterances are termed ‘action-oriented subjectifying’ (Wood & Kalinec, 2012), and are distinct from identifying utterances. Identifying utterances, or ‘identity-oriented subjectifying’, are a type of subjectifying utterance, showing talk of the person, like their features or attributes, for example, in the statement: ‘she is very good at mathematics’. Identifying talk does not refer to mathematical processes or objects.

Subjectifying entails description of what a person does, is or has. It thus has as its goal, social acceptance. As such, it underlies a ritualised practice (detailed in 3.3 to come). Both types of subjectifying listed serve to build or hold onto social bonds and to distance the learner from the mathematical object. However, action-oriented subjectifying has been shown to lead to mathematising in certain instances (op.cit), where a learner is encouraged to continue with the task, or returned to task when having become misdirected. The most obvious gap in existing literature is the relation of subjectifying to levels of performance. The literature does show that learners subjectify more frequently than they objectify (Wood & Kalinec, 2012).

3.2 Mathematics as a discourse

Sfard defines mathematics as "…a multi-layered recursive structure of discourse about discourse" (Sfard, 2007, p. 161), where mathematics is an autopoietic system that produces the things to which it refers, viz. its objects. Specifically, mathematical objects are defined as abstract discursive objects with distinctly mathematical signifiers (Sfard, 2008). The commognitive definition of a signifier is that it is a primary object with its realisation procedures. A realisation procedure pairs a signifier with another primary object, or the product of a procedure. This emphasises that object, signifiers and procedures are connected.

In this study, and following Sfard, I distinguish between three types of discourse evident in the learning of mathematics, namely: colloquial, informal and formal. Colloquial discourses are visually mediated by concrete material objects, existing independently of the discourse (Sfard, 2008). It is a spontaneous/everyday way of speaking (Vygotsky, 1986) that may be brought into a mathematical conversation. Colloquial narratives are endorsed by learners through empirical evidence or repetition. These are often non-mathematical actions, which learners engage in real life, with concrete objects. The formal discourse, by comparison, relies on symbolism, and an arsenal of special notation and algorithms, which enable us to communicate mathematically.   The use of symbols and words is rigorous. This is the distinguishing feature from an informal discourse which can have mathematical features and abstraction, while lacking the requisite rigour. The examination of prescribed school textbooks and curriculum statements appear to be directed to developing a formal or literate mathematical discourse, as a goal of school Mathematics.

While the types of discourse are presented as separate above, chapter two elaborated the need for the connection between the discourses in learning. The growth in mathematical discourse or learning is evidenced by developmental changes in discourse. This development occurs as a learner modifies her mathematical thinking, through a process of individualisation. Individualisation of a discourse refers to a gradual transition from observer of, to a fully active, autonomous participation in the discourse. The development is often a result of a commognitive conflict, where "learners use the same mathematical signifiers in different ways or perform the same mathematical tasks according to different rules" (Sfard, 2008, p. 161). Simply put, this could mean learners in interaction using a mathematical keyword differently to one another, resulting in one of the learner's changing the way they communicate in future.

Learning as a change in discourse results from this commognitive conflict, as well as from discussions where interaction forces a learner to realise differently in order to reach a well reasoned conclusion. In mathematising, learners change, because they realise that far more is possible in choosing an alternate discursive route. In most instances, alignment to the formal mathematical discourse provides the rigour for the change in discourse. Learners, when working together, become aware of ambiguities and inconsistencies, which result from their informal ways of reasoning. As a researcher in this study, changes in discourse became evident in two ways:

- the way in which learners use mathematical routines when working together, which include both words and symbols, and what these make possible for them to realise from these signifiers. That is, their process of mathematising (communicating about mathematical objects) in patterned ways. Deviation from the routine ways of mathematising provide a view of learner thinking.
- subjectifying, where learners talk about themselves or others and what they do as they participate in the given task.

Describing these aspects of learning at different performance levels allows a view into how learners have individualised mathematics at each of these levels. The prevalence of subjectification at each of these levels will show the distance of the learner from the object. As already stated, school mathematics has the goal of having learners reproduce the mathematical discourse that already exists, that is, to align learners with the historically established discourse. This is to prepare learners for the exploration of mathematics at tertiary level. Commognition seeks discursive patterns about mathematical objects themselves, and the way in which we talk of these objects. The rules of a discourse will give it its patterned nature. To this end, discursive literature identifies two types of learning: object-level and meta-level (Sfard, 2008). Object-level learning involves learning the rules for deriving narratives from those previously endorsed. These are explicit principles externally imposed by the formal mathematical discourse, and focus on the properties and behaviour of mathematical objects. This results in the expansion of the discourse in terms of the objects in focus. Object-level learning occurs when the learner is sufficiently familiar with both the objects and meta-rules that govern working with those objects.

School mathematics emphasises the learning of the meta-level rules of the discourse. Meta-level learning is shown in the means learners use to derive or explain object level rules. These become evident when learners talk about the actions of other participants and not the behaviour of the object itself. Meta level rules are involved when learners see patterns allowing them to class empirical evidence according to similarities they see. From this we can deduce the rules as tacit, contingent, variable and value-laden (Sfard, 2008). They are dynamic, worked and reworked by learners through their interactions, by following a knowledgeable other or in interaction with other learners. An example of meta-level learning is seen when learners make sense of the transition from counting numbers to integers. The properties of integers conflict with what learners already know about counting numbers. Sfard argues that these discourses are incommensurable. Choices have to be made about which properties of counting numbers would still apply to integers, and how other properties might then need to be redefined. Such rules are already established by formal mathematics, and learners are required to individualise them to make sense of them and to use the rules further.

Meta-rules can be classified in three ways: the applicability of a routine, the procedure involved, and the closing conditions or the result of the procedure (op.cit). It is within these rules that learners are able to exercise some creativity in their choices and uses of the rules from the selection available. Applicability of a routine and the way in which its closing conditions relate to the *when* of a routine, and the procedure relates to the *how*. Changes in the *when* of a routine, result in mathematical breakthroughs, which introduce new objects or expand existing discursive layers (Sfard, 2008). Having elaborated object and meta-level learning, it is unlikely that school Mathematical discourses, whose aim is to develop learner communication of established rules, would create new mathematical objects, or alter existing rules. School Mathematics exists to make learners competent in mathematical discourse by increasing their repertoire of available routines, and by providing an opportunity for applying extant rules to problems of increasing mathematical demand. It initiates learners into the exploration of mathematical objects and solving mathematical problems independently. This study attempts, through investigation of learners' mathematical routines, to contribute to thinking on why meta-level learning is so challenging to learners.

The work thus far has considered the broader aspects characterising mathematical discourses, namely the types of discourses and rules governing these. Commognition allows a way of zooming in on four characteristics of a mathematical discourse for closer reading. This process examines learner utterances for their most basic component parts, and the way in which these parts fit into a larger discursive structure. This again affirms my confidence in the commognitive framework as being able to convey the dynamism of the mathematical discourse in particular, by allowing a 'zoom in' and 'zoom out' on learner talk. Communication, like thinking, is not linear or one dimensional, but is a complex system, in which discreet parts fit together to convey meaning. The methodology of this study was to examine these characteristics across schools. The characteristics of discourse are:

i. Words: possibly the smallest verbal component part of learner talk. Here, focus was placed specifically on learners' use of key mathematical words. These are examined for: appropriateness of use in an utterance; whether the learner substitutes a colloquial word for a mathematical one, the meaning of which becomes evident in an utterance as a result of the use of an appropriate or inappropriate word; and the way in which keywords are transferred across different contexts. This enables a researcher to comment on the disciplined use of mathematical words in school Mathematical discourse.

ii. Visual mediators: these are visual representations of the mathematical objects upon which learners operate. Algebraic symbols and notation and the graphical representations included in this study provide examples. Commognition holds that there are specific ways in which learners look and work with these mediators and that these ways of working are easily embodied and automated. The visual can cue learners with specific discursive prompts, which provoke recall of specific knowledge and ways of working.

iii. Narratives: the goal of mathematics, as already stated, is to produce endorsable narratives. A narrative denotes "a sequence of utterances, spoken or written, framed as a description of objects, of relations between objects, or of activities with or by objects" (Sfard, 2008, p. 223). A narrative is considered endorsable if it can be derived from established mathematical means. Considered as mathematical narratives are axioms, theorems, and definitions. These are regarded by the mathematical community as its 'truths'. Narratives can be used alone or in relation to other endorsed narratives to make further deductions.

iv. Routines: from school mathematics, as a subset of formal mathematics, distinctive *patterned ways* emerge in which learners communicate about function. These mathematical regularities can be seen in the ways learners use keywords and visual mediators, derive new narratives or substantiate existing narratives, or create classes of information by similarities. Patterns also arise from other diverse influences, such as the recontextualisation of formal mathematics by the curriculum, the teacher and the classroom, and changes introduced by the learner herself, as she navigates objects and interprets their meta-rules.

The theoretical constructs described in this subsection raise several important questions in contexts of poor performance in the current study. How is mathematics recontextualised for poorly performing learners as evident in their common routines? What features characterise the routines of poorly performing learners and possibly contribute to the ways in which they form mathematical objects? The patterned features of learner discourse on function can be used as a means to characterise or describe what has come to be objectified as function for them. These patterned ways of working, called routines, enabled a view of the objects of function and what they came to signify to learners. The next section will therefore need to develop the construct routine further, as it may be observed in commognition, and the leverage the routine has allowed for in gaining insight into learner thinking.

3.3 Routines

In commognition, routines are defined as "a set of metarules that describe a repetitive discursive action" (Sfard, 2008, p. 208). These rules can be characterised as two subsets:

- The *how* of a routine, which describes the course of a patterned activity.
- The *when* of a routine, which describes a situation in which the performance of a routine is deemed appropriate.

(op.cit)

Examining these patterned ways of working for deviations is a means by which to examine learner discourse for alignment to the literate or formal mathematical discourse. The routines chosen by learners could be used to predict learners' discursive level (how they have objectified) and trajectory, as well as to assess the appropriateness of the routines chosen by learners. Research shows that for many learners the procedure is assigned importance, rather than the result (Sfard, 2007). Commognitively, this implies that the meta-discursive activity of reflection before, during and after closing the task, is not a priority of school Mathematics. Learners have been found to assess tasks or problems for cues they have encountered in their previous experience, where they impose a routine from their association with these cues (Nachlieli & Tabach, 2012). In most cases, especially in the secondary phase, experience is gained through the activities of the classroom. What initiates a particular discursive routine? Verbal prompts from the teacher or peers are largely responsible in terms of school mathematics. Other factors also play a role, such as variations in the use of keywords, visual mediators, routines and narratives in relation to a larger discursive context, and the nature of the activity learners engage (Sfard, 2007), as well as who is involved in the activity (Wood & Kalinec, 2012). Learners have been shown to revert to the most recent procedures they have encountered when solving problems (Sfard, 2007).

Mathematical routines aim to produce narratives about mathematical objects. The commognitive framework zooms in and thus delineates routines further by their outcomes:

- explorations produce endorsable mathematical narratives;
- deeds change objects through change in the environment; and
- rituals focus on reproduction of the *how* of a routine where the impetus is social reward.

Both deeds and rituals are necessary precursors to explorations. The goal of deeds is to transform physical objects, not to tell a story. As a result, they were excluded from scrutiny, as function is an abstract object, as opposed to physical object. I predicted that learners would communicate about this object in ritualised or exploratory ways at the point of exit from school mathematics. The methodology of this study enabled this distinction as learners were called on to elaborate their discursive choices. This distinction informs the choice of exploration and ritual as analytic tools in this study, to describe learner routines of the abstract object function. They can be seen as indicative of distinct ways of communicating – hence thinking – both exploratory or ritualised. The focus on routines examines the stories learners tell as they fill the mathematical object function and its connected processes with meaning. The study becomes an examination of learner thinking for exploration or ritual, and the way in which these are constituted. This distinction is now discussed in detail.

3.3.1 Exploration and ritual

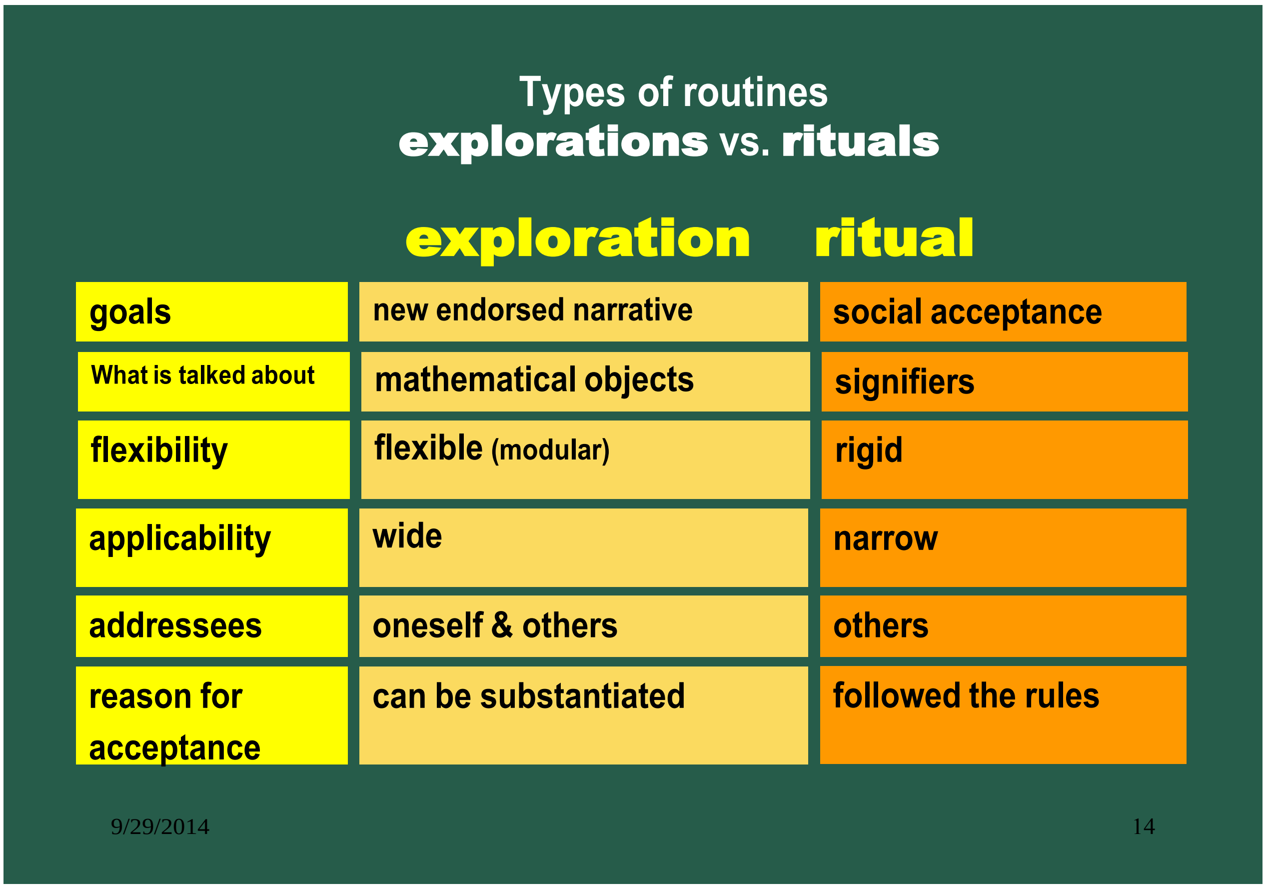
In commognition, ritual is a necessary precursor to exploration. This has informed the way I have set out this subsection. I have not handled each of these constructs as separate subsections, but together, in order to show the dependence then contrast of one with the other. Learner utterances were examined for evidence of objectification of function, and for the substantiation of closing narratives, to be considered exploration. The talk was of mathematical objects was key evidence. Exploration routines are applicable over wide contexts, and learners with this level of meta-thinking are able to use multiple, but equivalent means to substantiate their routines. Explorations do not depend solely on situational clues, or the need for social acceptance, but for what is made possible both with the mathematical object, and from it.

Rituals have, by contrast, the goal of creating and sustaining bonds with other people. They may be structured as a means for approval, and becoming part of a social group. In a way akin to ritual, learners sought attention, affirmation and approval from each other and the interviewer while they worked together. They were content to follow the lead discussant, and often act in agreement with the other. Imitation is a significant part of the discursive routine. Discursive actions are prompted by the other person, and are highly situated. As a result, they have a narrow range of applicability. Prompts offered by other participants are generally very specific, and extremely restrictive. Usually there is rigid emphasis on following rules: firstly, rituals that breakdown are repeated as opposed to corrected from the point of breakdown, which indicates that rituals enjoy a narrow range of applicability, limited to familiar contexts and recognisable cues; secondly, substantiations are not offered automatically, and are usually narratives that list the steps of the process (viz. the *how* of a routine, as isolated earlier).

At the meta-level of learning mathematics, new routines begin as ritual and gradually transform to exploration through a process of rationalisation. Imitation of a knowledgeable other is key to learning mathematics. The learner individualises the mathematical discourse by following the rules, rewording them, and reframing the contributions of other knowledgeable participants. In this, imitation of the knowledgeable other is a significant initiation into the discourse. Independent performance of the routine is still not possible initially. Learning occurs through scaffolded participation, structured and guided by the teacher in school Mathematics. Transforming ritual into exploration entails the learner's constant reflection on her performance, while simultaneously examining the rationale for the way in which the expert or teacher works. Vygotsky placed this ritualised discourse in the zone of proximal development of the learner when describing the period of individualising or rationalising. Proficiency, by means of reification of rituals, will transform ritual into exploration. The *how* is individualised well before the *when*. A learner whose performance is ritualised may harbour a collection of disconnected routines. Exploration, by contrast, looks for equivalence and groups routines that can be connected for their similarity. The implication is that the learner does not then hold a collection of disconnected routines, where the choice of one is random. The choice of routine, in explorations, is necessarily informed. The connection and link between rituals developing to exploration, allows compression of knowledge, where learners are able to both say and do more with fewer routines and words.

At a seminar at the University of the Witwatersrand in 2013, Sfard provided this useful summary of categories that distinguish ritual from exploration.

Figure 5 Characterisation of exploration and ritual.



This provided impetus for the current study, as it gave focus to learner discourse at the conclusion of the curriculum topic function in Grade 11. Learner discourse was thus examined for its routines, and these were categorised as shown above. Chapter 4 describes the research itself, its methodology and design, and how in the commognitive way, I operationalised and then refined the categories developed to describe learner thinking.

Chapter 4: Research design

4.1 Introduction

In this chapter, I describe the study design, and the way in which I proceeded to investigate learner thinking through learner discourse on function. This is an interpretive, qualitative study framed by the theory of commognition. One of the strengths of interpretation is that it focusses on the close relationship between the goal, the exploration, and the path taken to reach the goal (Mouton, 2012). The overarching methodology is informed largely by the way in which Sfard has worked with this as a theory of discourse. The theory of commognition has been described in Chapters 2 and 3. Aspects of commognition as they relate to the methodology and design of this study are described in section 4.4 of this chapter. Since the study has as its base that learning occurs through participation in a discourse, it proceeded through in-depth interviews and conversations with learners. It is necessary therefore to foreground the learners and context in which this study explored learner thinking. I will describe in detail, the schools, the sample of learners, the data collected, and most crucially, how the data was analysed.

4.2 The sample

4.2.1 The schools

This study used a purposive sample of learners drawn from six schools working in the Wits Maths Connect Secondary (WMCS) Project. The six schools were chosen from ten overall, because they each had a large enough complement of Grade 11 learners from which to choose, and relatively stable Mathematics departments, in the sense that learners would have seen little or no replacement of their Mathematics teacher in Grade 11; and the curriculum requirements for the teaching of functions would have been completed in the stipulated time. The symbiosis between the work of the PhD and the project work, which involved interacting with learners and teachers in the schools, helped to inform the choice of schools. For example, schools were also chosen according to a prediction as to whether they would provide the rich data I sought, and that learners would be competent to communicate in English as the language of instruction in Mathematics. Involvement with WMCS also informed me of the performance of these schools on the NSC examination, and visits to the schools gave an impression of motivation levels of teachers and learners. The six schools chosen were those best suited to the study's purposes. In order to take into account the context of the country's 11 official languages, I thought it prudent to learn more about the languages spoken in schools, in particular the classrooms selected in my study. Adler (2001) provided a useful classification of the language learning environments in schools as related to where they are located. She distinguished between two learning environments, namely additional language learning environments (ALLEs), and foreign language learning environments (FLLE). Intuitively, I presumed that the township schools were foreign language learning environments, where there would be little or no support for English in teaching and learning. I needed to confirm this, especially if I wished to explore learner communication. Conversation with all learners in this study before the paired interview, showed them to be multi-lingual in the main. ALLE was characterised by learners whose primary language is not English. English is an additional language for these learners, and there is support for English in and around the school. FLLE's in contrast, are institutions which are the only place where English is likely to be heard and used by learners, which is typically the case in rural South Africa. This distinction formed part of the description of the schools in the study.

School J situated close to the city centre, serves learners from the city centre and from townships around Johannesburg. The language of learning and teaching is English. In urban areas English is usually spoken more frequently. The two suburban schools, M and S, serve learners from the communities close to these schools, including a nearby township. Here too, the language of teaching and learning is English. The three remaining township schools, E, P, and T, use English as the language of teaching and learning, drawing learners from the surrounding community and from other SADEC countries. Lesson observations related to WMCS work, showed that teachers and learners are multilingual and code switch during lessons if necessary. While these schools are located in a township, further from the city centre, they are still surrounded by resources beyond the school itself where there is support for English use. This was unanticipated. It was indeed a surprise to find all the schools situated in the township, had resources like textbooks, study materials, workbooks, worksheets, and computer assisted learning programmes, all of which were in English. All schools in this study, can therefore be classified as additional language learning environments (ALLE) (Jill Adler, 2001). The six schools in the study could be categorised as follows:

School	Position in relation to Johannesburg City Centre	Distance from Johannesburg City Centre	Language learning Environment
J	Urban	< 1km	ALLE
M	Suburban	≈ 15km	ALLE
S	Suburban	≈ 15km	ALLE
E	Township	≈ 40km	ALLE
P	Township	≈ 40km	ALLE
T	Township	≈ 40km	ALLE

The description of schools included in this study sets forth the context of diversity and complexity from which learner mathematical discourses develop and grow. The proximity to the city centre is shown because it is a useful indicator of the way in which English can be found to be spoken more widely among learners during teaching and learning. Most learners in the study spoke more than two languages. Within a pair of learners, it was possible to find two learners who spoke different primary languages.

In Chapter 1 the significance of the issue of language received some introduction. While the politics and efficacies of language remain very important, in this study it would not have been practical enough to have learners communicate in their primary language, which was not English. The language diversity in classrooms involved in this study posed a practical challenge to find learners in the same performance grouping, speaking the same primary language. This is the reality of the diversity of languages in South African classrooms, with eleven official languages. Using primary languages in the interview would mean the interviewer would require translation into English, in order to probe learner responses further, which was deemed to be disruptive to the flow of talk between interviewer and interviewees. With two different primary languages, it was also possible that the learners would not fully understand one another during the interviews. The intention here was to focus on learners mathematical discourses, and how learners talk to each other in the language of teaching and learning, in the language that will be required for tertiary study in mathematics. English was settled on as satisfying this criterion, and as a second or third language for many learners it was a common thread through schools.

Learners were further welcomed throughout the interview to code switch freely in their conversations. This chapter shows evidence of the translation of learner talk from their primary language to English, which was closely inspected for its mathematical evidence.
The research design required that learners talk to one another, and English was thus expedient as a choice to support this design. Research themes relating language and meaning in mathematics are crucial and especially important in a South African context. Accommodating the diversity of languages in classrooms and the gains these lead to in learning mathematics are important directions for further study. However, the selection of English in this study, as a view into learner thinking, was a reasoned compromise. See the discussion in section 4.3 for the nature of the utterances involving translation from learners' primary languages into English.

4.2.2. The learners

To reiterate, the preliminary analysis examined learner performance on the 2009 NSC[8] so as to establish with a higher resolution, compared to assessment reports, the detail of the difficulties learners experience in mathematics. Performance over twenty questions related to function and algebra, were poor, marked by absence or error. This was a quandary relating to poor performance. Ben Yehuda et al. (2005) have observed that we lack conceptual tools for dealing with failure. This study therefore embarked on an alternate route, through discourse, to investigate learner thinking on the topic of functions (with its related algebra). Examining learner discourses within a commognitive framework provided the high resolution view into learner thinking on function and its related objects. Discourse becomes the subject of scrutiny and access into exploring learning. An initial conjecture as I set out on this path was that learners who performed well in mathematics would have a more objectified discourse. This informed the choice of learners for the study.

Foundational to this study is the assertion that commognition gives us the equivalence of 'thinking as communicating'. The aim of the research is to explore and describe Grade 11 learner discourses on function. It focuses on learner routines in functions, which it explores through semi-structured interviews with pairs of learners. Unlike the compelling absences that existed on

[8] NSC national senior certificate is the exit examination for Grade 12, which is the end of formal schooling in South Africa.

written examinations, an near silence, if you will, it was found that learners have developed a discourse on functions, and that they could respond to questions asked of each other, or asked by the interviewer. The absence of response seen on written tests did not occur in the interview setting. In fact, learners were very generous in their responses, and with sharing what they know. The ample data, provided opportunities for examining learner discourses of the same object in different contexts. Looking for alternate ways of examining learner thinking, as an approach to better understand learner performance labelled as *poor,* appeared to be justified at this point. The abundance of data was not anticipated. Contrast this with the scarcity of information on the Grade 12 examination scripts, characterised by absence and error, which has already been stated as the impetus for this study.

The collection of interview data was timed for after the point when topic functions had been taught in the schools. As per the curriculum requirements, the topic is completed and assessed in the June examination of the Grade 11 year. Learners were selected from the six schools based on their performance in this examination. The intention was to look at learner discourse at different levels of performance. Mathematics teachers assisted with providing a ranked list of learner's marks from the June Examination. This became available in August 2012, after the winter holidays. The initial plan was to look at a top, middle and bottom grouping of learners based on the assessment scores. The top was defined as Performance Group 1, referred to as PG1, comprising learners who had scored 80% and above. A survey of results showed that two schools had no learners in this category, and in three other schools, a group of four learners for the Card Matching activity (one of the activities planned to provoke learner communication as detailed below) could not be constituted. When the range of performance was adjusted down to 70% and above, the PG1 grouping became possible across all schools. After feedback from a conference and research seminar in 2012, PG2 was defined as being between 40-60% to capture learners who were passing mathematics, PG3 was defined as being below 30 percent. This is a characteristic feature of Mathematics performance in South African schools, namely a skewed performance curve in terms of poor performance.

Once performance groups were defined from the ranked lists, consultation with teachers about learners in these performance groups commenced. Teachers advised on learners who could communicate comfortably about their reasoning, and were reasonably fluent in English, basing this on their experience in the classroom. Data collection occurred, after it was deemed necessary to consider school and practical difficulties related to learners' availability. With the groups in place and learners identified, consent forms were sent to learners and parents of those learners involved, detailing the involvement in the study and requesting permission of parents for their child's involvement.

This study, the learners, and the schools they attended, encapsulated the complexity and dynamism of learning in environments of vibrant, thriving diversity and grave inequality. Yet, such entropic contexts also unexpectedly and inexplicably produce pockets of excellence. It stands to reason that these pockets of excellence ought to undergo further scrutiny so as to better to understand them and develop them.

4.3 Data sources

In an attempt to describe learner discourse in function, there were three principles underlying the development of the data sources~~:~~

i. to objectify function, learners must be competent in discourses of all its realisations;
ii. the discourse on function is necessary for the objectification of function; and
iii. for objectification to happen, learners practising the new discourse must reflect on what they are doing.

(Nachlieli & Tabach, 2012)

The paired interview schedule can be found in Appendix 1. The principles listed guided the following: first, learner discourses of the graph, table and algebraic representations were recorded from the semi-structured paired-interview; second, within these discourses, the flexibility of working across representations was a means to check if learner discourse subsumed each of the separate discourses under the single object function, where the question was asked as to whether learners regarded the table, graph and algebraic expression as being representative of the same thing; third, the structure of the interview allowed space for learners to explain their discursive actions and reflect on these, where, in instances in which they did not explain satisfactorily, they could be probed further.

As discussed in Chapter 1, South African classrooms are excitingly diverse in both culture and language. Included in this diversity were learners, not only from South Africa, but from other countries in Africa as well. Learners involved in this study, could speak a minimum of three languages, and a maximum of nine.[9] While this study did not look at language and culture of these learners directly, it acknowledges the importance these aspects play in learning. Acknowledging the diversity of the South African context, English is still stipulated in the National Curriculum statements as the language of teaching and learning. Learners (in this study) will write the National Senior Certificate Examination – the high stakes school exit examination – in English. From working in the six schools involved in the study, I found that teachers code-switched during lessons when they deemed it necessary or when prompted by a learner to do so. Learners were thus permitted to talk to each other during the tasks in that language with which they were most comfortable. This tacit permission was observed to have had two interesting results:

- Learners predominantly used English in their talk on the activities, which is possibly explained by the presence of the interviewer, for whom they would eventually need to explain the reasoning used to arrive at their final response. Additionally, this was to facilitate probing by the interviewer if responses needed clarity or extension.
- Learner's vernacular or primary language talk during the interview was transcribed with the assistance of a translator. The half hour of video recording was translated from a recording, where learners reverted to primary language more frequently. The result showed that talk involving vernacular, did not contain significant mathematical talk, or deviate from the mathematical explanations that were offered as final explanations. The vernacular talk largely showed an enquiry into how to approach a problem, affirmation of what a partner was saying or doing, and requests for clarity about the others' mathematical actions. In the main, the utterances were non-mathematical. For the reason of usefulness to the argument of this study, as well as limitations of time and finance, in consultation with peers, I took the decision not to translate all vernacular utterances, but only those most pertinent. Appendix 3 shows the translation on the transcript.

4.3.1 Description of measuring instruments[10]

There were two activities used to engage learners:

[9] This information was gathered in a pre-interview exchange with learners.
[10] The paired interview schedule of questions can be found in Appendix 1 and the card matching activity in Appendix 2.

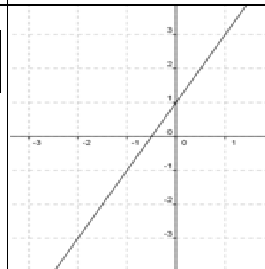
i. a card matching activity; and

ii. a semi-structured paired interview.

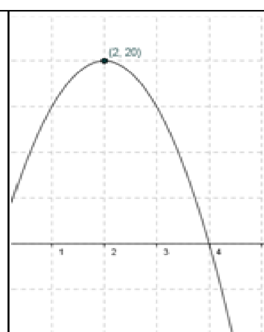
The description and purpose of each follows.

i. Description and purpose of the card matching activity. There were two purposes for this activity. Based on the principles listed in 4.2 above, relating to having learners competent in all the representations of function before they are able to objectify function, the first purpose was to access learner discourses across different representations, and the second purpose was for the selection of two learners from the group of four, to participate in the detailed paired interview that followed. The activity involved four learners working in a group, around a chart. The card matching activity (see Appendix 2) presented learners with a chart comprising five columns. Four columns for the various signifiers: algebraic expression, table (or numerical), graph or verbal.. The fifth column was for distracters or extra cards, which target areas learners commonly find reasoning through difficult. The rows showed various functions studied in school mathematics: two linear, four quadratic, three hyperbolic. Learners were presented with the chart on which one or two representations of each function were shown; the other cells in the row were blank. Blank cells had to be filled from a pile of cards, which contained the missing representations. The distracters provided useful opportunity to challenge learners, and to take them out of familiar representations typical of the school mathematics routine. For example, as illustrated below, an extract of the chart given to learners shows the distracter challenging the relating of the algebraic symbol $2x$ on Card 1, to the verbal expressions doubling and *squaring*, which appear on Card 5.

Figure 6 Extracts of the card matching chart showing strategies in development of the activity.

<table>
<tr><th></th><th>Algebraic</th><th>Numeric</th><th>Graphical</th><th>Verbal</th><th>Extra (may be correct or incorrect)</th></tr>
<tr><td>Linear 1</td><td>$y = 2x + 1$
1</td><td><table><tr><td>x</td><td>-3</td><td>-2</td><td>1</td><td>5</td></tr><tr><td>y</td><td>-5</td><td>-3</td><td>3</td><td>11</td></tr></table>2</td><td>
3</td><td>To find the output, double the input and add 1
4</td><td>To find the output, multiply the input by itself and add 1.
5</td></tr>
<tr><td>Linear 2</td><td>$y = -2x + 1$
19</td><td></td><td></td><td></td><td></td></tr>
</table>

<table>
<tr><td>Linear 3</td><td></td><td><table><tr><td>x</td><td>-3</td><td>-2</td><td></td><td></td></tr><tr><td>y</td><td></td><td></td><td>-3</td><td>-11</td></tr></table>12</td><td></td><td>To find the output, multiply the input by -2 and subtract 1
40</td><td></td></tr>
</table>

<table>
<tr><td>Quadratic 1</td><td></td><td><table><tr><td>x</td><td>-1</td><td>0</td><td></td><td></td></tr><tr><td>y</td><td></td><td></td><td>20</td><td>0</td></tr></table>13</td><td>
28</td><td></td><td></td></tr>
</table>

The first row was completed with learners so as to familiarise them with what was required in the activity. In the second row, we see a linear function, where the algebraic expression of the function was the given information. Learners were to place cards into the blank cells representing the table, graph and a verbal expression of the function. The fourth row of the activity shown gives learners both the table and graph of a quadratic function. You will notice additional structural features, in terms of the information given, and its presentation on the chart. To illustrate, values were omitted in the table on the fourth extract, where the graph is partially obscured. Learners were required to use the graph to find values that are missing on the table. The obscured x-intercept was significant in describing what learners could do. It also allowed a view towards how learners arrived at the algebraic representation of the function: did they use purely algebraic means by establishing the relationship from the table of values, or were algebraic algorithms applied to the specific features of the graph?

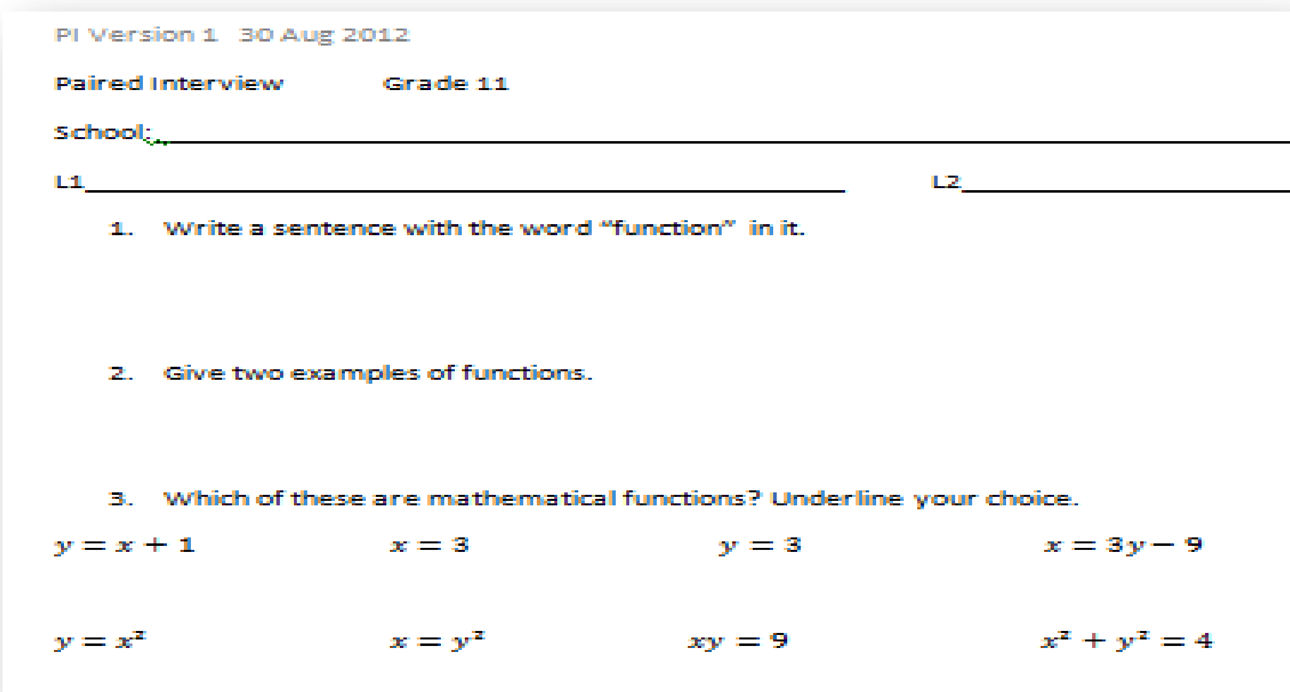
Learners worked in fours, according to the performance groups already discussed, and were instructed to talk to each other and offer a co-constituted, final response. The interviewer was available to answer questions for clarity of the instrument. Questions relating to the mathematics involved in the task were redirected to the group. This facilitated seeing the components of their function discourse, as well as illuminating learner thinking, whether verbal or written. From this group of four, two learners, who more easily explained their mathematical thinking, were selected for the paired interview that followed.

ii. Description and purpose of the paired interview. Going through the card matching activity as a precursor to the paired interview had certain advantages, some of which were not anticipated. One of these was the opportunity to get to know the learners and make them comfortable with the presence of the interviewer and audio-visual recording equipment. It was essential in this latter activity that learners were able to describe, substantiate, justify, err and correct their mathematical moves. Observations and discussions with learners prior to the activity were recorded as field notes. Primary language, additional languages and the distance of their home to school, were typically asked after in casual conversation before the activity commenced. All interactions with learners showed them to be enthusiastic in their participation, and very positive in their reflections after each activity. The paired interview formed the core of the data to be analysed.

The schedule of questions for the paired interview went through significant revision (see the original in Appendix 4). It speaks to the inexperienced researcher (myself), guided by a knowledgeable other, in the process of research supervision (paper forthcoming). From the governing principles of the methodology, it was essential to determine learner competency on all the realisations of function, and the level of objectification of function as a result of this evidenced competency. In addition, creating an opportunity for learners to reflect on their responses was important. Reflection entailed learners explaining, justifying, and substantiating their discursive moves (written or spoken). This appeared not to be part of their regular classroom and assessment practice.

The draft interview was piloted on a pair of learners in one of the schools. This was subsequently discussed with my supervisor, as well as Professor Sfard, who was visiting at the time. It was the common view that the interview schedule needed to be reworked to give learners more opportunity to respond, to encourage talk, and to include questions which were not traditionally asked in the school Mathematics classroom. Contrast the extract of the first interview schedule below, with the second revised version. The learning gained related to how to construct questions which open up the possibility for learners to talk spontaneously, and required them to think of their responses, as well as to hinder cue-based automated responses, except as part of a thoughtful response.

Figure 7 Version 1: Schedule of questions for paired interview.


PI Version 1 30 Aug 2012

Paired Interview Grade 11

School:

L1 L2

1. Write a sentence with the word "function" in it.

2. Give two examples of functions.

3. Which of these are mathematical functions? Underline your choice.

$y = x + 1$	$x = 3$	$y = 3$	$x = 3y - 9$
$y = x^2$	$x = y^2$	$xy = 9$	$x^2 + y^2 = 4$

Notice that:

- Questions 1 and 2, though appearing ‘open’, create the opportunity for non-mathematical responses. It invited colloquial, everyday responses. Learners talked of biological functions, for example.
- Question 3 can prompt responses for Question 2.
- Question 3 allows learners to respond by guessing.

The questions needed to give learners maximum opportunity to talk mathematically with each other. The initial questions in Figure 7 fell markedly short, even though learners were asked to discuss their written responses when complete. The contrast with the adapted schedule, shown in Figure 8 highlights the types of questions which encourage and entice learners into discussion, as well as the importance of having questions which are better directed towards the critical questions of the study.

Figure 8 Extract of the revised final schedule of questions for the paired interview.

Paired Interview Grade 11 Date:

School:____

L1________________________ L2________________________

Thank you for agreeing to talk to me about your mathematics learning.

I am particularly interested in the topic of functions. You have studied functions in Grade 10 and again in Grade 11.

I am sure you have interesting things to say about these and how you learned about them.

---

1. So let's begin with this scenario. Imagine your younger sister or a friend who is in Grade 9 now and so will be going into Grade 10 next year – and she is interested in mathematics and what it will be like in Grade 10.
   She asks you to tell her about the topic called functions that she knows she will learn.
   She asks you: Please tell me what I will be learning about when I do the topic of functions?
   What will you tell her?

   possible prompts

   - So, what kinds of things will I be doing in class when I am learning functions?
   - Is it an interesting topic? a difficult topic? What is difficult about it? What is easy about it?

2. Can you give me two different examples of functions – you can write them, on the sheet of paper
   - Prompt for other examples not mentioned in 1.

3. Can you think of any of these as representing a function? Place your choices in a separate pile.
   - Instruct learners to place in a pile on one side all the functions
   - A letter is used to number the card, it appears at the top left of the card so that the card oriented when being examined.

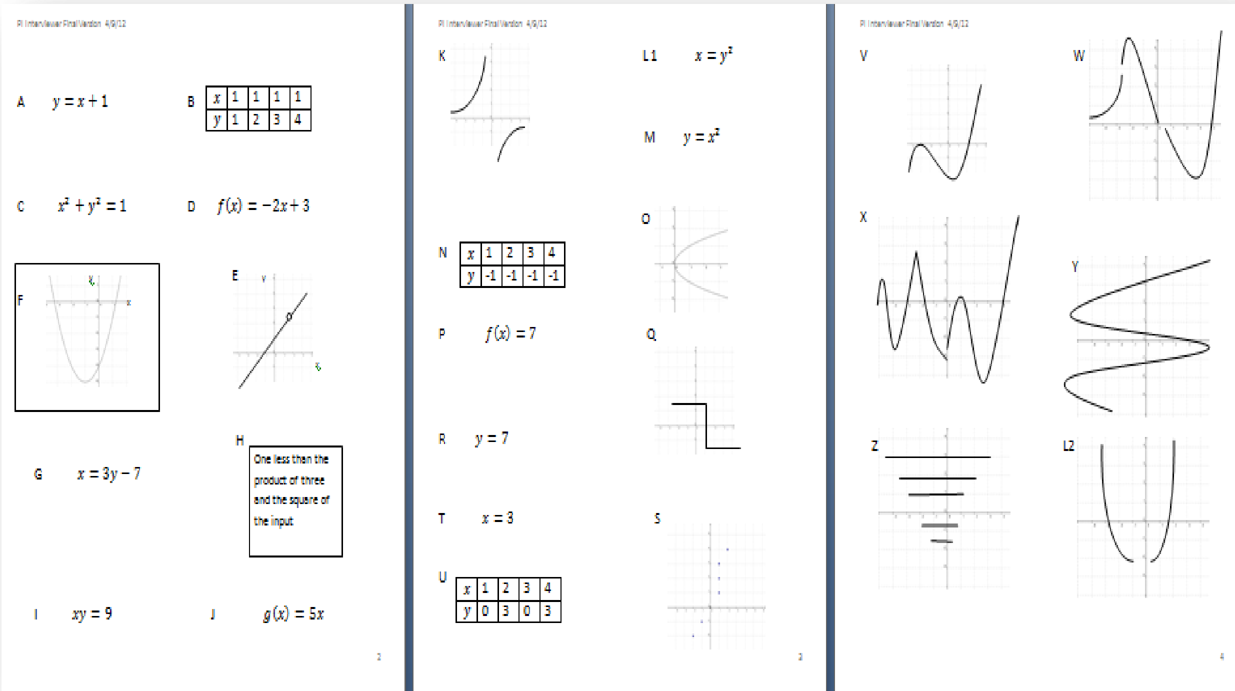
The revised, improved questions leveraged the following in the data:

- focused learners immediately on the mathematical object they were to engage;
- the subsequent question developed on the first, and extended learner thinking on the first;
- even without a formal mathematical discourse on function, learners were still able to participate in the discussion, where they were not observed to stall, without a response, as they had on the previous interview; and

- questions challenged learner thinking of the unfamiliar, to examine how they combine the unfamiliar with their existing discourses to make a coherent substantiations, where the balance between questions that, on the one hand challenge learners towards a coherent explanation and, on the other, those that inhibit mathematical discussion, prove critical to the observations of a novice researcher.

Question 3 on the paired interview, in particular, requires further elaboration. It mimicked the card matching activity, and learners at this stage were relaxed, hence confident, to volunteer their responses. Learners were given various relationships shown on cards below, which they were asked to separate into piles of functions and non-functions. Familiarity and experience with the card matching activity contributed to learners' understanding the need to explain their reasoning. They understood that they had to explain the routine chosen, as well as to justify why that particular routine was chosen. The graphical, tabular and algebraic expressions, shown on cards, challenged learners beyond the familiar functions of the school curriculum.

Figure 9 Extract from Question 3 on the paired interview.



The graphs of Question 3 offered scope for creativity and discussion. It saw learners work as a pair. Consensus on responses were emphasised and the discussions towards this contributed valuable data, especially where learners differed with each other. This was to visibilise commognitive conflict (discussed in Chapter 3), for the researcher. Graphs with unique features like discontinuities, proved excellent discussion points, as learners had not encountered these in school Mathematics before. Such examples facilitated discussion around how a function is defined, as well as thinking around the relationship between *x* and *y*, defined at a point. The usefulness of this lay in the extension of learners established discourse to objects with which they were unfamiliar. The algebraic expressions on the interview included aspects of formal algebraic notation and univariate relationships, noted as problematic for learners in the literature cited in Chapter 2. Question 3 was designed to represent as wide a variety of relationships as possible. The inclusion of representations that were unfamiliar held the possibility that learners would not respond. Most learners however responded to the 'unfamiliar' positively, encouraged by the other person in the pair.

Learner responses to the questions on the paired interview were separated into the representations: algebraic expressions, tables and graphs. These were coded, and codes were then aggregated for each representation. The methodology of this study foregrounded the dynamic process of research, namely of how important it is to fine-tune one's instruments, of the people involved in the process, of how complex human endeavours (such as learning in this case), can all interact in a complex mix, which is open to various ways of interpretation. The next section took the myriad of possible approaches to these complex themes and helped me to focus on the way that the data was examined.

4.4 Operationalisation

Operationalisation requires that we all act according to the proposed definitions (Sfard, 2008). I regard to be a strength of the commognitive framework to be rigorous in the way that the terms used in research are defined, I will now elaborate the terms I have used from commognition as relevant to this study, and how these were implemented in the analysis.

4.4.1 Operationalisation of theoretical/ commognitive terms

The commognitive framework lists four properties for a discourse defined as mathematical, namely: word use; visual mediators; narratives; and routines. Each of these properties have been detailed in Chapter 3. From the delineation of these properties and in examining routines, all the properties listed are examined as well. Word use, visual mediators and narratives, as they form part of learners' mathematical routines, gave the study a sharper contextual focus than examining each one separately. Accumulated into a routine, each contributed to a fuller picture of learners' mathematical thinking. The mathematics communicated in particular learner realisations, within a routine, provided the indications of exploratory or ritualised thinking. The distinction between explorations and rituals (discussed in detail in Chapter 3 and summarised in Figure 5 became the first level of operationalisation.

The semi-structured interview of a pair of learners permitted the transcripts to be chunked according to each question to which the learners responded. Within routines evident in learner talk, additional patterns presented within each broad category (shown in bold in figure10 which follows). For example, under mathematical objects, there were patterns that existed in learner discourse which permitted the category be described by specific indicators. These indicators were coded. The indicators arose as a combination of previous commognitive research, which informed how to look for an objectified discourse, as well as from multiple readings of the transcripts, across schools and performance groups, for patterns that arose. The organisation of the transcripts into the three performance groups, with the learner responses aligned, gave a view of learner responses across the levels of performance. The routines that learners employed commonly across schools, and in addition, routines that were contingent and spontaneously arose in the interview, were embedded in the transcript, arranged to performance levels, chunked into responses to specific questions.

4.4.2 Operationalisation: The analysis

The opportunity to contrast discourse across performance groups was telling. It was expected that the better performing learners had more to say about the object; they also responded in greater, more appropriate detail to probes, discussion and questions from each other and the interviewer. This was verifie when performance group responses to a particular question were set up alongside each other, the arrangement alone began to suggest a relationship between performance and the level of objectification. To develop the idea of level of objectification further, each category was mined further for finer resolution on what it could mean. The question that helped focus and detail each category was: *how could the category become visible in learner talk and written responses?*

This process, the development of the finer details within the categories, provided an extension to the existing parameters within the commognitive framework. It provided detail in each of the categories (see the bullets in the that follows). The indicators used arose from several readings of learner transcripts, combined with the synthesis of commognitive literature, supervision and inputs from seminar and conference presentations. Below is the table of indicators used to code learner utterances:

Codes distinguishing exploration from ritual.

Figure 10 Codes distinguishing exploration from ritual.

