PROSPECTIVE MATHEMATICS TEACHERS' TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE OF GEOMETRY IN A GEOGEBRA-BASED ENVIRONMENT

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A thesis submitted to the School of Education, Faculty of Humanities, University of the Witwatersrand, Johannesburg in fulfilment of the requirements for the degree of

Doctor of Philosophy

January 2017

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DECLARATION

I declare that this thesis is my own unaided work except as indicated in the acknowledgements. It is submitted for the Doctor of Philosophy degree at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Signed

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Date

DEDICATION

To Lethabo Kimberly, the beautiful sunshine in my life

To everyone to whom I owe for being "me"

ABSTRACT

This research study focused on exploring prospective teachers' knowledge of geometric reasoning in teacher preparation. Premised on the claims that learning mathematics is profoundly influenced by the tasks, by the learning context and by the tools that are used in mathematics instruction, mathematics prospective teachers' technological pedagogical content knowledge was examined. The technological pedagogical content knowledge (TPACK) framework was employed to study the prospective teacher's knowledge of circle geometry as proposed by Mishra & Koehler (2006). The main focus of the research was on investigating the empirical and theoretical questions of what characterizes aspects of prospective teachers' technological pedagogical content knowledge. These aspects were geometry content knowledge (CK), geometry pedagogical content knowledge (PCK) and geometry technological content knowledge (TCK). This exploratory multiple case study explores the TPACK of six mathematics prospective teachers enrolled in a second-year undergraduate mandatory mathematics methodology course in an urban South African university. Data was collected through prospective teachers' (PTs) responses to circle geometry tasks, interviews and screen cast recordings. Rubrics were employed as analytical tools. Duval's (1995) cognitive apprehensions and processes were engaged as interpretative tools to understand how the PTs responded to the CK, TCK and PCK tasks. The results suggest that prospective teachers' circle geometry technological pedagogical content knowledge constructed in a GeoGebra-based environment is characterized as weak emanating from weak geometry content knowledge (CK), weak technological content knowledge (TCK) and weak pedagogical content knowledge (PCK). The study has shown that a weak geometry CK was evidenced from the participating PTs' weak display of cognitive apprehensions and geometry reasoning processes. This study contributes to the current debates on teacher professional knowledge and on an understanding of frameworks for which teacher knowledge can be premised in South Africa. A model was developed for classifying and describing forms of mathematics connections in geometry knowledge at teacher preparation level.

ACKNOWLEDGEMENTS

The generosity of many people led to the fruition of this thesis. My deepest appreciation goes to the following:

1. My supervisor, **Prof. Margot Berger**, for the guidance and mentorship that you afforded me, for being meticulous throughout the countless revisions of the work and most of all for teaching me ‘how to listen’.
2. My co-supervisor, **Dr. Marguerite Miheso-O’connor**, for believing in my work, for being generous in sharing your expertise regarding rubrics, for understanding what I needed to hear and how to say it.

   Each of you has left an indelible mark on my life for which I can only express my sincere gratitude. May you continue to inspire and challenge many future students!
3. The mathematics prospective teachers who provided the opportunity for me to study their technological pedagogical content knowledge. Without your participation the fruition of my PhD would not have been possible.
4. My husband, Letsweletse and my children, Bontle, Moses, Tlamelo and Isabella for the encouragement and moral support, for listening to my frustrations and for your efforts to calm me down. Thank you Isa for your company, for listening to my complaints and for the scrumptious meals.
5. My mother and my siblings for your encouragement. This is for all of you.
6. My friends: Mark Winter, Rorisang Rammiki, Shadrack Moalosi, Ludo Mphathiwa and Felix Omal and many others for your support throughout this gruesome journey.

PUBLICATIONS ARISING FROM THIS STUDY

Refereed conference proceedings

Ramatlapana, K. A. (2016). Prospective Mathematics teachers' circle geometry technological content knowledge of teaching in a GeoGebra-based. In W. Mwakapenda, T. Sedumedi and M. Makgato. (Eds.) *Proceedings of the 24th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE)*, 12 – 15 January 2016, (pp. 208 - 220). Pretoria, South Africa.

Ramatlapana, K. A. (2014). Developing rubrics for TPACK tasks for Prospective mathematics teachers: A methodological approach. In M. Lebitso and A. Maclean (Eds.) *Proceedings of the 20th Annual National Congress of the Association For Mathematics Education of South Africa* (AMESA), 07 – 11 July 2014, (pp. 198 - 210). Kimberley: South Africa.

Ramatlapana, K. A. & Berger, M. (2013). Prospective Mathematics teachers' knowledge of teaching geometry in a GeoGebra-based environment through lesson plan development. In N. Mpalami and R. Letlatsa. (Eds.) *Proceedings of the 4th African Regional Congress of the International Commission on Mathematical Instruction (AFRICME)*, 11 – 14 June 2013, (pp. 140-150). Maseru: Lesotho College of Education.

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LIST OF ACRONYMS/ABBREVIATIONS

CAS: Computer Algebra Systems

CK: Content Knowledge

DGE: Dynamic Geometry Environments

DGS: Dynamic Geometry Systems

NAEP: National Assessment of Educational Progress

NCTM: National Council of Teachers of Mathematics

PCK: Pedagogical Content Knowledge

PK: Pedagogical Knowledge

PT: Prospective Teacher

TCK: Technology Content Knowledge

TPACK: Technological Pedagogical Content Knowledge

TPK: Technological Pedagogical Knowledge

CHAPTER 1
INTRODUCTION AND FORMULATION OF THE PROBLEM

1.0 Introduction

The purpose of this multiple case study was to explore and characterize aspects of prospective teachers' technological pedagogical content knowledge of geometry, constructed within a GeoGebra-based environment. The major focus of this chapter will be to formulate the problem of the study by providing a description of the study background, rationale, and statement of the problem. Further, the research questions that guided the study, as well as the structure of the thesis are outlined.

1.1 Background to the study

Understanding teacher competences has been the focus of research for some time. The issue of teachers' knowledge of teaching for high learner achievement has contributed to the conceptualization of the term *teacher knowledge* (Beswick & Watson, 2012). Through the works of Shulman (1986, 1987); Hill, Ball, and Schilling (2008) and Ball, Thames and Phelps (2008) various categories of teacher knowledge have emerged. Among the dominant categories of knowledge are content knowledge and pedagogical content knowledge, of which numerous studies have been done to establish an understanding of what the teacher needs to know to be able to successfully teach a subject. Within mathematics knowledge, there is an interplay between mathematics content knowledge and mathematics pedagogical content knowledge. For instance, there are various aspects that teachers and teacher educators need to consider to successfully teach geometry. Teaching geometry involves an understanding and appreciation of the history and cultural context of geometry, knowing how to recognize interesting geometrical problems and theorems, and geometry content knowledge, competence and proficiency (Jones, 2000; Jones, Lagrange, & Lemut, 2001). As such, I sought to investigate the prospective teachers' understandings as learners and teachers of geometry relating to content knowledge, pedagogical content knowledge and technological pedagogical content knowledge. Moreover, the introduction of technology to mediate the learning and teaching of mathematics has illuminated the crucial role of tools that impact on the understanding of mathematics.

Over the years, technology in education has had a great impact on the teaching and learning milieu. The rapid developments of new technology-based tools have gained widespread acceptance and use in the teaching and learning discourses. The USA National Council of Teachers of Mathematics (NCTM) has acknowledged the influential role of information and communication technology tools in teaching and learning of mathematics by developing standards that incorporate technology integration in mathematics (National Council of Teachers of Mathematics, 2000). It has been argued that technology has empowered learners through democratization of knowledge, participatory learning, authentic learning and multimodal learning (Lemke & Coughlin, 2009).

Despite the diversity in technology-based learning environments, questions have emerged about how technology can be integrated into mathematics teacher education. The development of prospective teachers' mathematical thinking processes is a major goal of any mathematics teacher education programs. Moreover, technology tools can be used to foster mathematical thinking processes such as conjecturing, justification, and generalization and so it is imperative that attention should be paid to prospective teacher (PT) learning and preparation to teach in technology-based environments. Teacher education programmes are proposing that undergraduate courses in mathematics for PTs integrate technology into teaching with activities that promote mathematical thinking (Adler & Davis, 2006). These activities should enhance PTs' thinking by developing mathematics habits of mind such as meta-cognitive skills and problem solving skills. Technology environments may boost mathematical thinking through visualization and abstraction.

The mandate of Mathematics Education programs is to deliberately ensure that they provide formal learning situations which prepare PTs to teach school mathematics as well as to develop classroom activities to address weak content knowledge that prospective teachers may display in teacher education programs (Peressini & Willis, 2004). Hence, the courses in these programs should include tasks that prompt mathematics thinking and reasoning, and that promote both teacher content knowledge and pedagogical content knowledge. Tasks for mathematics teacher education should, as pointed out by Bartolini Bussi and Maschietto

(2008, p. 206), “make PTs capable of planning and running effective classroom activities”. There is a gap in literature that focuses on prospective teachers with weak or no geometry content knowledge and pedagogical knowledge for teaching geometry in a technology-based environment. The study addressed that gap by examining what characterizes the prospective teachers’ pedagogical content knowledge for teaching circle geometry with GeoGebra (technological pedagogical content knowledge). The PTs’ circle geometry knowledge was developed within a secondary mathematics methodology course. The focus of the study was on the mathematical thinking processes of the prospective teachers as they learned or re-learned school geometry in the GeoGebra based environment.

Very little research exists that explores the complexities of South Africa’s prospective teachers’ geometry content and pedagogical content knowledge. Mathematics Education programs need to focus on both the PT as a learner of geometry and the PT as a teacher of geometry. In South Africa, some PTs lack prior knowledge of geometry because they have never learnt geometry at school. So there is a need to address both the content and the pedagogical aspects of PTs’ knowledge of geometry (De Villiers, 1997; van der Sandt, 2007; van der Sandt & Nieuwoudt, 2005).

The advent of new technologies has transformed the roles of the teacher, the student and the learner. Research has identified that limited technology knowledge and skills and technology pedagogical knowledge is a challenge for teachers who are confronted by the difficulty of integrating technology in a typical classroom (Hew & Brush, 2007). On the other hand, most learners and higher education students are proficient in using technologies such as mobile technologies. Some of the current cohorts of school learners are ‘digital natives’ whereas their teachers are ‘digital immigrants’. These terms were coined by Prensky (2001a) and both terms have spurred debate among technology integration researchers. Thinyane (2010) in her investigation into South African first year students has argued that students who qualify for the digital native title as defined by Prensky (2001a), do not all act and use technology in the manner that Prensky describes. Most current South African students (of the same age as many digital natives) lack experience in using technology. Many South Africa (SA) students are not digital natives in the Prensky sense; rather, although they may have some experience with mobile telephones they are newcomers to the use of many technological tools, such as computers.

Therefore, researchers and educators alike cannot ignore technology, the moving target. University students are currently faced with a demand for technology skills in undergraduate and graduate courses. Since most universities have technology resources where students have almost unlimited access to integrating technology into course offerings implies that students should be competent in technology usage.

Various university programs incorporate the use of technology, with no exception of mathematics education methodology courses as no exception. Methodology courses provide a meaningful context for technology integration where technology pedagogical content knowledge (TPACK) can be developed. A study by Angeli (2005 p. 394) suggests that preparing technology-competent teachers in teacher education programs is a “challenging and difficult issue that needs to be systematically planned and carefully considered”. The problem in South Africa is confounded by the fact that some of the PTs have never studied geometry at school and so need to learn the content as well. In addition, many of these students are not proficient with technology.

1.2 Rationale for the study

In my endeavour to pursue my interest in technology integration in mathematics learning, I examined PTs learning and re-learning mathematics and learning to teach mathematics with technology, specifically the GeoGebra software. As mediators of mathematics learning PTs should experience technology first if they are to incorporate it into classroom mathematics teaching and learning. It is worth noting that teachers’ beliefs in mathematics influence their decisions on pedagogical practices. It is essential to understand the beliefs that influence teachers’ decision to use technology as these may be barriers to using technology for instruction (Hew & Brush, 2007). In the same light, more research is needed in order to understand and improve mathematics learning in technological environments; particularly, what processes and actions should be illuminated and addressed when dealing with technological artefacts in mathematics instruction. In their study on South African teachers’ use of dynamic geometry software in high school classrooms, Stols and Kriek (2011) found that teachers’ behaviour towards dynamic geometry is influenced by the perceived usefulness of technology in the classroom. Teachers’ perspectives on teaching and learning mathematics in technology-rich environments should be illuminated and explored at teacher preparation level. Niess (2005) reiterates that teachers' decisions to implement technology into their teaching practice rests on their knowledge of technology, knowledge of mathematics, and knowledge of teaching.

The objective of this study was to examine PTs' knowledge within the context of school geometry content and pedagogical tasks developed in a GeoGebra-based environment. The PTs were enrolled in an urban university in South Africa. GeoGebra, like any dynamic mathematics software was preferred because of its roles in enhancing mathematics teaching, providing a foundation for deductive and inductive reasoning and enabling opportunities for creative thinking (Sanders, 1998). GeoGebra allows the user to dynamically construct, draw, visualize and adjust geometric objects from different perspectives using the drag mode. GeoGebra is an open-source software and that is an important consideration in a developing economy like South Africa. The Dynamic Geometry Environments (*DGE)* are known for developing visual skills which allow for experimental exploration of properties of figures and different orientations (Goldenberg & Feurzerg, 2008; Laborde, 2000; Mogetta & Jones, 1999). The use of DGE has contributed to geometry resurfacing in many countries' curricula. For instance, the inclusion of geometry in the 2012 Curriculum Assessment and Policy Statements (CAPS) curriculum of South Africa is an indication that South Africa recognizes the role of geometry in the curriculum. De Villiers (1996) attributes the past failure of traditional geometry education in South Africa to a curriculum presented at a higher cognitive level than those of the learners.

Goldenberg, Scher and Feurzerg (2008, p. 81) concur with Laborde (1992) that "geometry on a computer is different from geometry on a paper". In a dynamic geometry environment completing constructions and investigating its properties enhances students reasoning processes or knowledge in action by focusing on invariant properties while dragging elements of the figure (Mogetta & Jones, 1999; Owens & Outhred, 2006). DGE allow users to test geometric conjectures and to present dynamic illustrations of relationships or theorems but it cannot generate proofs of theorems. The implication is that DGE as a tool for learning geometry content knowledge has constraints and affordances. Geometry tasks developed in a technology-rich environment should be examined for their potential to develop PTs' mathematical knowledge. This knowledge is often found to be weak for learners in South

Africa (de Villiers, 1996). It is through the exploration with several tasks that inferences are made, deductions are drawn and techniques for solving such tasks are developed. Laborde (2001) posits that tasks developed in a DGE are either facilitated by technology or changed by the technology. Notwithstanding the role of technology in task development, it is imperative to understand the relationship between the teacher, student, task and technology. Olive et al (2010) contend that mathematical knowledge emerges through this interaction which is best understood in terms of a didactical tetrahedron, where teacher, student, task and technology are at the vertices of the tetrahedron.

One important justification for my study is the critical role and the potential that technology has on the teaching and learning of mathematics. As teacher educators we need to identify areas that advance mathematics knowledge for teaching, particularly paying attention to the promotion of the processes of instrumentation and instrumentalization when dealing with technologies to mediate learning in the mathematics classroom. I concur with Niess (2006) that teacher educators need to find means of addressing the question: What do teachers need to know and be able to do and how do they need to develop this knowledge for teaching mathematics in the 21st century? The epistemic goal of this study is to characterize aspects of PTs' technological pedagogical content knowledge and contribute to the knowledge and understanding of mathematics learning and teaching processes in technology environments. I reiterate that PTs in South Africa are learners of geometry and future teachers of geometry. Most PTs have not learnt geometry at school, with or without technology. The PTs are faced with the challenge of the technology infused environment which requires that they develop their technology content knowledge (TCK) and technological pedagogical content knowledge (TPACK) and their mathematical knowledge. As such, a deliberate move was made to specifically pay attention only to the TPACK constructs that have content (C) as the common denominator. It was deemed necessary to consider C since content knowledge is very weak among PTs and considering that CK necessary, albeit not sufficient.

1.3 Statement of the problem

It is evident that prospective teacher education has been under-researched in South Africa (Adler, 2004). Teacher educators require an in-depth understanding of the mathematical thinking of their students. This study was aimed at contributing towards the development of knowledge relating to mathematics PT learning and specifically contributes to the body of research into the teaching of mathematics PTs in methodology courses and mathematics knowledge for teaching as a research area. The pragmatic goal of this research was to explore ways of understanding the development of PT technology content knowledge, mathematical content knowledge and pedagogical content knowledge. This was achieved by means of PTs engaging with tasks that elicited visualization, construction and reasoning processes. It is through the observation and analysis of this engagement involving the use of the technological tool, GeoGebra, that I ultimately examined aspects of PTs' technological pedagogical content knowledge of circle geometry. The field of geometry was preferred largely for its potential for the advancement of mathematical meaning-making processes. I focused on circle geometry because learning circle geometry provides opportunities for the development of deductive reasoning, particularly within the context of proving theorems. Mathematics Education in South Africa needs an intervention for teachers who have never been exposed to school geometry. In the past two decades geometry has been in and out of the South African mathematics curriculum, implying that some learners were exposed to geometry whereas others were not. At the same time, teachers are not sufficiently prepared to teach geometry. Nakin (2003) exposes that the mathematics syllabi (between 1996 and 2000) of the six (6) Universities in South Africa in his study reflected an under-emphasis on geometry knowledge. This is attributed to the void impacted by the weak nature of or lack of school geometry knowledge. University graduates are thus often not prepared for the teaching of geometry in the schools.

The entire study took place within the context of PTs who mostly had weak mathematical understandings (Pournara, 2009) and weak technology skills (many PTs never used computers before coming to university). Bearing this in mind, re-learn school mathematics was integrated in the teacher preparation program. Using GeoGebra, PTs were provided with the experience of learning mathematics with technology; hopefully this helped them understand the value of a dynamic environment like GeoGebra for their own students to discover mathematics (Sherman, 2010). It is the intention of this study to inform mathematics teacher educators as they develop methodology courses for PTs of Mathematics Education programs.

1.4 Research questions

This study focused on the empirical questions of what characterizes prospective teachers' geometry content knowledge, geometry pedagogical content knowledge and geometry technological content knowledge in the context of a GeoGebra-based environment. These types of knowledge are different aspects of the PTs technological pedagogical content knowledge.

The study was guided by the following research question:
What characterizes aspects of prospective teachers' circle geometry technological pedagogical content knowledge? In particular,

1. What geometry content knowledge (CK) do the PTs display?
2. What technological content knowledge (TCK) do the PTs display?
3. What pedagogical content knowledge (PCK) do the PTs display?

1.5 Definitions of terms

The work proposed in the study necessitated operationalization of terminologies to suit the context of study.

- *Cognitive apprehensions* are several ways of looking at a drawing or visual stimulus (Duval, 1995).
- Comprehension of geometry involves three *cognitive processes*; the visualization process, construction process and reasoning process (Duval, 1998)
- *Content Knowledge (CK)* is teachers' subject matter knowledge.
- *GeoGebra* is an open source software that incorporates geometry, algebra and calculus in a fully connected DGS environment, by combining the basic features of DGS and Computer Algebra Systems (Hohenwarter & Fuchs, 2004).
- *GeoGebra-based tasks* are tasks for which the GeoGebra facilitates exploration and analysis (e.g., identifying relationships through dragging)
- A *geometric construction* is defined in this study as a drawing of a figure satisfying given conditions using GeoGebra. The product of the construction is referred to as a *GeoGebra-based construction*.
- A *diagram* is a visual representation of a figure.

- *Making mathematical connections* is the ability to recognize and make linkages between and among mathematical ideas.
- *Mathematics tasks* are what learners are asked to do to initiate an activity, the purpose of which is to stimulate thinking and reasoning (Mason & Johnston-Wilder, 2006).
- *Pedagogical Knowledge (PK)* is teachers' deep knowledge about the processes and practices or methods of teaching and learning.
- *Pedagogical Content Knowledge (PCK)* is knowledge of pedagogy that is applicable to the teaching of specific content.
- *Preparation-based mathematical connections* are connections that are made in the context of teacher preparation where the prospective teachers are both learners and future teachers of geometry.
- *Technology* refers to the computer as an artefact through which knowledge for teaching and learning mathematics may be advanced (de Vries, 2005).
- *Technological Content Knowledge (TCK) is* knowledge needed to understand which specific technologies are best suited for addressing subject-matter learning in their domains and how the content dictates or perhaps even changes the technology—or vice versa.
- *Technological Pedagogical Content Knowledge* (TPACK) is professional knowledge that teachers need to meaningfully incorporate pedagogy and technology within the content they teach (Koehler & Mishra, 2009: 9)
- *Visual explanation* is a description of that which can be visualized.

1.6 Structure of the thesis

This section presents a synopsis of the chapters of this thesis.

Chapter 1: Introduction and formulation of the problem

This introductory chapter provides a background to the study of PTs' mathematical knowledge. The background of the study situates the study within the area of mathematics teacher knowledge. The premise for this study as elaborated in this chapter is that it is imperative that attention should be paid to prospective teacher learning and preparation to teach in technology-based environments. I propose that there is a gap in literature that focuses on prospective teachers with weak geometry content knowledge and pedagogical knowledge for teaching geometry in a technology-based environment. The rationale for the study as presented in the chapter is that very little research exists that explores the complexities of South Africa's prospective teachers' geometry content and pedagogical content knowledge. Mathematics Education programs need to focus on both the PT as a learner of geometry and the PT as a teacher of geometry. One important justification for my study is the critical role and the potential that technology has on the teaching and learning of mathematics. The objective of this study was to examine PTs' knowledge in the context of school geometry content and pedagogical tasks developed in a GeoGebra-based environment. The epistemic goal of this study is to characterize aspects of PTs' technological pedagogical content knowledge and to contribute to the knowledge and understanding of mathematics learning and teaching processes in technology environments. The PTs are faced with the challenge of the technology infused environment which requires that they develop their technological pedagogical content knowledge.

Chapter 2: Literature review and theoretical framework

The chapter explores theoretical and empirical insights emanating from the discussion on teacher knowledge constructed in teacher preparation program that incorporates the use of technology. The structure of the chapter is such that it reviews the debates, claims, theories about teacher knowledge, prospective teacher technological pedagogical content knowledge constructed in teacher preparation and knowledge of geometry in the context of technology knowledge and pedagogical knowledge. It is argued in the chapter that integrating technology requires teachers to experience specific content areas in relation to specific technological tools. The research gap identified by this study is presented as a summary and conclusion of the chapter.

Chapter 3: Methodology

In this chapter, I discuss the methodological approach adopted in this study for exploring aspects of prospective teachers' technological pedagogical content knowledge of geometry constructed within a GeoGebra-based environment. An elaboration of the research design, data collection procedures and analysis is presented. This exploratory multiple case study described in the chapter was done by making inferences on how six participating PTs' think as they responded to the circle geometry tasks. A description of the participating PTs enrolled in a second-year undergraduate mandatory mathematics methodology course is elaborated. A detailed account of strategies employed to collect and analyse the data is provided. Rubrics were employed as analytical tools. These research tools were pilot tested to inform the major study.

Chapter 4: Analytic framework

The discussion in this chapter is focused on the analytical framework that I employed as a lens to explore aspects of prospective teachers' technological pedagogical content knowledge (TPACK) constructed in a GeoGebra-based environment. The major focus of this chapter is to interrogate how the TPACK theoretical framework was engaged as a frame of reference for analysing data. Inductive analysis was employed to develop the framework for data analysis as it emerged from an amalgamation of the TPACK theoretical framework and Duval (1998) cognitive apprehensions analytical framework for geometric reasoning. The study expanded the Duval analytical framework by extending it to include an analysis of teacher knowledge. In this chapter, the two frameworks which were used as lenses for deconstructing the tasks as a precursor to developing analytical rubrics for scoring the PTs' response to the tasks is explained. A description of the coding developed for the CK, PCK and TCK knowledge constructs is presented.

Chapter 5: Deconstruction of the tasks and rubrics

In this chapter my discussion is focused on the tasks utilized in the study and the analytical rubrics designed to examine the PTs' responses to the tasks. The tasks were designed to elicit technological pedagogical content knowledge (TPACK) constructs. In this chapter, I demonstrate how I deconstructed the tasks to provide a description that elaborates the critical components of the sub-tasks, the expectations of each sub-task and the TPACK construct that each sub-task tested. It also displays how the deconstruction of the tasks which is followed by a description of the rubrics were employed to qualify the responses to each task. The analytic rubrics were designed to capture TPACK-related evidence.

Chapter 6: Analysis by TPACK component: Prospective teachers' geometry knowledge (CK)

This chapter presents the results and findings relating to the aspect of content knowledge (CK) construct of the technological pedagogical content knowledge (TPACK). Both quantitative and qualitative analysis of the rubric scores of the individual cases' responses to the CK tasks is presented. Throughout this chapter and the subsequent two chapters, I discuss the trends within and across tasks and presented exemplary PTs' responses to written tasks and interview excerpts. In investigating 'what CK do the PTs display?' I have examined what the PTs' identified and recognized in the perceived figure and studied the types of connections that PTs made between representations, properties and theorems. In this chapter, I present and discuss the CK findings by using the PTs' responses (both from written tasks and from interviews). I bring forth what I considered prominent, absent or assumed by PTs within and across the CK tasks.

Chapter 7: Analysis by TPACK component: Prospective teachers' geometry technological content knowledge (TCK)

This chapter presents the results and findings relating to the aspect of technological content knowledge (TCK) construct of the technological pedagogical content knowledge (TPACK) framework in response to the second research question 'What technological content knowledge does the PTs display about GeoGebra-constructed geometric diagrams?' Both quantitative and qualitative analysis of the rubric scores of the individual PTs' responses to the TCK tasks and the PTs' scores across each task is presented. In this chapter, I present and discuss the TCK findings by using the PTs' responses (from screen cast recordings of the tasks and from interviews) to bring forth what I consider prominent, absent or assumed by PTs within and across the CK tasks.

Chapter 8: Analysis by TPACK component: Prospective teachers' geometry pedagogical content knowledge (PCK)

This chapter focuses on discussing the results and findings relating to the aspect of pedagogical content knowledge (PCK) construct of the technological pedagogical content knowledge (TPACK) framework in response to the third research question 'What pedagogical content knowledge do the PTs display?' As in the preceding chapters, both quantitative and qualitative analysis of the rubric scores of the individual PTs' responses to the PCK tasks and PTs' scores across each task is presented. An overview of how the pedagogical content knowledge construct was conceptualized in the study and a description of the analytical framework employed to interpret the responses to the PCK tasks is articulated. In this chapter, I present and discuss the PCK findings by using the PTs' responses (both from written tasks and from interviews). I bring forth what I consider prominent, absent or assumed by PTs within and across the PCK tasks.

Chapter 9: Summary of findings and conclusions

In this concluding chapter, I present a discussion of findings pertaining to aspects of prospective teachers' circle geometry technological pedagogical content knowledge. The findings from the research questions are interpreted with the discussion located within existing literature and Mathematics Education practices. A synthesis of the interplay of CK within the TCK and PCK constructs is explored to reveal the PTs' TPACK. The chapter ends by pronouncing the study's contribution, limitations and recommendations.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.0 Introduction

In this chapter I review literature related to the theoretical framework and the empirical studies that guided and informed this research study. The technological pedagogical content knowledge (TPACK) framework and its constructs are discussed in the context of mathematics teaching practice and teacher preparation. Relevant literature associated with the integration of the three knowledge domains of content knowledge, technology knowledge and pedagogical knowledge within the dynamic geometry environments in teacher preparation is questioned and discussed. Gaps of prospective teacher knowledge constructed in the contexts of re-learning school geometry, learning geometry with technology and planning to teach geometry with technology are identified. Since the TPACK framework is at the centre of this research, the literature reviewed directly relates to the theoretical framework, rendering it necessary to present a synthesis of both the literature review and the theoretical framework.

2.1 Integrating technology into mathematics teaching and learning

Technology in the teaching and learning of mathematics has been studied in several developmental research projects globally. Studies of computer use in school mathematics have largely examined innovations linked to developmental research projects. Many of these studies have investigated teacher participation and computer use in these developmental projects against the background of computer-based resources. For example, use of diverse interactive video materials to support a range of mathematical tasks at secondary level in England (Phillips & Pead, 1995), using GeoGebra to teach upper secondary level mathematics (Lu, 2008), and the influence of dynamic geometry software on plane geometry problem solving strategies (Aymemi, 2009). Jaworski (2010) studied the challenges of using GeoGebra as a tool directed at generating conceptual understanding through exploration and inquiry for undergraduate mathematics students. Niess (2005) investigated the development of prospective mathematics teachers' technological pedagogical content knowledge (TPACK) in a subject specific, technology integrated teacher preparation program. Collaboration and partnerships on projects and studies on technology in mathematics in higher education have recently been on the rise, with developing the use of technology to support teaching and learning being identified as a priority in most of these projects.

Although the technology community has advanced the benefits of integrating technology in education, there are discerning voices that have cautioned learning in technology-based environments. For example, research has shown that technology tools can engage students in authentic learning opportunities that enhance the development of basic and higher-order skills but United Nations Educational, Scientific, and Cultural Organization (UNESCO, 2008) warns that the success to integrate lies in the ability of the teacher to effectively integrate technology into classroom lessons. Drijvers and Trouchè (2008) have acknowledged the double jeopardy of teaching and learning mathematics in a technology-based environment, given the complexities of teaching and learning and the complexities of use of the technology tool. Mathematics teachers should be knowledgeable about mathematics content and pedagogy, in relation to technology integration in learning. Drijvers and Trouchè (2008, p. 364) elucidate on the *double reference* phenomenon which is the double interpretation of tasks by teachers and learners giving an example where "tasks that address mathematical concepts may be perceived to address how the computer environment would deal with such a task."

It goes without saying that technology may change the use, teaching and learning of mathematics. The new technology has not only made calculations and graphing easier, it has changed the very nature of the problems which are important to school mathematics and the methods mathematicians use to investigate them (National Council of Teachers of Mathematics, 1989). However, Dick (2008) warns that to facilitate mathematics learning, technology tools should conform to pedagogical fidelity, mathematical fidelity and cognitive fidelity. That is, technology tools should support the development of pedagogy, mathematics and cognitive development of concepts. Even several decades ago, scholars such as Salomon, Perking & Globerson (1991) argued that technology tools have changed the balance between accessing prior knowledge and constructing new knowledge with the scale tipped towards construction of new knowledge.

I understand integrating technology into the learning discourse as using technology within the existing curriculum. This implies that curriculum should be flexible in order to incorporate technology-based tools so that new learning environments that engage learners in constructivist approaches to learning may be developed. However, a simple combination of hardware and software will not make integration naturally follow (Earle, 2002). Needless to say, the essential role of technology in education is that "technology environments allow teachers to adapt their instruction and teaching methods to their students' needs" (NCTM, 2000:24). It is imperative to acknowledge that the role of the teacher changes in technology environment to that of a technology mediator. To mediate learning in a technology environment, NCTM (2000:25) "teachers select or create mathematical tasks that take advantage of what technology can do efficiently and well-graphing, visualizing, and computing".

Various research studies have been done on the integration of technology into mathematics teaching and learning. Isikal and Askar (2005) examined the effectiveness of spreadsheets and dynamic geometry software on mathematics achievement and mathematics self-efficacy. The results showed that using technology effectively as a learning tool improved students' mathematics achievement. The major benefit of integrating technology into teaching and learning of mathematics is that technology provides opportunities to engage students with different mathematical tasks and activities so as to develop mathematical skills and levels of understanding (Hollebrands, 2007). Further, it helps learners to "visualize certain math concepts better and add new dimensions to the teaching of mathematics" (Van Voorst, 1999:2). Studies have shown that teachers experience barriers in the integration of ICT into the classroom. Some of the barriers are lack of TPACK, lack of software, teacher resistance, and lack of vision as to how to integrate ICT in instruction (Jones, 2004; Snoeyink & Ertmer, 2001). Nonetheless, Hollebrands, Laborde, and Sträβer (2008) assert that teacher experiences with technology-based environments have been found to improve teacher knowledge of mathematics. This improvement is largely due to the four-stage processes proposed by Zbiek & Hollebrands (2008) namely: stage 1- teachers learn the technology, stage 2- teachers learn to do mathematics with technology, stage 3- teachers use technology with students, and stage 4- teachers attend to student learning in the context of technology. The participants of the study are prospective teachers who are both learners of technology and are learning to do mathematics with technology. Therefore, my study involved Zbiek & Hollebrands (2008) stages 1 and 2.

2.2 Technology and mathematics teacher education

The focus of teacher education is to prepare and develop teachers through robust programmes aimed at enhancing teacher professional knowledge. However, new trends in education are emphasizing the importance of learning with technology instead of learning from technology (Jonassen & Crismond, 2008). A study by the National Assessment of Educational Progress (NAEP) (1996) revealed that teachers who received training in the area of instructional technology are more likely, than those who had not, to use computers effectively. Teacher education programs should not just prepare teachers to handle software and other digital tools, but must relate the technology to mathematics content knowledge (CK), teacher pedagogical knowledge (PCK) and technology, pedagogy and content knowledge (TPACK). The types of teacher development programs have an effect on decisions that teachers make about use of technology for teaching purposes. Lederman and Neiss (2000) purport that teacher preparation programs often emphasize learning about technology instead of learning about integration of technology into classroom teaching.

I regard mathematics teacher education programs to be channels for producing teachers who are prepared to integrate technology into the mathematics classroom. It is through the intervention of mathematics teacher educators that PTs can make informed decisions on teaching in a technology-rich environment. According to the International Society for Technology in Education (2000), the challenge is for teacher education programs to produce and develop computer literate teachers who are confident in their ability to appropriately choose and incorporate instructional technology into their classroom teaching. Teacher educators should advocate for programs that emphasize the ability of teachers to make use of technology by effectively integrating technology into teacher education programs.

Research on the use of technology in education has prompted teacher educators to prepare teachers 'who can utilize technology as an essential tool for developing a deep understanding of the subject-matter and the pedagogy (Drier, 2001:173). Teachers need to master the new technologies as these evolve rapidly. Digital technology usage in some schools in South Africa can be seen as a step into taking advantage of availability of learning artefacts, but this study raises the question of teacher ability to infuse these artefacts into teaching practices. The rapid development of new technologies has reduced the life span of current technologies, requiring that users need to keep up with these developments. Mathematics educators have shown interest in incorporating DGE's within their undergraduate PTs' programmes. For instance, Angeli (2005) assessed PTs' technology competency in science PCK. Studies have examined PTs' experiences in technology-enhanced programs. Haciomeroglu, Bu and Haciomeroglu (2010) observed PTs participation in GeoGebra-based activities in a teacher education course. These studies bring forth the need for teacher educators to acknowledge the crucial role that technology has on teacher education and in understanding teaching and learning in different discourses. This study realised the dearth of research in technology integration in mathematics teacher education in South Africa. South African PTs are re-learning mathematics and are learning to teach mathematics with technology in line with the South Africa education system that endeavours to shift from a typical traditional classroom into a technology-rich classroom. In 2014 the Department of Basic Education rolled out a pilot project for using tablets in the Gauteng schools in pursuit of a paperless classroom. As mentioned in Chapter 1, as mediators of mathematics learning PTs should experience technology first if they are to integrate technology into classroom mathematics learning. Mathematics teacher education programs need to prepare PTs so that they are able to consider the mathematics content, the technology in use and the pedagogical methods employed in teaching the content. PTs are expected to integrate both mathematical knowledge and knowledge about the technology tools in mathematics teacher preparation within methodology courses. In such programs, knowledge is derived from experience for which I conjecture that teacher knowledge is influenced and framed by teacher practical experiences with tools.

2.3 Theoretical framework: Technological pedagogical content knowledge (TPACK)

Various researchers have acknowledged the complexities of integrating technology in teaching and learning (Angeli & Valanides, 2009; Artigue, 2002; Drijvers & Gravemeijer, 2005; Guin & Trouche, 1999; Laborde, Kynigos, Hollebrands & Strässer, 2006; Niess, 2005; Trouche, 2004). The complexities of integrating technology in teaching have led researchers to advance various models of integrating technology in Mathematics Education. Wang (2008) proposed a generic model, consisting of pedagogy, social interaction and technology. Niess et al. (2009) proposed a framework that describes and guides the process of teachers' learning as teachers develop their TPACK. According to Christou, Jones, Mousoulides and Pittalis (2006) solid theoretical frameworks that provide reliable innovative reference models are essential in informing the design of technology-rich learning environments. I bring forth two frameworks that are aligned with the crux of my study; the instrumental approach to the use of technological tools in teaching and learning, and the technological pedagogical and content knowledge framework. The instrumental approach and the TPACK are considered appropriate lenses for the study of prospective secondary mathematics teachers' knowledge development as they work on a set of GeoGebra tasks where such tasks are designed to advance both mathematics knowledge and technology knowledge. To understand the kind of knowledge teachers need within computerized environments, it is necessary to understand teachers' experiences as they relate to technology, that is, their instrumental genesis. I concur with Haspekian (2005:133) who argues that "from a teaching point of view, integrating a tool requires that the teacher simultaneously takes into account the different dimensions: the tool's features, the instrumented techniques and the concepts involved". Instrumental genesis acknowledges that instruments have a profound effect on the cognitive functioning of the user. According to Lagrange (2005), the cognitive structure is made of knowledge about the artefact and mathematical knowledge related to the domain of use.

It is essential to deliberate on literature related to the instrumental approach since it has a link to the teacher technological content knowledge (TCK), one of the TPACK constructs that are of interest in this research study. Premised on the Mishra and Koehler (2006) contention that PTs should know mathematics content and the manner in which content can be changed by a technology tool, the instrumental approach and TPACK provide a platform for examining teacher technology pedagogical content knowledge for teaching geometry. The instrumental approach, although back-grounded in this study, appropriately lends a critical view of the potentialities and constraints of GeoGebra for teaching and learning purposes in teacher education programs. For example, in their study which used the perspective of the instrumental approach, Drijvers and Gravemeijer (2005, p. 186), found that "students can only understand the logic of a technical procedure from a conceptual background". My study investigated PTs' TCK and the findings corroborated Drijvers and Gravemeijer's (2005) argument that users who have technical difficulties are more likely to have little or no grounded mathematical conceptual background.

The instrumental approach is a foundational theoretical framework for studying the use of technology tools (Guin & Trouche, 1999; Heid, 2002; Lagrange, 1999). This framework has been applied in studies on computer algebra systems (CAS) and dynamic geometry systems (DGS). Research on technology integration has revealed that mathematical knowledge is linked to knowledge of how to use the tool (Artigue, 2002; Laborde, 2003; Lagrange, 1999). The instrumental approach was developed by Vèrillon and Rabardel (1995). It resonates with Vygotsky's notion on tool use, in which tools are considered mediators of human activity. The approach has been applied by French mathematics educators such as Artigue (2002), Haspekian (2005); Kieran and Drijvers (2006); and Drijvers and Trouchè (2008) in their research on the integration of technology into the learning of mathematics.

The approach provides a psychological and socio-cultural framework for learning processes in a technological environment where it is understood that tools mediate between the human activity and the environment. It is an approach through which researchers can make sense of learners' use of technological tools and the potential impact of tool use on learners' mental processes in the context of mathematical activities. The instrumental approach involves three constructs, namely: the instrument, subject and object of activity. The instrumental approach has been utilized by various researchers such as Haspekian (2005) who studied the integration of spreadsheets into mathematics learning. Drijvers and Gravemeijer (2005) investigated the relationship between computer algebra use and algebraic thinking using the instrumental approach perspective, concluded that users who have technical difficulties are more likely to have no grounded mathematical conceptual background.

Researchers are in unison that the process of the instrumental genesis, which is described as a two way learning process in the technological environment, is a complex process that is dialectic (Artigue, 2002; Guin & Trouche, 1999). Instrumental genesis (*hereafter referred to as IG*) simply involves the two processes through which the subject acts on the instrument and the instrument acts on the subject's thinking. Hoyles and Noss (2003) refer to the process of IG as a relationship between tool and learner where the tool shapes the thinking of the learner, but the tool is also shaped by the learner thinking. IG has two processes; instrumentation and instrumentalization. These processes occur simultaneously in an interrelated two way direction.

Instrumentation is a subject-oriented process through which the subject conceptualizes the task through the effects of using the tool. The subject understands the task through the use of the instrument by developing techniques and schemes through the use of the tool. The tool shapes the actions of the subject. It is the process through which the potentialities or the constraints of the artefact are exposed (Artigue, 2002; Guin & Trouche, 1999; Trouche, 2004). For instance in relation to this study, if the structure of GeoGebra constrains the PT when solving a geometry problem, then the PT must change the activity or the execution techniques according to the structure of GeoGebra.

Instrumentalization is when the subject uses the instrument in specific ways. Instrumentalization is the construction of schema oriented towards the instrument, that is, the appropriation and transformation of the instrument by the subject. Compared to instrumentation, the instrumentalization process is artefact-oriented. The instrument becomes the means to solve the mathematical problem. Laborde (2003) contends that the use of the tool changes the way to do mathematics with a specific appropriation of the tool required. Monaghan (2003, p. 6) defines appropriation as "an everyday word associated with making something your own".

Researchers in the field of technology integration in teaching and learning employ the technological pedagogical content knowledge (TPACK) framework to study the development of teacher knowledge about technology integration (Lee & Hollebrands, 2008; Mishra & Koehler, 2006; Niess, 2005). Mishra and Koehler (2006) developed TPACK framework drawing from Lee Shulman's (1986) pedagogical content knowledge (PCK) framework. Teacher knowledge for technology integration is built on the interaction among three bodies of knowledge: domain-specific content knowledge, pedagogical knowledge, and technology knowledge (see Figure 2.1). For instance it is necessary to understand mathematical concepts and their inter-relationships so as to determine how these can be represented within the mathematics software. Confrey and Maloney (2008) further emphasize that teachers' content knowledge is transformed in the context of problem solving and multiple representations of concepts. TPACK focuses on the knowledge needed to teach well with technology.

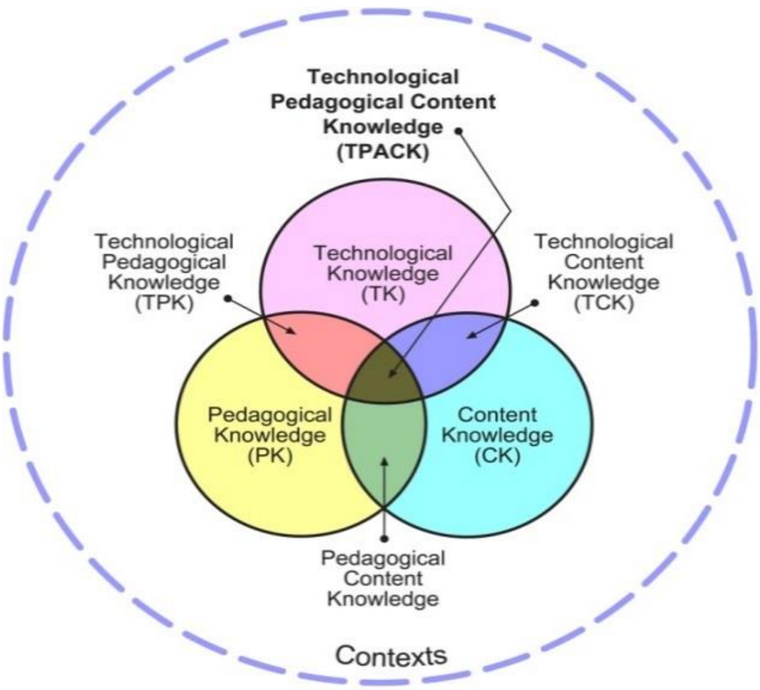


Figure 2.1: TPACK framework and its knowledge components

The technological pedagogical content knowledge (TPACK) framework is a prerequisite to effective integration of technology in education. Mishra and Koehler (2006) explicate that TPACK is the interaction of these bodies of knowledge, both theoretically and in practice, to produce the types of flexible knowledge needed to successfully integrate technology use into teaching. Mishra and Koehler (2006, p. 63) describe the knowledge constructs as follows:

Content Knowledge (CK) is teachers' knowledge about the subject matter that includes knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge.

Pedagogical Knowledge (PK) is teachers' deep knowledge about the processes and practices or methods of teaching and learning. It includes knowledge about techniques or methods used in the classroom; the nature of the target audience; and strategies for evaluating student understanding.

Technological Knowledge (TK) is the knowledge about various technologies, requires a deeper, more essential understanding and mastery of certain ways of thinking about and working with technology.

Pedagogical Content Knowledge (PCK) is knowledge of pedagogy that is applicable to the teaching of specific content. It covers the core business of teaching, learning, curriculum, assessment and reporting, such as the conditions that promote learning and the links among curriculum, assessment, and pedagogy.

Technological Content Knowledge (TCK) is knowledge needed to understand which specific technologies are best suited for addressing subject-matter learning in their domains and how the content dictates or perhaps even changes the technology—or vice versa.

Technological Pedagogical Knowledge (TPK) is the knowledge needed for a deeper understanding of the constraints and affordances of technologies and the disciplinary contexts within which their function is needed.

Technological Pedagogical Content Knowledge (TPACK) is the desirable knowledge needed for effective teaching with technology, requiring an understanding of the interactions among content, pedagogy, and technology knowledge on the basis in which these domains and contextual parameters interrelate. It is the intersection of the three knowledge domains that teachers need to implement the curriculum whilst supporting learner thinking and learning with technologies for specific content.

In the context of this study, the TPACK framework constructs were defined in this manner: ***CK*** is PTs' knowledge about circle geometry that includes knowledge of concepts, theorems and proofs; ***PK*** is the PTs' knowledge about the processes and practices or methods of teaching and learning geometry; ***TK*** is the knowledge about GeoGebra, that requires a deeper, more essential understanding and mastery of certain ways of thinking about and working with GeoGebra; ***PCK*** is knowledge of pedagogy that is applicable to the teaching of circle geometry; ***TCK*** is knowledge needed to understand how GeoGebra is best suited for addressing learning circle geometry; ***TPK*** is knowledge needed for a deeper understanding of the constraints and affordances of GeoGebra for teaching circle geometry; ***TPACK*** is the knowledge needed for teaching circle geometry with GeoGebra effectively.

Harris, Mishra and Koehler (2009) stress that TPACK is professional knowledge that teachers need to have in order to meaningfully incorporate pedagogy and technology within the content they teach. Koehler and Mishra (2009, p. 9) elaborate that this professional knowledge is about effective teaching with technology, requiring an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge to develop new epistemologies or strengthen old ones.

The TPACK framework acknowledges the complexities involved in understanding the relationship between technology, pedagogy and content (Harris & Koehler, 2009; Koehler & Mishra, 2009; Mishra & Koehler, 2006). The teacher knowledge phenomenon is drawing much debate, with various studies utilizing different approaches to measure the TPACK construct. Although there are quite a substantial number of research studies on TPACK, the framework is still under-researched, with questions raised on how to measure this phenomenon (Angeli & Valanides, 2009; Graham, 2011, Koehler & Mishra, 2009). Koehler and Mishra (2009) stress that to understand TPACK, one should view the three knowledge domains not in isolation but as interrelated. Classroom observations, interviews and document analysis are among the various techniques employed to study TPACK in relation to teacher proficiency in technology, technology adoption, perceptions, evaluation, and technology infused within a course (Angeli & Valanides, 2009; Koehler & Mishra, 2005; Koehler & Mishra, 2009; Schmidt & Shin, 2009).

Critics of the TPACK framework contest the definition and clarity of the TPACK construct based on the argument that TPACK is developed from the PCK concept, which took researchers decades to define (Angeli & Valanides, 2009; Archamboult & Barnett, 2010; Graham, 2011) and is still undergoing development. Graham (2011) contends that the lack of a more precise definition of the framework has major implications for understanding and measuring the TPACK constructs. For example, when defining the technological content knowledge construct some association with pedagogical knowledge is made although pedagogical knowledge has no link with technological content knowledge (Graham, 2011). The integrative approach towards understanding the TPACK framework dictates that TPACK should be understood as a combination of various types of knowledge. Such an approach suggests that the TPACK constructs cannot be viewed in isolation. Perhaps such lack of the precise definition of TPACK and its constructs creates the fuzziness of boundaries between the constructs.

Despite the fuzziness of the framework, I concur with Archamboult and Barnett (2010) who caution that researchers need to understand the relationship between the three domains of content, technology and pedagogy, acknowledging the complexities of these knowledge domains. TPACK provides a comprehensive frame for understanding the integrated system of thinking in investigating teacher knowledge. There is well documented research on the development of TPACK for teachers in practice but less is known about TPACK in the teacher education discourse and specific disciplines. I present in the next sections arguments for content knowledge (from a geometry cognitive point of view), pedagogical content knowledge and technological content knowledge in teacher preparation.

2.4 Geometry content knowledge (CK) construct

According to the TPACK framework, content knowledge (CK) is the knowledge of concepts, facts, and principles of the subject matter. The term "geometry content knowledge" is conceptualized in this study as a prospective teacher's ability to relate to diagrams, figural properties and theorems. The following sections discuss CK in terms of connections in geometry and understanding geometry from a cognitive perspective. The argument is based on the role of connections within the geometry structures and on Duval's (2006) conception that in order to understand learners' knowledge acquisition, one needs to analyse what learners produce in the process of learning mathematics (geometry).

