Master of Science by Dissertation:
Towards The Resolution Of Divergences In The Holographic Computation Of Extremal Correlators

Elie Danien RAJAOBELINA IARILALA (iarilala@aims.ac.za)

Supervised by: Robert De Mello Koch
University of The Witwatersrand, South Africa

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Declaration

I declare that this dissertation is my own, unaided work. It is submitted for the Degree of Master of Science in the University of The Witwatersrand, Johannesburg. It has not been submitted before for any degree of examination in any other University.

Elie Danien RAJAOBELINA IARILALA, Day of 09 February 2017.

Abstract

The main goal of this dissertation is to construct a better understanding of the subtleties that arise in the holographic computation of extremal correlators. It is well known that these computations, in the gravitational description, suffer from divergences, but the interpretation and origin of these divergences is unclear. The study starts with detailed computations of two- and three-point functions of a scalar field minimally coupled to gravity on Euclidean AdS_d space, three-point functions of two giant gravitons and one light graviton, and three-point functions of the Kaluza-Klein gravitons, using supergravity theory. Further, we also give the computation of these same correlators in the dual CFT. These involve novel techniques in the matrix model, including methods that employ Schur polynomials in the dual gauge theory analysis. By employing the usual AdS/CFT dictionary, we argue that extremal correlators are naturally related to collinear particles. There are divergences that arise in collinear amplitudes as a consequence of the fact that the particles momenta are parallel. We therefore reach the suggestive idea that the divergences in extremal correlator computations are linked to collinear divergences. Much remains to be done to really establish this connection.

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1. Introduction

Correlation functions are the basic observables of interest in quantum field theory (QFT) as well as in string theory. Their computation is not an easy task. One significant complication is that correlation functions are often divergent and this divergence must be understood and interpreted before the correlation function can be determined. Another key difficulty is that most computations of correlators must be performed using a weak coupling expansion. For this reason the computation of correlators in the strongly coupled limit of QFT is usually not possible.

The idea of the holographic principle, as realized in the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence (Maldacena, 1999; Witten, 1998; Gubser et al., 1998), has related correlators computed in string theory to correlators computed in field theory. A weak coupling expansion in the string theory then gives insight into the strong coupling limit of the QFT. However, the weak coupling string theory computations are still plagued by divergences. Our principal motivation in this study is to give a general understanding of the subtleties arising as a result of these divergences, in the holographic computations of extremal correlation functions. These subtleties were first found in Freedman et al. (1999), D'Hoker et al. (1999) and Caputa et al. (2012).

This dissertation is organised as follows:

In Chapter 2, we introduce a matrix model with one Hermitian $N \times N$ matrix M that is needed in Chapter 3 to construct a complex matrix model and and then extend it to a model with any number of matrices. This first chapter will consider a free matrix model and then an interacting one. Different approaches are used to compute the correlators in this model. First we use a generating function, then we use the so called Schwinger-Dyson equations and finally we use the techniques of ribbons graphs which are Feyman diagram techniques used to evaluate correlators in field theories. What makes the study of the matrix model interesting is that it is related to the dynamics of strings. We will give an argument that reaches this conclusion. We will show that the large N limit gives the classical limit of the string theory.

In Chapter 3, we present some basic group theory and representation theory that is used to compute the correlators of a complex matrix model with field Z. The representations of the symmetric group S_n using Young tableau play a central role for this, so one section of this chapter is dedicated to the construction of their matrix representations. The computation of correlators is achieved by working on the tensor product of n copies of the vector space on which the complex matrix field Z acts. On this tensor vector space the action of the symmetric group is defined naturally. The idea of projection operators in group theory allows us to build a set of projection operators that are related to the construction of the Schur polynomial operators. These polynomial operators are a complete set of orthogonal operators. Any multi-trace structure can be written as a linear combination of them. Another section is used to show how a complex matrix model can be obtained from a model with 2 Hermitian matrices M_1 and M_2.

In Chapter 4, we will use the idea of holography to compute two-point functions of scalar fields from supergravity on AdS_{d+1}. We will show that there is a subtlety in the computation since two different results will be obtained using two approaches which should be equivalent. We will decide which is correct by appealing to a Ward identity. The resulting two-point function is in agreement with the gauge theory. We will also present the computation of three-point functions of the gravity scalar field. All the ingredients that are developed in this chapter will be reused in Chapter 6.

In Chapter 5, the correlator of giant gravitons both on the S^5 and on the AdS_5 are computed. We perform the computations in the gauge theory using Schur polynomials and by using the $D3$-brane

Dirac-Born-Infield action on the gravity side. Here again, the computation is subtle since the integrals appearing in the computation are divergent. We present a regularization in which the values of the correlators from gauge theory are in agreement with the computation from the gravity side. This method of regularization is essentially chosen by comparison to the gauge theory. One may hope to do better, by providing an understanding of how and why the divergence arises. With this understanding one might be able to motivate what regularization should be chosen without appealing to the gauge theory. This is a key motivation for our study.