	Exploration		**Ritual**	
Goals	**Produces endorsable narratives** • Solves to derive new narratives; establishes purpose for solution and interprets • Asks , attempts to answer open questions	E6 E7	**Performed for social acceptance** • Reference to memory or authority • Works to goals set by others • Guessing	R1 R2 R15
What is talked about	**Mathematical objects** • Signifiers are abstract mathematical objects • Talk of specific features of mathematical objects • Symbols/ procedures are justified or related to the object	E1 E3 E5	**Signifiers** • Uses mnemonics/visual clues • Talks of/acts on symbols without their meaning • Misrecognition of form of algebra or graph; no meaning attached to symbol • Asks closed questions • Subjectification • Spontaneous/ everyday language	R5 R6 R13 R10 R11 R18
Flexibility	**Flexible** • Connects different representations; equivalence	E4	**Rigid** • Different representations are separate entities • Recycles old routines, inappropriately or appropriately • Concern with making errors • Difficulty with following rules	R4 R7 R9 R16

Applicability	**Wide** • Moves from process to object	E9	**Narrow** • Talks about actions and manipulations • Concerned with the final answer; Empirical proof	R3 R8
Addressees	**Oneself and others** • Questions and justifies narratives	E8	**Others** • Statements are not questioned or justified • Imitates others; tries/keeps pace	R12 R17
Reasons for Acceptance	**Can be substantiated** • Narratives, process logical/deduction; Justification	E2	**Followed the rules** • Emphasises rules and practice	R14

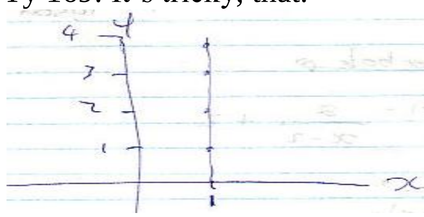
Chapter 3 noted that an exploration discourse is built through learners' initial ritualised participation. Through ritual a learner rationalises and individualises the discourse by gradually losing the scaffolding initially inserted by a teacher or knowledgeable other. The resulting exploration discourse has as its goal *to learn more.* The transition from ritual to exploration is marked by increased objectification. View examples of the assignment of each of the codes to transcripts in Appendix 5.

The following transcript compels explanation on two levels: first, the methodological and second, the analytical. First, the unit of analysis was learner realisations. Given a specific mathematical object, *what did learners realise from it*? All learner utterances were chunked according to the question on the interview schedule. How transcripts were arranged prior to coding is shown on an extract of one particular representation below. In the transcript extract shown in Extract 1 below, learners across three performance groups, PG1 to PG3 left to right, from school E is shown. To contextualise, learners were presented with a table of values B, and asked whether they saw a function on the table of values:

B

x	1	1	1	1
y	1	2	3	4

Extract 1 The contrast of learner talk across performance groups regarding Card B.

PG1	PG2	PG3
O 143: About this? Ja, I can agree ‘cause on… Can I plot this? *(to learner G. Starts sketching graph)* O145: Ja, this this this graph is the same as that one we are doing. So, if you can put a vertical line test… *(refers to graph)*  O146: It will touch it many times.	Ts 166: It would be an exponential *(pause)* *Interviewer: How do you know that?* Ts 168: Um, the *x*-value doesn’t change, and the *y*-value, well, it should be more like an exponential. Ty 169: It’s tricky, that.  *Interviewer: Is that exponential?* *(learners are looking at sketch)* Ts 171: No, it’s a straight line… *(pause)* *Interviewer: And is that a function?* Ts 173: No. *(Ty nods head indicating yes)* *Interviewer: Why?* Ts 175: It’s parallel to the *y*-axis. *Interviewer: You say it’s a function?* Ty 176: Yes. Ty 177: Because it has both *x*- and *y*-intercept. *Interviewer: Where’s the y-intercept?* Ty 179: Ok, no not intercepts but ah, it has the *x*-intercept and it has the coordinates, it has both the *x* and *y* coordinates. It forms a straight line. *(Ty places Card B in the function pile, Ts agrees with this move indicating with head nod)*	L96: I don’t think it’s a… I don’t think it’s a function. L97: ‘Cause, it doesn’t have, even an intersection, a *y*- intercept or an *x*-intercept. *(pause, probe, nothing further)*

In extract 1 shown, the question or object to be discussed was the values shown on Card B. The following methodological considerations were used in arranging the transcripts:

- Learners' written responses were inserted into the transcript, where they were referred to by learners, or where it would make sense to the reader of the transcript. In addition (though not shown), where learners may have represented their narrative by several sketches or algebraic symbols and claimed these to be in error, these representations were also included on the transcript. This was an attempt to capture learner thinking as a process in its entirety, including errors and moments of reflection and self-correction, and not just the end product.
- Certain learner responses, if vague or brief, were probed for further explanation. These were included in the relevant section/chunk of the transcript.
- Verbal utterances were inserted into the transcript verbatim. Care was taken not to alter utterances in any way. They were arranged according to the question they addressed. Learners did not work consecutively through the questions on the paired interview. Often they would refer back to or add onto responses they had previously given.
- O143 or Ts 166 refer to learners O and Ts and utterances 143 or 166. In the analysis, the full notation of an utterance appeared as E PG1 O 143 (E=school, PG1=performance group, O143= learner O's utterance 143). Lowercase script was used as the second letter referring to the learner if both learners in the pair had common first initials e.g. Ty and Ts.
- The interviewer probes were shown in grey and learner actions were captured in brackets. These are not allocated as an utterance number.
- Notice that utterances in PG1 and PG2 do not appear with consecutive numbering. In PG1, numbering skips from O143 to O145. This is because the interviewer question, prompt, observation and statements between 143 and 145 were not listed with a number.
- Numbers were also omitted if learners dealt with a matter outside the chunk in focus, e.g. actions not related to the mathematics, such as dropping a pencil and saying something as they picked it up, or a teacher coming to the door to enquire about something. These comments were edited out of the transcript completely in preparing transcripts for analysis.

- Mathematics related omissions in the transcript required some license and involved utterances about other functions on the task. It could happen that something a learner talks of in a particular question relates to another function in the activity. Learner discussion then switches to the other function. In this instance, one of two things occurred: the discussion which ensued was linked to the original function that was discussed. Here the discourse linking the functions was captured under the original function. If the discussion that ensued showed no link between the functions, but appeared as discussion of a single or particular function, followed by discussion of another, these discussions were captured separately under the respective functions concerned. This process of capture becomes visible when the utterances are not consecutively numbered.
- Transcripts were arranged according to learner responses to the different representations. Responses to all graphs, tables and algebraic representations were collated separately in these as main sections.

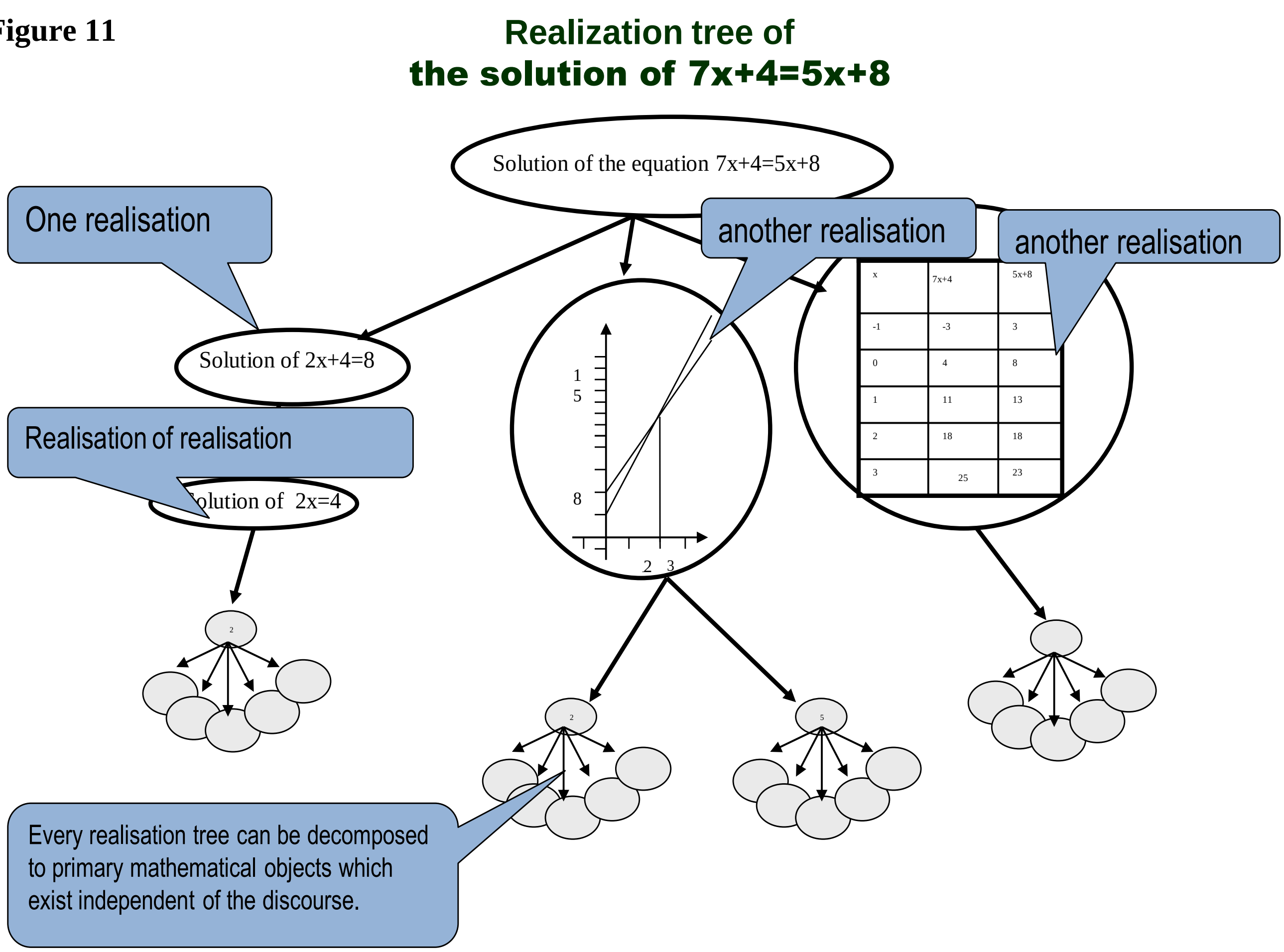
With the above arranged, the analysis of responses began using the table of codes for ritual and exploration.

Second, the discussion of methods related to analysing the data can now be discussed. The 18 original, separate transcripts were arranged according to schools, with each transcript reflecting the three performance groups belonging to the school, arranged alongside to each other. The arrangement of the data, per school, held several advantages. All learner utterances pertaining to a particular function representation became visible at a glance. Having read the transcripts separately three times over already, the need to preserve the verbal meaning in learner utterances and written communication was a priority. This facilitated analysis in the following ways:

- Within and across performance groups, across question comparison.
- Repetition in use of keywords or phrases, commonality across narratives and routines became instantly visible.
- Ritualised and exploratory thinking within a question became clear in the specific utterance, in the context of the question and as patterned practice across different levels of performance.

- Compression and expansion of discourse on an object was easily visible, simply by the number and length of the utterances on a question. It must be noted that objectification results in the economy of discourse – it disambiguates – and enhances the effectiveness of communication. Learners in PG1, for example, were able to say more mathematically with fewer words. PG2 and PG3 in comparison swung between long, subjectifying utterances around what they were doing, and PG3 particularly, had significant utterances referring to not being able to remember or not knowing what had to be done. The transcript setup thus seemed justified, as it provided an initial impression of learner communication, that perhaps would not have been so obvious or apparent had the transcripts been handled separately, in the way the data was collected.
- The next analytical move was to parse learner narratives into utterances that could be numbered. Narratives could comprise several utterances. With the unit of analysis defined as learner realisations, the analytical move was made to examine and classify each utterance for ritualised or exploratory thinking, as the learner worked towards a final realisation. The significance of the relationship between signifier and realisation(s) is illustrated in a Figure 12, taken from Sfard (2013a), presented as part of the seminar series, *Working with discourses commognitively*:

Figure 11



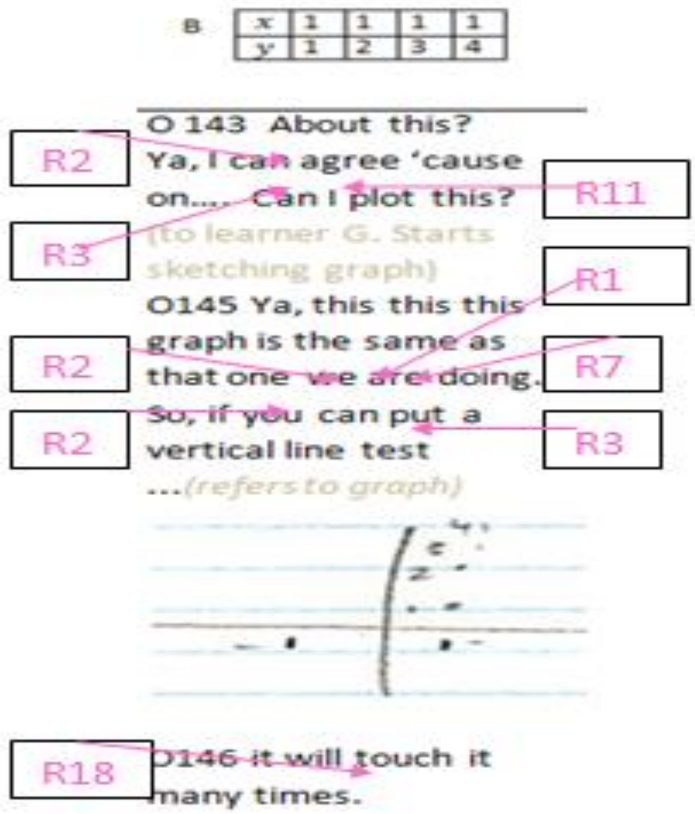
In Figure 12, the realisation tree depicted is effective in showing the rationale behind the parsing of utterances, when learners are given an equation to solve. When presented with such an equation, it has to be asked, “what does this collection of symbols signify for learners?” Learner realisations could involve an algebraic, graphical or tabular (numerical) approach. These are shown as primary realisations on the diagram. From these, we can see in the case of the algebraic approach that learners realise further on their primary (first) realisations. Methodologically, decisions were made concerning sequence of the realisation and breaking them up into analysable parts. In addition, operationalisation of the diagram included a substantiating step: once the primary objects were achieved at the bottom of the realisation tree, learners were asked to reflect on these primary objects and the means they used to arrive at them. Primary objects by definition cannot be realised any further in relation to the task. Substantiation called for learners to fill with meaning what $x = 2$ or the ordered pair $(2; 18)$ could mean in relation to the question asked. This was an important methodological inclusion, because it facilitated the distinction between ritual and exploratory thinking. Ritualised thinking became evident through learners repeating the process they had just executed, with little or non-related substantiations, where they were unable to reflect on their final answer. Asking learners to substantiate and justify their mathematical moves was also telling in the same way. It must be noted, that learner facial expressions, volume changes during utterances, and pauses in speaking, were not captured for analysis. They form part of detailed discourse analysis, which was able to examine how learners communicate mathematically and extend beyond the intentions and scope of this study. It ought to be remembered that I focused on what learners were saying mathematically, and characterising these as ritual or exploration.

Learner narratives were parsed into numbered utterances with consideration as to whether:

- the talked changed from one learner to another;
- a learner paused after an utterance to allow the other learner to continue; paused because she could not say anything further; paused to gather and think about what to say next; and
- to separate utterances according to perceived complete sentences or phrases. This allowed the route to a realisation or the realisation itself, to become clearer, where not every utterance contained a mathematical realisation.

With methods used to prepare the data for analysis exemplified, illustration of coding on extract of transcript E PG1 is shown below as related to table of values B shown:

Figure 13 Assignment of codes to learner utterances.



Challenges to consistency in coding arose in having to read learner utterances in relation to each other. This added dimension to the analysis, where the coded phrase or utterance was initially coded in isolation from the total narrative, and then examined in relation to all other utterances for what was realised by the learners, in relation to a particular question. Questions emerged, such as, did the narrative follow logical deduction? How and where did it break down when it did? Were ritual and exploration codes being assigned consistently? As stated, phrases were not isolated and coded; it was necessary to examine them in relation to each other. No mathematical communication relating to a particular question was ever discarded. This was to account for what learners were able to do with their realisations, especially for where they may not have had a fully developed formal mathematical discourse. These instances were particularly valuable for the ways that learners develop meaning by connecting what they already know, or have experienced. While the emphasis in this study is not on learner error, error frequently showed deviation from the formal mathematical routine. Such instances were provocative, particularly where learners erred in similar ways, where there is a sufficiency of rich data available gathered here that might initiate expanded exploration outside the scope of the current study.

Once ritual-exploration codes were assigned to utterances on the transcript as shown in Figure 13, they were tallied per code and per question. I repeated this process in a second filtration of data. Disparities were carefully considered, and I noted when codes had to be changed, as well as my motivation for changing them. This led to an aggregate of codes under each of the representations: tables, graphs and algebraic expressions. The aggregates were assigned the categories listed on the Figure 10. Presentation of these findings can be found in Chapters 5, 6 and 7 which follow.

4.5 Trustworthiness of the research

4.5.1 Description of data collection

This study initiated from trying to understand learners' poor performance in functions. In 2010, 2196 NSC 2009 scripts, of the schools involved in the WMCS project became available from the Department of Education. This study was permitted access to them, bound by the ethics of the broader project. The capture of scores, over twenty algebra and functions questions on the examination, highlighted overwhelming error and absence in learner responses. I have coined this 'the presence of absence'. This posed certain quandary: how could learners perform so poorly on a topic that enjoys longevity and significant presence in the secondary school curriculum and assessments from Grade 8? I reasoned that there had to be some other means (other than examinations) to establish the way in which learners think in functions, and to more precisely determine what the nature is of what learners know about this important mathematical object.

The research proposal was approved in March 2012, and final ethics clearance granted by May of 2012. Liaison with schools and teachers commenced immediately after the ethics clearance. Initial enquiry involved drafting timelines for schools to complete the curriculum requirements for functions. This was established as June 2012, with the final formal assessment as the June Examination. Teachers of the schools involved providing the examination results of their Grade 11 learners in early August 2012. Tentative performance groups were drawn up and then discussed with the teachers involved. On their recommendations, learners could be swapped for others, based on their performance as a mandatory requirement, and then on the learners' availability and proficiency in English to explain their mathematical reasoning. The range of the performance groups, as described earlier, was preserved in these decisions. Once learners were finalised for the study, ethics forms regarding the nature and extent of their participation in the study was detailed. These were signed by both learners and their parents before data collection commenced. These forms were collected from the schools and are part of the records of this study. Data collection commenced in August 2012 and ran till October 2012. Both the card matching activity and paired interview were conducted in a classroom, in the school concerned, from 14:30 (the end of the school day), till 17:00, where some interviews took longer than the time allocated. The card matching activity ran first. Learners were then selected from this activity for the paired interview, which commenced at the completion of the card matching activity across schools. The sole criteria for selection was learners who could explain their mathematical decisions. Field notes from the card matching activity noted details of learners such as the languages they spoke, and their frequent routines when they worked across the representations.

The pair of learners selected then moved onto the paired-interview. Unlike the card matching activity, here learners interacted primarily with each other to produce a common response to the interviewers' questions and probes. The card matching activity forced them to interact with each other and they understood this was expected of them. The paired interview appeared to expose far more in-depth talk from learners than did the prior activity. This was due to the interaction between the pair, the nature of the questions on the activity, and the probing of the interviewer. It proved to be an effective means to gather learners' discourses on the various function representations, and despite the formal discourse of the object function not being developed, learners were still able to offer colloquial and intuitive explanations for what they were thinking. There were a few instances in which learners left a question aside, without an attempt at mathematical explanation. This contrasted starkly with the absent responses on the written assessments I had already seen. The videoed interviews were transcribed between December 2012 and April 2013.

4.5.2 Handling of data

Transcripts were checked for accuracy of transcription. Learners written responses or additional notes on learner's responses were included on the transcript where they were deemed appropriate. Qualitative data analysis software, Nvivo,[11] was used initially to structure the transcripts. Learner utterances were separated and numbered into the three representations of functions that would be analysed. The interviewer utterances were formatted separately and not numbered, though they were recorded on the transcript.

As the analysis commenced in July 2013, regular supervision and consultation with discourse specialists suggested that the software not be used in analysis, since there was concern that it could obscure tones and nuances in the data, which were necessary in a qualitative, interpretivist paradigm, and indeed, although the software seemed to strip the data of its colour, it did systematically collect as in buckets, similar to the trends inferred in learner narratives. The manual coding that I used eventually did the same, but gave a more textured rendering of learner thinking.

The unit of analysis was learner realisations. The coding commenced and the manual process held great appeal, providing a total immersion in the data. It forced an intimacy with the process of learner thinking, by involving reading and rereading of learner utterances, on their own, and then in relation to one another. In contrast, the software involved the creation of nodes, which are themes, into which phrases and related utterances could be assigned. What this allowed for was the exploration and interpretation within a node, however, it at once sacrificed the picture of the realisation as a whole. The software allows for a lesser degree of nuanced reading on the part of the researcher than the manual process afforded, where, what it gained in objectivity and systematisation, it appeared to lose on opening up in the exploration/interpretation process. The connection between the video episode and the transcript was missing in the computerised coding process.

Reading and re-reading transcripts transports the researcher back into that space and time of the activity and one appreciates the expressions (facial and verbal), the tones, pauses, pitch of learner talk and reads transcripts with these in mind. The nature of qualitative research became apparent, in the handling of utterances, that could not be parsed so easily into nodes and the

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[11] **Reference for NVivo for Windows:** NVivo qualitative data analysis software, QSR International Pty Ltd. Version 10, 2012

bigger picture that somehow becomes opaque in a clinical approach to data. The computerised software approach was attempted, and then abandoned, except for the structure it imposed over the transcripts. In summation, the value of the total immersion in the data was made plain, even though this entailed far more work.

4.5.3 Validity and reliability

The nature of the analysis entailed looking for meaning in *what* learners communicated mathematically. This happened within a qualitative interpretivist frame. There were several opportunities to open up the progress of the study to scrutiny by the mathematics education community. Apart from regular scrutiny of the unfolding coding and its analysis by my supervisor, I presented my developing work at all opportunities made available at the university. Feedback from supervisors, respondents and fellow students has been valuable in injecting alternate perspectives, developing existing perspectives, and generally strengthening the analysis to move the work forward. The work was also given the opportunity to be presented at conferences over the past three years.

The dilemma of the importance to assign to language and learning in multilingual contexts, in relation to learner discourses, was raised and resolved at one such encounter as a result of the presentation of my work at the Saarmste Research School in 2012. An assigned mentor suggested that the vernacular utterances by learners be translated for the equivalent of 30 minutes of a given interview, so as to ascertain the mathematics contained in the primary or vernacular utterances (see Appendix 3 for the translated transcript). It became apparent, that many expressions did not pertain to mathematics, but were expressions of difficulty, or requests for help or clarity from a partner. This was a revelation, and recentred thought on the purpose of the study, namely, to explore the nature of learner utterances in function. The significant mathematical utterances were translated in cooperation between learners, when they were asked to explain their decisions. It would have been an advantage to be fluent in the various languages of the learners, but this was not possible in the context of the large sample of learners being worked with, or the diversity of languages spoken in these classrooms as typical South African classrooms.

Validity while working within the theory of commognition comes from the unambiguous definition of terms and their operationalisation. This implies that all terms used are defined and applied consistently. In this case, where interpersonal communication is regarded as a form of thinking, it is clearly defined by noting that it drops the entailments it derives from informal definition, and from the collage of multiple perspectives. With the definition of terms in commognitive theory requiring rigorous clarity, the need to be reliable in the assigning of codes became the next challenge. The coding of this study was subjected to consistent scrutiny during the process from PhD seminars, conference presentations, and with supervisor consultations regularly through the process. Transcripts, particularly the way they came to be arranged, supported the need for consistency. Inconsistency became easily visible across the performance groups. Each transcript was coded twice, in their entirety, to confirm consistency. No utterance or written response was excluded or cut from the transcripts. The data was saturated with the multiple reviews over time. Deviations in coding were revisited until they could be coded consistently. Notes detailed and justified the changed codes, and this contributed to the elimination of extraneous codes and the combination of codes which were similar.

While the study examined a wide spectrum of schools and learners in attempt to be representative of the South African context, it must be noted that the results are not generalisable. The schools included were already producing NSC results, have been described in economic-resource terms as average to poor, and are schools which exist across the spectrum of performance. While the findings of this study are not generalisable, they are generative (Mouton, 2012) on two levels:

- The examination of the data was thorough, systematic and comprehensive. Every learner utterance was carefully scrutinised. Outliers were included in the analysis and discussion. In fact they were valuable. Much discursive work picks the elements of discourse it wants to focus on. This restricts the parameters of enquiry from simply what becomes available. Every attempt was made in this study to preserve the fidelity of learners' realisations. This study in contrast, used varied means to view data, namely both a 'wide angle' zoom out, and 'finer resolution' zoom in.
- The study initiated a view into performance in Mathematics as this can be related to levels of discursive development.

The findings do, however, provide a useful window onto teachers about how learners, at different level of performance, form (or not) the object, function. Particularly interesting, was the way the object is formed from what appears as a lack in the development of a formal mathematical discourse. Learners who are unable to respond to examination questions show uncanny intuitive means in thinking of an abstract object, by connecting disparate, seemingly unrelated discourses. This study is a view into learner thinking in the six participant schools. I now turn to the heart of the thesis, where I will present a view of the data and the analysis which permitted an exploration into learner thinking.

Chapter 5: The algebraic expression

5.1 Introduction

The next three chapters develop a story of how learners communicate across multiple signifiers of function, namely: the algebraic expression, graph and table. This work explores and then describes the nature of learners' mathematical discursive routines in terms of ritual and exploration and in so doing characterises learner thinking. The task thus becomes to explore what this abstract mathematical object – function – is to learners, and the discursive means by which it comes to be formed in the way it does. Each of the following three chapters begins with a broad description (the zoom out) of the routines of the particular signifier, in each performance group. The close reading (the zoom in) characterising embedded detail within the routines in each representation follows, again inclusive of the communicative peculiarities of each performance group. Putting these dual, though inseparable views together, the zoom out and zoom in, aims to encapsulate the multiple dimensions and complexity in developing the discourse of the object. Commognitive theory assumes that function exists between *signifiers* and that it creates a discourse that "subsumes the discourses on algebraic expressions, graphs and tables" (Sfard, 2008, p. 211). The 'signifiers' commognition talks of is the 'representations' spoken of in most acquisitionist frameworks and in school Mathematics. Discursively 'signifier' offers a broader, more encompassing description. Whereas representation pertains to just the graph or table or algebraic expression, signifier encompasses the representation as well as all that is possible to realise from the representation. So, a signifier, together with all possible realisations from it, marks the growth of a discourse of an object.

5.1.1 The research questions

The parameters of mathematical discourse are vast, and its multiple layers complex. It was thus necessary to establish the boundaries that permit a zoom in on the data. Zooming in on routines enabled a closer and encompassing characterisation of learner utterances on function. Commognitive theory provided a means to zoom out on routines by making the distinction between ritual and exploration. To do this, the following questions guided the development and progress of this chapter:

1. What are the characteristic features of learner discursive routines at each of the performance levels?
2. How do these features contribute to the mathematical object function that comes to be formed for learners at each of the performance levels? What future realisations do they make possible?
3. How can the object that exists for learners be described at each performance level?

5.1.2 Detail of the Paired Interview

The aspects of the task analysed in this chapter involved learner communication of the algebraic symbolic form of various mathematical relations. The analytical focus is on learner routines pertaining to the algebraic expression. Figure 14 immediately below lists the algebraic expressions provided on the task, in no particular order. All learner communication on these, verbal, written, gestured or drawn, was coded. Here the intention was to ascertain whether learners:

- could see a function in the particular arrangement of algebraic symbols;
- could describe the features of the function or relation that become visible in the algebraic symbols; and
- had discursive means to discern the subsuming object.

To contextualise the learner responses being discussed in this chapter, relevant extracts from the paired interview schedule follow (the paired interview schedule is included in Appendix 1).

Figure 14 Algebraic expressions on the paired interview schedule.

$Card\ P: f(x) = 7$	$Card\ R:\ y = 7$	$Card\ T: x = 3$
$Card\ A: y = x + 1$	$Card\ G:\ x = 3y - 7$	$Card\ M:\ y = x^2$
$Card\ D:\ \ f(x) = 2x + 3$	$Card\ I: xy = 9$	
Card C: $x^2 + y^2 = 1$	*Card J*: $g(x) = 5x$	*Card L1*: $x = y^2$

There were notable features assigned to the cards in this aspect of the task, namely:

- single variable or univariate relations e.g. Card P, Card R

- the use of algebraic notation e.g. Card P
- expressions in the standard forms expected in school mathematics e.g. Card A
- non-standard forms of algebraic expressions e.g. Card I
- non-functions e.g. Card C
- inverses of functions studied in school mathematics e.g. Card L1

The next part of the analysis examined the way in which learners discussed interpretive type questions on the paired interview, involving algebraic symbolism. Figure 15 below, provides two examples of these types of questions, dealing with the equality of two functions, or when one function was greater than another. Learner responses to additional questions are presented on Table 2 in section 5.2, when learner routines are presented. The analysis provided in this chapter is inclusive of all instances on the paired interview, when learners responded using algebraic symbols or notation.

Figure 12 Two of the interpretive questions on the paired interview schedule.

4. Mr Muvhango told me about a function; let's call it g, where all the values of g are greater than 1. Please can you show me an example of such a function?

5. Here are two functions:

$$f(x) = 3x + 5 \qquad \text{and} \qquad g(x) = 17x + 2$$

   - Is there a number for x, for which both functions will have the same value?
   - How will you find this number x?
   - Can $f(x)$ be greater than $g(x)$?

The notable features worked into the questions asked above are:

- they are necessarily non-standard types of questions compared to questions asked on standard assessments;
- as a result, learners have to discuss their responses to arrive at a consensus;

- learners could respond to the question using any representation they chose. If they responded graphically to Question 4 for instance, they could be asked to provide an algebraic representation as well.

5.1.3 The chapter focus

This chapter deals with learner routines as these apply to the symbolic representation specifically. All utterances and written work, across the three performance groups were examined and coded. Despite the numerous constraints to participating in the discourse of this abstract mathematical object, learners show attempts at complexity and abstraction in discourse. Their discourse is characterised by the absence of the specialised and formal mathematical structure. Yet they do attempt extending their routine boundaries to exploration. Describing these instances was of particular interest, and can be seen later in this chapter. While the chapter appears to be set up as contrasting exploration with ritual such that they may be presumed to be directly opposed, they are in fact distinct yet inter-dependent, with exploration being a result (though not automatically) of the reified ritual (Sfard, 2008). It is also important to note that not every ritual is reified into an object, nor need it be. Evidence of learner talk, which while in the main ritualised, will show traces of or potential for exploration. Exploration marks the learner's move towards a more independent, agentive participation.

The nuances in learner discourse around the symbolic representation permit description on three levels derived from the questions asked above:

i. the nature of learner communication, using specific indicators to characterise ritual and exploration;
ii. the way in which various signifiers of function come to be realised or interpreted in each performance group; and
iii. the consequent impact of ritual and exploration on developing future realisations in each performance group.

5.2 The 'zoom out' on the discourse of algebraic expressions

5.2.1 Frequent Routines involving the Algebraic Expression

This subsection summarises the broad discursive routines of working with an algebraic expression found across performance groups. The overarching strategy involving the algebraic expression, across the six schools and all performance groups, was for learners to sketch the graph from the algebraic expression. The discussion that typically followed, involved the features which became visible on the resultant sketch. This is an indication that learners have not reflected on or individualised the significance and meaning of the symbols in the algebraic expression, and will be examined for its detail across performance groups later in the chapter.

The second most frequent routine, across all performance groups is linked to the way in which learners reason and individualise univariate functions. Cards P, R and T were particularly interesting, and they are referred to several times in the chapter. These special cases of the linear relation, where either x or y is zero in the general representation of the function $y = mx + c$, generate a horizontal and vertical line graph respectively, and posed certain challenge to learners as was frequently found in literature. Learner discourses around them were surprising and interesting at the same time. These relations disproved any initial preconceived assumptions that these were basic relations, and that Grade 11 learners would easily provide most, if not all realisations of these expressions as required by the Grade 11 curriculum. This quandary of research, where expectations do not become visible in the process or data, highlights an important way of working from a qualitative content analysis approach (Cho & Lee, 2014). Instances occur frequently in data which were unexpected and surprising. Literature had to be revisited to possibly account for, or reason through these. While learner responses to univariate functions are handled in greater detail as the chapter develops, the summary below shows the discursive routines associated with these objects:

- Utterances pertaining to the 'visible' variable (the variable shown in the expression) as the only variable which exists:

  M PG1 M316 Card T: $x = 3$: *There's no y-values*
- Utterances pertaining to the 'invisible' variable (the variable not shown in the expression) being zero:

  S PG2 P171 Card T : $x = 3$: *y equals zero*
- Utterances pertaining to the graphical representation:

  M PG3 S67 Card T: x = 3 : *You cannot plot this*

  J PG1 D458 Card R: $y = 7$: *It's like a spot on the thing* (refers to the y-axis)
- Utterances that relate to previous work:

  E PG2 Ty68 Card T: x = 3 : *An equation representing a line of symmetry*

All bullets above pertain to the algebraic expression that prompted learners into talk about these ‘special functions’. Interestingly, these specific algebraic representations are not realised as an object until they are transformed into a visual object, the graph, first. This can be deduced from the need to find a second variable (see M316 above); giving the *y*-variable a value, since it is not visible it must be zero P171 above; and S67 showing the need to plot the graph. Venkat & Adler (2012) has explained this as the lack of coordinated attention being given to both the representational objects and transformation techniques in school Mathematics (discussed as part of the antagonistic factors to learning in the introduction). In these utterances, learners are expecting to transform the expression into a recognisable form with two variables, to initiate the processes of substitution of values and the plotting of coordinates, which result in a graph.

What can be seen in terms of transformational activity, given the algebraic or symbolic representation, is that most learners default to a discourse on the algebraic representation of a function, insofar as they are able to sketch a graph through the process of constructing a table of values. The critical features embedded in the algebraic symbolism were discussed colloquially mostly, for graphs that are not linear. The parabola for example is described as ‘happy’ or ‘smiling’. This shows that the algebraic expression (with the exception of the linear function) is not fully objectified, but signifies spontaneous, everyday descriptions. Typically, utterances like those shown above were indications that learners may not yet see the algebraic expression as signifying an object. Hence, the equivalence between the algebraic expression and the graph can be lost. These were two critical disconnections obscuring the object. Learners used the plotting-process to access an alternate representation, the graph, on which the features are visible and thus easily identified and spoken of.

Literature notes that mathematicians use labels (commognitively, these are the keywords described earlier), for the objects they want to speak about. Few learners will individualise what the letters stand for, or why they are used (Mason, 2005). This pointed to a significant disconnect in learning found in the data. Learners are without the vocabulary of key mathematical words to identify or talk of features of the function expression. The linear function, in standard form, was the only exception. Keywords used by learners in connection with the linear function, the *y*-intercept and gradient, were identified directly from the algebraic expression, not requiring a graph to visually mediate the meaning of these symbols.

In summary, the broad discursive routines evident in the data, across performance groups, for the algebraic expression representing a function are:

- the algebraic expression is used as a tool to sketch a graph. Learners substitute values for x into the algebraic expression to find y values, and compile a table of these values;
- being represented as a graph remains the main criteria used by learners to identify a function;
- learners apply the vertical line test to the graph sketched, to discern a functional relationship;
- the algebraic symbolic representation is not an independent, self-standing signifier of a function in its own right for these learners, cueing a process;
- the symbols of the algebraic expression are generally not vested with meaning; and
- the absence of key mathematical words for the parameters of the algebraic expression is a hindrance to identifying the features of a function. While these are recognisable and spoken of when they are mediated visually on the graph, they are seldom spontaneously realised from the algebraic expression. This hinders the connection between the significance of the symbol in the expression and the feature as it appears concretely on the graph.

Table 2 which follows, shows the distinction between the broad discursive routines learners used in working with the algebraic representation across performance groups. It enables comparison and contrast at different levels of performance for the different symbolic expressions on the task.

Table 2 Frequent routines showing distinction across different performance groups.

<table>
<tr><td colspan="4">Discursive Routines of the Algebraic Expression across Performance Groups</td></tr>
<tr><td>Expressions</td><td>PG1</td><td>PG2</td><td>PG3</td></tr>
<tr><td>Univariate Expressions
$f(x) = 7$
$y = 7$
$x = 3$</td><td>• Identified as a linear function or a straight line.
• The variable not visible in the expression can have an infinite number of values.
• Features of the relation were discussed e.g. the gradient of the functions f(x)= and y= is zero.
• The expression can represent an equation of symmetry.
• The variable not visible in the expression has the value zero.</td><td>• The expression shows a point on the x or y axis.
• The variable not visible in the expression has the value zero, it cannot be any other value.</td><td>• Focus placed on the notation.
• Saw the equivalence of y = and f(x) =
• Decisions made as to whether the expression could be represented by a graph or not.</td></tr>
<tr><td>Linear Expressions
$y = x + 1$
$f(x) = 2x + 3$
$g(x) = 5x$
$x = 3y - 7$</td><td>• Identified the linear form from the expression whether in standard form or not.
• Objectified the features of the expression.
• Sketched the graph using the parameters of the expression.</td><td>• Talked of the features of the expression, the gradient and y-intercept.
• Used the expression to draw up a table of values.
• Transformed expressions to standard form.</td><td>• Given x values, y values can be found.
• In the non-standard form of the equation $x = 3y - 7$, 3 was identified as the gradient and the x is said to be on the wrong side.</td></tr>
<tr><td>Non-Functions
$x^2 + y^2 = 1$
$x = y^2$</td><td>• Guessed names for the expression: circle, hyperbola.
• Identified a circle.</td><td>• Transformed $x = y^2$ to $y = x^2$
• Identifies expression as a parabola.</td><td>• Had never seen such expressions and was not taught about them.</td></tr>
</table>

	• Attempted to transform the expressions to standard form $y =$ • Did not represent a function because the graph could be sketched.		
Hyperbola $xy = 9$	• Objectified identification of the features. • Talk of the general expression for hyperbolas and the significance of all parameters. • Talks of algebraic transformations of the expressions and their results.	• Transformed to $y =$ to identify a hyperbola, • Talked of the graph and not the expression. • Assigned values to x and y.	• Needed to plot the graph but did not know how.
Parabola $y = x^2$	• Objectified identification of the parabolic expression. • Identified in instances as an exponential expression from the 2.	• Used spontaneous descriptions for the features of the graph, 'smiley' etc. • Identified as parabola from the sketch.	• Used the expression to draw up a table of values to plot a graph.
Testing for a function	• Applied the vertical line test on the graph. • Identified expressions as functions from those labelled as such in the classroom.	• Applied the vertical line test to the graph.	• Held a vertical or horizontal line to a graph and looked for a single point of intersection as criteria for function.
Give an algebraic expression when $f(x) = x^2$ is moved a	• Provided algebraic expression.	• Provided algebraic expression.	• Conducted a pointwise plot of the expression.

unit upwards			
Interpret $f(x\text{-}2)$	• The graph moved two units to the left.	• Conducted a pointwise move of the points on the parent function.	• Multiplies $f(x)$ by $(x\text{-}2)$ gets $y = x^2 + 2$
Given $f(x) = 3x + 5$ and $g(x) = 17x + 2$ Can these expressions be equal?	• Offered a full algebraic and graphical solution. • Talk was objectified.	• $g(x)$ is always greater • Used empirical substantiation. • Talked of looking for a common point on a table or the graph.	• Looked for 'one numerical factor.'
Can $f(x) = 3x + 5$ be greater than $g(x) = 17x + 2$?	• Expressed solution as a region or interval of values on a number line.	• Substituted values into the expression. • Finds $x = 0$	• Substituted $x = 0$ in the expressions.
Can x^2 be greater than x?	• Tried empirical values – a selection of integers.	• Discussed the squaring as doubling.	• Tried empirical values – a selection of integers.

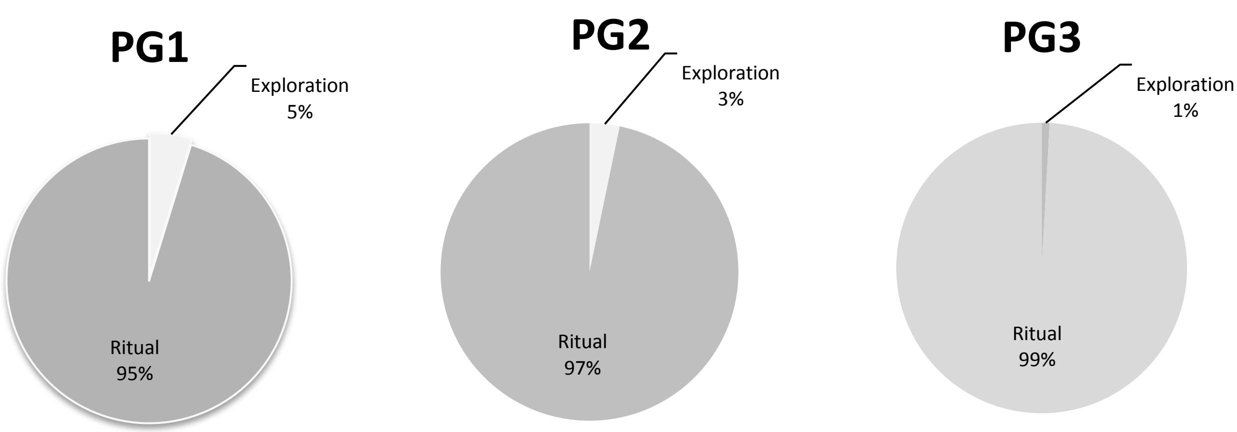
The distinction between learner routines showed that better-performing learners were more objectified in their communication, where they appear to be operating on objects rather than symbols. PG1 learners were able to reflect on functions and their processes and generalise these in certain instances. If we consider learning as building successive layers of discourse, increasing in complexity and abstraction, then we can surmise that better-performing learners are closer to individualising the discourse required by school Mathematics. They show refinement of process, knowledge of keywords to describe signifiers, reflection on processes for their results and degree of appropriateness, the encapsulation of common links between signifiers and objects. These contrast with the communication of poorly performing learners, who appear for the most part imprisoned in a singular process, and attempts to recall what they had learned.

Mathematics exists in picking up situational signals to recall and then execute a process, called 'the *how* of a mathematical routine', as described in Chapters 1 and 3. Reproduction of and reflection on processes are necessary for reification. New processes developing complexity are then added to those previously reified. Better-performing learners appear to enter these cycles of reification, and thus can realise far more than poorer-performing learners, who enter the inital cycle, remaining in that single loop. In addition, there appears little indication that poorly performing learners reflect on their mathematical moves, or justify what they had done and this could impact the opportunity for future realisations. Distilling learner routines appears to be a useful initial way in which to differentiate discourse at different levels of performance. The utterances in this section were examined as they formed part of a routine to provide a bigger picture of the patterned nature of learner communication.

5.2.2 Developing a picture of ritual and exploration across performance groups

In the previous section, whole routines were distilled as a first layer of description. This section now examines the discourse of these routines so as to classify them as part of a ritualised or exploratory practice. This is the second layer of description. The pie charts below show the accumulated frequencies of ritual and exploration codes embedded in learner utterances involving the algebraic representation, in the three performance groups:

Figure 13 Exploration and ritual per performance group.



The pie charts above were not surprising. It was expected that instances of ritualised discourse would far outweigh that of exploratory discourse in the data. As stated, new routines begin as rituals and gradually turn into explorations through a process of individualisation.

Individualisation is a transitory phase in ritualisation, where a learner participates in routines as part of a collective, most likely with a teacher and other learners in a classroom. Ritual is the form that routines take in Vygotskys' zone of proximal development (ZPD), where this process of individualisation, mentioned above, is termed rationalisation. This asserts the claim made by commognitive frameworks that rituals are a necessary "developmental precursor to explorations" (Sfard, 2008, p. 223). Discursive theory from Vygotsky and Sfard are additionally supported by this study's initial aggregation of codes. At the outset, this study acknowledges the importance of the contexts of learning in the six schools involved. Factors exist in each of the schools and in the cohort which contribute to this overwhelmingly ritualised aggregate. This was discussed in Chapters 1 and 4. They speak to learning contexts that are typical of most South African schools: the discourses of teaching and learning in school Mathematics, the nature of the school Mathematical discourse, which has as emphasis the *'how'* of a routine at the expense of the *'when'*, and the resources that are available to learners are some typical contributors.

Routines begin initially as a loose collection of realisations, which are individualised into an integrated discourse. Exploratory talk will often show learners are able to group realisations and routines which are related to each other. What was interesting in this study was that learners from all performance groups showed instances of exploratory discourse, despite the antagonistic factors to learning and the dominance of the ritualised practice of school Mathematics. How could these exploratory utterances be characterised across performance groups? A focus on PG3 communication helped this characterisation, by providing the least complicated view into learner thinking. The incidence of exploration occurred less frequently in this group, and these incidences could be reflected on to distil the most basic of exploration activity. PG3 showed exploratory talk in identification of objects and symbols, in objectified ways, as opposed to the relations of routines pertaining to objects. This was indication that certain rituals related to naming and a limited number of procedures were being reified. While this exploratory portion was small, and starkly depicted on the pie charts above, its occurrence was interesting in provoking investigation into the nature of this participation as it pertains to our poorly performing learners. Certainly, it can mark a significant point on which to build successive layers of discourse for these learners. Where, and around which aspects of the mathematical discourse do these instances arise? The significance of this for the broader study permits a view of the nature of the exploratory discourse in each performance group, and how it thus compares across performance groups. A broad description of the ritualised discourse is discussed first as it provides a window into the exploratory discourse.

Table 3 Ritual codes in each performance group for the algebraic representation.

Ritual:	PG1	PG2	PG3	Total of codes classified ritual
Total number ritual codes for the algebraic representation	701	1096	1122	2919
Total percentage	23.43%	36.63%	37.50%	97.56%

PG1 show the lowest number and proportion of ritualised codes on the algebraic representation compared to the other two groups (as shown on the table and pie charts above). This is because they were able to say more with fewer words through compression of their talk with symbolisation and reification. With a wider repertoire of routines, they could thus pick the most economical discursive means for the task required without being ambiguous. Generally, PG1 talk showed they were able to realise far more from the signifier than the other groups. Their utterances showed greater frequency of objectified talk. It ought to be recalled here that an object is a discursive construct which permits the connecting of processes, properties, routines, 'things' which are all linked to each other by a specific criteria. The word 'function', as a noun, or one such 'thing', subsumes all communication on algebraic expressions, graphs and tables representing a one-to-one or many-to-one relationship between the variables involved. Using key mathematical words in such objectified ways enabled PG1 to say more with less.

The objectifying potential of the discourse of PG1 is further supported by fewer subjectifying utterances noted in this group. Subjectifying utterances places the learner as central to the mathematics, as opposed to having the mathematics stand alone as talk of objects. School M's, PG1 and PG2 illustrate this, when responding to the question: "do you see a function in $f(x) = x^2$?" Both PG1 and PG2 could identify the symbolic expression as a function, named the expression, as representing a parabolic graph or being a quadratic, in five or fewer utterances, between the pair of learners involved. PG3, in contrast, took a 23-utterance exchange.[12] This suggested early on in the analysis that better-performing learners were more adept at using the

[12] Appendix 6 - Transcript for school M, showing utterances across performance groups, particularly PG3 M104-M127.

compressing apparatus offered by formal mathematics. The summary of the exchange of 23 utterances can best be articulated in the M-PG3 learners' written response: '*using a table method, a graph can be formed from x and y*' (Appendix 6). Transformation of the algebraic expression to table of values to the graphical representation was identified as a dominant ritual of PG3, and showed an entirely procedural orientation to the algebraic expression. Learners generally took any algebraic expression provided to them and made *y* the subject of the formula, resorting to the completion of a table of values, which was then graphed. Once graphed, and solely contingent on the possibility of graphing, the equation was classified as either a function, or not. Confirmed by Learner S in performance Group 3 of school M, utterance S114 (denoted in short by M-PG3-S114): '*…it has to be a function because a function has to be a graph'*. Subsection 5.3 of this chapter examines learners routines for discernment of a function.

In summary, the most frequent ritual across particularly the poorly performing groups related to being able to sketch the graph from the algebraic expression. They showed a strong processual orientation to the object. The graph was the critical condition defining a function relationship for these learners. The next section presents a broad description of what exploration codes show. The nature of the exploration routines of PG3 is again broadly discussed as it mimics the foundational exploratory routines of the better-performing groups. Table 4, which follows, shows the aggregation of exploration codes characterising the utterances across performance groups. It provides detail and supports the pie charts presented earlier.

Table 4 Exploration codes in each performance group for the algebraic expression.

Exploration	PG1	PG2	PG3	Total Exploration Codes
Total number codes indicating exploration for the algebraic representation	35	28	10	73
Total percentage	1.17%	0.94%	0.33%	2.44%

PG3 held a portion of the exploratory codes. Earlier in this chapter, the question arose about 'where and around what aspects of mathematical content these instances arise'. The coded utterances show exploration routines in PG3 revolving around talk:

- that connected different representations-equivalence of the algebraic expression and the graph (4 codes) ;
- of specific features of functions (4 codes);
- giving meaning to what symbolic representations signify (2 codes).

To illustrate this, two instances from the data have been chosen below:

1. T-PG3- B274: (points to equation on Card D: $y = -2x + 3$): *'...the equation of a straight line graph, its y equals mx plus c. And then here, ja here* [sic], *this is c, this is m, which is the gradient.*'

   This utterance B274 involved Learner B seeing equivalence in the graph and the equation (E4). The specific arrangement of algebraic symbols on Card D signified the graph of a straight line to the learner. The utterance showed further that the algebraic symbols *m* and *c* signified independent entities, where *m* was decribed as the 'gradient'. This was coded exploration code (E5) as 'gradient' was referred to as a noun, an entity. The symbol held meaning for the learner. Identification of objects is regarded as low-level objectified talk within the commognitive framework (Sfard, 2008).
2. T-PG3- B313: (refers to Card P: $(x) = 7$) :' ... *the gradient, its zero because... the graph is now in line with, with, with, with x'*[sic] . Then B315*: 'it's, it's, it's parallel to, to x... ja* [sic]. *It's parallel to x.* [The] *x-axis.'*

   Here we see learner B, realising relevant and key features of the graph from the equation. The algebraic expression on Card P signifies a line of zero gradient, parallel to the *x*-axis. This was coded E3.