Connections in geometry

Geometry ideas are organized in connected structures and this study contends that PTs' understanding and appreciation of the connectedness of the geometry concepts, knowing how to recognize interesting geometrical problems and theorems, can provide insight into their geometry CK and their ability to formulate and make deductions using geometry ideas. For instance, Koedinger and Anderson (1990) posit that strong competency in geometry can be recognized by the ability to use diagrammatic configurations to infer appropriate geometry CK in problem solving. Making mathematical connections is the ability to recognize and make linkages between and among mathematical ideas. Being able to make mathematical connections depends on a view of mathematics as a coherent structure comprising interrelated concepts. I am of the view that to understand or investigate one's knowledge of geometry, the descriptions of the forms of connections made in the geometry CK tasks reveal proficiency in geometry. In order to facilitate mathematical understanding and creative thinking, teachers should design good mathematical tasks that are used to achieve a variety of goals (Vale & Pimentel, 2011). The role of CK tasks is to stimulate students' cognitive processes (Hiebert & Wearne, 1993), initiate fruitful mathematical activity (Mason & Johnston-Wilder, 2006) and provide genuine learner engagement (Watson & Sullivan, 2008). But how are these connections identified, defined and classified? Literature on connections in mathematics is vast but I draw on the work of Businskas (2008) and Mhlolo (2012) about how mathematics teachers conceptualize mathematical connections in their practices. Businskas (2008) investigated and developed a model to describe and classify teachers' conceptions of mathematical connections they made when teaching different mathematics topics. Businskas' (2008:154) categories of mathematical connections are:

1. *Different representations:* connections made when the same concept is represented in two or more ways
2. *Implications:* connections made when *o*ne concept leads to another in a logical form, IF ... THEN...
3. *Part-whole relationships:* connections made when one concept is linked to another in some sense of part and whole.
4. *Procedures:* connections made when an algorithmic procedure is associated with a particular concept.
5. *Instruction-oriented connections:* connections made when mathematical objects are linked because they share some pedagogical purpose.

Deliberations on mathematics connections tend to dominate discussions of how mathematical connections are viewed within the mathematics education discourse. Ma (1999) maintains that connections link together concepts to a specific mathematical notion, which she refers to as concept knots. Businskas' (2008) conception of mathematical connections as constructed by the learner is of particular interest in that when viewed in this manner connections reveal how behaviours and thought processes are constructed and organized. Mhlolo (2012) developed a tool for identifying mathematical connections based on Businskas' (2008) classifications connections made by teachers in practice. Although Businskas (2008) and Mhlolo (2012) bring forward evidence of understanding connections made by teachers in practice, I maintain that connections should be explored in teacher preparation as well. Both researchers studied the connection across mathematics content areas, but my study focuses only on geometry. As mentioned earlier, geometry is organized according to structures. I raise some questions relating to connecting geometry ideas in teacher preparation and it is through such interrogations that this study developed categories of connections made by PTs when working with geometry tasks (see Chapter 6). How can connections made by prospective teachers when responding to geometry tasks be classified and characterized? What type of connections do the PTs employ to identify and describe geometric properties, theorems and representations? What do these mathematics connections reveal about PTs' geometry CK?

Adler (2004), a renowned researcher in mathematics teacher professional knowledge, strongly puts forward that opportunities should be made available for PTs to re-learn school mathematics in South Africa. This call is based on the contention that PTs' CK is weak (Pournara, 2009). Goos (2013) argues that school mathematics CK acquired at school is inadequate and should be revisited during teacher preparation. Yet again, the history of geometry knowledge in the post-apartheid South African school curriculum portrays a void impacted by the weak and lack of school geometry CK. This study identified the need to understand PTs' mathematics knowledge that is constructed in the contexts of learning or re-learning school geometry, learning geometry with technology and planning to teach geometry with technology. As mentioned in Chapter 1, very little research exists that explores the complexities of South Africa's prospective teachers' geometry CK and pedagogical content knowledge. Mathematics Education programs need to focus on both the PT as a learner of geometry and the PT as a teacher of geometry. It is well documented that South African Grade 12 learners have weak geometry CK (Atebe 2008; Feza & Webb 2005, Luneta, 2014). Whereas Nakin (2003), Padayachee, Boshoff, Olivier and Harding (2011) and Jansen and Dardagan (2014) provide evidence that undergraduates (engineering) are underprepared for university mathematics, there is not much mentioned about mathematics PTs. In South Africa, some PTs lack prior knowledge of geometry because they have never learnt geometry at school. So there is a need to address how the PTs learn the geometry content and the pedagogical aspects of PTs' knowledge of geometry (De Villiers, 1997; van der Sandt, 2007; van der Sandt & Nieuwoudt, 2005).

Geometric cognitive processes and apprehensions

Several frameworks for geometrical reasoning were proposed by research studies in the 1990's that aimed at understanding the processes of teaching and learning geometry. Jones (1998) suggests the van Hiele's (1986) model of thinking in geometry, Fischbein's (1993) theory of figural concepts, and Duval (1995) cognitive apprehensions for geometric reasoning. The van Hiele (1986) model is prominent among studies on geometry knowledge in South Africa. For example van der Sandt (2007), van der Sandt and Nieuwoudt (2005), Atebe (2008) and Luneta (2014) employed the van Hiele model of geometry thinking to study geometry knowledge at primary, secondary and tertiary education. The Duval model is of particular interest for this study as it is more concerned with understanding the development of cognitive processes as revealed when solving geometry problems (Duval, 1998, 2007). Duval (1995) suggests an analytic theory for analysing thinking processes involved in a geometric activity. Several studies refer to this theory (Torregrosa and Quesada, 2008: Gagatsis et al., 2010).

In its endeavour to promote mathematical understanding, teacher preparation should discuss possible ways of exploring learning using tasks developed in different contexts. Shimizu, Kaur, Huang and Clarke (2010, p. 4) suggest that "attention should be given to the analysis of cognitive demands enacted by tasks". In addition, Duval (1995), Laborde (2004) and Gagatsis et al. (2010) note that geometry tasks require an interaction with diagrams and the use of visualization to perceive the figures and their properties. How do the PTs appropriate their geometry knowledge in the context of integration of technology in teacher preparation? That is, teacher preparation should prepare PTs competent enough to visualize, construct and reason to reflect their knowledge and understanding of geometric processes. Diagrams, as representations, are a means to reasoning in geometry (Duval, 1995; Herbst, 2004; Laborde, 2004). Theoretically, geometry develops mathematical processes such as analytical, visual and logical thinking (Jones, 1998; Laborde, 2004; Duval, 2006; Goos, et al., (2010). According to Duval (1998: 38-39), from the cognitive point of view, learning geometry involves three cognitive processes; visualization, construction and reasoning (see Figure 2.2).

These cognitive processes can occur separately or simultaneously in a geometric activity or task. According to Torregrosa and Quesada (2008: 321) students “must coordinate the various cognitive processes and representational registers either from a mathematical or from a cognitive viewpoint in order to construct proofs in problem-solving”.

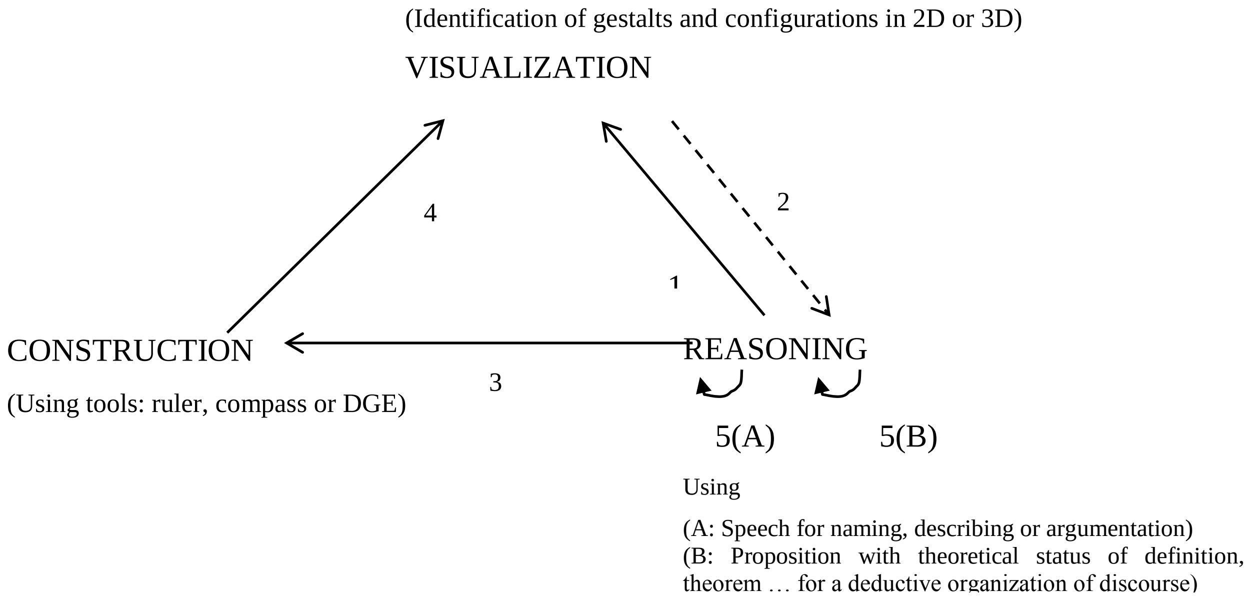


Figure 2.2: interactions of cognitive processes (Duval, 1998)

Figure 2.2 shows how the cognitive processes are connected. The arrows indicate how the processes support each other. Given a GeoGebra-based task like Task 1(c), (see Chapter 4), the construction process will require the three processes. Arrows 1, 3, 4 indicate that visualization of the figure through perceptual apprehension can be supported through reasoning about deconstructing the figure and its figural units; reasoning about the properties will support the construction with the GeoGebra construction tools. Arrow 2 shows that reasoning is not always supported by visualization. That is, what is seen does not always correspond with reasoning. Arrows 5A and 5B suggest that reasoning can possibly be independent of the other cognitive processes. The cognitive processes explained are:

Visualization processes: Several definitions of visualization are presented. According to Hershkowitz, Ben Haim, Holes, Lappan, Mitchelmore, and Vinner (1990:75) visualization is “the ability to represent, transform, generalize, communicate, document, and reflect on visual information”. Similarly, Presmeg (1997:304) purports that it is “the process involved in constructing and transforming visual mental image”. Whereas Duval (2002:322) refers to visualization as “the cognitive activity that is intrinsically semiotic, that is, neither mental nor physical…There is no understanding without visualization”. Further on, Duval (1999:13) explains that the visualization process “shows relations or, better, organization of relations between representational units”. Duval (1998) explains that the visualization processes involve space representation of a statement, heuristic explorations and verifications.

Construction processes: The processes involve actions where “geometrical configurations can be constructed according to restricted tools and mathematical properties of the represented objects” (Duval, 2002:232).

Reasoning relates to the discursive processes to extend knowledge, for proof and explanations (Duval, 1998:38). To reach a logical conclusion one needs to reason mathematically (geometry reason inclusive). Discursive processes include explanations of figural or geometric processes, using speech through descriptions and argumentations.

In recent years, interest in understanding visualization and reasoning in geometry has risen with much focus on figural representations. Several studies are built on Duval’s (1995) notion of apprehensions which links the cognitive processes within a geometrical situation. Originally, Duval used the term “grasp” of a geometric context, but “grasp” was later modified to “apprehension” of not only a geometric context but of a geometry figure or a visual stimulus. Duval’s (1995) cognitive apprehension theory was utilized in this study aiming at understanding the PTs’ visualization, construction and reasoning processes when making connections between representations, properties and theorems. The cognitive apprehensions are: perceptual, sequential, operative and discursive. See Chapter 5 for an elaboration of how the apprehensions were conceptualized in this study as an interpretative tool for understanding PTs thinking about geometry. Duval (1995) brought forward the four apprehensions to analyse and explain learner proficiency and/or competence when relating with figural representations. What are these apprehensions and how do they relate to the cognitive processes?

Perceptual apprehension: involves that which is recognized and discriminated at a glance in a figural representation. It is linked to the visualization process.

Discursive apprehension: the organized description of that which is perceived. For example, learners describe that which they can see. It involves connections between the identified configurations and mathematical principles through speech, discursive statements, language, symbols, etc. Charalambos (1997: 2093) contend that

> *"the mathematical properties in a figure cannot be defined of a simple visual confirmation. A figure is examined according to a "denomination" (We consider one ...), explanation, or a supposition which determine some properties precisely".*

The discursive apprehension is linked to the reasoning process. In this study, discursive apprehension is conceptualized as (a) the ability to connect configuration(s) with circle geometric principles, (b) the ability to provide good descriptions, explanations, argumentations, deductions, use of symbols, and reasoning depending on statements made on perceptual apprehension, and (c) the ability to describe figures through geometric language/narrative texts (Duval, 1995).

Sequential apprehension: relates to the cognitive process of construction but can also provide a basis for reasoning. It involves the sequence of construction of a figure or description of its construction relying on the mathematical and technical constraints of the construction tool. To sequentially apprehend a construct suggest the ability to describe or identify the order in which the figure was constructed depending on the mathematical properties of the configuration and the technical limits of the tool (see Figure 2.2).

Operative apprehension: when working with geometric objects, one can physically or mentally operate them through re-orientation, splitting into sub-figures or transforming the figure. The mereologic, optical and place way modifications are distinctive ways in which figures can be modified in this apprehension. Splitting or combining a figure and/or sub-figures is referred to a mereologic modification whereas optical and place way modifications are varying the size of a figure and varying its orientation. It is linked to visualization and reasoning processes.

Charalambos (1997); Jones (1998); Torregrosa and Quesada (2008); Chiang (2012) and Or (2013) are among scholars that have employed Duval's (1995, 1998) theory and opine that apprehensions intervene simultaneously or successively. For example, Duval (1995: 155) stresses that "operative apprehension does not work independently of the others, particularly of discursive apprehension". Both the perceptual apprehensions and the discursive apprehensions are required in the reasoning process of making connections between properties and theorems. There are situations where visualization is entrenched in the discursive process. I also base my rationale for the understanding of connections by acknowledging the position made by Torregrosa and Quesada (2008:2) that "discursive apprehension is the cognitive activity which produces a connection between the identified configuration and certain mathematical principles (definitions, theorems, axioms, etc.)". My study employed Duval (1995, 1998) cognitive theory to characterize PTs' geometry CK.

2.5 Technological content knowledge (TCK) construct

Globally, technology integration is widely accepted as a tool for mathematics learning and teaching particularly in contexts which are based on constructivist pedagogical model. Researchers such as Kaput (1992); Laborde, Kynigos, Hollebrands, & Strässer (2006); Kaput, Hegedus, & Lesh (2007); Heid & Blume (2008) have studied the use of technology in teaching and learning. The broad area of agreement in research is the potential role that technology has on learner achievement and the enhancement of mathematics learner thinking. The most common findings are that teachers are either reluctant to use technology or use it ineffectively. The reluctance of practicing teachers to integrate technology into teaching mathematics after undergoing professional development has recently led researchers to expose the complexities of the phenomenon (Drijvers, Doorman, Boon, Van Gisbergen, S. & Gravemeijer, 2007; Steketee, 2005). This exposure has been emphasized through focusing research on designing and examining technology-based activities that are purported to enhance mathematical thinking. Attention has and is been paid to the PT education programs. I concur with Angeli (2005) that the task of preparing PTs to become technology competent is difficult and requires many efforts aiming at providing them with ample opportunities during their education to develop the competencies needed to be able to teach with technology. Researchers acknowledge that mathematics methodology courses provide a meaningful context within which the integration of technology can be pedagogically situated in the teaching of subject matter (Angeli, 2005; Li, 2005; Niess, 2005).

Technological content knowledge (TCK) is the understanding of how both technology and mathematics content both aid and limit each other and address knowledge of how to represent content with emerging technology without considering a pedagogical context. According to Cox and Graham (2009) TCK is concerned with how content is represented with technology devoid of pedagogical context. Of all the seven constructs of the TPACK framework, TCK is the least researched (Hofer & Harris, 2012). However, I bring in assertions by Artigue (2002), Guin and Trouche (1999) and Trouche (2004) from the instrumental genesis point of views, that a display of TCK exposes the potentialities or the constraints of the artefact. For instance in relation to this study, if the structure of GeoGebra constrains the PT when solving a geometry problem then the PT must change the activity or the execution of techniques according to the structure of GeoGebra.

In their research on the link between research and software development, Sarama and Clement (2008:115) proposed that for any software to encourage mathematical thinking, its "learning trajectories should be based on models of cognition that have three components: goals, the developmental sequence specifying levels for goal attainment and instructional activities that facilitate learner growth". Laborde, Kynigos, Hollebrands, and Strässer (2006) further emphasize the importance of the interactions between students, instructors, tasks and technology in DGE.

Technology has been employed to enhance understanding of concepts in various domains of mathematics. According to Highfield and Goodwin (2008) geometry, algebra and calculus have been well researched as domains exploiting the potential affordances of technology. Technology used for teaching and learning calculus addressed gaps from the traditional approach through the conception of "dynamic approaches to numerical, symbolic, and graphical approaches, culminating in theories ranging from formal epsilon-delta analysis, which banished infinitesimals, to nonstandard analysis" (Tall & Piez, 2008:208). In their analysis of projects that investigated the use of technology in learning of rational number concepts, Olive and Lobato (2008:36) concluded that "technological environments have contributed to a significant expansion between conceptual analysis of rational numbers and to an understanding of relationship between children's whole number and rational number knowledge". In relation to algebra and technology, Heid and Blume (2008) contend that technology-based algebra curricula affect processes of mathematical activity, algebra content, and algebraic concepts and procedures. Heid and Blume (2008:423) suggest that approaches to teaching geometry within technology environment have changed the focus of the "traditional analytic and sequential approach of non-technological Euclidean geometry courses". Laborde (2003) contends that the use of the tool changes the way to do mathematics with a specific appropriation of the tool required. Monaghan (2003:6) defines appropriation as "an everyday word associated with making something your own". The approach to geometry tasks instruction "should enable students to effectively, meaningfully and purposefully employ geometry conceptual systems" (Battista, 2008:134).

Dynamic Geometry Environments (DGE) in mathematics education were popularized in the 1990's, with the evolution of Dynamic Geometry Software (DGS) like *Cabri*, *Geometer SketchPad* and *GeoGebra* in 2004. The DGE provide a platform for users to create, manipulate geometric constructions and explore underlying relationships in geometry conceptual systems. However, Aymemi (2009:8) contends that it is "argued that dynamic geometric environments tend to promote some types of empirical justifications and inhibit formal justifications". Initially DGE were regarded as tools for geometry but with time this progressed to their use for interactive geometry which is believed to have an impact on student learning in various domains (Goldenberg & Feurzerg, 2008). I take the instance of the use of dragging. Dragging, which is the main defining feature of DGS, allows for navigation and exploration of geometry concepts through multiple representation and interpretation. Research studies on dragging indicate that instrumentation processes address the critical relationship between drawing and figure, between spatial and theoretical representation.

GeoGebra description

GeoGebra is a free and open source software (FOSS) developed in 2004 by Markus Hohenwarter to support teaching and learning of mathematics. It incorporates geometry, algebra and calculus in a fully connected DGS environment, by combining the basic features of DGS and Computer Algebra Systems (CAS). GeoGebra offers two representations of objects through the algebra window and geometry window (see Figure 2.3). For any manipulation on the geometry representation there is a simultaneous change in the algebraic representation and vice versa. Among the attributes of GeoGebra are the ability (a) to specify the geometrical relationships between objects created on the computer and original constructions; (b) to provide visualization of different representations; (c) to be used in investigations to discover mathematics; (d) to be used for preparing teaching materials; and (e) to be used as a cooperation, communication and representation tool (Hohenwarter & Fuchs, 2004). Just like any other DGS, constructions within GeoGebra can be directly manipulated by using the 'drag mode' operation for exploration of conjectures.

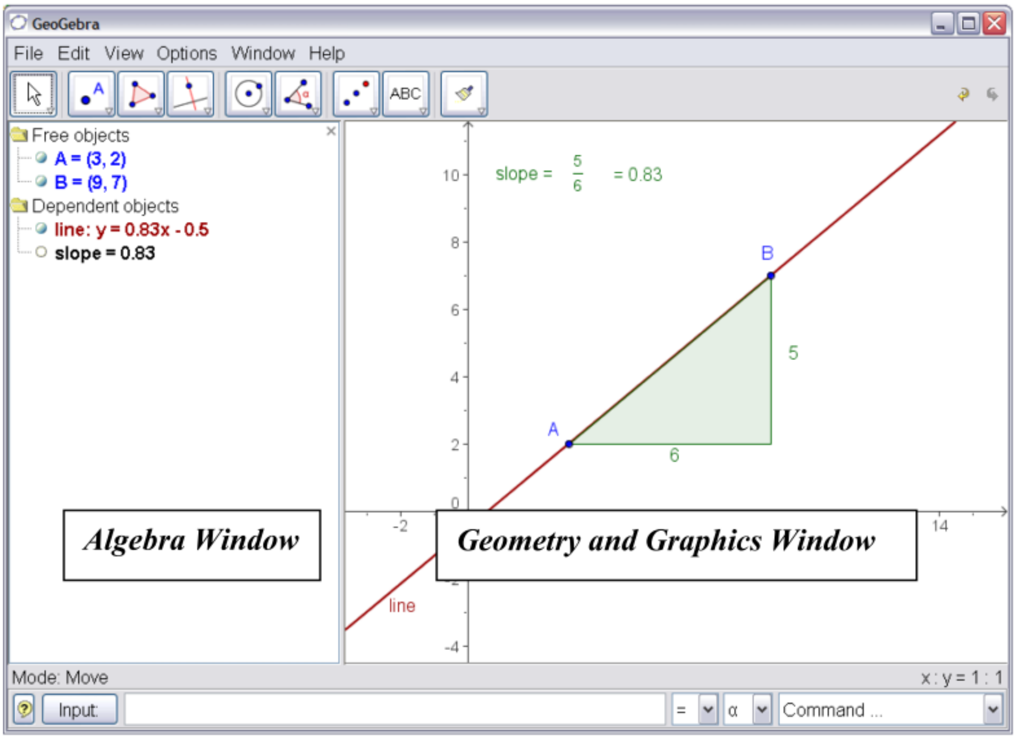


Figure 2.3: GeoGebra window

Research studies on the use of GeoGebra as a tool in the teaching and learning environment have been documented, focusing more on mathematics at middle and high school and on teacher professional development than on prospective teacher education. Learning school mathematics with technology is widely researched as compared to learning and teaching mathematics with technology at teacher preparation level. Lu (2008) investigated English and Taiwanese upper-secondary teachers' conceptions and practices regarding GeoGebra. His findings were that, to integrate GeoGebra into their teaching practices, teachers employed a wide variety of strategies in their preparation for teaching materials, presentation of mathematical content and concepts, classroom activities for interaction with pupils and investigation of mathematics.

GeoGebra has been praised as a tool for providing learners and teachers with a platform to enhance the visualization and reasoning processes. Studies by Guvan (2012); Bhagat and Chang (2015) have revealed the usefulness of GeoGebra as an effective tool for learning geometry. But the knowledge needed to use this technological tool requires, as suggested by Mogetta, Olivero, and Jones (1999: 99), “tackling a problem using dynamic geometry software involves interpreting the problem in terms of the menu items available within the software environment”.

2.6 Pedagogical content knowledge (PCK) construct

Understanding teacher knowledge has been in the forefront of many educational research fields. For the last two decades, researchers have developed models for understanding this phenomenon. Shulman (1986) referred to three dimensions of teacher knowledge; content knowledge, generic pedagogy knowledge and pedagogical content knowledge. Pedagogical content knowledge (PCK) is the knowledge of pedagogy applicable to the teaching of specific mathematics content. Drawing from Shulman’s (2006) definition, PCK comprises knowledge of mathematics content; knowledge of mathematics curriculum; and knowledge of teaching.

Various models of PCK have been developed. Cochran, De Ruiter and King (1993) proposed a model with four components; pedagogy, subject matter content, student characteristics, and the environmental context of learning. Ball, Thames and Phelps (2008) proposed a PCK model that partitioned Shulman’s (1987) model into knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Grossman (1990) contended that the four components of PCK are: knowledge and beliefs about the purposes for teaching a subject; knowledge of students’ understanding, conceptions and misconceptions of particular topics in a subject matter; knowledge of curriculum and curriculum materials; and knowledge of instructional strategies and representations for teaching particular topics.

It is clear from the discussion above that teacher knowledge is multidimensional. The difficulty to discern the different knowledge constructs is brought about by the complex web of relationships between knowledge constructs. For instance, Ball et al. (2008) propose the partitioning of the CK and PCK. Ball et al. (2008) categorize CK into three domains; common content knowledge, specialized content knowledge and horizontal content knowledge. Koehler and Mishra (2005) developed TPACK to acknowledge the relationships between content, pedagogy and technology and the contexts in which they function. Rollnick, Bennett, Rhemtula, Dharsey and Ndlovu (2008) proposed a model of four domains of teacher knowledge that interact to produce PCK for the manifestation of teacher knowledge. The four domains are: knowledge of subject matter, general pedagogical knowledge, knowledge of students and knowledge of context.

The models referred above provided lenses for understanding professional teacher knowledge (teachers in practice) rather than on teacher education (teacher preparation). Going back to Shulman's (1986) definition of PCK and the definitions brought forward subsequently by the likes of Baumert and Kunter (2006) and Gess-Newsome (2013), one cannot help realize that reference about PCK is made to practicing teachers in the context of enactment of teacher specific PCK than on prospective teachers. I differ with Loughran, Mulhall and Berry (2004) who argue that studies about prospective teachers provide insufficient knowledge about PCK. I bring forward a strong contention that if PCK is described as a merging together of content knowledge and pedagogical knowledge, then PTs' PCK and even PTs' TPACK can be defined and characterized.

But how has PCK been studied in different contexts? Science education has been in the forefront in the last two decades with research focussing on understanding of teacher pedagogical content knowledge scholarship. Park (2005) examined the nature and development of PCK of science teachers in their interaction with gifted learners. She employed different approaches to understand the role that learners play in organizing, developing and validating teachers' PCK. Magnusson et al. (1999) developed the PCK components' model of which researching knowledge about students' understanding of specific science topics became popular among science teaching research. For example see studies by Rollnick, Bennett, Rhemtula, Dharsey and Ndlovu (2008), Park and Oliver (2008), Gess-Newsome (2013).

Mathematics education has also followed suit in an effort to conceptualize mathematics knowledge by striving to understand the mathematics content knowledge and mathematics teaching instructional practices. For example, Ma (1999) concentrated on the profound understanding of fundamental mathematics; Adler and Davis (2006) made much accomplishment on the QUANTUM project which focused on understanding what and how mathematics for teaching is constituted in mathematics teacher education; Hill, Schilling and Ball (2004) researched on frameworks for mathematics knowledge for teaching; the COACTIV project by Baumert et al. (2006) terms of reference was to comprehend CK and PCK in processes of learning and instruction.

Content knowledge is premised to be a source of pedagogical content knowledge (Grossman, 1990; Kind, 2009). Several mathematics education studies contend that there is a correlation between CK and PCK (Brunner et al., 2008; Baumert et al, 2010; Tepner and Dollny, 2014; Evens, Elen and Depaepe, 2015) with CK as a pre-condition for developing PCK. Kleickmann et al (2013) examined the effect of CK and PCK on instructional practices by comparing the CK and PCK of mathematics teachers.

Debates on the above-mentioned studies argue that the CK and PCK constructs can be viewed as separate or mixed entities. Kahan, Cooper, and Bethea (2003) argue that although CK is a prerequisite for teaching, there is no guarantee that one with good CK had strong PCK. Hill, Schilling, and Ball (2004) acknowledge this by suggesting a merging of these bodies of knowledge into what they refer to as “mathematics knowledge for teaching” or MKT. Hill et al. (2008) demonstrate that the quality of instruction is determined by MKT, a notion that is supported by Brownlee, Purdie and Boulton-Lewis (2001). I pose the question: how then can PTs’ knowledge be characterized when CK and PCK are developed in the context of teacher preparation? Ramatlapana and Berger (2013) studied PCK of PTs developing lesson plans. The findings revealed that although PTs lacked pedagogical experience, they acknowledged that planning must reflect teacher knowledge of teaching strategies and especially the strategies for representing the content.

The investigations on the nature of PCK have culminated into complex and varied approaches to examining PCK. Of interest to this study is the analytical lenses employed to analyse PCK of mathematics teachers in qualitative design studies. Qualitative evidence-based studies have illuminated the existence of PCK in relation to CK but there is abundant evidence in quantitative studies that measured mathematics teachers' PCK. The most prominent tools used to capture instances of PCK in science education studies are the Content Representation (CoRe) and a Pedagogical and Professional-experiences Repertoires (PaP-eRs) developed by Loughran, Berry and Mulhall (2006). Content Representation (CoRe) codifies the teacher's understanding and representation across the specific content whereas Pedagogical and Professional-experiences Repertoires (PaP-eRs) is employed as a tool for reflecting on the teaching of the specific content.

With regards to the tools for measuring mathematics teachers' PCK, I draw upon Chick, Baker, Pham and Cheng (2006) framework which is of interest to this study. Earlier in this chapter, I have relayed and explicated how complex and multi-faceted teacher knowledge is. Hence, I collude with Chick et al. (2006) in their proposal to fuse the various components of PCK suggested by Shulman (1986, 1987), Ball (2000) and Ma (1999) and produce a solid framework for understanding teachers' PCK. Drawing from these PCK, Chick et al (2006) classify the various facets of mathematics teachers' PCK as;

(i) the clearly PCK: content and pedagogy are considered intertwined with components of this category including knowledge of teaching strategies, student thinking, curriculum and resources;
(ii) the content knowledge in a pedagogical context: geared towards mathematics content for teaching. The components of this category are Profound Understanding of Fundamental mathematics (PUFM), deconstructing content to key components, mathematical structure and connections procedural knowledge methods of solution; and
(iii) the pedagogical knowledge in the content context: focuses on generic pedagogy applied for specific content. The components of this category are goals for learning, getting and maintaining learner focus, classroom techniques and integrating technology

Chick et al (2006) acknowledge that there is an overlap among the components. Maher, Muir and Chick (2015) utilized the framework to examine PCK in secondary school mathematics lessons. Their findings were consistent with those of my study as there were indications that the PCK categories espoused in this framework were “often inextricably linked”. See Chapter 4 for how the Chick et al (2006) PCK framework was conceptualized for this study.

2.7 Chapter summary

It is perceived that teachers need specific type of knowledge to enable the integration of technology in teaching and learning (Schmidt & Shin, 2009). Moreover, I argue that integrating technology requires teachers to experience specific mathematics content domains in relation to specific technological tools. Thus it remains a matter of serious concern that there is need to explore how PTs construct their mathematical knowledge as they engage with technology. Several studies have employed the instrumental approach within the context of computer algebra software (Bretscher, 2010; Drijvers & Gravemeijer, 2005). There is need for undertakings that reveal the PTs’ knowledge within the DGE context. More specifically how do DGE tools such as GeoGebra influence the teacher content and pedagogical knowledge of school geometry in teacher preparation programs? How is PTs’ knowledge of circle geometry transformed as they work on tasks developed within a GeoGebra-rich environment? How is knowledge constructed in the contexts of re-learning school mathematics, learning mathematics with technology and planning to teach mathematics with technology? What characterizes such knowledge? The latter questions summarizes the arguments presented in this chapter that PT mathematics knowledge constructed in technological environments (DGE) is underexplored in research literature and is underrepresented in the mathematics teacher preparation milieu.

CHAPTER 3

METHODOLOGY

3.0 Introduction

This chapter provides the methodology employed in this study to explore aspects of prospective teachers' technological pedagogical content knowledge of geometry in the context of a GeoGebra-based environment. An outline of the justifications and descriptions of the research design and approaches, data generation strategies, data analysis procedures and ethical considerations that were used to examine participating PTs' circle geometry knowledge exhibited through the implementation of circle geometry tasks are articulated.

3.1 Research approaches and design

Any research design should be informed by philosophical and theoretical assumptions. This constitutes the research paradigm. The research paradigm in turn informs the methodology and the research design. The epistemological basis of this research was underpinned by the constructivist perspective, which postulates that learners construct knowledge and meaning from the experiences they are engaged with. Technology is often associated with human intervention with artefacts or tools, strongly suggesting that technology affects knowledge construction, teaching and learning. Premised on the contention by Mishra and Koehler (2006) that TPACK is not static but rather is flexible and socially constructed, this study adopted the social constructivist perspective. In the social constructivist approach, meanings (which are context bound) are constructed through multiple social interactions with the social world. Hence, teaching and learning are culturally and contextually bounded. It is incumbent upon the researcher to understand and interpret the meanings and knowledge of the participants as they engage with the reality about the social world (Crotty, 1998; Robson, 2011). The key proponents of social constructivism are Piaget and Vygotsky although their conceptualization of the paradigm differs. According to Piaget, learning is a process of continuous interactions between the learner and the environment. One the other hand, Vygotsky (1978) contends that a learners' cognitive development is influenced by their social-cultural-technological environment (Kivunja, 2014). Further on, Vygotsky (1978) mentions that in order to facilitate the construction of knowledge human action is mediated by tools and semiotics. Such knowledge is acquired through participation and engagement. Premised on the principles of social interaction and mediation through use of tools, it is the view of this study that knowledge is generated through the understanding of learning experiences of prospective teachers.

This study is premised on two claims held by social constructivists. First, that learning mathematics is profoundly influenced by the tasks, by the learning context and by the tools that are used in mathematics instruction. Second, that mathematical knowledge is developed through the relation produced by the interaction between content, pedagogy and technology knowledge. Knowledge acquisition is considered an active process of mental construction, modification or transformation of knowledge by an individual. An interpretative approach was suited for this study, specifically due to the assumption that knowledge is developed through social constructions with tools and shared meanings (Walsham, 1995) where individual's subjective experiences (epistemology) are realized. The implication is that reality is accessed through social constructions with tools and shared meanings (ontology).

The rationale for the adoption of the qualitative case study approach was that, this study aimed at capturing insights relating to ways in which prospective teachers' construct knowledge of circle geometry. It was not the intention of this study to measure variables or test hypotheses about PTs' knowledge as proposed in the positivist approach. Based on Yin's (1994:13) position that a case study design allows for a study to empirically "examine phenomena within its real-life context where boundaries between phenomenon and context are not clearly evident", this study utilized a case study design. I intended to explore PTs' thinking processes (phenomena) in learning geometry within a mathematics teacher education program (context). This study did not strive to measure the performance of the PTs' knowledge of circle geometry but to characterize their knowledge. To do this, I focused on gaining insight into these teachers' thinking processes as they responded to the circle geometry tasks. As such, the nature of the inquiry was appropriately suited to the case study design.

Teacher knowledge is multidimensional. The difficulty of discerning the different knowledge constructs is brought about by the complex web of relationships between the knowledge constructs. Hence, multiple cases within mathematics knowledge development can be examined through the case study approach to seek a range of sources of evidence of the knowledge constructs. I employed an exploratory multiple case study design. The case in this study is the TPACK of a participating mathematics PT. The study was perceived to be an exploratory multiple case study because it allowed for a deeper and detailed exploration of the PTs' TPACK, which as stated above, is complex. A variety of lenses into the various TPACK constructs were employed to study the multiple facets of teacher knowledge, implying that the case study was classified as an embedded case study. The rationale for a multiple case design was that mathematics knowledge development within a technology environment is influenced by the different components of PTs' technological pedagogical content knowledge (TPACK) and the use of the GeoGebra tool. The multiple cases were the different TPACKs (CK, TCK and PCK) of the different participating PTs.

The unit of analysis for this study was each participating PTs' technological pedagogical and content knowledge (TPACK). Since the study was an embedded case study, there were sub-units of analysis to be explored individually which were to be drawn together to reveal the participating PTs' TPACK. The sub-units of analysis were the participating PTs' circle geometry content knowledge (CK), the participating PTs' circle geometry pedagogical content knowledge (PCK) and the participating PTs' circle geometry technological content knowledge (TCK). A decision was made to focus on the TPACK construct that had content (C) as the common denominator. The critical interest of this study was to examine how the participating PTs' content knowledge which was purported to be weak manifested within the TPACK constructs. As mentioned in Chapter 2, a deliberate move was made to specifically pay attention only on the TPACK constructs which had content (C) as the common denominator. It was deemed necessary to consider C since content knowledge was very weak among PTs and considering that development of CK is a necessary, albeit not sufficient among PTs.

3.2 Research participants

This study purposefully focused on gaining in-depth understanding of the aspects of the PTs' TPACK with regard to circle geometry. The PTs were the primary participants for this case study. The PTs were enrolled in a second-year undergraduate mandatory mathematics methodology course that the researcher taught at an urban South African university. See a description of this course in Section 3.3 and Appendix B for the course outline. An open invitation was extended to all sixty-five (65) students to partake in the study in 2013. There was a verbal invitation extended during the geometry module lectures and an invitation through SAKAI, an eLearning platform. The invitation explained the purpose, procedures and intentions of the study (See Appendix A). Emphasis was made that under no circumstances would the students be coerced into participating in the study. Although the invitation was extended to all the students in the course, I intended to focus on a sample to pilot the tasks and the use of the screen-casting software, UltraVNC Addons, to record the PTs' interactions within GeoGebra. See Section 3.5 for an elaboration on the piloting exercise.

Only ten (10) out of sixty-five (65) students voluntarily agreed to partake in the study. The sample size of the participants (*herein referred to as PiPTs*) was considered manageable in terms of tapping into the insight of their geometric thinking as reflected in their responses to circle geometry tasks. No consideration was given to the PiPTs' performance in geometry and gender differences. Enrolment in the course was a critical criterion for participation.

I designed the tasks, facilitated the administration of tasks and conducted individual interviews a week after the implementation of the tasks. There were two categories of tasks, written tasks and GeoGebra-based tasks. See Chapter 5 for task design and descriptions. The researcher met with the PiPTs individually and distributed both tasks. It is at this meeting that instructions on how to complete the tasks and any further clarification regarding the nature of the tasks were discussed with each participant. The PiPTs were assigned written tasks to be completed at their own leisure. The intension was to source as much knowledge from the participants. The participants were specifically told that the tasks were not some sort of a test but a tool for capturing their content knowledge. The participants were instructed not to seek assistance when solving the tasks. With regard to the GeoGebra-based tasks, the participants individually worked on the tasks on the researcher's computer. The PiPTs' GeoGebra constructions were recorded using the screen-casting software, *UltraVNC Addon*, which was downloaded onto the researcher's computer.

For the main study, a call was made to the second year mathematics methodology 2014 student cohort. The intention was to employ a convenience sampling approach to identify the sample of the study. The approach to sampling dictated that I select the convenient sampling technique. Convenience sampling is a non-probability sampling technique. It was convenient for me to study the population that was easily accessible (students in my course). I was interested in learning about the mathematics student teacher preparation in the methodology course. I acknowledge the bias linked to the convenience sampling technique such as under-representation or over-representation of the population. To address the bias, I deliberately openly invited participation from all the students in the course to afford them the chance of participation and utilized the participants that were readily available. I also acknowledge that compared to a probability sampling technique, convenience sampling might have left out individuals who could have provided a richer understanding of the study phenomenon. I also acknowledge the inherent bias in convenience sampling that delimits the ability to make generalisations from the sample to the population of study.

The same procedure as in the pilot was conducted for both the selection of the main study participants and the collection of data. An invitation to partake in the study was put forward and only ten (10) out of sixty (60) students showed interest in participating in the study. Following informal discussion about the nature of the study, four out of these ten students decided to withdraw from the study. The tasks from the pilot study were re-designed and implemented by the final six (6) participants. I refer to these six participants as the participating PTs throughout this report. The demographics of the participants are presented in Table 3.1. The participants had the general characteristics of the population. The students in the course enrolled for two methodology courses. They either majored or sub-majored in any of the two subjects: Mathematics or Natural Science or Life Sciences All the participating PTs majored in Mathematics with Natural Science as their sub-major. The class average for Mathematics 1 and Methodology 1 were 65% and 68% respectively.

Table 3.1: case study PTs demographics

PT [1]	Major teaching subject	Sub-major teaching subject	Year 1 marks Mathematics 1	Methodology 1
Nkosi	Mathematics	Natural Science	79	86
John	Mathematics	Natural Science	73	65
Wisdom	Mathematics	Natural Science	60	62
Lesedi	Mathematics	Natural Science	63	64
Bonolo	Mathematics	Natural Science	55	75
Thabiso	Mathematics	Natural Science	62	68

The role of the researcher

I reiterate that knowledge which is constructed through mediation with tools and semiotics is acquired through participation and engagement. My role in the study was that of a participant-researcher. Adopting the case study dictated that I understand prospective teacher knowledge within its natural settings (within the methodology course). Owing to the contextual conditions I was positioned with dual roles of (i) the course convener and (ii) the researcher. As the convener of the course, my objective was not to study my own practice but to get an insight into my students' knowledge of teaching and learning school geometry. In other words, I had to contribute to the realization of the objectives of the course. The course acknowledged that the prospective teachers should be considered as both learners of geometry and teachers of school geometry. The course was developed with the intention of developing teacher knowledge of content, pedagogy and technology. As the researcher I had a second role of contributing to knowledge about PTs' learning within their local context. To understand prospective teacher tacit knowledge, it was crucial that I study this knowledge as a participant within the social context. I acknowledge the criticisms towards participation-

[1] Pseudonyms of PTs
[2]Further Education and Training phase constitutes secondary school Grades 10-12

observation. There are sources of bias relating to the positionality of the researcher. The lack of the researcher's objectivity may have influence over the participants' behaviours and the research may have an influence of the researcher' own beliefs. To deal with the ethical dilemma, participants were assured that their participation would not have an influence in their performance in the course. See Section 3.8 for further deliberation on ethical considerations. I have made several attempts to maximize the robustness of the research methodology; triangulation of data sources, incorporated evidence of PTs vignettes, synthesized with findings from the literature and objectively analysed the evidence by looking at within and across the cases.

3.3 Description of the Methodology Course: the study location

This Bachelor of Education (B.Ed) second year Mathematics Education course was specifically designed for mathematics major prospective teachers (PTs) preparing to teach the secondary school mathematics phase, referred to as Further Education and Training (FET)[2]. See Appendix B for the course outline. To enrol for B.Ed. mathematics programme, the minimum entry requirement is set at 65% pass for mathematics Matric examination. This requirement is lower than those of other degrees involving Mathematics courses at this institution and so, many B.Ed PTs might not be considered as mathematically able, nor as having mathematical potential (Pournara, 2009). Students enrolled in this course met two hours a week and must have passed a mathematics content course in first year and a first year secondary mathematics methodology course. The mathematics content course, referred to as Mathematics 1, aimed at deepening and broadening the PTs' mathematical knowledge of algebra, functions, trigonometry and geometry. The geometry module focused on shapes and their properties (lines, points, triangles, and quadrilaterals), geometrical constructions, congruencies and similarities. Technology was integrated into the course as a means of

[2]Further Education and Training phase constitutes secondary school Grades 10-12

exploring and communicating mathematical ideas (du Plessis and Parshotam, 2013). The first year mathematics methodology course focused on algebra and functions. The aim of the course was to provide PTs with the necessary background and insight into how to use and implement various teaching and learning strategies in the teaching and learning of mathematics in different classroom settings (Lampen, 2013).

This research study is located within the second year methodology course. The second year mathematics methodology course was theoretically and practically oriented to develop PTs' didactical knowledge, and it incorporated aspects of mathematics teaching that challenged PTs' mathematical thinking around geometry (see Appendix B). In order to pass this course, PTs were expected to demonstrate in relation to learning geometry, the ability to vis-à-vis:

- understand theories for learning and teaching geometry
- identify and select appropriate teaching strategies for given scenarios for learning geometry;
- select and design appropriate mathematics geometry learning materials for learners;
- integrate technology in teaching geometry (e.g. GeoGebra, Word, Sketch Pad)
- assess learners' written work on geometry and suggest appropriate remediation;
- relate learners' geometry misconceptions to appropriate theoretical ideas;
- reflect critically on their own practice as a school geometry teacher and relate this to issues dealt with in the course;
- engage competently with the geometry content covered in the course (Ramatlapana, 2011)

All students in the course were introduced to GeoGebra in their first year of study. GeoGebra was integrated into the second year methodology course structure. A learning trajectory was developed that engaged students in activities that were directed to enhancing their geometry content knowledge, geometry pedagogy content knowledge and knowledge of learning geometry with GeoGebra (Appendix B). The activities in the course included learning or re-learning circle geometry content, lesson plans' development and presentations of lessons activities on teaching circle geometry theorems with GeoGebra. Figure 3.1 presents an example of an assessment on teaching circle geometry towards the development of CK, PCK and TCK.

Lesson study is a process which includes planning a lesson, teaching and observing the lesson, debriefing the lesson, and revising the lesson.

You will create one mathematics lesson plan, in collaboration with your group members.
Design and present a 20 minute GeoGebra-based Grade 11 lesson on teaching circle geometry theorems. The lesson should incorporate ideas discussed during lectures. Technology based (GeoGebra, etc.)

The lesson plan should provide details and justifications for the sequence of questions and activities, key concepts that you want to communicate, misconceptions and common errors that you want to address.

Following feedback from the presentation, you will submit a revised lesson plan. The original lesson plan should be submitted prior to teaching the lesson. The revised lesson plan should be submitted a week after your presentation. All submissions should be online. A rubric for marking the lesson plans is posted on SAKAI.

Figure 3.1: Assignment 1

3.4 Data collection methods

A case study employs multiple techniques of data collection. Multiple sources of data sets are encouraged in a case study as they provide rigorous and empirically and theoretically grounded evidence and support triangulation of results (Cobb & Schauble, 2003). Yin (2003) advocates for the use of multiple sources of evidence to ensure construct validity. That is, do the sources of evidence measure what they are supposed to measure? In line with this rationale, the data generation instruments that were employed in the study were written tasks, GeoGebra-based tasks and interviews. These data generation strategies were employed in the piloting of the tasks and in the major study. The descriptions in the next section demonstrate that attention was paid to the design and procedures for administering each instrument.

3.4.1 Written tasks

Mathematics tasks are used as tools in research in Mathematics Education. To solicit case study PTs' knowledge of teaching and learning circle geometry, participating PTs were presented with tasks. A presumption was made that solutions to the tasks displayed the PTs' thinking. Therefore attention was given to task design, acknowledging the influence that piloting of tasks had on the implementation and the findings of the study. The participating

PTs were assigned four major tasks with sub-tasks which were either categorized as written or GeoGebra-based tasks. The participating PTs had access to both these types of tasks prior to submission. Refer to Table 3.3 for task specifications. The activities of the written tasks focused primarily on eliciting the three knowledge domains of pedagogy, content and technology in the context of circle geometry. See Chapter 5 for the design and descriptions of the tasks and Appendix C for the tasks and memoranda for tasks. The pedagogical tasks or subtasks comprised questions on the teaching of circle geometry. The content tasks were about solving circle geometry problems. The technological tasks were about using GeoGebra to construct and/or interpret GeoGebra-constructed geometric diagrams. The participating PTs were given a week to individually work on the tasks. The intention was to source as much rich responses as I could possibly get on the written work. The written work was done prior to the screen recorded GeoGebra-based tasks.

3.4.2 Screen recorded GeoGebra-based tasks

The study intended to explore teacher knowledge developed in a technology-rich environment. Due to the complex nature of learning in a technology-rich environment, it was not easy to observe individual participating PTs performing the tasks. The use of screen-recorded GeoGebra-based tasks was best suited to explore the participating PTs' content knowledge (CK) in relation to the GeoGebra tool. That is, participating PTs' technological content knowledge (TCK) was examined when PTs responded to circle geometry tasks that incorporated the use of technological tool. Among the four major written tasks were GeoGebra-based sub-tasks. See Table 3.3 for task specifications. The GeoGebra-based tasks were technological-based tasks that required the use of GeoGebra to construct and/or interpret GeoGebra-constructed geometric diagrams. A week after being given the GeoGebra based tasks, the participating PTs solved the GeoGebra-based tasks on researcher's computer in her office. The PTs' worked on the GeoGebra-based tasks outside lecture time.

Screen recording is highly recommended for this type of study because, as advocated by McDougall and Karadag (2008), it captures actual computer work activity by tracking the user's thinking processes. It allows the researcher to track the movements of the cursor during the construction process, record the elements of interest and explore the activities and interactions without disturbing the participating PT's attention to the task. The cursor movements can then be exported as videos or frames. See Figure 3.2 which displays a snap-shot of a video screen recording of Lesedi working on Task 1 (c). On the bottom right side of the Figure 3.2 the cursor is on Delete option in the dropdown menu to show that at 02:54 Lesedi selected the Delete option with the intent to delete point D.