PG3 showed restricted exploration presence in codes in the categories of mathematical objects, and in flexibility in working with the linear function specifically. It is worth noting that the codes present in PG3 were common to the other groups. PG3 codes relating to their goals in mathematics, the applicability of routines, who the mathematics is addressed to, and reasons for accepting narratives given or derived- remain ritualised for this group (discussed in 5.3.2).
The zoom out sets the background of the study with broad discussion of the rituals and explorations which exist in the dynamic interaction between pairs of learners when they

communicate about the object. Finer details of routines emerge as they are zoomed in on, and as the chapter progresses. The broad interpretation so far is that discourse around the algebraic representation is, in the main, ritualised across groups. Learners in all performance groups tend in specific and similar instances towards a more objectified and exploratory discourse. These instances are located most frequently when learners worked with the linear function. The objectified utterances pertained broadly to three areas:

- identification of the linear function from the arrangement of symbols in the expression;
- identification of the features of the function; and
- showing equivalence between the algebraic expression and the graphical representation.

Despite these limited occurrences, they still enable a starting point for a more nuanced description of learner participation in the discourse on function. Importantly, they identify, particularly for poorer-performing learners, a starting point on which teaching can capitalise. The presence of exploratory codes based on how learners at different performance groups were communicating, opened the analytic lens for classification of the exploratory codes. The codes for ritual and exploration as they occurred in learner utterances at different levels of performance can now be examined.

5.3 Frequent routine codes related to performance across schools: 'zoom out'

The purpose of this section was to provide a picture of the patterns of ritual and exploration as they occur across the three performance levels and schools. Trends in codes for those which occur most frequently across schools, frequent codes within a school and those codes which do not occur at all, help tell the story of learning about function across different contexts. In Table 5 which follows, the most frequent code occurs to the far left of the column, followed by the next frequent code in each of the schools and performance groups. Methodologically and analytically, this exposition of codes provides a focus for the nature of the ritualised and exploratory communication which follows. The collation of the most frequent routine codes allows discussion of the nature of the rituals and explorations, which occur most frequently in the schools of this study. The discussion which follows is still part of the zoom out, and offers a summary of contexts in which codes were found.

Table 5 Frequent exploration and ritual codes, across schools, for each performance group on the algebraic expression.

	Frequent Codes of Rituals			Frequent Codes of Explorations		
Schools	Most Frequent	Next frequent (within a margin of 5 codes)	Next frequent (within a margin of 5 codes)	Most Frequent	Next frequent (within a margin of 2 codes)	
PG1						
E	R3			E2		
P	R2	R11		E2	E4	
T	R3			E3	E4	
S	R2			E3		
M	R2			E4	E7	
J	R2			E8	E7	
PG2						
E	R2	R11				
P	R2	R7		E1		
T	R2	R3	R11	E3		
S	R2	R3		E3		
M	R2	R3		E2		
J	R2	R3				
PG3						
E	R2	R3				
P	R3	R2				
T	R11	R3		E3		
S	R3	R2	R11	E4		
M	R2	R3				
J	R11	R1	R3			

Table 5 above shows the dominance of ritual code R2 as most frequent across all performance groups and schools. R2 confirmed a strongly ritualised practice in adhering to goals set by others. This is described in commognitive theory as learners participating in the mathematical discourse to build and sustain relationships with one another. Being able to work mathematically appears as a result not to be assigned priority in learning. Such ritualised talk (detailed and substantiated in 5.4.1) was most frequent in the following instances:

- A learner deferring all decisions and actions to the other in the pair. Generally, the learner who appeared to lead discussions. Such talk occurred in instances where one learner followed another's mathematical moves. The participation appeared passive where one learner made no independent mathematical decisions; listened or restated what she had heard; made no contribution to the discussion.
- Mathematical decisions were seldom questioned, and seldom required justification.
- A learner would abandon her partially complete routine for no apparent mathematical reason but to follow the other.

These were strong indicators that learners participate in mathematics as one of the activities in which they can acquire social acceptance, where there is a need to build solidarity as opposed to dissent. Mathematical reasoning was thus based on the decisions, even opinions, of others and not on already endorsed mathematical narratives. This can contribute to the agency a learner develops in school mathematics, where mathematical decisions are based on the views of others. It showed that most learners are not yet independent participants in the discourse. The interaction in pairs showed overwhelming reliance on another (the other learner or researcher) to initiate and progress with the mathematical routine. This impacted the types of questions learners would ask, which were in the main clarity seeking. Again, this was indicative of an emphasis on social relationships, where it was more important to understand what the other learner was doing than to be understood or to make further realisations. The ritualised code R2 appeared most frequently across all schools and performance groups for the algebraic expression. From the contexts sketched above of the kinds of instances where this code arose, it became clear that R2 impacted the ways and levels at which learners engaged the mathematical discourse.

The prevalence of R2 across schools provoked further inspection for similar patterns in codes. The codes, which fell within an interval of 5 from each other on aggregate, were considered as the next frequent code. The next frequent codes, R3 and R11, again repeated across all schools and performance groups. It is pertinent to recall R3 indicated talk, which focused on an emphasis on actions and process; and R11 showed subjectification, indicating talk focused on the performer of the mathematics, rather than on the mathematics itself. Both these codes appeared inextricably linked in the data, as learners spoke their mathematics in terms of what they did to algebraic symbols: "*I moved the x to the other side*" is typical of such responses. It shows learners emphasising the actions they performed on algebraic symbols, almost to the exclusion of mathematical reasoning. This became particularly evident in the absence of response or poor response to questions, which required reasoning over process. It further suggested that the school mathematical discourse appeared to emphasise the accumulation of various processes and as a result developed parallel to and thus disconnected from the formal mathematical discourse.

A formal mathematical discourse would emphasise mathematical objects, connecting all signifiers and their realisations to these objects. Why do learners appear alienated from formal mathematical routines in favour of social ones? How do these social-acceptance-seeking routines impact their participation? Does this overt social discourse permit objectification and the exploration of mathematical objects? These are important questions which arose at this stage of the analysis. It does appear that rituals steeped in social acceptance are linked to the independence with which learners participate, and thus likewise to the depth to which they are able to reach for realisations on the algebraic expression.

The formal mathematical discourse with its specialisation, structure, rigour and syntax can present to learners not as a human endeavour, but as an external body of knowledge, which is disconnected from their spontaneous communication. Building these connections has already been discussed in Chapters 2 and 3 as not being a priority of school Mathematics, with its emphasis on the '*how*' of mathematical work. This combined with the need for social acceptance, appears to contribute to the overwhelming ritualised discourse seen in this study.

Agency, or the lack of it, can be seen in the grouping of the most frequent codes, R2 (adhering to goals of others), R3 (the emphasis on actions and process), and R11 (subjectification). The goal of learning is for the learner to gradually be released from the scaffolding of the knowledgeable other to increase her agentive participation. The pattern of frequent ritual codes shows that learners are not developing this independence. This must directly impact what and how learners individualise the discourse on function. While a ritualised discourse was expected, it was a revelation to see the stark confirmation through the frequent codes and particularly through the ways the codes appear linked to social aspects of learning across all levels of performance.

From the range of exploration codes shown on table 5, only the most frequent will be discussed in this section. The remaining are discussed in their categories (section 5.4) as the chapter develops. The instances of these codes were scarce, and it was difficult to establish trends or groupings, as was done for the ritual codes. The most frequent exploration code found was E3, which relates to identification of the features of the mathematical object in objectified ways. Across the three performance groups, the linear function and talk of its features appeared objectified, and contributed significantly to the aggregate of exploratory codes. This is possibly the result of the linear function being the first function that is introduced to learners. This function has the longest amount of engagement, in terms of time, across the curriculum for secondary schooling. A straight line is an abstract mathematical object and learners (across all levels of performance) have reified the identifying expression and features of the algebraic representation as self-standing objects, spoken of with the use of nouns. It is worth noting that only one school, school T, situated in a township, showed objectification of the features of the algebraic expression across all performance levels. What makes this possible? And, it raises other important questions: does individualisation of the mathematical discourse, and hence objectification, require longer time for engagement in the school curriculum? Or, can objectification occur through deliberate, sustained effort in the teaching process, which expresses and displays these objectified ways of communicating about the object? Both these questions, noted as important, are beyond the scope of this study, but are returned to in the concluding chapter.

Table 5 allowed a view into the most frequent routine codes that occurred across schools in very different settings. No distinction between performance levels was made. Learners' ritualised discourse showed emphasis related to establishing and keeping social bonds. Their restricted exploratory discourse related in the main to the linear function, its identification and the identification of its features from the algebraic expression. Examining the frequent codes for ritual and exploration across all groups has enabled a view of the way the discourse on function has developed for these learners and a description of their routines as a whole. At this preliminary stage of analysis and discussion, it appears that learners participate in a discourse of function characterised by what they can do with the algebraic expression. The objectified identification utterances appear to contribute to making the abstract object and its features concrete for learners when the expression is graphed. However, from both the frequent ritual and exploration codes in combination, it is not clear how learners could develop further realisations of the object.

5.3.1 Findings with explorations

Notwithstanding its limited presence within learner discourse, exploration is still important. From the tellingly disproportionate pie charts at the introduction of the chapter, a small percentage of learner communication can be attributed to exploration. The study proposes that an exploratory discourse manifests in improved performance in mathematics.

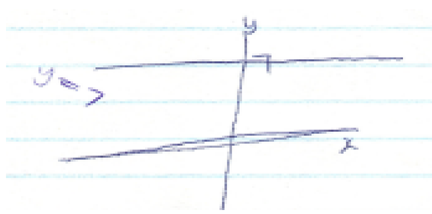
i. Better-performing learners engage exploration routines most frequently. In commognitive theory, routines characterised as exploratory, are divided into three types, namely: construction, substantiation and recall (Sfard, 2008). The exploration codes used in this study occurred under substantiation and recall only. The construction of a mathematical argument, from the problematising of function through its behaviour or features, remains outside the scope of school mathematics. Hence, it was not expected that the exploration routines of school mathematics would contribute to the construction of new knowledge or narratives in mathematics. The outcomes-based nature of the South African curriculum, in addition, shows little support for it. There is a focus in school mathematics, borne out in my data, and supported by existing research, that learners' focus constitutes the *how* of the mathematical routine. This means that the endeavour becomes more about successful completion of recognisable procedures than about the extension of the procedures to generate or investigate evolving complexity of mathematical objects. With this in mind, instances where narratives or routines were substantiated showing connection beyond the process to the object were taken as indication of objectified talk.

The semi-structured nature of the paired interview allowed for learners to be prompted (in most cases) for the rationale after the closing narrative. Learners were prompted for substantiation in instances where they undertook a numerical calculation, solved an equation, or engaged a definition or proof. Substantiation is a process by means of which one is convinced that a narrative can be endorsed. It was important to ascertain learners' realisations of the various algebraic representations in the interview. A good illustration of levels of substantiation across the three performance groups is shown in the extract 2 which follows. The utterances of PG2 and PG3 are presented to contrast the level of substantiation offered by PG1. In addition, it showed the stark distinction between exploration and ritualised discourse. The interactions of PG2 and PG3 were not assigned exploration codes, in contrast to PG1. In the extract, learners were asked what they saw on Card P: $f(x) = 7$, and were required to ultimately answer whether or not the card represented a mathematical function. The utterances relating to decisions on whether the expression is a function are not shown here. Utterances pertaining to the features of the algebraic representation of the function are shown below.

Extract 2 Contrasting extracts of performance groups to highlight exploration routines.

School/ Performance Group	**Learner Utterances**	**Comment**
T-PG1 **Exploration Codes:** **E3 E4**	M231: *Well $f(x)$ can be written as y. f of x is equals to 7 can be written as y equals to 7, which is a ...(*interrupted by Learner S). S232: *Straight line graph* (Learners go on to sketch the graph, without being prompted and the discussion that follows substantiates why they regard Card P:$f(x) = 7$ as a function).	In discussing Card P:$f(x) = 7$ Learner M makes comparison to Card R:$y = 7$. Sees equivalence in algebraic representations. Identifies the graphical representation from the algebraic. Both learners agree and are able to represent the function graphically without using a table of values. Rationale for exploration codes explained: The exploration codes assigned in the interaction was **E3** (attending to the key features of the mathematical object) where learners were able to sketch a straight line parallel to and above the x axis with 7 indicated as a y-intercept; and **E4** (sees equivalence in different representations), where learners see equivalence in different algebraic representations. In this case, between the algebraic and graphical representations.

T- PG2	V411 *I don't think I will be able to (draw the graph) 'cause...*	Discussing Card R: $y = 7$: Learners attempt to sketch graph. Utterances 412-416, which are omitted here, relate to the practicalities of doing this. For example, 'do you have a pencil?'
	J417 *There's going to be a point on there, I think.*	
	 	The sketch is presented with 7 dotted on the *y*-axis.
	V419: *Ja, so here, you don't have x, you only put er seven in y 'cause, you don't have x an x-axis.*	
	Interviewer: And *x* coordinates?	The interviewer asks about *x* coordinates?
	V426: *Er, they(x -coordinates) are not given.*	
	J433: *We put seven here...our x ja... our x is zero.*	

(No exploration codes)	V434: *We are not taught this function. As I've said, it's not a function So for us to draw a function, we will only draw the x–axis y-axis. Then put 1, 2, 3, 4, 5, up until 7 so that, you see, we will only put 7.*	Talk of the coordinates at the *y*-intercept. Learners talk of how they sketched the graph.
T- PG3	B253: *I don't really get what does f of x mean. When it's like this? Ja...when it's like this? But then, when it's like this…*(pause) *I think that.* B255: *Yes, when it's like on card D, I think f of x stand for y.* B282: *Let me try it. Since, heh, since I think that f of x it stand for y, that means, the graph will be like this. This is y, that's x. The graph will be like this. Ja. I'm done now.*  B288:... *it has only one x value, and uh, y x value.*	Refers to Card D: D $f(x) = -2x + 3$ Learners make associations with known narratives. Learners are prompted, by interviewer, to make a rough sketch of the graph. This was to encourage further talk.

	B290: It has only *y* value, of which is 7, and then an *x* value which is 0.	
	Interviewer: What makes that a function?	
	B298: *Because it's a straight line.*	Points to intercept on *y* -axis.
	Interviewer: So where's the mx term?	Interviewer invites learners to revisit the equation.
	N300: *It doesn't..* B301: *Ah, it doesn't have mx.*	
	Interviewer: Why?	Learners have previously mentioned the general equation of a straight line graph.
	B303: *Because it's, hai, I don't know*.	
	Interviewer: What does m stand for?	
	B305: *m, it's, it's a gradient. I mean gradient*.	Interviewer refers learners to the sketch of the graph above.
	Interviewer: What's the gradient of that line?	
	B311: *Well I think, I think now, gradient, it's, it's, it's zero.* B313:...*Because, now, the, the,*	<u>Rationale for exploration codes assigned</u>: E3 as learners talk of the features of the straight line algebraic expression, connected with the graph they have

	...ah, ah... *Okay,.. I think now, the gradient, it's zero because ..the graph is now in line with, with, with, with x. Ja.* B315: *It's, it's, it's parallel to, to x. Ja. It's parallel to x x- axis.* (not probed further)	sketched, E4. In comparison to the other two performance groups, it must be noted, these deduced utterances were prompted.

In the extracts above, PG1 learners are able to say far more mathematically with fewer words due to the compression that algebraic symbolism enables (Sfard, 2008). They have attached meaning to the symbols. They see the algebraic expression as representing a straight line and are able to substantiate this by drawing a rough sketch of the graph spontaneously. There was no need to prompt them to say or do more with the expression. The extract above shows PG1 interpreting the arrangement of symbols as signifying an object.

PG2, in contrast, did not recognise the $f(x)$ notation and did not respond to the algebraic representation. They also could not connect or see equivalence in the algebraic representations $y = 7$ and $f(x) = 7$. Their utterances pertaining to $y = 7$ are shown following PG1. The emphasis on process, the 'doing' something, namely, the sketching of a graph are indicative of an incomplete formation of the object. Learners realised one instance of the function only: that of a single point with coordinates (0;7), and not the defining algebraic relationship of the expression. They have interpreted the function expression as a single dot on the Cartesian plane. They relied on visible features (Sfard, 2008) in the algebraic expression $y = 7$. It is thus seen as a single point, where *y* is 7 and *x* absent in the equation, is taken as 0. The function expression does not signify an algebraic relationship to these learners, but signifies a single point.

PG3 worked in an interesting way in this extract. They, like PG2, did not realise the linear function from the algebraic expression. They instead looked through all other functions given to them till they found a representation they recognised, like Card D. Recognising this as the equation of a straight line, they recycled old knowledge to realise $f(x)$ as functional notation that replaces y. From this point, rewriting the expression as $y = 7$, they proceeded to a graphical representation. They plotted coordinates, and sketched a straight line. With prompting, they talked of the features of both the graph and equation, using them equivalently, as referents for each other. They discussed the features of the graph in objectified ways, but could not realise these features of the function from the algebraic representation. For this PG3 pair, the objectified talk did not arise in the primary object, the algebraic expression. Instead, the pair talked about the features of their secondary object the graph in objectified ways.

In summary, PG1 showed a more objectified notion of the algebraic representation, the primary object seen in compressed, reified discourse as well as the unprompted substantiation across representations. PG2 shows a discourse that emphasises process. PG3 showed talk dependent on external prompts, and only when the algebraic expression is visually mediated by the graph. PG3 were not able to build and sustain deductive argument as often as the other two groups. This could be picked up from the tensions that existed for the interviewer when relating to PG3, where it was required to prompt responses from learners or be left with no discussion of the mathematical object. Governed by the underlying methodological principle, to find out what learners could say and do, the prompting became necessary.

The extracts chosen showed PG1 had access to a more objectified discourse and the algebraic symbols signified far more to them than they did to the other two groups. They also were able to recall in greater detail previous narratives about the algebraic representation and the critical features held therein. This contributed to a discursive fluency that was less evident in PG2 and PG3. The extract shows PG2's unsuccessful attempt at recall, and PG3's attempt to establish connections to previous narratives as they went along. While recognising the way PG2 and PG3 worked and ultimately the connections they were able to establish, the approach appears tenuous and reliant on prompts and connecting perceptual patterns. Do the poorer-performing learners connect prompts to an object or process? It would appear at this early stage that PG3 learners connect prompts to processes with which they assosciate them. Recognition, recall and association, and not mathematical reasoning, are the drivers. Better-performing learners connect prompts to reified processes, and thus, to an object. Mathematical reasoning drives this.

Exploration routines in school Mathematics come from reproduction and substantiation of routines established in the classroom. The connection between ‘old’ and ‘new’ narratives is a process which implies that a level of individualisation of the discourse on function must occur. PG2 and PG3 showed discourse, which appeared to lack objectification as well as the connection of current narratives to those learned previously. These learners relied on the recall of disparate, disconnected processes which appeared to alienate them from being mathematical. In addition, they struggled with filtering relevance and meaning of the keywords and notation they used and working coherently within their mathematical routines.

ii. Better-performing learners show a wider range of exploration routines.

Figure 14 The range of exploration codes across performance groups.

PG1	E4 E3 E1 E7 E2 E9 E8
PG2	E3 E2 E1
PG3	E4 E3

Key for Exploration Codes	
Code	Code Meaning
E1	Signifiers are abstract mathematical objects
E2	Narratives are logically deduced by learners
E3	Speaks of specific features of mathematical objects that are relevant
E4	Sees equivalence and connects representations
E7	Seeks abstraction and generality. Asks open questions
E8	Questions endorsed/ derived narratives and attempts justification.
E9	Process is reified

Figure 14 shows the range of exploration codes in each performance group. The codes listed occurred within a margin of five, in terms of frequency, to one another. As a result, while E5 may have been a code assigned to some utterances of PG1, it did not fall within the range of five codes from the previous frequent code. It therefore will not feature in this discussion of most frequent codes, which examines what learners are doing *most* often.The range of exploration codes speaks, importantly, to the potential for learners to extend their current discourse. School Mathematics does not require the production of new narratives; it requires that learners reproduce and work with the existing formal narratives of the objects specified in the curriculum.

Developing exploration routines requires building an objectified discourse allowing learners to explore and form mathematical objects with which they can do far more. 'Doing more' is learning, synonymous with having far more realisations about an object, and thereby 'building fuller realisation trees'.

Learner mathematical thinking in school is directed by the curriculum. The curriculum specifies the objects and the extent to which they need to be realised. That is it specifies to teachers what must be taught, learned and assessed. Since the curriculum was not in focus and learner discourses were, detailed curriculum analysis was not undertaken. I have found resonance between the exploration codes of this study and skills, knowledge and values suggested in the curriculum and its related documents, despite these not having discursive focus. The way in which learners were communicating could thus be held up to the parameters of the curriculum so as to gauge learning. To illustrate, deduction mentioned explicitly in the National Curriculum as a skill, finds its equivalent counterpart in code E2; focus on specific features of functions, resonates in code E3; seeing equivalence and connecting representations, to E4 these are very much a part of the discourse of school mathematics and are explicit in curriculum and related documents (DBE, 2011b). So, while this study and the curriculum depart from different perspectives, there exist areas of learning over which they intersect.

Focusing on codes E1, E7 and E8. Frequent codes E7 and E8, situated in PG1 was expected, but E1, being more frequent in PG2 (compared to PG1), required examination. E1 speaks to objectification linked to identification of objects and features of functions. E1 was indeed present in PG1 and to an equivalent extent in PG2, but it was not among the most frequent routines for PG1 and hence is presented third in the table above. Other codes were more frequent than this one. On Card A below, PG1 showed a stronger orientation to doing something than PG2 who identified the function. E1 arose as dominant in PG2 because the group had a smaller range of exploration codes compared to PG1. Sometimes, identification was all that was possible for them. As discussed earlier, the extent of E1, in the data for PG2 on algebraic representations, arose in instances where learners were given the algebraic representation and the talk was of mathematical objects. PG1 andfor PG3, the algebraic representation was more often a signifier of a process, seen in the extract below. They did not automatically look at the algebraic expression to identify the function or its features. Extract 3 below shows contrasting objectified identification utterances from learners in the same school talking about Card A: $y = x + 1$:

Extract 3 Contrasting objectified identification utterances (process and subjectifying phrases have been highlighted for emphasis).

PG1	**PG2**	**PG3**
S –M27 : *So here its a function because if **you substitute** any value of x....*	S- F134: *I think we can safely conclude and say yes it is a function. Why because firstly it's a linear function...*	S- T104: *You can **use it to sketch** a graph. If you're given the x coordinate **use it to plot** a linear graph.*
Talk about process	Talk about an abstract object, the linear function. We see objectified use of a mathematical noun. Code E1	Talk about a process.

Instances of coded exploration and linked to identification of the object and its features appear frequently in the data. This talk illustrates different levels of objectification.

For PG1, apart from the narratives relating to the *how* of a mathematical routine, E2, E3, E4, showed learner talk that sought complexity/generality, and challenged substantiations within a pair, with greater frequency than the other two groups. To illustrate, in the interaction that follows on Card R, learners used various means to decide if an algebraic expression represented a function. The two learners in the PG1 pair decided that Card R: $y = 7$ was a function and justified their decision. J-PG1- D508 posed the following question: '*Does every function have a bx plus c? Does every function have an equation?'* This utterance was coded as an open question E7, with the intention to generalise about the algebraic representation and to compress the means they were using to determine whether or not the representation was a function. The asking of open questions was a strong indicator of exploration or potential for it. Exploration leverages complexity and abstraction as learners are forced to think beyond the familiar, practiced orientation of school mathematics, to *when* a procedure would be appropriate, and to expand opportunities for its use. The *when* facilitates the compression and subsuming of similar narratives of the object, and connections between different signifiers were more efficiently compared to the accumulation of several disparate *how* routines.

PG2 was comparable to PG1 in the frequency of exploratory utterances. In contrast, however, they showed no instances of attempting to ask open questions, to generalise. Like PG3, they showed far more instances of conceding a narrative with no need for further justification. The range of codes across performance groups appears to confirm a relationship between performance and exploration routines. Better-performing learners have a wider range of exploration routines. Exploration can be seen to expand learners' mathematical discourse, not only in the objects they can talk about, but also in how they talk about them. It encourages learners and steers them towards independent enquiry, like the asking of open questions, a goal of both learning and teaching mathematics. This agentive participation spurs growth of discourse on two levels, the development of discourse of the object itself, bound by the constraints of the regulatory meta-discourse of formal mathematical communication.

iii. Talk of the features of the object occurs across all performance groups. All groups could talk about the features of the algebraic expression in objectified ways. There was relative fluency across all performance groups with regards to the linear function. The extracts below show PG2 introduce an additional node to their realisation tree in the form of the graphical transformations on the parent function as a result of the algebraic transformation on the parent equation (see Card A below). The utterances below show learners from three performance groups examining the following cards:

Card G: $x = 3y - 7$ Card A: $y = x + 1$ Card M: $y = x^2$ Card D: $f(x) = -2x + 3$

Extract 4 Talk on Card G: $x = 3y - 7$

T-PG1-M295	*A straight line graph you have the x, the y intercept and then you have the, uh, the gradient, but this one you like, it's not like its put in order...*	Talk about the features of the straight line graph that is not in standard form.
T-PG1-M299	*You can rearrange it to be... it's... x is equal to 3 y minus 7 , it can be rearranged to be... 3 y is equals to x plus 7 , then y is equals to x over 3, then plus 7.*	Talk as earners rearrange the equation to its standard form.

PG1 discuss features of the expression of the linear function as self-standing objects, indicated by nouns in their sentences. The ‘things’ they identify: the *x*-intercept, the *y*-intercept, the gradient, are abstract, and cannot be seen or touched in our everyday experience. They are intangible. M299, shows the transformation of the expression, to a standard, familiar, patterned form, to which learners are more accustomed: $y = mx + c$. From the talk above, it would appear that the symbols of the algebraic representation have meaning. Learners transform the algebraic expression to its general form, $y = mx + c$; the patterned arrangement of symbols which communicate a linear function. In this form, the symbols can be attached to labels or keywords of the features they represent.

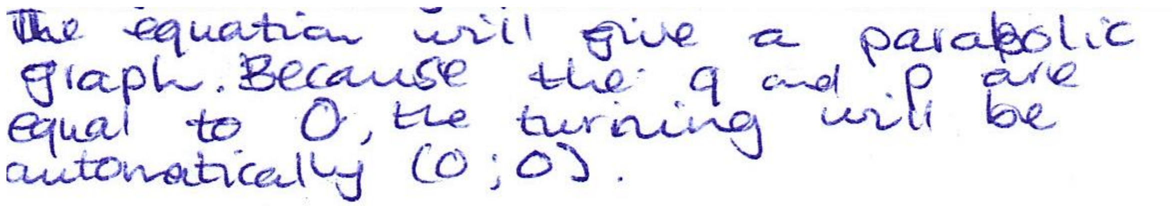
Extract 5 Talk on Card A: $y = x + 1$

T-PG2-V290	There's *y equal x and I think it's a straight line also...*	Related to the parent function $y = x$
T-PG2 -J291	*But it's going to shift by one unit...*	The transformation as a result of the addition of 1
T-PG2-V292	*Up one*	Specifies the transformation

Here, PG2 identify the function expression as a visual mediator. They discuss the transformation of the parent algebraic expression, $y = x$ to $y = x + 1$ through the secondary object of the graph. Formally, this is the algebraic transformation of $f(x)$ to $f(x) + 1$. This marks an additional node for realisations related to algebraic transformations. Given the algebraic expression, learners talked of the movement of the graph V290, ‘a straight line’, which shifts one unit up, due to the +1 change in the algebraic expression The change in algebraic symbolic representation results in an equivalent graphical change for these learners. The algebraic symbols are objectified as visuals on the graph for this group of learners. The graph appears as an equivalent representation of the algebraic expression. This occurrence, marking the new realisation node of algebraic transformation, did not occur across all PG2 pairs. It is highlighted here for occurring in that group. Similar talk on transformations occurred in PG1 as well.

The arrangement of symbols on Card M signified a parabola to these learners. They related the simple relationship on the card to an alternate, more generalised algebraic form of a parabola $y = a(x - p)^2 + q$, showing additional parameters a, p and q.

Figure 15 S-PG2


The equation will give a parabolic graph. Because the q and p are equal to 0, the turning will be automatically (0;0).

The meaning of the symbols relating to features of the graph are discussed, see the reference to the turning point. Again, the algebraic symbolic was visually mediated for these PG2 learners. The turning point, a direct, concrete, graphical referent would see equivalence in the function keyword, minimum. See the explicit reference to the graph 'turning' as opposed to the minimum value or turning point of the parabolic function. This extract conveys the multiple and complex layers in the discourse on function, and illustrates the ways in which learners slip discursively between their formal and informal layers to derive meaning from the space between these.

Extract 6 Talk on Card D: $f(x) = -2x + 3$

T-PG3-B274	*Well, what I see, like the equation of a straight line graph. The equation of straight line graphs, it's y is equal to mx plus c, and then here, ja here, this is c, this is m, which is the gradient.*	Related to the general equation of the linear function Indicates the -2 as *c* and 3 as *m*
T-PG3-N275	*So three stands for c?*	Seeking affirmation
T-PG3-B276	*Ja, and then this is y.* *Ja, that's what I, I think.*	Refers to $f(x)$

PG3 learners assign meaning to the arrangement of the symbols in a linear function and meaning to the individual symbols as well. Like the other cards discussed the algebraic expression is mediated visually with the graph. The features of the linear function were identified and filled with meaning.

The above selection of utterances coded E3 show that learners are able to talk of the specific features of the algebraic representation, across all groups The choice of the algebraic representation of the linear and parabolic functions was a deliberate choice for discussion as an objectified discourse occurred most frequently for these. Furthermore, all learners were also able to talk of how the parameters affect the graphical representation of at least one of these functions. Exploratory discourse was noted most frequently in the algebraic representation of the linear function, possibly because school mathematics uses the linear function to introduce the notion of function and its multiple representations from Grade Eight (DBE, 2011b). It has longevity in the curriculum in comparison. This subsection shows exploration routines for these learners existed in restricted ways relating mainly to the identification of functions and their features from the algebraic expression. The transformation of the algebraic expression and the consequent transformation relating to the graph showed an appreciation of generality and equivalence.

iv. Seeing the equivalence of representations and connecting these - the most frequent exploration code in PG1. Code E4 related to seeing equivalence among representations, and was most frequently found among the better-performing learners. Equivalence implies that are learner shows evidence that two representations, no matter how different they appear, are actually representing the same object. Without a discourse of the object function to begin with, learners still showed they could connect the representations to each other. It was difficult to see if they thought of these as equivalent. This section provides example of the equivalence when it arose, but also needed to show how easily learners transform from one representation to another. Of the 27 interactions in PG1, on the algebraic representation, 15 instances mentioned a graph immediately after viewing the equation, seen in three selected instances below.

Extract 7 The graph mediating the algebraic expression.

T-PG1- M231	*f of x is equals to seven which is a...*
S232	*straight line graph*

S-PG1- N35	*...x is equals to two and y is equals to two, it will, it will be like, I don't know a dead graph*
J- PG1- D488 **G489**	*It's like a, it's like a spot on the thing* *No, it's not a spot. It's a line.*

While the extract does not show a fully objectified mathematical discourse, it conveys the automatic association of algebraic expression with the graphical representation for all learners. Contrast this with Extract 8 below, where the graph was not an immediate realisation of the expression. Learners relate the given expression on Card I: $xy = 9$, to the general form of the expression for the family of hyperbolas.

Extract 8 Algebraic expressions belong in families.

	Card I: $xy = 9$
J-PG1 - G351	*It's a hyperbola...*
- G355	*Equals 9 over x.*
-G357	*It's a hyperbola, the equation of a hyperbola is, a over x plus .*

The word hyperbola suggested that the expression was being visually mediated; the learners do not call it a hyperbolic function. But they go on to the connect the algebraic representation to the general equation of a family of hyperbolic graphs. This realisation was interpreted as the extension of the given algebraic expression to a more general form of the expression, which describes the class of functions called hyperbolic. In this entire activity, the subsuming discourse of function was largely absent for learners. This raises an important question as to what learners do when they do not know precisely what a function is. However, in the context of the above, we see a specific hyperbola being subsumed into a class of all hyperbolas. It offers a generalisation for all arrangements of symbols which occur in this pattern. What all learners did have was a well-formed, function-specific discourse, restricted to keywords, visual mediators, narratives, and routines of several disparate functions. With these functions, they could talk of the equation as representing a non-specific relationship between *x* and *y*. This spanned their repertoire.

The linear, quadratic, and hyperbolic functions existed as separate entities that were not connected to each other by a subsuming discourse on function. Apart from two instances in PG1, no other group appeared to have generated a formal narrative for the object. As a result, this translated into two realisation nodes for learners when deciding if they saw a mathematical function on various cards. The first, the equation or algebraic representation providing a relationship between *x* and *y*, cued the completion of a table of values through the process of substitution, from which a graph is plotted. Clearly an entrenched ritualised practice, it was particularly pervasive across all PG3 groups. Familiarity with and recognition of the graph, a visual mediator, through the algebraic representation, was the sole criteria for declaring an equation a function or not. A global view (Even, 1998; Monk, 1988) of the function represented by the algebraic expression, through critical features held in algebraic symbols, was evident most frequently and spontaneously in the case of the linear function. The critical features of other functions only became clear once the graph could be viewed. This posed a challenge to learners when the algebraic expression of the function was not routine or recognisable from school mathematics. To generalise learner narratives on function: '*if I can sketch it, from the given equation, it's a function*'.

The second action on the part of learners, a consequence of this restricted discourse without the established notion of equivalence, was difficulty with univariate equations, such as $y = 7$ and $x = 3$. These posed challenge to substitution and plotting, the most frequent ritualised routine: which requires two variables. The presence of the second variable and its constant behaviour are not apparent to learners in a univariate relationship.

Extract 9 Equivalence and univariate functions.

S-PG1-written response	There are no x values
S-PG1-M71 **Later** **S-PG1- N86** **-M89**	*There's something there, there's no because... there's no x value.* *Ja, because there's no x.* *Because, they haven't given us any x values.* *It might be a function if we had the x value, but now we don't know what the x value is because... we don't know what the x value is.*

-N90	*Like, you know, what, like, I think, what we trying to say is, we haven't really been taught to look for things that aren't given to us. You know, like, if we are given coordinates, then we are able to know, like, which value is what, which value is what, you know. Or if we're given an equation, like y is equals to x plus 1, you know whichever value you substitute, you will be able to get a coordinate, you see, But now if… ja... we don't see an x value so we assume it's not there. Well, I assume if I don't see an x value, I assume it's not there.*

Extract 9 above illustrates a frequent routine regarding these special relations, where the algebraic representation occurs in one variable. Venkat &Adler (2012) suggest that school mathematics does not provide ways of dealing with these special cases (horizontal and vertical lines and lines through the origin) and the connections that they make possible. Commognitive theory calls these connections realisations, and looks for what is possible for learners to realise, within a representation, across representations and to the larger subsuming object. So even though it appears that learners can see the algebraic expression as equivalent to or from which the graph is made possible, the exploratory discourse of equivalence seldom progresses beyond this level. This is possibly due to an absence of discourse that connects these representations to the object. They are therefore unable to produce graphs for non-routine type expressions or to discern a function-relationship in such cases.

Data in this study, as for what is borne out in the broader literature, shows that representations of the object are backgrounded, while transformations, the algebraic processes connected to objects, are foregrounded (Artigue, 2011). This observation sums up the discourse of these learners. Transformations in school mathematics are a result of the manipulation of symbols with little or no rationale. An implication of Artigue's reference is that, just as learners appear to have developed objectified talk on the features of the graphical representation, they seldom move beyond this to reify process to have a discourse of the features that are held in the algebraic expression. Their objectifying related to identification of:

- The names given to symbols, e.g. m in $y = mx + c$, is identified as the gradient
- The arrangements of symbols in the algebraic expression signifying a function, which had to conform to the form $y = \cdots$ for the function to be identified.

Similar observations have been made in the wider literature, which also imply an emphasis on process, such as Dubinsky & Wilson (2013), Gripper (2011). Learner objectified discourse is based on the visible, identifiable features of the algebraic representation, with little evidence of reified processes. With this limitation, the absent variable will stymie the start of a transformation routine. It was also noted that keywords, which define the object function, such as domain or range, or words which relate the relation to the object, was absent in learner interactions. Words used by learners point directly to doing something to the algebraic representation or frequently to the inability to transform the algebraic representation into a familiar and recognisable form. This challenge was noted for all these learners pertaining to the univariate relation where learners could not initiate process.

Extract 10 The challenge of univariate functions.

J-PG1	Learners examine Card R: y = 7.
-D533	*But you can't give someone y is equal to 7. Mmm mmm It has no x value.*
-G534	*What has no x value?*
-D535	*That.*
-G536	*So then a, a function that has no x value is not a function?*
-D537	*You said it in; he said it in the beginning that it has an x and a y value.*
-G538	*Yes, I did say it.*
-D539	*Now, how…?*

PG1 appeared to visualise a graphical representation directly from the algebraic form after resolving it to standard form. These learners are able to realise and communicate features of the function, directly from the algebraic form. They could transform the expression, into an alternate, equivalent form about which they could say more or realise more. Whether they realised equivalence was much more difficult to observe. The other performance groups still worked with function as a process. This is noted as a precursor to the objectified notion (Sfard, 1991). Artigues' (2011) caution that learners seldom reify, which was seen with learners who did not go directly to the graph, who did not, as expected, discuss the features of the equation, but resorted to descriptions of the process of substitution into the equation to find *x* and *y* values that could be used to plot the graph. Through the visible features of a graph, the talk of the features of the function becomes objectified. A fully objectified notion of function from the algebraic expression would see learners' talk of the expression as belonging to a larger family of similar functions, which have defining features typical of that family. This was seen mostly in PG1.

v. Objectifying the features of an algebraic expression is the common code across all schools, and the most frequent code for PG2.

E3 describes talk of the specific features of the equation in objectified ways. The extract below, is typical of learner discussions around the parameters and form of the equations learners encountered. In examining Card M: $y = x^2$ the following exchange occurred:

Extract 11 The challenge of univariate functions.

S-PG2 -M438	*...let's first indicate that we found that parabola, okay? An equation.*
-F440	*Mm, mm... And knowing a parabola, we know that it touches the graph once on the vertical, using a vertical... so…*
-P441	*Okay, can we say, can we say that the y, er, the y values repeat themselves?*
-F442	*Repeat themselves, ja...*
- P443	*I'm also (inaudible)....talk about the p.*
-F444	*The p.*
-P445	*And the q.*
-F446	*Ja, because they are both zero, so the graph will automatically, oh ja, because they are both zero, the graph will automatically turn at zero. See at zero, zero , which is the turning point.*
-P447	*Mm, yes, turning point.*
-F448	*The turning point is zero, zero because both our q and our p is zero. So we found out it is a function.*

Again, the talk revolved around the features which became visible on the graph. In the extract above, utterance F440, learners applied the vertical line test to a parabola, which they deduced from the given expression on Card M. The vertical line test involves holding a vertical line to any graph sketched, if the vertical line intersects the graph at one point only, learners discern a function relationship. If the vertical line intersects at more than one point, the graph does not represent a function. This seems the single criteria that learners in PG1 and PG2 have to deduce a function. They offer no explanation for why this works. The deduction is mediated entirely visually, devoid of mathematical reasoning. In the extract above, learners talk about the *y*-values repeating themselves on either side of the axis of symmetry. The talk of values in relation to the axis of symmetry is informally communicated. The features of the equation arise from talk of the *p*, *q* and the turning point, where learners relate $y = x^2$, to a general equation of a parabola. These symbols signify the critical features of the function. Apart from mention of the keyword, turning point, the symbols *p* and *q* are not identified as related to the axis of symmetry or the minimum of the function. These are keywords. They are, however, assigned the correct coordinates to the turning point, in relation to the more generalised form of the algebraic representation. The single feature of the turning point and its symbolic place in the general parabolic algebraic representation, was the talk that was classified as indicating exploration.

To summarise, this subsection 5.3.1 attempted a broad description of learner exploratory utterances given different algebraic representations. While exploratory talk is a very small part of the classroom talk, it was interesting to examine the nature of the exploratory talk present in learner discourse when they communicated mathematically with each other. Better-performing learners showed the largest range and highest aggregate of exploration routines. Code E1, was interesting, as it appeared more frequently in PG2 than PG1. E1 related to talk of the algebraic representation as signifying an abstract mathematical object. In the data, the nature of code E1, related to the identification of functions and their features in objectified ways from the algebraic expression. Learners through all performance groups could discern and transform function expressions to the standard, general form. For PG3, this occurred most frequently with the linear function. Features such as the intercepts, turning points and asymptotes elaborated in school Mathematics could also be recognised and identified on relations that were not routinely studied in school Mathematics for PG1 and PG2. E1 represented the first level of objectification in discourse for learners, namely identification. The prevalence in PG2 could be accounted for in their trying to say as much as possible on the activity, relevant or not. PG1 in contrast identified functions and features if they were relevant to what they were doing.

Exploration that positioned the algebraic representation as signifying an object, or a 'thing', became visible through the presence of a noun in an utterance. To illustrate typical of instances occurring in the data, Card D: $f(x) = -2x + 3$, E-PG2-Ts 97: ' *...it's a straight line*'. This marked an important element of learner discourse on the algebraic expression. The expression is frequently realised as a graph across performance groups. PG3 were most processual in their approach to the function expression, in drawing up a table to sketch the graph. This they found difficult, with univariate expressions and functions not seen in school mathematics. The features of the graph, the secondary object were then discussed in objectified ways.

The equivalence of the algebraic and graphical representations was noted, specifically for the linear function across all performance groups. On this function, learners showed the greatest fluency in discourse. One possible explanation, already highlighted earlier, is that the linear function serves as an introduction to functions for these learners, three years prior. They thus have the greatest experience with this function. With the linear function, as with most others on the task we see the emphasis on transformation. The specific algebraic expression was transformed to general form when not given in this form. In summary, exploration routines remain at the level of identification and recall, and to a lesser extent, on substantiation.

5.3.2 Findings with rituals

The analytical lens used through this research process adapted and evolved as the study progressed. Perhaps the most important way in which this happened was to form a more inclusive notion of ritual and exploration, instead of viewing these routines as diametrically opposed. They are in fact mutually supportive in developing a mathematical discourse. It would have been naive to transpose learning into distinct and separated compartments. Learning or growing discourse happens on multiple levels with multiple influences. Rituals mark a natural tendency of human beings emphasising the social nature of learning. They are performed with others, and for others (Sfard, 2008). It is through the reification of rituals that learners can begin to explore mathematical objects. The coded data summarised as follows:

ii *All PGs showed codes R2, R 3, and R11 as most frequent rituals.*

iii *R2 followed by R3 was most frequent in PG1 and PG2.*

iiii *R3 followed by R2 was frequent in PG3, where R1 was significant in this group only.*

The coded data showed that rituals overwhelm learner mathematical discourses on function. The most frequent ritual code appearing across all performance groups is R2. This relates to discourse set to goals established by others, and speaks to the agency learners feel in their participation in the mathematical discourse. R3 is a code related to the mathematics being about process and transformation, was also significant. This level of discourse depicts participation in mathematics as following a set of rules, and largely rules that are set by others. This shows that formal mathematics is poorly embedded in learner discourse, making learner contributions easily changeable, and dependent on influences that are outside of mathematical justification. A certain consequence borne out in the data is that learners processed knowledge in discrete, disconnected pockets. Section 5.3.1 highlighted the general absence of an endorsed narrative for function and how this impacted exploration routines. This section permits a view of how this absence influences the discursive rituals learners use.

This zoom out on rituals can provide a broad picture of the how the frequent rituals across performance groups impacted the ways learners participate in the mathematical discourse:

- learners would change their particular narrative or course based an alternate opinion offered by the other learner, and seldom sought justification from their partner;
- they relied on verbal and circumstantial prompts from their environment;
- they seldom substantiated, tested or reflected on the narratives or solutions at which they would arrive;
- the emphasis on maintaining and forming social relationship meant that the mathematical object was displaced from focus; and
- poorer-performing learners placed emphasis on memory and experience when dealing with objects.

The implications of ritualised discourse summarised above raises the important question as to how the rituals which learners show on the task support the reification of processes and features towards building the discourse of function.

Not all rituals (or explorations for that matter) are equal and these categories in existing literature were too broad. The study took the broader categories of ritual and exploration, dissected the rituals and explorations as they occurred in the data, for defining characteristics of learner discourse. While the most frequent rituals present in learner discourses are evidenced in this section, albeit broadly, it became equally important to note codes that are absent, or less frequent. It is also apparent that groups of codes exist within the ritualised and exploratory codes assigned, which appear to make certain realisations possible, to appear as precursors to others, or to hinder further realisations. The intention was thus to check whether these codes or groups of codes facilitate transition to higher discursive levels. In light of this, the following two questions receive investigation here: first, can existing ritual codes be grouped to describe what may be required to transition from ritual to exploration? And second, by understanding the features of the ritualised routines, how can learners be assisted to transition to higher discursive levels?

The next 6 subsections begin the zoom-in view or the characterisation of rituals in relation to these questions. The analysis begins to look at the finer threads of learner utterances coded ritual to present a description of these. Using statistics arising from codes in the data, substantiated by extracts of learner talk, a picture of how the object is defined for learners begins to emerge. Commognition provides broader categories under which this characterisation occurred, namely: goals of routines invoked; what learners talk about; the flexibility and applicability of the routine; who the utterances are addressed to; and the reason for the acceptance of a routine. Chapter 4 provided the detail of the codes listed under each of these broader categories. These codes, descriptors of exploratory and ritualised communication, mined the data to saturation, serving the need to build detail around ritual and exploration through frequent occurrences. The development of these codes offers extension to the broader commognitive categories. Arising out of the combination of the data available on this study, discursive literature and the strong theoretical base, the codes intend to build on the work available. They are not definitive. They are, instead, a means to look at learner routines and contrast these using a performance lens.

5.4 Examining the categories describing routines: ‘Zooming in’

As argued in Chapters 2 and 3, all learning begins as ritual. We practice a specialised discourse through the way we see more knowledgeable people or resources engage the discourse. Thus, an initial ritualised practice gains us entry into the community practicing mathematics. Learning mathematics is more than that, however. It speaks to a basic human need to seek complexity, and to grow thinking according to the driving need we all seem to have to constantly improve. With this basic human will and need appearing inherent, it becomes necessary then to explore what makes learning mathematics so difficult for most learners. Using the idea of Vygotsky's zone of proximal development, this study sees the development as a change in discourse and the unreified ritual as bordering this zone. An investigation into the nature of learners ritualised discourse draws attention to the spaces that nudge thinking into this discursive zone of proximal development, making higher levels of complex communication possible.

5.4.1 Examining the Goals of Routines

Learner routines were scoured for the goal they represented. Did the communication of the pair show priority in producing a mathematical narrative, or was it seeking social acceptance? Identified goals were assigned codes from the list below. Each of these codes for exploration and ritual will be discussed in this subsection. The prevalence of each of the codes are shown on Figure 16 and are indicated as a percentage on Figure 17.

Figure 16 Codes describing the goals of a routine.

	Exploration Codes	**Ritual Codes**
Goals	Endorsed Narratives • E6 solves as a means to derive new narratives. Establishes a purpose for solution and interprets the process. • E7 seeks generality. Questions and answers are open, leveraging other objects.	Social Acceptance • R1 endorsed narratives come from memory or authority. • R2 adheres to goals set by others; satisfying needs outside of oneself. • R15 guessing.

Figure 17 Percentages of codes related to goals of routines.

Exploration						**Ritual**			
Goals	**Endorsed Narrative**					**Social Acceptance**			
		PG1	PG2	PG3			PG1	PG2	PG3
	E6	0	0	0		R1	1.20	1.27	3.01
	E7	6.85	0	0		R2	6.78	11.65	8.08
						R15	1.40	1.92	2.95

The percentages above are taken of a total of all exploration utterances under the heading exploration, and of the total of ritualised utterances under the heading ritual. Note that percentages, shown under exploration are as a percentage of the total of all exploration codes (73) across the 18 pairs of learners. The percentage rituals are shown as a total of all ritualised codes (2919) across the 18 pairs. As a result, because of the smaller number of exploration codes, the percentages of these appear higher. This accounts for why the exploration code E7 percentage seems higher, compared to the percentages of ritual codes. The tables' intention is to highlight codes which occurred frequently in the data, and not for quantitative comparison. PG1 were the only group who engaged in exploration with the algebraic expression. This was related to them seeking generality E7 as a goal. The other exploration code E6 was absent across performance groups.

i. E6 involves the goal of deriving new narratives and explaining the 'when' of routine-struggled for presence in exploration codes.

Since this was an absent code in the data, the implications of this for learning need to be discussed. What learners know is as much a product of what is present, as it is of what is absent in their communication. The absence of this code indicates an entrenched ritualised practice. As has already been discussed, learners appear to hold a collection of function signifiers and their associated routines. Without a formal narrative or definition of the object, these signifiers remain detached from each other and the object. Examining what motivates learners' choice of routine becomes important. So far, it appears that situational cues or prompts as opposed to mathematical reasoning drive progress through substantiations. This was particularly dominant among poorly performing learners. It could possibly also account for the lack of justification or reflection once learners have concluded a narrative. This is a critical and necessary mathematical skill, which evidences mathematical reasoning. The goal of school Mathematics is to reproduce endorsed narratives. Learning school Mathematics appears centred on established routines previously developed, generally through deduction, where the focus is essentially on the *how* of a routine. Learners have been taught how to do something, and they are constantly in search of the nature of what this is. This code allowed insight into how learners work with narratives they previously derived. The subsequent questions that arise are: do they combine or connect their existing endorsed narratives to seek new narratives? What means do they use to do this?

There are three types of exploration activity according to discursive literature, which link to code E6, namely: derivation, substantiation and recall (Ben-Yehuda, et al., 2005; Sfard, 2008). Derivation involves discursive procedures that give rise to new narratives. Substantiation is the way in which we endorse our previous narratives. Recall is remembering previous narratives so they can be used in specific instances. What we encounter in the extract below are learners in sustained attempt to recall routines from class work, from what they had been taught by their teacher, or from external motivating sources like assessments, for the purpose of substantiating their decisions. This shows that recall, in an exploration sense, is not initiated from their mathematical discourse or reasoning.

Extract 12 Recall.

M-PG1 -M23	*Er…, eish, is kind of hard, is one of those questions, is one of those questions you never see in Mrs. G's tests.*
S-PG3 -T95 **-L96**	*Remember last year when we were doing, is it parabolas?... and then ma'am said the… all the x values (was it x values or y values?)… that are in line with the graph are functions No, no... she said how to check if it's a function. Yo, man, I need my book. She said how to check if it's a function is… Yo, I've forgotten.* *Maybe I was absent.*

The emphasis on recall in this extract appears as an effort to recall from numerous sources, all non-mathematical (tests, what the teacher had said, notebooks), showing that these learners have not individualised the keywords, mathematical narratives and routines on function. Recall, starts initially as a ritualised practice, where a learner recalls words, narratives and routines of specialised knowledge which they use passively. Individualisation entails reflection on these key aspects of discourse used in communication. Reflection develops critical connections between learners informal and the formal discourse, between their various signifiers and the object, and between the various narratives they may already have of the object. The extract 12 above indicates that learners are not engaging the full potential of recall. Both groups shown do not engage mathematical discourse. PG1 placed an emphasis on recalling the types of questions they would have encountered on assessments, establishing that performance on assessments was a currency to this group. PG3 learners showed tensions they felt with recalling the mathematical discourse and conveyed an alienation from it. The source of their mathematical discourse was vested in non-mathematical sources, that is, the teacher, or the notes they had taken. Rereading similar extracts through the data gives the impression that the poorer-performing learners experience what is best described as an alienation from the mathematical discourse, where they seldom communicate mathematically. Their source of the mathematical discourse resides outside of themselves. This stands in contrast to better-performing learners, who prioritise recall for assessments.

The above argument establishes the importance of using recall for exploration. It enables learners to deduce and substantiate future narratives mathematically. The emphasis on recalling 'how' to execute a routine shows the importance with which learners regard the rules that govern a procedure. Remaining in recall discursively, however, is limiting. The data shows a lack of discursive means in deciding whether a routine was appropriate or not. This was seen when learners regarded the completing of an algebraic routine as closed or complete, even when explanation was called for. They were able to recall procedure, however prompts for justification were often met with a repeat of the routine. Sfard (2008), as already discussed in Chapter 3, argues that with regards to learning, changes in existing mathematical routines or the creation of new routines, result from changes in *when*. The absence of E6 shows that even our best learners, adept in the selection of mathematical routines taught in the classroom, are not engaging a discourse that develops and extends the mathematics of the classroom. They appear not to have engaged routines for their usefulness over varied contexts or given thought to extending routines to complexity beyond what was studied. This was observed to have a direct impact on how they connected the routines they knew to new tasks on non-standard type questions, where they struggled with selecting the appropriate routine if the familiar cues were not obvious.

The algebraic expression of the straight line and parabola, in particular, found learners discussing the features of the function in objectified ways. This seemed contrary to literature which showed that learners do not initiate the discourse of features or properties of functions (Nachlieli & Tabach, 2012; Ronda, 2009). However, progress beyond this level of exploration, developing reasoning and justification beyond identification seldom occurred. Questions illustrating this type of exploration could be: why do all straight lines conform to the patterned arrangement of symbols $y = mx + c$? Why does the c in $y = mx + c$ represent the y-intercept? Is there an arrangement of symbols from the general expression of the parabolic function which give rise to the turning point? Learners appear not to have developed these levels of complex enquiry critical to move to higher levels of discourse. As a researcher, I ask if learners have been taught what independent participation entails. The following aspects: the creation of new narratives; the connection between existing narratives; the connection between formal endorsed narratives and newly derived narratives in school mathematics; the discursive jump to reification; and the enquiry into why things are they way they are and why we do things the way we do, all appear to struggle for presence in these learners' mathematical discourse. These are too complex and specialised pedagogically, as well as too widespread and common across all levels of performance, to locate them as something learners are expected to develop exclusively on their own.

ii. Under goals, 6.85% of all exploration utterances involved E7 for PG1.

E7 pertains to the asking of open questions and the seeking of generality. Only PG1 showed evidence of this as a goal. The extracts discussed here relate to the talk in PG1 specifically. The incidence of this type of exploration was too low to make generalisable claims, but showed presence in the discourse of better-performing learners. The first selected extract shows learner attempts at generalisation, where J-PG1 deals with the domain of a circle, Card C: $x^2 + y^2 = 1$. Learners input the expression into a calculator to generate a table of values. This leads to Learner D questioning as to why the calculator identified certain values as undefined. As Learner G explained:

Extract 13 Learner attempts at open questions.

J-PG1 - G309	*...Let's say, for instance, we have a 2 here, 2^2, which is a 4. When we take this 4 this side, it's gonna give us a -3; $y^2 = -3$. To get y on it's own, you should root both sides, by rooting both sides, you gonna have the square root of -3, and you can't get that. It applies for each and every number. Even if you can take… every number.*
D311	*Thousands?…*
G312	*Yeah, thousands. It will still be the same.*
G318	*Yes. All integers, all real numbers.*

D319	***All integers, all numbers. Negative infinity to infinity***

In this extract, learners G and D sought generality based on the table of values generated by the calculator. They questioned and attempted to answer, what happens to values which did not lie in the domain of the function. This was coded as exploratory, as learners speak to the relation of the circle beyond the application of the vertical line test. This showed a discursive leap. While their talk presented certain limitations, where they did not discuss real numbers between -1 and 1, viz. the domain of the circle, it nevertheless showed potential for developing a subsequent layer of discourse. The strong processual orientation in utterance G309 and the absence of specialised keywords such as domain are also noted. Still, learners appear to have generalised, possibly supported by their readings on the calculator, that all values outside the domain of the circle will be undefined for the given algebraic expression. With little presence of this code across schools and performance groups, it can be assumed that such exploration is hardly present or developed in school Mathematics. Yet these learners display potential for it.

Such enquiry can leverage layers of discourse, including but not limited to this selection of topics required by the curriculum: discerning the domain and range of functions in general; values where functions are defined or not; and inverse functions, working with restricted domains. This illustrates the importance of the asking of open questions as a means to building generality and thus to connect between discrete pockets of knowledge. While there are definite limits to substantiation through a process and empirical instance, learners nevertheless attempt to generalise their findings to larger numbers in D311, and then integers and real numbers in G318. Definite constraints marking their attempts are the absence of keywords, formal mathematical narratives, and meta-discursive routines of how mathematists work for generality. Asking open questions appeared to reside with better-performing learners, yet this can be an important discursive skill for the extension or substantiation of existing narratives, which is a code absent in the previous section.

Attempts at generality extended also to the application of mathematical tools. E-PG1-G64, learner G talked about applying the vertical line test to a linear graph:'*they will always intersect at one point*', was a phrase which, in context, showed attempts at generality on the rationalisation of the vertical line test. Learner G extended this utterance with the comparison of the application of the vertical line test to a univariate relation, the vertical line, Card T: $x = 3$. This took a mathematical tool and attempted its application over multiple contexts. The substantiation offered is that the two lines (Card T and the vertical line) will not cut at one point only. It was the application of the vertical line test to a vertical line graph. Their talk showed that the lines would *'not cut at one point'*. Implicit in the talk was the notion of *coincident*. This absence of such specialised keywords through all interviews was significant, and was noted as a barrier to discourse development. It resulted in compensating non-mathematical mechanisms, which learner's used. Due to their everyday nature, these terms often conveyed ambiguity, which could further entrench learners' peripheral participation and alienation from the mathematical discourse. Ambiguity in this instance was seen when 'not cut at one point' could have been inferred as 'cutting at many, many points' or 'not cutting at all'. While their attempts often lacked the sophistication and rigour of a formal mathematical discourse, learners ought to be acknowledged for the spontaneity with which they reasoned and the levels of discourse and thinking these suggested.

This section examined the goals of learner communication for their focus on mathematics. In this description of exploratory discourse, which occurred in PG1 only, the focus was on unprompted inclination to generalisation, through the derivation of narratives or the asking of open questions. The code showing the derivation of narratives, E6, did not occur across schools or performance groups. The code describing attempts at generality through reasoning or open questions E7 occurred in PG1. It occurred without prompting, and while it may have been without the formal structure of mathematical proof, it showed the potential for abstract engagement of objects. While this located the learner on the periphery of formal mathematical reasoning, it illustrated the need for a knowledgeable other to connect learners' informal discourses to the more formal mathematical ones.

Further to this, a meta-discourse showing reasoning beyond the empirical was not found. This was evident in learners not being able to reproduce narratives, or deduce narratives from those given or prove to generalise their arguments. Attempts at generality appeared limited and intuitive among the better-performing learners, and absent for the poorer-performing learners. Realisations appear as halting, even 'stunted', where they often end after identification or a specific realisation. Specialised meta-discursive skills in building realisations from realisations and connecting these where possible are highly specialised ones, and these learners show the need to be taught these. The mathematical discourse is necessarily linked to how to work (or reason) mathematically and both these require demonstration by an expert.

iii. Ritual codes relating to the goal of social acceptance.

Figure 17 presented the following trends of these ritual codes:

- *R2 adhering to goals set by others was most frequent of all rituals among all groups;*
- *R15 gaining social acceptance through guessing showed increase from PG1 to PG3; and*
- *R1 showing emphasis on memory and authority for justification increased from PG1 to PG3.*

PG2 and PG3 ritualised goals revolved around social acceptance entirely found in codes R1, R2 and R15. The percentages of ritualised discourse increased from PG1 to PG3 across these codes. Poor performance could be linked to an increasingly ritualised practice. Learners appear driven by need to win approval and social acceptance from their peers and from authority. The drive is not to be more mathematical, but to belong to a community engaging mathematics. Code R2 showed a decrease in the percentage from PG2 to PG3. This appears contradictory to the conclusion above. To qualify this, PG2 had far more to say than PG1 and PG3 related to these codes. Much of their talk could be classified as non-mathematical, and without a *when*-filter. PG2 often picked up a cue and avalanched indiscriminately all that was possible to say about the cue. PG1 being able to reify showed compressed utterances, and could say more mathematically with fewer words, hence their lower ritualised aggregate. The economy and benefits of reification has already been discussed in section 5.2.2, where PG3 gave a 23 utterance response in comparison to the more compressed response of PG1.

In relation to these codes, PG3 had difficulty, in contrast to other performance levels, recalling relevant mathematics, rarely substantiating what they saw or did, hence frequently guessing. Guessing links strongly to the need for social acceptance and was found across all performance levels.

Extract 14 Utterances showing guessing.

E-PG3-M100	*We think it's not a function. ..Or it can be a function. We are not quite sure.*
J-PG1-G497	*Um... y = 7 is something like this.*
S-PG1-N108	*You know what...um.. it looks like a function, I think it is a function, but I don't know why...*

The question posed to learners in the given algebraic representation written as $y = \cdots$ or $(x) = \cdots$, was '*do you see a function in that expression?*' It was a question that encompassed two aspects: first, what they saw when they looked at the expression, and second, whether what they saw represented a function or not. The latter part of the question proved elusive to all learners who did not have a formal narrative for the object. The absence of the formal or endorsed narrative for function meant learners were unaware that functions could be subsumed into a class of mathematical objects sharing a defining characteristic. This blocked the connection of their informal discourse to the formal, and their developing discourse regarding each of the separate functions to the other. It prompted guessing throughout the activity about what the defining characteristic of a function could be.

Routines were based on familiarity with what had been taught in school, where the symbolic expressions for the linear, quadratic, hyperbolic functions were labelled functions, and learners relied on the vertical line test being applied to a graph. Learners consequently guessed in far more randomised ways when presented with functions or relations that were not part of school mathematics. Non-functions are not a key focus in school mathematics, except for the circle. Here too, they exist in the context of the domain being restricted to $-a \leq x \leq a, x \in R$ to create a semi-circle, which becomes a function. Aspects of the algebraic expression which may in parts resemble familiar features of functions studied in school cued a related guess.

In addition, guesses usually held further common features: M100 above shows how the guess usually included reference to the approval of the other person by including them in the decision (see the collective pronoun ‘we’), or the appeal to the authority of the teacher, assessments or textbooks, where uncertainty was also prefaced by the phrase ‘I think’ in learner utterances. Justification for why ‘I thought’ in a specific way was seldom substantiated. The next extract in Extract 15 picks up these shallow means of justification as they related to codes.

Extract 15 Means for Justification

S-PG1-N110	*... like, I don’t know how to put it, you know like most of the time in our tests, we’re not asked why things are functions you know, we just get what we get and then we put it down.*
T-PG1-M233	*Which is like... straight line graph from what we are learning, we are doing this sorts of graph where you have only one variable ...*
S-PG3- L182 **-T184**	*Ok the problem is that we didn’t, ok we did it, but then forgot, I don’t remember making thi …* *I say it’s a function and you say it’s not. Maybe. Make up your mind...*

These extracts of ritualised utterances showed how learners justified routines tied to social acceptance.

Further emphasising the drive for social acceptance, the interactional patterns show the way in which learners deferred too easily to the other person in the pair, convinced by justification that is not always mathematical, and in a sense, handing the mathematical authority over to that peer. This occurred across all groups, but most frequently in PG3, shown in Extract 16 below. This again emphasised the questions asked earlier about agency and the alienation learners may experience in participating in a mathematical discourse. If learners had different interpretations of a question, they seldom discussed both versions. The consensus which the methodology of this study hoped to achieve through learners talking to each other was compromised by what appeared as learners changing their opinions rather easily. An opinion is defined in the Microsoft thesaurus as a view, estimation, belief, judgement, or attitude. The moderation of the mathematical talk came from one of the pairs’ ability to recall more, or to recall differently and sound convincing. In Grade 11, iustification is expected to be mathematical.

Extract 16 Justification from deferring.

E-PG2- Ts113	*If he says it's not a function, I will say this one is also not a function because they are the same. It's just that f of x is replaced with y.* (Learners are comparing $y = 3$ with $f(x) = 7$)
T-PG1-M347	*You can have something like this.* (puts forward an idea but defers to other learner) *Ok...ok... what do you think it is?*
T-PG3-B294 **-N296**	*And then x, I mean y value, which is 7.* *Yes. It's like he says.*

To summarise, learner communication related to mathematical goals is significantly ritualised. The dominant goals relate to social acceptance confirmed by the prevalence of codes R1 and R2, occur across all performance groups. PG1 showed the only exploration routine attempting generality. The ritual nature of goal utterances appears to decrease as learners improve performance. Deeper study of the interaction between learners, focusing on the nuances of their communication, agency of the learner, and the understanding of power relations, could provide ways in which to explain this. These lie outside the parameters of the current study, but it is noted here that their frequency is significant. Goals which are not mathematical are seen to have an alienating effect on the learner, particularly for poorly performing learners whose interaction emphasise social relations and displace mathematics as primary purpose.

5.4.2 Examining *what is talked about* within routines

The following codes are discussed in this section.

Figure 21 Codes of what is talked about.

What is talked about	Mathematical Objects	Signifiers
	• E1 talk of abstract mathematical objects. Evident in objectified mathematics nouns. • E3 speaks of specific features of the object that are relevant. • E5 symbols are filled out with meaning. They signify a mathematical entity.	• R5 uses visual cues to remember process. • R6 talks of symbols rather than what is signified by symbols. • R10 asks/answers questions for clarification, affirmation. Questions are closed. • R11 emphasis on the person performing an action or engaging a process. • R13 misrecognition of the different representation and names representation incorrectly or not. • R18 uses spontaneous everyday language.

The prevalence of these codes in the data is listed below on Table 6. As stated earlier, the percentages shown under exploration are as a percentage of the total of all exploration codes (*n* =73) across the 18 pairs of learners. The percentage rituals are shown as a total of all ritualised codes (*n* =2919) across the 18 pairs. As a result, because of the smaller number of exploration codes, the percentages of these appears higher.[13] The tables' intention is to highlight codes which occurred frequently in the data, and not for quantitative comparison.

Table 6 Percentage codes for the routine category 'what is talked about?'

	Exploration					Ritual			
What is talked about	Mathematical Objects					Signifiers			
		PG1	PG2	PG3			PG1	PG2	PG3
	E1	6.85	5.48	0		R5	0.41	1.47	1.06
	E3	8.22	21.92	5.48		R6	0.75	1.40	1.61
	E5	1.37	0	1.37		R10	0.96	0.31	0.27
						R11	3.67	3.25	5.48
						R13	0.27	0.99	0.65
						R18	1.23	2.50	1.95

[13] When the exploration codes are combined with a ritual codes the percentage presence of exploration becomes too small to reflect on the table. These categories must be read separately for presence of exploration under exploration and percentages not compared across ritual and exploration.

i. Exploration Codes relating to talk of Mathematical Objects.

Examining learners discourse for utterances related to talk about process or object guided the analysis in this section. The focus was on what learners were able to realise from the algebraic expressions they were working with.

PG3 regarded the algebraic representation of a function as an indicator to the process of substitution, from which a sketch of the graph followed. This appears to be an overwhelming ritual in school Mathematics, discussed section 5.2. From this visual mediator, namely the graph, learners realise the object signified by the equation. This is seen in the written response from M-PG3 regarding the function $y = x + 1$:

Extract 17 Written response emphasising process.

Can still plot the graph and
can you the table to find the points

And from the paired interview:

Extract 18 Emphasis on process

MG-PG3-M61	*Ja, you can plot your graph. And you can still use the table to plot where each of the graphs stand.*

The algebraic expression cued the process of substitution. Process was indicated by the verbs 'plot, use, plot' in the utterance M61 and in Extract 19 below. The symbols as they stand in the expression do not independently hold meaning for what they signify to learners as a first realisation. PG3 learners mostly see the object through the graph. This was seen across schools. School S is provided as an additional example of this dominant routine in Extract 20.

Extract 19 Emphasis on process.

S-PG3-T104	*You can use it* (the table) *to sketch a graph. If you're given the x coordinates use it to plot a linear graph.*

S-PG3-T108	Later when asked about the table of values: *Sometimes you are given it* (the table) in *a question or you fabricate your own x coordinates.*

All extracts showed the repetition of the word 'plot' across different schools emphasising the process orientation to the algebraic expression. In Extract 19 , the verb 'fabricate' is an interesting replacement for the mathematical keywords 'independent variable'.

In contrast, PG1 and PG2 were able to realise the critical features of the expression from the meaning they had attached to the symbols in the expression itself. PG1 and PG2, in extracts which follow, spoke of the algebraic expression as signifying an object without engaging process and the features of the equation as signifying entities.

Extract 20 The algebraic expression as signifier.

M-PG1-M153	(Reflecting on $y = 3$) *A straight line graph*
-N156	*With the gradient of 0.*

Extract 20 between learner's shows talk of objects; the straight line and gradient are referred to as nouns. Learner M sees the graph and learner N sees the gradient, from the symbols of algebraic representation. Contrast this with two learners S-PG1 given the expression $x = 3$.

Extract 171 Implications of a process orientation

S-PG1 -N55	*x is impossible.... x=3(1) = 3.* (long silence) *I don't know the problem is, I've never seen some of these equations.*
-N59	*y's not there.*
-M63	*Same thing, y's not there.*

N55 shows recourse to process, the need for an *x* and *y* in an algebraic expression for substitution. This confirms the literature in Chapter 2 on special relations. Learners have difficulty realising values for the variable 'missing in the algebraic expression'. However, data presents, a useful contrast for the same learners in Extract 22 below. Learners discuss the significance of the symbols on a hyperbolic function $xy = 9$ relating it to the general form of the hyperbolic equation $y = \frac{a}{x-p} + q$. They talk of the meaning of parameters p and q, identifying these correctly as the vertical and horizontal asymptotes. Learners are thus able to generalise and objectify expressions, where both variables are present in the expression. The strong process orientation thwarts this on expressions of a single variable.

Extract 22 Symbolic meaning PG1.

S-PG1 -M176	*Hyperbola*
-N178	*Hyperbola without the q's and the p's and...*
-M179	*You thought the asymptote was 0?*
-N180	*Vertical and horizontal asymptote.*
-M182	*Both asymptotes are 0, so it's in this case...then it's a hyperbola.*

The contrast in the two extracts above could be related to the way that learners attribute meaning to algebraic symbols they can see. Learners transform algebraic representations from "embodied actions they perform on symbols, mentally picking them up and moving them around, with the added "magic" of rules" (De Lima & Tall, 2008, p. 3). The expectation of the presence of both *x* and *y* in an algebraic expression, to initiate substitution, '*y*'s not there' , points to the emphasis in school Mathematics of the *how*. In addition, this process-outcome approach induced by the *how*, contributes to a relative inflexibility in transformational activity, where learners are stymied if a visual clue (as in univariate relations) is absent, or the transformation process does not yield a recognisable outcome.

Venkat&Adler (2012) discuss, from their empirical evidence, the way in which particular teaching approaches of these special cases-horizontal and vertical lines- contribute to the features of the input and its representation not being attended to. Learners do not discern features of the univariate relations, because they have not developed a relationship notion of function. The relationship is easier to work with on the expression $y = mx + c$, where both variables are explicit. This possibly accounts for the fluency with which the algebraic representation of the standard linear function was handled. The special cases, however, provided interesting exchanges to reflect on. They highlighted the absences of the formal definition of function and a relationship orientation to the object. The discursive means learners use on to compensate for these absences emphasise process and are shown to be limiting. The high score in PG2 of code E3, pertaining to communication of specific features of mathematical objects needed examination. The best explanation comes through comparison of PG2 and PG1. The detail and fluency of the discursive sequence of S-PG2 in the written extract below, compares with the previous verbal exchange Extract 22 S-PG1-M176 to M182:

Extract 23 Symbolic meaning PG2.

$xy = 9$ is the Same as $y = \frac{9}{x}$, which is a hyperbolic function. Asymptotes vertical & horizontal are equal to 0. when $a > 0$ the graph will be in the 1st and 3rd quadrants

PG2 have reified transformation of the algebraic expression and the features of the function. They talked of the hyperbolic function, the asymptotes-differentiating between the horizontal and the vertical, and used the parameters to orient the graphs into quadrants on the Cartesian Plane. This extract was a good example to confirm the high score that PG2 achieved on code E3. PG2 had a more objectified discourse on what they could realise in terms of the features of the algebraic expression compared to PG1 and PG3. This related to far more objectified identification utterances than the other groups. This required explanation, as it appears as an anomaly on the statistical table at the start of this section.

This category showed PG2 to be more objectified than PG1. This was mainly due to their spontaneous identification of utterances. In most cases, they could substantiate the meaning of the symbols. However, the identification of key features, and the transformation of the algebraic representation to the standard form of the function, is not an automatic lever for complex functional discourse. It is what learners are able to do with the objectified aspects of the algebraic expression to move to higher discursive levels that becomes important. I argue that it is the beginning or initiation into exploratory thinking, where symbols are seen as entities and the abstract objects they bring into being, are a result. Having noted this as an initiation for higher levels of discourse, it still eludes PG3.

ii Ritualised Discourse of Mathematical Signifiers.

The next three subsections discuss the ritual codes which arose from talk of what the mathematics symbols signified for learners. A signifier is anything from which a realisation is possible.

a. .Everyday Language R18 and Visual Clues (R5).

The use of visual cues (R5) and spontaneous everyday language (R18) as memory devices to recall a process or feature was spread among all groups, but was most frequent in PG2. The graph of $y = 7$ discussed earlier was visualised using the metaphor of a 'dead graph', as opposed to being described visually as a horizontal line, or algebraically as an expression where the y-value remains constant. Similarly, the need to visualise the parabolic graph resulted in it being expressed as a 'sad face' or 'happy face', as shown in utterances below:

Extract 24 18 Everyday Language in mathematical descriptions

T-PG2 -V261	*Okay, ah, ne, you see, I think this graph, ah, will be like this, like see a smiley face. Ja, a happy one.*
-V269	*Ja, I think it's a…*
-J270	*Parabola.*
-J282	*It has a happy face, its gradient, it's not negative. If it was negative, it would not go...it will have a sad face. It will face down.*
-V283	*Down...like that*

Learners automatedly discerned the coefficient of the square term in the algebraic expression of a parabola and described this critical feature of the graph in colloquial terms. This discursive action happened frequently across schools. The visual mediator, a sketch of a smile or frown, created by using everyday language, replaced the mathematical keywords in this instance, maximum or minimum of a function. The use of the everyday terms like happy face, smiling, and sad face, was used widely across all schools and as explained earlier, and serve to obscure the object. They also contributed to obscuring of its features. Learner J had confused the parameter a (the numerical coefficient of the x^2 term) with the orientation of the parabola as concave down or up, and the 'gradient' of the parabola.

Additionally, the use of non-mathematical visual mediators and everyday language could contribute to poor development of future realisations on other functions, which may have a specific common feature. The incorrect use of the keyword 'gradient' and the meaning it has to the learner, will impact the development of the derivative in calculus, for instance. These codes showed the localisation of keywords to the function at hand. Keywords or signifiers like domain, range, maximum, minimum, which can describe functions globally, that is across classes, was not present for the symbolic representation. Such descriptions would enable learners to transfer the discursive keywords from familiar into unfamiliar contexts. These signifiers are also defined by their algebraic verification routines, which did not occur in the data.

Another frequent discursive move, which was also mediated visually, was the application of the vertical line test, used as the single defining criteria for a functional relationship. The vertical line test was part of the routines of PG1 and PG2. If the vertical line touched the graph once, indicative of a one-to-one or many-to-one correspondence between variables, the relation was declared a function. The notion of correspondence was absent. Decisions were based on the visual for all learners who applied the test. Algebraic justification for why the vertical line test would verify a function was absent in all learner talk and there were no questions as to why this tests works.

b. Talk of Symbols R6 and Misrecognition of Form (R13).

On both codes, better-performing learners in PG1 showed the least ritualised discourse, and PG2 showed as being most ritualised. Extracts from PG2 are selected here to illuminate this. The talk of symbols R6 rather than what they signify and misrecognition of form R13, played an additional role in obscuring the object. A good illustration of this is where learners (M-PG2-204-219) are given the expression $x = y^2$. They resolved this to $y = \sqrt{x} = x^{\frac{1}{2}}$ and declared this an exponential equation, because of the exponent $\frac{1}{2}$. What followed in their discussion was all the entailments of the exponential function, $y = a^x$, including the parameters of the expression and their effect on the appearance of the graph. The symbolic link for these learners was probably to the exponential equation $y = \frac{1}{2}^{x}$. Symbolic representations that are not filled with meaning lead to ambiguous and incorrect assumptions. Similarly, S-PG2-250-285, resolved the linear function, $x = 3y - 9$ to $y = -\frac{x}{3} - 3$, and concluded that this was a hyperbola, relating it to the general form of a hyperbola, $y = \frac{a}{x} - p$, 'because the hyperbola has a denominator'. The discussion which ensued was equally detailed to the prior one. Both these pairs of learners, from different schools, showed similar routines, the symbolism of their derived equation was related to general equation of the function in which they believed their critical feature resided.

The dichotomy of object-process was particularly visible in the way in which learners dealt with the univariate expressions $y = 7$ and $x = 3$. As previously described, they realised these relations through a process of plotting a point on the *y* or *x*-axes, respectively. Substantiations included describing the resulting graphs as a 'point'. The omitted variable, in each instance, was described as 'not being there' and was therefore assigned the value of zero. Hence, a single point was plotted on the relevant axis. Generally, the algebraic expression was read as an instruction to do something, rather than as made up of symbols vested with meaning individually or representing an algebraic relationship when in a particular arrangement. The key observations of this section were only possible because learners were given the opportunity to explain their mathematical actions. Examining learners discourse on specific tasks usually presented with the expected responses which mimic the classroom discourse. The opportunity to probe further illuminated the different sources that learners drew on from everyday experiences to justify their mathematics.

iii Closed questions (R10) and Subjectifying (R11).

The extract below exemplifies a frequent mode of questioning on the part of learners, and shows how the performer of the actions is emphasised in the discourse. Subjectifying was the code with the highest aggregate of all ritual codes across performance groups. Learners across all groups described what they did. Closed questions required a single answer or affirmation. In this data they were used to convey agreement strengthening social bonds. PG1 asked the most closed questions. Subjectified utterances are personalised and do not alienate the other in the pair. The potential for the discussion to extend beyond the specific answer sought was unlikely in such ritualised practice, where the extract below shows two learners discussing the addition of unlike terms, providing a good example of typical closed questioning techniques and subjectifying.

Extract 25 Closed questions and subjectification.

J-PG2-S439	*Also t**his**, like where did you get the -7x.*
I440	*I added -7 and -x.*
S441	*You can't. Can **yo**u…?*
I442	*Yes, **you** can...*
I443	*Why can't **you**?*
S444	*'Cause that… these are not the same,'cause, uh, here, it's like saying -1x - 7. Which are, **you** can't add **them**, 'cause **they're*** not like terms. ***You** can't.*
I445	*But **it** will give **you** -6.*
S446	*If **you** what?*
I447	*If **you** say what **you** just said.*
S448	*That's why **we** can't add or subtract **them**, 'cause **they** are not like terms.*

Most questions, like those shown above in Extract 25 across all performance groups, sought clarity about the actions just performed. They were questions about the *how*. They typically resulted in learners aligning to the discourse of the other person. They were questions that appeared to forge solidarity, which carried through to the end of the activity. The extract above, ends with learner I stating that he does not understand learner S's explanation, but allows learner

S's justification to stand as a common decision for the pair. This was a clear indication of ritualised talk, which prioritises social acceptance.

The other aspect that is emphasised in this extract is the focus on the performers of actions. Subjectifying utterances had the highest percentage in PG3 and the least in PG2. These were seen in the pronouns in each utterance and the absence of mathematical nouns. They have been highlighted for emphasis of their frequency. Indeterminate pronouns which refer to the mathematical symbols are also included here, where *it* in I445 and *them* and *they* in S448, are examples. These can result in ambiguity in learners' interpretations. Learners subjectify mathematical processes, where "I added"; "it will give you" stand as examples. An objectified discourse, characterised by alienation of the person and their actions, would be without these pronouns and actions, and would instead use mathematical nouns. For example, "the sum of..." "the result is". Learner talk showed these learners not to have reified processes yet. Rather, a process is described by their actions. Reification would allow learner I above, to relate the talk of process to past experiences, by naming the realisation of a pattern or commonality that was seen in all similar occurrences. It would simplify and compress learner descriptions. The fundamental endorsed narrative regarding the addition of like terms, in algebra, was not realised by learner I and had not established permanence in his discourse. He persisted in operating on algebraic symbols in a way that he did in arithmetic devoid of algebraic convention.

The ritualised, closed questions and the subjectifying in utterances in the extract, show that learners were responding to algebraic expressions of functions as a discourse of recall of process or the application of rules, across all performance levels. In addition, learners appear to exhibit a discourse of restricted or very little agency, particularly with PG3, whose subjectifying centred to a significant degree on expressions of frustration and discomfort at not being able to recall what was required of them mathematically. Like the three previous code groupings discussed, this grouping once again points to an entrenched ritualised practice among learners.

5.4.3 Examining Flexibility

The purpose of the discussion of the algebraic signifier or expression was to develop a description of what this signifier makes possible for learners to realise and if in these realisations of different component functions, learners show flexibility in the discourse of function as a whole.

Table 7 Codes for flexibility.

	Flexible	Rigid
Flexibility	• E4 connects different representations. Sees equivalence.	• R4 different representations are regarded as separate entities. • R7 recycles known routines/narratives. Appropriate or inappropriate. • R9 concerned with making errors or placing emphasis on avoiding them. • R16 difficulty in following rules.

Table 8 Percentage of codes related to flexibility of routines.

Exploration % (*n* = 73)						Ritual% (*n* = 2919)			
Flexibility	**Flexible**					**Rigid**			
		PG1	PG2	PG3			PG1	PG2	PG3
	E4	12.33	4.11	6.85		R4	0.14	0.38	0.62
						R7	0.48	2.16	0.86
						R9	0.03	0	0
						R16	0.07	0.03	0.03

The rules already established and endorsed in working with functions limit possible discursive moves (Sfard, 2008), where Sfard has noted that, “different ritual performances cannot be seen as interchangeable because they generate the same end product” (Sfard, 2008, p. 244). This study was timed at the completion of the topic of function in the school curriculum. Learners should therefore be equipped with a repertoire of the possible realisable options in any question involving functions. While it was interesting to see what learners were doing, it was more interesting to see how they deviated from the existing routine discourse. This could be seen when learners’ followed an algebraic algorithm, made numerical calculations, and resolved equations to their standard form. To arrive at the same point of closure to a problem would imply that learners were constrained by a set of rules. The examination of the algebraic expression for flexibility codes showed:

- Mostly PG1’s discourse showed equivalence between the algebraic representation and the graph. From the discernible features of the algebraic expression, learners were able to make a rough sketch the graph. This showed a global orientation to function (Even, 1990) discussed in Chapter 2.

- PG2's and PG3's took the equation as a prompt to substitute into the equation to generate the table to values, which could be used to sketch the graph. This is the pointwise orientation, discussed in Chapter 2 (Even, 1990).
- All groups showed evidence of identifying the symbols held in the general equation. This was most frequent across groups for the standard linear function. In particular, *m* and *c*, were identified as the entities they symbolised, the gradient and the *y*-intercept of the linear graph. Extract 26 shows the linear, parabolic and hyperbolic functions in general form being discussed.

Extract 26 Flexibility.

S-PG3 -T126	*Well, like last time, I said in these, um, the equations for graphs normally have a* pattern*. There's always a y-value, they always have a y-value, I mean y variable and x variable.* *Like the linear graph follows the* format *of y plus mx plus c. And then for a quadratic graph it would be, um,* y = *to, say,* $ab^2 + c$ *;something like that. And then for a hyperbola, it will be like a/b.*

The words 'pattern' and 'format' in the exchange show that learners are aware that the discourse on function is governed by the recall of rules and symbolic meaning, which give rise to the patterned general formula of functions. Here, Learner T attempts to recall, somewhat inaccurately, the equations of the parabola and hyperbola to connect the expression with the graph. The low percentages on R4 show that learners seldom in their discourse realise the equation as entity, without its connection to the graph. The recognition of the algebraic representation as a self-standing entity decreases from PG1 to PG3. This could be due to the output-orientation learners have been initiated into function with. It could also be due to algebraic symbols not being filled with meaning. While *x* and *y* are seen as variables, they are seldom spoken of as being related to each other in the algebraic representation. This suggests a limitation in the building of a relationship-orientation to function and disconnection from the larger subsuming narrative of the object. The implication is that the different component discourses remain disconnected also.

Critical features regarding the ‘square’ are loosely associated with the quadratic, and ‘quotient’ with the hyperbolic, are mentioned. This accounted for learners reaching the following flawed conclusions:

- $y = \frac{a}{b}x + c$ was a hyperbola, because of the quotient $\frac{a}{b}$
- $x = y^2$ learners were unable to deduce if the expression was a function. They realised that an expression with a x^2-term in it would usually signify a parabola. It was difficult for them to deduce the relationship when y was squared. They transformed the expression algebraically to $y = \sqrt{x}$. There was one of three decisions made at this point:
  – This was realised as related to work on a semi-circle, which had been the only time learners had encountered a square root in the algebraic expression for a semi-circle. This interpretation was found in PG1 only.
  – The exponent of ½ saw the expression classified as an exponential equation in PG2.
  – Mostly PG3 stalled completely in interpreting the expression or transforming it.

The patterned form of the general equation or symbolic expression was generally realised by learners as the visual mediator. Without the encapsulating discourse of function, and importantly, the notion of a particular relationship between variables, learners were not flexible in transposing discourse to symbolic expressions they could not recognise. Critically, there seemed to be no evidence in talk of why functions had their identifying symbolic expressions. This would have certainly contributed to the reification of the numerous algebraic transformations learners held. However, the significance of symbols, as they exist together in the patterned general equation, is absent in the discourse. Why do symbols come to be arranged in these patterns is critical to forming a relationship orientation.

R7 indicated recycling old routines. It showed in codes involving the application of the vertical line test to the sketch of any relation, it appeared as the second most frequent criteria that existed for learners to confirm a function (the first being the ability to sketch the graph), and it was a ritual that was applied frequently by PG1. In the extract which follows, PG2 explain why the vertical line test works to discern a functional relationship. The justification they give is description of a physical-visible act rather than a mathematical one.

Extract 27 Recycling routines without meaning.

S-PG2-F157	*Ja. When we test for a function, we test like this with a vertical line. Because if it was a horizontal, that means the parabola wouldn't be a function as it touches the x axis twice.*

The recycling of old routines in ritualised ways can result in uncertainty and indecision for learners. The quandary for learners, in this utterance, was as to whether it was a vertical or horizontal line test that is used to confirm a function. This occurred in both PG2 and PG3. It is indicative of a strong dependency on recall and not on reasoning. Examine the argument of elimination offered above: the conclusion learners reached was that the vertical line test worked, because "we know the parabola to be a function and the horizontal such as the *x*-axis will intersect the graph twice" , where, on those grounds, they disqualified the horizontal line test. This quandary described the difficulty of following rules not connected to a rationale. The function narrative was needed here. Had these mathematical explanations been connected to the rationale for the vertical line test, learners would not only have been aware of the existence of the vertical line test as it exists and works as a tool, but likewise aware of an expanded narrative that elaborates why it works. The use of old routines is useful if they can be applied over different problems. Repetition of routines may lead to reification. Learning routines in isolation and without meaning, blocks reification.

The approach used by learners limits the flexibility of the approaches used, as well as the options that are available to learners to expand discourse to a higher level. This is further supported by utterances when learners examine what they already know about a hyperbola:

Extract 28 Inflexibility from disconnection.

S-PG2 -F336	*Um, we don't have the, ja…*
-P337	*…the vertical line.*
-F338	*Ja, the vertical line asymptote, which will make the graph not to go beyond or to touch.*
-P339	*But then we should say that, because we said that the equation is for a hyperbola.*

Visible keywords from learners discourse on function, the ‘vertical line’, and from their discourse on hyperbola, ‘asymptote’ occurred in the extract above. They resolved these into a blended object, the *vertical line asymptote*. Ritual involving following rules without rationale or meaning showed learners filled rules with a blended meaning to justify their realisations. Such is the need for knowledge to be connected when learners attempt creative means to stitch component discourses together. These creative inventions usually happen at the expense of mathematical reasoning, are contingent, and become ambiguous if applied over multiple contexts.

Examining learners’ concern with making errors did not show up as significant in the codes (R9). This probably had to do with the enthusiasm they showed towards responding to all the questions. When they were unsure they were prepared to guess. They also showed little or no reflective means to self correct or correct one another.

The importance of connections, within and between component discourses, showed as a catalyst for exploratory talk, and was evident in the higher percentage (12.33%) of PG1 in code E4, namely the connecting of representations. For PG1, the symbols of the algebraic expression had meaning, and the algebraic expression seemed to be connected with a graphical representation in objectified ways. Both representations were spoken of as being synonymous, neither one needing to be transformed into the other via process. This connectedness of representations was strongest for PG2 and PG3 with regards to the linear function only. A possible reason that the exploratory utterances appear higher in PG3 than PG2 is that they exhausted all they could say about the linear function when it arose. In particular, they spoke of all the features of the graph in objectified ways. The level of objectification moved to a higher level in the PG2 and PG1, where they could say more about other functions, connecting features that were common across functions and speaking of the features of these various functions as they would change under transformation. PG1 showed fluency across the linear, quadratic and hyperbolic functions more frequently compared. They could also connect the features of functions to graphs they may not have encountered before. With the result, the x-intercept could be obtained algebraically from a given expression, could be spoken of in objectified ways, and could be identified from a wider selection of expressions.

Apart from the exploratory utterances on features of the algebraic expression being significant across groups, the majority of utterances across performance groups remain ritualised.

This has contributed to rigidity, where, even in utterances which are objectified, learners are unable to jump to higher discursive, where far more could be realised. Exploration remains the realm of what is familiar or easily connected to the familiar. Learners approach an algebraic representation with a set of routines, which they use to discern features or to sketch the graph. Flexibility enables learners to work efficiently in contexts which are unfamiliar by transferring discourse across.

5.4.4 Examining Applicability

Applicability relates to learners applying a particular routine to a task. This is the focus on the *when* of a routine, where, thus far, learner utterances have been examined largely in the context of the *how* of the routine. This section examined the array of routines learners could choose for a particular expression, and whether they justified their choice in objectified ways. Generally, the range of routines available to learners appeared restricted to one or two choices. It was in the ways that learners justified their choice that the exploratory and ritualised utterances could be coded.

Table 9 Applicability codes.

	Wide	**Narrow**
Applicability	• E9 - moves from process to object. Interpretation of process is offered.	• R3 - statements about actions and process. Remains in process(not reified)-no reflection on meaning at the end of process-spontaneous or provoked. • R8 - concern with the final solution. Proof/verification through a specific instances only.

Table 10 Percentages of codes related to applicability.

Exploration %						**Ritual %**			
Applicability	**Wide**					**Narrow**			
		PG1	PG2	PG3			PG1	PG2	PG3
	E9	4.11	0	0		R3	5.24	8.94	9.39
						R8	0.14	0.65	0.45

From codes of utterances, PG1 was able to move from process to object E9. PG2 and PG3 utterances remained ritualised in the application of a process. Applicability speaks to the *when* of a routine. Sfard (2008) subdivides the *when* of a routine into applicability and closing conditions. Applicability relates to circumstances when a routine is likely to be evoked. Closing conditions signal that a routine has completed successfully. Here, the way that a learner ended and reflected on a particular routine became important. In many instances, the routine applied depended on prompts from the other in the pair or from previous responses. The E9 utterances in PG1 pertained to identification of the function or the graph from the algebraic representation. The instances involved learners writing the equation in a form they recognised as standard for that family of functions. Once in standard form, they were able to realise the function and its properties. For example, J-PG1, given Card I: $xy = 9$, did the following:

Extract 29 Applicability

Written Response J-PG1	$\frac{xy}{x} = \frac{9}{x}$ $y = \frac{9}{x}$
J-PG1 -G35	*It is a hyperbola.*

Here we see the algebraic transformation written down and an unsolicited response to identify the function. What marked this utterance as objectified was the verb 'is', emphasising an object (G35). PG 1 were also able to identify and transfer the features of a function across to other functions. For PG2 and PG3, this occurred mainly with the linear function. In contrast to PG1, PG2 and PG3 utterances showed both these groups to have been prompted by recognition of the form of the equation from previous experience of sketching the graph (a process). Both prompts being situational in nature, this can be seen as contributing to the aggregate of codes for R3, which deal with talk of actions and manipulations.

Extract 30 Ritual application.

J-PG2 - I398	*Never did it.*
- I391	*Never done it, but…*
- S392	*It's not a function, because here it says xy equals to nine, and uh, a equation for a function, uh, it's y equals to nine, not x is equals to something.*

Extract 31 Ritual application.

JS-PG3-*S335*	*xy is equals to nine. That is, that is definitely not a function, ma'am.*
- *S337*	*We've never drawn anything.*

The latter two extracts show the discourse as being tied to a process. Learners spoke of 'never' having 'done' the equation or the graph. The expression given as $xy = 9$ was also in a form they did not recognise. PG1, in contrast, was able to apply the routine algebraic transformation solving for y in the expression. The closing conditions of the routine, described the function in an objectified way, see J-PG1-G35 above. The PG2's mediated the symbols iconically, J-PG2-S392, a function cannot be 'x equals...', speaks to the form of the algebraic expression. PG3 attempted to mediate the object colloquially through the graph, but are unable to recognise the graph from the expression.

R8 dealt with utterances that show concern for the final answer. Data shows that these two codes R3 and R8 appear to occur together frequently as in the following case:

Extract 32 Applicability: Emphasis on the final answer.

J-PG3-S267	*Yes, ma'am, it's like a quadratic one. You can substitute x to that, by three, then you gonna get your final answer.* (Learners discuss Card T: $x = 3$)

Learners in PG2 and PG3 appear more frequently to expect that a process will have an outcome, a final answer recognisable by its form, rather than by its correctness. For example, the critical features of a parabola will most likely be integers. A process is deemed applicable if it yields these 'neat' values. For many students "the procedure rather than the results is the gist of classroom mathematics" (Sfard, 2008, p. 211). Established procedures are rule bound and close in predictable ways. For the PG2 and PG3 learners, the recall of a procedure, with emphasis on identifiable syntax producing recognisable final answers, was the goal. Reflection on the final answer was non-mathematical.

5.4.5 Examining who is addressed

This grouping of codes speaks to the primary goal of ritualised discourse, which is social acceptance. The focus is on who is addressed in a given utterance.

Extract 33 Codes for Addressees.

	Oneself and Others	**Others**
Addressees	• E8 questions and justifies endorsed or derived narratives.	• R12 routines and answers are not questioned or justified. • R17 imitates the other person, keeps pace with the other.

Extract 34 Percentages of codes related to what/who is being addressed.

Exploration						**Ritual**			
Addresses	**Oneself and Others**					**Others**			
		PG1	PG2	PG3			PG1	PG2	PG3
	E8	2.74	0	0		R12	0.48	0.48	2.02
						R17	0.34	0.14	0

The need for social acceptance was seen in the ease with which learners agreed with each other and took up opposing narratives without challenging them. It is significant that PG1 showed the only exploration utterances in the group. PG2 and PG3 utterances sought affirmation or clarity. Challenges to narratives put forward were not pursued, and the viewpoint of one learner generally carried to the end of the discussion. This was discussed in 5.4.1 in detail. By contrast, PG1 attempted to justify their routines, asked for clarity, and participated in an evolving dialogue. It is within these recalling and substantiating narratives that we see PG1 show a far more expansive array of realisations compared. For example, J-PG1-(485-568) Card R: $y = 7$ signified the following five realisations for learners:

- the equation represents a straight line;
- the gradient of the straight line is zero;
- justifies this by the omission of the *mx* term in the equation;
- talks of substituting various *x* values, integers only; and
- realises that the possible *x* values are infinite.

These realisations evolved through conversations between two learners. It progressed as each realisation became a signifier for a new realisation. Justification and substantiation was sought from within mathematics, and not from affirmation by the other person. Contrast this with the response on Card R: $y = 7$ from P-PG2-S (114-118): "we can say *x* is a function or not a function, then of course *y* will also not be a function. It doesn't have *x* coordinates". In this utterance, we see one learner respond. His response is situated as it related to a previous response on Card T: $x = 3$. The resultant sequence of utterances did not stem from or refer to mathematically endorsed narratives. He used no mathematical means to justify the decisions he made. What was also apparent was the absence of response from the second learner.

When working with Card P: $f(x) = 7,$ PG2 and PG3 show R12 in routines where narratives were not questioned, but accepted.

Extract 35 Acceptance of narratives

T-PG3- B288	*It has only one x value, and uh, one y value.* (Points to intercept on y-axis)
-B290	*It has only y value of which is 7 and then an x value which is 0.*
Interviewer	Do you agree, N?
-N296	*Yes. It's like he says.*

R17, where learners imitated each other, was also found in PG2 and PG3:

Extract 36 Imitation.

E-PG2-Ts113	*If he says it's not a function, I will say this one is also not a function because they are the same.* (the functions being compared are $y = 7$ and $x = 3$)

Imitation is an important part of ritualised behaviour that allows learners access to a new discourse as they progress from initial passive participation, to full agentive participation. Through thoughtful imitation, a knowledgeable other decreases scaffolding of the discourse as the learner develops independence (Caspi & Sfard, 2012). The algebraic representations, discussed above, have been part of classroom discourse, and are included in curriculum documents. Imitation is important, as the learner copies what an expert does, and learns to work in this way. They reword narratives, repeat routines and decide what changes and what is to be kept constant in successive implementations. These successive attempts allow modifications in learner discourse, and focus and compress the discourse around critical features of the object, viz. they reify it. This raises questions as to how school Mathematics supports learners in the individualisation of the mathematical discourse towards reification. Learners' attempts at individualisation appear to be random, disconnected, and generally unguided. Socially, they tend not to disagree or amend narratives offered by one another. They choose to conform and agree in conversations. This shows their participation, largely as passive and dependent on others.

5.4.6 Examining reasons for acceptance

This section looks at the reasons learners find a narrative or routine to be acceptable. The focus is on the justification of the routines which were offered by learners. The codes that apply are:

Extract 37 Codes of reasons for acceptance.

Reason for Acceptance	**Can be Substantiated** • E2 Narratives are logically built / deduced by learner; Outcomes can be justified.	**Followed Rules** • R14 emphasises following rules and the importance of practice.

Extract 38 Percentage of codes related to the reasons a narrative is accepted.

Exploration					Ritual			
Reasons for Acceptance	**Can be Substantiated**				**Follows Rules**			
		PG1	PG2	PG3		PG1	PG2	PG3
	E2	5.48	6.85	0	R14	0.41	0	0.10

PG1 and PG2 deduced new narratives on the algebraic representation from previously endorsed narratives. School mathematics focuses on building a deductive argument. The score of 0 in PG3 requires an examination of what these learners were doing when they saw an algebraic expression:

- They identified the signifier, but did not make endorsable realisations from it which was relevant in the given instance.

Extract 19 Identification and absence of realisations.

MG-PG-3-S84	*x equals three y minus nine.* (reads the equation)
M85	*That's a graph. Yes.*
- S86	*Yes.*
- M87	*Yes, it's an equation.*
- S88	*Ja.*
- M89	*Ja. Because you can still find x and y,… Ja...*
- M91	*… to plot your points, because you can take the y to the other side…*

- They verified through rules or visual means:

Extract 40 Verification through rules or visual.

S-PG3-T155	*This one also doesn't have a slope. They say the gradient is undefined. There's no slope to… .Ok let's use logic here. You can* ***walk on*** *a horizontal plane, right? That doesn't have a gradient, but then for this one, where the gradient is undefined,* ***you can't walk going up like this****… ja, I think that's how I look at it.*
	Some time later
-T157	*Umm… undefined… huh?*
-T159	*Something that* ***you can't believe*** *or work out.*

- They appealed to authority or memory as justification:

Extract 41 Appeal to authority.