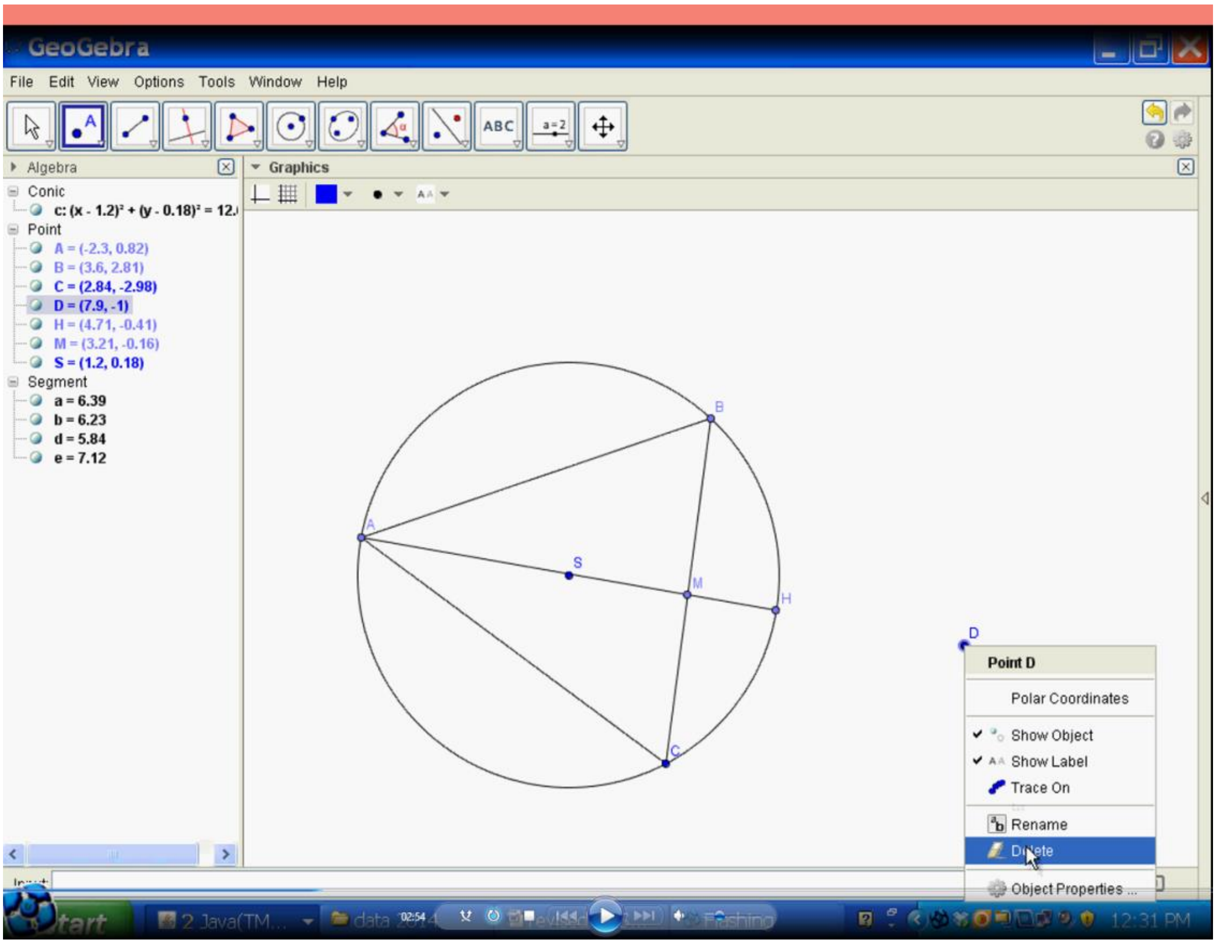


Figure 3.2: A snap-shot of Lesedi's video screen recording of Task 1 (c)

As mentioned earlier, the screen-casting software, *UltraVNC Addon*, was used to capture the work. The participating PTs' work was captured (recorded) whilst they were working on the tasks in the GeoGebra platform. The captured work was converted to video recordings. For example, the screen-recording for all the participating PTs on Task 1 (c) was between 4 minutes and 20 minutes long.

3.4.3 Semi-structured interviews

The study employed 90 to 120 minute semi-structured interviews as a means to probe into PTs' responses to the written tasks and the GeoGebra-based tasks (see Appendix C). Semi-structured interviews were preferred because they are flexible in approach to gaining insight into PTs' thinking and to tap into their circle geometry knowledge. The participating PTs were individually interviewed three days after the completion of the GeoGebra-based tasks. This time period gave me the opportunity to acquaint myself with the PTs' scripts of the written tasks and the screen recordings videos of GeoGebra-based tasks in preparation for the interview. The interview focused on the participating PTs' explanations about the solution processes for all the tasks. The interviews were designed to encourage participating PTs to generate narratives on their experiences relating to implementation of the tasks. The semi-structured interviews allowed the researcher and the participants to engage in a dialogue allowing for probing of responses by the interviewer. The one to one interview included playback of the video of screen-cast recording episodes where the participating PTs described their thinking to the researcher. The interviews were employed as a means of data triangulation that aimed at displaying participating PTs' knowledge of geometry content, knowledge of pedagogy and knowledge of technology. The interview also focused on the participating PTs' response to the tasks as reflected in the participating PTs' written scripts. It involved narrating of participating PTs' thinking during the process of answering each task. Questions like "take me through the solution to the task" to solicit participating PTs' response to the tasks; "what were you thinking when you wrote this answer?" to focus on participating PTs' thinking about the tasks; and "why did you delete the segment" to focus on participating PTs' interaction with GeoGebra, were used to probe the participating PTs thinking processes. Audio-recording of the interviews was conducted to assist the researcher to store the data in its original form for the analysis at a later stage.

3.5 The pilot study

The pilot study was guided by Sierpinska's (2004) position that the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching is considered essential in mathematics teaching and learning. I focused on an intact group of PiPTs to pilot both written tasks and GeoGebra-based tasks. The piloting exercise was intended to inform the design, reliability and construct validity of the tasks and the use of the screen-cast recorder. Attention was paid to the tasks and the screen-cast recorder because these were the intended data collection instruments that the study would use as a means to get insight into the PiPTs' knowledge of teaching and learning of circle geometry. The design of the tasks was informed by the objective of the study: to characterize participating PTs' TPACK. As such, the tasks were designed to address all the constructs of the TPACK framework. The tasks that elicited CK, PCK and PK were written tasks whereas the tasks which elicited TK, TPK, TCK and TPACK were GeoGebra-based since they incorporated the technology knowledge domain. There were six tasks designed to elicit the TPACK constructs as illustrated in Table 3.2. The matrix shows that the tasks elicited at least one construct. For example, five tasks elicited the CK construct whereas Task 6 elicited all the constructs.

Table 3.2: Matrix for tasks specifications in pilot

Nature of the task	TPACK construct that the tasks focuses on	Task 1				Task 2	Task 3		Task 4		Task 5			Task 6
		(a)	(b)	(c)	(d)	(a)	(a)	(b)	(a)	(b)	(a)	(b)	(c)	
Written tasks	CK	√	√			√	√	√	√	√	√		√	√
	PK							√				√		√
	PCK							√				√		√
GeoGebra-based tasks	TK			√		√		√		√			√	√
	TCK			√	√	√		√		√			√	√
	TPK							√						√
	TPACK							√						√

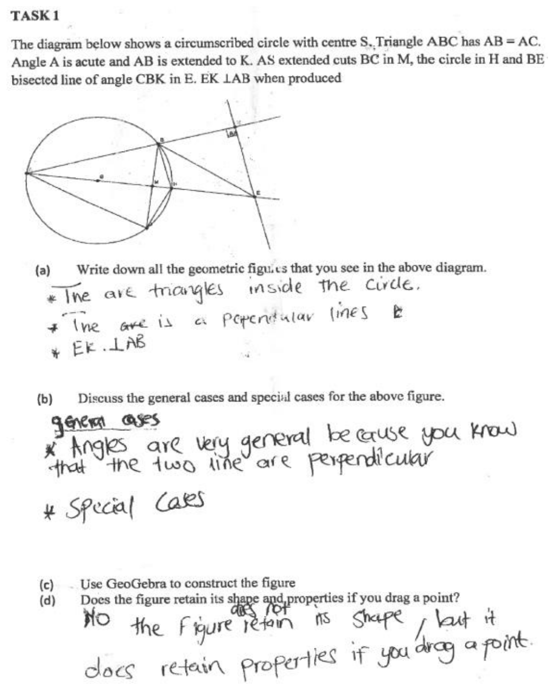
Note: √ means that the task elicits the TPACK construct

As mentioned already, the question items and format of the tasks were scrutinized for construct validity. Were they measuring what they were supposed to measure? The rigorous scrutiny of the tasks also provided an opportune moment to develop and refine the analytical rubrics for the major study.

3.5.1 Modifications to Tasks

As previously mentioned, the exploration of the participating PTs' knowledge of circle geometry was done by probing into the participating PTs' thinking displayed in the participating PTs solutions to the TPACK tasks. These tasks were deliberately designed to elicit the TPACK knowledge constructs. A reflection on the pilot tools revealed that there were faults in the task design. Some tasks were not explicit in terms of the TPACK construct intended to elicit whilst other tasks were struck off (see Table 3.3). To better discern participating PTs' TPACK, the critical components of the tasks were addressed. The structure of some questions was revised as evident in Task 1. Figure 3.3 (see also Appendix C) shows a comparison of responses to Task 1 of a PiPT and that of a participating PT. Some critical components of the tasks before piloting were found wanting since there was a lack of explicitness in the item descriptions and in what the expectation of the questions were. The sub-tasks of Task 1 were either re-constructed or deleted. For example Task 1 (b) was rephrased with the change stemming from the ambiguity of the meaning of the terms 'special cases' and 'general cases'.

Task 1 before pilot (PiPT script)



TASK 1

The diagram below shows a circumscribed circle with centre S. Triangle ABC has AB = AC. Angle A is acute and AB is extended to K. AS extended cuts BC in M, the circle in H and BE bisected line of angle CBK in E. EK ⊥AB when produced

(a) Write down all the geometric figures that you see in the above diagram.

* The are triangles inside the circle.
* The are is a perpendicular lines
* EK.⊥AB

(b) Discuss the general cases and special cases for the above figure.

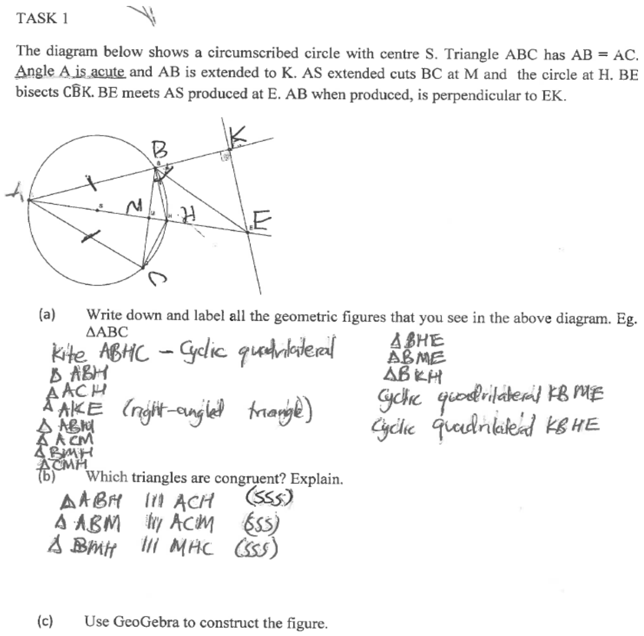
general cases
* Angles are very general because you know that the two line are perpendicular
* Special Cases

(c) Use GeoGebra to construct the figure

(d) Does the figure retain its shape and properties if you drag a point?

No the Figure does not retain its shape, but it does retain properties if you drag a point.

Task 1 after pilot (participating PT script)



TASK 1

The diagram below shows a circumscribed circle with centre S. Triangle ABC has AB = AC. Angle A is acute and AB is extended to K. AS extended cuts BC at M and the circle at H. BE bisects $C\hat{B}K$. BE meets AS produced at E. AB when produced, is perpendicular to EK.

(a) Write down and label all the geometric figures that you see in the above diagram. Eg. ΔABC

Kite ABHC – Cyclic quadrilateral
ΔABH
ΔACH
ΔAKE (right-angled triangle)
ΔABM
ΔACM
ΔBMH
ΔCMH
ΔBHE
ΔBME
ΔBKH
Cyclic quadrilateral KBME
Cyclic quadrilateral KBHE

(b) Which triangles are congruent? Explain.

ΔABH ||| ACH (SSS)
ΔABM ||| ACM (SSS)
ΔBMH ||| MHC (SSS)

(c) Use GeoGebra to construct the figure.

Figure 3.3: Comparison of Task 1 with PTs' responses before and after the pilot study

Piloting informed the focus of the study and the study procedures. One challenge that I encountered during the process of collecting pilot data was the PiPTs not responding to all the tasks as required. All data was considered valuable particularly since the tasks elicited different knowledge constructs, so it was of great importance to have responses for all the tasks. There was a huge amount of data collected from the PiPTs who could not fully comprehend the tasks. A decision was made to cut down on the tasks so that data collected could be manageable during the analysis process. See Table 3.3 for task specifications after piloting. As such, a deliberate move was made to specifically pay attention only to the task that focused on the TPACK constructs that have content (C) as the common denominator. It was deemed necessary to consider C since content knowledge was found to be weak among

South African PTs and considering that CK was a necessary, albeit not sufficient, aspect of maths teaching (see Chapter 1 and Chapter 2).

Table 3.3: Matrix for tasks specifications after piloting

Nature of the task	TPACK construct that tasks elicit	Task 1			Task 2		Task 3		Task 4
		(a)	(b)	(c)	(a)	(b)	(a)	(b)	(a)
written tasks	CK	√	√		√		√	√	
	PCK					√			
GeoGebra-based tasks	TCK			√					√

Note: √ means that the task elicits the TPACK construct

3.6 Reliability and validity of the Data

The rigor of qualitative research is meant to generate and sustain the readers' trust and confidence in the research findings (Opie, 2004). Claims made in a case study should be authentic and credible by ensuring that the research instruments are valid and reliable. I needed to confirm that the tasks as the main tools for the study were testing what they were intended to test and if inferences about the participating PTs' TPACK made from the participating PTs' performance scores were valid. The validity of the tasks was achieved by looking for content and construct related evidence. I employed rigorous task analysis to ensure the validity of the tasks.

Aided by critical readers, the tasks were checked for the validity and reliability of inferences made through use of rubrics. The critical readers were both the supervisors of this study. The components of the task items were critically assessed if they elicited the TPACK construct that were supposed to be testing. Section 3.5.1 provides evidence that tasks were modified after a rigorous task analysis during the piloting stage. The analytical rubrics (see following section) were employed to analyse the tasks both in the pilot and major study. The criteria for the rubrics were rigorously scrutinized for construct validity by the critical readers. Construct validity refers to the extent to which the assessment tool claims to measure a construct. The use of written tasks, screen recording and interviews were considered as multiple sources of data collection that could enhance reliability and validity. In this study the rationale for focusing on participating PTs' performance scores and explanation or descriptions of their thinking when responding to the tasks was seen as a measure to ensure that the tasks assessed participating PTs' TPACK.

3.7 Data Analysis

Often TPACK development has been studied through the use of Likert-type scales, appropriating the use of pre- and post-tests to measure the development. Acknowledging the weaknesses of the Likert instrument and taking into consideration the design of the study, I decided to employ the use of rubrics to analyse participating PTs' responses. This analytical method used grounded theory approach in developing the descriptions of the rubrics. Clement, Chauvot, Philipp and Ambrose (2003) contend that rubrics serve a dual purpose (i) providing insights into written responses and (ii) use of numerical scores to statistically analyse responses. A rubric is a guideline that describes the characteristics of the different levels of performance used in scoring or judging a performance. An analytic rubric was preferred because it allowed for different levels of achievement of performance criteria to be determined.

The participating PTs responses were scored according to the analytic rubric that I designed to capture TPACK-related evidence. The rubrics are referred to in this study as the TPACK rubrics. The development of the TPACK rubrics was drawn from Miheso-O'Connor (2011), who employed the use of rubrics to measure pedagogical content knowledge proficiency in teaching mathematics. As such, the design of the rubrics was guided by the question "What would the participant need to know or be able to do to successfully respond to this task?" The TPACK rubrics used specific scores based on a five-point qualitative scale (ranging from 0 to 4) to capture the participating PTs' proficiency in the three main knowledge domains of content, pedagogy and technology and to provide insights into the participating PTs' responses. To generate the descriptions, I conducted an item analysis of each task according to the criteria that I developed from the two sources of evidence: TPACK constructs as conceptualized in the study and Duval (1995) cognitive apprehensions on geometry reasoning. Each task was first categorized according to the Duval's geometry cognitive apprehension and the TPACK construct that it was testing. Then, the process of developing categories for the descriptions or criteria for each performance level followed. The categories in rubrics were not exhaustive. The process started off with developing broad categories which were then refined inductively from the data suggesting that rubrics emanated from the categories of all the actual responses.

The TPACK rubrics had to be specific and explicitly address the expectations of the tasks, implying that the constructed rubrics were to be a guideline to analysing the participating PTs' responses. The descriptions were built from the expected ideal solutions of each task. That is, each rubric was specifically designed for a specific task. I utilized a five-point qualitative scale ranging from a score of 0 for non-response and/or incorrect response to a score of 4 for a correct response. The description for level 4 was based on the ideal correct solution, where all traits in the description were realized. In some instances, examples had to be given as a guide for some descriptions to clarify where certain responses would fit. The rubrics were scrutinized for both content and construct validity in the pilot study. See chapter 4 for further descriptions of the rubrics and coding of responses. A rubric was developed for each of the sub-tasks resulting in 8 rubrics for the major study (out of the original 13 rubrics used in the pilot study). An analysis of the tasks was essential in determining the reliability and validity of the items. A robust evaluation of the quality of items was expected to strengthen the arguments about what characterizes aspects of participating PTs' TPACK for learning teaching geometry in a technology-based environment.

Inter-rater Reliability of rubrics

Inter-rater reliability was considered when establishing the reliability and consistency of rubric scoring. A second rater was employed to assist in scoring the responses. The rubrics of the pilot study were rigorously revised before and after piloting several times with critical readers. I took into consideration before the pilot exercise that constructing rubric descriptions without the data at hand should be flexible to accommodate all possible responses. The development of rubrics was a lengthy process that required a negotiation that would cater for all possible strategies for the solutions. Distinguishing between cases required a negotiation between the theoretical and the practical. This process necessitated mediation between item analysis of the tasks and descriptions of the TPACK rubrics that focused on the TPACK constructs. The tasks and the rubrics were rigorously tested for coherence, reliability, and validity during this process. To test for validity and reliability I ensured that the descriptions were explicit and appropriate for each level. There was also a need for coherence between the expectation of the task and the TPACK rubric descriptions. The task item analysis process involved examining item format, item performance scoring and item wording. This effort resulted in improvements in the performance level criteria and the holistic scoring of the rubrics and the elimination of some sub-tasks from the pilot study.

Analysis of tasks

The TPACK rubrics were employed to analyse both written and GeoGebra-based tasks. The overall possible score of the participating PTs' responses for the modified tasks ranged from 0 to 32 based on the performance levels 0 to 4 of the scoring rubrics. The objective of this study was to characterize participating PTs' knowledge of geometry in terms of PCK, TCK and CK. To do this required qualifying the nature of the aspects of TPACK that the participating PTs displayed. In determining the participating PTs' competence in knowledge of geometry, I aligned the levels of coding for the quality of the knowledge displayed to the performance levels of the rubrics. Table 3.4 shows how the quality of knowledge was linked to performance levels. The quality of the performance levels were categorized as poor for level 0, nearly acceptable for level 1, acceptable for level 2, definitely acceptable for level 3 and high for level 4. However, the quality of participating PTs' TPACK was categorized as faulty, partial or adequate.

Table 3.4: linking quality of knowledge with performance levels

Quality of PTs' TPACK		TPACK rubrics performance levels	quality of the performance levels category
0 (faulty)	←	0	Poor
1 (partial)	←	{1 {2	nearly acceptable acceptable
2 (adequate)	←	{3 {4	definitely acceptable high

A quantitative summary of scores for the responses for each case (TPACK construct) was presented. Descriptive statistics were used to analyse the participating PTs' performance scores within and across the tasks for each case. Frequencies of scores were used to interpret the patterns of responses. Duval's (1995) analytical theory of cognitive apprehension was employed to understand the participating PTs' visualization, construction and reasoning processes. That is, Duval (1995) cognitive apprehensions and cognitive processes were used to interpret participating PTs' responses to all tasks. As an example, see Section 6.4 for an elaboration of how the cognitive processes were linked to cognitive apprehensions to determine forms of connections.

The interviews were audio recorded and transcribed. The interview transcriptions were used as a means of triangulating the rubric scores and getting insight into the written responses and substantiate trends illuminated by the performance scores. Two outputs were produced from the GeoGebra-based tasks; (i) a GeoGebra file of the construction, and (ii) screen cast video recording of the construction process. In the case of the GeoGebra file, the PTs' constructions as represented in the GeoGebra algebraic view, the graphic view and the construction protocol were analysed for evidence of TCK. The screen cast video recordings were analysed in two ways; (i) frames or snap-shot captured, and (ii) tracking the movements of the cursor and keyboard entries. The screen cast video recordings of the GeoGebra-based tasks were also transcribed with codes developed according to the Duval (1995) cognitive apprehensions and cognitive processes.

3.8 Ethical Considerations

This study strived to abide by the ethical considerations for research conducted in South Africa and ensured that ethical procedures were followed to protect and respect the rights of the participants. Ethical clearance to conduct the research was sought from the Head of School in the university and obtained from the School of Education Human Ethics Committee. Detailed information on the research and the research process was provided to the participants. The participants were accorded the opportunity to view their marked scripts; the screen cast video recordings and audio-recorded interviews. Written and verbal informed consent was obtained from the participants prior to the start of the study. Confidentiality and anonymity were maintained before and throughout the study, with a leeway for participants' to withdraw from the study at any time. The researcher assured the participants that participation in the study would not have any effect on their performance in the course. It was necessary to deal with any conflicts that might arise from issues of lecturer-student power-related tensions. The researcher and the participants engaged in a relational dialogue where clarity was given on the benefit of the study to the researcher, the participant and the methodology course. The dialogue offered opportunities for better understanding of the prospective teacher knowledge construction.

3.9 Chapter summary

This chapter sought to present the methodological approach adopted for exploring aspects of the six participating PTs' TPACK. An elaboration of the research design, data collection procedures and analysis were presented. This exploratory multiple case study described in the chapter sought to explore aspects of prospective teachers' technological pedagogical content knowledge of geometry constructed within a GeoGebra-based environment. The exploration was done through examining PTs' thinking processes as they responded to the circle geometry tasks. Data was collected through responses to tasks, interviews and screen cast recordings. Rubrics were employed as analytical tools.

CHAPTER 4

FRAMEWORK FOR ANALYSING PTs' GEOMETRY TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE

4.0 Introduction

In this chapter I will focus my discussion on the analytical framework that I employed as a lens to explore aspects of prospective teachers' technological pedagogical content knowledge (TPACK) constructed in a GeoGebra-based environment. The major focus of this chapter will be to interrogate how the TPACK framework was engaged as a frame of reference for analysing data for the study. I will elaborate on how using the inductive approach, a framework for data analysis emerged from an amalgamation of the TPACK framework and Duval's (1998) cognitive apprehensions' analytical framework for geometric reasoning. The study expanded the Duval analytical framework by extending it to include an analysis of teacher knowledge. The two frameworks were used as lenses for deconstructing the tasks as a precursor to developing analytical rubrics for scoring the PTs' response to the tasks. Refer to Chapter 5 for the elaboration on how the tasks were deconstructed. The purpose of developing the frameworks was to provide an analytical tool to be employed in analyzing the PTs' geometry knowledge. Further on, a description of the coding developed for the CK, PCK and TCK knowledge constructs is articulated. Throughout the chapter, I use Task 1 to show how the analytic tools were put into action in the coding process.

4.1 The TPACK as a conceptual framework

As mentioned in Chapter 3, this study was premised on the claims that, firstly, learning mathematics is profoundly influenced by the tasks, by the learning context and by the tools that are used in mathematics instruction. This claim is extended to all domains of mathematics. I contend that PTs' geometry thinking is profoundly influenced and framed by PTs' practical experiences with tasks, tools and the PTs' learning context. Secondly, that

PTs' geometry knowledge is developed through the interactions between content, pedagogy and technology knowledge. See Figure 4.1 for the conceptual framework.

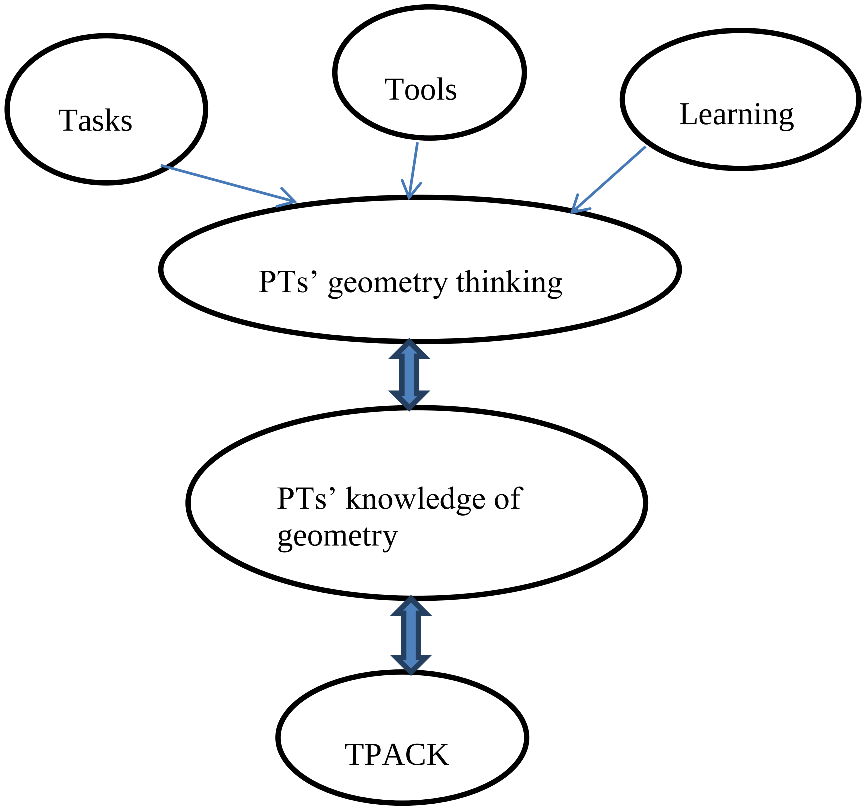


Figure 4.1: conceptual framework

The technological pedagogical content knowledge (TPACK) is a prerequisite to effective integration of technology in education. Mishra and Koehler (2006) explicate that TPACK is the interaction of content, pedagogy and technology bodies of knowledge, both theoretically and in practice, to produce the types of flexible knowledge needed to successfully integrate technology use into teaching. I employed the technological pedagogical content knowledge (TPACK) framework to study the teacher knowledge of circle geometry as proposed by Mishra & Koehler (2006). See a detailed elaboration about TPACK in Chapter 2.

4.2 The TPACK as an Analytical Framework

The first point of analyzing the PTs' responses to the tasks was to conceptualize the TPACK constructs according to the context of my study. Drawing from Mishra and Koehler (2006, p.

63) descriptions of the TPACK constructs, and through inductive analysis, I developed an analytical tool to describe the knowledge constructs as they relate to my study (CK, TCK and PCK). Table 4.1 presents the TPACK analytical framework I developed for analyzing PTs' knowledge. The table shows how the constructs were conceptualized with corresponding evidence of each construct. Mishra and Koehler (2006, p. 63) describe content knowledge (CK) as the "teachers' knowledge about the subject matter that includes knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge". CK was contextualized in this study as the PTs' knowledge of circle geometry concepts, theorems and proofs. The indicators for this construct acknowledge how the knowledge about circle geometry is organized and presented. For example, the PT is regarded as exhibiting knowledge of circle geometry when a connection is made between properties, theorems and representations.

Table 4.1: TPACK analytical framework

TPACK constructs	This construct as conceptualized in the study is about…	Indicators for the construct (the PT exhibits this knowledge when the PT...)
CK	knowledge of circle geometry concepts, theorems and proofs.	identifies and recognizes in the perceived figure several sub-figures; makes connections between geometry representations, properties and theorems; provides justifications to organize and connect circle geometry concepts, theorems and proofs.
TCK	knowledge of how GeoGebra and circle geometry influence and constrain one another; knowledge of how circle geometry can be changed by GeoGebra.	uses GeoGebra to make connections between concepts(pragmatic role of GeoGebra); recognizes how GeoGebra is used within the understanding of geometry; identifies aspects of circle geometry in GeoGebra constructions (epistemic role of GeoGebra); produces and describes a construction of a diagram with GeoGebra; configures and re-configures diagrams with GeoGebra.
PCK	knowledge of learner circle geometric thinking; knowledge of pedagogy that is applicable to the teaching of circle geometry; knowledge of circle geometry representations.	evaluates learner geometric thinking; describes teaching strategies; provides and uses multiple representations; explains geometry knowledge in meaningful ways; addresses any shortcomings or misconceptions.

Mishra and Koehler (2006, p. 63) describe technological content knowledge (TCK) as the "knowledge needed to understand which specific technologies are best suited for addressing subject-matter learning in their domains and how the content dictates or perhaps even changes the technology—or vice versa". In this study, TCK is defined as the knowledge of how GeoGebra and circle geometry influence and constrain one another and how circle geometry knowledge can be changed by GeoGebra. The indicators for this construct acknowledge that technology has an influence on subject-matter, which is referred to as content knowledge in this study. For example, the PT exhibits circle geometry technical content knowledge when the pragmatics and heuristic roles of GeoGebra are evident in the response.

Mishra and Koehler (2006, p. 63) describe pedagogical content knowledge (PCK) as

> *"The knowledge of pedagogy that is applicable to the teaching of specific content. It covers the core business of teaching, learning, curriculum, assessment and reporting, such as the conditions that promote learning and the links among curriculum, assessment, and pedagogy".*

However, I defined PCK as (i) the knowledge of what makes circle geometry concepts difficult or easy to learn, (ii) knowledge of learner circle geometric thinking, (iii) knowledge of pedagogy that is applicable to the teaching of circle geometry, and (iv) knowledge of circle geometry representations. The indicators for this construct acknowledge that pedagogy has an influence on content knowledge. For example, the PT exhibits circle geometry PCK when there is evidence in the response that the PT evaluates learner geometric thinking, describes teaching strategies, provides and uses multiple representations, explains geometry knowledge in meaningful ways, and addresses any shortcomings or misconceptions.

4.3 The Duval (1995) analytical framework for cognitive apprehensions

The second point of analysing the PTs' responses to the tasks was to conceptualize the cognitive apprehensions as interpretative tools for the TPACK constructs. Duval's (1995) cognitive apprehensions were employed as interpretative tools to discuss how the PTs responded to the CK, TCK and PCK tasks. The tasks were classified as cognitive questions since they focused on PTs' circle geometry knowledge and its conceptions. Duval (1995) describes apprehensions as several ways of looking at a drawing or visual stimulus. Table 4.2 presents Duval's analytic framework for the four cognitive apprehensions. The cognitive apprehensions are perceptual, discursive, operative and sequential apprehensions. The table illustrates how each apprehension was categorized to characterize geometry knowledge which in this study is the PTs' ability to relate to diagrams, figural properties and theorems.

Table 4.2: Duval (1995) analytical framework for cognitive apprehension as conceptualized in this study

category	Description of category (the apprehension is characterized as the …)	Indicators for apprehensions (the PT exhibits this type of apprehension when the PT….)
Perceptual apprehension	• ability to identify at first glance figures and recognize in the perceived figure several sub-figures.	• lists figures/shapes. • labels the figures/shapes.
Sequential apprehension	• ability to organize or produce a construction of a figure, depending on the technical affordances and constraints of GeoGebra and knowledge of geometrical properties. • ability to describe a construction of a figure, depending on the technical affordances and constraints of GeoGebra and knowledge of geometrical properties.	• Produces a GeoGebra construction protocol.
Discursive apprehension	• ability to connect configuration(s) with geometric principles. • ability to provide good description, explanation, deduction, use of symbols, reasoning depending on statements made on perceptual apprehension. • ability to describe figures through geometric language/narrative texts.	• describes the various ways to model or illustrate the theorem. • demonstrates the ability to provide an explanation of the concept or the procedure for the proof. • provides an explanation of general or specific instructional strategies for teaching the tan-chord theorem.
Operative apprehension	• ability to perform operations on the figure or its subfigure, either mentally or physically • ability to introduce several strings of figures from a given figure (configuration) • ability to modify the figure that appeared at the first glance (reconfiguration)	• describes a theorem with geometric reasoning. • links the theorem to information given in the diagram. • detailed description with clear explanation that modifies the figure that appeared at the first glance.

Within this study, perceptual apprehension was described as the ability to identify and recognize figures at a glance. Evidence for the perceptual apprehension was realized when the PTs listed and labeled figures perceived from a diagram. A GeoGebra construction protocol was evidence that the PT sequentially apprehended the figure to produce or describe a construction on the GeoGebra user interface. Producing a construction protocol depended on the technical affordances and constraints of GeoGebra and the PT's own knowledge of geometrical properties. To discursively apprehend a diagram indicates an ability to make connections between the configurations and geometry principles. The connections are evident through geometric language/narrative texts displayed within good descriptions, the appropriate use of geometry symbols and reasoning made through perceptual apprehension. An example of an indicator for this apprehension is when a PT describes the various ways to model or illustrates the theorem; this was considered as evidence of a discursive apprehension. The ability to configure and reconfigure a diagram is a description of operative apprehension category. Evidence of the operative apprehension was when the PT modified the figure.

As mentioned earlier, I expanded the Duval (1995) analytical framework for cognitive apprehension by extending it to include an analysis of teacher knowledge. Table 4.3 shows how the cognitive apprehensions were linked to the TPACK constructs in the process of interpreting the PTs' responses for each task. To understand the linkage, an explanation of what is involved for the comprehension of geometry is necessary. According to Duval (1998), there are three cognitive processes involved in the teaching and learning of geometry; the visualization process, construction process and reasoning process. Duval (1998) posits that these processes are linked to the cognitive apprehensions in that geometry thinking comprises of visualization of geometry objects, construction of geometry objects and reasoning about geometry objects. For example, a perceptual apprehension requires one to visually process geometry objects, suggesting that to perceive an object one must recognize and identify its configurations. Sequential apprehension entails a process of detailing the procedures for producing or describing a geometry construction according to the restrictions of a tool. These apprehension processes are dependent on reasoning about the geometric objects. Hence, tasks that elicit the TPACK constructs can be interpreted through the use of cognitive apprehensions.

Table 4.3: specification of the apprehensions in the TPACK constructs within the tasks

Nature of the task	TPACK construct that tasks elicit	Task 1			Task 2		Task 3		Task 4
		(a)	(b)	(c)	(a)	(b)	(a)	(b)	(a)
written tasks	CK	PA	DA		DA		DA	DA	
	PCK					DA			
GeoGebra-based tasks	TCK			SA					SA

Note: PA means perceptual apprehension; DA means discursive apprehension; SA means sequential apprehension

Table 4.3 shows that the CK and PCK tasks were interpreted using the perceptual apprehension (PA) and discursive apprehension (DA) whereas sequential apprehension (SA) was utilized to interpret the TCK tasks. All the written tasks (CK and PCK) required the PTs to make visual interpretations and to reason deductively whilst the GeoGebra-based tasks (TCK) required the use of GeoGebra to construct and/or interpret GeoGebra-constructed geometric diagrams. Duval (1995) emphasizes that the apprehensions can be used separately or simultaneously. For example, sequential apprehension in some cases might involve operating on the diagram (operative apprehension), suggesting that OA was back-grounded. Hence the apprehensions illustrated in Table 4.3 are those that were considered to be dominant when the PTs interacted with the diagrams and/or with GeoGebra.

4.4 Analysing CK

The CK construct was conceptualized in the study as the knowledge of circle geometry concepts, theorems and proofs. The objective of the study was to characterize the CK that the PTs displayed. Therefore, the coding for CK was drawn from two categories: (i) identifying and recognizing in the perceived figure several sub-figures, (ii) making connections between geometry representations, properties and theorems. These categories were considered appropriate since they could be classified as mathematics processes of making connections, representations and reasoning.

Identifying and recognizing in the perceived figure several sub-figures category

As mentioned in Section 4.3, cognitive apprehensions were employed to interpret PTs' TPACK construct. This category was coded under the cognitive category of perceptual apprehensions. Listing and labelling codes were developed for perceptual apprehension. The themes employed for listing and labelling of figures were identified. These themes were (i) systematized listing and labelling according to shapes, (ii) unsystematic listing and labelling, and (iii) systematic listing and labelling of triangles.

There were five sub-tasks tasks that elicited visualization and reasoning within the CK construct. To illustrate how the CK tasks for this category were analysed, I use Task 1(a) as an example (see Section 6.3). The PTs exhibited knowledge of CK when they identified and recognized in the perceived figure several sub-figures. This evidence of CK was coded according to Duval's cognitive apprehensions as a perceptual apprehension. Further on, the response was demarcated into two sub-themes. What was perceived at a glance was examined as to whether it was systematically or unsystematically presented. The first sub-theme was to determine if the identified figures were systematically or unsystematically presented. That is, was there any system used to identify the figures? The second sub-theme was the system of labeling. I determined whether labeling was systematic or unsystematic.

Making connections between geometry representations, properties and theorems category

This category was coded under the perceptual and discursive apprehensions. The cognitive processes required for making connections in the context of this study were visualization and reasoning. Perceptual apprehensions followed by discursive apprehensions are required in the process of making connections between representations, properties and theorems. I conceptualize mathematics connections as a tool that the PT uses to organize and describe their thinking when dealing with circle geometry. Hence, the types of connections that PTs made when interacting with geometry tasks shed light into the CK the PTs display.

Whereas Businskas (2008) and Mhlolo (2012a, 2012b) refer to practice-based mathematical connections**,** in this study I referred to these connections as teacher preparation-based mathematical connections because these connections are made in the context of teacher preparation where the prospective teachers are both learners and future teachers of geometry. For this study, I based my analysis on the PTs as learners of geometry. Drawing from Businskas (2008) and Mhlolo (2012a, 2012b) and through grounded analysis, I developed categories to describe the types of connections that PTs make. See Table 4.4 for the categories for the types of connections. I used inductive analysis to determine the categories of connections basing this on the expectations of the tasks, the geometry concepts within the tasks and the forms of connections. Further on, the forms of connections were demarcated into sub-themes: visual, systematic organization, implications and theorem application connections.

Table 4.4: categories for types of connections

Cognitive processes	Forms of connections	Indicators (we know this when there is use of …)
Visualization/ reasoning	Visual connections	words, symbols and figures to make connections between and among different representations.
Visualization/ reasoning	Systematic organization connections	words, symbols, propositions, figures and figurative units to organize geometric concepts or objects e.g. organizing geometric objects in terms of general and special cases.
Reasoning	Implication connections	properties, theorems, justifications and definitions to make logical connections between different geometric statements.
Visualization/ reasoning	Theorem application connections	a specific theorem A to solve problem B.

As mentioned earlier, to explore PTs' knowledge of circle geometry, I probed into the PTs thinking displayed in the PTs solutions. Table 4.4 indicates that the forms of connections are drawn from the PTs cognitive processes. Each form of connection had specific indicators. As such, an example for the categories for the form of connections termed 'visual connections' was among three different types of representations: verbal, figural and symbolic. An illustration of situations of responses for each form of connection is given in each description. The coding for each form of connection is discussed below.

Coding the connections

In Chapter 5, I discuss how I deconstructed the tasks using the following three components: (a) the critical components of the task, (b) the ideal actions required in completing the task, and (c) the CK construct(s) intended to be addressed by the task or the sub-tasks. This action was essential for the rigorous item analysis exercise that paved the way for building the rubrics used to summarize the PTs' responses to the tasks as seen in Section 6.2.1. I then pegged the forms of connections to the performance levels of the rubrics for the tasks. See Table 4.5 for an interpretation of how the quality of connections was linked to the rubrics performance levels. In determining the PTs' competence in knowledge of geometry, I aligned the levels of coding for the quality of the connections to the performance levels of the rubrics.

That is to say, I pegged the quality of connections (levels 0, 1, 2) with the performance levels of the rubrics (levels 0, 1, 2, 3, 4). In so doing, levels 3 and 4 of the performance rubrics were classified as level 2 of the quality of connections. Levels 2 and 1 of the performance were classified as level 1 of the quality of connections. Level 0 was classified as level 0 of the quality of connections.

Table 4.5: linking quality of connections with performance levels

Quality of connection levels		Rubrics performance levels	CK competence
0 (faulty)	←	0	Poor
1 (adequate)	←	{1 {2	nearly acceptable acceptable
2 (strong)	←	{3 {4	definitely acceptable High

The quality of connections within the tasks coded level 0 was classified as *faulty* knowledge of the relevant circle geometry, level 1 as *adequate* knowledge of the relevant circle geometry and level 2 as *strong* knowledge of the relevant circle geometry. A connection in the faulty category indicates that the PT shows poor understanding of the specific aspect of circle geometry. A connection in the adequate category indicates that the PT shows an adequate understanding of the specific aspect of circle geometry. A connection in the strong category indicates that the PT has good understanding of the specific aspect of circle geometry.

To illustrate how the CK tasks for 'making connections' category were analysed, I will again use the coding for visual connections category as an example (see Table 4.6). The PTs exhibited knowledge of CK when they made connections between geometry representations, properties and theorems. As mentioned earlier, forms of connections were identified as visual, systematic organization, implications and theorem application connections with each form of connection having specific indicators. Table 4.6 presents analysis of visual connections made between two different types of representations: verbal and figural registers. In this case, the PTs had to make connections between the figural register and the verbal register.

Table 4.6: visual connections made between the verbal and figure(s) registers

PT	verbal and figure(s)				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	1	1	2	1	2
John	2	2	2	1	2
Wisdom	1	2	2	2	2
Lesedi	1	1	2	2	2
Bonolo	2	1	0	1	2
Thabiso	1	2	0	1	2

Note: 0, 1, 2 denote quality of connections levels

A connection that qualifies to be at level 2 provides an explicit link between the figure and its verbal description as presented in the task. A visual connection scaled at level 2 is for responses that score at performance level 3 or 4 in the analytic rubrics. The PT would have identified a figure from the diagram and from its verbal description as given in the task. A verbal description with detailed properties of the figure identified, clearly illustrates that the description articulates that which was perceived. Using Task 1 (a) as an example, a response such as '*ΔABM →right-angled triangle'* shows that a figure (triangle) was identified from the diagram, labelled for specificity (*ΔABM*) and described using its properties (right-angled triangle). This connection was classified as a strong connection.

A connection that qualifies to be at level 1 provides a less explicit link between the figure and its verbal description as presented in the task. Level 1 visual connection between the verbal and figural register is pegged at analytic rubrics performance level 1 or 2. A verbal description with less detailed properties of the figure identified does not illustrate that which was perceived. For example, a response such as '*Isosceles triangle'* provides less details of what is seen. It is not specific as to which triangle is being referred to. This connection was classified as an adequate connection. In contrast, a level 0 visual connection between verbal and figural registers is assigned to a poor response in terms of performance rubrics. The PTs' response at this level displays faulty knowledge of the relevant circle geometry. Analysis of the quality of connections is presented in Chapter 6 (Section 6.4). Each category was coded specific to the indicators. A description of the coding for each form of connection was presented according to the indicators.

4.5 Analysing TCK

The TCK construct was conceptualized in the study as the knowledge of how circle geometry concepts may be represented with GeoGebra, the knowledge of how GeoGebra and circle geometry influence and constrain one another and the knowledge of how circle geometry can be affected by the use of GeoGebra. The objective of the study was to characterize the TCK that the PTs displayed. Therefore, the coding for TCK was drawn from two categories: (i) construction of geometric diagrams with GeoGebra, (ii) verbal description of geometrical diagram constructed with GeoGebra. These categories were coded under the perceptual and sequential apprehensions. The cognitive processes linked to these apprehensions are the construction and reasoning processes.

Construction of geometric diagrams with GeoGebra category

This category was classified as the construction process of the cognitive processes. The ability to correctly produce a construction with GeoGebra was an indicator for PTs' TCK. Sequential apprehension guided the analytical process for the PTs' constructions. To characterize the PTs' TCK in this category, I examined their GeoGebra files and screen-cast recordings for the process used to construct the figure with GeoGebra. In the GeoGebra file, I focused on the output of the GeoGebra algebraic view for text inputs of the construction processes, the output of graphic view for the geometric representations of the construction and the construction protocol for the step-by-step construction processes. The screen-cast recording provided a visual process of the actions made during the construction process.

The algebraic view contains the numeric and algebraic representations of free and dependent constructed objects. To analyse the algebraic view, the number of outputs were identified and then classified according to object type. On the other hand, the graphic view contains geometric representations objects. These can be drawn or created and modified using the construction tools. To analyse the graphic view, the objects drawn were identified. The codes for analysing the construction protocol were the order of construction and the number of steps taken to construct of the geometric objects. Screen recording captured the actual construction process by tracking the movements of the cursor and the PTs' interaction with the GeoGebra construction tools and the GeoGebra menu. To understand the construction process, I analysed the actions of the cursor as the PTs were constructing the objects. Codes for the actions and the time taken to complete the construction were determined.

Verbal description of geometrical diagram constructed with GeoGebra category

This category was coded under the cognitive perceptual and discursive apprehensions. Discursive apprehension guided the analytical process for the PTs' descriptions. The ability to verbally describe errors in a GeoGebra-based construction was an indicator for PTs' TCK in this category. To respond to the task that featured in this category, the PTs interacted with a learner's GeoGebra file to discursively identify and describe the errors in the GeoGebra-constructed diagram. There were five themes developed from the statements of the description with each addressing what the PTs could or could not describe. These themes were based on all four apprehensions as described in Table 7.6.

4.6 Analysing PCK

The PCK construct was conceptualized in the study as the prospective teachers' knowledge about teaching circle geometry. The objective of the study was to characterize the type of PCK that the PTs' have. The cognitive process linked to this apprehension is the reasoning process. The PCK tasks elicited knowledge of geometric reasoning in teacher preparation with the hope of establishing the PTs' geometric reasoning skills in pedagogical contexts. The descriptions were to reveal a discursive apprehension of connections between configurations and mathematical principles through narratives

I employed the Chick, Baker, Pham, & Cheng (2006) model to analyse the types of PCK that the PTs exhibited in a hypothetical mathematics learning environment for teacher-preparation. Chick et al. (2006) framework unpacks how PCK is evident in teaching. Table 4.7 shows the three PCK categories with indicators for each sub-category. The Chick et al. (2006) framework was modified and adapted as an analytical tool for PTs' PCK.

Table 4.7: A modified Chick, Baker, Pham, & Cheng (2006) framework for analysing PCK

PCK Category	Evident when the PT …
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching the proof of the tan-chord theorem
Learner Thinking	Discusses or addresses learner ways of thinking about the proof of the tan-chord theorem
Learner Thinking-Misconceptions	Discusses or addresses learner misconceptions about the proof of the tan-chord theorem
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate the proof of the tan-chord theorem
Explanations	Explains the proof of the tan-chord theorem
Knowledge of Examples	Uses an example that highlights the proof of the tan-chord theorem
Knowledge of GeoGebra	Discusses/uses GeoGebra to support teaching of proof of the tan-chord theorem
Curriculum Knowledge	Discusses how the tan-chord theorem fit into the curriculum
Purpose of Content Knowledge	Discusses reasons the tan-chord theorem being included in the curriculum or how it might be used
PCK Category	Evident when the PT …
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics (PUFM)	Exhibits deep and thorough conceptual understanding of identified aspects of the proof of the tan-chord theorem
Deconstructing Content to Key Components	Identifies critical mathematical components within the tan-chord theorem that are fundamental for understanding, applying and proving of the tan-chord theorem
Mathematical Structure and Connections	Makes connections between the tan-chord theorem and other circle geometry concepts
Procedural Knowledge	Displays procedural skills for proving the tan-chord theorem
Methods of Solution	Demonstrates a method for proving the tan-chord theorem
PCK Category	Evident when the PT …
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for learners' learning
Getting and Maintaining Learner Focus	Discusses or uses strategies for engaging learners
Classroom Techniques	Discusses or uses generic classroom practices
Integrating technology	Discusses or uses GeoGebra as a pedagogical tool

Based on the notion that teacher knowledge is multi-faceted, Chick et al. (2006) developed a framework that fused together elements of PCK as proposed by various PCK researchers. Among these researchers are Shulman (1986, 1987), Ball (2000) and Ma (1999). Chick et al. (2006) trimmed the elements of PCK into three categories; (i) Clearly PCK, (ii) Content Knowledge in a Pedagogical Context, and (iii) Pedagogical Knowledge in a Content Context. These categories were adapted and modified for this study as proposed in Table 4.7.

In line with the Chick et al. (2006) model, the attributes of the PCK were modified into three categories: (i) the ability to demonstrate how pedagogy and circle geometry are intertwined, (ii) the ability to deconstruct circle geometry knowledge in a pedagogical context, and (iii) the ability to describe pedagogical knowledge in the context of circle geometry. These categories were coded under the discursive apprehension. Coding was determined for the various PCK sub-categories as displayed in Table 4.7. The PCK construct was analysed qualitatively and quantitatively. A deductive approach was utilized to classify the main categories. The process required establishing whether a specified sub-category was evident in the description; it was coded as a 'yes' if evident or 'no' if not evident. The patterns of the attributes for each PCK main category were then interpreted as a response to the type of PCK that the PTs displayed.