S-PG3 -T95	*Remember last year when we were doing, is it, parabolas, and then Ma'am said the, all the x values (was it x values or y values?) that are in line with the graph are functions? No, no she said how to check if it's a function. Yo man, I need my book. She said how to check if it's a function is… yo, I've forgotten.*

There appeared to be no mathematically based deductive process involved in these selected utterances. Appeals to authority, recall and perceptual accounts appeared to suffice. As a result, the substantiations were non-mathematical, vague and porous. Learners in PG3, had difficulty recalling narratives and this impacted their discursive fluency. In most cases, the stalling at recall meant that narratives had to be reconstructed from any available means, mathematical or not. The goal of this category was to see narratives deduced logically. This ritualised discourse, on the other hand, showed narratives which relied on the recall of rules and the emphasis on practice. The utterances involving recall are shown below and involve the application of the vertical line test:

Extract 42 The vertical line test.

E-PG1- G64	*It's like this. Ok, now, if I can put a vertical line, it doesn't cut that one point. Remember a function, a function cuts at one point.*

Learners showed competence in the application of the vertical line test. Why the test can be used to discern function, is left unexplained by all learners. Probing learners for explanation of this narrative shows up as a reliance on the recall of a rule. No utterances gave a mathematical justification for the test. It was a purely ritualised practice.

PG1 would attempt a justification through logical deduction. They also sought mathematical means for justification rather than agreement from their partner. The other performance groups, in most cases, did not spontaneously substantiate their narratives. For the most part, learners showed strongly ritualised, patterned ways of working and an easy deference to their partners. Their justifications decreased in mathematical reasoning from PG1 to PG3. PG3 made little attempt to question, disagree or conflict with the other's narrative.

5.5 Conclusion

This chapter examined learner realisations of the algebraic representation, zooming in on specific codes to perform a close reading. Frequent patterns in codes have made it possible to group component pieces of the analysis dealing with exploration into larger, encompassing, descriptive frames. These larger descriptive frames emerged through patterns across all performance groups under which the codes fell. Learner mathematical discourse, on zooming out, appeared as spontaneous/everyday talk, objectified talk, as well as an interesting overlap between these two types of talk. Objectification was located in the attempts at the formal mathematical discourse by learners. While ritualised discourse was overwhelming, the statistical aggregates of codes showed learners do make an effort to shift to a more formal discourse with the algebraic expression. These attempts at objectification, evident in exploration codes, appeared most frequently among the better-performing learners. However, their infrequency and lack of sophistication indicated that these were spontaneous attempts by learners. The formal mathematical discourse thus appears as not being a priority of school Mathematics.

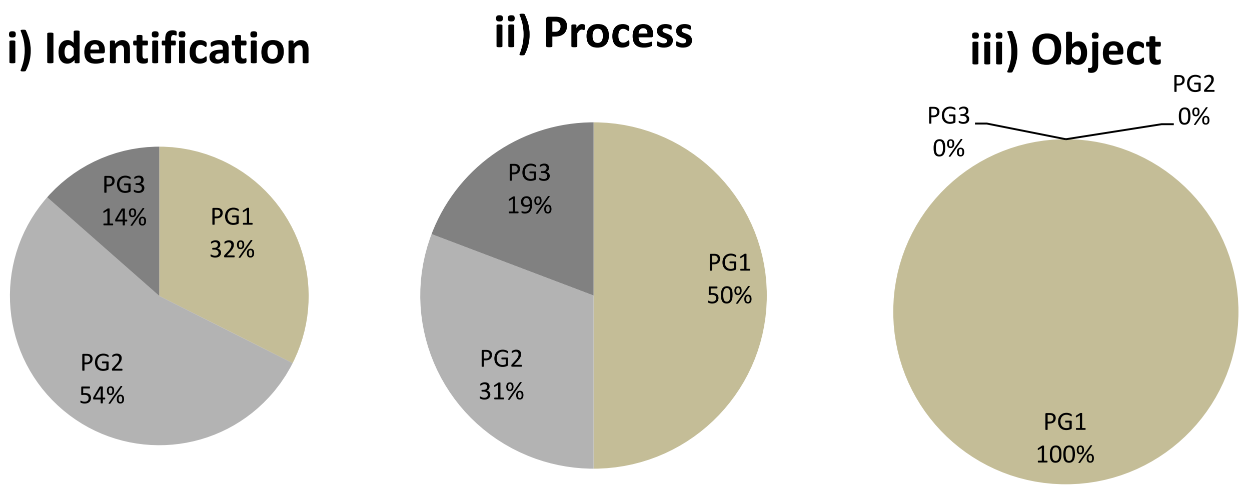
From the frames suggested above, it appears that learners engaged three levels of objectification talk in their exploration routines on this study: identification of signifiers, talk of process and talk of objects. Through several iterations of codes placed in each of the three categories, the wider framing descriptions persisted. No code fell outside of these three broad descriptive frames. This delineation of exploration codes helped build description of where learner thinking on the algebraic expression was located. The classification summarised in Table 11 shows the framing of exploration codes into categories.

Table 11 Classification of exploration codes.

Exploratory Utterances		
Identification	**Talk of Process**	**Talk of Object**
E1 Talk of abstract mathematical objects. Evident in objectified mathematics nouns.	E2 Narratives are logically built / deduced by learner; Outcomes can be justified.	E7 Seeks abstraction and generality. Questions are open.
E3 Speaks of specific features of the object that are relevant.	E4 Connects different representations. Sees equivalence.	E8 Questions and justifies endorsed or derived narratives
E5 Symbols are filled out with meaning. They signify a mathematical entity.	E6 Solves as a means to derive new narratives. Establishes a purpose for solution and interprets the process.	E9 Moves from process to object. Interpretation/ justification of process is offered.

The framing of codes in this way helps to build a classification of learners' exploration routines as well as highlight possible connections between performance and exploration. The pie charts which follow show performance against the incidence of objectified talk in each of the frames listed on the table above. Note that the percentages shown are as a total of the codes per category of identification, process and object, for each performance group. The pie charts show definitively that talk related to objects was found in PG1. The conclusion that can therefore be drawn is that better-performing learners were able to explore the algebraic expression as a signifier for function.

Figure 21 Exploration Routines categorised



To structure the discussion of this chapter, these categories are henceforth discussed in view of the levels of objectification evident in the learner communication observed.

i. Identification utterances. This category examined the talk of the algebraic expression, in terms of identifying the function, the component symbols that comprised the expression, and the features of the relation, as nouns. This was the indicator of objectification. It signalled an initial exploratory routine. Typical utterances under the identification framing were of the type: "that's a linear function" or "it's a straight line" or "m represents the gradient of the straight line". Included in these utterances, was talk of the features of the function. For example, the quadrants in which the hyperbolic function would lie, was related by learners to the parameter a in the general expression, $y = \frac{a}{x-p} + q$. Identification was the most frequent evidence of objectification, in PG2, across all functions studied in school mathematics. PG3 objectified utterances involved in the main, identification of the linear function and its features from the symbols in the general expression $y = mx + c$. A possible reason the PG1 utterances on identification appear lower than PG2, is that PG1 offer a compressed discourse, saying more with less, and their utterances focused on what was necessary. PG2's tendency towards ritualised discourse showed in the need to list all that they could recall, which may not have been relevant at the time. This is a critical distinction emerging in talk about the algebraic expression:

PG 1appear to have been more discerning about the *when* of mathematical routines when compared to PG2, and their talk was largely relevant and appropriate to what they observed and were doing. PG2 recalled or identified everything that they could without a ‘relevance filter’. This is, in essence, the nature of ritualised behaviour. Mathematical reasoning is backgrounded for PG2. An additional critical distinction between PG1 and other groups was they could transfer key identifying words across to non-standard functions.

It was significant that the discourse on univariate functions, such as $x = 3$, proved to be problematic for all groups. Learners had difficulty identifying the domain, range or listing coordinate pairs which satisfied these expressions. Again, the prevalence of the specialised discourse was so low, suggesting that it was overlooked in school mathematics, across school and learner groups. As a result, for PG2 and PG3, such relations existed as a single point plotted on either the *x* or *y* axis. They mediated these objects concretely, through a sketch of a point on the Cartesian Plane, to talk about them and assign them meaning. Specialised mathematical knowledge, evident in mathematical words or phrases, such as description of the domain and range, what a function relationship means, and discourse on the special features of the general expression, was limited across groups.

PG1, in comparison, with a wider repertoire of identification routines, and had a more global realisation of the algebraic expression. Expressions represented in non-standard ways, such as $x = 3y - 7$, would be transformed to the standard form $y = mx + c$, before they could be identified as a linear function or straight line, in PG2 and PG1 This was a good indication of process reified to object, discussed in the following section.

In summary, identification frames the objectified talk which arose across all performance levels. It is suggested that the keywords for naming certain functions and their features are satisfactorily established across all performance levels. Learners could correctly name what they are attending to and use these keywords in context.

ii. Process utterances. Process-related utterances included numerical calculations like substitution into the given algebraic representation, the transforming of an algebraic expression, the use of the calculator for calculation or verification, using algebraic means or formulae to identify the features of a function. While PG1 had the highest percentage of process utterances, they also held a wider repertoire of processes, flexibility of these, and thus recourse to alternate process-means to justify their outcomes. PG3 process routines largely involved numerical verification. PG2 showed a narrower repertoire of routines than PG1. Univariate functions stymied process, in that learners were unable to solve through algebraic reasoning for the missing variable. This was perhaps the most significant indicator that functions were not fully developed from a relationship orientation for these learners. For all groups, an equation had to be resolved to the standard form $y = [\dots]$, before the function or graph could be identified, or its key features named. This the learners found difficult in univariate functions.

The process leading to the identification of function was determined in one of three ways: whether a graph could be drawn from the algebraic representation; whether the resultant graph intersected a vertical line once (the vertical line test); and whether the equation conformed to the general form of equations that were studied in school. The distinction here between ritual and exploration, was the result of the accompanying explanations learners offered to substantiate the algebraic process used or calculation conducted. Process was indicated by action oriented discourse, involving verbs. For example, a process-oriented utterance would read, 'add x to y'. Note the verb *add*. The equivalent objectified utterance, indicated by the codes in this section, would read as 'the sum of x and y'. Note the absence of verbs, and the noun *sum*. This would indicate talk of an object as opposed to a process. This section focused on the reified processes learners used to work with functions and to justify the algebraic expression as a function.

iii. Object utterances. PG1 spoke of mathematical objects, which contrasted with the subjectifying tendencies of the other performance groups who spoke of human action. This appeared to confirm research that showed that the elimination of talk of human action in mathematical discourse contributed discursive changes that were linked to improved performance (Ben-Yehuda, et al., 2005). PG1 were able to describe equivalence between the different representations, qualify the different algebraic transformations with a rationale for the chosen one, and choose an algebraic means of substantiation which illuminated the required object without ambiguity. Essentially, what distinguished them from other groups was a thoughtful reflection on an algebraic outcome, for what it meant at that point, along with the ability to self-correct. PG1 were open to extending the significance of their outcome, through an open question or narrative that raised the level of the discourse in terms of generality and abstraction. Their agency was tied to mathematics, and narratives were questioned and co-constructed in learner interactions. Algebraic symbols signified entities. The learners saw the equivalence of different routines, which resulted in the same outcome. The definition of function did not exist as a narrative for most PG1 learners. Two out of the six PG1 pairs were able to tentatively put together a sentence communicating a relationship between *x* and *y*. The informal narrative that existed for most learners occupied space in the overlap of formal and informal discourse, based on their experiences with the word ‘function’. The object appeared as a series of component, and disconnected discourses, each with its unique identifying features and processes. The overlap was inhabited by learners attempting to spontaneously generalise these disparate discourses according to perceived (and not mathematical) connections.

To conclude, framing learners’ exploration routines as located in identification, process and object, was useful in order to see how performance linked to exploration in particular and description of learners routines in general. Moving towards reification of process and more objectified talk appears to be connected with better performance. It appears as logical starting point for developing discourse in PG2 in particular. Building the need for justification, where learners explain their choices and reflect on these, appears as a need across all groups of learners. The ritual codes already existed in their description categories, where the overwhelming ritual observed across all schools was that the object function was the graph. As a consequence, if the algebraic expression could be graphed, it would be declared a function. To achieve this end, learners resorted to the process of creating a table of values from the algebraic expression and graphing this. Ritual codes which came up strongest related to goals of social acceptance and subjectification. Both these appear as barriers to objectification and accessing higher levels of complexity.

Seeing how learners work with other signifiers, the graph and table will build a fuller picture of learner thinking on the object, function. It proved fascinating in this chapter to examine what learners did without having the formal narrative for the object. The analysis above saw learners, nevertheless, attempt building a unifying discourse. It was encouraging to observe that they looked for a means to group these algebraic expressions so as to call them functions, where the graph became the defining characteristic. For these learners, their ritualised routines showed that the object resided in one of its signifiers. The formal definition would have given learners a means to connect, integrate and subsume the varied narratives they held on various signifiers, into a single object.

Chapter 6: Tables

6.1 Introduction

The table of values is a signifier, which can be realised as an algebraic formula or as a graph (in this study termed the algebraic expression, and graph, respectively). On the first discursive level, a table of values can make several features of the relation explicit or implicit- for example, the *y*-intercepts and *x*-intercepts are deduced from where either a *x*-value or *y*-value are zero; an increasing or decreasing relationship; the type of correspondence exhibited between the variables (e.g. one to one); and – pertinent to school Mathematics – functions can be deduced as linear, quadratic or cubic. Perhaps the most important challenging level of thinking expected in Grade 11 was learners using the table of values to find an algebraic expression for the relation represented by the values.

In school Mathematics, learners are initially oriented to functions through plotting a graph from the table of values. This occurs in Grade Nine, two years prior. It is called a 'pointwise' orientation in literature and was discussed in Chapter 2. For the purposes of this chapter, this was regarded as an initial discursive level. Two other discursive levels derived from how learners are expected to interpret tables in school Mathematics (DBE, 2011b) were considered. The second discursive level was talk that involved discerning features of the function from the table of values. The third discursive level involved generalising the relationship between variables from the values on the table. Nachlieli & Tabach (2012) regard the table to be an encapsulated collection of ordered pairs, each showing reification of the process, represented by the algebraic expression of the function. Graphs and algebraic expressions of functions give a generalised representation of a function, while a table helps you find information on particular values (Even, 1998; Mason, 2005).

This study commenced with the assumption that working with specific values, in a pointwise manner, would be easy for learners. Where did the challenge arise? To explore the observations of research cited above, this chapter looked for what learners did with tables in terms of their mathematical routines. To this end, it examined instances in working with tables, when learner communication showed exploration codes. Was the table seen as a mathematical object or just a tool that would make the graph or algebraic expression it represented visible?

These signifier-specific questions are then contextualised into to the broader research questions already asked in Chapter 5.

1. What are the characteristic features of learner discursive routines at each of the performance levels?
2. How do these features contribute to the mathematical object function that comes to be possible?
3. How can the object that exists for learners be described at each performance level?

The following tables were available to learners on the task:

B

x	1	1	1	1
y	1	2	3	4

N

x	1	2	3	4
y	-1	-1	-1	-1

U

x	1	2	3	4
y	0	3	0	3

Given a table of values, learners were asked what they saw in the values and ultimately if they saw a mathematical function. Tables B and N provided opportunity for learners to discuss a function relationship between the values and to describe that relationship algebraically. Table U was a non-standard type of table, and was intended to determine which properties learners could discern from it, as well as how they interpreted the relationship of values it described.

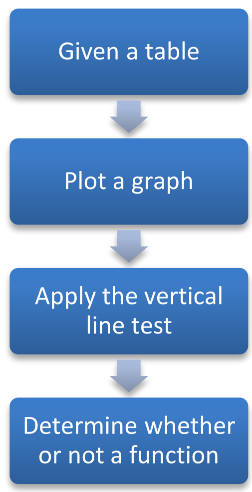
As in Chapter 5, the zoom out involved the broad description of learners' routines when they worked with a table of values. This allowed the broad classification of learner thinking in terms of ritual or exploration. The routines were then zoomed in on and coded in terms of describing what learners did or said, as part of characterising the nature of their routines.

6.2 Frequent routines zoom out

6.2.1 Findings overall

Before the examination of routines for ritual and exploration codes, it is worth examining some of the broader routines found in each performance group. The most prevalent routine is captured in the Figure 23 below:

Figure 23 Table routine



With this broad routine established, it was interesting to see how it was applied across performance groups.

Table 12 Broad frequent discursive routines for the table across performance groups.

<table>
<tr><th colspan="4">Table Routines</th></tr>
<tr><th></th><th>PG1</th><th>PG2</th><th>PG3</th></tr>
<tr>
<td>N <table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>y</td><td>-1</td><td>-1</td><td>-1</td><td>-1</td></tr></table></td>
<td>Discerns relationship with or without graph.
Plots the points.</td>
<td>Notices y-values are -1.
Plots points and joins them.
Identifies a non-function from graph.
Identifies graph as a scatter plot.</td>
<td>Plots points.
No discursive action. Never seen this graph before.</td>
</tr>
</table>

B x: 1 | 1 | 1 | 1 y: 1 | 2 | 3 | 4	• Discerns relationship with or without graph. • Full objectified description. • Plots the points. • Uses the vertical line test. • Identified as a function.	• Plots points and joins. • Non-function. • Identified as an exponential. • Talks of features: *x* and *y* intercepts. • There is no equation behind this, it has 1 coordinate.	• Identifies non-function by guessing from table. • Functions have intersections and x and y intercepts. • Plots and joins. All straight lines are functions.
U x: 1 | 2 | 3 | 4 y: 0 | 3 | 0 | 3	• Plots points. • Identifies a non-function from graph. • Identifies features :y-intercepts.	• Plots. Joins points. • Identified as sine wave.	• Plots. Joins points. • Identified as sine wave. • Reference to assessments.

Learner communication on the table of values was, in the main, ritualised, with six out of 296 codes showing exploration. This was consistent with the context of the curriculum and the approach to working with tables in South African school Mathematics. The tables that were produced by learners from the algebraic representation in Chapter 5 to sketch the graph are not included in the results of this chapter. Here, learner discourses around given tables on the paired interview,[14] and what these made possible to realise was examined. PG1 produced the only exploratory codes:

- E1-talk of tables as signifying abstract mathematical objects;
- E3 –talk of specific features of the table; and
- E4-talk that connects the table to other representations.

Of which, E4 was the most frequent code.

[14] See Appendix 1 for the paired interview schedule of questions that contained the tables referred to here.

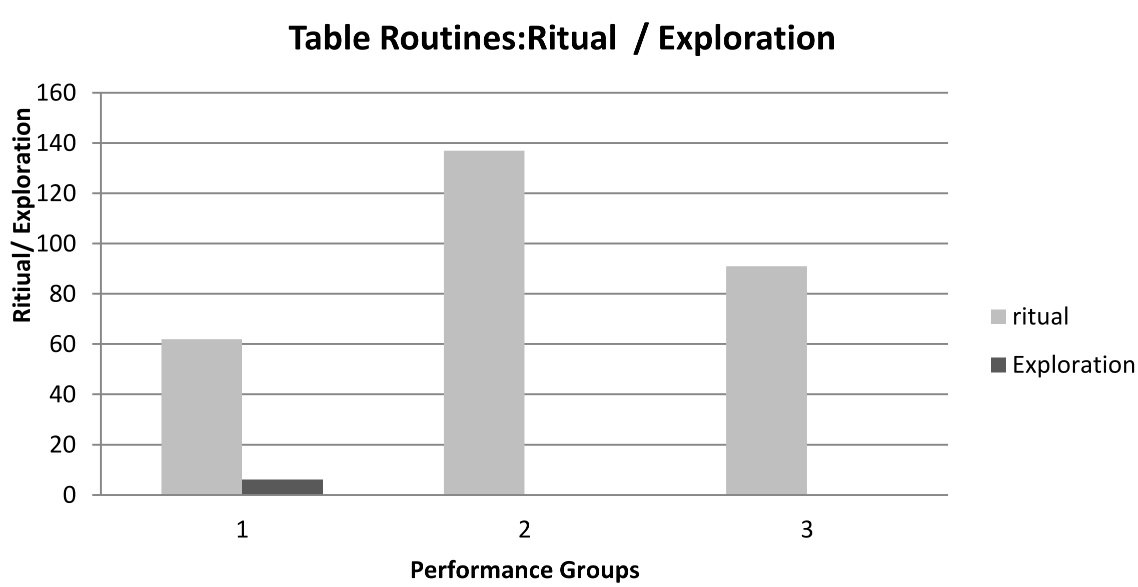
The four most frequent ritualised codes overall were:

- R11-talk that is subjectifying (22.76%)
- R2- talk showing following the goals set by others (21.03%)
- R3- talk about actions and manipulations (14.14%)
- R7- talk showing recourse to old routines (13.10%)

In aggregate, these account for over 70% of the rituals involving work with tables. All other codes aggregated as less than 10% of the total ritual codes. Codes R14, R16, R17 were not evident in the data. Like the algebraic expression, talk on tables was mainly ritualised, with PG2 again showing the highest aggregate of ritualised codes. Frequent ritual codes will be discussed in detail as part of the main categories of ritualised codes in section 6.3. The aggregate of utterances and codes of these utterances pertaining to tables were significantly smaller than for the talk on the algebraic expression. In addition to there being fewer tables than there were algebraic equations, learners had far less to say and do with each table showing a limited repertoire of routines. The exploration codes, as a result, presented even smaller. This permitted a discussion of the combination of ritual and exploration codes across performance groups in this section (6.2) the zoom out. The data in Chapter 5 was vast in comparison, required thorough inspection, and did not permit this kind of scrutiny.

The figure below shows the ritualised and exploratory aggregate of codes for a table of values across the three performance groups:

Figure24 Table routines as aggregate of codes for ritual and exploration.

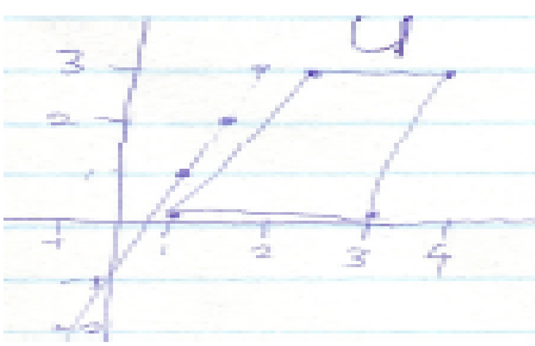


6.2.2 PG3

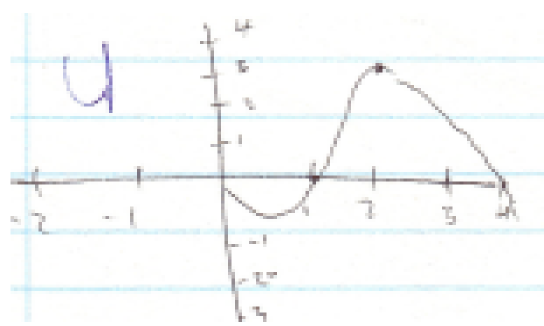
In PG3, learner talk is entirely ritualised (91 ritual codes and 0 exploration). The learners utterances showed a limited number of discursive moves. As with the algebraic expression in Chapter 5, these will have to be investigated in the context of learners not having a formal narrative for the object. The dominant routine in PG3 showed learners using a table exclusively to sketch a graph. The table presented coordinates to be plotted. Plotting by learners showed a level of indecision, inconsistency, and incorrectness, particularly when it came to Card U. There was no indication that a table of values represented a relationship between variables *x* and *y*. The table of values was thus not a signifier for an algebraic symbolic expression or for a function. Equivalence of the table to other representations was not evident. There was no talk of patterned behaviour of the *x* or *y* values on their own or these as connected to each other. Features of the table, like the *x* values increase or (1; 0), represents an *x*-intercept, were not discussed. The talk showed process utterances as learners plotted the coordinates on the Cartesian Plane. Their illustrations for Card U are shown below:

Figure 25 PG3 illustrations of Card U

J-PG3 Illustration A

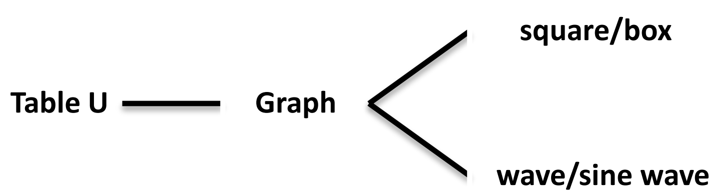


and later, J-PG3 Illustration B



The talk around these two sketched representations derived from Table U is summarised as:

Figure 26 How a table of values is visually mediated for PG3.

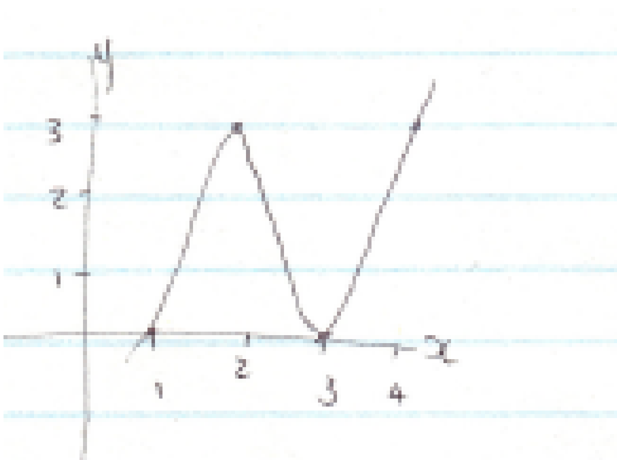


U

x	1	2	3	4
y	0	3	0	3

While it could be thought that this visual mediation of the table of values was isolated to PG3 of one school it was not. Accompanying their sketch P-PG3 identified the graph from a previous assessment P-PG3-Mo 442: "(I've seen this) *in my exam question paper"*.

Illustration C Card U sketched by P-PG3



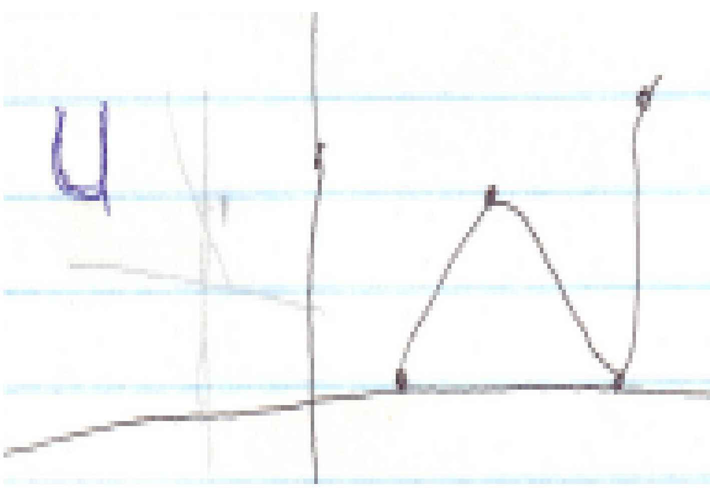
As for PG1's frequent reference, in Chapter 5 on the algebraic expression to assessments, PG3 similarly referred to assessments as directing them to what needs to be learned. Assessments like the June examination, referred to by Mo 442, conveyed messages of what is important to learn in school Mathematics. Split functions are not part of the curriculum, and are not taught or assessed in school Mathematics. The graph was identified as a sine graph from the learner's incorrect representation of it in Illustration C, and resonated with an assessment the learner had previously encountered. The table of values in this case did not communicate a correspondence to learner. Learners joined the points to create a visual, which was identifiable from their previous experience from the classroom and in assessments, as sine graph; whereas, from everyday experience, which was identifiable as a square, box or wave. These were consistent descriptions across schools.

Across PG3, all points plotted on the Cartesian Plane were joined into a shape that was recognisable in response to the question asked of them: ‘Do you see a function represented on the table?’ To the learners of this group, being a function was contingent on the graph being sketched as the defining criterion. In school Mathematics, $y = sinx$ is described and studied as a part of functions. Despite the values depicted on the table of card U, clearly not representing trigonometric ratios for angles and lines, PG3 sketched a sine wave and identified a function, due to visual similarities in the way the graphs appear, and based on their recent classroom experience.

6.2.3 PG2

PG2 had a total of 137 codes assigned to their utterances (all ritual, no exploration). They produced the maximum number of codes as they had far more to say than other groups. Like the other groups, without the object function being fully developed, they relied on the table primarily to sketch the graph. In general, for PG2, all tables on the activity were identified as functions because ‘you can use them to sketch the graph’.

Illustration D Table U sketched by J-PG2.



Comparing the J-PG2 sketch of Card U above with the sketches of PG3 (Illustrations A, B and C) shows a scarcity of detail on the rough sketch from PG2 compared to those of PG3. The position of the last point to the right of Illustration D is curious. When asked for the motivation for joining the points J-PG2-I346 replied: “I wanted to see what shape the graph

[was]”. The shape signified the sine graph to this group of learners as well. The talk among learners did not show mention of the critical features of the table, such as the *x*-intercepts (indicated by the *y*-values being zero on the table), or features of graph as they become visible once the graph was sketched. The features of the sine graph were not transposed onto the sketch either. Like PG3, the graph outcome was realised with familiarity to work just covered in school Mathematics – in this case, trigonometric graphs. Without joining the coordinates, there were two PG2’s who described the visual as a scatter plot (also a recent topic covered in the classroom), and the linear and horizontal graphs (from cards B and N) were related to those encountered in linear programming. This confirms literature which suggests that learners cope in Mathematics by associating with routines through discursive clues related to their past experience (Nachlieli & Tabach, 2012).

As seen with PG3, narratives and deductions offered by this group showed a low level of correctness. To examine ‘correctness’, instances where learners veered off endorsed narratives or mathematical routines became relevant, and these instances accounted for a total of 30 from amongst their total utterances about tables. They arose in incorrect identification of signifiers, or from a routine which was not appropriately used, or was abandoned without closure. Learner narratives of the table were porous and interchangeable, as they inconsistently applied keywords and routines. PG2, like PG3, showed little means of reflection or self-correction. While PG3 had fewer utterances, they had nine noted instances in which they were incorrect, compared to the 30 for PG2.

6.2.4 PG1

In direct contrast, PG1 had no incorrect utterances noted for this signifiers. The exploration routines of PG1 revolve around identification (E1 and E3) of the function and its critical features and process (E4), which connects the table to its other representations. PG1 showed the least number of codes on utterances about tables, 62, again they are able to say more with less. As noted, there were no notes on the incorrectness in their talk.

To illustrate the consolidation and compression of their discourse, Extract 43 below shows PG1 learners' attempts at exploration when examining Card B.

B

x	1	1	1	1
y	1	2	3	4

They were able to move from the signifier table to graph without the process of plotting points. This indicated they had reified process.

Extract 43 Reification of points on a table.

J-PG1 – G394	*x = 1.*	This was coded as learners connecting representations table with symbolic. Coded E4.
-D397	*It's a straight line.*	The features of the graph are discussed without a sketch. Coded E4.
-G398	*So it's not.*	
-D399	*It's not. Function.*	Signifies an abstract mathematical object. Coded E1.

The fluency of the above discussion was not evident in other performance groups. PG1 learners did not need to use the table to sketch the graph. They automatedly realised a symbolic expression to describe the relationship of values seen on the table, see G394. D397 stated the features of the relationship, and they progressed on to realising the straight line as not a function in G398 and D399, respectively. The final realisation in D399 does not give further justification for the decision. The table was realised as an algebraic expression, the algebraic expression realised as a graph. Chapter 5 provided sufficient evidence and discussion of the application of the vertical line test pertaining to this group. Their deduction that this table did not represent a function was most likely based on using the vertical line test. There was no evidence in the data that learners could realise a function correspondence from values on a table. Similar routines to the one describes above, accounted for the aggregate of exploratory codes attributed to this group. The compression of talk in PG1 was possible through the use of symbolism and reification. These facilitated three realisations for PG1: the equation, the graph and the verification of a non-function. Without lengthy subjectifying descriptions of process and detail, the talk of PG1 was indication of the economy of an objectified discourse. Relating back to theory, this was a good illustration of the horizontal development of a discourse on function which has subsumed discourses on tables and graphs (Sfard, 2008, 2012a). It also marked the consolidation of the vertical discourse on the class of linear functions for PG1 learners (op.cit). Horizontal and vertical discourse development was discussed in chapter 2 (2.4.2). Such objectified discourse related to a table of values occurred twice for PG1 in two schools.

6.2.5 Range of Ritual Codes per Performance Group

Table 13 Range of ritual codes per performance group

Ritual Codes	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10	R11	R12	R13	R14	R15	R16	R17	R18	Repertoire of Codes
PG1																			12
PG2																			13
PG3																			11

Ritualised discourse showed a comparable range of codes across all performance groups: R14 -the emphasis on rules and practice; R16 - having difficulty following rules; R17 - imitating others, were ritual codes not present across all performance groups. There were few rules to remember for plotting coordinates and learners have, over the past two years, individualised this skill. This possibly accounted for the absence of these ritual codes. The ritual codes present, emphasised what has become very apparent in the data on tables, across the six schools: tables are used to sketch graphs. The features of the table of values, the features of the table linked to their significance on the graphical representation, a relationship between the *x*- and *y*-coordinates, appear not to have been explicitly developed as part of the school mathematical discourse. This was inferred from the unstructured, informal meta-discourse of learners as they worked with tables when they were not plotting the coordinates. Most learners did not appear to look at the ordered pairs as reification of an algebraic relationship between *x* and *y*. Coordinates marked the location of a point on the Cartesian Plane, in much the same way one would place numbers onto a Bingo board. It is worth noting that the curriculum mentions a multiple-representational approach within the different functions (DBE, 2007). It does not, however, provide teachers with guidance on how to teach for equivalence or for connection to function, the subsuming object. Its insertion into learner discourse appears to be assumed in teaching, as an almost accidental pedagogical outcome.

Further observations could be made across the performance groups in relation to the ritual codes:

- R8 - showing concern with the final answer is present in PG1 and not in the other two groups. (discussed in 6.3.4). Recall this was a code absent across groups for the algebraic expression.
- R13 - showing recognition of form, was not found in PG1 but in other groups (discussed in 6.3.2)

The observations above initiates the distinction between the different levels of performance in terms of the ritual codes.

6.3 Frequent ritual codes zoom in

In their communication about tables, learners showed significant aggregate in ritual codes across performance levels. The zoom in focuses on the nature of the ritualised talk on tables of values. There were three levels of discourse sought in the ways that learners worked with tables of values:

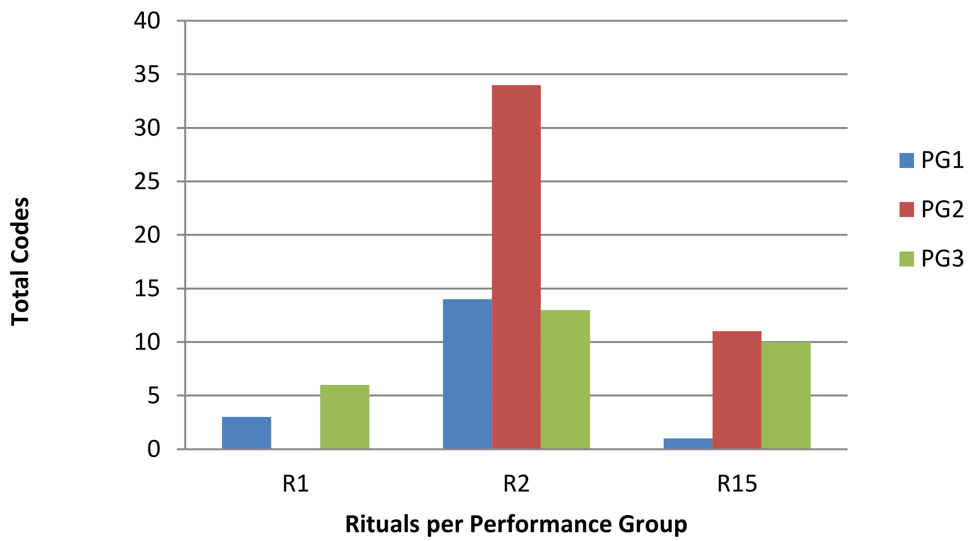
Level 1 - the plotting of points;

Level 2 - the features of the relation evident on the tables of values;

Level 3 - a generalised algebraic expression for the values shown on the table.

6.3.1 Goal of Learner Talk

Figure 27 Ritual codes for social acceptance.



Ritual codes related to Social Acceptance
• R1 - endorsed narratives come from memory or authority • R2 - adheres to goals set by others; satisfying needs outside of oneself and mathematics • R15 - guessing

The most frequent ritual R2 involving tables, across all performance groups, involved a strong need for social acceptance. The guiding rationale in assigning codes involved looking at what the learner wanted to achieve at the end of a narrative or process. The utterance earned a ritual code if the learner could show no evidence to justify or reflect on the routine used, but instead looked to the other learner in the pair or to an external source for affirmation. The extract below shows the talk between two learners in PG2, as they examine the cards showing different tables.

Extract 44 Social acceptance PG2

P-PG2-S163	*Er, the table, this is* ***what we usually do when we are doing*** *er, ...exponential functions*	Reverts to a familiar procedural cue related to the table
-R164	*And a straight line.*	Affirms Learner S and adds on
-S165	*And a straight line.*	Restates Learner R
-S166	*You draw the table before we plot...*	

The process above, while extracted from one school, illustrates broadly the ritual followed by most learners. A table signified a process and not an independent object with its own features, rules and further possible realisations. Since this appeared as the only routine available to learners, they confirmed this process with each other, as what they had to do for the exponential graph and the straight line R1. They remained at discursive Level 1, the plotting of coordinates.

The second extract shows learner talk around Card B:

B

x	1	1	1	1
y	1	2	3	4

Learners are engaged in the follow-up question, deciding if the table represents a function.

Their reasoning entails a comparison of Card B with Card N:

N

x	1	2	3	4
y	-1	-1	-1	-1

They based their decision about Card B on their earlier decision for Card N.

Extract45 Social Acceptance PG2 comparison of Card B and N

J-PG2 **-S380**	*I still think it is a function.*	Refers to Card B
-I381	*But if you say this* (Card N) *is a function, then that* (Card B) *has to be a function.*	Recycles an old routine
-S382	*Then why do you say it's not a function?*	
-I383	*Because the same reason why we said this is not a function. Because, if you look at this graph* (Card B), *it's using the same point constantly, the same x-value, and it's just going on top of it, like that, and, it's not a function. No, no, no. It's not a function.*	Refers to a sketch of Card B
-I384	*It's not a function.*	Restates decision
-S385	*I'll say it's not.*	Changes discursive position to agree with Learner I

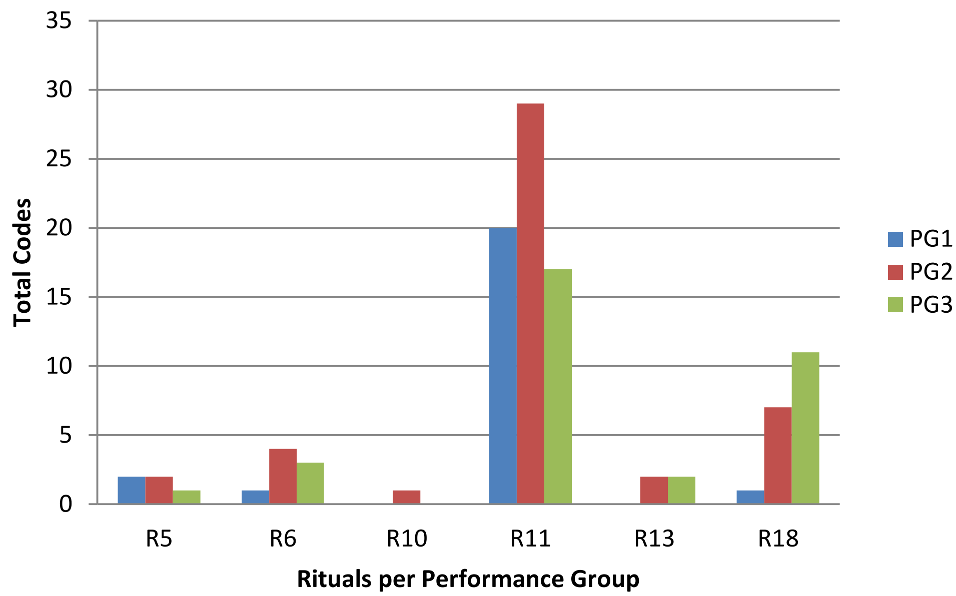
At the end of the discussion, learner S accepted the explanation offered by I383, and changed his position from S380 to S385. Learner S did not show or seek justification for decisions that were made or changed. Utterance I383 was also a good example of the informal discourse that develops around situational or visual cues when the formal discourse has not been filled in for learners. Learners appeared to have no mathematical means to realise and communicate even the most obvious differences between the two tables, Card B with constant x values and Card N with constant y values. The keyword 'constant' is not part of the observations they made yet it is an everyday word conveying the same mathematical meaning. It appears as a word less technical or specialised when compared to mathematical words like asymptotes, which these learners have been shown to use. The gist of the exchange between learners shows their reasoning justified by influences outside of the established mathematical discourse, where decisions made for one table were applicable to the other R2. They changed their mathematical statements based on what the other learner said or did R2. Critically this also shows a lack in the formal development of the discourse on the table.

Instances of guessing R15 were not seen as frequently as expected, given that learners had based the decision of whether a table represented a function on a collection of criteria based on personal experience of each of the functions studied in school. The demand of the task with tables signalled a process of plotting, which learners had as an established routine. They did not need to guess to achieve the first level of discourse expected. They showed little awareness of the other levels on which a table could be worked with, and hence could not extend complexity from what they could already do. When it was clear that a learner's justification was not isomorphic with the collection of endorsed narratives of the established discourse, such utterances were coded as guessing. The utterance Extract 46 - I383 above, intimated a one-to-many correspondence, "it's using the same point constantly, the same x-value, and it's just going on top of it"; this was from what the learner observed and not a formal mathematical narrative. It was interpreted as informal talk by the learner to describe what was seen on their graph, and not as guessing. Guessing codes were assigned mainly to the how learners decided the table represented a function of not.

With the restricted discourse learners appear to have of the table, they reached Level 1 of the mathematical goals set out at the beginning. The significantly high aggregate of R2, coupled with R15 codes, shows that they rely on each other once again, rather than the mathematics to justify what they do. The discourse involved in the relational aspects brought out on a table does not appear to have been developed with these learners. Learners' attempts at consistency in what they do were noted in this extract, as in the previous chapter. However, establishing this consistency was by non-mathematical means, see Utterance I381: "But if you say this (Card N) is a function, then that (Card B) has to be a function". The gist was a social one, referring to what a learner had done previously. There were no mathematical choices at play on decisions being made about the object. The absence of the formal definition of function once again became stark in instances such as these. Without a connection to the formal mathematical object, decisions were random and socially motivated, where the search for endorsement was social rather than mathematical. Better-performing learners PG1 showed a less ritualised discourse for social acceptance. PG2 were highest on this code. This was, again, due to their unobjectified discourse. PG3 had very little of the mathematical discourse which they could recall, and thus, appear to have sought social acceptance far less than PG2.

6.3.2 What is talked about

Figure 28 Signifiers or objects



Signifiers

- R5 - used visual cues to remember process.
- R6 - talked of symbols rather than what is signified by symbols.
- R10 - asks answers questions for clarification, affirmation. Questions are closed.
- R11 - emphasised the person performing an action or engaging a process.
- R13 - misrecognised the different representation and named representation incorrectly or not.
- R18 - used spontaneous everyday language.

Here, the talk was of signifiers as opposed to mathematical objects. Sfard (2008) has defined a signifier as an object for which there are realisation procedures. Talk coded ritual exemplified procedures and not the object. For these learners, using data from the codes outlined above, a table did not signify a relationship between two variables, but signified two ritualised routes. First, a table signified a physical structure, which arose from process cues to substitute into an equation. Second, it was a visual cue, which housed coordinates that had to be plotted. The notion that it could be an equivalent representation of a function, as a graph or expression, did not come through in the data. The table was an intermediary tool, which facilitated access to the other representations. These realisations were termed ritualised, because the discursive choices learners made were not based on a form of mathematical reasoning that connected to the object, viz. function. Learners were not able to see a function directly from the values on the table.

Overwhelmingly, code R11 dominated all other codes in this category. It was most frequent in PG2. While R11 pointed to a form of objectification, the discursive focus was on the person performing actions and not on processes or objects (see Chapter 3). This was seen in the data, where learners emphasised how they had performed their actions in phrases such as "I did..", "I took..", and "I moved...". A good illustration of this can be found in the extract shown previously, P-PG1-S163, where learner S said: "Er, the table, this is ***what we usually do when we are doing*** er... exponential functions" [sic]. A common pattern in their subjectifying was the repeated use of the word 'we' to show a co-ownership of the mathematising. The seeking of a social endorsement. Objectified talk of an exponential function as a fully fledged existing 'thing', an object described as a noun, is contrasted with the talk above of an exponential function as the subject of a process that 'we do', or 'we are doing'. The person and their actions, which usually involved the recollection of rules, showed these foregrounded, and a porous mathematical discourse with the object as a subtext emerge. This resulted in a clear disadvantage for these learners, when taking into consideration that mathematics is a discourse which develops through building on preceding layers. Layers of discourse can be built on, for successful participation, only if learners words and processes are filled with meaning, and these can be connected and combined with others to develop the succeeding layer. What appears to happen in the discourse on tables, is that learners are given the tables much in the same way that they would be given a recipe for a cake, or the technical steps involved to operate a complex machine. The plotting of points is the highest level of complexity most learners seem to achieve with the table.

R5 involved use of visual cues to remember process. The table itself, by its structure, became an obvious cue to plot. This was coded as ritualised. Recall, learners were asked what they saw on the table of values and if it represented a function to them. The spontaneous plotting of values was thus coded ritual. Visual clues provided by the table dealt with the value of zero being among either the x or y values or both. Learners in PG1 could identify these as the y- or x-intercepts of the graph before it was sketched. In this way, reaching the first level of mathematical discourse expected- the identification of features. PG2 and PG3 could not identify features of the table. The second dominant ritual of determining a function from the graph by holding a vertical line to the graph and checking for not more than one intersection between both, held for the table as well. The table, like the algebraic expression previously, was visually mediated. In short, the table showed one dominant ritual, the plotting of a graph. This secondary object became the source of further realisations.

R6 involves talk of symbols, rather than of what was signified. Evidence of this type of talk was highest among learners in PG2 and PG3. Take for example PG3, who used the word intersection which is not related or even possible to see on the Table B:

B

x	1	1	1	1
y	1	2	3	4

It was a word which probably had been used in relation to graphs learners had plotted in the past. Or, it referred to the intercepts. It was used in this context without meaning, indicative of ritualised practice:

Extract 46 Intersection.

E-PG3-L97	*'Cause it doesn't have, even an intersection, a y- intercept or an x-intercept.*	The intersection is mentioned in addition to the intercepts and was not seen when coded to indicate an intersection with the axes.

The code R6 was used to code words and symbolic representations, which were used out of context and clearly did not signify their mathematical meaning to learners. Symbolic misrepresentations were more frequent in Chapter 5. The symbolism here related to learners depicting a relation like $x = 1$ as a coordinate. Keywords associated with the table show poor development and algebraic symbolism showing a functional relationship was also largely absent.

Code R10 describes the asking of closed questions for clarification or affirmation. This occurred in PG2 only, and was not a frequent code for the table. This probably related to the entrenched ritual of learners to plot the graph from the table and to not see the table as an equivalent representation of a function, like the equation and graph, with its own features and possible deductions. An example of a question asked in PG2 related to whether the resultant graph had to be sketched to scale (T-PG2-V158).

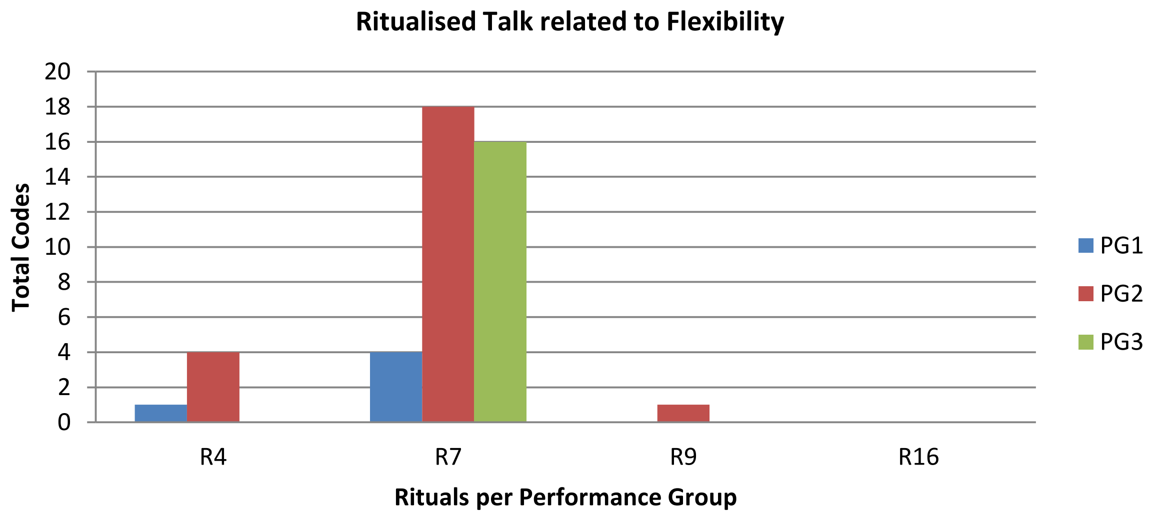
Code R13 related to the recognition of form and pertained in all instances to the resultant sketch from the table. PG2 and PG3 learners reflected on the sketch and were prompted by cues from topics that were recently discussed in school. T-PG2-V166 related Table N to a scatter plot, for example. In illustrations at the beginning of the chapter, learners joined dots to form recognisable graphs like the sine curve. This situatedness is a strong indicator in the literature of ritualised learning (Nachlieli & Tabach, 2012; Sfard, 2008).