4.7 Chapter summary

In this chapter my discussion was focused on the frameworks employed in the study. The aim of the study was to characterize aspects of prospective teachers' technological pedagogical content knowledge (TPACK) constructed in a GeoGebra-based environment. The amalgamation of the Duval's (1995) framework for analysing PTs' apprehensions and Mishra and Koehler (1986) TPACK framework were found to be useful for exploring aspects of prospective teachers' circle geometry technological pedagogical content knowledge. The description of the frameworks was followed by a description of the themes and categories, and as well as the coding that was informed by the analytic framework and the interpretative framework. I elaborated on how the Chick et al. (2006) PCK framework was conceptualized.

CHAPTER 5

DECONSTRUCTION OF THE TASKS AND RUBRICS

5.0 Introduction

The purpose of this case study was to explore aspects of prospective teachers' technological pedagogical content knowledge of geometry in the context of a GeoGebra-based environment. The major focus of this chapter is to describe and discuss the tasks and rubrics utilized in the study. The deconstruction of the tasks precedes the descriptions of the rubrics for each task.

5.1 Features of the tasks

In terms of TPACK, mathematics teacher knowledge for technology integration is built on the interaction of content knowledge, pedagogical knowledge, and technology knowledge. The tasks selected for this study had elements of these three bodies of knowledge. Although the main emphasis of the tasks was to intertwine content, pedagogy and technology, I designed the tasks according to Stylianides & Stylianides (2010) and Biza, Nardi, & Zachariades' (2007) recommended features of mathematics pedagogy and content tasks for PTs. The technology tasks were planned with reference to Laborde's (2001) recommended features. See Appendix C for the tasks and memoranda for tasks.

Stylianides & Stylianides (2010) propose that the nature of mathematics tasks for preparing teachers should engage participants in mathematics content, link mathematical ideas suggested by theory or research, and engage participants in mathematical activity from the perspective of a teacher of mathematics. Similarly, Biza, Nardi, & Zachariades (2007) suggest that the structure of tasks should explore (i) subject-matter knowledge, (ii) types of pedagogy and, (iii) types of didactical practice that describe feedback to learner's response.

The technological features of the tasks were structured as suggested by Laborde (2001: 293). He categorizes tasks in a dynamic geometry environment as

(1) tasks for which the technology facilitates but does not change the task (e.g., measuring and producing figures); (2) tasks for which the technology facilitates exploration and analysis (e.g., identifying relationships through dragging); (3) tasks that can be done with paper-and-pencil, but in which new approaches can be taken using technology (e.g., a vector or transformational approach); and (4) tasks that cannot be posed without technology (e.g., reconstruct a given dynamic diagram by experimenting with it to identify its properties – the meaning of the task comes through dragging). For the first two types, the task is facilitated by the technology; for the second two, the task is changed by technology.

The tasks comprised of a series of content-based and pedagogical-based questions involving typical problems at the level of South African Grade 11 geometry. I reiterate that attention was paid to the CK, TCK and PCK constructs of the TPACK framework. The knowledge competencies drawn from the TPACK framework that the participants were expected to demonstrate in response to the proposed tasks were:

- Demonstrate the skills and understanding for interpreting mathematics learner thinking (PCK)
- Demonstrate pedagogical skills for planning to teaching school geometry (PCK)
- Demonstrate an understanding of grade 11 Euclidean geometry theorems and proofs with application of different approaches to the proofs in a GeoGebra-based environment (CK and TCK)
- Demonstrate an understanding of use of GeoGebra in solving circle geometry tasks. (TCK)

5.2 Deconstructing tasks

In deconstructing the tasks, I addressed three components: (a) the critical components of the task, (b) the actions required to complete the task, and (c) the TPACK construct(s) addressed by the task or the sub-tasks. Table 5.1 provides a description of how the TPACK constructs were operationalized in this study.

Table 5.1: Knowledge constructs as operationalized in the tasks

TPACK constructs	Is present when the PT demonstrates….
CK	knowledge of circle geometry concepts, theorems and proofs
TK	understanding and mastery of certain ways of thinking about and working with GeoGebra
PK	processes and practices or methods of teaching and learning circle geometry
TCK	knowledge of how GeoGebra and circle geometry influence and constrain one another knowledge of how circle geometry can be changed by GeoGebra knowledge of how GeoGebra can be used to facilitate the learning of circle geometry
PCK	knowledge of what makes concepts difficult or easy to learn knowledge of student thinking knowledge of pedagogy that is applicable to the teaching of circle geometry
TPK	knowledge of the constraints and affordances of GeoGebra in teaching circle geometry
TPACK	knowledge of the interplay between teaching circle geometry with GeoGebra using appropriate pedagogical strategies

I have employed the TPACK constructs and Duval's (2004) *apprehension* and *cognitive* perspectives of geometric reasoning as a lens for deconstructing the tasks. Duval's four cognitive apprehensions of 'perceptual', 'sequential', 'discursive' and 'operative' provided a framework for understanding geometric reasoning, visualization and construction processes utilised when the PTs responded to the tasks. See Chapter 2 for an elaboration of cognitive processes.

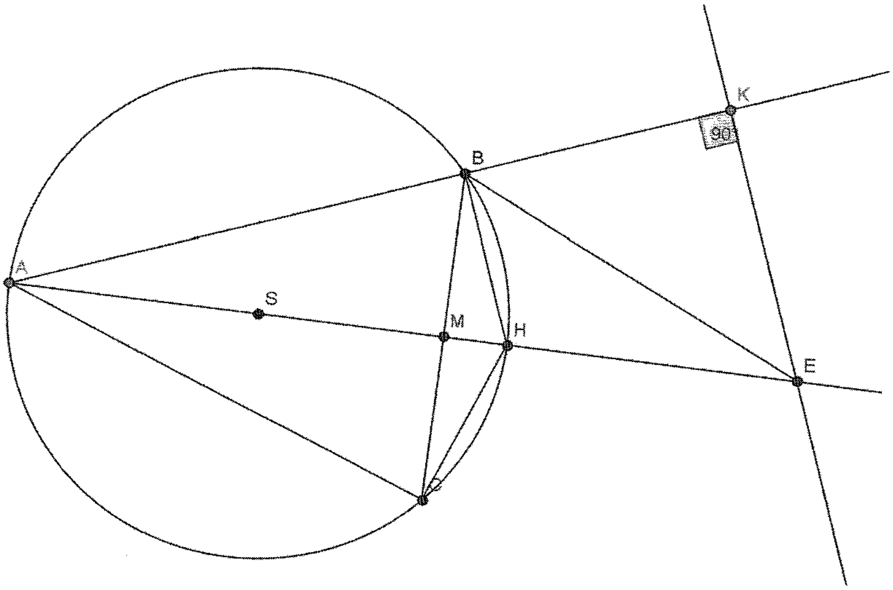
As mentioned in Chapter 3, the PTs responses were scored according to the analytic rubrics designed to capture TPACK-related evidence. I utilized five-point qualitative scale analytical rubrics basing on the PTs' responses to the tasks. An analytic rubric was preferred because it allowed for different levels of achievement of performance criteria to be determined. The different levels incorporated PTs' thinking in relation to the cognitive apprehensions and the TPACK constructs. I used a reverse method in determining the descriptions or criteria starting with performance level 4 building down to performance level 0. The description for level 4 was based on the ideal correct solution, where all traits in the description were realized. The rubrics had to be specific and explicitly address the expectations of the tasks. The descriptions developed were built from the expected ideal solutions devised in the memorandum. In some instances, examples had to be given as a guide for some descriptions to make clear where certain responses would fit.

In designing the rubrics, I was guided by the question “What would the participant need to know or be able to do to successfully respond to this task?” The rubrics were developed through inductive and deductive processes by capturing the PTs’ performance in the three main knowledge domains of content, pedagogy and technology. As I indicated in Chapter 3, I started off with broad categories and these were then refined so that all data could be categorised. To generate categories and codes, I read through and grouped all the responses for each task and sub-task according to the descriptors. See Section 4.3 for an explanation of how the TPACK construct and cognitive apprehensions informed the grouping of the responses. The three sources of evidence for the descriptors were: TPACK constructs as conceptualized in the study and the Duval (1995) model of *apprehensive* and *cognitive* perspectives on geometry reasoning. Each task was first categorized according to the Duval’s geometry apprehension and the TPACK construct that it is testing.

5.2.1 Deconstructing Task 1

Task 1 comprised a series of content-based and technology-based questions involving typical problems based on Grade 11 geometry level, requiring the participants to identify, describe and construct geometrical objects. The purpose of the task was to provide a platform from which I, as a researcher, can infer the participants’ knowledge of geometry properties, generalities, or theorems. See Figure 5.1 for Task 1 and Appendix C for Task 1 and its memorandum. The major purpose of the task was to provide opportunities for application of the cognitive apprehensions and cognitive processes for geometric reasoning.

The diagram below shows a circumscribed circle with centre S. Triangle ABC has AB = AC. Angle A is acute and AB is extended to K. AS extended cuts BC at M and the circle at H. BE bisects $C\hat{B}K$. BE meets AS produced at E. AB when produced, is perpendicular to EK.



(a) Write down and label all the geometric figures that you see in the above diagram. E.g. ∆ABC
(b) Which triangles are congruent? Explain.
(c) Use GeoGebra to construct the figure.

Figure 5.1: Task 1

The critical components of the task

The mathematical object of the task was to compose and decompose figures within a given diagram using an understanding of geometrical concepts and spatial representations derived from the figure. Task 1 was based on the argument by Gagatsis, Deliyianni, Elia, Monoyiou and Michael (2010:37) that "geometrical figures are simultaneously concepts and spatial representations". This argument suggests that "diagrams in two-dimensional geometry play an ambiguous role: on the one hand, they refer to theoretical geometrical properties, while on the other, they offer spatio-graphical properties that can give rise to a student's perceptual activity" (Laborde, 2004:1). The major purpose of this task was to make a mathematical argument when interacting with the diagram. Herbst (2004) argues that interacting with diagrams provides an opportunity to make reasoned conjectures. The task required the visualization, construction and reasoning processes to be enacted. The task required a perceptual apprehension of the diagram in order to identify and discriminate the figures from a given diagram

The action required to complete the task

The task required the PTs to make visual interpretations, to reason deductively and to do constructions. The task provided an opportunity to explore the PTs' prior knowledge regarding definitions, properties, theorems and constructions of geometric figures, deductions that could be made about these figures and the ability to transform a static drawing to a dynamic construction. Tasks 1(a) and (b) examined the ability to discriminate and recognize in the perceived figure several sub-figures and as such this task was concerned with examining the PTs' visual spatial ability: the mental ability to manipulate objects and their parts in a two dimensional space. Task 1(b) solidified the deductions made in (a). Tasks 1(a) required a perceptual apprehension. Tasks 1(b) required a discursive apprehension. Tasks 1(c) required a sequential apprehension. Task 1 (c) provided the PTs with opportunities to explore construction strategies and to solidify the idea that these constructions were based on geometric properties identified in (a) and (b). In this task PTs invented strategies for constructing a perpendicular bisector, a cyclic quadrilateral, isosceles triangle, etc., by building more sophisticated GeoGebra constructions, such as inscribing an isosceles triangle in a circle.

The TPACK construct(s) addressed by the task

The task comprised content-based and technology-based questions. The task was testing the TPACK constructs of CK and TCK. Tasks 1 (a) and (b) examined the CK that required geometry competences. A conceptual understanding of aspects of circle geometry should be identified by making connections between concepts. Task 1(a) required perceptual apprehension of the figure in order to identify sub-figures. Task 1(b) required a discursive apprehension of the figure in that the PPTs needed to give justifications as to why the relevant triangles were congruent.   Task 1(c) examined the TCK that requires competence to use GeoGebra to mediate geometry proficiency. Task 1(c) required sequential apprehension to deal with the knowledge of how to represent circle geometry properties within a GeoGebra environment. The PT was required to identify the geometrical relationships between the objects created in the dynamic and static environments. To successfully do the identification, PTs needed to visualize the different configurations of the figures and use GeoGebra construction tools such as the 'drag mode' tool to sequentially explore the conjectures.

The rubrics for Task 1

Task 1 (a)

This task tested PTs' geometry content knowledge. The PTs were required to "write down and label all the geometric shapes/figures that you see in the above diagram". To avoid misunderstandings, an example was indicated to lead the respondent towards the expected answer. In performance levels 4 – 1, the descriptions reflect that the PT correctly identified and labelled the figures mentioning at least the three figures. See Table 5.2 for rubrics for Task 1(a). Although I expected the PTs to know the basic figures i.e. circle, triangle, quadrilaterals, level 1 catered for responses that mentioned two (2) figures correctly regardless of the type of figure. I considered that labeling could be a constraint to some respondents. There are at most 17 figures that one can recognize in the perceived figures and several subfigures so an interval of number of figures had to be determined for the 4 levels. As noted earlier, the lowest number of figures should be 3 and the maximum for a response that considered the figures built from the three basic figures is 17. However, an exceptional case would be an inclusion of semi-circles and circle segments. This statement qualifies the at most 17 figures identified.

Table 5.2: rubrics for Task1 (a)

level	Description
0	No shape/figure identified
1	Correct identification and labelling of 3 figures even if similar e.g. all triangles
2	Correct identification and labelling of 4 - 9 figures with three major shapes :circle, triangle, quadrilaterals inclusive
3	Correct identification and labeling of 10 - 16 figures including three major shapes :circle, triangle, quadrilaterals inclusive
4	Correct identification and labeling of at most 17 figures including three major shapes :circle, triangle, quadrilaterals inclusive

Task 1(b)

This task tested PTs' geometry content knowledge. The PTs were required to show and explain "which triangles are congruent". In levels 4 – 1, the descriptions reflect that the PT correctly identified the congruent triangles based on the recognition that AH was given as the diameter of the circle. See Table 5.3 for rubrics for Task 1(b). The mathematical statement given in the responses for these levels should reflect both the visualization and reasoning process. However, a correct identification or configuration of the diagram to show congruency may not necessarily be aligned with the correct reasoning or explanations. As such, the explanations were coded with respect to the levels as correct, incomplete correct, faulty, and no explanations. For instance, level 3 differs with level 4 in that the level 3 response provides a correct but incomplete explanation.

Table 5.3: Rubrics for Task 1(b)

level	Description
0	incorrect identification of pairs of congruent ∆s or no response
1	Correct identification of 3 pairs of congruent triangles; no explanations
2	Correct identification of 3 pairs of congruent triangles ; Faulty explanations
3	Correct identification of 3 pairs of congruent triangles ; incomplete correct explanations
4	Correct identification of 3 pairs of congruent triangles. Correct explanations using geometric reasoning, recognizing in reasoning that AH is diameter.

Task 1(c)

This task tested PTs' geometry technological content knowledge (TCK). The PTs were required to "Use GeoGebra to construct the diagram". In this task there was interplay between knowledge of GeoGebra and geometry knowledge. The intention was for the descriptions to capture both knowledge of GeoGebra and geometry knowledge. The response for the task required a proper use of GeoGebra, suggesting that in constructing the diagram with GeoGebra, there were three possibilities; a correct construction, an incorrect construction or no construction, See Table 4.4 for rubrics for Task 1(c). A level 4 description reflected a correct construction at a glance, suggesting that during the construction process, a complete exploitation of the affordances of GeoGebra was realized, resulting in a short concise sequence of construction. A level 3 description showed a correctly constructed diagram but using a long sequence of construction. A level 2 description was for an incorrect disjointed construction that indicated less exploitation of affordances of GeoGebra. At this level there was no systematic approach to the construction with a possibility of disorientation when a point was dragged. A systematic approach would optimally use GeoGebra as a dynamic geometric tool. At level 1 an attempt to construct was made but did not necessarily produce the required diagram, reflecting some technical knowledge but lack of geometry knowledge.

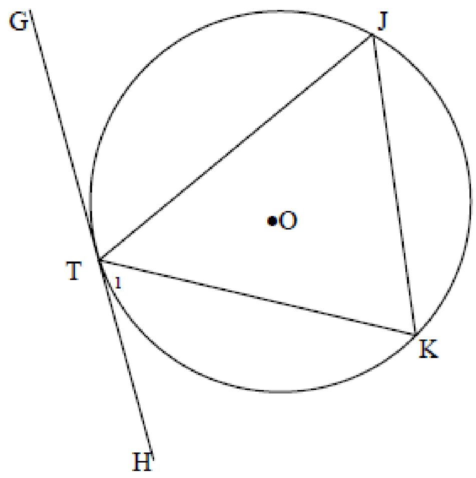
Table 5.4: Rubrics for Task 1(c)

level	Description
0	Inability to use GeoGebra
1	Some figure drawn, missing other details e.g. ΔABC not isosceles
2	incorrect disjointed construction, less dependent on GeoGebra, no systematic approach to construction, possibility of disorientation when point is dragged
3	Correct construction at a glance, complete dependent on GeoGebra, long sequence of construction
4	Correct construction at a glance, complete dependent on GeoGebra, short sequence of construction

5.2.2 Deconstructing Task 2

Task 2 was a content-based and pedagogical-based question. See Figure 5.2 for Task 2 and Appendix C for Task 2 and its memorandum. The content-based sub-question involved a typical problem based on a Grade 11 geometry level, requiring the participants to prove a circle geometry theorem in several ways. The pedagogical-based question required knowledge and skills in applying this kind of task in a mathematics classroom situation.

In the diagram below O is the centre of the circle. GH is a tangent to the circle at T.
J and K are points on the circumference of the circle. TJ, TK and JK are joined.



(a) Prove the theorem that states that $\hat{T}_1 = T\hat{J}K$ using four different methods (four constructions)

(b) How would you handle this problem in a classroom environment?

Figure 5.2: Task 2

The critical components of the task

The critical component of the task was the ability to "do proofs" and recognize the multiple methods of proving the tan-chord theorem. The PT needed to understand and have knowledge of producing statements with reasoning to connect the statements to a conclusion in the process of providing a proof. The PT was expected to provide a logical argument by producing a sequence of statements with justified reasons. The multiple methods required the PTs to recognize how other Euclidean geometry theorem were prerequisites of the tan-chord theorem, suggesting that to be able to produce multiple methods of the proof requires remembering theorems and knowing how to link and apply them to the tan-chord theorem. The pedagogical aspect of the task acknowledged that the PT should have knowledge of modelling the various instructional strategies applicable in teaching of proofs. The PT must demonstrate the ability to provide an explanation of the proof. Generally, the task required a discursive apprehension. The task required both visualization and reasoning processes.

The actions required to complete the task

The task required multiple methods of proving the same theorem. It required a recall and an application of known theorems, definitions, and postulates to construct and justify specific statements for a particular method. To complete the task required a demonstration of the ability to provide an explanation of the proof. Task 2(a) and Task 2(b) required a discursive apprehension of the figure using knowledge of the theorem proof.

The TPACK construct(s) addressed by the task

Task 2(a) elicited CK by requiring four methods of proofs of the tan-chord theorem whereas Task 2(b) elicited PCK by necessitating an explanation of this technique in the classroom situation. Each method allowed the PTs to provide a representation of statements with reasons and a construction that was linked to these statements. The ideal response to Task 2(b) should describe or demonstrate the various ways to model or illustrate the theorem. The demonstration should encompass the ability to provide an explanation of the concept or the procedure for the proof. The demonstration should discuss or utilize the general or specific instructional strategies for teaching the tan-chord theorem.

The rubrics for Task 2

Task 2 (a)

Task 2(a) required the PT to provide four different proofs of the tan-chord theorem by producing statements with reasons. Task 2(a) tested CK regarding proficiency in knowledge of circle theorems. Of note is the realization that good knowledge of the proof of the tan-chord theorem requires knowledge of several other circle geometry theorems. See Table 5.5 for rubrics for task 2 (a). The description for performance level 4 required all the correct four methods of proofs and statements, namely; (i) proof by congruency and proof by using these theorems (ii) angles in the same segment theorem, (iii) angles subtended at the centre and circumference theorem, and, (iv) equal tangents theorem. A correct response showed a construction of $T_1 + T_2 = 90^o$, a construction of tan ⊥ diameter/radius with all constructions supported by correct geometric reasoning and statements. The PTs had to construct T_2, which was not illustrated in the diagram, in order to justify the tangent-chord theorem. A level 3 performance had the same description as level 4 except that the PT provided three correct methods of the proof. The criteria for level 2 catered for a response that provided a construction of 1 or 2 correct methods of proof with correct geometric reasoning and statements, suggesting that if, for instance, 1 proof with correct statements was provided then it was categorized as a level 2 performance. However, if 1 proof was provided with not all but some correct statements and flawed construction of $T_1 + T_2 = 90^o$ then it was categorized as a level 1 performance.

Table 5.5: Rubrics for Task 2(a)

level	Description
0	No construction of proof
1	construction of 1 proof using any one of the methods; flawed construction of proof that T1 + T2 = 90º; correct construction of tan ⊥ diameter/radius, some correct geometric reasoning and statements to re-think of how to explain the T1 and T2 and the tan perpendicular to diameter
2	construction of 1 or 2 correct methods proofs; correct construction of T1 + T2 = 90º; correct construction of tan ⊥ diameter/radius, correct geometric reasoning and statements
3	construction of 3 correct methods of proofs; correct construction of T1 + T2 = 90º; correct construction of tan ⊥ diameter/radius, correct geometric reasoning and statements
4	construction of 4 correct proofs using four methods :congruency and three theorems (<s in same segment, <s subtended at the centre and circumference, equal tangents); correct construction of T1 + T2 = 90º; correct construction of tan ⊥ diameter/radius; correct geometric reasoning and statements

Task 2(b)

Task 2 (b) required the PT to situate the task of providing four different proofs of the tan-chord theorem in the classroom teaching environment. Task 2(b) tested the PCK relating to proficiency in knowledge of teaching circle theorems. The PT was expected to demonstrate the ability to explain the theorem using different representations and teaching strategies. See Table 5.5 for rubrics for task 2 (b). At level 4 the PT displayed the ability to give a description of how to do the correct constructions of the proof with correct geometric reasoning and statements and provided detailed description of at least two teaching strategies. Categories for level 3 description required the PT to discuss at least one correct solution of the proof and a description of one teaching strategy. However, if the discussion and the description were not detailed then the criterion was allocated as a level 2 performance. The criteria for a level 1 performance was a response that demonstrated flaws in its explanation or demonstrating a minimal understanding of the theorem whilst level 0 response illustrated an unsatisfactory effort to describe the method for classroom environment.

Table 5.6: Rubrics for Task 2(b)

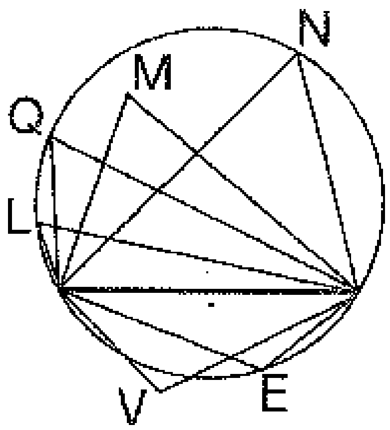
level	Description
0	effort to describe method for classroom unsatisfactory
1	Flawed explanation that demonstrates a minimal understanding of the theorem.
2	discussing one possible correct instructional method; one teaching strategy mentioned but not detailed
3	discussing at least one possible correct solution; detailed description of one teaching strategy e.g. question and answer
4	Ability to give a description of how to do the correct constructions with correct geometric reasoning provided; discussing possible solutions; detailed description of at least two teaching strategy with clear demonstration of the need for the practical approach

5.2.3 Deconstructing Task 3

Task 3 was a content-based question. See Figure 5.3 for Task 3 and Appendix C for Task 3 and its memorandum. The content-based (CK) sub-questions involved typical problems based on a Grade 11 geometry level, requiring the PT to display knowledge, application and interpretation of a task involving ‘angles in the same segment subtended by same chord/arc at the circumference theorem’.

Suppose question 1 below was part of a geometry lesson:

Question 1
In the diagram below, L, Q, N and E are points on the circumference.
Which of the angles are equal?



(a) What are the main mathematical ideas in the question 1 above?
(b) Produce a solution to the question.

Figure 5.3: Task 3

The critical components of the task

There were two components to Task 3; (i) the ability to understand the mathematical ideas in question 1, and (ii) producing the solution to the question "*which of the angles are equal?*" The critical issue in addressing the components of the task was the ability to discursively apprehend the diagram. That is, the PT needed to make a connection between the configuration(s) and the geometric principles of the diagram in order to provide the mathematical idea behind the question of the diagram and make interpretations about the role of the diagram in understanding the theorem. The task required both the visualization and reasoning process to be enacted.

The actions required to complete the task

The task required the PT to recognize that a figure could depict various relations of an object in relation with other objects. The PT needed to apprehend in the figure the relationship between the common chord, segments and angles (a perceptual apprehension). A discussion about question 1 should focus on the relationship between the angles subtended by the same chord whether in the same segment, different segments or not inscribed. A PT with a good knowledge of this relationship would be able to dispel any misconceptions related to the theorem. The PT was required to produce a solution to the question which required knowledge of the theorem. Task 3(a) required an explanation of the ideas that were foregrounded by question 1. The ideas should include the angle in the same segment theorem, its converse and misconceptions. Task 3(b) required a solution of question 1.

The TPACK construct(s) addressed by the task

Task 3 comprised content-based questions. The task elicited the TPACK construct of CK. Tasks 3 (a) and (b) examined the CK that required geometry competences. A conceptual understanding of aspects of circle geometry should be reflected when PTs make conjectures between concepts, to answer the question relating to understanding and recognition of a circle geometry theorem. The respondent must provide a description that demonstrates knowledge of angles in same segment theorem and its converse. Statements should be justified by appropriate reasoning. The response must address misconceptions of the theorem.

The rubrics for Task 3

Task 3 (a)

An ideal response required an explanation of the ideas that were foregrounded by question 1. A level 4 performance displayed a correct description of the angle in the same segment theorem and its converse with correct geometric reasoning. See Table 5.7 for rubrics for Task 3(a). The components of the description encompassed the key words: *same segment, subtended by same chord/arc, inscribed, angles on circumference*. The discussion about the idea included a mention of the angles that sought to address possible misconceptions of the theorem. A detailed description of the diagram provided clear reference to the positions of the points in relation to the chord that subtends all the angles. However, when reference was made only to $\hat{L}, \hat{Q}$, and $\hat{N}$ then the description was pegged as a level 3 criteria. Further, the description was brought down to level 2 when the converse was not mentioned. A level 1 performance described a response that had a flawed explanation that demonstrated a minimal understanding of the theorem. A flawed explanation would include M because it was on the same segment as L, Q, and M or include E because it was on the circumference (inscribed). An unsatisfactory description and/or an explanation that included V as satisfying the condition for the theorem were pegged at level 0.

Table 5.7: Rubrics for Task 3(a)

level	Description
0	effort to describe the diagram unsatisfactory; explanation includes V or all points
1	flawed explanation that demonstrates a minimal understanding of the theorem such as including M, E
2	correct description of the angle in the same segment theorem with correct geometric reasoning and no mention of its converse; detailed description of the diagram with clear reference to the positions of the points in relation to the chord that subtends $\hat{L}, \hat{Q}$, and $\hat{N}$
3	correct description of the angle in the same segment theorem and its converse with correct geometric reasoning; detailed description of the diagram with clear reference to the positions of the points in relation to the chord that subtends $\hat{L}, \hat{Q}$, and $\hat{N}$
4	correct description of the angle in the same segment theorem and its converse with correct geometric reasoning; discussing possible misconceptions; detailed description of the diagram with clear reference to the positions of the points in relation to the chord that subtends all the angles

Task 3 (b)

In levels 4 – 1, the descriptions reflect that a solution that the PT provided should be justified. See Table 5.8 for rubrics for Task 3(b). Justification in this context involves the correct use of terms explicitly given in the theorem. However, I noted that a correct identification of the equal angles might not necessarily be aligned with the correct justification. As such, the justifications were coded with respect to the levels as correct, incomplete correct, faulty, and no justification. For instance, level 3 differed with level 4 in that the level 3 response provided a correct but incomplete justification. A level 4 described an ideal response where the PT provided a correct identification that $\hat{L} = \hat{Q} = \hat{N}$ with justifications using geometric reasoning and /or recognizing in reasoning around the common chord. Across the PTs, their justifications explicitly stated the key essential words: same segment, *same segment, subtended by same chord/arc, inscribed, angles on circumference*. The criteria for a level 2 description was (i) faulty justifications, (ii) with inclusion of E in the justification, (iii) and/or only used the chord in justification, (iv) and/or the same segment was not considered in the justification.

Table 5.8: Rubrics for Task 3(b)

level	Description
0	incorrect identification that $\hat{L} = \hat{Q} = \hat{N}$ or no response , or inclusion of M, V
1	Correct identification that$\hat{L} = \hat{Q} = \hat{N}$. no justifications
2	Correct identification that$\hat{L} = \hat{Q} = \hat{N}$. Faulty justifications, inclusion of E, using chord in justification; same segment not considered
3	Correct identification that$\hat{L} = \hat{Q} = \hat{N}$. incomplete correct justifications
4	Correct identification that$\hat{L} = \hat{Q} = \hat{N}$. Correct justifications using geometric reasoning, recognizing in reasoning the common chord, key words explicitly stated

5.2.4 Deconstructing Task 4

Task 4 was a technological content knowledge-based (TCK) question. See Figure 5.4 for Task 4 and Appendix C for Task 4 and its memorandum. The task required the knowledge and skills to interpret learner thinking during a construction of a diagram transformed from a verbal description to a GeoGebra construction. See Appendix C for Task 4 description and a screen-shot of Jane's construction. The technological content knowledge-based sub-question required knowledge and skills to utilize GeoGebra when identifying learner errors in the constructed diagram.

Jane used GeoGebra to construct a diagram using the description below:

AB is a vertical diameter of a circle with centre O.

P is any point on the circle closer to A than B.

The perpendicular to AB at O meets AP produced at M.

OM and BP intersect at K.

BM cuts the circle at T.

Draw radius OP.

Attached is Jane's GeoGebra construction of the diagram. Click here for the GeoGebra file.

(a) What is wrong with Jane's construction? (hint: use drag mode, construction protocol)

Figure 5.4: Task 4

The critical components of the task

The main characteristics of the task was the interpretation of a construction resulting from a production of a diagrammatic version of objects mentioned in verbal descriptions with the aid of a technological tool, GeoGebra. The critical components of the task were the recognition or understanding of the key geometric concepts that a learner would use to transform a verbal description into a graphical representation. There is interplay of various domains of knowledge to perform this task as suggested by Weiss & Herbst (2007).

The actions required to complete the task

The errors to be recognized in the construction of the diagram were; (i) M is constructed as arbitrary point, (ii) OM drawn is independent of AB, and (iii) the order of construction of P is incorrect. The task required the ability to confirm that the description as provided in the learner's construction was correct. This confirmation involved manipulation of the constructed figure instead of just inspection. The PT was expected to analyse the construction through the dragging mode to check the robustness of the construction to confirm the properties in the construction and determine what changed and what stayed the same. The PTs were to provide a description that demonstrated knowledge of geometry definitions and/or properties of these geometric words (*perpendicular, vertical diameter, intersects, produced*, *closer to than),* knowledge of how the properties of a diagram aided in the construction of a diagram, the disposition to translate statements to a diagrammatic register, and the knowledge of a construction procedure. The task required all four apprehensions and cognitive processes to be enacted.

The TPACK construct(s) addressed by the task

The TPACK construct addressed in Task 4 was technological content knowledge (TCK). Task 4 (a) examined the TCK that required competences in geometry and use of GeoGebra. The task examined the PTs' conceptual understanding of geometry properties in a GeoGebra environment where a learner was expected to produce a diagram bound by a specification that transformed a verbal description into a graphical representation. The PT's knowledge of GeoGebra affordances and constraints would be explicit in the descriptions of the learner's errors in the constructed diagram.

The rubrics for Task 4

Task 4 (a)

This task elicited PTs' geometry technological content knowledge (TCK). The PTs were required to identify "what is wrong with Jane's construction?" In this task there was interplay between knowledge of geometry definitions and/or properties, knowledge of how the properties of a diagram aided in the interpretation of a diagram, the disposition to translate statements to a diagrammatic register, and the knowledge of a construction procedure. See Table 5.9 for rubric for Task 4(a). The interplay suggests that all the Duval's four cognitive apprehensions of geometric reasoning are required in diagnosing the errors in the learner's construction; the perceptual ability to recognize the errors, the sequential organization of the construction, the description of the learner's errors through a discursive apprehension of the figure and the operating on the figure to ascertain the learner's errors.

The description for a level 4 performance was a response that was informed by the PT's own construction to ascertain the correctness or errors that Jane could have made. Such a response reflected an ability to give a detailed description of errors in Jane's construction supported by correct geometric reasoning. The PT's reasoning was centred on the recognition that in the figure, OM is not perpendicular to AB. The PT should have used the drag mode and/or used the navigation bar to check the correctness of Jane's construction. The criterion for level 3 was for a response that ascertained the errors by only working from Jane's construction, instead of comparing with their own construction. If the reason why OM was not perpendicular to AB was not given in the response, then the response was regarded as a level 2 criterion. The criterion for level 1 was a response that, indicated that only through a perceptual apprehension, identified that $A\hat{O}M \neq 90°$ as seen in the diagram.

Table 5.9: Rubrics for Task 4(a)

level	Description
0	No response
1	recognize as given in the construction that $A\hat{O}M \neq 90°$
2	Use of the drag mode to check the correctness of the Jane's construction; use of navigation bar; no justification why OM is not perpendicular to AB
3	ability to give a detailed description of errors in Jane's construction with correct geometric reasoning provided; Use of the drag mode to check the correctness of the Jane's construction; use of navigation bar; gives reasons why OM is not perpendicular to AB
4	Constructed own diagram, ability to give a detailed description of errors in Jane's construction with correct geometric reasoning provided; Use of the drag mode to check the correctness of the Jane's construction; use of navigation bar; gives reasons why OM is not perpendicular to AB

5.3 Chapter summary

In this chapter my discussion was focused on the tasks utilized in the study. The aim of the study was to characterize aspects of prospective teachers' technological pedagogical content knowledge (TPACK) constructed in a GeoGebra-based environment. I deconstructed the tasks and provided a description that elaborated the critical components of the sub-tasks, the expectations of each sub-task and the TPACK construct that each sub-task tested. The cognitive processes and cognitive apprehensions enacted in each task were discussed. The deconstruction of the tasks was followed by a description of the rubrics employed to qualify the responses to each task.

CHAPTER 6

ANALYSIS BY TPACK COMPONENT: PROSPECTIVE TEACHERS' GEOMETRY CONTENT KNOWLEDGE

6.0 Introduction

In Chapter 4, I presented the analytic framework proposed to examine the TPACK aspects of CK, TCK and PCK. This chapter presents the data analysis relating to the aspect of content knowledge (CK) construct of the technological pedagogical content knowledge (TPACK) framework in response to research question 1:

What circle geometry knowledge do the PTs display?

The chapter begins by providing an overview of how the sub-unit of analysis (the content knowledge (CK) construct) was conceptualized in the study (Section 6.1). A discussion of the descriptive summary and the quantitative analysis of the rubric scores of all the participating PTs' responses to the CK tasks follows (Section 6.2). Further, an inductive analysis of the PTs' visualization (Section 6.3) and reasoning (Section 6.4) competencies employing the Duval's (1995) cognitive apprehensions is articulated. For each section, I provide the overall results in a tabular form, followed by a discussion of the overall results for all the participants. Each section is concluded by a summary of findings. When discussing the results explicitly, I refer to Nkosi, Wisdom and Lesedi, whose responses were considered rich in detail and typical of other responses. Throughout this chapter and the subsequent two chapters, I discuss the trends across and within each task and present participating PTs' responses (Nkosi, Wisdom and Lesedi) to written tasks and interview excerpts to support the findings. However, there are instances that required examples of responses of the three other PTs to strengthen the arguments.

6.1 Sub-unit of analysis: PTs' geometry content knowledge (CK)

The sub-unit of analysis for this chapter was the participating PTs' geometry content knowledge (CK) displayed in the CK tasks. The CK construct was conceptualized in the study as the knowledge of circle geometry concepts, theorems and proofs (see Section 5.2). Knowledge of geometry required for the successful completion of the CK tasks required two main thinking processes: (i) identifying and recognizing figures, (ii) making connections between geometry representations, properties and theorems. The exploration of the participating PTs content knowledge of circle geometry was done by probing into the PTs thinking displayed in the participating PTs solutions to the TPACK tasks that were deliberately designed to elicit the TPACK knowledge constructs. I employ the argument that geometry thinking requires the processes of visualization, construction and reasoning (Duval, 1998).

Geometry knowledge in this study included a student's ability to relate to diagrams, figural properties and theorems. Diagrams, as representations, are a means to reasoning in geometry (Duval, 1995; Herbst, 2004; Laborde, 2004). I employ Duval's (1995) cognitive apprehensions as interpretative tools to discuss how the participating PTs responded to the tasks. As mentioned in Chapter 3, 4 and 5, these cognitive apprehensions are perceptual, discursive, operative and sequential apprehensions. I use these apprehensions to analyse how the participating PTs interact with diagrams acknowledging that variation of diagrams induce different levels of proficiency in geometry. The CK tasks elicited visualization and reasoning, so I deliberately established the participating PTs' visualization and reasoning skills. To understand the participating PTs' visualization and reasoning processes in responding to circle geometry tasks, I was guided by the following sub-questions.

(1) What do the PTs identify and recognize in the perceived figure?
(2) What types of connections do the PTs make between representations, properties and theorems?

Sub-question 1 relates to visualization whereas sub-question 2 relates to reasoning. In deliberating on the sub-questions aimed at measuring the participating PTs' CK and aided by evidence from the quantitative analysis, I use the PTs' responses (derived from interviews and written answers to tasks) to provide insights into what I consider prominent, absent or assumed CK knowledge of participating PTs within and across the CK tasks. Answers to these questions are provided as summaries at the end of Section 6.3 and Section 6.4.

6.2 Analysis of Rubric Scorings of CK tasks

The tasks in which CK was foregrounded were Task 1(a), Task 1(b), Task 2(a), Task 3(a) and Task 3(b). The responses to the written tasks were scored using rubrics. As described in Chapter 5, rubrics were used as a tool to measure the participating PTs' CK. The rubrics used specific scores based on a five-point qualitative scale (performance level 0 to performance level 4) to capture the participating PTs' proficiency in the three main knowledge domains of content, pedagogy and technology. For each rubric the quality of the performance levels were categorized as follows: level 0 as poor, level 1 and level 2 as adequate, and level 3 and level 4 as good knowledge of geometry. See Table 4.5 in Chapter 4 for the coding of quality of connections and performance levels.

6.2.1 Analysis of PTs' performance across CK tasks

A summary of the scores for the cases is presented in Table 6.1. The table presents data of six PTs; Nkosi, John, Wisdom, Lesedi, Bonolo and Thabiso. There were five (5) tasks examining CK, giving an overall mark ranging from 0 to 20. The overall mark was essential in determining the overall CK performance score for each participating PT. The frequencies of the rubric scores are included in the discussion to indicate the scoring pattern of PTs' performance level across the tasks. The mean and standard deviation are provided to interpret the individual PT's scores. Refer to Chapter 5 for rubrics for each task.

I will use Nkosi as an example to illustrate the scoring patterns as presented in the Table 6.1. An account of Nkosi indicates that he attempted all the tasks with performance scores ranging between performance levels 1 and 3. This implies that the qualities of his responses were scored either adequate or strong. The classification of the five tasks that he attempted indicates that Task 1(b) and Task 3(b) each scored 1, Task 2(a) scored 2 and Task 1(a) and Task 3(a) each scored 3. The frequency distribution provides an account of scoring across all the performance levels of the rubrics. Nkosi's frequencies suggest that his CK competence is between adequate and strong.

Table 6.1: Scoring of PTs responses **across** and **within** the CK tasks

PT		Rubric scores /4 for each sub-task					Summary across the tasks			
		Task 1		Task 2	Task 3		mark /20	%	Mean ($\bar{x} = 2.3$)	SD (SD=0.205)
		(a)	(b)	(a)	(a)	(b)				
Nkosi		3	1	2	3	1	10	50	2	1
John		2	4	2	3	1	12	60	2.4	1.14
Wisdom		3	3	2	3	2	13	65	2.6	0.548
Lesedi		3	2	2	4	4	15	75	3	1
Bonolo		2	3	1	1	2	9	45	1.8	0.837
Thabiso		2	3	1	2	3	11	55	2.2	0.837
Summary **within** the tasks	mark /24	15	16	10	16	13				
	%	63	67	42	67	54				
	Mean	2.5	2.667	1.667	2.667	2.167				
	SD	0.548	1.033	0.516	1.033	1.169				

Note: 1, 2, 3, 4 denote performance level score

The distribution of scores ranged from 1 to 4 as expected. Looking across the unit of analysis, the following was observed: six PTs scored a 1, two PTs scored a 4 with Lesedi scoring two of the 4's. Scores 2 and 3 were the most prominent scores to be attained by the PTs across all the tasks. A score of 4 as reflected in Table 1 suggests that John and Lesedi are the only PTs to provide model answers. John answered Task 1 (b) correctly; Lesedi answered Task 3 (a) and Task 3(b) correctly. Bonolo had the lowest overall mark of 45% and Lesedi scored the highest overall mark of 75%.

The overall mean and standard deviation were 2.3 and 0.205 respectively, suggesting that three PTs (John, Wisdom and Lesedi) scored above the mean and rest of the PTs scored below the mean (between 1.8 and 2.2). For example, Thabiso's overall mark and mean were 55% and 2.2 respectively, which is slightly below the overall mean score of 2.3. Of the 30 responses, 13 (43.3%) displayed strong geometry CK competence while 17 (57.7%) displayed adequate geometry CK competence.

6.2.2 Analysis of PTs' performance within CK tasks

Reference is made to the summary of the participating PTs' scores across each task as presented in Table 6.1 above. The analytical rubrics were employed to qualify the responses to each task. A general overview of the table indicates that all the five tasks were attempted, with 1 as the lowest score and 4 as the highest performance score attained in a task. Task 1 (a) is the only task that was not scored at 1 whilst Task 1(b), Task 3(a) and Task 3(b) were all scored at performance level 4 by two PTs (John and Lesedi).

The percentage mark for each task, as attained by the participating PTs ranged between 42% and 67%. Task 2 (a) had the lowest mark of 42%. The scores for Task 2(a) ranged between 1 and 2 with four of the participating PTs attaining a 2. The mean and SD of Task 2 (a) are 1.667 and 0.516 respectively, confirming that the quality of responses for this task was poor and hence there was a slim variation between scores in this task. The mean and SD of both Tasks 1(b) and 3(a) were 2.667 and 1.033 respectively, confirming that the quality of responses for these tasks was strong. There was a variation between scores in these tasks with each task realizing a model score of 4 and reflecting that four of the six PTs' scoring was strong.

Although Tasks 1(b) and 3(a) both received the highest percentage mark, only two PTs (John and Lesedi) answered one of these tasks correctly performing at level 4. Since Task 1(a) and Task 3(b) provided contrasting trends, I use these as examples to explain the characteristics of performance for each task as displayed in the table. The overall mark for Task 1 (a) was 63%, with scores ranging from adequate (score 2) to strong (score 3). This demonstrates that none of the responses reflected poor performance. The classification of the score of the PTs' responses to Task 1 (a) are; 3 PTs scored 2 (adequate) and 3 PTs scored 3 (strong). The mean is 2.5 and SD is 0.548 suggesting that there was not much variation in performance within this task. On the other hand, the overall mark for Task 3 (b) was 54%, with scores ranging from adequate (score 1) to strong (score 4). That is, none of the PTs' scored 0 (poor). The classification of the scores of the PTs' responses to Task 3 (b) are; 2 PTs scored 1 (partial) and 2 PTs scored 2 (partial), 1 PT scored 3 (adequate), 1 PT scored 4 (adequate). The mean is 2.167 and SD is 1.169 suggesting that there was a great variation in performance within this task as compared to the other four tasks.

In general, focusing on the frequencies of the scores attained across the tasks, the most frequent scores were 2 and 3 as shown in Table 6.2.

Table 6.2: frequencies of scores across the tasks

		Frequency of scores across each performance level				
		0	1	2	3	4
Task 1	(a)	0	0	3	3	0
	(b)	0	1	1	3	1
Task 2	(a)	0	2	4	0	0
Task 3	(a)	0	1	1	3	1
	(b)	0	2	2	1	1

The frequencies in Table 6.2 present the occurrences of scores attained by the number of PTs in relation to each task. For instance, the performance scoring pattern of Task 1(a) reflects that of the six participants in the study, three PTs scored at performance level 2 and three PTs scored at performance level 3. The trends in the scores suggest that half of the PTs performed at level 3 or higher in Tasks 1(a), 1(b) and 3(a) whereas 5 or more PTs performed at level 2 or less in Task 2 and 3(b). Task 2(a) is the only task which the PTs did not score higher than level 2. Frequency of performance scores across the tasks strongly indicates that the PTs performance in circle geometry can be classified as between adequate and strong. However, this statement can be qualified by an in-depth analysis of the performance of each task based on the answers to the two sub-question stated above.

Section 6.2 Summary of quantitative findings within and across CK tasks

The rubric scores attained by the participating PTs were quantitatively analysed within and across the tasks. Section 6.3 provides a summary of descriptive statistics of the sub-unit of analysis. The summary showed that overall variation of scores is low, suggesting that the PTs had similar abilities with below than acceptable knowledge of circle geometry knowledge. The conclusion is based on the contention that the expected average performance for the CK tasks should be 4 but the attained average is 2.3, signifying below expected knowledge levels of circle geometry. The rubric scores in the descriptive summary in this section provide statistical features of the participating PTs' individual performance. However, an interpretation of the scores within and across the tasks in Section 6.3 would bring forward an insight into the participating PTs' CK.

6.3 Sub-question 1: Identifying and recognizing the perceived figures

Generally, working with geometry tasks requires an interaction with diagrams and the use of visualization to perceive the figures and their properties. The PTs should have the competence to visualize and reason to reflect their knowledge and understanding of geometry (Duval, 1995; Gagatsis et al., 2010; Laborde, 2004). The commonality of the CK tasks was that all the tasks required cognitive apprehensions to deal with the knowledge of circle geometry properties and theorems. Apprehension in this context refers to the several ways of looking at a drawing or visual stimulus (Duval, 1995). The major focus of the interaction with the diagram was to identify and describe concepts as perceived in the diagrams. For instance, Task 1(a) particularly required a perceptual apprehension of the diagram in order to identify and describe figures with similar or contrasting properties. Task 1(a), Task 1(b) and Task 3(a) evoked perceptual apprehension. See Section 4.4 for the conceptualization of perceptual apprehension. The findings of the participating PTs' responses to each task embedded within this sub-question are presented. The discussion that follows will apply for all the findings that answer the sub-question "*What do the PTs identify and recognize in the perceived figure*?" This section discusses the visualization process whilst the reasoning processes discussion will follow in Section 6.4.

Results for Task 1

Task 1(a) required a perceptual apprehension of the diagram. The PTs were expected to identify and discriminate the figures from a given diagram. To do this required reflecting on an understanding of geometrical concepts and spatial representations derived from the figure (see Task 1 in Section 5.2.1 and Appendix C). The task provided an opportunity to explore the PTs' prior knowledge regarding definitions, properties and theorems. Generally, this task was concerned with examining the participating PTs' visual spatial abilities by listing and labelling that which they can identify.

The figures that the participating PTs were expected to recognize and identify were circle, semi-circle, segments, triangles and quadrilaterals. I expected the PTs to specifically identify a circle, 2 semi-circles, 8 segments, 6 single triangles, 5 compound triangles made up of two single triangles, 1 compound triangle made up of three single triangles, 1 compound triangle made up of four single triangles and 3 quadrilaterals (see Table 6.3). The identifications indicated that the PTs could see a circle, a semi-circle, triangles, quadrilaterals and others. A summary of the Table 6.3 shows that half of the PTs did not see a circle, 5 out of 6 participating PTs saw single triangles, only one PT could not identify compound triangles comprising 2 single triangles, half of the PTs saw compound triangles comprising 3 single triangles, 4 out of 6 PTs identified compound triangles comprising 4 single triangles, only one PT saw the semi-circle and all PTs failed to see circle segments but were able to recognize quadrilaterals. The total numbers of figures identified by each PT were between 5 and 14 figures with triangles featuring most prominently. The triangles were seen by all the PTs.

Table 6.3: PTs' identification of figures (perceptual apprehension)

PT	Number of observed figures that PTs identified									
	Circle E=1	Single triangle E=6	compound triangles comprising of …			Semi-circle E=2	Segment E=8	Quads E=3	others	Total figures Identified E=27
			2 single triangles E=5	3 single triangle E=1	4 single triangle E=1					
Nkosi	1	5	1	0	1	0	0	2		10
John	0	5	1	1	0	0	0	1		8
Wisdom	1	6	4	1	1	0	0	1		14
Lesedi	0	5	2	1	1	0	0	3	1	13
Bonolo	0	0	3	0	1	0	0	1		5
Thabiso	1	1	0	0	0	1	0	1	2	6

Note: E is the number of figures expected to be identified;

I use responses by Nkosi and Lesedi, who both scored a 3 to illustrate how the figures were identified. Nkosi identified 10 figures: a circle, 5 single triangles, 1 compound triangle made up of two single triangles, 1 compound triangle made up of four single triangles and 2 quadrilaterals. Lesedi identified 13 figures: 5 single triangles, 2 compound triangles made up of two single triangles, 1 compound triangles made up of three single triangles, 1 compound triangle made up of four single triangles, 3 quadrilateral and 1 non-existing figures.