R18 related to colloquial discourse, where communication in spontaneous, everyday language- and was the second most prevalent code in the data for PG2 and PG3. Within this code, PG3 was strongest. Learners described the graph for Table U above as a 'wave' (J-PG3-Z249). T-PG2-V166 justified joining the points plotted as they would be 'scattered' if they were not joined. P-PG3 (ML 80 and Mo 81) described 'parabolas' from their sketch of Card U as 'kind of a smile' or 'a smiley and the sad' for the shapes they saw on their sketch. As seen earlier, these descriptions related to the graph, a secondary object and not to the table from which they originated.

To summarise, learners' talk on the table showed emphasis placed on what they did with the table. Without mathematical keywords and phrases, they resorted frequently to colloquial and everyday descriptions for what they saw on the graph. The table was generally not seen as an equivalent representation of a function, but was a tool to help make this decision. The description related to objects was covered in section 6.2.2 earlier, where the objectified discourse of PG1 was discussed. This presented the only exploration routines evident for the table.

6.3.3 Flexibility

Figure29 Ritual codes for flexibility.



Rigid • R4 - different representations are regarded as separate entities. • R7 - recycles known routines/narratives. Appropriate or inappropriate. • R9 - concern with making errors or emphasis on avoiding them. • R16 - difficulty in following rules.

Flexibility related to the varied ways that learners may have responded in a given situation. It was difficult, though interesting, to try to explain why learners did not communicate in ways that were expected. R16, which emphasised rules and thus restricted the available responses from learners, was not evident in the data. This probably related to very few formal mathematical rules being available to them in the interpretation of a functional relationship or the realisable features of the functional relationship from the table. Learner talk did not need to emphasise the following of rules when engaging the learners' procedural approach to plotting a graph. R9 arose from learner talk on sketching graphs to scale. This was the only instance in which learners expressed the need for accuracy in the sketching of a graph from a table. This indicated that the pair remained entrenched in the ritual associated with a table from two years prior.

The most frequent ritual in this category was R7, related to recycling old narratives and routines. PG2 and PG3 dominated this code. The distinction between R7 and R16 was that R16 looked for statement of and difficulty in application of formal mathematical rules that were endorsed. R7 looked to learners referring to previous ways of working or processes and applying these in a particular instance. The following observations were made: all tables were plotted to be realised further as a graph; the coordinates were joined because that is what was always done in the classroom; in most cases the resulting straight line, even the vertical, were functions because the straight line was known to be a function; and all functions recognised from graphs were those studied at school as part of the topic, viz. functions. Typical utterances are shown below.

Extract 47 Recycling old routines.

E-PG1 -G139	*I did it, I did it in the way that I learn it. I couldn't really tell, and when I see it here, it's like, oh wait, I remember a graph.*	After plotting Table N of a straight line $y = -1$
-O140	*From Grade 10 stuff.*	Relates to previous routines

The processual- emphasising talk of PG2 can be seen in the extract showing a frequent routine.

Extract48 Process rituals.

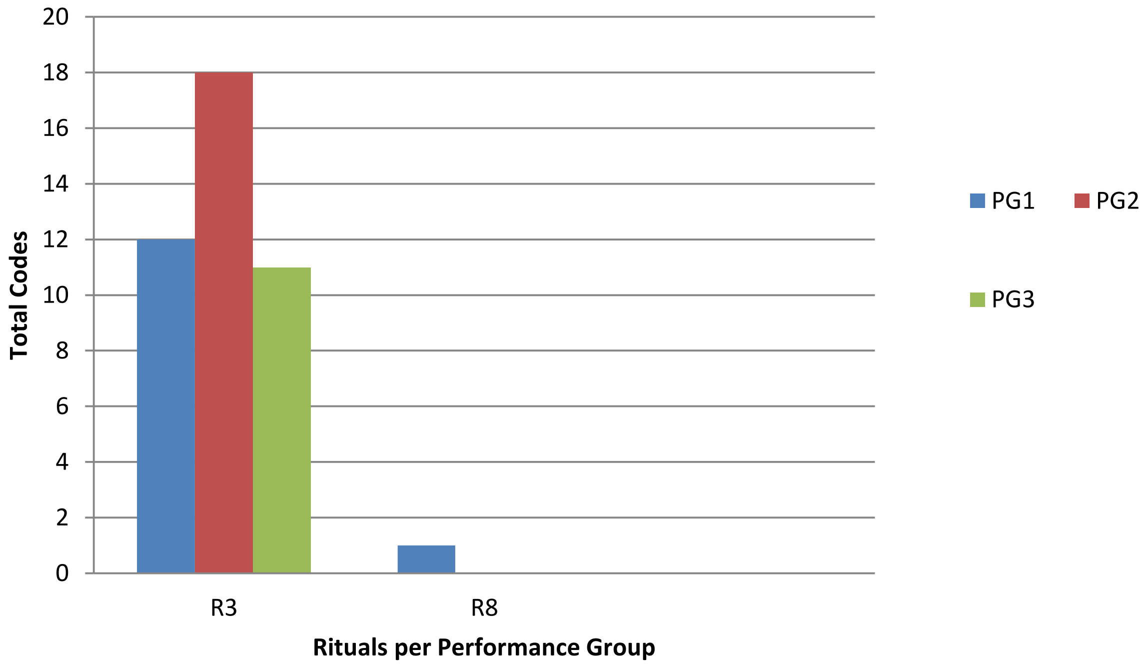
T-PG2-154	*You see, in Card B, uh, we think there is an equation behind here, so to get this equation, to get, ah, no. Ja, you see, this one, what makes it an equatio… ah, ja, I think you can plot*	
-J155	*That graph, because you have the coordinates.*	
-V156	*You have coordinates.*	

R4 dealt with learners' attempts to connect representations. PG1 and PG2 showed this coded presence. This does not mean that PG3 were able to connect representations successfully and see equivalence among them. They did not provide a discourse that could be coded as this ritual, whereas PG1 and PG2 showed that there were instances where they regarded representations as independent of one another. All learners from all performance groups did not see a functional relationship directly from the values on the table, but from the sketch of the graph. This was the most direct way of concluding that learners did not see equivalence of the table and graphical representations. Yet again, this can be attributed to the poor formal development of the discourse of tables, and a poor connecting discourse between the two representations.

An overall conclusion for this code is that learners do not have a repertoire of routines with which to communicate about tables. This was evident in low aggregate of codes for this category. The ritualised talk about tables revolved around using tables as tools to sketch graphs. The realisations that discern features of the table do not occur, and subsequent deductions of the properties of the functions' other representations happened in isolation. The realisations possible from tables were limited, as were the routines learners were required to draw upon for a table of values. Flexibility was restricted, as most learners appear limited to just a single routine.

6.3.4 Applicability

Figure 30 Ritual codes for applicability



Narrow
• R3 - statements about actions and process. Remains in process (not reified) - no reflection on meaning at the end of process-spontaneous or provoked. • R8 - concern with the final solution. Proof/verification through a specific instances only.

Learner routines regarding tables show narrow applicability. This is deduced from a dominance of talk that focused on the actions they performed or the processes they invoked. The added dimension of reflecting on an outcome was frequently absent for all groups, where talk centred on process. This is a strong indicator that the table as an object, together with the processes involved in working with it, had not been reified for learners. Narrow applicability can be related to the sequential nature of the curriculum that does not allow for the parallel development of other related discourses. Chapter 5 observed that learners did not have a global approach to the algebraic symbolic form, and the same can be said for tables, where such an approach would have helped learners reify the table and its features, increasing the applicability of their routines.

Code R3 examined instances of process in learner talk. The most frequent phrase used by learners referred to plotting the graph from the table. There were 15 such utterances. A selection of utterances from each performance group is shown below:

Extract 49 Emphasis on Process PG1.

E-PG1-133	*Eh, I can formulate a, an equation, yes an equation out of this thing. Which is, even* ***if I can plot*** *this things, I'll get a vertical, a straight vertical line that is parallel to the x-axis.*	
-G134	***If I plot this****, this um, point. Oh...!*	
-G135	***If I plot*** *this point, I'll get a straight line, that is parallel to the x axis.*	
-G136	*Okay. He's saying i****f he can plot*** *these co-ordinates on a graph…*	
-O137	*Ok. He's saying,* ***if he can plot*** *these, this is one, two, three, four. .... Okay, he's saying, this is what he's going to get.*	

Extract 50 Emphasis on Process PG2.

T-PG2-J249	*Uh, ...having that I* ***can plot****, uh, a graph using those coordinates, in the Cartesian plane.*	

Extract 51 Emphasis on Process PG3.

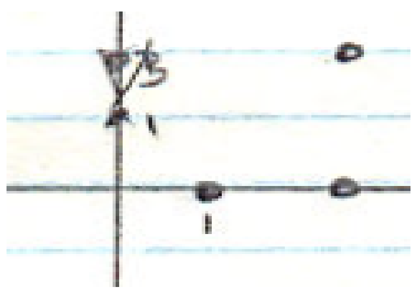
J-PG3-I331	*'Cause we saw that it has the x-values and y- values, so once you plot them…*	

These extracts are purposively chosen from different schools and from the three performance groups such that the emphasis on using a table to plot a graph comes into view. The focus was on doing something, where, thereafter the talk focuses on the features of the graph. This was prevalent across all performance levels.

R8 reveals insight into talk that indicates a drive for closure. This was evident only in PG1. The drive shown in the extract below, was to plot a graph from a table (the prompt) so that the vertical line test could be applied and the conclusion, whether a function or not, would represent the closing condition.

Extract 52 Drive for closure.

<table>
<tr>
<td>E-PG1-G162</td>
<td>Ja, because, ja. It'd end up being like this.
Er, I take these two points, 1 and 1 and 3…
it's 1, and it's 1, and it's 3...
It'd end up being like this.
And if I put a vertical line, then I end up having two points.

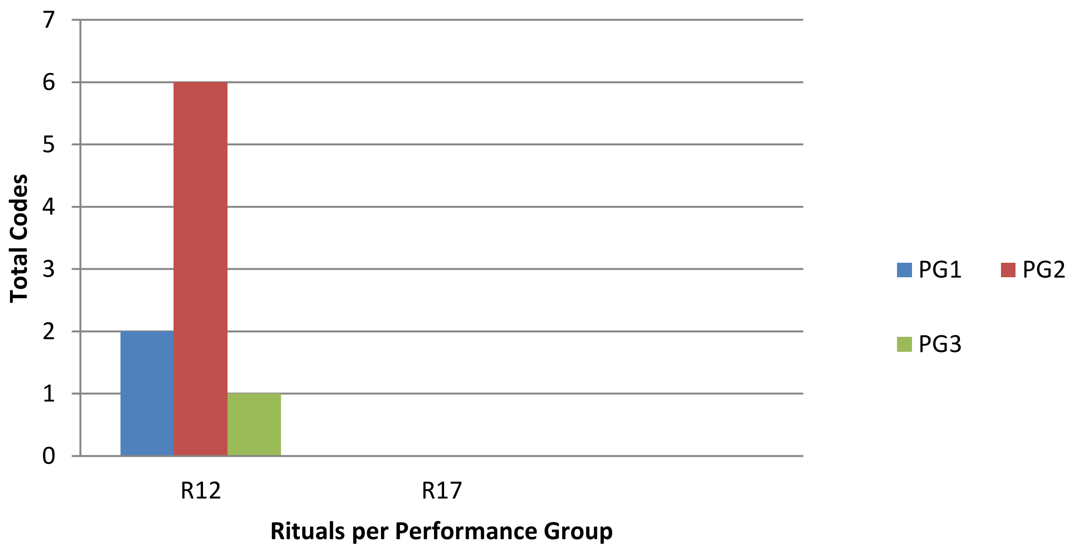

U
<table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>y</td><td>0</td><td>3</td><td>0</td><td>3</td></tr></table></td>
<td>Card U is discussed. Learner G had deduced earlier that Card U represented a function. The utterance shows a response to Learner O, asking for explanation of the decision.</td>
</tr>
</table>

The boldness of the statement "it'd **end up being like this**", indicated concern for reaching a conclusion. It also highlighted the disadvantage of reliance on a particular representation to answer all questions. If the graph was incorrectly sketched, it could force learners into incorrect conclusions, where connecting discourse between the features of the table and the graph would have provided a means for learners to verify the conclusions they had made.

Narrow applicability was strongly marked by the ritualised dependence on sketching graphs by all groups of learners. The narrowness of this applicability cannot be attributed to learners alone, as they consistently showed that the specialised discourse on tables has not been fully developed for them. Like the discourse on graphs, the narrow applicability of learner routines emerged as a disconnection between these representations and the discourse on function. Developing discourse on tables would expose the relationship between variables and co-variation, which appears absent in current learner discourse.

6.3.5 Addressees

Figure 31 Rituals showing who is addressed.



Others

- R12 - routines and answers are not questioned or justified
- R17 - imitates the other person, keeps pace with the other

The discourse related to R12 showed the need for social acceptance and affirmation. The closing of a procedure without justification and reflection, with link to a mathematical form of endorsement, indicated that the discourse was more for social acceptance than mathematical reasoning. Learners were seen to agree or disagree with the other learner, without question or the need for substantiation. There appeared an easy acceptance of the others' narrative. This ritualised behaviour was highest for PG2. The talk in code R12 revolved around learners stating a table was a function, and not providing justification for this response. Not providing justification was particularly prevalent in most PG3 utterances.

R17 was absent, possibly because the routines in dealing with tables did not show a wide range, and learners generally agreed with the sketching the graph. This was the common accepted routine across all groups, hence the absence of this code.

The discourse around tables therefore addresses the other learner in the pair and appears motivated by the drive to provide a response to the questions asked. These were seldom motivated or linked to a mathematical reasoning.

6.3.6 Reasons for acceptance

Figure32 Reasons for acceptance.

Followed Rules
• R14 - emphasises following rules and the importance of practice.

The figure shows code R14 did not feature in the aggregates. The options or realisations from the table were very limited for these learners, where these were governed by process and not guided by rules. Establishing the rule governing the values on the table would possibly have expected learners to follow some algebraic rules. However, this level of discourse was not present in the data.

6.4 Conclusion

The routines involving tables were restricted when compared to the algebraic expression, where learners had an expansive repertoire of what they could say of, and do with the algebraic symbols. The table signified a process of plotting points which were joined. Learners, in the main, used tables to plot graphs from which they could identify features and the functional relationship. Like in the algebraic expression, the table routines showed an absence of certain keywords connected to functions represented by a set of values, like domain, range, maximum, minimum, increasing or decreasing. If these were present, they might have contributed to the connection of the table to other signifiers. To illustrate, the domain is easily seen on a table and transfers to building talk about the domain of the graph, which was seen as problematic on NSC scripts in the preliminary analysis initiating this study. This again builds progressively to the algebraic verification of domain from the algebraic expression. *Where would this expression be defined*, would be a typical query. This suggests a higher discursive level, and would require a complete algebraic verification, where the two prior realisations, from the table and expression, could have been mediated visually. Algebraic verification would require learners to reason using algebraic symbols. This serves to illustrate the importance of building learner discourse around keywords and their meaning.

Another critical absence in learner talk showed that learners did not deduce a relationship between the values on the table. They had not reified the process depicted by the coordinates. Values were not discussed as a correspondence or mapping either. Hence, the notion of covariance was not apparent, and yet is critical for the formation of the object.

PG1 showed the only objectified talk in identifying features of the table, which contributed to their exploration codes on the table. They could identify the *y* or *x* intercepts of the graph from the table. The frames for exploration codes, developed in Chapter 5, showed in identification for the table signifier. PG2 and PG3 had not reified the processes embedded in the table. This was confirmed in all NSC examination reports as something learners find difficult. The entrenched pointwise orientation appears to be rigidly established, and learners from all performance groups show dependence on it for working with tables. It is clear from the little learners could say and do with the table that they did not see its equivalence with the algebraic expression. It was used primarily as a tool, except for better-performing learners, who showed instances of objectification in identification. Discourse of the table is poorly developed across performance groups, and contributes to the disconnection from the encapsulating object.

Chapter 7: Graphs

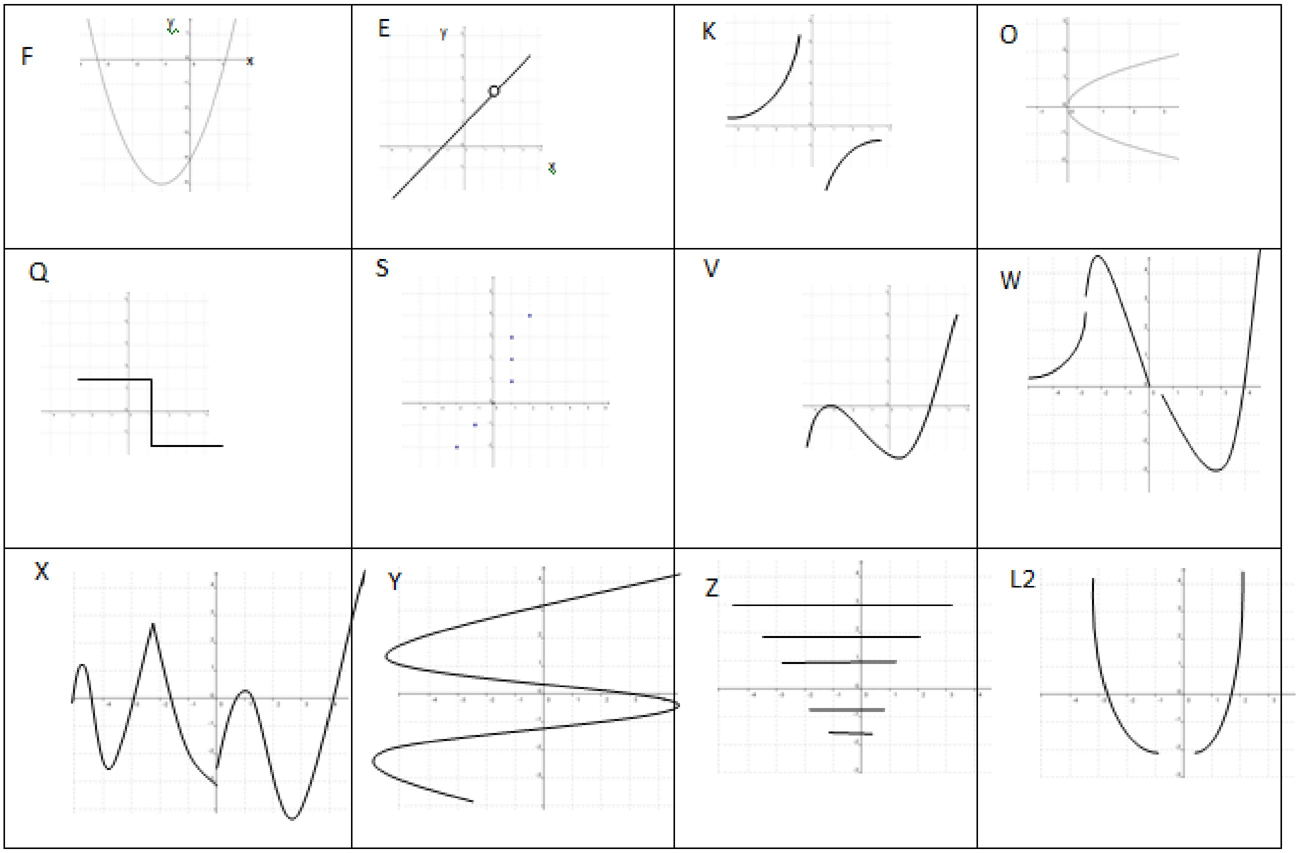
7.1 Introduction

In this, the third and final signifier being analysed, learner discourse on the graphical signifier is explored for learner routines as they work with graphs. Communicating in objectified ways usually marked a learners attempt at exploration. This chapter has the added task of providing a global view of learner routines of all representations discussed in Chapters 5 and 6. It will thus have to present an encapsulated discursive view of how the object function has come to be formed for learners. There are two questions which guided the chapter development: what is the nature of learner discourses of the graphical representation? Does the discourse of the graphical representation connect with other representations to form a unified and objectified notion of function? As in previous chapters, these questions are guided by the broader research questions of the study, reiterated here:

1. What are the characteristic features of learner discursive routines at each of the performance levels?
2. How do these features contribute to the mathematical object function that comes to be formed for learners at each of the performance levels? What future realisations do they make possible?
3. How can the object that exists for learners be described at each performance level?

The following graphs were available to learners on the task. Learners were asked what they saw on the card, and if what they saw represented a function.

Extract 53 Graphs on the paired interview.



The cards held the following design features to encourage learners to talk about them:

- graphs which were familiar from school mathematics;
- graphs which were unfamiliar;
- specific features like discontinuities; and
- a selection of function and non-function relationships.

This chapter will show that talk of the graphical signifier appears to be more objectified in comparison to the algebraic expression and table. It did not invoke recall of a process as automatically as the latter two signifiers. Perhaps through its visual presentation, learners are able to identify and talk of the graph through its specific, visible features. The current chapter has progressively revealed the way in which learners identify and describe functions and their properties or features in objectified ways from the graph. This, again, was in the context of what learners say or do with a graph when they do not know exactly know what a function is.

7.2 Frequent routines zoom out: rituals and exploration

This subsection provides a zoom out, a description of the prevalent routines across performance groups. It provides the initial description of routines across performance groups. The broad routines learners used when they worked with the graphs shown in the prior extract are summarised on table 14 below, for each performance group:

Table 14 Summary of routines.

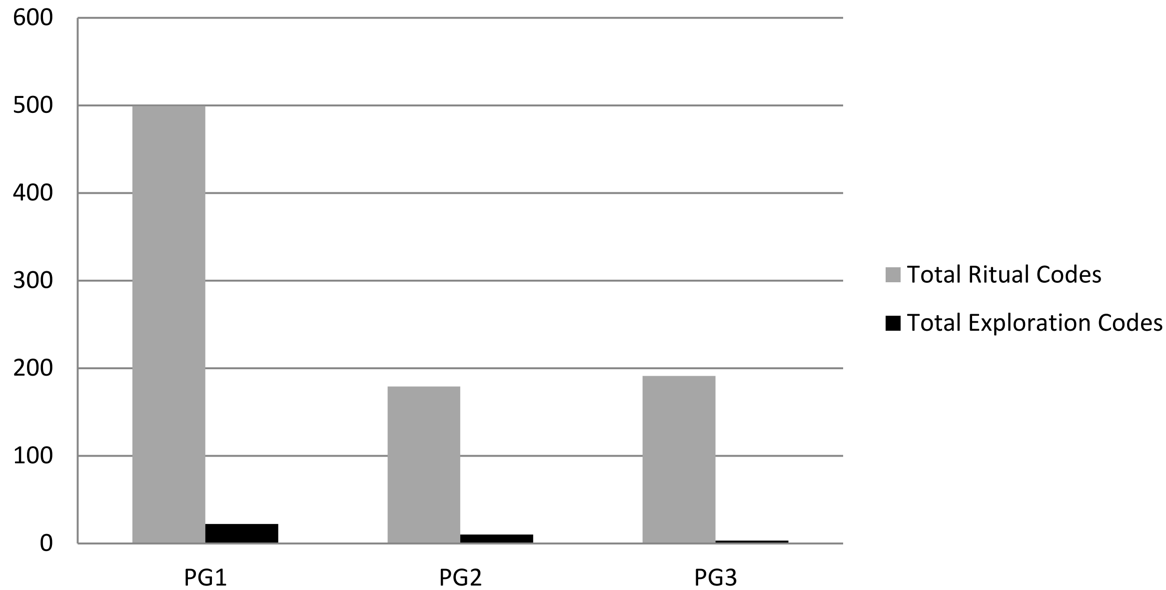
Graph Routines		
PG1	**PG2**	**PG3**
• Identification of graph (studied in the classroom) and their features in objectified ways. • Extends these features to graphs not seen before. • Show the significance of symbols as they relate to features of the graph. • Use keywords applicable to the graph in ways that convey their meaning and significance. • Can see equivalence between the graphical and algebraic expression representations. • Discuss the transformation of graphs in connection to their algebraic transformation.	• Identification of graph (studied in the classroom) and their features in objectified ways. • Extend this in limited instances to graphs not seen before. • Show the significance of symbols as they relate to features of the linear, parabolic, hyperbolic functions. Inconsistently. • Use keywords applicable to the graph in ways that convey their meaning and significance in limited instances. Often in passive and associative ways. • Attempts to relate the graph to its defining expression. • Do not filter what is required by the task. List all associations they can make. • Strong subjectification and focus on process.	• Identification of graph (studied in the classroom) and their features in objectified ways. This is limited compared to PG2. • Frequent ambiguous referents. • Incomplete sentences. • Little or no evidence of justification. • Few routines, which are recycled. • Frequent use of spontaneous/everyday labels. • Symbols in notation are empty. • Strong subjectification.

• Engage in longer communication chains with each other. • Attempt to generalise and build complex ideas. • Seeks mathematical justification for deductions made. • Wide repertoire of routines.		

These routines were broadly characterised as exploration and ritual to begin the picture of what the graph signified to learners.

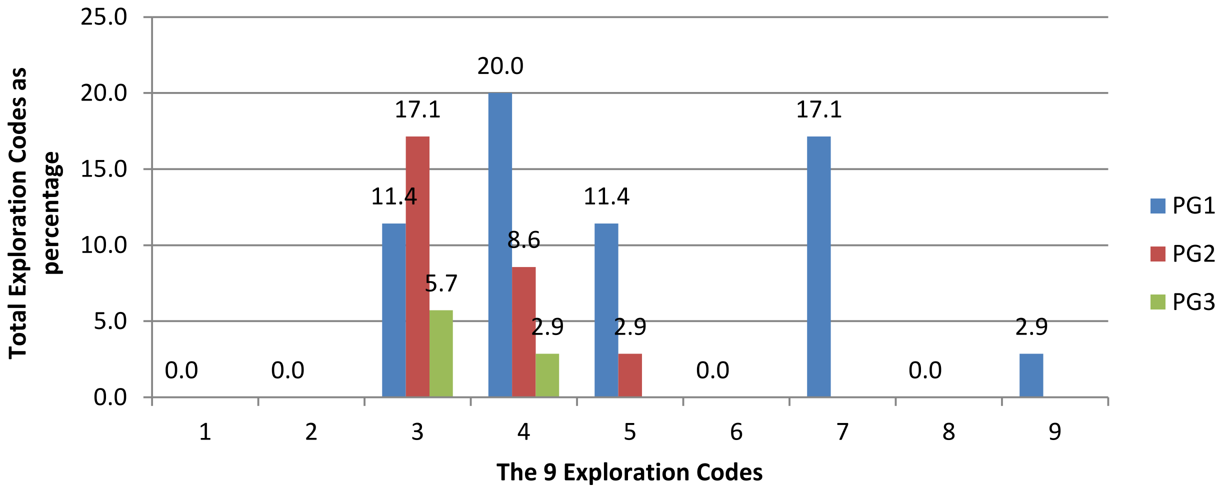
7.2.1 Findings Overall

Figure 33 Graph routines: exploration and ritual across performance groups



i. Explorations. The figure above shows learner communication was again in the main ritualised, with 35 out of 869 codes showing exploration. Interestingly and unexpectedly, compared to utterances on tables in the previous chapter, PG3 emerged with exploration codes on the graphical representation. This was unexpected, and will be discussed later. As expected, the widest range of exploratory utterances occurred in PG1: E3, with talk of specific features of the graph; E4, with talk that connected the graph to other representations; E5 with talk where symbols signified entities; E7 with learners having asked and attempted to answer open questions; E9 talk showing the move from process to object. PG1 held 22 of the total 35 exploration codes assigned across all performance groups in the graph activity. The range of exploration codes per performance group and their percentages are shown below.

Figure 34 The range of exploration codes per performance group.



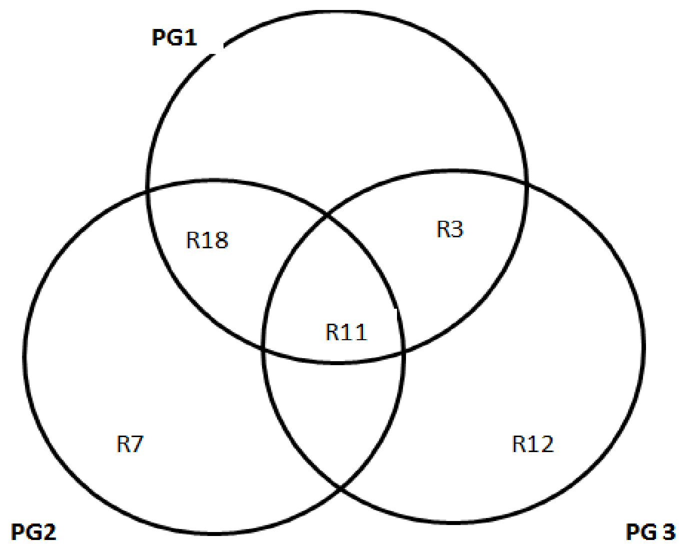
The figure shows exploration code E3 and E4 occurred in all performance groups. E3 occurred with the greatest frequency across all performance groups; E4 was the only exploration code, where the aggregate of PG2 exceeded PG1; exploration codes E7 and E9 occurred in PG1 only; and E1, E2, E6, E8 were not evident in learner utterances.

The frequency of E3 and E4 shows that learners spoke of features of the graph in objectified ways. On the graphical representation, these features were visually mediated and learners were thus able to comment on the features as entities because they could see them. Given a physical object, learners are able to describe the various features it possesses (Slavit, 1997). To explain the connection between representations E4, the connection between the algebraic expression, table and the graph, has already been established in Chapter 5 and 6. With limited objectification observed on the algebraic expression and practically none on the table, the graph appears a little more encouraging especially for learners who struggle in mathematics.

Identification was again strong for this signifier; the distinction being, in this chapter, that learners were able to identify far more graphs from their features than for the algebraic expression. Graphs studied in school mathematics appeared easier for learners to talk about, where they were able to connect the linear graph to its algebraic representation more often and with ease compared to other graphs. E4 represented the highest code aggregate for the graphical representation, and was the most frequent type of exploration in PG1. PG2 and PG3, in contrast, spoke of the specific features in objectified ways before they connected representations. In relating the graph to other representations, learners also related it to its identifying expression and the general expression of that family of graphs. The specific identifying expression of the non-standard type graphs posed a challenge. Learners could recall the formula for the general expression of the parabola for instance, but finding the specific parameters related to the graph, from its identifying critical features, proved difficult for learners, and was seldom attempted.

ii. Ritual. It was interesting to observe that the most frequent ritual code, across all performance groups, was R11, which relates to subjectification (talk that emphasises the performer of actions rather than the actions or the mathematical object). This was a frequent code for the previous two signifiers as well. The three most frequent routine codes, per performance group, are presented in the Venn diagram below:

Figure 35 Venn diagram showing the three most frequent ritual codes per performance group.



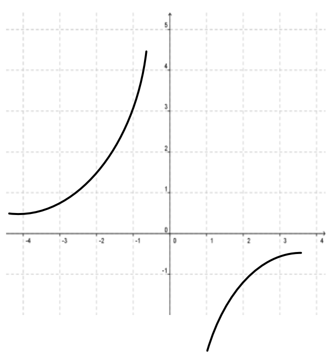
Code	Ritual Talk related to	Code Category Key
R11	Subjectification	What is talked about
R18	The spontaneous/everyday	What is talked about
R3	Actions and manipulations	Applicability
R7	Recycling old routines	Flexibility
R12	Routines and narratives that are not questioned	Addressees

Subjectification was again significant across all groups, where it showed learners inserted description of their actions in the processes they executed, or they asked or responded to calls for assistance from each other. It was a strong confirmation of the hypothesis that learners were oriented to communicating a process or to each other, rather than talking about the objects with which they were working. This ritualised talk appeared to obscure the object, and the focus frequently veered off mathematical content to social interaction. The frequent codes showed occupation of 5 of the 6 categories of ritual codes, emphasising an overtly ritualised practice. The examination of the different performance groups can now follow.

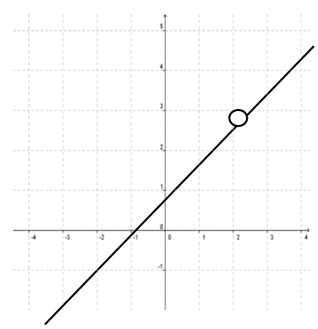
7.2.2 PG3

There were three exploration codes aggregated in PG3 learner utterances for codes E3 and E4. These involved talk around specific features of mathematical objects and talk showing attempts to connect the graph to its other representations. Recall that the connection to the algebraic expression did not occur when these PG3 learners worked with tables in Chapter 6. The exploration codes arose in talk around the two cards shown below:

Card K:



Card E:



As in previous chapters, learners were asked what they saw on the cards, and required to discuss whether or not this represented a function. The exploratory utterance coded E3 involved Card K (talk of specific features of mathematical objects).

Extract 54 Exploration in PG3.

E-PG3-M117	*Er... it is a function, whereby we have two…*	Justification for function left incomplete. Special mathematical words not evident.
-M118	*…which is a hyperbola.* *Yes, it's a hyperbola.*	**Coded E3** when related to prior utterance. The object is related to the two (parts) of the graph.
-M119	*Having it in the second quadrant and the fourth quadrant...* *which is a negative one.*	Utterance omits special mathematical word. Relates to the sign of the coefficient *a* in the general equation $y = \frac{a}{x}$ in relation to the position of the hyperbola. **Coded E4**

This extract aptly shows the dynamic nature of coding discourse. The first aspect to note is the limitation imposed in such cases where phrases from utterances are regarded in isolation. M118 above, taken in isolation, appears as a fully objectified utterance, and would be coded E1 (the graph signifies an abstract mathematical object that is a function and a hyperbola). However, it must be noted that the discourse ought to be examined in context of the total extract and with the multiple links to what learners drew on as they progressed through the activity. The result was that transcripts may not present chronologically, as linked utterances are placed next to what was being talked about. Often learners referred backwards to a function already discussed, connecting the present card with another they had already seen, as they remembered something for the function they were currently dealing with. Relating utterances in this way provided a fuller picture of what learners were thinking about the specific object. The extract above shows learners identify the graph as an object, an entity, which has specific features. Objectified talk is evident in the way they are shown to have spoken of features of the graph and relate these to the symbols in the algebraic representation. Talk around Card E confirmed that this pair of learners is able to discuss the specific features of mathematical objects.

Extract 55 Talk of function properties.

E-PG3-M181	*Yah, it's a function. This one is definitely a function.*	Correctly identifies the Graph E as a function without justification.
-M182	*Looking at the, um, the graph.* *The slope, um, it's positive, you can be able to find the gradient, and you know the gradient... to be positive.*	Subjectification rituals are frequent. Exploration was coded in the talk of the specific features of the mathematical object. **Coded E3**

The extract above shows talk of the linear function and its features. It is worth noting that both the linear and hyperbolic functions are discussed in detail, with respect to their features and behaviour in transformation, multiple times in the transcripts. However, in this case, the pair of learners avoids discussing the discontinuity. Functions are typically introduced in the school curriculum through specific function types (Carlson, et al., 2008). The discussion by PG3 learners, pertaining to graphs that were 'unconventional', that is, not specified in the curriculum, did not show additional exploration codes in PG3.

Of their aggregate of 191 codes, PG3 had 64 utterances, which were *incomplete or ambiguous*. These related to learners being unable to recall narratives, and not having the appropriate mathematical words to describe features or processes or properties of a function, as in M117 above. Learner M paused after the word 'two', unable to describe or name the separate parts of the hyperbolic graph on Card K. An interesting implication of this absence of specialised/ keywords in learner discourse was seen in ambiguous utterances containing indeterminate pronouns like M119, "having ***it*** in the second and fourth quadrant" and "which is *a negative **one***". The "it" in the first utterance pertains to the parts of the hyperbola and their position on the Cartesian plane. Learners identified the parts of the graph as 'it'. The second selected utterance, "which is *a negative **one***" reveals that ambiguity results in vagueness about the object. Is the hyperbola identified as "a negative one", by virtue of its position, in the second and fourth quadrants? Or, is this negative related to the parameter "***a***", in the general equation of the hyperbola? The task is to interpret this consistently across transcripts as a researcher, but more importantly, to note the implication of this restricted communication between learners and the impact the ambiguous references can have on learning. How the other learner in the pair interpreted this, would have been interesting to uncover. Such ambiguity, involving the absence of keywords, was not questioned by learners in PG3 for clarity, as these learners worked with whatever they thought the ambiguity to mean.

The three frequent ritual codes in PG3 were R11 (talk related to subjectification), R12 (talk related to routines or narratives that are not questioned or justified) and R3 (talk related to actions and manipulations). Overwhelmingly, subjectifying utterances revolved around:

i. learners focused on how they would perform procedures:

Extract 56 Procedure

J-PG3-S163	*Like, something, like this, no, hai, it's not ma'am, 'cause it's something like this where we, you plot here, plot there, plot there, plot ja, there, and there. Yes, ma'am. So, it's… not… I... think not.*	Note the emphasis on the performer of the action and the action.

ii. what they could recall from previous experience:

Extract 57 Recall

J-PG3–S130	*Ay, then ma'am I... I've never seen such, I don't wanna lie.*	Never having seen a graph before was a frequent criteria to discern a non-function.

iii. ambivalence and uncertainty

Extract 58 Uncertainty

E-PG3–L160	*I can say it's a function, but it's my first time seeing this graph.*	The subjectifying talk makes learners decisions seem tentative.
-L161	*Because the vertical line test will cut once, that's the reason why I'm saying it's a function.*	
-M163	*Yah, I think she's right, but I'm not quite sure.*	Routine is not questioned.

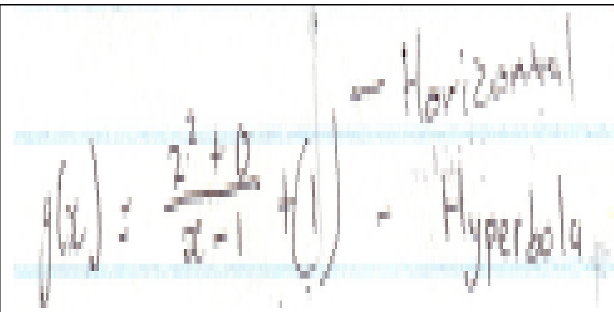
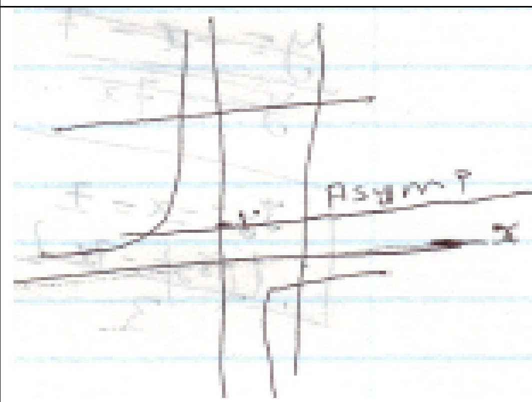
These examples of subjectifying take the emphasis off the mathematics and place the focus on the person and the actions they perform. Learners appear to have prefaced utterances with expressions such as, "I think", "I can say", "I don't remember doing", "it looks like" etc. Such utterances caused learners to appear to have been unsure and tentative about their mathematics, even if these utterances were mathematically correct (see iii above). The mathematics PG3 communicated was fluid and changeable. This was seen in instances where:

- the vertical line test, could interchangeably be applied as a horizontal line test, depending on the orientation of the graph e.g. E-PG3-L176
- *f(x)* notation signified parabolas and *g(x)* straight lines to learners e.g. P-PG3-Mo135.

7.2.3 PG2

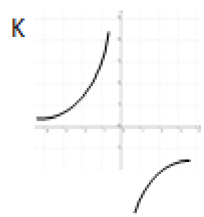
There were 10 exploration codes assigned to PG2. Their range of codes was similar to PG3 (E3 and E4) with the addition of code for E5 (talk where symbols signify entities). E3 was evident, in the use of mathematical words, which were used to describe the graphs. Like PG1, they used mathematical words such as *asymptote* and *undefined* correctly and in context, when compared to PG3. PG3 appeared to use such words in an associative context only, meaning that learners had a repertoire of keywords that they associated with the particular graph or feature. 'Asymptote' was one such word. It was used in most contexts in which hyperbola was seen or mentioned, whether it was relevant or not. This showed an association of the particular function or feature with this defining word. Usually, there were no utterances to convey meaning of such mathematical words. Discursive research terms this 'passive' use of the word (Nachlieli & Tabach, 2012), and learners used the word because they had heard it before in particular contexts and as part of particular phrases. The meaning of the word ‘asymptote’ and its associated routines involving the graphical representation were not evident in PG3. By way of contrast, PG2 attempted to communicate meaning of such words, for example, asymptote was referred to as “a line the graph will not touch”. Talk was found which extended this by relating it to *x* values or the domain where the graph would be undefined. PG2 explained this in relation to the graph and referred to the asymptote in the symbols of the algebraic representation. Such instances, served to describe how they moved between representations, from graph to equation. The selected example from J-PG2 shows this:

Extract 59 Linking signifiers.

-S235	*We have the x asymptote and we have the y asymptote, I mean, ja.* *One is horizontal and the other is vertical…*	
	…so if you look at this equation over here, this is the horizontal asymptote, this one here… (points to the graph) *…so you come to one, add um, no, actually one would be here...* (inaudible) *…and you cut through one...* (points to the graph) *…then you call that a horizontal asymptote.*	

The talk in the extract which follows showed instances of connecting representations E4, the graph and algebraic expression, as well as features of the graph E3, where these were linked to specific symbols in the expression E5. The algebraic representation offered was not a correct estimation of Card K or of the sketch. The talk arose unprompted in connection with Card K. It contained appropriate keywords, which were related to the features of the hyperbolic graph. There were three instances in which PG2 learners suggested an equation for Graph K. In all cases, it was a hyperbolic expression for Card K:

Extract 60 Connecting Signifiers

		
T-PG2- V351 **-J352**	*This one can't touch zero due to...* *...the equation.*	$y = \frac{2}{4+x^2}$
E-PG2- S235	*...so if you look at this equation over here, this is the horizontal asymptote, this one here. So you come to one, ad um, no, actually one would be here…*	$g(x) = \frac{x^2+2}{x-1} + (1)$ – Horizontal - Hyperbola
J-PG2- S260	*...Ja, uh, this will determine whether the asymptote, since it's a positive, it will be two units up from the original thingy, then which means, uh, the asymptote will be on, would lie on, lie on positive two…*	$\frac{2}{x} + 3$

While PG2 learners were able to represent the form of the hyperbolic graph algebraically, they were not entirely correct in representing the parameter *a*, which, in each instance, could be

-2 rather than 2. What this highlighted was the strength of the methodology and analytic framework of this study, which provided the means to accurately examine the links that learners made in their mathematical discourse in detail. Conventional tests and examinations, which evaluate learners, are not able in such substantive terms, to investigate the connections that learners make. Learners used mathematical words and symbols correctly, and their talk showed the potential to extend this mathematical discourse to future and more complex realisations. E-PG2-S235 above provides comprehensive detail of what the graph signified to learners. This can be compared to the frequently incomplete sentences in PG3, who could not recall relevant keywords.

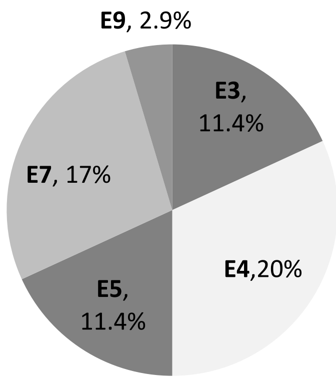
Through examining their written responses, it can be seen how learners have associated the graph to algebraic symbols and the established meanings of these symbols. It was clear that within the compartment, PG2 learners call ‘hyperbola’, existed a recognisable graph, with recognisable features, described using key mathematical words, but also the symbolic representation to which learners could connect the visible features of the graph. They, however, remain passive (Nachlieli & Tabach, 2012) in their word use, as the symbolic representations they listed, were not correct representations or estimations of what the graph is. ‘Passive’, shows that learners used these associated hyperbola keywords in ways that they have heard them used before.

The routines that would assist in the development of a symbolic representation of the function, hyperbola, appear not to be adequately established. Routine-driven discourse on graphs would show that learners have the ability to recognise and perform a restricted collection of routines, which are associated with the word, hyperbola (Nachlieli & Tabach, 2012). Additionally, PG2 learners appear still to be at the passive discursive stage, as they do not reflect on their responses or seem able to correct each other’s responses. To support this description further, learners show that they are able to make at most a single realisation from the graphical representation.

7.2.4 PG1

The exploration codes evident in PG1 discourse in graphs, comprised 62,9% of the total exploration codes shown.

Figure 36 Percentages of exploration codes in PG1 for graphs.



E3 - talk of specific features of mathematical objects

E4 - talk that connects representations

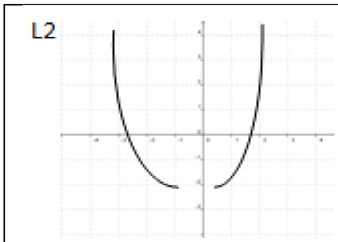
E5 - talk where symbols signify entities

E7 - the asking and (attempted) answering of open questions

E9 - talk that moves from process to object

Compared to PG2 and PG3, PG1 learners were able to ask and attempted to answer open questions E7, which came up as their second highest aggregate of codes. They were also able to move talk from process to object, E9. Both these codes were absent in PG2 and PG3. The contrast of PG1 with PG2 talk in Figure 34 provides a way in which to examine the asking of open questions on Card L2.

Extract 59 Open questions.



J-PG1-		**T-PG2-**	
D908	*Does the function go on for infinity, or what?*	V472	*Because, because of here.* (points to the discontuity) *Ja, this one...* *It doesn't.. the* ***graph****.. you <u>can't define</u>* ***it****.* *You can say it's a ...****hyperbola****, ja.. er... hyperbola cause it doesn't meet here.*

G909	*Yes.*		*(the discussion ceases here)*
D910	*These ones don't go on for infinity.* *(at the discontinuity)*		
D919	*Space!... space… space.* *So if you say it goes on for infinity, that means there shouldn't be any spaces…*		
	(The discussion continues to G1045)		

The extract above contrasts the ways that learners, in two performance groups, attempt to make sense of the discontinuity, without having the mathematical jargon to refer to it. Based on the separate parts of the hyperbola, and despite the shape looking more like that of a parabola, PG2 relate the Card L2 to the hyperbola. The discontinuity, the shape and arms of the graph, were critical features they noticed; which cued a link to the hyperbola. Learners from both groups know the graph is not defined at the discontinuity. The orientation of graph L2 being different to that of a hyperbola, is not communicated. PG1, in contrast, discussed the endpoints of the discontinuity or 'space' between the arms and this further prompted the discussion of infinity. They successfully identified the arms of the graph, which continue to infinity; discussed restrictions pertaining to the equation as a result of the discontinuity; discussed transformations of the graph and possible changes of parameter; speculated about the possible outcomes of transformation on both a parabola and hyperbola; and finally, discussed if the graph represented a function or not. This represents a string of seven realisations for PG1.

What also stood out in the PG1 exchange was that learners were able to communicate and build on from each others' utterances as they interacted. Often, this was prompted by a question as in D908 above. PG2 talk, in contrast, patterned an almost one-way dialogue, between a speaker and a passive listener. Learners alternated these roles, not building on previous utterances. PG2 asked fewer questions of each other and these were generally closed. PG1 showed a wider repertoire of mathematics words and routines and showed the dynamism of mathematics as a single realisation becomes the spark of future realisations through questions, narratives and routines.

To conclude, observations of the three performance groups, on various graphs, PG2 and PG3 learners are restricted by the absence of specialised mathematical word or phrases, which could help build richer descriptions and explanations aligned to those endorsed by the school mathematical community. Poorer learners were seen to communicate more frequently in indeterminate pronouns and prepositions as a result. PG2 above showed the ambiguous referents: “here”, “this”, and “it”. These obstruct clarity. PG1, in comparison, used far more mathematical words and were able to use these mostly in phrase-driven ways (Nachlieli & Tabach, 2012).

Learners in PG1 had a wider repertoire of mathematical words and routines; and these contributed significantly to their being able to explore unfamiliar graphs. This could be seen in the constant or fixed phrases that related to restrictions, undefined values of the domain, infinity, and descriptions and processes related to certain functions. Expanded mathematical word usage enhanced the precision and economy of what learners were able to say mathematically. While PG1 were able to initiate new nodes on their realisation trees, that is realisations of realisations, they too showed incompleteness when the phrases were ambiguously framed by vague referents. The consistency of the absence of specialised knowledge, through the six schools, and across performance groups, points to problems that are difficult for a learner to influence on her own.

7.2.5 Repertoire of Ritual Codes per Performance Group

Table 15 Range of ritual codes for graphs across all performance groups.

Ritual Codes	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10	R11	R12	R13	R14	R15	R16	R17	R18	Range of Codes	Total Codes
PG1																			14	499
PG2																			12	179
PG3																			17	191

The activity on graphs was structured so that learners did not have to rely on recall of rules or procedures as with the other representations. Data showed that their talk centred on the visible, identifying features of the graph. When they proposed an equation as representing a graph, it was more of estimation, as the critical values on some graphs were obscured on the cards with which they worked.

The table above, of codes from ritualised utterances, serves also to point to the ritual codes which were not present when learners communicated about graphs. As expected, PG3 had the widest repertoire of ritual codes. R16, viz. talk of difficulty following rules, was not present in any groups' data. PG2 broke the trend established on the algebraic expression and table and appeared the least ritualised in their communication. They were not as communicative as the other groups on this signifier. Their responses were absent, or brief, seldom offering justification for their routines.

Codes were examined with the following question in mind: *how does PG1 communicate about graphs that the other two groups do not?*

The single code that stands out is R13, absent in PG1, and present in the other groups and dealing with recognition of form. From the data, PG1, in identifying graphs they were familiar with, spoke in objectified phrases. Objectification was linked strongly to identification of graphs and their features, for this group, which was less so the case for the other two groups. This resulted in PG2 and PG3 showing ritualised code 13, and PG1 not. On aggregate of codes alone, PG1 had far more to say about graphs than did the two other groups. In addition, PG1 also showed usage of far more mathematical words and phrases in connection with graphs. They could say more about what they saw, in less ritualised ways. Their ritualised talk arose mainly through their high incidence of subjectification.