An in-depth analysis of the PTs' responses was essential in determining and understanding the techniques employed for listing and the type of figures identified. Listing the figures reflects a visual explanation of the perceptive process. Duval (1995) suggests that there are specific laws by which one organizes what one visualizes. This implies that to list the figures the PTs had to mentally deconstruct the diagram and such deconstruction required perceptual apprehension. Interviews were used to determine the PTs' thinking in their decision to list the figures. Nkosi, Wisdom and Thabiso recognized the circle and the larger triangle among all the figures in the diagram. These three participating PTs are singled out because they were the only ones to list the circle. The excerpt below provides a strategy that sheds light on why Wisdom recognized a circle and its role in the construction of the diagram:

Kim: tell me when you started drawing this did you, did it matter what you started with the triangle or the circle. Or you assumed that you had to start with the circle?

Wisdom: I had to start with a circle

Kim: why?

Wisdom: because everything is being done inside the circle

Kim: inside the circle?

Wisdom: yah

Wisdom is singled out because he provided a unique response in which he demonstrated that the circle was the most prominent figure that he recognized. That is all other figures were drawn relative to the circle. Were the triangles listed common for the participating PTs? There is an indication that the PTs saw single triangles much more than other figures. It shows that Wisdom identified all the 6 single triangles whilst three PTs (Nkosi, John and Lesedi) listed 5 of the 6 triangles. I noted that Thabiso could not recognize the larger triangle but is the only PT that identified the semi-circle. The PTs had to reconfigure the triangles by regrouping the single triangles into compound triangles.

Clearly, in the process of identifying the figures, the participating PTs reconfigured some figures. It seems the PTs could see the single triangles except for Bonolo who recognized the reconfiguration but no single triangles. It is not clear what strategy was used for listing the single triangles; as such it was difficult to describe the strategy that was used for identifying the single triangles that were left out. Although the participating PTs could identify single triangles, they had difficulty in discriminating the diagram to see the compound triangles. Of the seven triangles that were compound, five PTs could identify 4 or less compound triangles. However, the participating PTs who identified four compound triangles could easily identify single triangles but missed triangles BME and BHC and quadrilateral BKEM. The participating PTs inability to recognize and identify compound triangles confirms Duval's (1995) observations where he ascertains that reconfiguration that involves the use of one figure twice has a potential to inhibit visibility of reconfiguration. He claims that learners are of the view that two or more triangles stuck together will not be triangles unless you judge them individually.

Duval (1995) makes clear that visibility of the reconfigurations occurs most prominently where the combination of the figures forms a familiar shape. It is evident that the participating PTs recognized the quadrilaterals at a glance because the PTs listed at least one quadrilateral and gave specifications of the type of quadrilateral. All the PTs recognized quadrilateral ABHC from a reconfiguration of the triangles ABH and ACH. It is interesting to note that the PTs deduced from knowledge of quadrilaterals and their propositions that there was a cyclic quadrilateral conforming to the features of a kite but failed to mention the circle (half of the PTs).

In determining the characteristics of the participating PTs' knowledge of geometry, a convincing argument to situate the PTs' understanding of geometrical concepts and spatial representations derived from a diagram was necessary. The variation (0.548) between the participating PTs responses for this task suggests that in some situations there was commonality in thinking. The participating PTs did not recognize that within the diagram, there were figures that were enclosed by the arcs and chords. Three out of six PTs did not mention the circle. Thabiso stands out as the only PT that identified a semi-circle and none of the PTs identified the segments. Could it be that the definition of figures refers to the figures classified only under the ordinary shapes or polygon? Thabiso listed an isosceles triangle and the scalene triangle. This participating PT could see the big shapes but not small figures.

The task required the participating PTs to list and label figures that they could identify. Labelling generally was a follow-up of listing in that PTs had to label what they listed. The table presents evidence of labelling or not labelling in relation to listing. All the PTs except Thabiso were able to label what they could identify. Of the six figures that Thabiso identified, he labelled only two triangles. I established two categories for labels: labelling of triangles and quadrilaterals and labelling of the circle. Labelling of triangles and quadrilaterals was further split into clockwise and anti-clockwise. Table 6.4 reveals that of the 10 figures that Nkosi labelled, 4 figures were labelled clockwise, 5 anti-clockwise, the circle was listed and a system was followed in listing or labelling. Lesedi's 5 figures were labelled clockwise, 7 anti-clockwise, the circle was not listed and a system was followed in listing or labelling. Although Thabiso identified 6 figures he failed to provide labels for 4 of these figures in spite of the diagram being fully pre-labelled.

Table 6.4: PTs labeling of figures

PT	Labels				
	Labeling system for triangles and quadrilaterals		Labeled Circle	systematic Listing	Number of Figures identified
	clockwise	Anti-clockwise			
Nkosi	4	5	Yes	Yes	10
John	6	2	n/a	No	8
Wisdom	10	3	Yes	Yes	14
Lesedi	5	7	n/a	Yes	12
Bonolo	5	0	n/a	No	5
Thabiso	2	0	No	No	6

Note: n/a means not applicable

In listing the figures, the participating PTs' thinking varied in their responses. The variations for listing were (i) systematized listing according to shapes, (ii) unsystematic listing, and (iii) systematic listing of triangles. The PTs who listed the figures systematically presented the greatest number of figures. Perhaps there is a link between identification and knowledge of figures? For example, Wisdom and Nkosi listed systematically according to shapes. Due to the systematic listing of triangles, Wisdom identified more figures than Nkosi. Although Lesedi did not list according to the types of shape, she had a system for listing triangles. In general, a coordinated system of labeling was observed. Those PTs who used both clockwise and anticlockwise had the highest listings.

Task 1(a) was concerned with examining the participating PTs' visual ability to manipulate objects and their parts in a two dimensional space. Table 6.4 illustrates that the participating PTs could identify certain geometric figures and their properties. They were able to see figures as single entities rather than as a configuration of these single entities. From this observation, it can be inferred that the participating PTs had difficulty identifying the figures through reconfiguration. Duval (1995, p.155) suggests that "seeing requires discerning the original figure to allow reconfiguration". That is, although this task elicited perceptual apprehension, there was an overlap of different apprehensions. Reconfiguration necessitated an operative apprehension in order to identify the figures. The participating PTs' perceptual apprehension of the diagram is weak due to their failure to recognize and discriminate all the figures and sub-figures through mental modification of the diagram.

Task 1(b) required the participating PTs to identify the congruent triangles in the diagram (see Section 5.2.1 and Appendix C). To accomplish the tasks required that both visualization and reasoning processes be enacted. The element of identification requires a visual observation and as such, a perceptual apprehension of the diagram precedes participating PTs' reasoning for the triangles that are purported to be congruent. The task required a discursive apprehension of the diagram in order to identify and describe the perceived congruent triangles. In this apprehension, the PT needed to connect the configurations of the figure with the properties of such figures in order to provide an argument as to why the pairs of triangles identified were indeed congruent. The statements that were provided by the participating PT described their perceptual apprehension. The reasoning statements also described the perceived figure through use of geometric language and symbols. Reasoning and connections are discussed later in this chapter.

There were four pairs of congruent triangles that the participating PTs were expected to identify. Table 6.5 provides a summary of the PTs' identification of the congruent triangles and the performance scores attained for this task. The classification of the score of the PTs' responses to Task 1 (b) were: one PT scored a 1, one PT scored a 2, 3 PTs scored a 3 and 1 PT scored a 4. The scoring of this task indicates that all the PTs correctly identified at least one pair of congruent triangles but differed in the number of identified pairs of triangles. See Chapter 5 and Appendix C for the memorandum where an explanation for the pairs of congruent triangles is given. The numbers of identified congruent triangles in the participating PTs' responses as shown in Table 6.5 are: Bonolo identified 1 pair of congruent triangles, Thabiso identified 2 pairs of congruent triangles, Nkosi, Wisdom and Lesedi identified 3 pairs of congruent triangles and John identified 4 pairs of congruent triangles.

Table 6.5: PTs' identification of congruent triangles (sequential apprehension)

PT	Identification made on perceptual apprehension Order of listing of pairs of congruent triangles ΔABH ≡ ΔACH (1) → ΔABM ≡ ΔACM (2) → ΔMBH ≡ ΔMCH (3) → ΔKBE ≡ ΔMBE (4)	No. of triangles listed	Rubric score
Nkosi	1 → 2 → 3	3	3
John	1 → 2 → 3→ 4	4	4
Wisdom	1 → 2 → 3	3	3
Lesedi	1 → 2 → 3	3	2
Bonolo	1	1	1
Thabiso	1 → 3	2	3

An in-depth analysis of the participating PTs' responses was essential in determining and understanding the techniques employed for listing the figures identified. Listing the figures is a visual explanation of the perceptive process. That is, one can list that which they can visualize. Duval (1995) suggests that there are specific laws in which one organizes what they visualize. To list the figures, the participating PTs had to mentally deconstruct the diagram and such deconstruction required perceptual apprehension. Basing on the assertion by Mason & Johnston-Wilder (2006) that congruence is a property involving the relationship between two objects, the PTs had to determine a specific technique for identifying the perceived congruent triangles.

The participating PTs' competence with respect to knowledge of properties of the triangles, knowledge about angles and triangle theorems was reflected in the participating PTs' ability to shift between the given verbal description and the diagrammatic representation. The shifting was essential to fit the properties that were not specified in the verbal description but could be implied in the diagram. For instance, the congruency in the task had to be determined based on the circumscribed triangle ABC as the first point of reference whilst noting that AH was the diameter even though it was not stated in the verbal description. Upon establishing this fact, the PT had to note that the order of listing, as explained in Chapter 5, the congruent triangles is for ∆ABH ≡ ∆ACH as congruent pair (1) followed by ∆ABM ≡ ∆ACM as congruent pair (2) followed by ∆MBH ≡ MHC as congruent pair (3) then ∆KBE ≡ ∆MBE as congruent pair (4). This means that to identify pair (2) required establishing the congruence of ∆ABH and ∆ACH. The congruency of pair (3) required recognition of pair (1) and pair (2). All the PTs identified the pairs of congruent triangles in this order except for Thabiso who skipped pair (2) by identifying pairs (1) and (3). John identified all the four pairs of congruent triangles in the right order.

Table 6.6: PTs' identification of congruent triangles

PT	Identification made on perceptual apprehension	
	Triangles identified in 1 (a) that are among the congruent triangles in 1 (b)	triangles not identified
Nkosi	∆AMB, ∆CHM, ∆BHM, ∆BEK,	∆MBE, ∆ABH, ∆ACH, ∆ACM
John	∆ABM, ∆AMC, ∆BME, ∆BKE, ∆BMH, ∆CMH	∆ABH , ∆ACH
Wisdom	∆BMH, ∆ABH, ∆BKE, ∆ABM, ∆AMC, ∆AHC, ∆CMH	∆MBE
Lesedi	∆ABH, ∆ACH, ∆BME, ∆ABM, ∆ACM, ∆BMH, ∆CMH	∆KBE
Bonolo	∆ABH, ∆AHC	∆ABM, ∆ACM, ∆MBH, ∆MHC, ∆KBE, ∆MBE
Thabiso	∆ABM	∆ABH, ∆ACH, ∆ACM, ∆MBH, ∆MHC, ∆KBE, ∆MBE

Note: Expected to identify ΔAMB, ΔCHM, ΔBHM, ΔBEK, ΔMBE, ∆ABH, ∆ACH, ∆ACM

Further analysis was done to provide insights into the participating PTs' reasoning in relation to the visualization process. Drawing on Duval's (1995) suggestion that there are specific laws in which one organizes what they visualize, I decided to explore if there was a link between the figures identified in Task 1(a) and the way these figures were organized to identify the congruent triangles. Do the triangles that the participating PTs perceive influence the way they organized the triangles to identify congruent triangles? Table 6.6 illustrates how the PTs identified the congruent triangles. As noted above relating to Task 1 (a), the participating PTs saw single triangles much more than other figures and this in some way might have affected their ability to recognize congruent compound triangles. Among the list of congruent triangles were compound triangles ∆ABH, ∆ACH and ∆MBE. These triangles were not seen by half of the participating PTs. The triangle that was least identified was ∆MBE. Most of the listed congruent triangles were not identified in Task 1(a). For instance, John identified only 7 of the 13 triangles in Task 1 (a) but identified all the 4 pairs of congruent triangles, indicating that some of the identified congruent triangles were not mentioned in Task 1(a). Thabiso, on the other hand, identified only 1 of the 13 triangles in Task 1(a) but recognized 2 pairs of congruent triangles. Although only four of the six PTs provided reasons for their identifications, these findings confirm that the PTs understood the meaning of congruency. I suggest that the PTs relied on their knowledge of congruency and geometric properties rather than on their visual perception to judge whether the identified triangles were indeed congruent.

Results for Task 2

As mentioned in Section 5.2.2, the critical component of the Task 2 was the ability to "do proofs" and recognize the multiple methods of proving the tan-chord theorem. The participating PTs were to produce statements with reasoning to connect the statements to a conclusion in the process of providing a proof. Task 2 required reasoning but not necessarily visualization processes to be enacted. Therefore, discussion relating to Task 2 is presented in section 6.4.

Results for Task 3

Accomplishing the task required enactment of both the visualization and reasoning processes (see Task 3 in Section 5.2.3 and Appendix C). Task 3(b) solution required an enactment of what was recognized in Task 3(a), and as such, only the results of Task 3(a) are discussed under this category. Task 3(a) required the PT to explain the critical components in a geometry question. The idea that the participating PT brings forward should reflect an understanding and recognition of a circle geometry theorem (angles subtended on the circumference by the same chord in the same segment are equal). In interpreting the geometry question, the participating PT should recognize that a figure can depict various relations of an object in relation to other objects (see Section 5.2.3). The task required a discursive apprehension, meaning that it required making connections between configuration(s) and the geometric principles of the diagram in order to provide the mathematical idea behind the question.

Reference is made to Table 6.1 above. There were some variations between scores in this task with 4 of the 6 PTs scoring a 3 or better. The variation of the score of the PTs' responses to

Task 3 (a) are one PT scored a 1, one PT scored a 2, three PTs scored a 3 and one PT scored a 4. The discussion of the PTs responses to this task is confined to the identification and descriptions of the perceived figures as reflected in the reasoning in the statements. Table 6.7 below provides a summary of the participating PTs' identification of circle geometry concepts of the angles, chord and segment and the participating PTs' performance scores attained for this task. The concepts were identified from the written statements that explained the 'main mathematical ideas'. For instance, Nkosi responded that *"This shows that angles on the same segment are equal, only if they are on the circumference"*. This statement presupposes that Nkosi identified all the angles by use of *only if*. Therefore Table 6.7 coding reflects that Nkosi considered from the diagram the inscribed angles, the angles within the circle and the angles outside the circle. He mentioned the segment but omitted the same side of the chord. Lesedi, on the other hand, responded that "*To address the misconception that angles that are equal and subtended by the same chord have to be on the same circumference not inside the circle or outside the circle*". Lesedi identified all the angles in relation to the chord, although she is not explicit about the segment which she refers to as the 'same circumference'. Lesedi mentions the misconceptions that are depicted by the figures, indicating that the description is drawn from perceptually apprehending the diagram.

Table 6.7: PTs' identification of concepts in the task

PT	Identification made on perceptual apprehension					Rubric score
	Angles			Chord	Segment	
	Inscribed	Within the circle	Outside the circle			
Nkosi	X	X	X		X	3
John	X	X	X	X		3
Wisdom	X	X	X	X		3
Lesedi	X	X	X	X	X	4
Bonolo	X			X		1
Thabiso	X			X		2

The scoring of this task indicates that all the PTs identified the correct theorem. However, the response statement about the concepts of the theorem differed across the PTs. Table 6.7 shows that, in general, all the participating PTs at a glance saw that in the diagram there were angles inscribed on the circle, angles within the circle, angles outside the circle, a chord and segment. All the PTs except Bonolo and Thabiso recognized the inscribed angles and acknowledged the other angles. Four PTs did not mention the segment and failed to identify those angles within the circle and outside the circle that were fulfilling the conditions for the converse of the theorem.

Section 6.3 Findings: What do the participating PTs identify and describe in the perceived figures?

Section 6.3 presented and analysed what the participating PTs could identify and describe in the perceived figures. The analysis focused on the PTs' competence to identify and describe that which they could visualize and reason about. Perceptual apprehension was employed to interpret how the PTs listed, labeled and described the figures. To examine what the participating PTs identify and recognize in perceived figures, I made judgments on 'what could the PTs see' and 'what could the PTs not see?' Tasks 1(a), 1(b) and 3(a) were analysed to understand PTs' competence relating to the identification and description of perceived figures. Refer to Chapter 5 for an elaboration of tasks deconstruction. In Tasks 1(a) and (b), the participating PTs' competence on knowledge of properties of the triangles was reflected in most participating PTs' ability to shift between the verbal description and the diagrammatic representation. The results showed that when interacting with the diagram, the participating PTs had difficulty in discriminating the diagram to see the compound triangles and yet were able to recognize the quadrilaterals at a glance because the combination of the triangles formed a familiar shape. The participating PTs could not recognize that within the diagram, there were figures that were enclosed by the arcs and chords. The participating PTs' perceptual apprehension of the diagram is regarded as weak if they fail to recognize and discriminate all the figures and sub-figures. Although the participating PTs were able to see and name what they recognized, they were able to see figures as single entities rather than as a configuration of these single entities. The results however strongly indicate that the participating PTs understood the meaning of congruency and that they relied on their knowledge of congruency rather than on their visual perception to judge whether the triangles identified were indeed congruent or not.

In Task 1(a) and (b), the participating PTs' perceptual apprehension of the diagram was considered as weak since they failed to recognize and discriminate all the figures and sub-figures. Although the participating PTs were able to see and name what they recognized, they were more likely to see figures as single entities rather than as a configuration of these single entities. Further the PTs' perceptual apprehension had an impact on the discursive apprehension in Task 3. There were indications that participating PTs had the ability to identify the correct theorem although the ability to identify all the concepts of the theorem differed across the participating PTs.

6.4 Sub-question 2: Making connections between geometry representations, properties and theorems category.

In this section an in-depth understanding of the specific forms of connections that participating PTs made when solving circle geometry tasks and the answers to the sub-question "*What types of connections do the PTs make between representations, properties and theorems?*" has been sought. In describing the forms of connections for proficiency in geometry in the CK tasks, I draw on the work of Businskas (2008) and Mhlolo (2012) about how mathematics teachers conceptualize mathematical connections in their practices. Refer to Section 2.4 in Chapter 2 for a discussion on mathematical connections and Section 4.4 in Chapter 4 for a discussion around analysing the connections made between representations, properties and theorems in this study. I conceptualize mathematics connections as a tool that the PT uses to organize and describe their thinking when dealing with circle geometry. I adapt Businskas (2008) types of practice-based mathematical connections made through **different representations, part-whole relationships, implications** and **procedures** to describe the prospective teacher preparation-based mathematical connections made through geometric representations, properties and theorems. These comprise:

- Visual connections made through use of different representations of geometrical objects
- Systematic organization connections made through the structure of geometric properties and theorems
- Implication connections made through logical reasoning with geometric properties and theorems
- Theorem application connections made through the application of theorem(s) to make conjectures when dealing with specific circle geometry problems.

An example of situations of a response for each form of connection is given in each description. See below for a further description of the categories and the discussions and analysis for each form of connection. I present an analysis of each form of connections within each task followed by examples of each form of connection and its coding. As mentioned earlier, I discuss the tables to get a general idea of all the participating PTs' responses but refer to responses by Nkosi, Wisdom and Lesedi when discussing the results explicitly. However, there are occasions where exemplary responses from other PTs are presented.

6.4.1 Visual connections

I contend that the visualization process evokes visual connections to be made between and among different representations of geometric notions. A visual connection displays an organization of relations between and among representations within the visualization process. Duval (1995) contends that access to mathematical objects is through their semiotic representations where there is a link between the different registers of semiotic representation. He contends that semiotic representations show relations or organization of relations between representational units. Guided by the conceptualization of the terminology 'different representations' and 'different registers' as espoused by Duval (1999), I classified the different forms of semiotic representations as verbal, figural and symbolic registers (see Figure 6.1) and categorized the connections between and among these registers as visual connections. Refer to Table 4.4 in Chapter 4 for the indicators for visual connections. These were connections made between the verbal and figure(s), connections made between symbols and figures, connections made between different figures, and connections made between definitions and figure(s). For example, to identify figures in Task 1(a), Lesedi's response in Figure 6.1 shows connections between and among verbal, figural and symbolic registers.

Kite ABHC – Cyclic quadrilateral
ΔABH *ΔBHE* *ΔBKH*
ΔACH *ΔBME*
ΔAKE (right angled triangle)
ΔABM
ΔACM
ΔBMH
ΔCMH
Cyclic quadrilateral KBME
Cyclic quadrilateral KBHE

Figure 6.1: Lesedi's response to Task 1 (a)

For instance, "Kite ABHC – Cyclic quadrilateral" shows that when identifying the figures, Lesedi shifted between the verbal descriptions of the diagram (verbal register), the different configurations within the figure (figural register) and concluded that ABHC is not only a kite but a cyclic quadrilateral as well. She appropriately made a symbolic representation of the kite (ABHC) (symbolic register). However, her score of 1 for visual connections made between the figural register and the verbal register suggests that she was not explicit in her description of the rest of the figures that she identified.

The next two sections present discussion of visual connections made between the verbal and figural registers, and connections made between symbols and figural registers. The visual connections were more pronounced in these categories.

6.4.1.1 Visual connections made between the verbal and figural register

Reference is made to Table 6.8. As mentioned in Section 6.4.1, visual connections were made between different registers. In this case, the participating PTs had to make connections between the figural register and the verbal register. When responding to tasks, the participating PTs had to make a transition from verbal register to figural register.

Table 6.8: Connections made between the verbal and figural registers

PT	verbal and figural registers				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	1	1	2	1	2
John	2	2	2	1	2
Wisdom	1	2	2	2	2
Lesedi	1	1	2	2	2
Bonolo	2	1	0	1	2
Thabiso	1	2	0	1	2

Note: 0, 1, 2 denote quality of connections levels

Table 6.8 shows that in the verbal and figural registers category, more strong connections were made than weak and faulty connections. The results concur with Duval's (1995, 1999) arguments that the connections that the participating PTs made between different representations strongly suggest coordination between their verbal registers and figural registers.

Results for Task 1

To illustrate the coding for this category of connections, I present examples and meanings of codes 2 and 1. Table 6.8 shows that in Task 1(a) two PTs (John and Bonolo) made level 2 connections in this category. Figure 6.2 illustrates John's response to Task 1 (a). John's response is considered exemplary as compared to that of other PTs

ΔABM →right-angled triangle
ΔAMC →right-angled triangle
ABHC – quadrilateral – kite (cyclic quadrilateral)
ΔBME- right-angled triangle
ΔBKE - right-angled triangle
ΔABE- isosceles triangle
ΔBMH- right-angled triangle
ΔCMH - right-angled triangle
ABHC – quadrilateral – kite (cyclic quadrilateral)

Figure 6.2: John's response to Task 1 (a)

John's response was scored 2 because he provided an explicit link between the diagram and his verbal description of it. Not only did he identify the triangles (figural register) from the diagram and from the verbal description given in the task, he also provided a verbal description with detailed properties of the figure identified (verbal register). The response also shows a connection to the symbolic register but this is not mentioned as the focus is only on verbal and figural registers. In contrast, in Figure 6.1, Lesedi provided a less detailed description of what is seen. As such Lesedi's connections in Task 1 (a) were scored at 1. Listing and labelling the geometric figures require a connection between the verbal registers and figural registers but Lesedi's response above clearly illustrates a weak connection between these registers.

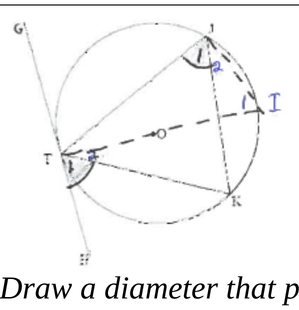
Compared to Task 1 (a), half of the PTs made strong connections in Task 1 (b), which does not come as a surprise because as presented in Table 6.1, more than half of the PTs performed at level 3 or above. These PTs gave a correct identification of at least 3 pairs of congruent triangles but with incomplete explanations. The frequency of scores indicate that John and Wisdom both had the highest scores for making strong connections (level 2) between the verbal representation and the figural representation. The excerpt below about Task 1 shows that to confirm his interpretation of the diagram, Wisdom made connections between verbal registers and figural registers.

Wisdom: *Okay, its like this, its like this and what I have done I had* ***to go back*** *to the description of the words to* ***confirm*** *(that). Okay, yah absolutely, this line at K was produced at AB which is at A. So this is like a confirmation of the words that I had to confirm - that did I draw what I am supposed to draw?*

The excerpt refers to Wisdom's interaction with Task 1. Wisdom indicates that he made shift between the verbal representation and the figural representation in order to identify the figures and confirm the congruency. It clearly shows that to confirm his interpretation of the diagram, Wisdom made connections between verbal registers and figural registers.

Results for Task 2

The performance scores for Task 2(a) ranged between 1 and 2 with four of the participating PTs attaining a 2, indicating that the quality of responses for this task was poor. See the PTs' performance scores in Table 6.1. However, a deliberate decision was made to interpret the visual connections within those poor scores. The visual connections between the participating PTs' verbal registers and figural registers for this task ranged between 0 and 2, suggesting that the connections varied between faulty and strong. Nkosi, John, Wisdom and Lesedi made strong connections. For example, see Nkosi's response in Figure 6.3. Nkosi operatively apprehended the diagram in order to make connection between the geometric principles and the identified configurations.



Draw a diameter that passes point T till the circumference

Let $T\hat{I}J = I$

$H\hat{T}I = 90°$ *(Tangent ⊥radius)*

$T\hat{J}I = 90°(< \ in\ semi-circle)$

$J_2 = T_2$ *(< in same segment)*

$T_1 + T_2 = J_1 + J_2 = 90°$

Therefore $J_1 = T_1$

Figure 6.3: Nkosi's response to Task 2 (a)

Bonolo and Thabiso were the only PTs that registered a score of 0 in Task 2 (a), demonstrating that *faulty* competence in the knowledge of circle geometry is linked to lack of coordination between their verbal registers and figural registers. Thabiso contends that

Thabiso: *well, I can produce, eh .. different diagrams to show the theorem .. but I cannot prove this theorem, I don't know how…I can identify equal angles.*

Thabiso's response above clearly illustrates a weak connection between these registers.

Results for Task 3

From Table 6.8, we see that all the participating PTs made stronger connections in Task 3 (b) than Task 3 (a) in in terms of connecting the figure with the verbal registers. To illustrate the coding for this task, I present Task 3 (a) which presents a less detailed description of what is seen. As such Thabiso's connections in Task 3 (a) were scored at 1 (Figure 6.4).

Angles on the circumference subtended by the same chord are equal

Figure 6.4: Thabiso's response to Task 3 (a)

Task 3 (a) required a visual explanation of all objects in the figural register, suggesting that the participating PT was expected to organize the figure in order to verbally describe it. Thabiso provided a generic description of the 'angle in the same segment' theorem that did not specifically describe the figure as presented in the task. In contrast, refer to Figure 6.8 for Lesedi's response to the task. There were strong indications in Lesedi's response that in solving Question 1 in Task 3, Lesedi made a connection between the figure and its verbal description. However, this response did not make explicit the connections between the symbols and figure as will be explained in Section 6.5.1.2.

6.4.1.2 Visual connections made between symbols and figures

Reference is made to Table 6.9. As mentioned in Section 6.4.1, visual connections were made between different registers. In this case, the participating PTs had to make a connection between the symbolic registers and the figural registers. Table 6.9 shows the levels of connections by task.

Table 6.9: Connections made between symbols and figure(s)

PT	Symbols and figures				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	1	0	2	0	2
John	2	2	2	0	2
Wisdom	1	2	1	2	2
Lesedi	2	0	2	0	2
Bonolo	1	1	0	0	2
Thabiso	1	2	0	0	0

The classification of the scores of the participating PTs' connections demonstrates that the most frequent scoring was at level 2 and 0, implying that most connections were strong or faulty connections rather than weak. John made most strong connections whilst Thabiso made the most faulty connections between symbols and figure(s). It is most evident in Task 1(b), Task 3 (a), and Task 3(b) that the PTs have difficulty with making connections between symbols and figures.

Results for Task 1

To illustrate the coding for this category of connections I present examples and meaning of code 0, a faulty connection. The participating PTs responses to Task 1 (b) indicate that through abstraction, all the participating PTs identified the pairs of congruent triangles but some did not appropriately use the congruency symbols and correct description of the triangles. Figure 6.5 shows Nkosi's response to this task.

Δ ABM and ΔACM

Δ ABH and ΔACH

ΔBMH and ΔCHM

Figure 6.5: Nkosi's response to Task 1 (b)

In response to the task, the participating PT was expected to provide a mathematical statement that reflected correct identification or configuration of the triangles to show congruency and to give reasons. Nkosi's response shows that he identified the congruent triangles but used "and" instead of the congruency symbols "≅" or "≡". The use of the connector “and” is considered inappropriate. The third pair of the purported “congruent” triangles was as a matter of fact not appropriately labelled to conform to the congruency principles. Nkosi also gave no justifications as to why triangles were congruent. On the other hand, Lesedi (see Figure 6.6) identified the congruent triangles but used “*III”* instead of the congruency symbols “≅” or “≡”. Her justifications were also faulty.

ΔABH III ACH (SSS)
ΔABM III ACM (SSS)
ΔBMH III MHC (SSS

Figure 6.6: Lesedi’s response to Task 1 (b)

The connections between symbolic registers and the figural registers made by Nkosi and Lesedi were classified as faulty. I noticed that this was in contrast with the strong connections they made between verbal and figural representation.

Results for Task 2

Scoring levels in Table 6.2 indicate that there were more strong connections made between symbolic registers and the figural registers in Task 2(a). Three of the six participating PTs’ connections were scored at 2 indicating that strong connections were made between the symbolic registers and the figural registers. Refer to Figure 6.3 that displays Nkosi’s response to Task 2(a). As demonstrated in Section 6.4.1.1 the configurations in his response strongly suggest an operative apprehension. There is evidence of connections being made between the symbolic registers and the figural registers. There is a transformation of the figure by splitting into sub-figures whilst ensuring the correct use of symbols to identify such reconfigurations. On the one hand, Wisdom’s response in Figure 6.7 was scored at 1 since there was use of incorrect symbolic registers in conjunction with correct figural registers. It clearly illustrates that there was a weak connection between these registers.

Since AB is perpendicular to CD, we draw in the radii AB and AF. Since the tangent is perpendicular to the radius,

$$C\hat{B}F = 90° - F\hat{B}A$$

= ½$\left(180° - F\hat{B}A - A\hat{F}B\right)$
= ½($F\hat{A}\boldsymbol{B}$**)**
= $F\hat{E}B$

Figure 6.7: Wisdom's response to Task 2 (a)

Results for Task 3

As mentioned in Section 6.4.1.1, the participating PTs made more faulty connections in Task 3 (a) than Task 3 (b). Task 3 (a) was dominated by the participating PTs' faulty connections except for Wisdom who made strong connections. The task required an interpretation of the ideas put forth by the geometry question. The participating PTs explanations were mostly verbal descriptions of the theorem without explicit reference to particular symbols. For example, Lesedi's response in Figure 6.8 was;

To address the misconception that angles that are equal and subtended by the same chord have to be on the same circumference not inside the circle or outside the circle

Figure 6.8: Lesedi's response to Task 3 (a)

Although her response was correct, Lesedi did not use symbols to identify the specific angles, chord and arc that she was referring to. The use of symbols in explanations exposes the respondent's thinking and understanding of the concepts. In contrast, Figure 6.9 displays Wisdom's response which is more explicit.

Angles that touch the circumference, which are subtended by the same cord are equal. For example in the diagram above, $\hat{L} = \hat{Q} = \hat{N}$ *. All these angles are subtended by the same cord x and y and touches the circumference.*
Because shows different angles being subtended by the same cord $\overline{xy}$ *. Some of these angles touches the circumference like*$x\hat{L}y$,$x\hat{Q}y$,$x\hat{N}y$,$x\hat{E}y$*, and other angles are not/don't touch the circumference e.g.* $x\widehat{M}y$, $x\hat{V}y$

Figure 6.9: Wisdom's response to Task 3 (a)

Wisdom makes direct reference to the perceived objects. Despite that there were errors in his work (for example, not mentioning that angles have to be in same segment), Wisdom's cognitive processes are clearly articulated. The PTs were successful in making connections between the symbols and figures in Task 3(b) (see Table 6.9). This was evidenced by their ability to provide specific symbols that clearly illustrated the angle in the segment theorem. The PTs were able to apprehend the diagram by connecting the identified symbolic representation of the angles with the figural representation. I conclude that the PTs overlooked the underlying symbolic registers when making visual connections.

6.4.2 Systematic organization connections

I use the work of Duval (1995, 1999) to argue that the cognitive processes of visualization and reasoning are essential in understanding geometric thinking. Duval (1995, 1999) posits that the use of different registers and movements within registers promotes understanding of mathematical concepts. I suggest that this implies the importance of some kind of structure to systematically organize the relationships within the different registers (words, symbols, propositions). Refer to Table 4.4 in Chapter 4 for the indicators for systematic organization connections. For example, to prove congruency in Task 1(b), Nkosi's response in Figure 6.3 shows a systematic organization of the geometric concepts or objects (words, symbols, propositions). The sequencing of the steps indicates a deliberate organization of thoughts. Nkosi makes a conjecture that ΔABH and ΔACH are congruent. He then provides reasoning in a logically organized structure in an explanation that connects words, symbols and propositions to prove congruency.

I argue that making connections is a requirement for the systematic organization of geometric concepts or objects. The categories for the systematic organization connections were classified as connections between (i) figures and figural units, (ii) between properties and theorems, and (iii) between definitions and properties. A figure is composed of figural units. For example, line segments and points are figural units of a triangle.

What follows in the next two sub-sections is a discussion of the connections made between figures and figural units and connections made between properties and theorems. I deliberately chose these two forms of connections because all the CK tasks elicited knowledge to organize geometric objects in terms of general and special cases about figures, properties and theorems. The systematic organization connections were more pronounced in these categories.

6.4.2.1 Systematic organization connections made between the figure(s) and figural units

Reference is made to Table 6.10. As mentioned in Section 6.4.2, some kind of structure was needed to organize the relationship between the figural registers. In this case, the participating PTs had to make connections between the figures and figural units. Table 6.10 shows the levels of connections by each task.

Table 6.10: Connections made between figures and figural units

PT	Figures and figural units				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	2	1	2	1	1
John	1	2	2	1	1
Wisdom	2	2	0	2	1
Lesedi	1	1	2	1	1
Bonolo	1	1	1	1	1
Thabiso	1	1	1	1	1

The classification of the scores of the participating PTs' connections demonstrates that the most frequent scoring was at level 1, implying that most of the connections between figures and figural units were weak. The table reflects that in the figures and figural units category of the systematic organization connections, more weak connections were realized than the strong and faulty connections in the figural registers. Overall, Wisdom made most strong connections whilst Bonolo and Thabiso made the weak connections in all the tasks.

Results for Task 1

As mentioned earlier, Task 1(a) examined the PTs' visual explanation abilities. The task required the listing and labelling of identified figures and as such, the participating PT's organizational abilities were illuminated. Table 6.10 shows that Nkosi and Wisdom made strong connections between figures and figural units as revealed by the manner in which the figures identified were organized.

To illustrate the coding for this category of connections, I present examples and meaning of code 2, a strong connection. A strong systematic organization connection must reveal a deliberate systematic sequencing of figures, properties and theorems. A systematic organization between the figures and figural units is evident when there is a logical organization of figures and their figural units. Figure 6.10 displays an excerpt of Wisdom's response to Task 1 (a).

ΔBEH
ΔABE ΔBMH
ΔAKE ΔBCH
ΔABH ΔBKE
ΔABM ΔBHE
ΔAMC ΔCBH
ΔAHC ΔCMH
Circle at centre S with points B, H, C and A at the circumference.
Cyclic quad ABHC

Figure 6.10: Wisdom's response to Task 1 (a)

Wisdom strategized when listing and labelling. The excerpt shows that he identified the basic shapes and then used the figural units to list the subsets of the basic shapes. It is because of this strategy that amongst the participating PTs, he was able to identify the highest number of figures and figural units as displayed in Table 6.10.

Results for Task 2

As mentioned in Section 4.4 in Chapter 4, a systematic organization is evident when words, symbols, propositions, figures and figurative units are used to organize geometric concepts or objects. Table 6.10 shows that there were more level 2 connections than level 1 and 0 that were realized between figural and figural units in Task 2(a). Nkosi, John and Lesedi connections were categorized as level 2 because there were considered as strong connections between figures and figural units. For instance, Lesedi made strong connections between the objects. Figure 6.11 displays Lesedi's organization of geometric objects in terms of properties and theorems.

Construct a diameter (perpendicular to the tangent)
$\hat{T}_1 + \hat{T}_2 = 90°$ *(rad ⊥ tan)*
Construct a chord (JI)
$\hat{J}_1 + J_2 = 90°$ *(angle in a semi-circle)*
$\therefore \hat{T}_1 + \hat{T}_2 = \hat{J}_1 + J_2$ *(both* $90°$
But $\hat{J}_1 + J_2 = 90°$ *(angle in the same segment)*
$\therefore \hat{T}_1 = \hat{J}_1$
$\therefore \hat{T}_1 = T\hat{J}K$

Figure 6.11: Lesedi's response to Task 2 (a)

In order to organize her thoughts, Lesedi starts by de-configuring the figure to systematically make a connection between configurations and sub-configurations and to be able to identify the properties and theorems related to configurations. Wisdom's response was classified as level 0 connections since it was categorized as a faulty connection. Although there is evidence of operative apprehension, the statement and thoughts display that faulty connections were a result of a weak sequential apprehension.

Results for Task 3

All the PTs made weak connections between the figures and figural units in Task 3 (a) and Task 3 (b) except Wisdom who attained a level 2 score in Task 3 (a). Figure 6.12 presents an excerpt from Lesedi's response to Task 3 (b):

$\widehat{Q} = \widehat{N}$ *(angles in same segment)*
$\widehat{L} = \widehat{N}$ *(angles in same segment)*

Figure 6.12: Lesedi's response to Task 3 (b)

Lesedi's score of 1 indicated an inadequate understanding of the connection between sets of equivalent entities. She was able to recognize the theorem applicable for the task and identified the equal angles but could not make a deduction that connects $\widehat{Q}$ and $\widehat{L}$.

I conclude that the analysis of connections made between figures and figural units is weak. This finding concurs and brings insight into what was concluded in the sub-research question

1. The major finding was that the participating PTs' perceptual apprehension of the diagram was weak due to their failure to recognize and discriminate all the figures and sub-figures through mental modification or organization of the diagram.

6.4.2.2 Systematic organization connections made between the properties and theorem(s)

Reference is made to Table 6.11. As mentioned in Section 6.4.2, some kind of structure was needed to organize the relationship between the figural registers. In this case, the participating PTs had to make connections between properties and theorem(s). The table below shows the levels of connections by task.

Table 6.11: Connections made between properties and theorem(s)

PT	Properties and theorems				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	2	1	2	1	1
John	2	2	2	1	1
Wisdom	2	2	1	2	1
Lesedi	2	1	2	2	1
Bonolo	2	2	0	1	1
Thabiso	2	2	0	1	1

The classification of the scores of the PTs' connections demonstrates that the most frequent scoring was at level 2 and 1, implying that most connections varied between strong and weak connections rather than faulty connections. Moreover, strong connections made an edge over weak connections.

Results for Task 1

Task 1 (a) demonstrates that although the participating PTs had a weak apprehension of the diagram they in fact had knowledge of properties of geometric figures and theorems. Identifying figures required knowledge of theoretical geometrical properties of the figures and theorems. All the participating PTs attained level 2 connections in Task 1 (a). Evidence indicates that the PTs had knowledge of the characteristics of the identified figures, which they were able to identify using definitions and theorems. For example, other than the basic figures, all the participating PTs identified the cyclic quadrilaterals, the kite and congruent triangles in Task 1 (b). However, the same cannot be said about Task 3.

Results for Task 2

The table shows that in the properties and theorem(s) category of the systematic organization connections, Bonolo and Thabiso made faulty connections in Task 2 when proving the tan-chord theorem. Thabiso confirms that "*I cannot prove the theorem"*. This finding is consistent with the findings on these two participating PTs' weak ability to make connections between verbal and figural representation. I therefore make a claim that weak competence in linking figures with verbal descriptions suggests lack of knowledge of geometric properties. John, Wisdom and Lesedi realized the most level 2 connections.

Results for Task 3

To illustrate the coding for this category of connections, I present an example and meaning of code 1, a weak connection. I will make reference to John's exemplary response. Figure 6.13 displays John's response to Task 3 (a).

> *For angles subtended by the same chord to be equal they must be angles* ***which touches*** *the circumference of the circle. These angles have* ***to be on the circumference*** *of the* ***major circles****.*

Figure 6.13: John's response to Task 3 (a)

The highlighted phrases provide inappropriate descriptions and definitions of the angles in the same segment theorem and related misconceptions. John gives the converse of the theorem by relating properties of angles to the location of these angles in the circle in a systematically organized manner. However, the statement is incoherent with inappropriate descriptions and definitions of the angles.

In general, connections made between the systematic organization of geometrical properties of the figures and theorems for Task 3 (a) and Task 3 (b) were found to be weak. I consider the discursive apprehension of most participating PTs for both Task 3 (a) and Task 3 (b) to be poor as a result of the poorly done connection of properties in the "angle in the same segment" theorem. The weak competence was portrayed from the definitions and language used to describe the perceived figure (see Figure 6.13). The excerpts below shed light on John's reasoning about the descriptions of the mathematical situations.

Excerpt 1

John: then I noticed that when you do others, it starts to make sense may be its English, the problem is English. ….

Kim: oh, ok, let me understand the sentence, you didn't interpret it mathematically, that was the problem.

John: yah, the mathematical language.

Excerpt 2

John: and IM, it's a cyclic quad. Theory of tangent is the same theory that we had that when it's on the circumference, I don't know how to say it but then this angle will be equal to…

Kim: equal X?

My judgment about these excerpts is that the difficulty in describing figures through geometric language/narrative texts as stressed by Duval (1995) hindered the PTs' ability to change from figural registers to verbal registers. The participating PTs have knowledge of circle geometry properties and theorems but they lack knowledge to appropriately systematically organize geometric language to describe the properties and theorems.

6.4.3 Implications connections

The discursive apprehension is the inability to establish a logical relationship between the mathematical principles and the identified configurations. The logical relationship suggests that connections between and within properties, definitions and theorems are developed and enhanced through descriptions, explanations and argumentation. The process ultimately leads to deductive reasoning where there is use of connectors such as the "**let**", "**if-then**" and "**therefore**". Implications are made in connecting the premise and the conclusion. Refer to Table 4.4 in Chapter 4 for the indicators for implications connections. For example, to prove a theorem in Task 2, Nkosi's response in Figure 6.3 in Section 6.4.1.1 illustrates an implicit use of the terms or notions of "**let**", "**equivalence**" and "**therefore**" to connect the geometric principles and the identified configuration in a logical argument that leads to a conclusion.

Thus the discursive apprehension is relevant to implication connections. These forms of connections involve cognitive processes of reasoning. The connections made through implications establish the logical relationship between the mathematical principles and the identified configurations. The logical relationship suggests that connections made between and within properties, definitions and theorems are developed and enhanced through descriptions, explanations and argumentation.

What follows in the next two sections is a discussion of the connections made between definitions and figure(s) and connections made between properties and justifications. The implication connections were more pronounced in these categories.

6.4.3.1 Implication connections made between the definitions and figure(s)

Reference is made to Table 6.12. As mentioned in section 6.4.3, implication connections were made between and within properties, definitions and theorems. In this case, the participating PTs had to make a connection between definitions and figure(s). The table below shows the levels of connections made between definitions and figure(s) within tasks. I conceptualize definitions in this study as verbal descriptions of geometric objects or figures.

Table 6.12: Connections made between **definitions and figure(s)**

PT	Definitions and figure				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	1	1	2	2	1
John	2	2	2	2	1
Wisdom	1	1	2	2	1
Lesedi	2	1	2	2	1
Bonolo	1	2	0	1	1
Thabiso	1	1	0	1	1

Findings in Table 6.12 reveal that in the definitions and figure(s) category of the implication connections, the most frequent scoring was at level 2 and 1. That is, most connections were strong or weak rather than faulty. Moreover, weak connections made an edge over strong connections. John made most strong connections whilst levels of connections varied for the other participating PTs.

Results for Task 1

In both Task 1 (a) and Task 1 (b) logical conclusions were determined by the way the participating PTs characterized the figures. There were more weak connections than strong connections made between definitions and figure(s) in these two tasks. For example, in Task 1 (b), defining congruency means describing and relating the characteristics or properties of figures precisely. In other words, descriptions and explanations reveal connections made when making deductions. In Task 1 (b) a strong connection (level 2) is revealed by a meaningful description of the identified congruent triangles through the use of appropriate symbols, a correct order of labelling the congruent triangles and appropriate justifications. For example, Nkosi's response in Figure 6.5 in Section 6.4.1.2 demonstrates a logical conclusion that correctly characterized the congruent triangles. The connections were classified as strong connections.

Results for Task 2

Task 2 (a) responses revealed that participating PTs had faulty and strong knowledge of the tan-chord theorem. Bonolo and Thabiso made faulty connections in Task 2 when proving the tan-chord theorem. These two participating PTs provided different diagrammatic representations of the theorem. Figure 6.14 illustrated Bonolo's response of the proof. Bonolo provided a sketch that did not conform to the characteristics of the diagram in the task. For instance, circle S is not a perfect circle; DC and BC are not tangent to the circle.

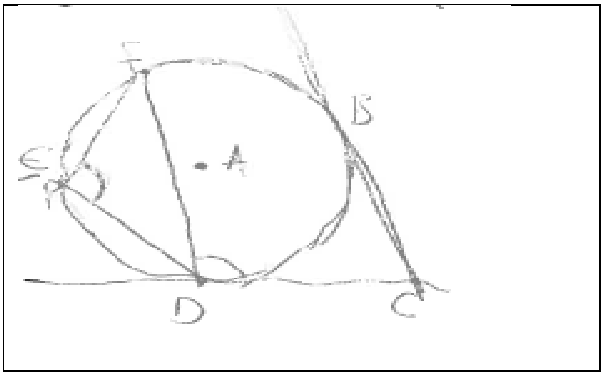


Figure 6.14: Bonolo's response to Task 2 (a)

The faulty connections that Bonolo and Thabiso made when apprehending the diagrams probably contributed to the weak connections between the properties of figures and the theorems.

Nkosi, John, Wisdom and Lesedi displayed strong knowledge of definitions of figures and figural units in Task 2 (a) and Task 3 (a) and were able to establish logical relationships between representations, properties and theorems. Figure 6.3 displays Nkosi's response to Task 2 (a). Nkosi starts by making propositions about the reconfiguration. He demonstrates implications made when making connections between the premise and the conclusion. Nkosi demonstrates the use of "**let"**, "**equivalence**", "**therefore**" and symbols in connecting the geometric principles and the identified configuration in a logical argument that leads to conclusion. Nkosi reconfigured the diagram to guide in the construction of the proof. This gesture was common among the participating PTs as reflected in Wisdom's excerpt below that makes reference to Task 2. I needed to get insight into the reasoning about making constructions before the proving processes.

Kim: *yah, so what challenges do you think they will meet when constructing the proofs?*

Wisdom: *aah I think that different proofs require different constructions so if learners do not know the different proofs obviously they will have a challenge when constructing the different ...aah... shapes because first they will have to prove that yah I can do it this way so this way I need this kind of a diagram, so if they don't know different ways to proof they will have difficulties constructing it. So basically they need four different ways of how to prove a theorem precisely because the more they know its easy to construct that diagram.*

Results for Task 3

Task 3 (b) was a follow up of Task 3 (a). Task 3 (b) required the participating PT to provide a correct identification of equal angles with justifications using geometric reasoning about the 'angle in the same segment theorem'. The participating PTs responses were limited in the identification of the angles and justifications. The connections made by all the PTs between the definitions and figure(s) were coded at level 1 implying that the use of symbols and the identification of angles displayed weak knowledge of the theorem.

6.4.3.2 Implication connections made between the properties and justification(s)

Reference is made to Table 6.13. As mentioned Section 6.4.3, implication connections were made between and within properties, definitions and theorems. In this case, the participating PTs had to make a connection between properties and justification(s). The table below shows the levels of connections made between properties and justification(s) within tasks. See Figure 6.16 for an example of connections that Bonolo made between properties and justifications.

Table 6.13: Connections made between **properties and justification(s)**

	Properties and justifications				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi		1	2	1	1
John		2	2	1	1
Wisdom		2	2	2	2
Lesedi		1	2	2	2
Bonolo		2	1	1	1
Thabiso		2	1	1	1

Justification in this study is conceptualized as communicating a link between the properties and theorems. The communication should reflect knowledge of properties of geometric objects in a logical relationship. For example, justifying congruency means making deductions by describing and relating the characteristics or properties of figures.

The table shows that in the properties and justification(s) category of the implication connections, scoring was at level 2 and 1 throughout all tasks, implying that connections made indicate that the participating PTs' knowledge of circle properties and justification(s) is weak or better. The table shows that, generally, the participating PTs made the same numbers of strong connections as compared to weak connections. There were no faulty connections registered. Wisdom made the strongest connections while Nkosi, Bonolo and Thabiso made the weakest connections.