The zoom out summarises the following observations:

- PG1 showed the widest range and highest aggregate of exploration codes;
- PG3 held more objectified utterances on this representation than they did on the algebraic expression and the table;
- objectified talk included talk that connected representations- the graph with the algebraic expression;
- learners were able to transfer features, like the *x*-intercept, off graphs studied at school to non-conventional graphs, where this was more frequent for PG1;

- specialised keywords, formal mathematical narratives, were limited and decreased as performance levels decreased. PG3 showed this in incomplete sentences, or ambiguous referents for keywords.
- PG3 learners relied on recall of past routines they had learned. Poor recall stalled their progress showing that routines were memorised as opposed to individualised; and
- subjectification R11 was significant across all groups.

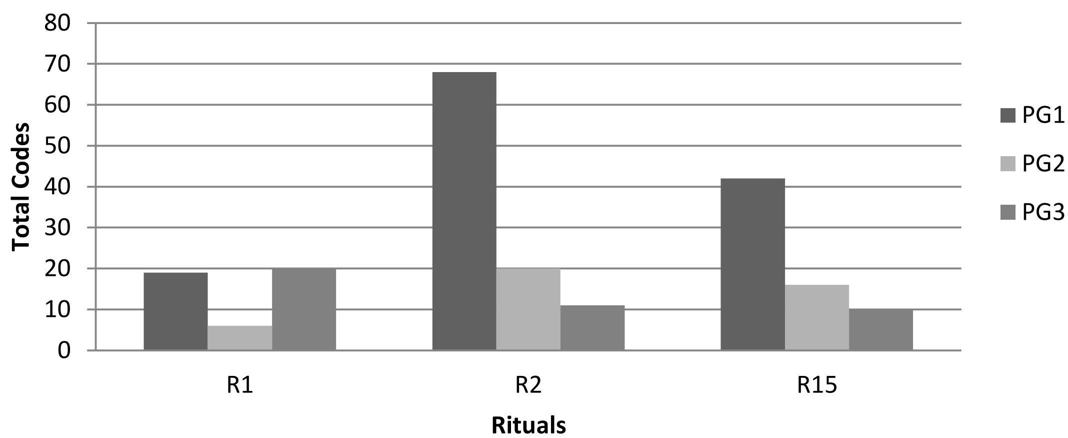
These ritual codes can now be characterised.

7.3 Frequent rituals codes: Zoom in

This section examines the ritualised talk for the ritual categories already defined as learners dealt with the graphical signifier.

7.3.1 Talk about Goals

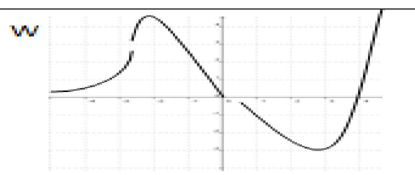
Figure 37 Goals



Goals related to social acceptance
• R1 - endorsed narratives come from memory or authority • R2 - adheres to goals set by others; satisfying needs outside of oneself • R15 - guessing

The most frequent ritual, across all performance groups was R2, which related discourse goals to social acceptance. There were distinct ways in which these occurred across the performance groups. Incidence of tacit agreement decreased from PG3 to PG1. PG3 learners could frequently make a statement, without it being questioned or engaged by their partner. This accounts, in part, for why PG1 had a significantly higher aggregate of codes compared, where PG1 can be seen to have had a wider spectrum of interactional patterns relating to goals. These can be summarised as: spoken agreement or disagreement on narratives with and without justification for doing so; requests or questions for clarity or justification from each other; frequent use of collective pronouns to suggest a decision at which learners jointly arrived. The extract below illustrates some of these occurrences.

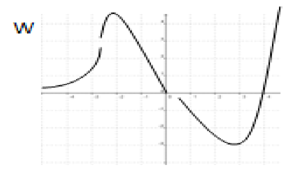
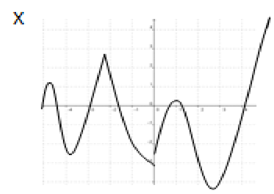
Extract 60 Ritual interactions

J-PG1		Learners discuss their decision if Card W represents a function.
G734	*We haven't decided*	Collective pronoun
D737	*Undefined.*	Speaks to learner G.
G738	*See, from this part to this, it's a hyperbola, from this to that, what is it? That's the question.*	
G739	*Not a function.*	Decision
D740	*It's not a function.*	Agrees and revoices Learner G.
G741	*Ja, we think it's not a function.*	Restates with collective pronoun.

In the extracts which follows, Extract 63 illustrates the way in which learners compensate for the formal mathematical discourse, which they do not have. Extract 62 prior, shows learners in agreement that the hyperbola is a function. While the discontinuity may have been the distracter, they were not compelled to agree with each other, yet this they did, conclusively, by G741. Agreement without mathematical verification was characteristic of this ritualised code.

A second extract that differentiates J-PG1 from other groups, shows learners able to ask questions of each other and work with one another's responses. Learners are, again, working with Card W.

Extract 61 Questioning PG1.

D710	*Not a function*	Learners scan Card W 
G711	*You think it's the same?*	(as Card X)  Learner G points to Card X
D712	*I think it's the same. This one card here, it's that here.*	Talk about the discontinuity as 'that'
G713	*So you think, eh eh, a graph that cuts is not an equation?* *It's not a function.*	

By contrast, typical utterances of PG3 learners, from the same school as the PG1 pair above, show curtailed interactional patterns where they seldom sought justification from each other. This appeared as an almost passive agreement.

Extract 62 Tacit agreement PG3.

J-PG3-Z200	*Hai, card W's not a function.*	
-S201	*W's not a function.*	

Both Extracts 61 and 62 from polar performance groups, shared a common thread, the absence of mathematical justification in learner talk. Data showed this to be particularly prevalent in PG3. The transcript extracts 63 and 64 both show a single turn having taken place in the conversation to confirm the mathematical decision, the equivalent of a restatement. Restatements were also frequent in PG3 talk. There is no single instance, in the data, where these learners challenged each other's decisions or justifications. PG1 showed they could question each other's moves. This preserved social bonds with the absence of conflict or dissent. The drive for social acceptance was further seen in frequent references to the interviewer during the activity. Affirmation appeared tied to perceived authority in Extract 63 below:

Extract 63 Talk directed to interviewer.

J-PG3-Z330	*I think it's a exponential. It can be a parabola, ma'am. I think it's a parabola.*	

Despite instruction being made explicit at the start of the activity that mathematical decisions were to be made in conversation with each other, learners still frequently sought input from the interviewer. PG1, in contrast, did also engage in talk which showed several turns where they questioned each other and attempted justification of their decisions. In these cases, the ways they used collective pronouns showed their co-ownership of mathematical decisions, and they were able to cross-reference with other instances where they have made similar justifications. These attempts to apply and refer to or recall endorsed knowledge was seen most frequently in PG1 in the main.

Talk in PG2 showed significant instances of restatements of the others' narrative. Other than this, they did not show rituals, which were different from those discussed for the other groups.

Extract 64 Restatement.

J-PG2 -I273	*A hyperbola is two pieces.*	
-S274	*Two pieces.*	

PG2 frequently revoiced, but tended not to change or extend the utterance of the other. Imitation is an essential part of learning especially at the start. It is problematic if discourse remains at imitation, since learners are to gradually move to independent participation in the discourse. This was not evident in PG2 and less so in PG3. PG2 used collective pronouns, in similar ways to PG1, to show that there was joint ownership of responses and decisions made. Utterances such as: "you know", "we don't have", "we call", "we know", "in our equation", "we put it here", were coded under R2. They seldom showed discussion of, or questioned a narrative, or showed joint participation in the deduction of narratives as the inclusive pronoun suggested.

To summarise, R2 was the most frequent code, across all performance groups. PG1 have an expansive mathematical discourse compared to other groups, as well as longer deductive chains that were also intertwined with a social one. PG2 and PG3, in comparison, refer to mathematical objects in a passive way (Nachlieli & Tabach, 2012) more often, once identified, learners have very little to say about the objects. They also show significant incidence of incomplete or incorrect utterances compared to PG1. Learners seldom communicated with each other, but to each other, and this tendency decreased from PG2 to PG3. PG1's mathematical discourse pertains to their being able to have and extend conversation of the objects; to invoke appropriate routines that involve the mathematical word; to question and seek clarity of the objects being spoken of; and to use the mathematical word in utterances that show the word's relevance and meaning. They had a wider repertoire of routines, and could realise far more when they talked of the different functions. Intertwined with the mathematical talk, like the other groups, was talk that sought or offered affirmation of utterances, phrases which described agreement and co-ownership of decisions being made, viz. the social discourse mentioned earlier.

Code R15 related to guessing. PG2 and PG3 frequently declared they had not seen the function before. PG1 attempted most questions even when the function or its features were new to their experience. They were able to connect what they had previously learned to features which looked similar, for example, they linked the discontinuity to values for which the algebraic expression would be undefined. PG1 also had an incomplete informal definition of a function and they used the vertical line test when discerning a function. This tool enabled them to respond to more graphs, even the more unconventional types shown on the activity. Typical utterances in PG1, which were coded R15, involved:

- Connecting responses to previous realisations. E-D712: "I think this one is similar to that", showing the intention to align responses to other cards in the activity. Similarities were realised in the graph, its features or peculiarities (like discontinuities). This attempt at connecting responses was seen in PG1 in the main. They sought patterns in signifiers and based decisions on these, in terms of treating them uniformly.

- Phrases that were not definitive, or portray confidence in a decision, such as, E-G742: “ja, we think it’s not a function” or E-G404: “...I dunno” to complete a response, ES D405 responds: “is it... possible?” This pointed to a discourse that was partially developed. Hence, learners closing statements portrayed indecision and doubt. They did realise that more was expected of them mathematically, but did not know what that was. There was awareness that meta-rules of functions exist, and that these were to be used to substantiate realisations. They voiced discomfort particularly for when they could not make the required substantiation.
- Invention as in J-G591: “yes, I would say because on the Cartesian Plane there’s only one line, not...several lines” and J-D1171: “the mathematicians skip them”. Both extracts came from utterances related to the yet undeveloped discourse on discontinuity (referring to card X). These could be read as an appeal to authority to validate a realisation. The first referred to routines learners were familiar with from school Mathematics.

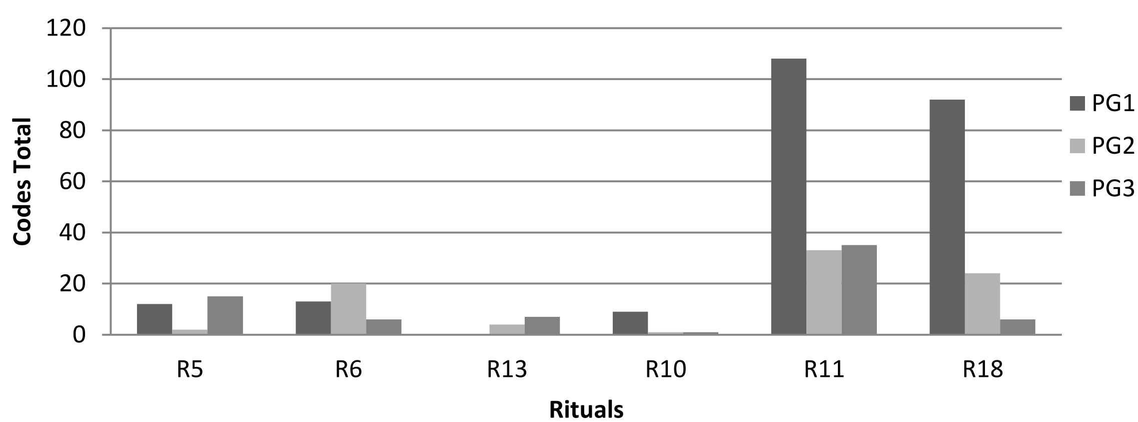
PG2 prioritised labelling functions and features they saw, J-I2260: “what do you call this?” Without the mathematical keyword word, the talk was halted. This resulted in a closing statement without further justification, such as J-S319: “I think it’s not...”. Often, the mathematical word served as a prompt for learners to recall associations they had made to that word, whether relevant in that instance or not. Their priority it seemed revolved around recall of a word and its associated routines. They discriminated least when it came to decisions of relevance, speaking to the neglect of the when component of routines.

PG3 depended more on what they had ‘seen’ before, before they could talk about a function compared to PG2. J-S195: “I’ve never seen such...” and E-ML63: “it looks like it”. The seeing, served as justification for mathematical decisions made. Guessing”I think it’s an exponential,....parabola”. Like PG2, PG3 prioritised identification of object or feature, where, generally, the object was identifiable by means of key features. PG2 had a limited number of routines associated with object. They recalled these routines as constant phrases, with the emphasis on the *how*, whether relevant in that instance or not. For both these performance groups, the *when* of the routine did not feature as a consideration. Put simply, PG2 had associated phrases connected with the object, and PG3 had associated words that they relate to the object. PG1 had a wider repertoire of fully developed routines associated with graphical signifiers. Commognitively, they possessed more nodes on their realisation trees. Again, the challenge for them was *when* the routine would be applicable. This added a dimension to the guessing (R15), which was not evident as frequently in PG2 and PG3.

R1 related to memory and authority, was seen across learner groups referring to questions, which were asked in examinations and tests at school. PG1 often assigned importance to the mathematics emphasised in assessments. There was also the use of the elusive pronoun ‘they’: E-PG1-D941: “maybe **they** can give you restrictions” and E-PG1-G683: “...**they** did it like this”. The ‘they’ in these utterances could be referring to a teacher, assessments or the textbook, and this could not be determined with certainty. The second frequent occurrence across all groups was the appeal to the interviewer for validation of a statement or written response. When directed to talk with each other, particularly PG2 and PG3 exhibited discomfort.
With functions represented as graphs, the expectation was that, since the abstract object was visible as a picture, the discourse should be reified, tending towards objectification with greater ease. The level of objectification remains mainly at identification for the graph across performance groups. The data shows discourse which has goals that tend towards recall of mathematical words and phrases in PG2 and PG3, and routines in PG1.

7.3.2 What is talked about

Figure 38 Signifiers.



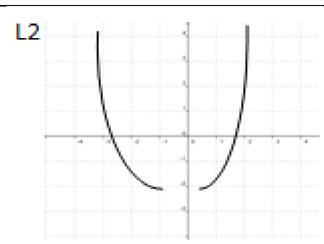
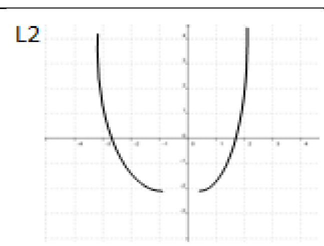
Signifiers

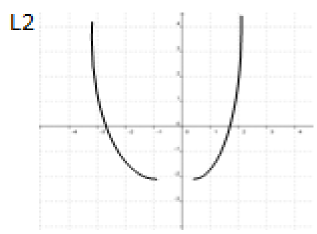
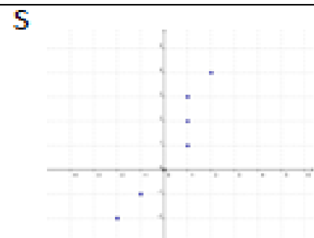
- R5 - uses visual cues to talk of features or process.
- R6 - talks of symbols rather than what is signified by symbols.
- R13 - misrecognition of the different representation and names representation incorrectly or not.
- R10 - asks/answers questions for clarification, affirmation. Questions are closed.
- R11 - emphasis on the person performing an action or engaging a process.

For expediency, R5, R6 and R13 have been grouped as they deal with symbols or visual clues that signify an object or an action to learners. These codes were discussed in detail in proceeding signifiers, with interesting patterns and deviations to be discussed here.

All groups easily identified functions studied as part of school mathematics. It was interesting to observe the way in which they identified and spoke of graphs that were unconventional. Learners infused these unconventional graphs with meaning, from the features of graphs they had studied and from their everyday experience. Below is a selection of utterances from all performance groups:

Extract 65 Talk of features.

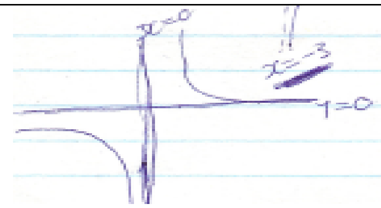
E-PG1		G205*: But, but it looks like a parabola. That's why that's why I said that it's a it's a function.* O206: *Ja, I thought it's a parabola.*
T-PG2		V472*: You can say it's a the hyperbola, ja.. er.. hyperbola 'cause it doesn't meet, here*

J-PG3		S120: *It looks like a parabola…* Z120: *But it isn't 'cause it's not joined together* S121: *…not joined together.* Z121: *So, it's not a function.*
E-PG3		M175: *A scatter plot. Um, maybe let's say you have four there, and there and there, you can obviously not go there, and then if you see this.. .go hand in hand with each other, then you can join them. That's what I know about the scatter plot.*

Specific features of the graphs trigger specific realisations for learners. The contrast between PG1, PG2 and PG3, was seen in the way that they talked of the discontinuity without having this word or its meaning filled out. PG1 focused on the entire visual and its features, and confirmed a parabola from these. Knowing that a parabola was identified as a function from school Mathematics, they deduced the graph a function, without the formal narrative of what that entailed. They made connections to past experiences. PG2 focused on the discontinuity, a critical 'new' and interesting feature of the graph. The graph they had studied which was not continuous was the hyperbola, and they thus identified the form of Card L2 as a hyperbola. PG3 compared, recognised the form as close to a parabola. They too did not have the mathematical word for a discontinuity. They used this unknown feature as the determining criterion for Card L2 not being a function. The absence of formal endorsed narratives hindered their realisations by pushing them towards spontaneous everyday realisations and tenuous connections with what they had learned in the classroom. Realisations of the signifier 'discontinuity' exist in an intuitive and spontaneous way, related to how the discontinuity appears on the graph described in everyday words, such as gap or break. These tenuous mathematical links reinforce a ritualised discourse as the discussion becomes about personal untested interpretations.

Learners were able to relate the orientation and features of the graph to the parameters, or symbols of the algebraic representation. Talk of symbols was coded R6. These symbols or parameters related to the following features: the intercepts of various graphs, the position of the hyperbola, the quadrants that were occupied by the graph, the asymptotes, the parabola's orientation as concave up or down, or the slope of the straight line.

Extract 66 Properties and features.

E-PG1-G794	*'Cause let's say for instance, we have the hyperbola here, and here. Now in this case, our asymptote is our axis, which is x = 0, and y = 0. But once we add this p or a q, these value changes.*	
P-PG3-Mo86 **-ML87**	*The graph smiles, and if it's negative, the graph…* *…the graph, it's sad.*	*Learners talk of the parameter a in the general equation of the parabola* $y = ax^2 + bx + c$

These utterances highlight the ritualised discourse on the meaning in symbols for learners. The first, gives algebraic symbolism to the hyperbola graph, its asymptotes, and the effect of parameters p and q, tied to the vertical and horizontal transformation of the graph. The symbols p and q relate to the symbols of the general expression for a hyperbola as presented in school Mathematics. The PG3 exchange, emphasises the everyday as it related to the appearance of the graph, "smiles" and "sad", and the use of indeterminate pronouns like "it" to refer to the coefficient of x^2. A second interesting occurrence in PG3, related to symbols, involved algebraic notation. Learners ascribed functional notation $f(x)$ and $g(x)$ to specific graphs:

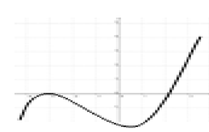
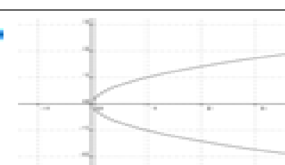
Extract 67 Functional Notation.

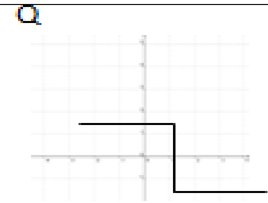
P-PG3-Mo135	*g of x, we, we have g of x in, when we doing parabola graphs, ne. We have f of x and g of x and then f of x is the, um, the graph, the graph that smiles, and that makes the sad part. Ja. And then the g of x, sometimes it's a straight line graph, sometimes it's a line, um, like this one.*	

$f(x)$ was related to the parabola in the statement "the graph that smiles, and that makes the sad part". $g(x)$ was realised as "sometimes" being a straight line. Again, this pointed to the discourse of function being partially developed in terms of the individual functions studied in school mathematics and their connection to the specialised notation of function. Notation, like related formal endorsed narratives, must be connected with the definition of the object. The significance of notation, how a function is represented algebraically, develops thinking about a function as an input-output relationship, showing the process of co-variation; to developing function as a mathematical object. Reasoning and communicating about functions requires notation (Doorman & Drijvers, 2011). PG2 and PG1 used notation in expected ways. With the subsuming discourse partially developed, notation appears to have developed incidentally. This was seen in the way that learners tried to fill the notation symbols with meaning, through connections with the experiences from class work and assessments involving functions. The data suggests that learners associate the parabola with $f(x)$ possibly because teachers, learning resources and assessments frequently use $f(x)$ to identify a parabola.

Developing discourse as increasing discursive layers through participation, necessarily requires a formal mathematical discourse be introduced by a knowledgeable other. This enables learners to speak, initially through imitation then independently, with the economy and precision of the endorsed narrative and the formal structure of the discourse. This gives the discourse economy, and removes ambiguity in communication. The crucial development of a connecting discourse of function, that refines the colloquial and informal discourse of the individual functions, and connects it to the endorsed narratives of the encapsulating object, does not, it would appear, to be within the range of experience of these learners. The collection of disparate discourses which learners appear to have are unstable, changeable and situational, seen for example, in PG3 when they inspected Card V.

Extract 68 Card V PG3.

E-PG3-M199	*...besides the testing of the vertical and horizontal line when looking at the graph, I can say it's quite like, that of, uh... W.*	V 
E-PG3-L127	*Card O, It's a function because it's a parabola. If you use the, um, verti… uh, uh, it's not a vertical line, sorry, it's a horizontal line test. Eh, if you use it to test this if it's a function, it's gonna be a function.*	O 

E-PG3-M166	*This side, it's a function. Vertical line test, even if you can come this side, or come this side, it will cut once. You can put a vertical line test wherever. If I put it here it cuts once, once, once. But I don't know this kind of graph.*	

The first two utterances show the situatedness of the vertical line test for learners as it easily evolved to the horizontal line test, so that a familiar conclusion could be reached. As discussed regarding the algebraic expression, the significance of the vertical line test (like the significance and meaning embedded in notation) has not been connected to definition of function. As a result, we have the tool and the formal definition appearing to work towards the same learner goal, that is, producing a means to reach conclusions of the object. This was an excellent illustration of ritualised learning and application of routines disconnected from their meaning and purpose. The third extract, shows the detailed application of a tool. In this instance, it raised the question: had learners possessed even a rudimentary narrative defining function, such as 'for every *x* there is a unique *y*', would this have tied in with the application of the vertical line test on the graph Q? Learners are explicit about the fact that a vertical line would touch the vertical line of the graph, at one point only. In summary, R5 shows that the vertical, sometimes horizontal line test is a criterion that PG3 learners used to discern a function relationship. A vertical line held to a graph, and the noticing of the number of times it intersects a graph, is a visual cue for discernment of function. The narrative for the PG2 and PG1 learners appeared to be, 'a function is cut by a vertical line once'.

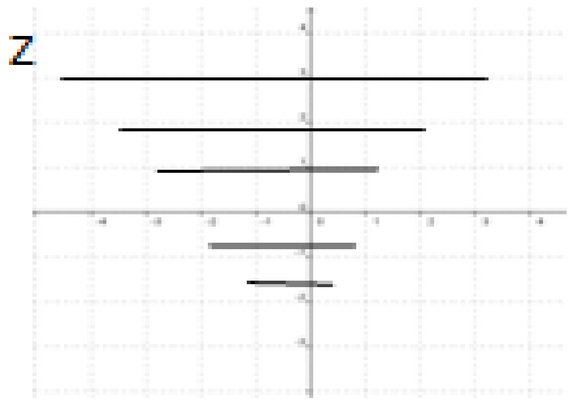
Code R10, showed that few closed questions were asked, and that they involved seeking specific details or clarity of the visible features of the graph. For example:

Extract 69 Closed questions.

J-PG1-G713	*Wha…, what are the coordinates for this?*	*Refers to the discontinuity*

PG1 was the only group who incorporated attempts for generality in their talk when discussing the discontinuity and the hyperbola. Questions were open to incorporating discussion of undefined values of the function and infinity related to the domain. For the other groups, the discontinuity posed the greatest hurdle to mathematical communication. Learners emphasised that it was not something that they had worked with in the classroom, or seen before. For poorer-performing learners, the discontinuity became the criteria for saying a graph was not a function.

As seen in the previous representations, learners objectify themselves in talk on graphs; R11 is the most frequent code. Interestingly, the subjectification evident in talk of graph was tied to a process. The process could be mathematical, or relational. The mathematical talk featured the moves a learner makes to resolve an algebraic process. Examine the exchange between two PG1 learners, who are discussing Card Z:



Extract 70 Subjectifying.

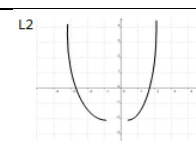
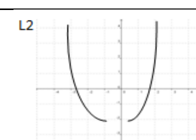
J-PG1-D459	*Let's say we have an equation, it's sort of a function, plus 7 = y. You, you can't add a number here and expect this one to look the same as the other one.* *Like, let's say this is the equation of this graph, when you say x, let's say x is two, two plus seven equals to nine , then we replace x with something else, and you say three, this one, this one is changing, but the other one's not. So when you say three plus seven here, equals to something else. It's not the same thing. This one… six, nine, and here it's ten.*	Written as they talk: $x+7=y$ $2+7=9$ $3+7=10$

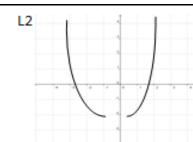
The algebraic process shows a strong tie to a human action, as learners attempt to assign equations to the graph shown. This focus of a person performing an action hinders objectification. These lengthy, subjectifying, process utterances, shown above occurred in PG1. They predominated in the PG2 and PG3 discourse. Mathematics was less about reasoning and more about communicating what was done. While their mathematical actions may not have been correct, it communicated the tendency in PG1 to relate the different representations. Subjectifying foregrounds the voice and actions of the person, while backgrounding the object.

As with the other representations, learners appear to be process-orientated, looking for ways to do something and describing this doing. Here, learners substituted values for *x* into an incorrect expression for the graph. The lines on Card Z, a non-conventional graph, appear to lie a unit above each other. The substitution of *x* values mimicked this by increasing the *x* values in increments of one. In this example, subjectifying talk appeared strongly tied to process. Talk of features of the graph and their relation to algebraic notation are prefaced with phrases that portray uncertainty. This occurred in all groups, with the need to preface utterances with statements like, “I think”, “I can say”, “I’ve never seen”. Such phrases were used to relate their feelings of confidence or uncertainty in what they were saying, and as conversational bridges, to relate to each other or invite the other to make a contribution in a non-threatening way.

The spontaneous/everyday talk, R18, occurred in the way that learners related what they were doing, or how they described mathematical objects or replaced mathematical words of which they were unsure. With regards to unknown mathematical words, it was particularly interesting the way that learners from all groups described the discontinuity when they saw it on the graph.

Extract 71 Spontaneous talk of the discontinuity.

J-PG1-G907	*Okay, I’m not sure about the breaking part, but these parts are functions.*	
J-PG2-I275	*The thing about the hyperbola is, in standard form, it’s always something like this. This is how it looks like.*	

-I276	 *And, uh, they are always, both, it's like a pair, they both make a function. One represents half, and the other one represents half which equals to one. That's the simplest way, ma'am. 'Cause you can't say I have one shoe, and it's one pair of shoes. You need both to make a pair.*	
J-PG3-S120 **-Z120**	*It looks like a parabola…* *But it isn't, 'cause it's not joined together.*	

No groups appear to have encountered a graph with a discontinuity, which resembled a hole or a gap in the graph, though they had worked with discontinuous graphs such as the hyperbola and tangent. PG2 above drew on very concrete similarities within their experience. The discomfort that L2 evokes in I276 stems from the graph not being oriented as a hyperbola. See the $\frac{1}{2}$'s inserted into the first and third quadrants of the learners sketch I275, and the similarity drawn to a pair of shoes. Each part of the graph is described as a half of a whole, drawing on experience with the hyperbola, which comprised two parts. PG2's peripheral discourse is far removed from the formal mathematical narratives. Yet they too, like PG1, spoke of a 'break' or the graph being 'broken' when discussing Card W. PG1 by contrast, offered an expanded discourse and speak directly of the graph, with a repertoire of words that describe what they saw in the visual. Calling the discontinuity a 'space' occurred more frequently than the word 'holes', in PG1 utterances. They connected what they saw, linking the discontinuity to being a restriction on the algebraic expression of the graph, and to further talk of areas where the graph approached infinity.

Talk of infinity is considered an advanced layer of discourse (Kim, et al., 2012). While the discourse is not entirely aligned to a formal mathematical narrative, it conveys the areas that are open to exploration and abstraction, which learners have initiated on this graph. They applied the discourse on features of graphs in school Mathematics, to all other graphs which may hold similar features. Showing the talk across all groups for Card L2, highlights, through contrast, the range of connections possible in the discourse of the different groups. PG3, while able to connect L2 to the parabola; do not go beyond this in their discussion. The opportunity for an extended exchange between learners in PG3 was characterised by short closing statements and little talk, which invited exchange.

In describing processes, learners use informal talk, dominated by indeterminate pronouns (highlighted below), action verbs, and colloquial expressions for the way in which graphs can be transformed. Expressions such as "it comes this side" and "this goes there", are examples of this. Reified narratives to describe processes were infrequent, replaced with casual, conversational, everyday phrases learners would ordinarily use to communicate with one another. This possibly emanated from the approach to mathematical procedures in the classroom, being taught as an act of doing, a discourse of the 'how' of a procedure.

Extract 72 Disconnection between the graph and algebra.

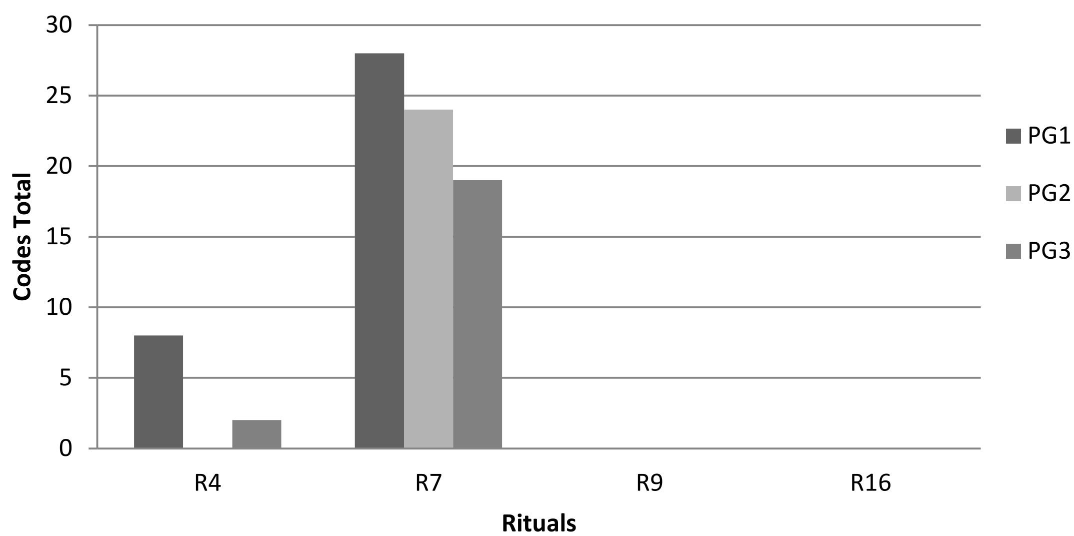
J-PG1-G978	*Mmmhmm. For lik,e a, from like, a hyperbola. Hyperbolas are in two different quadrants, but* ***they*** *move simultaneously. If you add or if you subtract,* ***they*** *move.* ***It*** *only changes when you reflect… Which is,* ***it*** *comes* ***this*** *side, but still, either way,* ***they*** *still move… You cannot have* ***this*** *side of the hyperbola going* ***there*** *and* ***this*** *one going* ***there****.*	

Extract 72 shows that the graphical procedures developed separately as an act of doing, not accompanied by the formal algebraic connection for how expressions or graphs are transformed simultaneously by each of these representations. Transformations simultaneously relating the graph and algebraic expression occurred for the straight line, and to a lesser extent, on the parabola for this group. This was an opportunity for teaching and learning to build notions of equivalence through simultaneous processes.

In summary, the discourse when learners' talk of graphs remains ritualised, emphasising their actions on signifiers. Data shows a strong tendency for learners to emphasise what they did to signifiers and this is conveyed in a relaxed, conversational way, with little evidence of connection to a mathematical narrative. Learners thus respond to features of the graph as signifiers for an action they need to initiate. The connection between the algebraic expression and its links to the graph was seldom initiated in talk.

7.3.3 Flexibility

Figure 39 Flexibility Codes

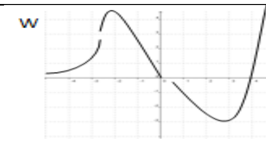
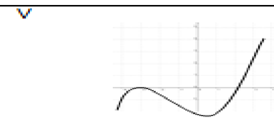
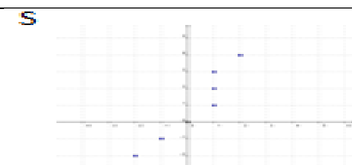
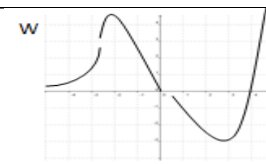
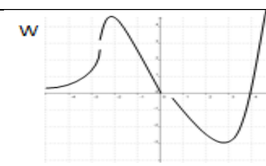


Rigid

- R4 - different representations are regarded as separate entities.
- R7 - recycles known routines/narratives. Appropriate or inappropriate.
- R9 - concern with making errors or emphasis on avoiding them.
- R16 - difficulty in following rules.

The ritual code R7 occurred most frequently across all groups in this category. The ease with which they could transfer the meta-rules from one signifier to another was examined. It was evident in instances where learners related the graphical signifier to others with similar features to similar algebraic processes. This was easiest to do with graphs that were studied in the classroom, particularly those most current. The talk which followed transferred to the features properties or rules of the familiar graph to the new signifier. PG1 show the highest aggregate, because they had the widest range of routines when compared to other groups.

Extract 73 Visible features transfer.

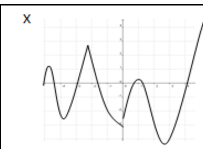
J-PG2-I 178	*This is a, you know, the sin cos graph.*	
J-PG1-G406	*Mmm. It's not a parabola. It's a sin cos, it's a sin cos graph.*	
E-PG3-M175	*A scatter plot. Um, maybe let's say you have four there, and there and there, you can obviously not go there, and then if you see this... go hand on hand with each other then you can join them. That's what I know about the scatter plot.*	
E-PG3-L191	*Um, eh, I can say it's forming, uh, many parabolas. It's continuous.*	
E-PG2-Ts81	*I think it's a function because I saw… er ...it almost looks like a parabola and then it comes and then its cutting, and there's a parabola and then its cutting; I think it's a function.*	

This confirms extant research, which showed that students assume that functions are linear or quadratic where the assumption is unwarranted; where, for the students in the study concerned, any u-shaped graph was deemed a parabola (Carlson, et al., 2008). The last two extracts confirmed the results cited in the literature above, and moreover, it was found here that learners make associations with the most recent discourse they were developing. Here, learners related

Card W to statistical and trigonometric graphs, the wave appearance of sine and cosine graphs, or the scatter plot. The visual presentation cued learners to make connections with visual images they had recently encountered, where the visual signifier allowed such comparisons and transfers to happen. The decisions though were not mathematically based and alternate interpretations such as the scatter plot could not be justified in the function context. Without reasoning be made explicit, learners flexibility is best described as rigid, where decisions were based on observation and intuition in most cases.

R4 regarding different signifiers as separate entities occurred to a limited extent. Learners generally associated the graph with a particular equation, whether or not they were able to find it. Parabolas, straight lines and hyperbolas were linked to the general equation of these functions. The disconnect between the graph and the equation was found when learners worked on non-standard graphs. Typically, in such cases, they talked about these graphs as not having an equation.

Extract 74 Linking representations (signifiers).

J-PG1-G662	I don't think there's an equation for such a graph, I don't think so…	

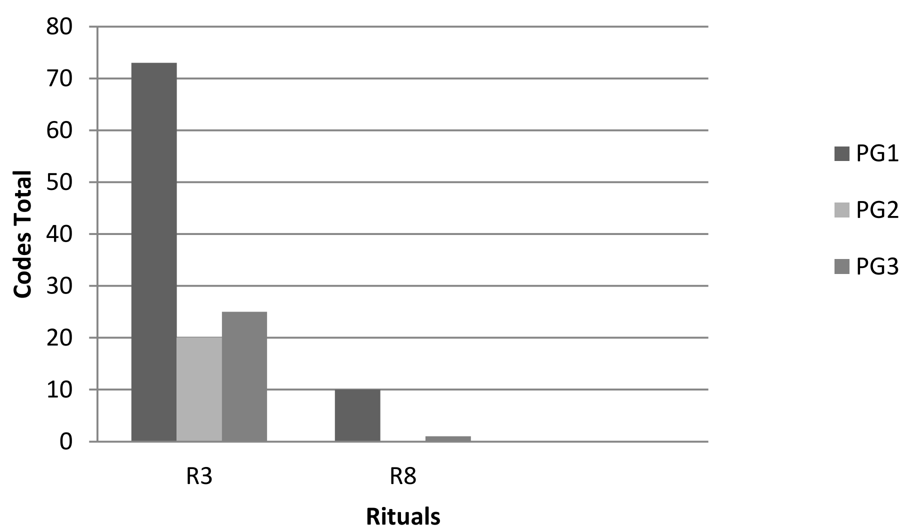
These showed learners were aware that the graphical representation was tied to an algebraic identity. Compound functions are not part of school Mathematics. By saying they did not think there was an equation for Graph X above, probably meant they were unable to find the defining algebraic expression for card X. They were not rigid in talking of the graph in isolation from its other representations, and familiar graphs were frequently associated or identified with their general expression. However, learners across groups seldom sought the identifying expression for a non-standard graph, despite identifying some of the component parts correctly. They did not have formal rules to do this.

The absence of R16, viz. showing difficulty in following rules, was possibly due to learners having a discourse around the significant features of a graph, these being visual and not requiring algebraic verification. Better-performing learners spoke automatedly of the visible features of the graph. PG1 elevated this discussion to the transformation of the presented graph off the standard position on the Cartesian plane, and provided an algebraic expression to describe this.

In summary, learners appeared less rigid and ritualised on the graphical signifier compared to the others. Critical features of the graph cue learners to talk about the function. Features such as the *y*-intercept or asymptotes were transferable across the various graphs for better-performing learners. It was encouraging to note that all pairs showed they could move flexibility between the graphical and algebraic representations in certain instances. These were limited for poorer-performing learners. It can be deduced that the visual of the graph invites talk on what learners can see and do with this signifier. Flexibility in building complexity or extending what they knew was not to be found, especially in the discussion of unfamiliar graphs.

7.3.4 Applicability

Figure 40 Codes for applicability.



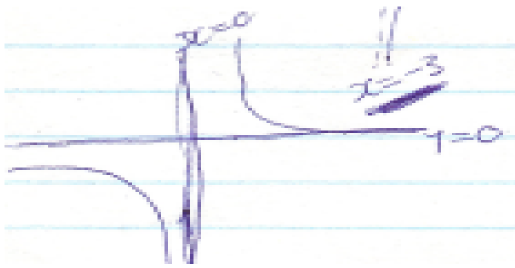
Narrow

- R3 - statements about actions and process. Remains in process (not reified) - no reflection on meaning at the end of process -spontaneous or provoked.
- R8 - concern with the final solution. Proof/verification through a specific instances only.

Code R3 shows that the features of a graph prompted learners to particular ways of working. The ways of working show little deviation across groups and suggest that these routines are part of the school mathematical discourse. Ritualised thinking is indicated in instances which were unfamiliar, where learners were expected to explore and apply the processes in which they were fluent. Routines involving talk of the intercepts and asymptotes appeared reified. These were applied across different and unfamiliar graphs. Applicability examined the use of keywords across different graphs. Learners across groups transferred familiar keywords across different contexts. It was interesting how learners communicated when they did not have the keywords that were needed e.g. function, infinity and discontinuity.

In process utterances, learners incorporated a description of their actions. The 'how' of the discourse on graphs is dominant and learners saw this as a mathematical justification. Coding looked for closing conditions on a routine that showed *when* a process held relevance, or when it could be applied across various contexts. This was an indication of learners seeking wider applicability for what they were doing. Routines related to assigning an equation to graphs from school Mathematics, viz. finding intercepts and asymptotes, were consistently attempted by all groups. With the larger number of utterances, compared to PG2 and PG3, PG1 presented with this code significantly more often. They could apply their routines over wider contexts, and they had a wider repertoire of routines than the other groups. The applicability of routines became obscured with the talk of process, and these became restricted to specific instances in PG2 and PG3. Procedures reified to objects did not show frequently in the data except for PG1, related to the transformation of a particular graph and the descriptions of the image.

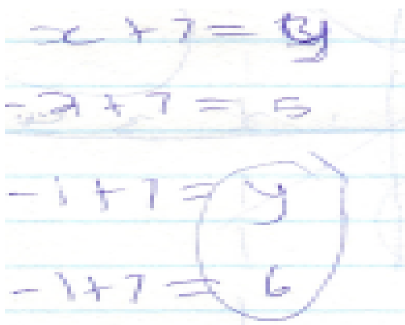
Extract 75 Applicability over different contexts.

J-PG1-G794	*’Cause, let’s say for instance, we have the hyperbola here, and here. Now in this case, our asymptote is our axis, which is x equals to zero, and y equals to zero. But once we add this p or a q , these value changes* 	

Extract 75 emphasises process talk of learners engaged in performing actions. “*x* equals to zero, *y* equals to zero, once we add this *p* or *q*”, are all phrases that show a process orientation, with emphasis on actions. Compare “*x* is equal to zero” with the objectified statement “*x* is zero” or “parameters *p* and *q* transform the graph in the following ways”. The ways that learners communicate mathematically can be linked to the discourse of the classroom, their primary exposure to mathematical communication. School Mathematics, from its curriculum and learning resources, shows strong association with verbs, such as solve, manipulate, simplify. The emphasis in school Mathematics lies primarily on activity (Drijvers, Goddijn, & Kindt, 2011). As with working flexibly, discussed in the prior chapters, the development of an objectified discourse decreases the process orientation and subjectification, thus allowing a learner to generalise and apply routines over a wider range of contexts. These ritualised patterns stymie the growth of wider application, and thus the opportunity for exploration.

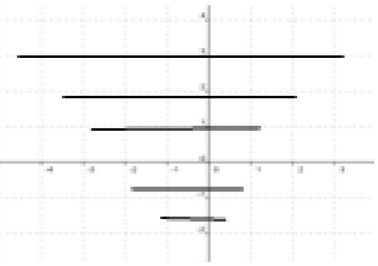
R8 is discussed from Extract 78 and looks at the means learners use to justify their routines.

Extract 76 Means for justification.

J-PG1-D474	*It's like, it's like what I said, ma'am, when, when you have a function, it sort of goes up and down, up and down, and here you cannot have an equation where, let's say, negative two is, is, uh, ja, let me show you something…*	
J-PG1- D475	*Eh, negative two is x like I said before. x plus seven equals to three.* *Then you say negative two plus seven is equals to y,y,y,y...* *Negative two plus seven is equals to seven minus two, is five.* *y is equals to five.*  *And then you say, eh, you replace this negative two with negative one, negative one plus seven is equals to Y. Negative one plus seven is equals to six. They both have to change when one is changing. That's what I think.*	

In D474, a function is described as performing a physical, visible action, of going up and down. D475 was an attempt to illustrate co-variation using specific values for *x*, namely 1, then 2. Learner D showed the *y* value change from 5 to 6, coded R8. This was seen as an attempt at justification. Both these realisations arose out of a diversion in learners' discussion around whether Card Z represented a function or not:

Z

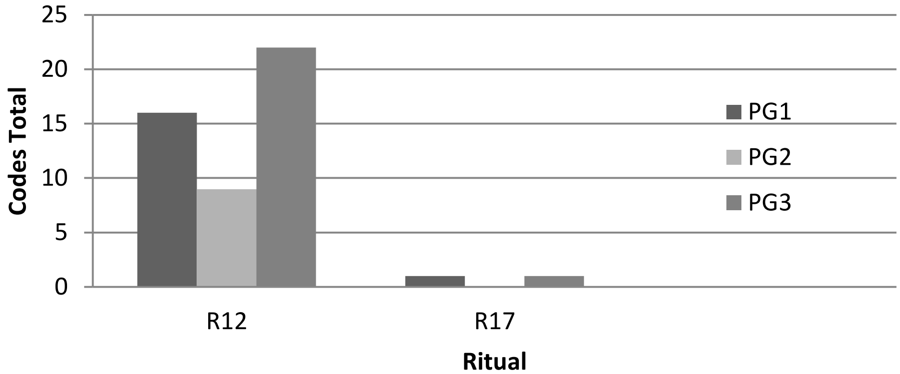


Card Z, posed challenge to learners, described as a Christmas tree by PG3. They recognised the component lines individually but not the compound graph. They attempted to explain that co variation was a condition for the function to appear as it does. They saw that change in the value of x to give a value for y, resulted in a 'new line'. While these learners had used the vertical line test previously as a tool, they did not, in this instance, use it to decide on a function or not. Utterances show that having not seen such as graph before became the criterion on which this decision was made.

Learners' substitution of values for x into a familiar linear function $x + 7 = y$ was an incongruity difficult to account for against the clearly horizontal lines shown in the function. The algebraic representation above did not provide calculated values that estimated any of the values of the horizontal lines. While these instances may need further investigation, it was interesting to see learners revert to known routines, whether or not they prove applicable. How they chose these, is something which remained unexplained. Substitution is one such routine, frequently called on as a means for verification. This ritual can be seen as the need to do something mathematical, whether relevant or applicable. Codes R3 and R8 were both present in the extract.

7.3.5 Addressees

Figure 41 Codes for addressees.



Others

- R12 - routines and answers are not questioned or justified.
- R17 - imitates the other person, keeps pace with the other.

Learner utterances again show a strong tendency towards social acceptance. PG3, showed more frequent instances of passive participation and easy acceptance that the other groups. Each relied on the other for statement that was seldom questioned or discussed, and often restated. By contrast, PG1 engaged each others' responses, sought explanation, differed with and challenged each other and sought mathematical justification. This could be seen in the duration and number of turns in a learner exchange. PG3's exchange pattern after viewing a card could be generalised as a learner isolating a feature of a graph, declaring the graph a function or not, followed by agreement by the second learner. PG3 learners engaged no formal discourse of the object. Their talk was dependent on familiarity with the graph or its features and their identification.

Extract 77 Talk for others: Restatement.

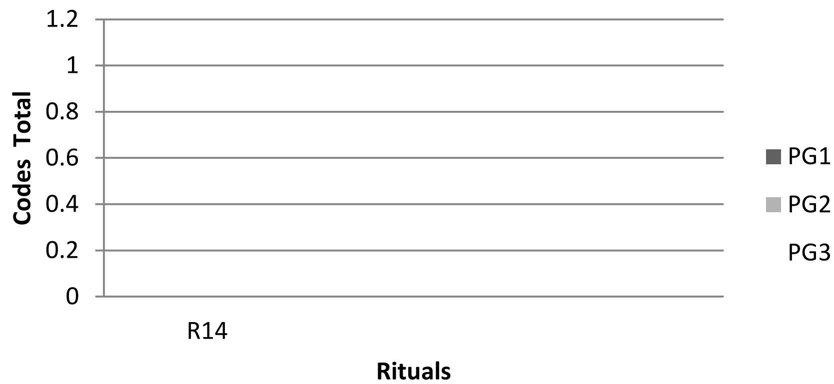
J -PG3-S120	*It looks like a parabola…*	
-Z121	*But it isn't 'cause it's not joined together… not joined together.*	
-S122	*…not joined together.*	
-Z123	*So, it's not a function.*	

In Z121, learner Z discerned an unfamiliar feature of the graph, and stated this. In S122 the second learner paused thoughtfully and then restates the observation. Despite his initial observation (S120) that the graph could represent a parabola, S abandoned this observation, and did not question the closing utterance (Z121). As in this instance, PG3 talk showed frequent restatement as a ritual. In addition, PG3 showed short definitive utterances, often closing the possibility for further question or discussion. This contributed to the high incidence of R12.

PG1 had more to talk about on all graphs, in terms of the graph or its features. They also talked to each other, exchanged information and built on each others' observations. A hyperbola could spark realisations about its parameters and position related to the quadrants, and the effect of parameters on the transformation of the graph, including discussion of infinity. They had an extensive path of realisations with realisations of these realisations. R12 was indicative of incidence where they could say very little, and then defer to their partner to conclude a decision. There was a tacit social contract, that not knowing or uncertainty spelt two choices; restatements of what was said, or allowing the decision of the other learner to carry. Both these are intrinsically social and not mathematical means for verification. Learners sought affirmation from each other rather than from the correctness of the mathematics. R12 and R17 were codes that were closely linked as a result. Learners showed a discourse establishing and affirming social relationships, as opposed to, a discourse grounded in mathematical objects.

7.3.6 Reasons for Acceptance

Figure 42 Codes emphasising rules.



Followed Rules • R14 - emphasises following rules and the importance of practice.