Results for Task 1

Task 1 (a) was excluded from this category because the task did not explicitly require participating PTs to make justifications on the perceived figures. Task 1 (b) required the PTs to connect the configurations of the figure with the properties of such figures in order to provide a justification for congruency. The justifications provided by the participating PTs indicate that more connections were made at level 2 as compared to level 1.The justifications were linked to strong knowledge of deductions about properties of figures and definitions. Figure 6.15 is an excerpt from Bonolo's response to Task 1 (b) showing the level 2 connections.

ΔABH and ΔACH are congruent
$A\hat{B}H = A\hat{C}H$, *angle on a semi-circle = 90°*
side AH = shared side
AB = AD
therefore SAS

Figure 6.15: Bonolo's response to Task 1 (b)

Bonolo makes a proposition about the two congruent triangles ΔABH and ΔACH. Then, in a logical manner, he organizes his thinking around some deductive system of axioms, properties and theorems.

Bonolo: *I must ah ah visualize the triangles first. Do they look the same?*
Kim: *what do you mean, the same?*
Bonolo: *in terms of mirror reflection.*
Bonolo: *ahhh. It's a kite. They share the same properties… in the semi-circle*

Implication statement in Bonolo's working is prominent in the last statement "**therefore SAS**". The postulate justifies the congruency of the identified triangles.

A level 1 response displayed a less explicit justification for congruency. A response by Lesedi was classified in this category.

Kim: *explain which triangles are congruent. Can you tell me… how you got this (referring to the written answers)*

Lesedi: *I said triangle ABH is congruent to triangle ACH because this side is equal to this side; and then this distance is equal to this distance; and then they both share the equal side AH*

Kim: *how do you know that BH is equal to HC?*

Lesedi: *ahh, because they say AB is equal to AC so I saw that this side is equal to this side and then the ray (AE) is the angle bisector of angle A*

Lesedi: *the reason is the same for all the triangles; equal sides and share common side, because of the isosceles triangle.*

The dialogues above give an insight into Lesedi's thinking when responding to the task. Her written response shows listing of three pairs of what she considers congruent triangles, all connected by an inappropriate congruency symbol. Her verbal explanation suggests that she had some knowledge of congruency but did not correctly justify the geometric facts that she mentioned from the perceived figures.

Results for Task 2

The participating PTs' responses to Task 2 (a) should reflect understanding of various circle theorems (see Section 5.2.2 for Task 2). A strong perceptual apprehension of the figure illustrates that the statement $\hat{T}_1 = T\hat{J}K$ can only be verified by proving the tan-chord theorem. Herbst and Miyakawa (2008: 469) contend that "a proof may tell us why the statement is true, as well as what ideas that statement connects or requires by virtue of being true or in order to be true".

The justifications of the proof should reflect its logical structure linked to geometric facts about the theorem. Task 2 (a) required the PT to show multiple methods of the proof but the participating PTs provided only one method. Therefore the overall performance score for this task was the lowest. The reasons for producing one method of the proof was expressed by John that

John: *yes, the problem is the logic behind this there is so many pieces of theorems and you don't know how to put it in order.*

Moreover, an in-depth analysis of the method that was provided indicates that more connections were made at level 2 as compared to level 1, suggesting that the participating PTs made strong connections between properties and justifications. Four of the six PTs' made level 2 connections, demonstrating a complete understanding of this method of proof. Nkosi, John and Lesedi used method 2 of the memorandum while Wisdom used method 4 of the memorandum. See John's response to this task in section 6.4.3.1. All the four PTs executed a logically structured justification for the application of the tan-chord theorem in the perceived figure. In each response, a proposition is given after a reconfiguration of the diagram is done. Then a formal argument is established to validate that $\hat{T}_1 = T\hat{J}K$. This cannot be said about Bonolo and Thabiso. They provided reconfigurations of the diagram but failed to make an argument to validate that the two angles are equal. See Thabiso's response in Figure 6.16. Thabiso provided four different scenarios to illustrate, and identified the congruent angles of the theorem but did not provide a proof to justify why the angles were congruent. The connections that Bonolo and Thabiso made were coded at level 1.

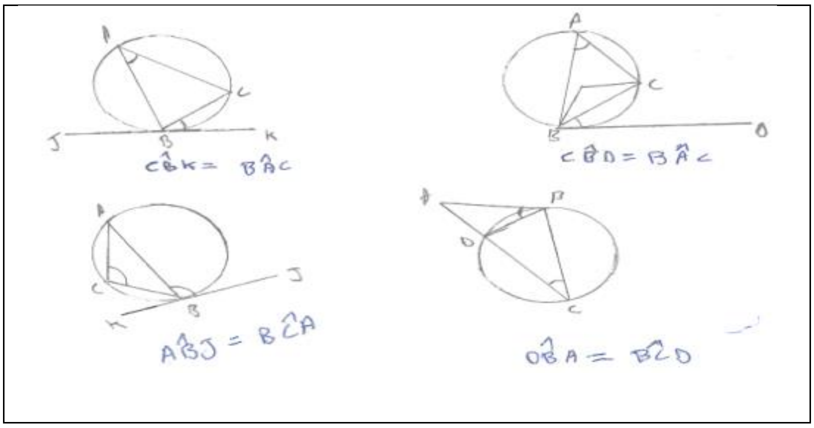


Figure 6.16: Thabiso's response to Task 2 (a)

Results for Task 3

The critical components of Task 3 (see Figure 5.3) were the ability; (i) to understand the mathematical ideas in question 1, and (ii) to produce the solution to the question: "*which of the angles are equal?*" The participating PT must apprehend in the figure the relationship between the segments, chord and angles and relate to the angle in the same segment circle geometry theorem. In this apprehension, the PT should focus on the perceived relationship between the angles subtended by the same chord either in the same segment, different segments or not inscribed. The response statements about the mathematical idea should reflect that the PTs identified angles, segments and chord in the geometry question. A response provided by the PTs should reflect a perceptual apprehension of the diagram in the geometry question in order to make a discursive statement. In responding to Task 3, the PT were expected to give justifications that explicitly stated the key essential words; *same segment, subtended by same chord/arc, inscribed, angles on circumference, converse and misconceptions*. The PTs were expected to express the theorem in natural language. To accomplish the tasks required a clearly organized convincing logical argument reflecting knowledge of the theorem.

As mentioned earlier, justification should reflect knowledge of properties of geometric objects in a logical relationship. Table 6.14 shows that in Task 3 more connections were made at level 1 as compared to level 2, indicating that even though at a glance, the participating PTs recognized the idea depicted in the question they could not logically communicate this idea. For instance, when looking at Nkosi's response in Figure 6.17, he identified the theorem that was depicted in the question but the argument showed a flawed description of the conditions for the theorem to apply in any situation. The connections that Nkosi made were coded at level 1.

*This shows that angles on the same segment are equal, **only if** they are on the circumference*

Figure 6.17: Nkosi's response to Task 3 (a)

Nkosi gave the connector "**only if**" as a justification for the equivalent angles but failed to emphasize that the angles should be subtended by the same arc or chord. However, Nkosi correctly identified the equal angles as $\widehat{Q} = \widehat{N} = \widehat{L}$ but did not provide a justification for this. Overall, when making connections between and within properties, definitions and theorems, there is a strong indication that PTs can make justification in proofs but are weak in providing descriptions and explanations to reveal the connections that are made when making deductions. This argument concurs with the finding in the systematic organization section.

The PTs portray weak competence of geometric language ability to describe the perceived figure.

6.4.4 Theorem application connections

The forms of connections in this category are those connections that involve recognition of which theorem is appropriate to apply to the situation at hand. That is, a connection is made when it is acknowledged that theorem A is applicable for solving B. Theorem application requires logical reasoning. The reasoning is guided by a premise that leads to a conclusion about a specific theorem to apply for a specific case. Refer to Table 4.4 in Chapter 4 for the indicators for theorem application connections. For example, to recognize and identify equal angles in Task 3 (b), Wisdom's response in Figure 6.18 shows the connections made through the application of a specific circle geometry theorem to make a statement about the specific angles.

$\widehat{N} = \hat{L}$ .... *angles are subtended by the same chord.*

Figure 6.18: Wisdom's response to Task 3 (b)

A theorem is conceptualized in this study as a statement that has been proven about the characteristic or property of a geometry object. The next two sections present a discussion of theorem application connections made between properties and theorem(s) and connections made between figure(s) and theorem(s). The two forms of connections provided a remarkable pattern for theorem application connections. The theorem application connections were more pronounced in these categories.

6.4.4.1 Theorem application connections made between the properties and theorem(s)

Reference is made to Table 6.14. As mentioned in Section 6.4.4, the participating PTs had to make a connection between properties and theorem(s). The table below shows the levels of connections made between properties and theorem(s) within tasks.

Table 6.14: Connections made between properties and theorem(s)

PT	Properties and theorem				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	1	1	2	1	1
John	2	2	2	1	1
Wisdom	2	2	2	1	1
Lesedi	1	2	2	1	1
Bonolo	2	1	1	1	1
Thabiso	2	2	1	1	1

The table reflects that in the properties and theorem(s) category of the theorem application connections, scoring was at level 2 and 1, implying that connections made between properties and theorem(s) indicate the participating PTs' knowledge of circle properties and geometry theorems is weak or better. Moreover, weak connections made an edge over strong connections. Wisdom and John made most strong connections whilst Nkosi and Bonolo made most weak connections.

Results for Task 1

When responding to Task 1 (a), the participating PTs made connections between properties and theorem in the process of identification. The first point of entry for the participating PTs was to define the figures through knowledge of properties of basic geometric objects. A further classification of the figures required the use of theorems by making reference to the properties of the identified figures. More connections at level 2 than connections at level 1 were made in this task, indicating that the participating PTs had a strong knowledge of theorem(s) applicable in the identification of the properties of geometric objects in the task. Nkosi and Lesedi made connections at level 1. Lesedi's response in Figure 6.1 shows an explicit application of the theorems using the properties of the identified figures. However, she made an error by identifying quadrilaterals KMBE and KBHE as cyclic quadrilaterals. The vertices of the quadrilaterals are not inscribed, suggesting that the properties for the cyclic quadrilaterals do not hold.

Task 1(b) required an identification of congruent triangles, signalling the use of properties of triangles that meet the condition for the congruency theorem. Table 6.14 illustrates that the participating PTs made strong connections within this category. The explanation or justification given by the participating PTs suggests that there were connections made between the properties and the theorem. The congruency theorem was used to identify the triangles, resulting into more level 2 connections realized than level 1 connections. This result concurs with another interesting finding in this study (section 6.2.1) which indicates that most PTs gave a correct identification of at most 3 congruent triangles but with some of the correct explanations incomplete in justifying the congruency.

Results for Task 2

Task 2 (a) required the participating PTs to prove that two angles were equal. To do that required an identification and application of the tan-chord theorem. Proving a theorem necessitates a logical organization of facts to maintain the truthfulness of the statements. The bases of the statements are the definitions, properties, propositions and postulates. The responses by Nkosi, John, Wisdom and Lesedi strongly suggest that they recognized that the tan-chord theorem should be applied in proving the congruent angles. Although the participating PTs provided only one of the four required proofs, they made strong indications that they had knowledge of properties appropriate for proving the tan-chord theorem.

Results for Task 3

Task 3 (a) and Task 3 (b) required the participating PTs to identify a theorem by making reference to the diagram. Table 6.15 shows that the PTs' connections were at level 1. As discussed in the systematic organization connections section earlier, the participating PTs' understanding in linking figures with descriptions suggests lack of knowledge of geometric properties in relation to the "angle in the same segment" theorem.

6.4.4.2 Theorem application connections made between the figures and theorem(s)

Reference is made to Table 6.16 below. As mentioned in Section 6.4.4, theorem application connections were made between and within figures, properties, definitions and theorems. In this case, the participating PTs had to make a connection between figure(s) and theorem(s). Table 6.15 shows the levels of connections made between figure(s) and theorem(s) within tasks.

Table 6.15: Connections made between figure(s) and theorem(s)

PT	figures and theorem				
	Task 1		Task 2	Task 3	
	(a)	(b)	(a)	(a)	(b)
Nkosi	1	1	2	1	1
John	2	2	2	1	1
Wisdom	1	1	2	2	1
Lesedi	1	1	2	1	1
Bonolo	1	1	1	1	1
Thabiso	1	1	1	1	1

To illustrate the coding for this category of connections I present an example and meaning of code 2, a strong connection. Task 1 (a) required the participating PTs to identify the figures using the angle theorems, triangle theorems and circle theorems. This is a form a perceptual apprehension. More level 1 than level 2 connections were realized in this task. John's response was scored 2 because he provided a more detailed description of what was seen indicating that there was a more explicit link between the diagram and the properties relevant to the theorem. See Section 6.4.1.1 above for the excerpt from John's response in Figure 6.2. John identified "*ABHC – quadrilateral – kite (cyclic quadrilateral)*". I consider this response as an attempt to connect the figure to its definition and properties. The response also clearly indicates that a further more precise definition of the quadrilateral, that is, the "cyclic quadrilateral" demonstrates that figural processing occurred. Compared to John, the other PTs were not explicit in their identifications. A mere listing of figures did not provide adequate insight into the thinking involved in connecting figures to theorems (cyclic quadrilateral theorem, triangle theorems). Their responses were thus classified at level 1.

The classification in Table 6.15 of the scores of the participating PTs' connections over all tasks demonstrates that the most frequent scoring was at level 1, implying that most of the connections between figure(s) and theorem(s) were weak. The table reflects that in the figure(s) and theorem(s) category of the theorem application connections, more weak connections were realized than the strong and faulty connections. John made most strong connections whilst Bonolo made most weak connections.

Results for Task 1

Task 1 (b) dealt with identifying congruent triangles and with providing justifications for these identifications. More level 1 than level 2 connections were realized in this task, indicating that the participating PTs made weak connections between figure(s) and theorem(s) in the process of identification. The discussion above has shown that the analysis of the participating PTs' written responses suggests that the participating PTs had an understanding of the congruency concepts. However, I needed to get insight into what figural processing occured when making connections between the figure(s) and the theorems. Responses by Nkosi, Wisdom, Lesedi, Bonolo and Thabiso were not very explicit in terms of linking the figure(s) with theorem(s). John's response in Figure 6.19 shows a convincing connection between the figure and the theorems. John provided a figure that has been extracted from the diagram and a verbal description of the identification and proof of congruent triangles. Clearly, the response explicitly highlights that in the figural processing John made strong connections between the figure and the triangle and circle theorems.

(a) For Δ ABH and ΔACH

$A\hat{B}H = A\hat{C}H = 90°$ ($\angle$ in semi-circle = 90°)

AH common

AB = AC given

$\therefore \Delta ABH \equiv \Delta ACH$ reason RHS

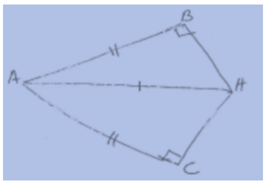


Figure 6.19: John's response to Task 1 (a)

There was a strong coordination of the figural register with the theorems in John's response. This did not occur in other PTs' responses. The visualization processes illuminated in the PTs' responses was largely on working with the special relationship between the sides of the triangle in order to identify the congruent triangles rather than on applying the triangle and circle theorems.

Results for Task 2

Task 2 (a) was concerned with the application of the tan-chord theorem to prove that two angles were equal. Table 6.15 shows that connections that Nkosi, John, Wisdom and Lesedi made were classified as level 2 connections. These PTs made reference to the figures in order to link the figure with the theorem. The participating PTs proposed a reconfiguration of the diagram to pave way for the theorem that could be applied to show that the two angles were congruent. For instance, Wisdom states that *"Since OT is perpendicular to GH, we draw in the radii OT and OK … Since the tangent is perpendicular to the radius".* Wisdom acknowledged that the theorem could be applied if the diagram conformed to the conditional statements of the theorem. Connections made between the figure(s) and theorem(s) by Bonolo and Thabiso were scored at level 1. Bonolo provided figural processing and produced three figures which he assumed show that the two angles were equal. Thabiso on the other hand provided four similar constructions of the tan-chord theorem that illustrated the two congruent angles. See Figure 6.16 for Thabiso's response. Bonolo and Thabiso reflect a lack of understanding of how to use a construction to visually represent the proof of a theorem. Although Bonolo and Thabiso recognized the theorem applicable in this situation, they failed to prove the actual tan-chord theorem.

Results for Task 3

Task 3 (a) and Task 3 (b) required the PTs to explicitly put forward that the figure illustrated a detailed description of the "angle in the same segment" theorem and its converse. A logical explanation was essential in determining if in the process of visually apprehending the figure the participating PTs concluded that the "angle in the same segment" theorem was applicable in this context. The PTs' explanation needed to provide a connection between the figure and the theorem. Table 6.15 shows that connections that Nkosi, John, Lesedi, Bonolo and Thabiso made were classified as level 1 connections. In processing the figure, the PTs were selective in their descriptions and appeared to focus only on acknowledging the equal angles. Below is an excerpt of Thabiso' thinking.

Kim:	*what is the idea?*
Thabiso:	*learners are to make conjectures and prove them…if they are correct also come up with a theorem*
Kim:	*what theorem?*

Thabiso: *the diagram helps to make connections with the task of proving which angles are equal that are found in the circumference.*

Thabiso: *I mean that angles on the circumference subtended by the same chord are equal*

Although Thabiso could identify the theorem, he was not specific about the angles that he made reference to. This trend was common across the PTs who scored at level 1. The PTs did not provide clear reference to the positions of the points in relation to the chord subtending all the angles. Connections made by Wisdom were categorized as level 2 connections. Unlike the other PTs who provided an implicit description of the theorem, Wisdom was more specific in terms of naming the identified objects. See 6.4.1.2 for Wisdom’s explanation. In his explanation, Wisdom first identified the theorem that was applicable in this situation. He then proceeded by explicitly highlighting the concepts, specific angles and properties that convincingly showed that he had a good understanding of the “angle in the same segment” theorem. In this task, Wisdom reflected a strong discursive apprehension. I conclude that in general, the PTs’ discursive apprehension of the diagram shows weak connections between the figure(s) and the theorem(s).

Section 6.4 Findings: What types of connections participating PTs made between representations, properties and theorems?

Section 6.4 presented and analysed what types of connections participating PTs made between representations, properties and theorems. Perceptual apprehensions followed by discursive apprehensions were required in the process of making connections between properties and theorems. I also based my rationale for the understanding of connections by acknowledging the position made by Torregrosa and Quesada (2008:2) that “discursive apprehension is the cognitive activity which produces a connection between the identified configuration and certain mathematical principles (definitions, theorems, axioms, etc.)”. In this study, discursive apprehension was conceptualized as (a) the ability to connect configuration(s) with circle geometric principles, (b) the ability to provide good descriptions, explanations, argumentations, deductions, use of symbols, and reasoning depending on statements made on perceptual apprehension, and (c) the ability to describe figures through geometric language/narrative texts (Duval, 1995). See Table 4.2 in Chapter 4 for a comprehensive illustration of how the discursive apprehension was conceptualized.

An in-depth understanding of the specific forms of connections that PTs made when solving circle geometry tasks was sought. The process of making connections between geometry representations, properties and theorems as premised in this study was achieved through perceptual apprehensions preceded by discursive apprehensions. Inductive analysis was employed to determine the categories of connections based on the expectations of the tasks, the geometry concepts within the tasks and the forms of connections.

The participating PTs made strong connections between verbal and figure(s) category of visual connections, indicating strong coordination between their verbal registers and figural registers. The most connections that PTs made between symbols and figures varied between faulty and strong connections indicating that the PTs either had difficulty or were efficient in connecting symbols and figures. The participating PTs identified the congruent triangles but not all participating PTs appropriately used the congruency symbols and the correct description of the triangles. The participating PTs overlooked the underlying symbolic representations when making verbal descriptions of the “angle in the same segment” theorem. The PTs explanations were mostly without explicit reference to particular symbols.

The participating PTs’ connections made between figures and figural units category of the systematic organization connections reflect a weak competence in linking figures with figural units confirming that the participating PTs’ perceptual apprehensions of the diagram were weak. The PTs failed to recognize and discriminate all the figures and sub-figures through mental modification or organization of the diagram. The participating PTs demonstrated a strong ability to make connections between properties and theorem(s) but they lacked the knowledge to systematically organize geometric language to describe the properties and theorems.

The participating PTs’ connections made between definitions and figure(s) category of the implications connections reflect a weak competence in linking the properties of figures to the theorems. Despite this observation, some participating PTs were able to establish logical relationships between representations, properties and theorems in two tasks. The use of symbols and identification of angles displayed weak knowledge of the “angle in the same segment” theorem. Generally, the PTs made the same numbers of strong connections as compared to weak connections between the properties and justification(s). Justifications provided by the PTs were linked to strong knowledge of deductions that relate to the characteristics or properties of figures. However, the participating PTs' weak geometric language suggests weak abilities to make implication connections during the process of making deductions.

The connections made in this category revealed that the participating PTs had strong knowledge of theorem(s) applicable to the identification of properties of geometric objects. Some of the explanations or justification given by the PTs either had errors or were correct with incomplete explanations. Nonetheless, the participating PTs recognized the specific theorem applicable for the specific context(s). In general, the PTs' discursive apprehension procedure showed weak connections between the figure(s) and the theorem(s). The PTs' thinking involved in connecting figures to theorems was not explicit particularly in the recognition and identifications of figures, indicating that the participating PTs made weak connections between figure(s) and theorem(s) in the process of identification. During figural processing, the PTs made reference to the figures in order to link the figure with the theorem. However, the participating PTs were selective in their descriptions and appeared to focus on properties rather than on applying the triangle and circle theorems.

6.5 Chapter summary

The purpose of this investigation was to explore the participating PTs' circle geometry knowledge by probing the participating PTs' thinking displayed in the PTs' solutions to the TPACK tasks. In investigating "what CK do the PTs display?" I was guided by two subsidiary questions. Firstly, I examined what the PTs' identified and recognized in perceived figures. Then, I studied the types of connections that participating PTs made between representations, properties and theorems. In this chapter I presented and discussed the sub-unit of analysis, PTs' CK, by using the participating PTs' responses (both from written tasks and from interviews) to bring forth what I considered prominent, absent or assumed by the participating PTs within and across the CK tasks. The next chapter presents analysis of the sub-unit of analysis of TCK of participating PTs.

CHAPTER 7

ANALYSIS BY TPACK COMPONENT: PROSPECTIVE TEACHERS' GEOMETRY TECHNOLOGICAL CONTENT KNOWLEDGE

7.0 Introduction

The previous chapter presented the data analysis and findings relating to the aspect of content knowledge (CK) construct of technological pedagogical content knowledge (TPACK). The major focus of this chapter is on presenting the results relating to the technological content knowledge (TCK) construct of the technological pedagogical content knowledge (TPACK) framework in response to research question 2:

What technological content knowledge do the PTs display about GeoGebra-constructed geometric diagrams?

In line with the previous chapter, I provide an overview of how the sub-unit of analysis (TCK construct) was conceptualized in the study (Section 7.1). A descriptive summary and the quantitative analysis of the rubric scores of all the participating PTs' responses to the TCK task will follow (Section 7.2). Thereon, a typological analysis with an inductive sub-analysis of the PTs' construction (Section 7.3) and reasoning (Section 7.4) competencies employing Duval's (1995) cognitive apprehensions is articulated. For each section, I provide the overall results in tabular forms, followed by a discussion of the overall results for all the participants. Each section is concluded by a summary of findings. The discussion of the findings is focused on answering the two sub-questions aided by evidence from the quantitative analysis presented earlier for each task. Sections 7.3 and 7.4 are focused on Task 1(c) and Task 4(a) respectively. When discussing the results explicitly, I refer to responses by Nkosi, Wisdom and Lesedi. As mentioned in Section 6.1, I discuss the trends across and within each task and present participating PTs' responses (Nkosi, Wisdom and Lesedi) to a GeoGebra-based task and interview excerpts to support the findings.

7.1 Sub-unit of analysis: PTs' circle geometry technological content knowledge (TCK)

The sub-unit of analysis for this chapter is the participating PTs' circle geometry technological content knowledge (TCK). The TCK construct was conceptualized in the study as the knowledge of how circle geometry concepts may be represented with GeoGebra (see Section 5.2). In this study, the circle geometry technological content knowledge required for the successful completion of the TCK tasks comprised two aspects: (i) construction of geometric diagrams with GeoGebra, (ii) verbal description of geometrical diagram constructed with GeoGebra. The exploration of the participating PTs' TCK was done by probing into the participating PTs' thinking displayed in the participating PTs' solutions to the TCK tasks. These tasks were deliberately designed to elicit knowledge of how GeoGebra and circle geometry influence and constrain one another and how knowledge of circle geometry could be effected by the use of GeoGebra.

Duval's (1999) cognitive apprehensions notion is used as interpretative tools to discuss how the participating PTs responded to the TCK tasks. These cognitive apprehensions are perceptual, discursive, operative and sequential apprehensions. Refer to Section 4.3 for an elaboration of the cognitive apprehensions. I use these apprehensions to interpret how PTs interacted with GeoGebra when they used GeoGebra to reproduce pencil-and-paper diagrams and to describe a GeoGebra constructed diagram. Since the TCK tasks that the participating PTs responded to elicited knowledge of GeoGebra constructions and reasoning, I seek to gain insight into participating PTs' GeoGebra construction skills and geometric discursive skills. To understand the PTs' construction and the discursive processes in responding to circle geometry tasks, I was guided by the following sub-questions

1) What do the GeoGebra constructions reveal about the participating PTs' knowledge of circle geometry constructed in a GeoGebra environment?
2) What types of descriptions do the PTs give about geometrical diagrams constructed with GeoGebra?

A geometric construction is defined in this study as a drawing of a figure satisfying given conditions using GeoGebra. The product of the construction is referred to as a GeoGebra-based construction. A diagram is a visual representation of a figure.

7.2 Analysis of Rubric Scorings of TCK tasks

The tasks that elicited TCK responses were Task 1(c) (for sub-question 1 above) and Task 4(a) (for sub-question 2 above). See Chapter 5 for the tasks and their descriptions as elaborated in Section 5.2.1 and Section 5.2.4 respectively. In this section I present the descriptive analysis relating to the performance scoring for individual PTs and the performance scoring across and within the two tasks.

7.2.1 Analysis of TCK scores for individual cases

Table 7.1 presents data for the six PTs; Nkosi, John, Wisdom, Lesedi, Bonolo and Thabiso. There were two (2) tasks testing TCK, each marked out of four. The overall mark was essential in determining the overall TCK performance score for each participating PT. The mean and standard deviation are provided to interpret the individual PT's scores. The scores ranged from the poor performance (score 0) to high performance (score 4).

Table 7.1: Scoring of the PTs' responses across and within the TCK tasks

PT		Rubric scores /4 for each sub-task		Summary of scoring across the tasks			
		Task 1(c)	Task 4(a)	mark /8	%	Mean ($\overline{X} = 1.667$)	SD ($SD = 1.155$)
Nkosi		0	4	4	50	2	2.828
John		0	2	2	25	1	1.414
Wisdom		0	4	4	50	2	2.828
Lesedi		0	3	3	37.5	1.5	2.121
Bonolo		1	1	2	25	1	0.000
Thabiso		3	2	5	62.5	2.5	0.707
Summary of scoring **within** the tasks	mark /24	4	16				
	%	17	67				
	Mean	0.833	2.667				
	SD	1.602	1.211				

A general observation across the responses shows that the scores ranged between 0 and 4 across the tasks. Four of the six PTs scored a 0 for Task 1 (c) and, two PTs scored a 2 for Task 4 (a). The observed absent scores were 2 and 4 for Task 1 (c) and 0 for Task 4 (a). A score of 4 as reflected in Table 1 indicates that Nkosi and Wisdom provided model answers for Task 4 (a). Across both tasks, Nkosi and Bonolo had the lowest mark of 25% whilst Thabiso scored the highest mark of 62.5%. The overall mean and standard deviation were

1.667 and 1.155 respectively, indicating that three PTs (Nkosi, Wisdom and Thabiso) scored above the mean and rest of the PTs scored below the mean (between 1.000 and 1.500). For example, Lesedi's overall mark and mean were 37.5% and 1.500 respectively, indicating a performance slightly below the overall mean score of 1.667. Although John and Bonolo had the same mean of 1, their standard deviations differed due to the pattern of scoring across the two tasks.

7.2.2 Analysis of PTs' scores within TCK tasks

Reference is made to the summary of the PTs' scores within each task as presented in Table 7.1 above. A general overview of the table indicates that all the two tasks were attempted, with 0 as the lowest score and 4 as the highest performance score attained in a task. Four PTs (Nkosi, John, Wisdom, Lesedi) scored 0 for Task 1 (c), whilst Task 4 (a) performance scores ranged between 1 and 4 with two PTs (Nkosi and Wisdom) attaining performance at level 4.

Section 7.2 Summary of quantitative findings across and within the TCK task

Task 1 (c) was scored the lowest at 17% (mean score). The scores for Task 1 (c) ranged between 0 and 3. The mean and SD of Task 1 (c) were 0.833 and 1.602 respectively, confirming that the quality of responses for this task was poor. Task 4 (a) scored the highest mean score at 67%. The mean and SD of Task 4 (a) were 2.667 and 1.211 respectively, suggesting that in general the quality of responses for this task was below average.

The rubric scores in the descriptive summary (see Table 7.1 above) provided descriptive analysis of the participating PTs' individual performance. The overall performance of the participating PTs indicates that the variation of scores was low, suggesting that the PTs had similar abilities exhibiting with a weak knowledge of circle geometry TCK. The conclusion is based on the contention that the ideal average performance for the TCK tasks should be 4 (teachers should be able to do tasks without error). The attained average is 1.667 indicating poor knowledge of the construction of diagrams with GeoGebra (Task 1 (c)) but adequate knowledge to describe a geometrical diagram constructed with GeoGebra (Task 4 (a)).

7.3 Sub-question 1: Construction of circle geometry diagrams with GeoGebra

I draw again on the notion that PTs should have the competence to visualize, construct and reason to reflect their knowledge and understanding of geometry (Duval, 1995; Gagatsis et al., 2010; Laborde, 2004). Task 1(c) particularly required a perceptual apprehension followed by a sequential apprehension in order to construct the diagram in the GeoGebra environment. See Section 5.2.1 for a deconstruction of Task 1(c).

The discussion that follows relates to the findings about Task 1(c). To address the sub-question "*What do the GeoGebra constructions reveal about the PTs' knowledge of circle geometry constructed in a GeoGebra environment?*" I examined what the PTs could or could not construct with GeoGebra.

The critical component of Task 1 (c) was the ability to reproduce a pencil-and-paper diagram using GeoGebra. The GeoGebra-based construction was expected to reflect the participating PTs' ability to transform the pen-and-pencil diagram and verbal statements from a static environment to a dynamic construction on GeoGebra. When interacting with GeoGebra, the participating PTs were expected to do the following, not necessarily in this order: (i) draw a circumscribed triangle ABC where AB=AC; (ii) draw line AS which when extended cuts line BC at M and the circle at H; (iii) draw line BE which bisects angle CBK; (iv) draw line BE which meets line AS produced at E; and (v) draw line AB which when produced is perpendicular to line EK. The PTs' constructions produced within the GeoGebra user interface were studied for the TCK evidence. As such, I examined the PTs' constructions as represented in the GeoGebra algebraic view and the graphic view. The algebraic view illustrates the text input in the construction processes whereas the graphic view provides the visual component of the construction. I also examined the participating PTs' construction protocols for the step-by-step construction processes and the screen-cast recordings. These different data sources are discussed in the subsequent sub-sections. Each section is concluded by a discussion of findings culminating from the analysis of the sub-sections.

7.3.1 Analysis of the algebraic view of Task 1 (c)

One of the affordances of GeoGebra, to both learner and researcher, is the multiple representations of an object. A GeoGebra default screen shot shows an algebraic view and graphic view (Figure 7.1). An object can be represented in algebraic form on the algebraic view window. The algebraic view contains the numeric and algebraic representations of the constructed objects presented in alphabetical order but not necessarily according to the order of construction. Table 7.2 shows a summary of the participating PTs' constructed objects as represented in the algebraic view. The expected number of constructed objects according to the model solution in the graphic view of the ideal construction were: 2 angles, 1 conic figure, 3 lines, 8 points, 2 rays and 5 segments.

Table 7.2: Summary of PTs' objects representations on the algebraic view of Task 1 (c)

PT	Number of drawn objects represented in algebraic view					
	angle E=2	conic E=1	Line E=3	point E=8	Ray E=2	segment E=5
Nkosi	1	1	1	11	2	5
John	1	1	3	8	0	5
Wisdom	0	1	5	8	0	6
Lesedi	1	1	1	14 (6)	3	7
Bonolo	1	1	3	11 (3)	2	8 (2)
Thabiso	1	1	2	9 (1)	2	7

Note: (i) number in brackets represents the number of objects deleted in the graphic view but visible in algebraic view; (ii) E is the expected number of constructed objects to be represented in the algebraic view

Generally there were variations between the number of expected objects and the actual number of objects the participating PTs constructed as seen in the algebraic view. Much variation occurred in the number of lines, points and segments presented in the participating PTs' constructions. The expected number of lines was three but the number of lines that the PTs constructed ranged between 1 and 5. The expected number of points in the construction was 8 but those of the PTs ranged between 8 and 14. However, some participating PTs (Lesedi, Bonolo and Thabiso) had 9 or more points in the graphic view but the results show indications that some points were later removed in order to meet the construction process requirements. See Figure 7.1 for a display of Lesedi's algebraic view and graphic view.

The algebraic view allows for objects to be removed from the window provided these objects are free objects that are not dependent on other objects. For instance, Lesedi's algebraic view showed that she plotted 14 points which she later trimmed to 8. There were also some variations realized in the number of segments drawn by the PTs.

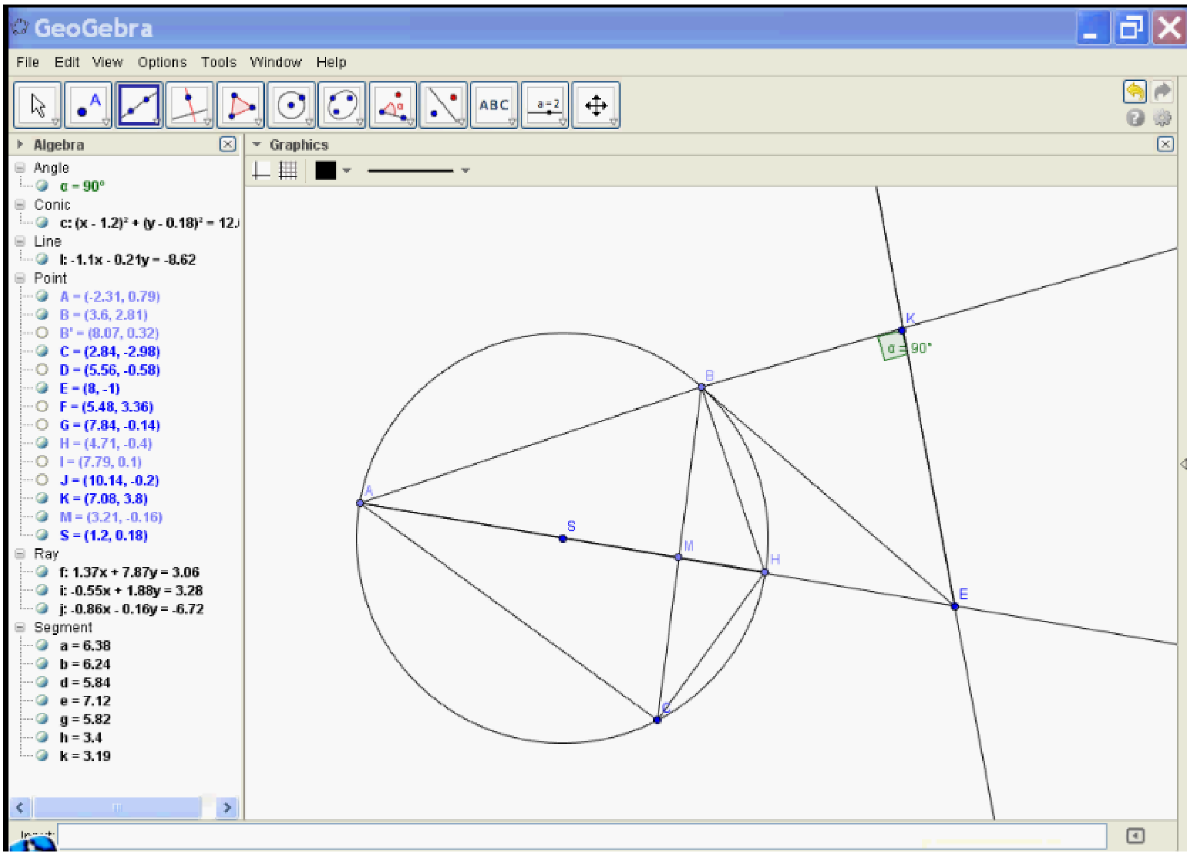


Figure 7.1: Lesedi's Task 1(c) GeoGebra construction

Task 1 (c) required a construction of two angles, acute Angle A and Angle BKE which is 90°. Table 7.2 shows that all the participating PTs except Wisdom constructed only one angle, Angle BKE. Wisdom did not construct this or any other angle. The algebraic view shows an object when it is constructed in the graphic view. Thus it could be seen that the acute Angle A was not constructed but was visible by default in all the constructions. The participating PTs did not confirm the acute angle through its measurements. My conversation with Nkosi sheds light regarding the failure to construct Angle A.

Nkosi: I didn't consider that…if you just gave me this and don't give me the description?

Kim: Yes

Nkosi: Yes, I would just look at the diagram.

Kim: you'd just look at the diagram? But suppose you'd also looked at the description, what would you have changed in your construction?

Nkosi: the accuracy.

Kim: *Because your only concern was the 90 degree*

Nkosi: *yes because that's the only one I could see from the diagram*

Kim: *suppose it was mentioned in the diagram that AB=AC, would you have made sure that they were accurate?*

Nkosi: *yes, then I was going to make sure that they were accurate.*

Since the accuracy of measurements of Angle A was not given in the diagram, the PTs concluded that this angle did not warrant due construction but could be visually recognized and classified as acute. All the participating PTs constructed one conic figure as expected but that could not be said about the lines. The task required a construction of three lines: angle bisector BE, angle bisector AE and a perpendicular to AB produced. The participating PTs produced between 1 and 5 lines which were not necessarily the required lines. For instance, as is discussed in the next section, none of the participating PTs constructed the angle bisector BE. Overall, there were two rays in the model diagram. Half of the participating PTs constructed the two expected rays whereas John and Wisdom did not construct the rays at all.

The analysis of the algebraic view strongly indicates that the participating PTs' knowledge of how circle geometry concepts may be represented with GeoGebra is weak. The GeoGebra-based constructions revealed that in general, all the expected objects were drawn but did not meet all the construction requirements. All participating PTs lacked the competence to produce a construction of a figure on GeoGebra by constructing the expected number of objects. For example, the construction output on the algebraic view showed that extra points were drawn than the required. The variations on the number of objects indicate that the participating PTs were focussed on re-producing the diagram on GeoGebra without taking into consideration the properties of the objects. I conclude that the participating PTs' TCK is weak resulting from the inability to organize or produce a construction of a figure by transforming geometric statements from a static environment to a dynamic construction employing GeoGebra as a construction tool.

7.3.2 Analysis of the graphic view of Task 1 (c)

The analysis of the algebraic view provides an overview of the number of objects constructed but does not explore how the construction requirements were met. This sections and subsequent sections give an insight into what objects and object properties were constructed by the prospective teachers.

The graphic view of the GeoGebra user interface provides a visual component of the construction or drawing. The geometric objects are displayed in this view where the objects can be drawn or created and modified using the construction tools. Table 7.3 shows a summary of objects the participating PTs constructed in order to meet the construction requirements of Task 1 (c) as explained in Section 7.3.

Table 7.3: Summary of PTs' objects constructed in the graphic view of Task 1 (c)

PT	construction requirements met					
	Circle S	Triangle ABC	Angle bisector BE	AB when produced, is perpendicular to EK	AB = AC	Points A, B, C passing dragging test
Nkosi	√	√	*X*	√	*X*	No point
John	√	√	*X*	*X*	*X*	No point
Wisdom	√	√	*X*	*X*	*X*	No point
Lesedi	√	√	*X*	√	*X*	No point
Bonolo	√	√	*X*	√	√	one point (A, B, or C)
Thabiso	√	√	*X*	√	√	any two points (A, B, or C)

Note: √ indicates construction requirement met; X indicates construction requirement not met

Generally, some construction requirements were met to produce the objects. At a glance the graphic view shows that all the PTs constructed circle S and triangle ABC. One given property of triangle ABC was that AB = AC, implying that triangle ABC was isosceles. Table 7.3 demonstrates that despite the participating PTs constructing the triangle, their construction did not satisfy this property. Only Bonolo and Thabiso constructed the two congruent sides AB and AC indicating that these PTs successfully constructed the required triangle. Although the algebraic view suggests that the participating PTs constructed at least one line, none of these lines could be classified as the angle bisector BE. However, 4 of the 6 PTs constructed the line perpendicular to AB produced. All the participating PTs struggled to construct the perpendicular to AB produced affirming that the PTs could not exploit the technical affordance of GeoGebra.

7.3.3 Analysis of construction protocols

The construction protocol of the GeoGebra user interface provides a textual representation of the order and steps of construction or drawing of geometric objects. The construction protocol was employed to analyse how the constructions and drawings were organized. This analysis provides an insight into how the constructions and drawings were sequentially apprehended. A sequence has to be followed using GeoGebra in order to make the construction. When discussing the findings for the construction protocol, Table 7.2, Table 7.3, Table 7.4 and Figure 7.2 were the point of reference. When analysing the sequences of construction, I draw on Duval's (1995) position that the order of construction depends on either the mathematical properties that are represented and/or the technical limits of the tools which are used. I considered how the specific properties that should be extracted from the static diagram were sequenced in the construction. The order of construction was corroborated with the examination of the videos of the screen recording. This strategy was essential in addressing the limitations of a construction protocol. A construction protocol does not show steps that are deleted during the construction process. The implication is that the construction protocol provides some but not complete access into understanding PTs' geometrical activity.

The model construction with a short accurate protocol and the points A, B, C passing the dragging test was 2 minutes long with 20 construction steps. The model sequence of construction depended on the geometric object properties and the GeoGebra construction tools. This sequence was ideal in that it provided a short sequence by exploiting the affordances of GeoGebra. The sequence of the objects for construction was as follows: (1) ΔABC where AB=AC; (2) Circle S; (3) AS extended cuts BC at M and circle at H; (4) BE bisects $C\hat{B}K$; (5) AB extended to K. (6) AB produced perpendicular to EK; (7) BE meets AS produced at E.

I use Nkosi as an example to illustrate the construction protocol and its constructed diagram. See Figure 7.2 for his construction protocol and Figure 7.3 for the GeoGebra construction. Figure 7.2 displays Nkosi's construction processes. The construction protocol provided a detailed account of how Nkosi sequenced the objects which are summarized in Table 7.2. The GeoGebra construction protocol provided an insight into how the constructions were sequentially apprehended.

No.	Name	Definition	Value
1	Point A		A = (3.76, 1.5)
2	Point B		B = (-0.38, 2.66)
3	Circle c	Circle through B with centre A	c: $(x - 3.76)^2 + (y - 1.5)^2 = 18.49$
4	Point C	Point on c	C = (-0.46, 2.35)
5	Point D		D = (2.4, 1.76)
6	Ray a	Ray through C, D	a: 0.59x + 2.86y = 6.43
7	Point E		E = (3.5, 3.56)
8	Ray b	Ray through C, E	b: -1.21x + 3.96y = 9.84
9	Point F	Point on b	F = (9.76, 5.48)
10	Point G		G = (11.86, -1.36)
11	Line d	Line through F, G	d: 6.84x + 2.1y = 78.28
12	Point H	Point on c	H = (6.08, -2.12)
13	Segment e	Segment [C, H]	e = 7.91
14	Point I	Intersection point of c, b	I = (6.78, 4.56)
15	Segment f	Segment [I, H]	f = 6.72
16	Point J	Intersection point of c, a	J = (7.97, 0.62)
17	Segment g	Segment [H, J]	g = 3.33
18	Segment h	Segment [J, I]	h = 4.13
19	Angle α	Angle between C, F, G	α = 90°
20	Point K	Intersection point of d, a	K = (11.48, -0.11)
21	Segment i	Segment [I, K]	i = 6.63

Figure 7.2: Nkosi's Task 1(c) construction protocol created with GeoGebra[4]

The objects as named in the construction protocol correspond with the objects in the Algebraic View as seen in Figure 7.3 and Table 7.2. The Algebraic View lists the objects in alphabetical order but not necessarily in the order in which they were constructed. Definition refers to the description of the geometric properties of object in relation to other objects. For example, in Step 3, the Circle C is defined in relation to centre A and point B.

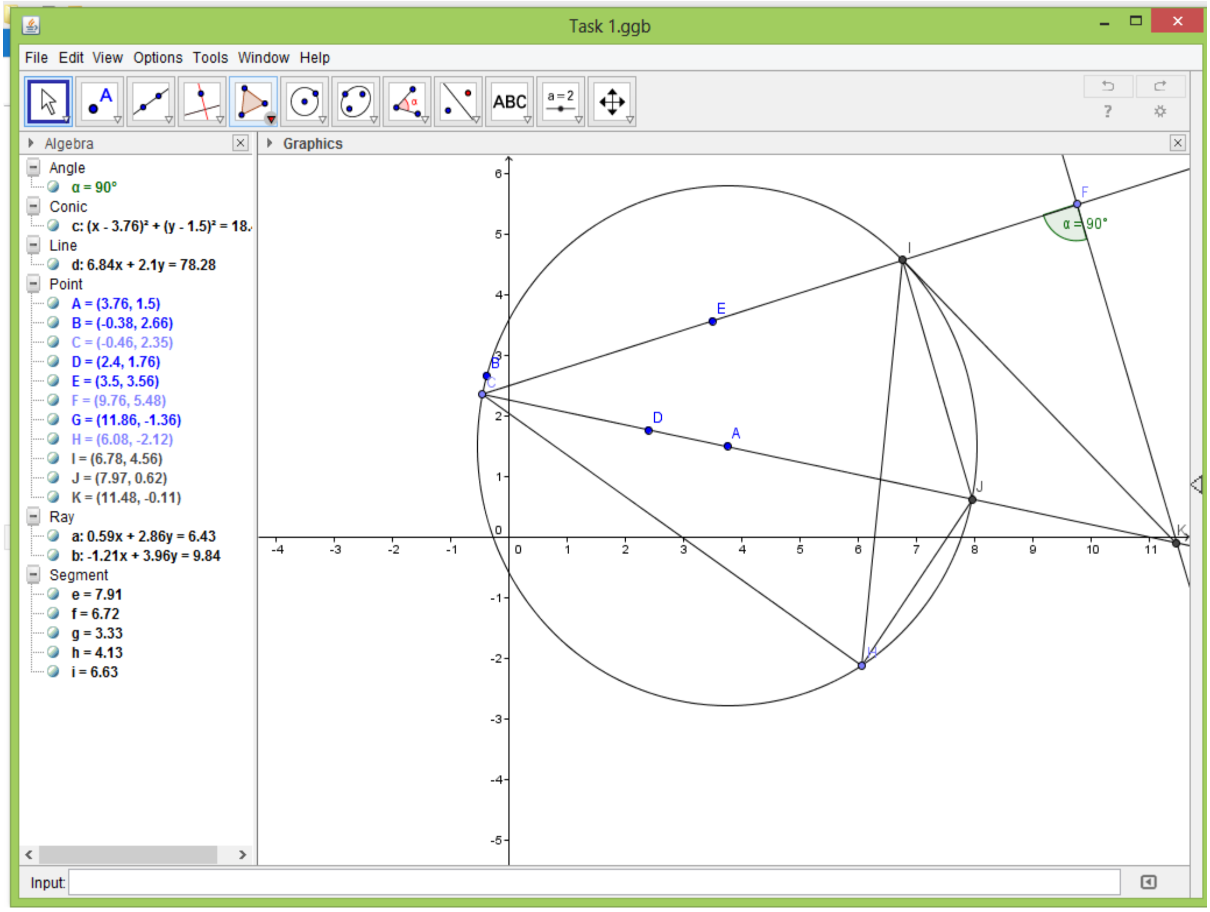


Figure 7.3: Nkosi's Task 1 (c) GeoGebra construction

Nkosi's construction protocol had 21 steps done in 4 minutes 53 seconds. He first constructs the circle with centre A through point B. I categorized this action as sequence 1(Table 7.4). In sequences 2 and 3, he draws ray *a* through C and D and ray *b* through C and E. He constructs line *d* through FD and a segment e through C and H followed by a construction of segment *f* which connects I and H and segment *g* which connects H and J (sequence 4). In sequence 5 he makes a manual construction of a 90° angle through C, F and G. Next, he constructs K which is the intersecting point of line *d* and ray *a*. He finally constructs segment *i* which connects I and K (sequence 6). Nkosi did not construct the angle bisector. He used his own labels instead of the labels given in the diagram.

Table 7.4 considers the sequence of construction, the number of construction steps and the time each PT took to construct the diagram.