Discourse on graphs involved learners' talk of what they observed on the visual mediator. These observations appeared not to hinge on rules, or procedures, which they had to remember and restated. Rather, they discerned features of the graph, which were visible and spoke of these. This differed for example, from when learners viewed the algebraic representation and were required to initiate processes to sketch a graph in order to make the object visible. Learners responded to the visual mediator easily. Here learners were able to estimate the equivalent algebraic expression for graphs they were familiar with, and utterances showed no reference to rules or practice, hence R14 was not present in the ways that learners communicated about graphs.

7.4 Conclusion

The graphical signifier held the widest range and the highest aggregate of exploration codes compared to the expression and table. Learner routines, nevertheless, still showed up as predominantly ritualised. Talk on the graphical signifier, revolved around the features of the graph and what could be realised from these. These were spoken of as abstract objects, without prompting. This occurred more frequently for better-performing learners. Identification of the graph signifier and its features counted as the first level of objectification. The second level, which became evident, marked the transformation of the graph.

The spontaneity with which better-performing learners could associate a graphical signifier with its algebraic expression suggested that they saw equivalence. PG3 showed this equivalence, but were restricted to the linear function in the exploration codes. The graph appears as the easiest route to building an objectified discourse, as poorer-performing learners displayed competence here. PG3 showed exploration codes for objectified identification and the equivalence of signifiers. If this is examined in the context of the school Mathematics routine trajectory, algebraic expression (A) to table (T) to graph (G), learners, from this data, appear able to realise the object in the final signifier (G). Graphs thus provide a logical entry into the first layer of discourse for the building of complexity with functions. While the trajectory (A-T-G) described manifests in an overtly processual orientation to function, seen in Chapter 5 and 6, the discussion in this chapter shows that the reverse trajectory (G-T-A) could be an inroad to exploration. Learners showed a wider range of exploration routines with the graph. Teaching needs to exploit this competence.

The pedagogic move through all combinations of the sequence (A-T-G) or (A-G-T) could leverage several advantages, where it might: establish flexibility between the signifiers; reify processes involved; and draw out the equivalence of these signifiers. The goal of this would be to build towards meaning in the formal narrative (or definition) of function. It was encouraging to see learners being able to enter exploration routines, from their predominantly ritualised practice, having had a lack of, or a poor discourse of this subsuming object. It is worth imagining what could be possible for these learners to learn, with a discourse of the function object that is able to connect all the disparate signifiers, rules, and routines they currently hold.

The absence of keywords, mathematical words and phrases which are used correctly in context, was widespread across all groups. Importantly, their meaning and significance also appeared without the requisite mathematical depth. Better-performing learners spoke more precisely using key mathematical words appropriately compared to other groups. This compressed lengthy everyday descriptions meant that they could say far more with fewer words. Ambiguity marked the communication of poorer-performing learners. Keywords have added advantage that they can be transferred across multiple graphs. A *y*-intercept would describe a critical feature on any graph, not just a straight line or a parabola. The economy of correct mathematical keywords and narratives needs attention for these learners.

Picking up on earlier notions of identification, process and object developed as categories for describing exploration (see conclusion Chapter 5), the graphical signifier locates exploration in identification and objectification. The graph did not entail process for these learners. It marks a pedagogic opportunity to establish algebraic process routines as linked to the graph. The turning point for example can be algebraically verified, and not just read off a graph. Mathematical deduction seems not to be embedded as a means for verification for these learners. Yet it is a vital school Mathematical practice, making the reproduction and extension of routines possible. This is a necessary tool for exploration.

For the learners in this study, the graph was the function. Learners in PG1 and PG2 could use the vertical line test to discern this relationship. The test applied was vacant of mathematical meaning. No learner, even from the better-performing group, attempted to rationalise why it would work. This ritual driven use of mathematical tools and routines, and the passive use of key mathematical words (Nachlieli & Tabach, 2012), marks the proto-stages of discourse development. The question has to be confronted as to why our learners show significant presence in ritualised routines and passive word use, even towards the end of their engagement with functions.

Chapter 8: Findings and Conclusions

8.1 Introduction

Literature emphasises that rituals are pervasive in school Mathematics and they should be as they are a necessary part of learning. They were particularly pervasive in the data that I have seen. Sfard provides the useful distinction in mathematical routines between ritual and exploration. I have used these as a means to describe the ways that Grade 11 learners think about and engage mathematical objects. Ritual is a necessary route to developing exploration of mathematical objects. But, if as I have found, learners remain in a ritualised practice, it can have grave consequences for learner performance, as illustrated in Chapter 1. The purpose, therefore, of mathematical rituals is to induct learners into the formal mathematical discourse and to prepare them thereafter, for independent exploration of the mathematical objects. The problem I have shown is that learners from the six South African schools where I have worked in this study remain ensnared in ritualised practice.

To investigate this further, I worked from the assumption that learners who perform better in mathematics would have more objectified, mathematical communication than would those learners who performed poorly. To this end, I mined the discourses of Grade 11 learners on tasks involving functions, by first separating them into groups based on their performance in school assessments. The findings across the three signifiers: algebraic expressions, tables and graphs, discussed separately in prior chapters (5, 6 and 7) can now be collated, to provide a unified picture of the object for learners across performance levels.

Table 16 below provides this summary. It shows all frequent exploration codes descending in frequency and the three most frequent ritual codes per performance groups. Engaging in independent exploration is a goal of learning. For this reason, and noting the slim opportunity to develop exploration routines in school Mathematics, all exploration codes are shown. Marking learners' attempts at complexity and abstraction, they are interesting for patterns within performance groups and also across them. Only the three most frequent ritual codes are shown to develop discussion around patterns in ritual practice. This hinges on logic which says that repetitiveness will show the established rituals. The patterns emerging across schools are most compelling, and they give a picture of the kind of learning school Mathematics develops.

Table 16 Routines for Function.

Function																	
Algebraic Expressions						Tables						Graphs					
Exploration			Ritual			Exploration			Ritual			Exploration			Ritual		
Performance Group						Performance Group						Performance Group					
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
E4	E3	E3	R 2	R 2	R 3	E4			R 11	R 2	R 11	E4	E3	E3	R 11	R 11	R 11
E3	E2	E4	R3	R3	R2	E1			R2	R11	R7	E7	E4	E4	R18	R7	R3
E1	E1	E5	R11	R11	R11	E3			R3	R3	R2	E3	E5		R3	R8	R12
E7	E4											E5					
E2												E9					
E9																	
E8																	
E5																	

The table enabled me to speak of participation in the mathematical discourse according to two aspects:

- The first, to contrast the nature of mathematical routines of the learners at different levels of performance: low, medium and high. I hoped that this distinction would come to characterise necessary connection between different levels of performance, so that we (mathematics teachers and researchers) might examine ways to facilitate ascendency to better levels of performance.
- The second, and perhaps my most compelling need, was to describe the discursive routines of poorer-performing learners, in ways that did not describe them as deficit or incapable. ‘Learners’ can’t do maths’ is a phrase often heard, to which I stand opposed. This provided the rationale to focus on their routines. I followed the logic to work from what they already know and do mathematically, to begin thinking on how to help them improve.

I found the broad commognitive definitions of ritual and exploration helpful, but had to move beyond this broader understanding in order connect performance and routine, at the level of detail required when trying to distinguish between three differently performing groups of learners. I did not want to locate failure in the learner, and took special care not to do this in the ways that I described what learners do. However, I wanted to examine the nature of failure held up to a reference for successful participation in mathematics. This I argue has been achieved in Table 16 above.

I have already stated that the ensnarement into prolonged, stagnant ritualised discursive practice is problematic. Notice how, when we move focus to discursive routines, rituals or explorations, we no longer evaluate the learner as (in)capable, but what we teach and learn, how and what we do, stands to scrutiny. Each learner is unique and what this contribution does is provide a disaggregation of ritualised thinking in ways that separate different forms of ritual from the ways that learners come to participate differently in the discourse. In addition, I have also been able to disaggregate exploratory discourse, to look at traces of where this occurred, and to offer some description and interpretation. Knowing where our learners are in terms of discursive skills, and being able to determine which of these have developed easily and those which pose a challenge to learners, is I argue, the point from which to depart in order to make any credible recommendations for learning.

Too often, in the contexts of poverty and disadvantage in which I have worked as professional developer and teacher, poorly performing learners in particular are taken backwards in our remedial efforts to a randomly determined starting point, where we deem they need to begin learning again. Couple this with a teaching approach which waters-down mathematics to make it easier for learners to understand, and we have a sinister impact on performance. Encouragingly, Table 16 shows that these learners have already objectified some of the features of functions E3 and that they can work with equivalence E4. Building on these established discursive layers would provide a logical starting point for developing complexity. Knowing frequent routines directs learning and teaching towards the potential for reification.

The section that follows discusses the key findings under the six headings which I have used to characterise learner discursive routines as they exist at the end of school Mathematics. To encourage you to read further, let me first summarise the main observation garnered from a broad sweep of the data over the three representations of function: *learners were voluminous around their communication of the algebraic expression; they could say or do very little other than plotting from the points on the table; and they thought that the graph depicted a function.* Locating the frequent codes of Table 16 in the established routine categories, I can now begin developing the object, as it came to exist at different levels of performance, through learners frequent routines. The focus on the most frequent codes means that not all codes present will be discussed, just the frequent ones. Frequency as a discerning criterion gives a view of what learners do and think more frequently. This most likely describes how they have individualised the discourse on function overall, and what has become established for them. The next section gains focus in the question: what do the frequent codes of Table 16 show us about learners' efforts to connect with the object?

8.2 Findings

8.2.1 Goal of learner's communication: To produce endorsed narratives or for social acceptance

	Exploration		**Ritual**	
Goals	**Produces endorsable narratives** • Solves to derive new narratives; establishes purpose for solution and interprets. • Asks , attempts to answer open questions.	E6 E7	**Performed for social acceptance** • Reference to memory or authority • Works to goals set by others • Guessing	R1 R2 R15

- E7 was the single most frequent exploration goal occurring for PG1. This indicated, specifically, a need to generalise among the better-performing learners.
- R2 indicating a ritualised practice occurred as the frequent code across all performance levels.

Learner talk was directly related to what they thought their goal was. Data confirmed the goal as talk, which meets social needs across all performance levels, but was particularly prevalent among poorer-performing learners. The inducement into a social practice impacts the ways learners endorse what they do, and the consequent levels of justification and substantiation which become available in this restricted practice. Independent agentive participation shows in how a learner has individualised mathematical discourse to reproduce or deduce mathematically endorsable narratives. An overt social emphasis such as this, saw the endorsement come from attempts to recall (implying reliance on memory and not reasoning), and from engagement with others, in which affirmation and approval were sought. The most basic level of recall became visible discursively in learners passive use of keywords and a limited selection of routines executed. Poorer-performing learners showed a slimmer repertoire of keywords and routines than better-performing learners. Better-performing learners showed individualisation of keywords and routines evident in the meaning they ascribed to these aspects of their mathematical discourse. Individualisation ensured that these key elements could be applied over wider and even more demanding tasks. Recall ensured that poorer-performing learners remained dependent on each other for keywords, and on the 'how' of a mathematical process.

Perhaps the most provocative aspect of this study lay in how learners talked about an object for which they did not have the established formal discourse. This has to be examined from questions which arose in earlier chapters, and were answered separately in terms of each signifier: the algebraic expression, table and graph, in earlier chapters. Table 16 consolidates the frequent routine patterns of the individual signifiers and permits the same questions to be adapted now, with reference to the complete object. As a result, the question evolves to: what are implications for the mathematical object formed, when learners have social acceptance as a goal? How does social acceptance as a goal show in learner performance?

To better understand what something is, we can look for what it is not. Ritual stands to exploration as one such contrast. Social acceptance as a goal of discourse stands in contrast to the drive to produce endorsable mathematical narratives. This contrasts essentially mathematical goals with social ones. Better-performing learners are shown to have mathematical goals. These learners showed attempts at generality. Generality contributes to connecting disparate signifiers to the object. While learners did not show the meta-discursive means for building generality through the specialised language, syntax or ways of working of the mathematician, discursively it marked a point of entry to begin exploration of the object. Better-performing learners ask telling questions about the object. In contrast, data shows that poorer-performing learners seldom ask questions, or dismiss or abandon questions to which they may not know the answer. Thus, mathematical goals determine not only how learners participate in the discourse, but also the levels of discourse to which they are able to extend. PG1 appears to understand that school mathematics provides narrative and routines, and that they share partial responsibility in growing complexity.

The distinction between PG2 and PG3 showed PG2 being able to recall far more than PG3, but without a ‘when’ -filter. The recall of routines was a source of anxiety for PG3. PG2 seldom used their signifiers towards extending to the next likely discursive level. They showed short deductive chains when they did. Their engagement with each other showed that they were prompted frequently by what the other learners said. This was a general indication of not individualising what they had learned. The ways they recalled mathematics showed they were still routine to phrase driven in their utterances. While this communication did comprise informal mathematical utterances, it suffered the tension induced by fickleness in the ways mathematical decisions were made, where these were based on opinions, and seldom on reasoning.

PG3 claimed very little of the mathematical talk compared with the smallest repertoire of routines, and significantly more utterances, expressing their difficulty in remembering the mathematics. They could recall far less and they expressed this often as an anxiety. The admissions of not knowing what to do, or not understanding what was being asked, gave them a sense of mutual support, which arises generally from people sharing a common problem. Their realisation strands too were short, often colloquial. Speaking with everyday referents in observation and justification, further emphasised social bonds. They too seldom challenged each other, but conformed to a decision, often without needing justification. This showed a generally passive participation in engaging the object. Their colloquial and socially linked utterances, I feel, would serve to alienate them further from the mathematical discourse.

In summary, all learners participate in a social, rather than in mathematical discourse. When a mathematical signifier arises, it is mediated through their interactions, rather than through the mathematical narrative. The social influence keeps learners on the periphery of the mathematical object. PG3 are furthest away. PG1 have not uncovered the object, but attempts at generalisation mark initial attempts to objectify.

8.2.2 What was talked about: Mathematical objects or signifiers

The codes below define this category.

Table 17 Objects and Signifiers Codes

What is talked about	**Mathematical objects**		**Signifiers**	
	• Signifiers are abstract mathematical objects. • Talk of specific features of mathematical objects. • Symbols/ procedures are justified or related to the object.	E1 E3 E5	• Uses mnemonics/visual clues. • Talks of/acts on symbols without their meaning. • Misrecognition of form of algebra or graph; no meaning attached to symbol. • Asks closed questions . • Subjectification • Spontaneous/everyday language	R5 R6 R13 R10 R11 R18

It was encouraging to find all exploration codes present across the three signifiers. R11 and R18 were the most frequent ritual code across all performance groups. It must be kept in mind that exploration codes were significantly lower in frequency than ritual codes. I discuss the exploration codes to fulfil my initial aim of describing what learners can do. These codes describe the discursive paths learners have carved out to access the object. The most frequent ritual code, on the other hand, shows what learners are frequently doing, and thus what has to be developed for objectification and exploration. Under these potentials and constraints, ‘what learners talk about’ comes into focus in order to describe how the function-object exist for learners.

Learner talk centred on what they did with algebraic symbols in this category. In these discussions, the better-performing learners spoke unprompted, of what the symbols signified. Mathematical processes were prioritised, as opposed to mathematical objects. This became apparent in talk at the closure of a process, where the outcome did not signify further realisations for learners. In such instances, learners were not able to justify the process or substantiate the outcomes which it made visible. Further realisations from the conclusion of a process were usually not made without prompt. Possible conclusions from the chosen processes remained unexplained. Symbolic expressions, it seems, existed to be manipulated in standard, familiar processes.

Learners seldom discussed properties of expressions and tables, if these existed. They emphasised the syntax of the process as opposed to what that process may have made possible to realise about the object. However, utterances show that symbols were not entirely without meaning. This was especially evident across performance groups for the linear function, for the algebraic expression and graph. Learners could identify keywords and names associated with some symbols. This was where most of the objectified talk was located.

i Rituals: R11 and R18 appear most frequent:

The high frequency of these codes highlights the need for emphasis on building an objectified discourse in teaching. It further emphasises the need for building connection between learners' previous spontaneous discourse and the developing formal discourse.

ii Exploration: E1, E3, E5 are frequent codes:

Objectified talk occurred most frequently in this category because the features of the signifier and symbols which were involved were identified as nouns. This is the first level of objectification. PG3 remained at this level. PG2 and PG1 were able to reify certain processes, which they associated with the representation. This can be regarded as a next level of objectification. This marked a clear distinction between the better-performing learners and PG3. All groups were able to give meaning to some of the abstract symbols and keywords that are related to functions, where:

- this category showing the widest range of exploratoration codes;
- talk of the features of the function was the most frequent code across all performance groups;

- the features of the algebraic expression and the graph showed greater incidence of objectified talk compared to the table. The features of the function were difficult to discern from the table for all learners; and
- PG1 showed attempts at abstraction in their objectified talk.

Looking at the ways in which learners approached the tasks showed overwhelming evidence that learners' emphasised themselves and their actions in their attempts to mathematise. This relates to the social aspects of learning mathematics and characterises an overtly ritualised practice. Learners participating in a discussion began almost immediately to survey the processes they could recall and to talk about each other's actions, rather than mathematical realisations from the information that was provided. Recall of a routine is indeed important in mathematics, but mathematics extends beyond it. It involves learners in the selection of appropriate routines and the application of these, as a necessary first step. The emphasis on process and routine, without critical decision making about their relevance, was an overall characteristic of the ways learners approached tasks, namely: 'that I am doing something appears to override why I am doing it'.

The second step and the most challenging, highlights the learning paradox, discussed earlier: learners are to participate in a discussion of mathematical objects of which they remain uncertain in their understanding. Through this participation objects begin their mathematical existence for learners. This is the aim of exploration routines, which build the object and complexity around it. The highest level of exploration for this group of learners showed in the identification of features and objects and the reification of processes.

School mathematics can thus be considered as a major contributor to the high levels of signification, due to the emphasis on *how,* at the expense of *when,* in mathematical routines. Learners thus do not pursue developing higher discursive levels of complexity and abstraction around an object. This was deduced from the prioritising of process over reasoning, in curriculum documents, and seen in its manifestation in learner talk in the data. The emphasis on signification across the expression, table and graph further suggests a relationship between working with signifiers (as opposed to objects) and the high incidence of subjectification. The emphasis on process translated into communication about 'what I did with the signifiers' for learners.

At this point of synthesising the communication of the object, the connection became plain between the main routine categories and how an insidious ritualised practice can stunt mathematical growth. A significant portion of subjectifying utterances, for example, related to learners' difficulty in remembering what to do, being confused about appropriate mathematical actions, and describing their intentions or processes using verbs to show actions on symbols. This tied in with the ritualised goals of social acceptance, where learners communicated with each other in reinforcing social bonds. The solidarity of shared common experience from similar difficulties like those mentioned above, obscured and displaced attention away from the mathematical object, particularly for learners who were not performing well in mathematics from the outset. Poorer-performing learners try to remember multiple and varied disconnected routines (as might be required to memorise a dictionary). Apart from there being no rational means to do this, it is also a daunting task when you attribute it currency. This could also have contributed to the uncertainty and indecision poorly-performing learners expressed as they commenced a task. They felt under pressure to remember routines disconnected from the mathematical object, and they had no means to link to the narratives and routines.

The prevalence of spontaneous talk (as opposed to formal mathematical talk) again supported the notion of displacement from the mathematical object. Spontaneous talk was not expected at the levels it was found for these Grade 11 learners, particularly the learners who perform well. While this appears as a criticism of the learning evident so far, it is only partially so. This is because spontaneous language can be used as a tool to bridge or connect with the formal mathematical discourse. The knowledgeable other, the teacher in this case, would largely be responsible for the transformation of one to the other, by increasing learners' participation in the mathematical discourse. In my assessment, this is not prioritised in teaching and learning, and stems from a view of learning and teaching which constructs the mathematics learner from the very beginning as deficit. Evident in learner talk showing similar critical absences and incorrectness. When we assume that learners are incapable, we construct environments for learning which rely on over-simplification and drawing inappropriately on learners' spontaneous experiences. For school mathematics, discourse development or learning appears to have stalled here. While this may provide for teachers and learners an environment in which they falsely assume to be participating in a common mathematical discourse, they are actually engaging talk of a hybrid, colloquial form of the object. This hybrid object lacks the formalisation, regulation and disambiguation of the mathematical discourse and obscures its future development. The approach withholds the tools for exploration on the assumption learners may not cope with the development of complexity. The ways we approach complexity in learning account more for the difficulties in learning than the complexity itself. It does not enable connection between what a learner thinks of the object and what a learner does, and the refining of both these crucial learning processes are treated incidentally. The gradual building of this connection possibly accounts for better-performing learners, showing more evidence of the formal mathematical discourse than the other groups.

My summary for this section relates to the question: what do the rituals and explorations embedded in what learners talk about, contribute to the formation of the object. There are several levels to forming an object: all performance groups show that they can identify key features and properties of functions. Reification of processes increased from PG2 to PG1. As reification increased, so did what learners could do and say about the object. The mathematics of poorly-performing learners relied on identification, which cued related processes. They remained in signification. This was perhaps their biggest block in developing complex thinking on the object. To poorer-performing learners, the object remained a collection of disparate and disconnected processes. For better-performing learners, the word 'function' could be used as a self-standing noun, or as part of phrases which indicated there existed a connection between the different functions. Within these individual functions occurred pockets of objectification. This was the best they could do, without a formal narrative of the object to which to connect the individual functions.

8.2.3 Flexibility: Flexible or rigid

Routines which resulted in the same endorsed narratives can be used interchangeably. This is flexibility. I looked for flexibility when learners were able to use different procedures to get the same end result; when learners used routines in ways that were unexpected, and sought to understand how they justified these choices.

A summary of the most frequent codes showed:

i. Rituals:

- Learners' recycling old routines, R7, appeared as the most frequent code across all performance groups. The ritualised practice was especially stark in instances where learners recycled old routines in inappropriate ways.

ii. Explorations:

- All performance groups showed exploratory code E4 for the algebraic expression. For PG2 and PG3 these were limited to the linear and less frequently to the quadratic function.
- Only PG1 showed this code across expressions, tables and graphs.
- Only PG1 were able to reflect on the closing conditions of a process and the relevance of a process itself.

Rituals were largely related to learners picking up perceptual clues from the task and attempting to match these with familiar procedures they could recall. Flexibility required learners seeing equivalence across the three signifiers, but also equivalence in different algebraic procedures. Objectification relies on reified processes becoming objects, where learners transform from process to object. Transformations were noted, as if in reverse, for poorer-performing learners when they moved from symbolic to concrete, in their talk. The realisation of abstract mathematical objects as concrete objects has a direct influence on the flexibility with which routines can be applied, and what these made possible for learners to realise further. Seeing parabola as a 'smile' obscured developing the notions of the minimum and the gradient of the graph at a point. This has implications for developing future objects like the derivative in Grade 12.

The movement from process to object was coded as exploratory in the data and this occurred most frequently in PG1. The exploration codes noted across all performance groups for the algebraic expression came from the expression being a signifier for the table and the graph. This order of realisations appeared as a frequent and automated routine. It conveyed equivalence among these signifiers. Flexibility and objectification was less evident in learners moving from the graph to the algebraic expression. Only PG1 showed they could move to the algebraic expression, indicating the changes in multiple parameters of the linear and hyperbolic functions. Parabolic functions moved off the standard position, and not turning on the *y*-axis, posed challenge to this group.

In my opinion, the flexibility in moving between representations is located in the teaching orientation to function. Again, if the emphasis is on process as opposed to the relationship/equivalence orientation to begin with, learners will show automation in specific instances, and mostly in a single direction, as described. Flexibility is described as moving through all representations, not restricted to a single direction. It is not an intuitive move, but requires deliberate attention by the knowledgeable other. While I do see flexibility in working certain routines with functions, it appears rigidly formed, restricted to specific instances for these learners. Better-performing learners showed flexibility in adapting their routines to non-standard tasks and questions compared to poorly performing learners. They also showed flexibility:

1. in following through with routines which are established and described above; or
2. in having a wider repertoire of algebraic transformation routines with which to justify their decisions.

In summary, better-performing learners showed wider flexibility with their mathematical keywords and routines. This enabled a multi-directional approach to the object. This resulted in these learners being able to realise far more on the individual functions on the task.

8.2.4 Applicability: Wide or narrow

Applicability relates to the 'when' of a mathematical routine, which this study has shown was seldom present in learners discursive routines. Applicability specifies when a mathematical routine can be used. Usually, learners scanned a mathematical task for prompts, which directed their subsequent discursive course. Applicability was tied to the way that a learner would conclude the routine, called the closing conditions. The data in chapters 5, 6 and 7 shows the emphasis that learners placed on concluding a routine as opposed to reflection on its appropriateness. The challenge to applicability for learners appears to be from the choice of equivalent routines, especially in predominantly ritualised environments. This study found that learners could easily switch to an alternate routine that is suggested by the other in the pair, with little or no convincing required. Without justification being a part of learners' discursive practice, learners showed little investment in the routines they chose. Routines were frequently abandoned if learners could not progress further with the routine, or if an alternative routine was suggested.

Summary of findings:

i. Rituals:

- There was emphasis on action and process R3, and concern with ‘doing something’, indicative of ritualised practice. This was cued by visual clues or verbal suggestions. This code was most frequent across all performance groups and all signifiers.
- The second frequent code, related to the drive for closure of the chosen process.
- There was little evident reflective focus on the final outcome in working with tables and graphs. It was worth noting for PG2 and PG3, when working with the algebraic expression, because learners emphasised the recall of familiar routines, cued but unfiltered. Applicability appeared tied to recall, as learners easily abandoned their response to a question if they could not recall the routine they were associating with it.

ii. Explorations:
   - Moving from process to object was found in PG1 for expressions and graphs. This confirmed the applicability of the chosen mathematical routine. It usually involved reification of process to object.
   - PG1 learner justifications, when they occurred, showed learners were working with objects.

Learners were driven by two actions in the ritualised practice:

- what should be done; and
- how it should be done to completion, in ways that were familiar and recognisable, to produce outcomes which themselves concluded in ways which were familiar and recognisable.

Learners were required to justify the choices they made. They generally had a repertoire involving talk of procedures or properties that they associated with the algebraic expressions and graphs. These were easily applied across all performance groups. PG1 had the widest repertoire of algebraic routines compared to the other groups. Routines related to the table had one dominant application procedure, which involved the plotting of points. The table was not viewed as an object showing a function relationship but was a tool to plot points. The activity of all three performance groups concentrated around the recall of a familiar routine from cues in the task or from the other learner. PG3 showed the most ritualised utterances on the algebraic representation related to applicability, PG2 on the table and PG1 on the graph. The higher aggregate in applicability ritual codes implied that learners were constrained by the mathematical choices they had available. The high aggregate of ritual utterances for PG1 on graphs resulted from learners having a wider range of routines available, where, however, they described and justified these by emphasising their actions and process, not by mathematical verification.

PG1 showed exploratory talk on the graph and this focused on the properties that became visible from the graph. These were objectified and transferred appropriately across different functions. PG3 did not see the properties, and were most oriented to process on the algebraic expression where their talk focused on what they did. Learners in PG1 showed incidence of objectified talk of the algebraic expression in similar ways described above. Aside from plotting, there were no realisations from the table.

PG1 and PG2 showed that they were able to transfer the properties and features of graphs they had studied in school Mathematics to the unconventional graphs on the task. They could qualify decisions based on the features being visible on the graph. Only PG1 could transform the graphical representation by translation and reflection, and extend these to transformations in the algebraic expression. This was bidirectional as they could also relate a change in the parameters of algebraic expression to the resultant changes to the graph. PG3 when probed for justification of their routines, often repeated the routine or frequently explained they were unable to justify their choices. Their mathematics reasoning initiated from the 'how' of a mathematical routine derived from cues they saw in the task or from talking to one another. From evidence relating to applicability, it appears that learners choose from a narrow range of mathematical routines. Choices are cue-based. Better-performing learners were able to:

- justify their choices of routine, but, more importantly,
- reflect on their solutions and choose an alternate route if required for purposes of correction or to check the solution at which they arrived. It ought to be noted however, that this self-correction and reflection were not frequent.

Learners who were performing poorly were not able to filter their narratives for correctness; they relied on their partner for this purpose. Justifications were as a result non-mathematical. This makes an examination of who is being addressed in routines an important focus. In summary, applicability made paths to the object visible and not the object itself directly. Again, better-performing learners showed a wider range for applicability of their routines.

8.2.5 Addressees: Oneself and others, or others exclusively

These codes became visible in the ways that learners communicated what they were doing to each other. Talk was directed more to the person they worked with as opposed to talk about the mathematics. The goal of learners in this study appeared not to become more knowledgeable in mathematics, but to establish and keep social relations, reinforcing the emphasis in goals discussed earlier.

Summary of findings:

i. Rituals:

- R12 was the most frequent ritual across all groups, where they do not seek justification of the routine;
- PG3 had the highest aggregate of R12 in algebra and graphs; and
- R17 ties strongly to R12 where learners change the course of their routine to follow what the other in the pair is doing.

ii. Explorations:

- PG1 could use alternate algebraic means to justify what they were doing.

Only PG1 showed exploratory thinking in the ways they used mathematics to justify what they were doing. PG3 did not seek or attempt justification of their routines in algebraic expressions or graphs. PG2 and PG3 had difficulty reflecting on and explaining the reasons for their choice of routine with one another. Their communication focused on detailed description of what they had done. Interjections and questions were seldom addressed by a mathematical response, but reiteration of the process, or recall from a lesson or what a teacher had said or what was covered in assessments. Explanations did not appear to emanate from individualised mathematical reasoning.

Poorly-performing learners were too easily convinced by what the other person in the pair was doing, and easily accepted explanations offered. This speaks to the low confidence poorly performing learners appear to have in the subject; they seldom conflicted or contradicted the routine of other person. Questions arose from the need to understand what was being done so they felt a part of the process. Justifications, particularly among poorer-performing learners, related to appeals to authority. References to the teacher, schoolwork, the textbook, were often sufficient to continue with a chosen routine. This low confidence also showed in the ways that these learners viewed learning and authenticating their decisions as coming from sources external to themselves. They could only learn if they were taught. Mathematics was something they did, as opposed to something they learnt. Learners appear not to know how to learn mathematics.

As performance levels decreased, so did the use of mathematical keywords and the formal mathematical discourse. Poorer-performing learners spoke colloquially in what can be described as a social conversation as opposed to a mathematical one. They showed a disconnection with the formal mathematical discourse and they seldom spoke to one another. They showed an isolated participation in the mathematical discourse. Better-performing learners appear to engage each other in their routines more often. This opens up the possibility of learning from each other. The ritualised code in this category shows an alienated learner from others and consequently from the discourse. Again, this obscures the path to the object.

8.2.6 Reason for acceptance can be substantiated or followed the rules

The ritualised communication stemmed from learners using non-mathematical means to justify a routine. Data shows that simply stating that were applying a rule would suffice as justification of a mathematical result. Elaboration or reasoning of the meaning conveyed by the rule was not present.

Summary of findings:

i. Rituals:

- PG1 learners justified a routine by saying they followed the rules more frequently compared to other groups;
- R14 was not among the most frequent ritual codes although it did occur across all groups;
- PG1 and PG3 emphasised rules with the algebraic representation only; and
- the code was absent in all PG2 utterances across all representations.

ii. Explorations:

- Learners in PG1 and to a lesser extent PG2 could build algebraic narratives through logical deduction using algebraic symbols. This was restricted to work in algebraic expressions only. It did not extend to learners using algebraic means to justify features evident on a table of values.

As stated earlier, learners in PG3 showed concern when recalling routines, and hence when recalling rules. They often were not able to justify what they were doing due to difficulty in remembering rules and the disconnection from the object by which they had learned these rules. These learners had not made the discourse their own by reasoning the place and purpose of rules and hence, held no agency within it. All groups regarded rule-following in mathematics as important. Rules would be part of the endorsed narratives of mathematics. While rules are learned and initially practiced passively, reasoning their rationale and application would be a necessary condition for exploration.

Through the ways in which they participated in the discourse on function, PG1 learners had internalised rules encountered and could apply them without prompts. PG1 could deduce one rule from another or select a relevant rule to facilitate further deduction. Both PG1 and PG2 could show longer deductive strands than PG3. PG3's peripheral participation in the mathematical discourse has stunted the building of the formal discourse. They were not able to connect their existing discourses, colloquial or informal, to newly introduced rules. They applied new rules as a form of imitation of what they had experienced in the classroom. Each new rule thus existed in isolation of all others.

Poorer-performing learners would benefit from a teaching approach which builds on and connects their colloquial mathematical discourses to the formal mathematical. They also need an environment which encourages their participation in the formal mathematical discourse. One way of doing this would be to ask learners to justify their reasons, or to explain the rules they use. Specialised learning begins with thoughtful imitation of a knowledgeable other. Exploring ways of scaffolding knowledge efficiently for poorly-performing learners needs to be examined by teachers and research.

8.3 Discussion

The problem is persistent poor performance in Mathematics. The thesis started with the expectation that learner communication would, in the main, be ritualised. This was duly confirmed. The reason for it is is due to school Mathematics emphasising the *how* of a mathematical routine. Since school provides the learner with entry into the formal mathematical discourse, it is hardly surprising that discourses develop the way that they do. A coherent and reasoned mathematical discourse relies on connections between the developing discourse and those past. The connection between these is often assumed to be automatic, starting in the curriculum which lists topics as separate and disconnected. Teachers consequently maintain these separate compartments in teaching.

Examining learner communication for their mathematical routines, and describing these in terms of ritual and exploration, provided a useful analytic lens into learner thinking, while emphasising the disconnection between discursive layers. A zoom out, or broader examination of rituals and explorations as part of the larger discourse, shows that an objectified discourse, which attempts to be as close as possible to the formal mathematical discourse, is related to improved performance. We see this confirmed by the PG1, the group of learners who are successful in mathematics. This finding confirms existing literature (Ben-Yehuda, et al., 2005; Bills, 2002).

An objectified discourse appears to support the development of exploration. This is because learners are then exploring mathematical objects and not trying to complete a mathematical process as a goal. The best way to comment on objectification in this study is to look at successful performers PG1, contrasted with those who are not passing mathematics PG3. PG2 is located in the space between these two groups. Objectification became visible on three levels of discourse in this study:

i. identification of functions, their features and properties;
ii. reification of the processes associated with functions; and
iii. talk of objects.

PG2 and PG3 learners remained mainly at the level of identification. PG1 showed evidence of all three levels. Utterances which refer to memory and recall seem to be strongly associated with identification. This was most prevalent in PG3, who found difficulty in recalling names and the syntax involved in processes. PG2 attempted to justify their routines on the second level of objectification. They could execute a process but they justified it in two ways: a repetition of the process, or a mathematical interpretation of the process in the context of the task. PG1 showed in comparison, that they had a repertoire of endorsed narratives, which they could execute with flexibility and offer interpretations in situations in which they were unfamiliar.

The most ritualised routines belonged in PG2. The ritualised patterns in PG2 and even more so in PG3 were steeped in subjectification. This was the most frequent code across the study as a whole, and can be taken as a significant part of the way that these learners communicate mathematically. This shifted attention away from the mathematical objects and to the performer of the actions. Subjectification was thus an obstacle to exploration for all these learners. Another way that subjectification appeared was through references to the sources of the routines that learners enacted. PG1 picked from a repertoire of mathematical rules, but PG2 and PG3 referred to rules which they remembered from class work, assessments, and textbooks. This shows a strong tie to mathematics as a social activity, a community into which they wanted acceptance, through appeals to authority. The source of mathematical narratives is thus vested in a person, or the circumstances under which the narratives are learned. Assessments are noted as an impetus and indication for what needs to be learned. As a result, learner narratives are changeable to fit the assessment demands.

There are many distinctions in the ritualised practices of the learners. Perhaps the most important one to come out of this study is that the discourse of object function has not been formalised and encapsulated for these learners across schools and performance groups. PG1 grappled with an informal sense of the object, but possessed an array of routines compared to the other groups. To PG2 and PG3, the object function resides in the graph. This results in learning pockets of disconnected routines related to a specific representation. Many poorly-performing learners seldom progress beyond this, overwhelmed by a sense of too many representations and too many rules.

The parameters of school Mathematics, in addition, establish the rigid boundaries in which learners are to engage the objects. School Mathematics appears the most obvious cause for the segmented ways in which learners think when it comes to functions. It is a school's responsibility to provide a knowledgeable other, who serves as a mirror of the formal mathematical discourse, and shows how to engage it. Combined with the antagonistic factors to learning mathematics, the work of the knowledgeable other becomes all the more complex.

8.4 Contributions

In this study I have examined the learning of mathematics in the context of rampant poor performance. I sought a description of learners' discourses of the mathematical object, function, at different performance levels. Far too often, in literature and life, the persistent problem is located with the learner. As a teacher, philosophically and morally I had difficulty with this. Having passed through apartheid education, both at the secondary and tertiary levels, I encountered the apartheid philosophy, which defined learners of colour by limitation, particularly in learning of mathematics and the sciences. I was told by my chemistry lecturer at university that we were deemed not to be “genetically predisposed” to accommodating the levels of complexity and abstraction called for in such disciplines. I struggled to learn chemistry after that.

When we depart from a position which constructs learners as capable, we are forced to examine learning by alternate means. Using Sfards’ discursive theory of commognition, and selecting her lens on routines, enabled an alternate view of learning. Learning is defined by Sfard as “an individualised form of interpersonal communication” (Sfard, 2008, p. 81). Through participation in the mathematical discourse, we develop through levels of abstraction and complexity. This view of learning resonated strongly with me both on a personal and professional level, as a teacher of mathematics. Locating my study in repetitive patterns (routines) in learner communication, I sought to access learner thinking in function, through the nature of learners’ discursive routines. The theory provides the distinction in routines between exploration and ritual. I could thus describe learner thinking broadly as being ritualised or exploratory.

To build detail around learner routines, I was guided by the following questions:

1. How does learner thinking, evident in the routines they use, illuminate poor performance in functions?
   1.1 What are learner routines in function at different levels of performance?
   1.2 What are the characteristics of learners rituals and explorations at different levels of performance?

2 Did learner routines on function include exploration and what did this look like?
   2.1 Where and around which aspects of the mathematical discourse do instances of exploration occur?
   2.2 What were features of learners rituals?

3. What can a focus on routines tell us about the object, function?

I mined my rich learner data through each of their categories, and found certain distinctive features emerged. Theoretically, these would need to be developed further and they extend beyond the scope of this study. Methodologically, though, it became clear very early on, that there are vast differences in the discourses of learners who perform well in mathematics and those who do not. Philosophically, this sat more comfortably with me. As a teacher, I could develop description of learners' mathematical discourses and steer clear of harmful, limiting descriptions of learners themselves. Constructing learners as capable beings, especially in South Africa, where most learners speak multiple languages in diverse and dynamic learning environments, I could extract characteristics of failure and success from learners' mathematical discursive routines.

My development of indicators within ritualised and exploratory discourse was empirically based and the study does make decisive claims about learning. Methodologically, however, I opened a view which contrasted failure and success. It offers a way to look at discourse related to levels of performance.

There are several contributions that this study makes to the field of mathematics education research, in general, and discursive work, in particular:

i. It describes what exploration discourse looks like in dominant ritualised practices.
ii. Within the commognitive framework, ritual and exploration are two broad categories, used to describe thinking. In order to make sense of school mathematical practices, particularly in the South African school environment, there was need for a more elaborate understanding of what ritualised practice looks like. My study extends the theory by offering a description of what ritualised and exploratory discourses look like. Due to a dominance of ritualised practice in South African schools, I hope to initiate the impetus for deeper insights into these practices.
iii. It holds up a contrast of the discourse of failure in mathematics to the discourse of success.
iv. It presents a methodology for looking at performance with respect to discursive levels.
v. Finally, it establishes the case for the importance of exploration routines in school mathematics for their link to successful performance in Mathematics.

8.5 Recommendations

This study can make the following recommendations to policy, practice and research:

8.5.1 Policy

- The teaching of function needs to orient towards a function as a relationship. School mathematics teaches the different functions as separate topics. Learners use the word 'function' as a description of each of these packages, and in particular, related to the graph only. With the emphasis on the relationship between variables or co-variation, learners would be able to encapsulate the different functions and their representations into the object, function.
- Diverse function types can be incorporated into school Mathematics to offer learners a broader experience with functions. This must include the different representations with the emphasis on equivalence and including the properties of the table as a signifier. The table was discussed by learners at a very basic level in this study. Yet, it has great potential to be introduce into algebra as the learners' first experience with expressing generality. It deserves greater emphasis in policy documents.
- Describing a teaching approach which encourages the independent participation by the learner as soon as possible in secondary school Mathematics. Chapter 1 discussed the observation, certainly in most South African classrooms where learners work with little agency and certainly seldom without the instruction and presence of a teacher.
- Allocation of time in the curriculum and teaching schedules to developing independent participation by learners in the mathematical discourse. This is different to the current emphasis on remedial work and re-teaching. It must have as its aim working mathematically through problems of increased demand and challenge. This has two discursive benefits, the first of which is that it raises the level of demand for learners and grows the basic human need to build complexity and abstraction. Learners gain from exposure to non-routine type questions. This encourages learners to work with or develop exploration routines. The second discursive benefit is that it allows learners to watch an experienced mathematist, the teacher, work with and talk about mathematical objects or engage exploration routines.
- A learner's first language is crucial within the space of developing connected knowledge. Teachers will benefit from having an African language as part of their Mathematics teacher training. It is not possible to have teachers speak all eleven of official languages, but speaking one is a start. It will provide crucial insights into how learners, learn

mathematics often in a language which is not their first. This could impact how teachers build their explanations, and how they manage the connection between learners colloquial and informal talk with the formal mathematical discourse.

8.5.2 Practice

- We treat classrooms as homogeneous environments in which the poorly performing learners are taught in the same ways that successful learners are. In addition, the school and education department apportion the blame of failure to the learner. When we approach learning as a development of a discourse, and progress as measured through transitions through different levels of discourse, the philosophical shift places the emphasis on knowledge and learning, as opposed to the learner. This data shows that better-performing learners benefit from discussion with each other. They entered exploration routines more often. In an overcrowded classroom and tightly packaged curriculum, better-performing learners are a teaching and learning resource. They are a knowledgeable other in situations of limited and strained resources. Poorer-performing learners could benefit from working with a peer.
- Teaching needs to acknowledge the place of learners' colloquial and informal discourses and maximise these in the transition to a formal mathematical discourse. The complexity of multiple first languages being present in a classroom, in combination with poor performance, sees teachers often creating new jargon ('smile', 'frown' as descriptions of the parabola, for example) or inventing short methods distinct from formal mathematics, which is deemed too difficult for learners. Data showed that learners did not communicate in key mathematical words and narratives expected at Grade 11 level. The bridge between the colloquial/informal discourse and formal discourse is a strategic one. The formal necessarily builds on the discourse that learners bring to the classroom. Emphasis needs to be placed on correct use of the mathematical discourse and processes are to be linked to the objects they signify.
- There is much to be gained from an objectified discourse. All possible realisations can then be connected to each other or to the object they build towards. This implies that attention in teaching must be given to building connectedness between what is currently being taught, in relation to what has been taught. This appears absent, or difficult to do, if mathematical objects are broken down into discrete topics, which are each treated

separately as discussed in Chapter 2. This lack of connectedness in teaching impacts the development of flexibility and applicability in learners mathematical routines discussed earlier in this chapter. Learners are thus unable to transfer mathematical routines across contexts and to different objects. This is necessarily a barrier to exploration and independent participation goals of teaching mathematics.

8.5.3 Research

Commognition has as its strength the rigour of the definition of its keywords and the operationalisation of the terms it uses. At the start of a research career, the frame has contributed to my clarity with the definition of the words I have used and hence the meanings I wished to convey. It has impacted the ways that I look at other research as a consequence. Like mathematics is a discourse, this was my initiation into the discourse of research and it showed that learning is possible if we learn the rules of communication for a particular community.

To this end, the role of the knowledgeable other in the chosen research frame was invaluable as there are multiple levels involved in the enquiry. Developing a rigorous description of the encompassing role this expert plays in the research process would illuminate, disentangle and demystify a process fraught with confusion for the student. The involvement of Sfard, for instance, in the development of my research questions for this study, opened up the original questionnaire to a much greater degree for learner talk. It provided rich data for working within the commognitive framework.

I also have to examine what a commognitive approach enabled in this study. The first aspect was that it emphasised rigour in the definition of key research terms. This ensured that terms we use easily in conversation, like thinking and learning, were clearly defined as were the key constructs within the theoretical frame. The second notes, that the findings of the study were located in six schools and across 18 pairs of learners. While this provided an initial view of learner routines across schools and performance levels, and certain common ritual practices persisted across these different settings, I am careful not to generalise my findings. I am cautioned by the fact that this is a relatively new framework, there was not a vast collection of literature to draw upon and certainly my extension of it will need further testing in research. The strong theoretical framework of commognition provided an anchor for my analytical work and thus the findings derived as a result. The weight of the findings in this study comes from their strong theoretical base, but also from the multiple ways in which the most frequent patterns emerged as common across schools and groups. I therefore generalise these tentatively in this thesis, with the view that they pinpoint necessary starting points for future research of this type. They initiate conversation on tackling poor performance by alternate and systematic ways by allowing detailed description of learner routines, noting about them that which might hinder accessing the object.

My selected focus on different routines of ritual and exploration gave new dimension to the way that I had previously viewed learning. Routines are a basic human practice, where we seek patterns to make sense of things. They are seen and used by experienced mathematicians to grow knowledge. Teaching learners to be aware of and to attend to mathematical routines has implications for learning mathematics as well as for learning about learning. The focus on routines in this study finds its contribution in exposing dominant routines and their characteristics as taken from school Mathematics. It begins to tackle the challenge in meta-level learning which learners experience. In attempting this, perhaps the most challenging research aspect of commognition for me was the initiation into the theory. I anticipate that as it gains ground, its constructs, keywords, narratives will be distilled for their simpler, core meaning. Also, further work will provide varied ways in which researchers have individualised and extended the theory, the appeal of which lay in its rigorous, well-defined research principles and its contribution to how we learn and view learning.

8.6 Suggestions for future research

The learners in this study show limited access to the formal, endorsed narratives of mathematics. They appear overwhelmingly bound in recall and process. I speculated about several competing tensions which may have contributed to this and discussed these in Chapters 1 and 4. Commognition provided an alternate way of looking at learning mathematics as a means to address persistent poor performance. Some gaps in literature arising from this study are listed below, and could benefit from further investigation under a commognitive lens:

- discourse as it relates to different levels of performance;
- the transition from ritual to exploration and the means to cross this divide in learning;
- the discourse of failure;
- participation in the mathematical discourse in multilingual classrooms;

- learning mathematics in contexts of poverty of environmental and economic resources and expertise;
- subjectification (which appeared as a dominant code in this research);
- building pedagogic strategies for an objectified discourse and exploration routines for learners; and
- teaching strategies for increasing mathematical demand without alienating learners.

Using a relatively new approach to understanding learning, there were several questions which arose during the course of the study. These are listed below as themes or questions which this work could take further:

- the role of a knowledgeable other in the development of the methodology of a study;
- the role of language in the participation in a mathematical discourse;
- how exploratory discourse can develop in ritualised contexts;
- a reexamination of the importance and placing of the formal mathematical narratives in school mathematics;
- effective means of scaffolding which allow a learner to move from ritualised participation to independent exploratory participation;
- links between discursive levels and performance; and
- using learners' colloquial discourses effectively to support the development of the formal mathematical discourse; and
- developing an objectified practice from an overtly subjectifying one.

These themes taken further in research could build a fuller picture towards understanding performance and participation in mathematics in South Africa.

8.7 Self Reflection

My research has made a significant contribution to my growth as a teacher and as a researcher. Commognition provided the gaze into learning as I tried to look into the persistent problem of poor performance in mathematics. As a researcher, it emphasised the importance of thorough and unambiguous operationalisation when working discursively. As a teacher, it invoked the need to seek effective means of developing learner discourses. Often I would oversimplify the mathematical discourse, to invite learners into participation within it. I have come to realise that bridging this colloquial developing discourse with the formal mathematical, has to be deliberate teaching effort. Its neglect possibly accounts in part for the difficulties learners experience in meta-level learning. Viewing mathematics as a connected discourse, many of its routines are applicable over various contexts and fields of its knowledge, and these routines are necessarily connected to its objects. Many learners, even better-performing ones, engage a ritualised practice due to learning without connection. Connection implies that discursive layers gain significance and can be built and extended from those which already exist for learners.

As I conclude this study, I became aware of new work from authors referred to often here. They examined the role of the definition in developing a mathematical object (Tabach & Nachlieli, 2015). Similar to my study, learners worked in pairs. Contrasting with my study: learners engaged a formal definition and worked on developing meaning for the definition with an expert or knowledgaeable other. This was important meta level learning especially where learners' meanings were held in conflict to each other. The learners from my study develop some discourse of the object function, it appears without the formal mathematical definition. Without the guidance of an expert through discursive conflict, my better performing learners showed they were able to make skilful deductions from objectifying the properties and nature of various functions and encapsulating these into themes. This enabled them to discern a function. While their means were far from being rigorous or even mathematical, better performing learners were still able to make the distinction. This was encouraging as the contrast of the two studies showed the learners in my study tended to wards an objectified disourse and exploration routines largely unguided by experts. It speaks strongly to possibility.

I became aware through the research process of the crucial role that language plays in learning. It was wonderful in interaction with learners to hear the diverse sounds and languages of the classroom. Language offers a bridge between learners' informal experience and the formal discourse of mathematics. While I have tried to show in this study, through translation of learner utterances, from their first language to English, that their talk in their first language was largely not mathematical, I was struck by a moral dilemma. As a South African and a teacher, I have not prioritised the learning of an African language as part of my practice, and have missed on this enriching my experience of learning and teaching mathematics. I have resolved that this will be my next pursuit after this study is complete.

Finally, our access to technology and knowledge has grown exponentially. How we teach learners to be efficient in learning mathematics becomes very important in this context. Growing the school mathematical discourse for advanced levels in reasoning, and creating an independent drive for complexity and abstraction, prepares our learners well for their future.

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