Table 7.4: Summary of PTs' construction protocols for Task 1 (c)

PT	Number of construction steps (expected N=20)	Time taken to construct (expected T=02:00)	Sequencing of construction objects						
			Circle S	ΔABC	AB extended to K.	AS extended cuts BC at M and circle at H.	BE bisects $\mathbf{C\hat{B}K}$.	BE meets AS produced at E	AB produced perpendicular to EK.
Nkosi	21	04:53	1	5	3	2	-	6	4
John	20	04:51	1	3	4	2	-	6	5
Wisdom	20	21:07	1	2	4	3	-	6	5
Lesedi	27	08:56	1	2	5	3	-	4	6
Bonolo	25	11:55	1	3	6	2	-	5	4
Thabiso	22	04:29	1	3	4	2	-	6	5

Note: N is the expected number of steps; T is the expected time taken to construct; 1, 2, 3, 4, 5, 6 is the order of construction of the objects; - means object not constructed

The table shows that the number of construction steps for the five students ranged between 20 and 27. The time taken, as seen in the screen recording, to construct was not consistent with the number of steps. Wisdom produced 20 construction steps as shown in the construction protocol but took the longest time to complete the construction, indicating that he took longer to manipulate his construction. The recording included the times when the participating PT was thinking and not necessarily interacting with GeoGebra.

The sequence of constructions in Table 7.4 shows that all the participating PTs constructed the circle first and BE last. When quizzed on the reasoning about constructing the circle first, different versions were given. Nkosi explained that "*because it's easier to draw this quadrilateral because it's a cyclic*". Wisdom alluded that "*because everything is being done inside the circle"* he had to start with the circle. Lesedi lamented that;

Kim: what was your intention? Why start from the circle, why not the triangle

Lesedi: I wanted to start with the circle so that I can draw the diameter first. If I start with the triangle first without a circle I wouldn't know where my centre will be

Kim: your centre be? Oh. Ok. So your concern was the centre?

Lesedi: yes

Clearly Lesedi was thinking about the affordances and constraints of GeoGebra. She is aware that GeoGebra does not have a diameter construction tool so one needs to construct a line through the centre of the circle in order to draw a diameter. Later on she mentions that *"I dragged H down, then I had to drag to make sure that the circle pass through centre S".*

Thabiso provided an exemplary response to Task 1(c) as compared to the other PTs. The performance scores in Table 7.1 placed Thabiso's score of 3 as the highest of the PTs. Thabiso's sequence of construction was the shortest with well executed linking of GeoGebra affordances with geometric principles. Although he could not construct the angle bisector, he is the only PT that used the perpendicular line construction tool to produce a perpendicular EK.

The construction protocol was employed to analyse how the constructions and drawings were organized. According to Duval (2002) the construction processes involve actions where "geometrical configurations can be constructed according to restricted tools and mathematical properties of the represented objects" (p.232). This analysis provided an insight into how the constructions and drawings were sequentially apprehended. The sequence of construction displayed in the participating PTs' construction protocol strongly suggests limited knowledge relating to the affordances of the GeoGebra tools. A sequence has to be followed using GeoGebra in order to make the construction but none of the constructed diagrams the PTs produce met the requirements. The drawings/constructions did not pass the drag test with the perpendicular EK drawn rather than constructed. None of the PTs constructed the angle bisector, suggesting that the point E was not plotted in the correct position.

7.3.4 Analysis of screen recordings of PTs working on GeoGebra-based tasks

The PTs were screen-recorded whilst they were performing the construction tasks. Screen recording captured the actual construction process by tracking the movements of the mouse and the PTs' interaction with the GeoGebra construction tools and the GeoGebra menu. I examined the transcripts of the videos of the screen cast recordings. As mentioned in Section 7.3.3, the screen recording corroborated the construction protocol. I was determined to find a connection between the participating PTs' knowledge of geometric properties and the affordances and constraints of GeoGebra in representing these properties. As such, since the indicators for TCK were the ability to produce and describe a construction of a diagram with GeoGebra, the actions made during the construction process were employed as analytical tools for the screen recordings. A deductive approach was utilized to identify and classify the actions.

Table 7.5: PTs' actions during the construction process

PT	Number of construction actions on GeoGebra							
	Selects	draws	inputs	Drags	deletes	renames	total	Time taken to construct
Nkosi	9	9	7	5	0	0	30	04:53
John	18	10	10	0	4	8	50	04:51
Wisdom	53	25	22	22	21	0	143	21:07
Lesedi	23	11	11	5	2	7	59	08:56
Bonolo	28	14	20	0	16	1	79	11:55
Thabiso	13	14	6	0	1	7	41	04:29

Select actions

In order to construct or draw, the participating PTs had to select the tools that were appropriate for the construction/drawing of a particular object and/or make selections from the GeoGebra menu. I classified each selection as a 'select action'. Students should have selected the object based on the properties of the objects. On some occasions the PTs selected a wrong construction tool. This action was reversed by selecting the un-do icon in the GeoGebra menu. A correct construction of the diagram on GeoGebra, according to the memo, required 19 construction tools selections. Table 7.5 shows that Nkosi made the least number of selections at 9 in less than 5 minutes whereas Wisdom had the highest selections at 53 in about 21 minutes. Table 7.5 suggests that the PTs who took less time made fewer tools selections. The implications of the results from the select actions indicate that the link between the participating PTs' knowledge of geometric properties and the PTs' knowledge of affordances and constraints of GeoGebra in representing these properties was questionable.

Draw actions

The drawing action focussed on the construction/drawing of segments, lines, rays and circle in the graphic view. A correct drawing/construction of the diagram on GeoGebra required 11 drawing actions. The participating PTs' drawing actions were just about the same as the required number of drawing actions except for Wisdom who made 25 drawing actions. Worthy of mentioning is that the GeoGebra allows for a selected construction tool to be used several times on the graphic view. Therefore the selection actions do not in any way determine the number of drawing actions. The results from the draw actions imply that the participating PTs' strong perceptual apprehension of the static diagram informed their choice of action. Clearly the PTs were knowledgeable about the GeoGebra construction tools needed for the construction.

Input actions

The input action refers to actions required for plotting the points and inputting the angle in the construction. The requirements were that 8 points had to be plotted and an 90° angle to be inserted in the diagram. The participating PTs had either more inputs (John, Wisdom, Lesedi, and Bonolo) or fewer inputs (Nkosi and Thabiso). The implications of the results from the input actions indicate that there was a disconnection between participating PTs' strong perceptual apprehension of the static diagram and use of the GeoGebra construction tools to construct the dynamic diagram (sequential apprehension). More inputs sign-posted lack of technical skills or knowledge to transform geometric properties in a GeoGebra environment.

Dragging actions

GeoGebra allows for dragging of objects in the graphic view to show how these objects transform. Just like any Dynamic Geometry Environment, dragging a geometric object (e.g. point, line) in a GeoGebra interface indicates or confirms whether its properties are maintained or not. I called the drag test utilized by the participating PTs the 'drag action'. The objects were dragged to explore and check if the object maintained the geometric properties and whether the dynamic diagram had all the properties of the static diagram. The model construction required at least 1 dragging action. In the model construction, the perpendicular EK had to be dragged to intersect with angle bisector EB at point E. But Table 7.5 demonstrates that dragging occurred or did not occur at all in some constructions. John, Bonolo and Thabiso did not employ the dragging affordance of GeoGebra. Nkosi, Wisdom and Lesedi construction recordings show that dragging actions were performed. Wisdom had the most dragging actions. Although he was determined to re-produce the static diagram, Wisdom was aware of the potentials of GeoGebra for confirming that indeed the construction was correct. He asserts that;

Wisdom: *I wanted to see whether, it would, I wanted to answer this question whether when you drag it changes shape so I saw "kuti" eeh if I drag it changes, when you drag any point right it changes so I wanted to make sure that it doesn't change.*

Clearly, Wisdom wanted to confirm the relationship between the whole figure and its figural components in a GeoGebra environment. To accomplish this, the PT required a good knowledge of the properties of the diagram. He was not concerned about the time it took him to do the construction but instead explored with the drag tool to investigate the geometric objects in the GeoGebra platform. The implications of the results from the dragging actions indicate that the participating PTs' discursive apprehension of the static diagram did not inform their choice of action in sequentially apprehending the diagram. Clearly most PTs were knowledgeable about the GeoGebra construction tools needed for the construction as interpreted in the input actions but most failed to employ the dragging action to confirm the correctness of their constructed diagram.

Delete actions

The delete actions were employed to clean up the construction of extra and unwanted objects. The objects deleted were lines, angles, points, rays and segments. Wisdom had the most delete actions at 21 followed by Bonolo with 16 delete actions. Nkosi made no delete actions.

Rename actions

In order to reproduce a pencil-and-paper diagram on a GeoGebra environment the PTs had to rename the object labels as given in the static diagram. GeoGebra assigns name labels to construction objects but renaming of labels is permissible. There were 9 input actions expected to be performed. Table 7.5 shows that Nkosi and Wisdom used the GeoGebra-assigned labels and did not rename the objects.

Section 7.3 Findings: What the GeoGebra constructions revealed about the participating PTs' knowledge of circle geometry constructed in a GeoGebra environment?

Section 7.3 presented and analysed what the GeoGebra constructions revealed about the participating PTs' knowledge of circle geometry constructed in a GeoGebra environment. Sequential apprehension was employed to interpret the participating PTs' competence in constructing geometry diagrams within a GeoGebra environment. The construction produced was expected to reflect the PTs' ability to transform the statements from a static environment to a dynamic construction employing GeoGebra as a construction tool. I examined what the PTs could or could not construct with GeoGebra. Data sources were the GeoGebra algebraic view and the GeoGebra graphic view, the GeoGebra construction protocols and the screen-cast recordings.

The GeoGebra algebraic view in the participating PTs' constructions showed that, in general, all the expected objects were drawn but did not meet all the construction requirements. The GeoGebra graphic view showed what PTs constructed in order to meet the construction requirements. Generally, not all construction requirements were met to produce the objects. All the PTs constructed circle S and triangle ABC. Despite that the participating PTs constructed the triangle their constructions did not satisfy the congruent-sides property. Only two participating PTs successfully constructed the 'required' triangle.

The GeoGebra construction protocol provided an insight into how the constructions were sequentially apprehended. A short sequence demonstrated less dependence on GeoGebra, indicating technical ability in the use of GeoGebra. Dependency on GeoGebra means relying on GeoGebra's ability to specify the geometrical relationships between objects and their configurations. Hence the higher number of construction steps strongly indicated more dependence on GeoGebra. Most constructions did not pass the drag test and the perpendicular EK was constructed manually by 4 out of 6 participating PTs. This suggests a limitation on the knowledge of the affordances of the GeoGebra tools.

The screen recording captured the actual construction by tracking the movements of the mouse and the participating PTs' interaction with the GeoGebra construction tools and the GeoGebra menu. The actions or manipulations that the PTs performed in the construction process were analysed. The PTs' movements of the mouse were tracked as they selected the construction tools and the GeoGebra menu from the toolbar. The participating PTs had to select the tools that were appropriate for the construction of a particular object and/or make selections from the GeoGebra menu. Of the required 19 construction tools selections, the participating PTs made between 9 and 53 select actions. Thus the PTs who took less time to construct the diagram made fewer select actions. The PTs' drawing actions were just about the same as the required number of drawing actions except for one PT (Wisdom) who made 25 drawing actions. I noted that the select actions were not consistent with the number of draw actions. The PTs had either more inputs than the required number of inputs or fewer inputs. The objects were dragged to explore and check if they maintained the geometric properties and whether the dynamic diagram was an exact replica of the static diagram. The construction recordings show that 3 of the 6 participating PTs performed the dragging actions. These actions were mainly to confirm the relationship between the figure and its figural units in a GeoGebra environment. The PTs used the delete actions to clean up the construction of extra and unwanted objects. The delete actions ranged between 0 and 21. Since GeoGebra assigns name labels to constructed objects, the participating PTs had to rename the objects as given in the static diagram. Two participating PTs used the GeoGebra-assigned labels and did not rename the objects.

7.4 Sub-question 2: Description of a geometrical diagram constructed with GeoGebra

To gain an understanding of the participating PTs' reasoning processes about circle geometry concepts presented within a GeoGebra environment, this study was guided by the sub-question "*What types of descriptions do the PTs give about geometrical diagrams constructed with GeoGebra*?" To examine the PTs' descriptions about the construction errors in a GeoGebra-constructed diagram, I was guided by the two questions: (i) what could the PTs describe? (ii) What could the PTs not describe?

A description of the geometrical diagram constructed with GeoGebra required a discursive apprehension of the GeoGebra-constructed diagram. As mentioned in Chapter 6, perceptual apprehensions followed by discursive apprehensions are required in the reasoning process of making connections between configurations and mathematical principles. The discursive apprehension was necessary to describe the diagram through geometric language/narrative texts and statements. The statements should reflect the participating PTs' perceptual apprehensions: how they identified the configurations and the geometric properties that have been translated into geometric objects in a GeoGebra environment.

The discussion that follows relates to the findings about Task 4(a). The critical component of Task 4 (a) was the ability to describe errors in a geometrical diagram constructed with GeoGebra (see Section 5.2.4 for Task 4). The errors which the PTs identified in Jane's construction should make reference to the order of construction of P and should emphasize that, in Jane's construction, M was constructed as an arbitrary point and not dependent on O and A. In Jane's construction OM was not constructed perpendicular to AB. The expectation of the Task 4(a) was that the participating PTs should, when describing the learner errors, demonstrate; (i) knowledge of geometry properties of these geometric words (*perpendicular, vertical diameter, intersects, produced, closer to than),* (ii) knowledge of how the properties of a diagram aid in the construction of a diagram, (iii) ability and the disposition to translate statements to a figural register, (iv) knowledge of construction procedures and (v) knowledge of dragging process and its uses. Table 7.6 shows a summary of the categories of descriptions from written responses that the PTs used (even if not explicitly) to identify errors in Jane's construction.

Table 7.6: Summary of the categories of descriptions used by PTs when discussing Jane's errors

	The description given is about				
PT	geometry properties	how the properties aid in the interpretation of a GeoGebra constructed diagram	Translation of verbal to diagrammatic representations	construction procedures	Manipulating the diagram
Nkosi	√	√	√	√	√
John	√	√	√	*X*	*X*
Wisdom	√	√	√	*X*	√
Lesedi	√	√	√	√	*X*
Bonolo	√	√	√	*X*	*X*
Thabiso	√	√	√	*X*	*X*

Note: ***yes*** *means Note: √ indicates that the type of description is reflected in the PTs' response;X indicates that the type of description is not reflected in the PTs' response*

The discussion that follows is that of the five categories of descriptions used in table above.

7.4.1 Geometry properties category

The PTs' description of the errors in Jane's diagram must reveal their knowledge of the relationship between geometric properties and their representations in the GeoGebra environment. Indicators for the knowledge of geometry properties in the description must comprise of any if not all of these geometric words: *perpendicular, vertical diameter, produced, intersects, closer to than.* Table 7.6 shows that all the participating PTs made reference to the geometric properties as represented in the construction. The most common utterance showed that the PTs perceptually recognized that the angle given at the intersection of AB and OM was not 90˚. For instance, Bonolo indicated that the line *"AB is not perpendicular to line OM"*.

7.4.2 Knowledge of how the properties of a diagram aid in the construction of a diagram category

Task 4(a) required movement from the verbal description to the construction of the diagram as per these statements. As such, a connection was required between the properties and the GeoGebra constructed diagram. Generally, this category addressed the discursive apprehension. The description in this category must reflect the participating PTs' understanding of the connection made when enacting the properties in a GeoGebra environment. That is, were the verbal specification transformed to figural representations? All the PTs' descriptions displayed this connection. An assertion by Nkosi that "*The perpendicular at O could have been on the other side of the circle and never meet PA produced"* describes a link between the knowledge of properties and its spatio-graphic representation. The statement suggests possible ways of constructing the perpendicular to meet the stated requirements. Bonolo's description that "*the line AB is not perpendicular to line OM"* was not very explicit but the statement was informed by what was seen in the diagram.

7.4.3 Ability to translate statements to a diagrammatic register category

The description should reflect that the participating PT has an understanding of how the diagram was constructed using the stated construction requirements. The speech or narrative provided by the participating PT should show that the participating PT recognizes the connections that Jane made between the verbal statement and diagrammatic registers. Generally, this category addressed the discursive apprehension. The use of appropriate geometry language in the description reflects on the participating PTs' interpretation of the learners' understanding of the connection between properties and objects as represented in the GeoGebra environment.

7.4.4 Knowledge of a construction procedure category

The description in this category should reflect that the participating PT has knowledge of the procedures for constructing the diagram. The description must show that the participating PT understands what Jane did in organizing the construction on GeoGebra and the errors that were committed in the construction process. Generally, this category addressed the sequential apprehension. Only two participating PTs (Nkosi and Lesedi) referred to the procedure for construction in their descriptions. For instance, Lesedi responded that *"Jane didn't use the perpendicular construction that is why her angle was not 90°"*. She acknowledged that there should be a procedure for the construction. When pressed further for clarification of her statement, Lesedi mentions that "*The first mistake that she did was in her sequential construction of the perpendicular to .. AB .. does not… is not at O. It is close but this causes the error above. All the other constructions are correct"*.

In contrast, the description given by John, Wisdom, Bonolo and Thabiso do not give consideration for the construction procedures that Jane followed and that resulted with errors in the diagram. Although Thabiso suggests that *"It is evident also that AB is not vertical, it is skew"*, I consider this statement as speculative that is arrived at by visual inspection of the diagram. I base this purely from the fact that Thabiso did not operate on the figure in determining that AB was not vertical. As seen in the next category, Thabiso did not manipulate Jane's diagram to ascertain the correctness of the construction.

7.4.5 Ability to manipulate the diagram through dragging category

This category is used to explain how the PT operated on the diagram to ascertain that the diagram requirements were met on GeoGebra. Generally, this category addressed the operative apprehension. To operatively apprehend a diagram in a dynamic platform involves the modification of the figural units, which can be possible through the use of the dragging mode. Therefore, dragging is important for this study because it incorporates technological content knowledge and technological knowledge of the user. All the participating PTs described at least one error in the diagram but only two participating PTs (Nkosi and Wisdom) considered the use of the drag test. The drag test was used to determine the correctness of the construction. Nkosi states that *"If you drag point M around the circle then we see that the construction of Jane will not stand"*. Wisdom manipulates the diagram and concludes that *"When you use a drag mode, dragging either point A or B, the diagram changes or lose its intended angles, e.g. " The perpendicular to AB at O meets AP produced at M" constrain becomes invalid when you drag"*.

Section 7.4 Findings: What types of descriptions do the participating PTs' give about geometrical diagrams constructed with GeoGebra?

Section 7.4 presented and analysed the types of descriptions that the participating PTs' gave about geometrical diagrams constructed with GeoGebra. Inductive analysis was employed to develop the categories of the descriptions. Data sources were the written responses and the screen-cast recordings. Operative apprehensions followed by discursive apprehensions were employed to understand the participating PTs descriptions of GeoGebra constructions.

In the descriptions, all the PTs made reference to the relationships between geometric properties and their representations within the GeoGebra environment. The PTs perceptually recognized that the angle given at the intersection of AB and OM is not 90˚. All the participating PTs' descriptions displayed a connection between the verbal descriptions and the construction of the diagram. The participating PTs gave statements that strongly suggested that the descriptions were informed by what was seen in the diagram. This means that the PTs displayed knowledge of how the properties of the diagram aided them in the construction of the diagram. The geometry language used in the description revealed the participating PTs' understanding of the connection between properties and objects as represented in the GeoGebra environment. This notion demonstrates that the participating PTs had the ability to translate written statements (verbal register) to a diagrammatic register. The descriptions had to acknowledge that there should be a sequential organization of the construction, that is, a procedure for the construction. However, only two participating PTs referred to the procedure for construction in their descriptions. This indicates that most PTs did not operate on the figure to ascertain the correctness of the construction. The participating PTs relied on their perceptual apprehension rather than on operatively apprehending the learner's diagram. Only two of the six PTs gave statements that explain how they operated on the diagram to ascertain that the diagram requirements were met on GeoGebra. The use of the drag test to determine the correctness of the construction was clearly not taken into consideration.

7.5 Chapter summary

The purpose of this investigation was to explore the PTs' technological content knowledge within the context of circle geometry by probing into their thinking as displayed in their solutions to the TCK tasks. I was guided by two sub-questions in investigating 'what TCK do the PTs display?' Firstly, I examined what do the GeoGebra constructions reveal about the participating PTs' knowledge of circle geometry constructed in a GeoGebra environment. Then, I studied the type of descriptions that the PTs gave about a geometric diagram constructed with GeoGebra. The findings indicate that the participating PTs' sequential apprehension of the static diagram in a GeoGebra environment was weak. None of the GeoGebra-based constructions met the construction requirements with most PTs unable to execute a correct sequence in order to correctly construct the dynamic diagram (Task 1 (c)). In contrast, some of the participating PTs' description of GeoGebra constructions strongly indicated adequate knowledge of the connection between the geometry properties and affordances and constraints of GeoGebra (Task 4 (a)). Inductive analysis was employed to categorize the participating PTs' descriptions. Refer to Chapter 9 for further discussion of the TCK findings.

CHAPTER 8

ANALYSIS BY TPACK COMPONENT: PROSPECTIVE TEACHERS' GEOMETRY PEDAGOGICAL CONTENT KNOWLEDGE

8.0 Introduction

The previous chapter presented the data analysis and findings relating to the aspect of technological content knowledge (TCK) construct of technological pedagogical content knowledge (TPACK). The major focus of this chapter is the presentation of the results relating to the aspect of pedagogical content knowledge (PCK) construct of the technological pedagogical content knowledge (TPACK) framework in response to research question 3:

What pedagogical content knowledge do the PTs display?

The chapter begins by providing an overview of how the sub-unit of analysis (PCK construct) was conceptualized in the study and a description of the analytical framework employed to interpret the responses to the PCK tasks (Section 8.1). A discussion of the descriptive summary and the quantitative analysis of the rubrics scores of the individual participating PTs' responses to the PCK tasks and PTs' scores across each task follows (Section 8.2). Then, a comparison and discussion of the individual case PCK findings and the cross-case PCK findings is presented (Section 8.2.1). As mentioned in Chapters 6 and 7, I provide excerpts of responses by Nkosi, Wisdom and Lesedi to support the findings and only bring the examples of responses of other PTs to strengthen the arguments. Thereon, an inductive analysis of the participating PTs' PCK employing the Chick, Baker, Pham, & Cheng (2006) model for analysing the types of PCK that the participating PTs exhibit is articulated (Section 8.3). For each section, the overall impression of the results is provided in a tabular form, followed by a discussion of the overall results for all the participants. Each section is concluded by a summary of findings.

8.1 Sub-unit of analysis: PTs' geometry pedagogical content knowledge (PCK)

The sub-unit of analysis for this chapter is the PTs' circle geometry pedagogical content knowledge (PCK). The PCK construct was conceptualized in the study as the prospective teachers' knowledge about teaching circle geometry. To understand the participating PTs' PCK, I adapted the Chick, Baker, Pham, & Cheng (2006) position about the PCK construct. See Chapter 4 for the elaboration of the framework.

The circle geometry pedagogical content knowledge required for the successful completion of the PCK task (Task 2(b)) comprised three thinking processes: (i) the ability to demonstrate how pedagogy and circle geometry are intertwined, (ii) the ability to deconstruct circle geometry knowledge in a pedagogical context, and (iii) the ability to describe pedagogical knowledge in the context of circle geometry. The exploration of the PTs' PCK was done by probing the PTs thinking displayed in the written descriptions in Task 2(b). The descriptions reveal a discursive apprehension of connections between configurations and mathematical principles through narratives. Since the PCK tasks that the participating PTs responded to elicited knowledge of geometric reasoning in teacher preparation, the objective of the analysis was to establish the PTs' geometric reasoning skills in pedagogical context. To understand the PTs' reasoning processes in the descriptions within the PCK task, I was guided by the following sub-question:

What do the descriptions in Task 2(b) reveal about the type of PCK that the PTs' have?

8.2 Analysis of Rubric Scorings of the PCK task

Outlined in this section is a presentation of results of the descriptive analysis of the performance scoring for individual PTs and the performance scoring within and across this task. A general observation across the unit of analysis is that the scores ranged between 2 and 3 across the task.

8.2.1 Analysis of PCK scores for individual cases

Table 8.1 presents data of six PTs: Nkosi, John, Wisdom, Lesedi, Bonolo and Thabiso. As mentioned in Section 5.2.2 in Chapter 5, Task 2(b) elicited PCK. The mean and standard deviation are provided to interpret the individual PT's scores (see Table 8.1). As mentioned in Section 5.2 in Chapter 5 all rubric scores ranged from the poor performance (score 0) to high performance (score 4).

Table 8.1: Scoring of the PTs' responses to the PCK Task 2(b)

PT		Rubric scores /4 for Task 2b)
Nkosi		2
John		3
Wisdom		3
Lesedi		3
Bonolo		2
Thabiso		3
	mean	2.667
	SD	0.516

Table 8.1 indicates that the participating PTs scores ranged between 2 and 3. See Chapter 5 for the criteria for performance levels 2 and 3 of Task 2 (b). Four of the six PTs scored a 3. The observed absent scores were 0, 1 and 4. Nkosi and Bonolo had the lowest scores of 2 each. The criterion for a score of 2 is a response that discusses one correct instructional method and mentions one teaching strategy that is not detailed. The overall mean and standard deviation were 2.667 and 0.516 respectively, suggesting that four PTs (John, Wisdom, Lesedi and Thabiso) scored above the attained mean and two PT (Nkosi and Bonolo) scored below the attained mean. For example, Thabiso performed at level 3, which is slightly above the overall mean score of 2.667.

Section 8.2 Summary of quantitative findings across and within the PCK task

The overall performance of the participating PTs indicates that the variation of scores was low, suggesting that one could conclude that the participating PTs had similar abilities. The desired average performance for the PCK tasks should be 4 but the attained average is 2.667, signifying an adequate knowledge about teaching circle geometry. On the rubric descriptions, an average score of 2.667 indicates a slightly below adequate performance that signals a description that is not rich in details.

The rubric scores in the descriptive summary in Table 8.1 provided statistical features of the PTs' individual performance. However, an interpretation of the scores within the task provides an insight into the qualitative implication of the scores of PTs' PCK. The next section presents the PCK findings to the sub-question supported by evidence from the quantitative analysis presented above for each task.

8.3 Types of PCK that PTs exhibit

Teacher knowledge encompasses several bodies of knowledge as proposed by various researchers such as Shulman (1986), Grossman (1990), Mishra and Koehler (2006) and Ball et al. (2008). All are in agreement that there is a body of knowledge required for teaching, referred to as PCK. I employed the Chick, Baker, Pham, & Cheng (2006) model to analyse the types of PCK that the participating PTs exhibit in a hypothetical mathematics learning environment for teacher-preparation. Refer to Section 4.6 in Chapter 4 for an elaboration on how Chick et al (2006) unpacked ways in which PCK is evident in teaching. The next sections elaborate how the PCK construct was analysed qualitatively and quantitatively. The descriptions given in the responses to Task 2 (b) were used to identify the attributes of the PCK using a modified Chick et al. (2006) PCK analytic framework.

The framework was employed to identify the attributes of the PCK as displayed in the participating PTs' responses to Task 2 (b). See Section 5.2.2 in Chapter 5 and Appendix C for Task 2 description. Task 2 (b) required the PT to situate the task of providing four different proofs of the tan-chord theorem in the classroom teaching environment. The participating PTs were expected to describe the various ways to model or illustrate the theorem. Their descriptions were to demonstrate an ability to provide an explanation of the concept or the procedure for the proofs by utilizing general or specific instructional strategies for teaching the tan-chord theorem. As such the response to Task 2(b) should have all the elements of the PCK as elaborated in the framework.

A deductive approach was utilized to classify the categories. The process required establishing whether a specified sub-category was evident in the description or not. If present, it was coded as a ‘yes’; if not evident it was coded ‘no’. I deconstruct the description in Table 8.2 to illustrate how the ‘yes’ and ‘no’ are used. To illustrate how the coding was done in the ‘Clearly PCK’ category in Table 8.2, I give examples of response by Nkosi who scored at level 2 (Figure 8.1).

> *I would let the children go through the proof with me and give them a chance to try to understand it on their own*
>
> *I would also use colours to label similar angles*
>
> *For proving of equal angles, I would make sure I go through the reason carefully so that they understand better*

Figure 8.1: Nkosi’s written response to Task 2 (b)

The sub-categories labelled as ‘yes’ (in Table 8.2) mean that Nkosi uses a specific strategy for teaching the theorem. He intends to incorporate the use of colours as pedagogical tools that enable visualization of the concepts. It is through this intention that he makes connections between different representations necessary for proving the theorem. Nkosi addresses the learner knowledge of thinking about the theorem. That is, providing strategies that ensure that the learners are engaged in the understanding of the proof. By such, he describes the generic classroom practices for learning within this activity through which learners are the focus of the interaction.

Nonetheless, the ‘no’ means that there is no evidence that other sub-categories of PCK are considered. Nkosi is not explicit in discussing learner misconceptions and the cognitive demand of the task. Although he provides ways of illustrating the ‘similar angles’, neither examples to highlight and model the theorem nor situate the theorem in the curriculum are mentioned. There is no explanation to demonstrate ways in which the proof is modelled in GeoGebra or otherwise. It is not clear if Nkosi has a thorough understanding of the theorem because he does not discuss the methods of solution and reasons that he stated.

Table 8.2 presents a summary of the PCK attributes as displayed in the participating PTs' responses. The table is based on Chick et al.'s (2006) PCK framework. An elaboration of the patterns of attributes for each PCK category follows in the next sections.

Table 8.2: Summary of PCK attributes displayed in PTs' responses to Task 2 (b)

PCK category displayed	Is the PCK category evident in PT's response?					
	Nkosi	John	Wisdom	Lesedi	Bonolo	Thabiso
Clearly PCK						
Teaching Strategies general	Yes	Yes	Yes	No	No	Yes
Teaching Strategies specific	Yes	Yes	Yes	Yes	Yes	Yes
Learner Thinking	Yes	Yes	Yes	Yes	No	Yes
Learner Thinking-Misconceptions	No	No	No	No	No	No
Cognitive Demands of Task	No	No	Yes	Yes	No	Yes
Appropriate and Detailed Representations of Concepts	Yes	Yes	Yes	Yes	Yes	Yes
Explanations	No	Yes	No	No	No	No
Knowledge of Examples	No	Yes	No	No	No	No
Knowledge of Resources (GeoGebra)	No	No	Yes	No	Yes	No
Curriculum Knowledge	No	No	Yes	No	No	No
Purpose of Content Knowledge	No	No	No	No	No	No
Content Knowledge in a Pedagogical Context						
Profound Understanding of Fundamental Mathematics (PUFM)	No	Yes	Yes	Yes	No	Yes
Deconstructing Content to Key Components	No	Yes	Yes	Yes	Yes	Yes
Mathematical Structure and Connections	No	Yes	Yes	Yes	Yes	Yes
Procedural Knowledge	No	Yes	Yes	Yes	Yes	Yes
Methods of Solution	No	No	Yes	No	No	No
Pedagogical Knowledge in a Content Context						
Goals for Learning	No	No	Yes	Yes	Yes	Yes
Getting and Maintaining Learner Focus	Yes	Yes	Yes	Yes	Yes	Yes
Classroom Techniques	Yes	Yes	Yes	Yes	Yes	Yes
Integrating technology	No	No	Yes	No	Yes	No

Note: Yes means PCK sub-category evident; No means PCK sub-category not evident

8.3.1 Analysis of Clearly PCK category

This category of teacher knowledge for teaching content is directed to situations where pedagogy and content are "completely intertwined" (Chick et al., 2006:298). The clearly PCK category comprised knowledge of teaching strategies, learner thinking, cognitive demand of the task, concept representations, resources and curriculum. Reference is made to Table 8.2 to interpret the Clearly PCK category.

In general, there were more instances for the 'no' PCK code than the 'yes' PCK code in this category. This indicates that there were more elements of PCK absent in cases where pedagogy and content are completely intertwined. Nonetheless the participating PTs were explicit in exhibiting the need for teaching strategies to approach the theorem in the mathematics classroom environment. Aspects of the task were mentioned with a clear demonstration of how the proof could be modelled in teaching. Figure 8.2 shows Lesedi's response to the task that was scored at level 3.

> *I will start by reminding the learners that an angle in a semi-circle is 90°. And remind them about the theorem of tangents and the theorem that angles subtended by the same chord are equal. Then give them this activity and ask them to construct any necessary lines that will help them in answering the questions*

Figure 8.2: Lesedi's written response to Task 2 (b)

Lesedi implicitly provided a strategy for teaching the theorem. She attended to learner thinking by linking previous knowledge of theorems that were associated with the tan-chord theorem. She identified the aspect of the tasks that affected the complexity of the task. Lesedi contends that "*I wanted to remind the learners of previous theorems"*. She suggested an operative apprehension of the diagram by "*ask*ing *them to construct any necessary lines"* to simplify and modify the figure for easy accessibility and believed that "*using four different methods to prove the theorem will accommodate all learners in understanding the theorem"*.

However, this interplay strongly suggests lack of acknowledgement of other PCK facets. This is a feature which was common to all the participating PTs. Lesedi did not explicitly contemplate the use of pedagogical resources to support understanding. Whilst she required the learners to "construct", there was no mention of tools for construction. She did not provide examples to highlight the theorem to deal with misconceptions.

In general, although the participating PTs took learner thinking into consideration, learner misconceptions were not addressed. There were no explanations or examples given to highlight neither the proof of the theorem nor a mention of how the theorem fitted in the curriculum. Two of the six PTs incorporated the use of GeoGebra in their descriptions.

8.3.2 Analysis of circle geometry knowledge in a pedagogical context PCK category

The PCK category in this section focuses on knowledge of a particular content area as displayed in a pedagogical context. Chick et al. (2006) contend that the teacher must have a deep conceptual knowledge of the content and how to deconstruct its key components in a pedagogical context. Such deconstruction should reflect teacher knowledge of the content structure.

Table 8.2 demonstrates that in general there were more 'yes' than 'no' codes for the Content Knowledge in a Pedagogical Context PCK category. The participating PTs were explicit in exhibiting the CK that illustrated an understanding of the theorem in a learning environment. Wisdom's response for this category, which scored all 'yes' was considered exemplary. Below is Wisdom's response to the task which was scored at level 3.

I would first make sure that learners first understand all their terminologies first that are related to the theorem e.g. what is a tangent chord etc. Secondly learners at this stage are expected to have proved other theorems because we might correlate each to prove this one as well since we are supposed to give 4 different approaches to this problem

Figure 8.3: Wisdom's written response to Task 2 (b)

The PCK that Wisdom exhibited expressed knowledge of the tan-chord theorem organizational structures and an understanding of teaching the theorem. He suggested a strategy that considered the connection between the theorems and the need for the learners to demonstrate knowledge of the proof. This response reveals the connection and interplay between the content and pedagogy knowledge domains. Connections were made between the critical components of the theorem and other circle geometry concepts. Nonetheless, the classroom learning situation depicted, did not address the method of proving the theorem.

8.3.3 Analysis of pedagogical knowledge in the context of circle geometry PCK category

Chick et al. (2006: 298) refer to this category as the "teaching knowledge that is applied to a particular content area". It describes the pedagogical knowledge that the teacher displays when teaching circle geometry.

Table 8.2 demonstrates that in general, there were more 'yes' than 'no' codes for the sub-categories of the Pedagogical Knowledge in a Content Context PCK category. See Figure 8.2 for an excerpt of Lesedi's response which was scored at level 3.

Generally, the participating PTs were explicit in strategically focusing on the learner, learning and teaching practices. Nonetheless, the pedagogical knowledge displayed did not address use of tool in teaching circle geometry. For instance, Thabiso listed the teaching resources for the construction of the theorem but could not discuss how these tools could be incorporated in teaching. Thabiso's goal was for learners to understand the relationship between the two angles. He demonstrated how he would engage the learners in a geometry learning context, where the role of developing construction skills is appreciated.

Section 8.3 Findings: What do the descriptions in Task 2(b) reveal about the type of PCK that the PTs' have?

Section 8.3 presented and analysed what the descriptions revealed about the type of PCK that the PTs' had. In order to characterize the PTs' geometry pedagogical content knowledge (PCK), I was guided by a sub-question that deliberately established the PTs' knowledge for teaching geometry. The PCK task that the PTs responded to elicited reasoning skills. Data sources were written tasks and interviews.

Task 2 (b) was designed to test the PCK in a teacher-preparation environment. Situating the task of providing four different proofs of the tan-chord theorem in the classroom teaching environment dictated that the PT discursively apprehended the geometry content in a pedagogical environment. A discursive apprehension suggests an ability to provide statements based on connections between configurations and geometry principles, narratives, good descriptions and appropriate geometry language. These statements were to contain all the necessary facets of circle geometry PCK. An in-depth understanding of the types of PCK that PTs exhibited was intended to understand the PTs' PCK of circle geometry. A modified Chick, Baker, Pham, & Cheng (2006) framework was employed to identify the presence and absence of the PCK facets demonstrated in the PTs' responses. The next paragraphs summarize the three categories within which the PCK facets were examined.

The Clearly PCK category, where pedagogy and content should be completely intertwined revealed a lack of acknowledgement of all PCK facets. The PTs were explicit in describing the teaching strategies to approach the geometry theorem and demonstrated how the proof could be modelled in teaching. However, learner misconceptions, explanations, examples, how the theorem fitted in the curriculum and the integration of resources were not mentioned.

The Content Knowledge in a Pedagogical Context PCK category revealed that the PTs fore-grounded content by deconstructing the key components of circle geometry in a pedagogical context. The PTs were explicit in exhibiting knowledge of the tan-chord theorem and an understanding of teaching the theorem but could not produce the varied methods of proving the theorem. This was attributed to weak knowledge of CK as evidenced in Chapter 6.

The Pedagogical Knowledge in a Content Context PCK category, where PK is fore-grounded in a circle geometry context, revealed that the PTs were explicit in strategically focusing on the learner, learning and teaching practices but omitted the integration of pedagogical tools in teaching circle geometry.

8.4 Chapter summary

In this chapter, I presented an analysis of participating PTs' PCK. The purpose of this analysis was to explore the participating PTs' pedagogical content knowledge of circle geometry by probing into their thinking as displayed in their solutions to the PCK task. I was guided by a subsidiary question to investigate "what PCK do the PTs' display?' I inspected and classified the PCK that the participating PTs exhibited in a hypothetical mathematics learning environment for teacher-preparation using the Chick et al. (2006) PCK framework. A discussion of the participating PT's CK, TCK and PCK findings follows in the next chapter.

CHAPTER 9

SUMMARY OF FINDINGS AND CONCLUSIONS

9.0 Introduction

The previous four chapters presented the results and findings relating to the participating PTs' content knowledge (CK), technological content knowledge (TCK) and the pedagogical content knowledge (PCK) constructs of technological pedagogical content knowledge (TPACK). In this chapter, I summarise the findings in relation to the research questions and give an interpretation of these findings in relation to the reviewed literature. Secondly, in the conclusion, I consider the focus of the study, its contributions to the broader field of mathematics education, in connection with the existing literature. Finally, I present the limitations of the study and then suggest areas for further research.

9.1 Research question 1: What geometry content knowledge do the PTs display?

In order to characterize the PTs' geometry content knowledge (CK), I was guided by two sub-questions

1. What do the participating PTs identify and recognize in the perceived figures?
2. What type of connections do PTs make between geometry representations, properties and theorems?

The overall performance of the participating PTs on the CK tasks indicates that the variation of scores was low, with performance scores clustered around performance levels 2 and 3. However, performance levels 2 or less were more frequent indicating that the participating PTs had similar abilities with partial knowledge of circle geometry. The observed and expected patterns from the two sub-questions stated above validate that indeed the participating PTs' geometry CK was poor.

The participating PTs' perceptual apprehension was considered weak. The results have shown that PTs' could not recognize and discriminate all the figures and subfigures through mental modification of the diagram. The participating PTs showed weak competence in linking figures with figural units. Further the participating PTs' perceptual apprehension had an impact on the discursive apprehension.

The forms of prospective teacher preparation-based mathematical connections that PTs made between geometric representations, properties and theorems were generally weak. The visual connections made between symbolic, verbal and figure(s) representations were strong, indicating strong coordination between their verbal registers and figural registers. However, connections between the different representations and the properties and theorems were weak for the reason that the systematic organization connections, implication connections and theorem application connections were all weak. The participating PTs established logical relationships between representations, properties and theorems in some tasks but the systematic organization of geometric language to describe the properties and theorems was generally weak. The PTs' ability to connect configuration(s) with circle geometric principles was considered weak. Drawing from Duval's explication of discursive apprehension, the PTs' "ability to provide good description, explanation, argumentation, deduction, use of symbols, reasoning depending on statements made on perceptual apprehension, and the ability to describe figures through geometric language/narrative texts" was weak.

In this study, the analysis for CK was focussed on the perceptual apprehension and discursive apprehension with the operative apprehension back-grounded deliberately. Duval (2004) contends that in order to analyse any form of visualization "the existence of several registers of representation provides specific ways to process each register". This finding confirms Koedinger and Anderson's (1990) contention that strong competency in geometry can be recognized by the ability to use diagrammatic configurations to infer appropriate geometry knowledge in problem solving. The participating PTs displayed lack of what Duval (2004) and Gagatsis et al. (2010)

refer to as competence in apprehending the geometric figures perceptually. A weak perceptual apprehension has a strong link with a weak discursive apprehension. Duval (1995, 2004, 2006) contends that the synergy of processes of visualization, reasoning and construction are essential for proficiency in geometry. Hence, the results show that the participating PTs' encountered difficulty merging the cognitive processes of the visualization and the reasoning when responding to the CK tasks.

9.2 Research question 2: What technological content knowledge do the PTs display?

In order to characterize the participating PTs' geometry technological content knowledge (TCK), I was guided by two sub-questions that deliberately established the PTs' knowledge of how GeoGebra and circle geometry influence and/or constrain one another and how knowledge of circle geometry can be effected by the use of GeoGebra. The TCK tasks that the participating PTs responded to elicited PTs' GeoGebra construction skills and geometric discursive skills.

1. What do the GeoGebra constructions reveal about the PTs' knowledge of circle geometry constructed in a GeoGebra environment?
2. What types of descriptions do the PTs give about geometrical diagrams constructed with GeoGebra?

The quality of responses for the construction task was poor, with most scores at performance level 0. The quality of responses for the description task was adequate, with scores ranging between performance level 1 and 4. The participating PTs' performance on the construction task was faulty in terms of their ability to organize the construction of the diagram with GeoGebra but adequate when describing a GeoGebra-constructed geometric diagram. The observed and expected patterns from the two sub-questions stated above validate that indeed the PTs' geometry TCK was considered to be below average. The participating PTs' TCK is conceptualized in this study as the knowledge of circle geometry in the context of a GeoGebra environment. The interplay between the two knowledge domains required that when determining the quality of the response, I considered if the PTs' incorrect figure was due to weak geometry knowledge or to weak knowledge of GeoGebra or both.

The constructions revealed that the participating PTs identified and extracted from the static figure the objects to be constructed. The required objects were not all constructed as expected implying that not all constructions requirements were met. The construction protocol showed how the constructions were sequentially apprehended. There was little dependence on GeoGebra to organize the construction. Such lack of dependency indicates that the participating PTs did not utilize the affordances and constraints of GeoGebra when making connections between the construction and geometric principles. The participating PTs' had technical constraints and not geometrical constraints. They knew the properties and could identify the figures and figural units as evidenced in their responses to the CK tasks and in their discussions about the TCK tasks. But they did not have the technical knowhow to construct the diagram in GeoGebra.

In the descriptions, all the participating PTs made reference to the relationship between geometric properties and their representations in the GeoGebra environment. Their statements strongly suggested that the descriptions were informed by what was seen (perceptual) in the diagram. However, the descriptions strongly indicated that the PTs did not operate on the figure through, for example, dragging to ascertain the correctness of the construction. The failure to do this confirms a lack of awareness of GeoGebra technical affordances and constraints. Some participating PTs did not advantageously employ features of the technological tool, which Artigue, (2007) and Noss, (2001) propose invite the user to undertake an action such as dragging upon it. For example, the input actions indicated that there was a dis-connection between participating PTs' strong perceptual apprehension of the static diagram and use of the GeoGebra construction tools to construct the dynamic diagram (sequential apprehension). A third of the participating PTs incorporated the use of GeoGebra in their descriptions, which was no surprise since the TCK displayed was weak. This confirms that the findings by Harris and Hofer (2011) and Harris et al. (2009) that teachers focus on content-based pedagogy rather than on affordances and constraints of the technology.

Duval's four cognitive apprehensions of geometric reasoning were required in the construction and description of GeoGebra-based diagrams. The participating PTs' inability to produce a GeoGebra construction of a pen-and-pencil diagram indicates a weak sequential apprehension. The participating PTs' descriptions of GeoGebra-constructed figures indicate a weak operative apprehension that impacted on the discursive apprehension. As stated above, the PTs did not operatively apprehend the figure to inform the descriptions. According to Mogetta, Oliviero and Jones (1999: 99) "undertaking the construction involves making explicit the starting points and the relationships between them". Hence, the results show that the participating PTs' encountered difficulty merging the cognitive processes of the construction and the reasoning when responding to the TCK tasks.

9.3 Research question 3: What pedagogical content knowledge do the PTs display?

In order to characterize the PTs' geometry pedagogical content knowledge (PCK), I was guided by a sub-question;

What do the descriptions reveal about the type of PCK that the PTs' have?

The sub-question deliberately established the participating PTs' knowledge for teaching geometry. The PCK task that the PTs responded to elicited reasoning skills. Data sources were written tasks and interviews.

The overall performance of the participating PTs on the PCK task indicates that the variation of scores was low, with performance scores clustered around performance levels 2 and 3. As such the PTs were considered to have similar abilities with a below adequate knowledge of circle geometry pedagogical content knowledge. The observed and expected patterns from the sub-question stated above, validate that indeed the participating PTs' geometry PCK was weak.

As stated earlier, the circle geometry pedagogical content knowledge required for the successful completion of the PCK task (Task 2(b)) comprised three thinking processes: (i) the ability to demonstrate how pedagogy and circle geometry are intertwined, (ii) the ability to deconstruct circle geometry knowledge in a pedagogical context, and (iii) the ability to describe pedagogical knowledge in the context of circle geometry. The participating PTs demonstrated that they had difficulty blending the content with pedagogy. For example, the participating PTs were able to 'deconstruct the content to key components' but failed to acknowledge the purpose of the content and give explanations about learning the theorem. Furthermore, the responses on the purpose of content were in contrast to those of teaching strategies. The indication is that to PTs the teaching strategies are of importance rather than the purpose of the content that they teach.

9.4 Main research question: What characterizes aspects of prospective teachers' circle geometry technological pedagogical content knowledge constructed in a GeoGebra-based environment?

For over a decade, researchers have contended that teacher-preparation programmes have an influence on teacher use of technology in practice. For example, Angeli (2005) and Crompton (2015) contend that preparing PTs to become technology efficient and competent is difficult but necessitates providing them with ample opportunities at teacher education. This study served the purpose of understanding prospective teachers' cognition of circle geometry technological pedagogical content knowledge. This understanding was done through the use of written tasks and GeoGebra-based tasks to measure PTs' levels of geometry competency. That is, tasks were used as tools to describe PTs' TPACK. Of particular interest was the TPACK constructs of CK, TCK and PCK. In examining these constructs or aspects of TPACK, the ultimate objective of the study was to determine participating PTs' knowledge of geometry, knowledge of geometry in a GeoGebra-based environment, knowledge of GeoGebra for teaching and the knowledge to responds to learners' issues related to geometry when using GeoGebra.

Although the three previous chapters (Chapters 6, 7, and 8) illustrated how each construct was examined, the goal was to ultimately synthesize the results to determine how the PTs' TPACK could be characterized. Koehler and Mishra (2009) stress that to understand TPACK, one should view the three knowledge domains not in isolation but as interrelated. In agreement to Koehler and Mishra (2009), Crompton (2015: 242) recommends that TPACK "involves a number of variables, independent of each other and contextually bound, that need to be brought together in order to be effective". As elaborated in Chapter 3, the constructs were explored individually but were to be drawn together to reveal the PTs' TPACK.

CK and TCK

My interest in this study was to examine how the participating PTs' CK, which is purported to be weak, is evident within the TCK and PCK constructs. The study has shown that weak geometry CK emanated from participating PTs' display of weak cognitive apprehensions and geometry reasoning processes. The study has shown that participating PTs with weak circle geometry CK had difficulty in perceptually and discursively apprehending diagrams, figural properties and theorems in both the static and dynamic spaces. This study has confirmed Duval's (2012) claim that there is a link between what is seen and what is uttered about that which is seen. Chapter 6 provides evidence that the participating PTs encountered difficulty when giving visual explanations of what was perceived and difficulty in seeing figures as a configuration of single entities. This finding concurs with the claims by Duval (2011) and Michael – Chrysanthou, and Gagatsis (2013) that perception can be an obstacle when shifting from configurations and re-configurations. Clearly, the evidence indicates that if PTs had difficulty mentally modifying the figure (operative apprehension) then this was likely to impact on their ability to organize figures in both the static and dynamic spaces (sequential apprehension). Chapter 7 provides evidence that a weak CK contributed to a weak TCK in terms of merging the geometry principles with affordances and constraints of GeoGebra to produce diagrams in a dynamic space. The findings conforms to Drijvers and Gravemeijer (2005) argument that users who have technical difficulties are more likely to have no grounded mathematical conceptual background.

Duval (1995) theory on cognitive processes was useful for this study because it shed light into the connections that PTs made between registers in different spaces. Fusing the cognitive processes of constructions and reasoning shed light into the participating PTs' cognition of the role of GeoGebra in learning and teaching geometry. There was evidence of the pragmatic and epistemic roles of GeoGebra when the participating PTs used GeoGebra to make connections both in the static and dynamic spaces. The use of GeoGebra to produce a diagram was weak, indicating a weak pragmatic view of the use of the technological tool. On the other hand, the display of the epistemic role of GeoGebra was adequate. This conclusion was supported by the participating PTs' minimal use of the drag mode to check or verify the conjectures about the properties of the configurations. This finding confirms Hölzl (2001) claims that students mostly utilize the drag mode to modify the appearance of the construction than to make heuristic explorations of such constructions.

CK and PCK

As previously mentioned, the focus of the study was on examining prospective teachers' knowledge of geometric reasoning in teacher preparation with the hope of establishing the participating PTs' geometric reasoning skills in pedagogical contexts. This knowledge was defined in this study as PTs' circle geometry PCK. In examining the participating PTs' PCK, I employed the Chick et al. (2006) framework to characterize the PTs PCK. As elaborated in Section 5.4, this framework was developed, drawing from Shulman's (1985:47) contention that "to be a teacher requires extensive and highly organized bodies of knowledge". These bodies of knowledge are the knowledge of content and pedagogy. Drawing from Rollnick et al. (2008: 1365) definition of PCK as that "knowledge that teachers create by transforming their content into a teachable form", this study confirmed the manifestation of CK within PCK.

Using Chick et al. (2006) PCK framework reveals that PTs' CK had influence on their PCK, reiterating findings by Baumert et al. (2010) which show that there is a correlation between content knowledge (CK) and pedagogical content knowledge (PCK). The patterns of attributes displayed by the participating PTs in each PCK category demonstrated weak knowledge needed to teach circle geometry emanating from their weak CK. The participating PTs' descriptions of the hypothetical learning situation revealed a weak discursive apprehension of connections between configurations and mathematical principles in the pedagogical context. The participating PTs identified specific teaching strategies without considering learner misconceptions. Due to weak content knowledge, the PTs could not produce examples to relate different methods of solutions, suggesting that there was lack of consideration of knowledge of the learner and knowledge of content representation. The knowledge of learner and knowledge of content representation are regarded as major components of PCK.

The link between the weak CK and PCK was revealed when the participating PTs could not provide explanations or examples to highlight either the proof of the theorem or a mention of how the theorem fitted in the curriculum. Gal (2005) highlights that teachers with pedagogical skills for visual perceptions are well equipped to provide a variety of strategies to deal with geometrical objects. I presume that due to the PTs' lack of the ability to provide different proofs for the tan-chord theorem, they were less likely to provide explanations about the theorem.

9.5 Conclusion

In this concluding section, I reiterate the focus of the study, state the themes emanating from the findings and elaborate on their contribution to current thinking. The discussion of the study contributions and recommendations for further research culminates with limitations of this study.

9.5.1 The focus of the study

The argument for this study was that integrating technology in teaching mathematics necessitates that teachers (through the use of tasks) experience specific mathematics content areas in relation to specific technological tools, particularly at teacher-preparation level. My position is supported by Özgün-Koca, Meagher and Edwards (2010:19) who propose that;

"Using advanced technologies in methods classes puts pre-service teachers in the position of being learners. This allows them to pay explicit attention to developing their TCK, which in turn encourages them to reflect on their PCK and CK".

Chapter 1 brought into light the complexities of mathematics teacher education in South Africa. The prominent complexity is that the PTs in South Africa are not only learning technology and the teaching of mathematics but they are also re-learning mathematics and sometimes learning geometry for the first time. This study has confirmed studies by van der Sandt (2007) and van der Sandt (2008) about the inadequacy of content knowledge of mathematics teachers in South Africa. Henceforth, the thesis for this study is that

Prospective teachers' circle geometry technological pedagogical content knowledge constructed in a GeoGebra-based environment is characterized as weak. This is a result of weak geometry content knowledge (CK), weak technological content knowledge (TCK) and weak pedagogical content knowledge (PCK).

9.5.2 Study contributions

Research on the TPACK is relatively young as compared to other bodies of knowledge constructs in the teaching profession. This study made contributions to knowledge, methodology and theory of understanding the mathematics teacher professional knowledge at teacher preparation level, which is argued to be fluid.

Methodological contributions

In this section I offer my reflections on my role as a researcher and teacher educator to share the methodological contribution offered by this study. In my quest to understand technology integration in mathematics learning, I set out to examine PTs re-learning of mathematics and learning to teach mathematics with technology, specifically using GeoGebra software. Of utmost importance was to examine what knowledge the PTs should have to be able to teach geometry with technology-rich environments like GeoGebra. There is a plethora of research on the development of TPACK of practicing teachers with most of these studies focusing on measuring the development of TPACK. The common tools for measuring TPACK in these studies are observations, interviews, questionnaires and pre/post-tests (Angeli & Valanides, 2009; Koehler & Mishra, 2005; Koehler & Mishra, 2009; Schmidt & Shin, 2009). I broke ground and chose a different route. I studied and characterized PTs' TPACK by focusing on their thinking processes when solving geometry tasks that elicited the TPACK construct of interest. The output and thinking processes were captured in written tasks and GeoGebra-based tasks. Three contributions made were; (i) developing tasks that elicited the TPACK constructs (Chapter 5), (ii) using screen-cast recording of PTs' thinking processes as they interacted with GeoGebra (Chapter 6), and (iii) developing rubrics that were employed as tools for the analysis of the TPACK constructs (Chapter 4). All this suggests that tasks were the determining factor for this study. Learning mathematics for understanding requires teachers to have some knowledge of the epistemological, cognitive and instructional aspects of school mathematics.

I developed a model to classify and describe forms of mathematics connections in geometry knowledge at teacher preparation. See Table 9.1. These connections were linked to the geometry cognitive processes.

Table 9.1: categories for types of connections

Cognitive processes	Forms of connections	Descriptions of the forms of connections
Visualization/reasoning	Visual connections	connections made through use of different representations of geometrical objects
Visualization/reasoning	Systematic organization connections	connections made through the structure of geometric properties
Reasoning	Implication connections	connections made through logical reasoning with geometric properties and theorems
Visualization/reasoning	Theorem application connections	connections made through the application of theorem(s) to make conjectures when dealing with specific circle geometry problems.

In comparison with Businskas' (2008) study where teachers were interviewed and were very general in talking about mathematical connections, in my study PTs were given tasks to work on and their thought processes were examined for mathematical connections. The use of tasks contributed to knowledge on how to deal with tasks as tools for research. Refer to the modification of tasks in Chapter 3 (Figure 3.3). The explicitness of tasks is essential in ensuring that the item description measured what was intended. Content validity is crucial when tasks are used as research instruments.

Contributions to practice

What knowledge does this study bring to mathematics teacher education? There were three contributions realized: (i) contribution to mathematics methodology course design, (ii) contribution to mathematics connections focusing on geometry, and (iii) contribution to the study of TPACK at teacher preparation.

The use of tasks and rubrics shed light into how tasks for mathematics methodology courses should be designed. The results have demonstrated that weak PTs' knowledge emanates from gaps in content knowledge. The aspects of TPACK constructs examined in this study strongly indicate that teacher education programmes should put in place structures that deliberately endeavour to develop PTs that are capable of integrating technology in the teaching and learning of geometry. The programmes should address the three critical knowledge domains (CK, TCK and PCK). Teacher education needs to understand prospective teachers' thought processes in order to provide them with a meaningful education.

Learning environments should consider task design as it is premised to influence learner activities (Ainley & Pratt, 2002). Research has demonstrated that activities that utilize technology have the potential to influence the acquisition of techniques for solution to tasks and a better comprehension of mathematics content (Guin & Trouche, 1999; Hoyles, 2001; Lagrange, 1999). Hence the crucial contribution of the use of tasks in this study.

Classroom activities should thus address these components (Bartolini Bussi & Maschietto, 2008). Mathematics tasks are what learners are asked to do to initiate an activity (Mason & Johnston-Wilder, 2006), the purpose of which is to stimulate thinking and reasoning. I consider tasks as the backbone of a mathematics activity as they determine the success and failure of realizing the objectives of the activity.

Literature reveals that different types of mathematical tasks prompt different kinds of activities; indeed the design of activities and the choice of tool to be used in these activities are significant in mathematics learning (Horoks & Robert, 2007; Hoyles, 2001). Doyle (1983, p. 161) argues that "tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information". Drawing from Stein, Grover, and Henningsen (1996) suggestion that teachers select and set up the kinds of tasks that reformers agree should lead to the development of students' thinking capacities, this study applied this notion to PTs.

Solving tasks is embedded within the interaction of mathematical meaning systems of symbolism, visual display and language. Geometry is one such area where these meaning systems are illuminated and provides an opportunity for teacher concept explanations, symbolism and visual display of geometry figures.

Mathematics tasks and rubrics are also used as tools in research in Mathematics Education. It is evident that the design, analysis and empirical testing of mathematical tasks whether for the purposes of research or teaching is considered essential in mathematics teaching and learning. Sierpinska (2004) analysed research reports from studies on Mathematics Education and revealed that 85% of these studies used mathematics tasks as tools for their research, an indication of the crucial role that tasks play in research. However, Sierpinska (2004) cautions that researchers should substantiate a rationale for task selection if mathematics tasks are regarded as tools of research on a par with other research methodological tools. The use of rubrics to understand and measure PT knowledge was considered as a contribution to knowledge.

Theoretical contributions

The TPACK framework was extremely useful because it shed insight into the difficult problem of advancing technology in a teacher learning context that is shrouded by inadequate knowledge of mathematics content. Koehler & Mishra (2009: 9) elaborate that TPACK professional knowledge is about (i) "requiring an understanding of the representation of concepts using technologies; (ii) pedagogical techniques that use technologies in constructive ways to teach content; (iii) knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face" . The theory also contributed on the basis of understanding teacher technology integration which Mishra and Koehler (2006) argue has the potential to promote the development TCK and PCK.

The study adapted Duval's analytical framework by extending it to include an analysis of knowledge (TPACK) in teacher preparation. See the analytical framework in Chapter 4. The rationale for adapting the Duval's (1995) analytical framework was that characterising the different apprehensions may help in analysing PTs' responses to geometry problems. The two frameworks were used as lenses for deconstructing the tasks as a precursor to developing analytical rubrics for scoring the PTs' responses to the tasks. The TPACK theory contributes towards understanding South African mathematics teacher behaviour in technological learning environments and in teacher preparation to teach in technological environments.

Contribution to literature on mathematics teacher education in South Africa

This study contributes to the current debates on teacher professional knowledge and an understanding of frameworks for which teacher knowledge can be premised in South Africa. Scholars have pointed the shortcomings of the TPACK framework and continue to develop new frameworks that militate against the complex nature of teacher knowledge. Mishra and Koehler (2006: 1047) "believe that any framework, however impoverished, is better than no framework at all". Graham (2011: 1958) concludes that with reference to the TPACK framework " theoretical work has not been adequately articulated" whilst Angeli & Valanides (2009), Archambault and Barnett (2010) and Graham (2011) suggest a closer inspection of the 'fuzzy' boundaries of TPACK constructs.

The complexities of South Africa's prospective teachers' geometry content and pedagogical content knowledge are articulated in Chapter 1. Various researchers have advanced the acknowledgement of how the vacuum created by lack of geometry knowledge in the curriculum has marginalised South African students from the development of advanced understanding of mathematics (Padayachee et al. 2011; Jansen & Dardagan, 2014). Notwithstanding this, scanty research exists that explores South Africa's prospective teachers' technological pedagogical content knowledge in the area of geometry. It was the intention of this study to focus on both the PT as a learner of geometry and the PT as a teacher of geometry in order to make a contribution to the development of mathematics teacher education programmes. This study premised that teachers' perspectives on teaching and learning mathematics in technology-rich environments should be illuminated and explored at teacher preparation level, hence building on the works of Stols and Kriek (2011). Stols and Kriek (2011) examined South African teachers' use of dynamic geometry software in high school classrooms, found that teachers' behaviour towards dynamic geometry is influenced by the perceived usefulness of technology in the classroom.

9.6 Limitations of the study

In Chapter 3, I categorically emphasized that the objective of this study was not to measure the performance of the PTs' knowledge of circle geometry but to characterize their TPACK. The critical interest of this study was to examine how the participating PTs' content knowledge which is purported to be weak manifests within the TPACK constructs. Hence the study was limited to certain aspects of TPACK, which are, CK, TCK and PCK.

The claims made in this case study assumed that the research instruments were valid and reliable. The main tools for the study were testing what they were intended to test and inferences about the participating PTs' TPACK performance scores were considered valid for the PTs that participated in this study and the particular tasks. Only a few very specific tasks were used in this study. More tasks and different tasks may well have elicited different types and levels of CK.

This study was limited to a mathematics methodology course offered at a university in Gauteng, South Africa. The non-probability sampling technique (convenience sampling) was preferred because it allowed me to study the population that was easily accessible (students in my course). I acknowledge the bias linked to the convenience sampling technique such as under-representation or over-representation of the population. The selection of the six primary participants of this study was through their willingness and interest to participate in the study that provided a platform to reflect on their interest in technology and in learning geometry. The choice of participants was to address the bias in convenience sampling that delimits the ability to make generalisations from the sample to the population of study. Therefore the findings of this exploratory case study are restricted to a small sample, and cannot be generalized to the overall population (Yin, 2003).

9.7 Recommendations for further research

Avenues for further study are suggested as follows:

(i) As mentioned in Section 4.1, this study is premised on the claims that PTs' geometry knowledge is developed through the interactions between content, pedagogy and technology knowledge. The results suggest that prospective teachers' circle geometry technological pedagogical content knowledge constructed in a GeoGebra-based environment is characterized as weak emanating from weak geometry content knowledge (CK), weak technological content knowledge (TCK) and weak pedagogical content knowledge (PCK). Further research is needed to confirm this finding. The direction of the research can be maintained but with a larger sample size that has different attributes.

(ii) This study recommends that teacher perspectives on teaching and learning mathematics in technology-rich environments should be explored at teacher preparation level. Further research is needed that explores South Africa's prospective teachers' technological pedagogical content knowledge in all domains of school mathematics. Further studies are recommended that can address what is considered effective TPACK in teacher preparation.

(iii) Another route recommended for further studies is to explore how the TPACK theory can contribute towards understanding South African mathematics teacher behaviour in technological learning environments and how teacher education programmes can prepare prospective teachers to teach in technological environments. The productive use of technology in teacher preparation programmes in South African is under-researched. Dynamic geometry can provoke curricular change at teacher preparation.

(iv) This study was delimited to the geometry content area. What are the possible challenges and limitations of extending the exploration to other domains of mathematics employing similar design? Further exploration in other mathematics areas such as functions, trigonometry and calculus in needed. A study that focuses on, say learning functions with GeoGebra, can highlight complexities that are peculiar to the learning and teaching of functions.

This study set out to investigate PTs' knowledge within the context of school geometry content and pedagogical tasks developed in a GeoGebra-based environment. The undertaking was achieved and in the process several questions for future research were raised in relation to the use of tasks as research instruments. What are challenges of task design in the technological environment and paper and pencil environment in teacher preparation? What are the challenges of developing tasks that develop TPACK at teacher preparation? How does GeoGebra as a DGE in learning geometry provide opportunities for engaging in cognitive reasoning? What types of tasks developed in a technology-based environment are appropriate for prospective teachers with weak content knowledge? I remain asking these questions as I conclude this study.

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APPENDICES

APPENDIX A: PARTICIPANT INFORMATION SHEET

Research Title: Prospective mathematics teachers' technological pedagogical content knowledge of geometry in a GeoGebra-based environment

You are being invited to take part in my PhD research study. The study is located within the EDUC 2198: Secondary Mathematics Methodology course, which is one of the courses you have registered for. Before you decide, it is important for you to understand why the research is being done and what it will involve. Note that participation in the study is voluntary. Please take time to read the following information carefully and discuss it with others if you wish. Please ask if there is anything that is not clear or if you would like more information. Take time to decide whether or not you wish to take part. Please contact me if you are interested.

Thank you for reading this.

In my endeavour to pursue my interest in technology integration in mathematics learning I want to examine the development of PTs' technology pedagogical content knowledge by exploiting school geometry tasks developed in a GeoGebra-based environment. GeoGebra, like any dynamic mathematics software is preferred because of its roles in enhancing mathematics teaching, providing a foundation for deductive and inductive reasoning and enabling opportunities for creative thinking. As mediators of mathematics learning PTs should experience technology first if they are to incorporate it into the classroom mathematics. Through the manipulation of GeoGebra, PTs will be provided with the experience of learning mathematics with technology to understand the value of a dynamic environment like GeoGebra.

In order to do this research I need volunteer participants to complete written tasks and GeoGebra-based tasks. I will screen–record participants working on the GeoGebra-based tasks in my computer, which has the screen-cast software for screen recordings. I invite you to participate in this research project by agreeing to be screen–recorded while doing the tasks. I would like all sorts of students to participate in this project. I will also want to interview the participants about their experiences on working on all the geometry tasks. These interviews will take place outside lecture time. I also want to be able to use data from the screen-recording to study how the relationship between the student and GeoGebra evolves when solving geometry tasks.

I hope to publish my research in national and international journals and to present my work at national and international conferences. You will never be identified in any presentation; indeed if you do participate, I will use pseudonyms for you throughout the data collection, analysis process and presentation of research.

Please do not feel obliged to participate – there is no penalty or negative consequence if you do not. If you do want to withdraw from the research at any point, you may do so without penalty. I guarantee you that any data that I collect will have no bearing whatsoever on your grades for any course modules in your B.Ed programme, nor on my attitude to you, nor any other aspect of your life at the University of the Witwatersrand now or in the future. Indeed when data is transcribed from the audiotapes, written assignments and screen– recordings, I will use pseudonyms for you.

If you do accept the invitation to participate

- I will audiotape and screen–record you working on GeoGebra-based tasks.
- I will use data from your written tasks.
- I will interview you about your experiences on working on the geometry tasks.

Thank you.

APPENDIX B: METHODOLOGY COURSE OUTLINE

WITS SCHOOL OF EDUCATION
DIVISION OF MATHEMATICS EDUCATION
BACHELOR OF EDUCATION 2014

COURSE	EDUC 2198 SECONDARY METHODOLOGY: MATHEMATICS 2
STUDENT GROUP	B Ed. (Senior Phase and FET)
COURSE COORDINATOR	Kim Ramatlapana
CONTACT PERIODS	Monday 5; Friday 4 and 5
WEBSITE	https://cle.wits.ac.za

COURSE OUTLINE
The aim of this course is to build on your introductory experiences of teaching and learning mathematics. We will focus on current trends in teaching and learning maths, both locally and internationally. We will continue to challenge our own mathematical thinking at all times. This means there will be lots of mathematics in this course. The course will deal with practical and theoretical aspects of mathematics, and the teaching and learning of mathematics. Particular attention will be given to the South African context and to the climate of change that pervades education in the country at present. The mathematical content dealt with in the course will focus on FET.

The course has links with School Experience and focuses on the following roles of the teacher:

- Learning mediator
- Interpreter and designer of learning programmes and materials
- Assessor

COURSE OUTCOMES
The aims of this course are that by the end of it you will:

- Have had a meaningful learning experience which enables you to integrate theory and practice.
- Understand the structure and nature of the South African Curriculum and Assessment Policy statements for FET mathematics
- Have developed mathematical content knowledge and pedagogical content knowledge
- Be clear of what is involved in teaching a particular aspect of the mathematics content
- Become a competent, confident and creative reflective mathematics teacher
- Be able to critically analyze and evaluate own and others' pedagogical practices
- select and design appropriate mathematics learning tasks for learners in FET
- assess learners' written work and suggest appropriate remediation
- relate learners' misconceptions to appropriate theoretical ideas
- reflect critically on their own practice as a mathematics teacher and relate this to issues dealt with in the course
- Be able to integrate technology in teaching mathematics

The following mathematical content will be explored:

- Trigonometry
- Functions
- Geometry
- Statistics

WORK AND ASSESSMENT PROGRAMME

BLOCK 1: GEOMETRY			
Week	Activities	Homework	Assessment
1 11-15 FEB	Welcome, Introductions, Course outline and expectations; course assessment, consent for practice-based research -Discussion on the three kinds of knowledge that are crucial for teaching school mathematics: • Content knowledge; • pedagogical content knowledge; • technology pedagogical content knowledge	Prepare for Test (content) Reading material -Ball, Thames & Phelps (2008). Content Knowledge for Teaching What Makes It Special? Hohenwarter, J. and Hohenwarte, M (2008). Introduction to GeoGebra. http://www.geogebra.org/book/intro-en.pdf	
2 18-22 FEB	Focus on • Planning to teach a mathematics lesson • Teaching a mathematics lesson • Content in a mathematics lesson • Evaluating a mathematics lesson		Content Test 1 18 FEB
3 24-28 FEB	Using technological tools to teach mathematics. -Prepare a GeoGebra-based Grade 11 lesson on teaching a circle geometry theorem. The lesson plan should provide details and justifications for the sequence of questions and activities, key concepts that you want to communicate, and common errors that you want to address. Following feedback from the discussions, modify the lesson script.	Group preparations for micro teaching	
4 4-8 MAR	Teaching circle geometry theorems -Van Hiele theory		
5 11-15 MAR	Teaching circle geometry theorems -Examine geometry tasks and discuss the impact they may have on students' learning experiences		Content Test 2 11 MAR
6 18-22	Teaching circle geometry theorems		

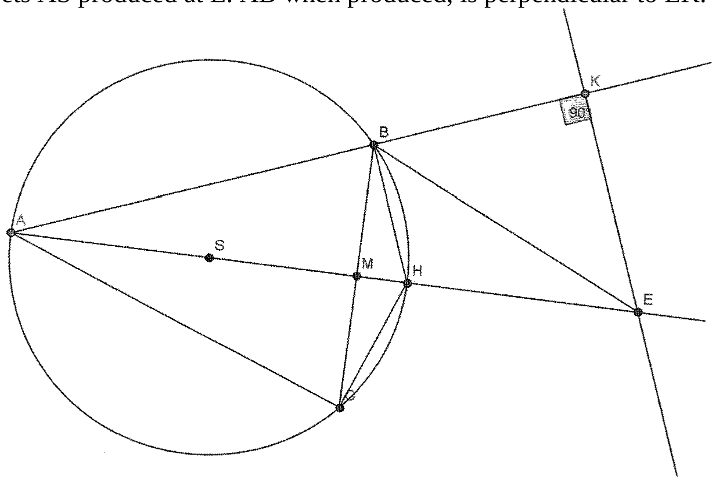
MAR			
7 25-29 MAR	RESEARCH BREAK		
BLOCK 2: TRIGONOMETRY			
8 1-5 APRIL	Does language interfere with mathematics learning?	Orton (1987)	
9 8-12 APRIL	Why do some learners perform better than others? Mathematics content area: unit circle -Develop conceptual understanding and proficiency of Grade 10, 11 trigonometry	Orton (1987)	Content Test 3 8 APR
10 15-19 APRIL	Mathematics content area: trig ratios -Develop conceptual understanding and proficiency of Grade 10, 11 trigonometry		
11 22-26 APRIL	Mathematics content area: trig functions -Develop conceptual understanding and proficiency of Grade 10, 11 trigonometry		
12 29-3 MAY	Mathematics content area: Pythagorean theorem -Develop conceptual understanding and proficiency of Grade 10, 11 trigonometry		
13-15 8-24 MAY	SCHOOL EXPERIENCE		
16 27-31 MAY	Examine TRIGONOMETRY tasks and discuss the impact they may have on students' learning experiences		Assignmen t 1 Apr 26
17 3-7 JUNE	Problem solving -Watch and analyze a video "THE EXAM" that depicts a problem solving situation (teaching and learning dynamics). Approaches to mathematics instruction		
18 10-24 JUNE	EXAMS		TEST
19 – 21 25-12 JULY	WINTER BREAK		
BLOCK 3: FUNCTIONS			
22-25 16-8 AUG	SCHOOL EXPERIENCE		
26 12-16 AUG	Assessment: The Nature of Mathematical Tasks (***Functions for teaching***) -Discuss the nature of mathematical tasks and the teacher's role in instruction. -What are the different types of tasks a teacher might design and what are the cognitive demands of such tasks? -How does the design of the mathematical task encourage or constrain thinking? Using the Mathematical Tasks Framework (Stein, Smith, Henningsen & Silver, 2000) and Stein & Smith (1998) mathematics task analysis guide to examine various mathematical tasks and discuss the impact they may have on students' learning experiences.	Reading for this session: -Stein, M.K., Smith, M.S., Henningsen, M.A. & Silver, E. (2000).	

27 19-23 AUG	Mathematics content area: Functions for teaching -Consider and analyze Grade 10-12 function tasks -Developing memo for tasks; provide justification for mark allocation		Content Test 4 19 AUG
28 26-30 AUG	Teaching the slope of a function		
29 2-6 SEPT	RESEARCH BREAK		
30 9-13 SEPT	STATISTICS What does it mean to do statistics -Mathematics content area: Statistics for teaching -Develop conceptual understanding and proficiency of statistics -Draw upon misconceptions in learning statistics -Examine statistics tasks and discuss the impact they may have on students' learning experiences Developing concepts of data analysis		
31 16-20 SEPT	Reasoning with statistics: data collection -Examine statistics tasks and discuss the impact they may have on students' learning experiences		
32 23-27 SEPT	Reasoning with statistics: data analysis -Examine statistics tasks and discuss the impact they may have on students' learning experiences		Content Test 5 23 SEPT
BLOCK 4: PROBABILITY			
33 30-4 OCT	Understanding measures of centre and variability -Interpreting results		Oct 2
34 7-11 OCT	Exploring concepts of probability		
35 14-18 OCT	Teaching probability		Assignmen t 2 Oct 18
36 21-25 OCT	Teaching probability EXAM EQUIVALENT discussion		
37-40 28-18 NOV	EXAMS EXAM EQUIVALENT due 8th November		

APPENDIX C: TASKS AND MEMORANDA FOR TASKS

Task 1

The diagram below shows a circumscribed circle with centre S. Triangle ABC has AB = AC. Angle A is acute and AB is extended to K. AS extended cuts BC at M and the circle at H. BE bisects $C\hat{B}K$. BE meets AS produced at E. AB when produced, is perpendicular to EK.

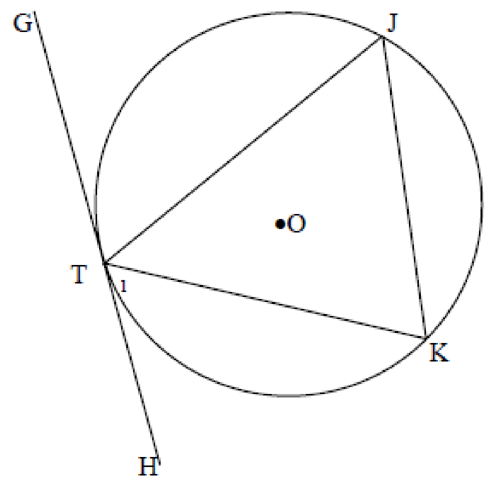


Question	Memo
(a) Write down and label all the geometric figures that you see in the above diagram. E.g. ΔABC	**Circle S,** 2 semi-circles, 4 segments **Triangles:** ΔABM, ΔACM, ΔBMH, ΔMHC, ΔBHE, ΔBKE (all single triangles); ΔABC, ΔABH, ΔAHC, ΔBHC, ΔBME, ΔABE, ΔAKE (all composite triangles) **Quadrilaterals**: ABHC, BKEH, BKEM (accepts kite ABHC, cyclic quad ABHC)
(b) Which triangles are congruent? Explain.	tests knowledge of congruency. Required to show that: (a) ΔABH ≡ ΔACH AB = AC given $\hat{C} = 90^\circ$ (∢ in semi-circle) $\hat{B} = 90^\circ$(∢ in semi-circle)

	$\boldsymbol{AH\ common}$ $\therefore\ \Delta\boldsymbol{ABH} \equiv \Delta\boldsymbol{ACH}$ (SAA) (b) ΔABM ≡ ΔACM CM=BM (ΔMHB ≡ ΔMHC AC=AB (given) $C\hat{A}M = CBM$ (ΔABH ≡ ΔACH AM common ∴ΔABM ≡ ΔACM (c) ΔMBH ≡ ΔMHC HC=HB (ΔABH ≡ ΔACH) $C\widehat{H}M = B\widehat{H}M$ (ΔABH ≡ ΔACH) HM common ∴ΔMBH ≡ ΔMHC **(d)** Construction of Triangle ABC AS extended cuts BC at M and the circle at H. BE bisects $C\hat{B}K$. BE meets AS produced at E. AB when produced, is perpendicular to EK
(c) Use GeoGebra to construct the figure.	Construction of Triangle ABC AS extended cuts BC at M and the circle at H. BE bisects $C\hat{B}K$. BE meets AS produced at E. AB when produced, is perpendicular to EK

Task 2

In the diagram below O is the centre of the circle. GH is a tangent to the circle at T.
J and K are points on the circumference of the circle. TJ, TK and JK are joined.



Question	Memo
(a) Prove the theorem that states that $\hat{T}_1 = T\hat{J}K$ using four different methods (four constructions).	Method 1 The task requires a construction of the method. A consideration should be made in transforming the statements and reasoning with the construction. The requirements for this method are: • Identification of the radii • Proof of the angles in a triangle • The application of the tan-chord theorem to show that $T_3 + T_2 = 90^{\circ}$ • Concluding that $\hat{T}_1 = T\hat{J}K$

Construct OT, OJ and OK

$\hat{T}_1 = \hat{J}_1 = x$ (radii)

$\hat{T}_2 = \hat{K}_1 = z$ (radii)

$\hat{K}_2 = \hat{J}_2 = y$ (radii)

$2x + 2y + 2z = 180°$ ($\angle$ sum Δ)

$x + y + z = 90°$

$x + y = 90° - z$

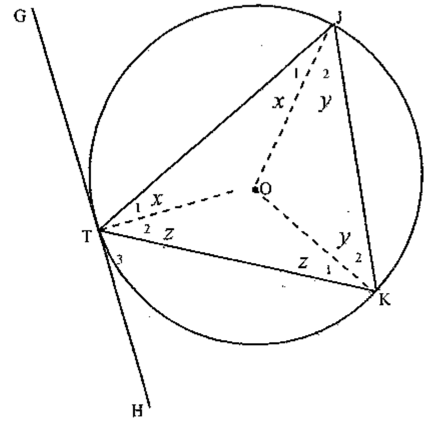
$O\hat{T}H = 90°$ (rad $\perp$ tan))

$\hat{T}_3 = 90° - z$

$= 90° - (90° - (x + y))$

$= 90° - z$

$= T\hat{J}K$



Method 2

The task requires a construction of the method. A consideration should be made in transforming the statements and reasoning with the construction. The requirements for this method are:

- The application of the tan-diameter theorem to show that $T_1 + T_2 = 90°$
- The application of the angle in semi-circle theorem to show that $J_1 + J_2 = 90°$
- The application of the angle in same segment theorem to show that $J_2 = T_2$
- Concluding that $\hat{T}_1 = T\hat{J}K$

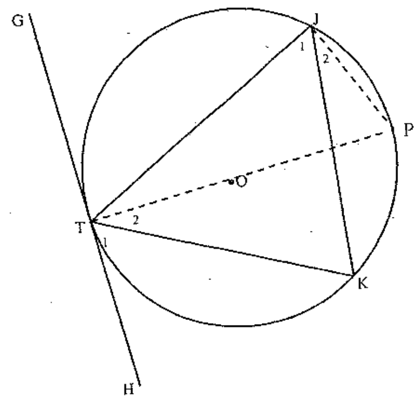
Draw diameter TP.
Join P to J.

$\hat{T}_1 + \hat{T}_2 = 90°$ (tan ⊥ diameter)

$\hat{J}_1 + \hat{J}_2 = 90°$ (∠ in semi-circle)

$\hat{J}_2 = \hat{T}_2$ (∠ in same seg)

$T\hat{J}K = \hat{T}_1$

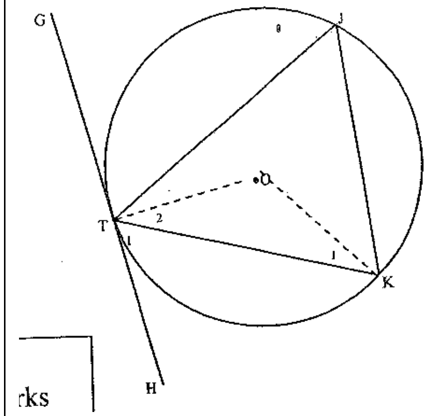


OR

Method 3

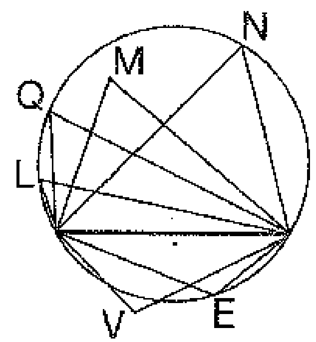
The task requires a construction of the method. A consideration should be made in transforming the statements and reasoning with the construction. The requirements for this method are:

- Identification and construction of radii OT and OK
- Proof of angles in isosceles triangle to show that $T_1 = K_1$
- The application of the angle in tan-radius theorem to show that $T_1 = 90° - x$
- The application of the angles at the centre and the circumference subtended by same chord theorem to show if twice $T\hat{O}K = T\hat{J}K$ then $T\hat{J}K = 90° - x$
- Concluding that if $T\hat{O}K = T\hat{J}K$ then $\hat{T}_1 = T\hat{J}K$

	Draw radii OT and OK Let $\hat{T}_2 = x$ $\hat{K}_1 = x$ (∠ opp = radii) $\hat{T}_1 = 90° - x$ (rad ⊥ tan) $T\hat{O}K = 180° - 2x$ (∠ sum Δ) $T\hat{J}K = 90° - x$ (∠ circ cent) $T\hat{J}K = \hat{T}_1$ (= $90° - x$)  Method 4 The task requires a construction of the method. A consideration should be made in transforming the statements and reasoning with the construction. The requirements for this method are: • Identification and construction of GT extended to H and tangent KH at K • The application of the angle in tan from common point theorem to show that $K_1 = T_1$ • Proof of angles in isosceles triangle to show that $T_1 = K_1$ • The application of the angle in tan-radius theorem to show that $T_1 + T_2 = 90°$ • The application of the angles at the centre and the circumference subtended by same chord theorem to show that $T\hat{O}K$ = twice $T\hat{J}K$; meaning that if $T\hat{O}K = 180° - (T_1 + K_1)$ then $T\hat{J}K = ½[180° - (T_1 + K_1)]$ • Noting that since $T_1 = K_1$ then $T\hat{O}K = 2\hat{T}_1$ • Concluding that if $T\hat{O}K = ½T\hat{J}K$ then $\hat{T}_1 = T\hat{J}K$
(b) How would you handle this problem in a classroom environment?	describe or demonstrate the various ways to model or illustrate the theorem. The demonstration should encompass the ability to provide an explanation of the concept or the procedure for the proof. The demonstration should discuss or utilize the general or specific instructional strategies for teaching the tan-chord theorem

Task 3

Suppose question 1 below was part of a geometry lesson:

> Question 1
>
> In the diagram below, L, Q, N and E are points on the circumference.
>
> Which of the angles are equal?
>
> 
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Question		Memo
(a)	What are the main mathematical ideas in the question 1 above?	The respondent must provide a description that demonstrates knowledge of angles in same segment theorem and its converse. Statements should be be justified by appropriate reasoning. The response to address misconceptions of the theorem
(b)	Produce a solution to the question.	$\hat{L} = \hat{Q} = \hat{N}$ applying the angle in the same segment theorem.

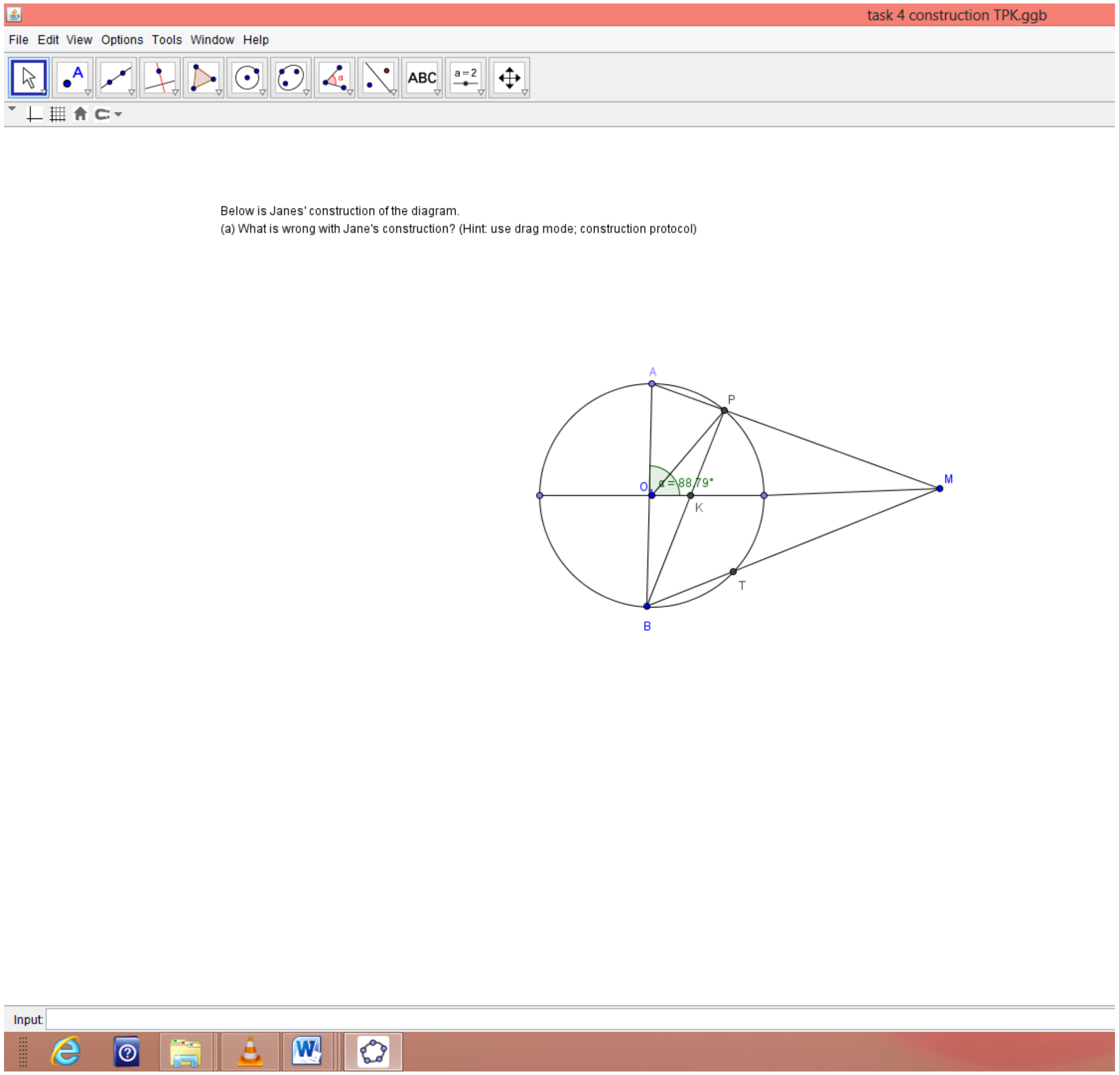
Task 4

Jane used GeoGebra to construct a diagram using the description below:

AB is a vertical diameter of a circle with centre O.
P is any point on the circle closer to A than B.
The perpendicular to AB at O meets AP produced at M.
OM and BP intersect at K.
BM cuts the circle at T.
Draw radius OP.

Question	Memo
Attached is Jane's GeoGebra construction of the diagram. Click here for the GeoGebra file. What is wrong with Jane's construction? (hint: use drag mode, construction protocol)	Errors in the construction of the diagram 1. M constructed as abitrary point 2. CD is independent of AB 3. Order of construction of P The respondent must provide a description that demonstrate knowledge of geometry definitions and/or properties of these geometric words (*perpendicular*, *vertical diameter*, *intersects*, *produced* , *closer to than)*, knowledge of how the properties of a diagram aid in the construction of a diagram, the disposition to translate statements to a diagrammatic register, and the knowledge of a construction procedure.

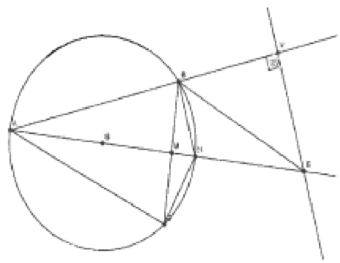
A SCREEN SHOT OF JANE'S CONSTRUCTION



APPENDIX D: NKOSI'S TASK 1 script

TASK 1

The diagram below shows a circumscribed circle with centre S. Triangle ABC has AB = AC. Angle A is acute and AB is extended to K. AS extended cuts BC at M and the circle at H. BE bisects $C\hat{B}K$. BE meets AS produced at E. AB when produced, is perpendicular to EK.



(a) Write down and label all the geometric figures that you see in the above diagram. Eg. ΔABC

ΔAMB, ΔACB, ΔCHM, ΔBHM, ΔBEM, Δ BEH, ΔBEK, Δ AEK

Circle with centre S

quadrilatals ABHC, KEBM

(b) Which triangles are congruent? Explain.

Δ ABM and Δ ACM

Δ ABH and Δ ACH

Δ BMH and Δ CHM

(c) Use GeoGebra to construct the figure.

APPENDIX E: **EXCERPT OF NKOSI'S TRANSCRIPT**

Kim: I want us to look at the tasks, as you have responded to them. So we look at each task and recordings and go through the recordings as well to see how you responded, to have an idea of how you responded. The first one ***Task 1(a****); it was to write all and label all the figures you see in the diagram. Tell me, how did you come up with these figures?*

Nkosi: I think it was the shapes and I listed and the shapes I can see. I can mostly see triangles and one quadrilateral. And I also saw a circle.

Kim: you saw a circle?

Nkosi: yes

Kim: now, let's start with a circle. You can only see one full circle? Or can you also see parts of the circle

Nkosi: no I only see one full circle

Kim: the line AH, what do you call it?

Nkosi: diameter

Kim: so if I have a diameter, what do we call this region?

Nkosi: which region?

Kim: the region above that is enclosed by the circle above the diameter.

Nkosi: I don't know, I wouldn't call it the major circle. I don't know. What is it called?

Kim: A semi-circle.

Nkosi: a semi-circle? Ohhh.

Kim: and the one below is also a semi-circle.

Nkosi: it is.

Kim: and we also talk about segments. Do you see segments there?

Nkosi: yes they are

Kim: Are they not figures?

Nkosi: no

Kim: you didn't think of them as figures?

Nkosi; no, not as geometric figures

Kim: okay in your own understanding, what do you understand about the word geometric figures?

Nkosi: I understand more like geometric shapes

Kim: shapes?

Nkosi: yes

Kim: okay. Fine, so according your response, for you the geometric shapes meant looking at the triangles, the circle and the quadrilateral.

Nkosi: yes

Kim: now let's look at the triangles, you had 1, 2, 3, 4, 5, 6, 7, 8 triangles. Are those the only triangles that you see?

Nkosi: yes, those are the only ones I see.

Kim: how did you identify them?
Nkosi: I started inside the circle, It is this one, 1,2 and this one,3,4 and then I said 5,6 and then 7,8 then on this line, this one is 10
Kim: you can come up with a pattern. You can come up with single triangles that one have one single triangle within. Okay, so those are 1, 2, 3, 4, 5, 6 right?
Nkosi: yes
Kim: then you can come up with triangles that have two triangles within. You have 1,2,3,4 right?
Nkosi: yes
Kim: then you can list triangles that have 3 triangles within, 1,2,3. Then you can also have a triangle that composes of 4 triangles within which is the big triangle. So all in all you should have 13 triangles.
Nkosi: are they those ones over here?
Kim: yes, did you see that? With quadrilaterals, how many did you see? Two?
Nkosi: yes
Kim: which are A, B, H, C right?
Nkosi: yes
Kim: and then K, E, B, M.
Nkosi: Yes
Kim: Okay, now is there any special about A, B, H, C?
Nkosi: I forgot the term
Kim: Cyclic
Nkosi: yes
Kim: okay, now, question (b), which triangles are congruent? Now I want you to explain to me because you have identified three pairs of triangles. Now I want you to explain to me how you come up with the conclusion that these two are congruent triangles?
Nkosi: This triangle is equal to that triangle because they have equal sides; they share the same common side
Kim: which is?
Nkosi: AM
Kim: Okay
Nkosi: and these two sides are equal
Kim: which are B, M and C, M. why?
Nkosi: it's segmented the…. This is a chord that is cut by a diameter, that cuts it into equal parts and then the angle here is both 90
Kim: okay, angle B, M, A is 90 = A, M, C
Nkosi: so, with the side's side angle I can see that they are congruent.
Kim: okay, now these other two? Angle A, B, H and triangle A, C, H
Nkosi: Okay, you can see these two lines are equal, B, H and C, H
Kim: why would they be equal?
Nkosi: because this line here, there's another congruent triangle inside here BM = CM right? Then the angles here are equal.
Kim: they are both equal to what?
Nkosi: 90 degrees and then they share a common side

Kim: which is?
Nkosi: M, H, therefore B H is = to C H
Kim: which means that you have to start with this one first
Nkosi: yes
Kim: This means that you need to show that BMH is congruent to CHM for you to be able to prove that ABH is congruent to ACH. So you started with ABM being congruent to ACM and then from there you go to BHM being congruent to triangle CHM. Then from there you can conclude that ABH is congruent to ACH, that's what you are saying right?
Nkosi: yes
Kim: okay. Question c) you were to construct this. Tell me, when you constructed, what did you consider?
Nkosi: The angles
Kim: the angles?
Nkosi: Yes. For example the 90 degree angle
Kim: so your plan was to make sure that you have the 90 degree angle first? What was your plan?
Nkosi: my plan was to start with the circle then after the circle I…
Kim: why did you start with the circle?
Nkosi: because it's easier to draw this quadrilateral because it's a circle. Then I drew point A here
Kim: okay. So you draw a circle, centre A and B the circumference. Take me through what you are doing here.
Nkosi: now I want to draw a ray, because I can see there's a ray here. Then I drew a ray that goes through the red point centre. Then I drew it and I'm going to draw another ray which goes through B. then I see that the rays are finished so I should draw a segment now.
Kim: which segment?
Nkosi: Segment LX or LCO. Now I want to draw a line KA. Now I want to go the segments, then I join those two points at the circumference.
Kim: what's going on?
Nkosi: I'm just making sure that it looks the same and I joined those two points.
Kim: what does 'looks the same' mean?
Nkosi: the same with this one, they are similar. Then I just join the lines and move that point..
Kim: you want to move this point?
Nkosi: yes I wanted it to look like that.
Kim: why were you moving it?
Nkosi: because I see here the gradient is the same
Kim: so you wanted it to look as exactly like it did on the diagram because it seems like AE is horizontal
Nkosi: yes. Then I drew in the 90 degrees and I could see that it wasn't a 90 degree so I made it a 90 degree which was difficult
Kim: but then again when you have two line segments that meet at a 90 degree, what does it say about those two lines?
Nkosi: they are perpendicular

Kim: *yes they are perpendicular, so you have your 90 degree.*

Nkosi: *yes, so let's see what is needed. Just wanted to remove the E and it becomes one short line segment IE.*

Kim: *Now tell me when you drew this; didn't you refer to the description there?*

Nkosi: *no, not at all*

Kim: *why not?*

Nkosi: *because, I'm not drawing it by hand so I know for example that these two lines would be equal, this would equal to that BM will equal to that line*

Kim: *in your diagram, they will be equal?*

Nkosi: *yes*

Kim: *now let's look at your GeoGebra file to see if they are equal because you are saying that triangle ABC, we know that from the description that it is an isosceles triangle right? So in your own thinking, when you draw this, the description of your diagram should fit exactly what's in there?*

Nkosi: *Yes*

Kim: *Okay. Now let's see in your triangle CHI it is an isosceles triangle. We expect CI to equal to CH. So now which line here is CI?*

Nkosi: *I don't know*

Kim: *this one is E, okay? This one, the 7.9 one and this is B which is 6.43*

Nkosi: *I didn't consider that*

Kim: *and you also didn't consider that IK in your diagram should be the bisector for angle HIF because you are given here that B bisects CBK. Was that the question that was misleading you and led you to think that? Suppose you were just given this diagram then the question says, Construct the diagram on GeoGebra. Would you still not look at the description?*

Nkosi: *if you just gave me this and don't give me the description?*

Kim: *I also give you the description and then I say draw. Suppose all these other questions were not there and I said construct this figure*

Nkosi: *on GeoGebra?*

Kim: *Yes*

Nkosi: *Yes, i would just look at the diagram.*

Kim: *you'd just look at the diagram? But suppose you'd also looked at the description, what would you have changed in your construction?*

Nkosi: *the accuracy.*

Kim: *Because your only concern was the 90 degree*

Nkosi: *yes because that's the only one I could see from the diagram*

Kim: *suppose it was mentioned in the diagram that AB=AC, would you have made sure that they were accurate?*

Nkosi: *yes, then I was going to make sure that they were accurate.*