In Chapter 6, we will discuss the computation of a three-point function of the scalar gravity field dual to the chiral primary operators in gauge theory. Extremal and non-extremal correlators are considered following D'Hoker et al. (1999). The computation of the extremal correlators is achieved in two different ways on the gravity side by considering an analytic continuation of the non-extremal case and secondly by using the bulk-to-boundary propagator in Fourier space which involves a modified Bessel function as we have seen in Chapter 4. Here however, its index is an integer.

In Chapter 7, motivated by the connection between R-charge of operators in gauge theory and angular momentum of the dual particle states in string theory, we suggest that the divergences that appear in extremal correlators maybe related to collinear divergences. We explain how the cancellation of IR- and collinear-divergences (achieved by summing over degenerate states) may be useful for understanding the divergences in extremal correlators. We conclude this work in Chapter 8.

Appendices A and B show how to contract the indices of symmetric traceless tensors, and give the relevant properties of spherical harmonics that are needed for the analysis carried out in Chapter 6.

2. Correlation functions from Matrix model

2.1 Matrix model

The central set of ideas explored in this dissertation involve the duality between Yang-Mills theories and string theories. This discussion can be carried out, in a simpler setting, by considering a matrix model (Corley et al., 2002; Balasubramanian et al., 2002; de Mello Koch et al., 2007b) instead of Yang-Mills theory.

In the following, we are going to give an argument supporting the idea that any matrix model is a theory of strings. We start with a discussion of the computation of correlation functions in scalar quantum field theory. With this discussion complete, we then enumerate the modifications to the quantum field theory of a scalar field needed to build the matrix model. Then we will compute correlators using the generating function as in field theories. After this we derive the Feyman rules, called ribbon graph diagrams, from which we can explain the dynamics of the matrix model.

First of all, the generating function in field theories is defined as

$$Z = \int [d\phi]\, e^{iS + \int d^4x J(x)\phi(x)}. \tag{2.1.1}$$

Using the functional derivative defined by

$$\frac{\delta J(x)}{\delta J(y)} = \delta(x-y), \tag{2.1.2}$$

we easily obtain

$$\frac{\delta Z}{\delta J(y)} = \int [d\phi]\, e^{iS + \int d^4x J(x)\phi(x)} \phi(y). \tag{2.1.3}$$

With the generating function we are able to compute the correlation function or correlator, which is defined by

$$\langle \ldots \rangle = \int [d\phi]\, e^{iS} \ldots \tag{2.1.4}$$

In the above, $\ldots$ stands for any product of observables. From (2.1.3), correlators are given by taking derivatives of the generating function, evaluated at $J = 0$, as follows

$$\langle \phi(x_1)\ldots\phi(x_n)\rangle = \int [d\phi]\, e^{iS}\phi(x_1)\ldots\phi(x_n) \tag{2.1.5}$$

$$= \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} Z[J]\big|_{J=0}. \tag{2.1.6}$$

To obtain the matrix model, we make the following modifications in the above discussion:

(1) Move to Euclidean space: $iS \to -S$.

(2) Consider a matrix valued field: $[d\phi] \to [dM]$ with M a Hermitian matrix such that $M = M^\dagger$. The integral over the measure $[dM]$ means we have to integrate over all possible Hermitian matrices. So we have to integrate over

- all possible diagonal elements m_{ii}. This correspond to N real integrals,
- all possible imaginary parts of the elements m^i_{ij} above the diagonal, which contributes a further $\frac{1}{2}N(N-1)$ real integrals,
- all possible real parts of the elements m^r_{ij} above the diagonal, corresponding to $\frac{1}{2}N(N-1)$ real integrals.

We thus have

$$[dM] = \mathcal{N} \prod_i^N dm_{ii} \prod_{i<j}^N dm^r_{ij} dm^i_{ij}, \tag{2.1.7}$$

where $\mathcal{N}$ is a normalization for the measure.

(3) Move to 0-dimensions: we imagine the universe is a single point.

We will see that this toy model correctly captures the ideas of gauge/gravity duality.

2.2 Free matrix model

For free theory, the generating function takes the form

$$Z_0[J] = \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right)+\mathsf{Tr}(JM)}, \tag{2.2.1}$$

where

$$\frac{dJ_{kl}}{dJ_{ij}} = \delta_{ik}\delta_{jl} \tag{2.2.2}$$

and

$$\frac{dZ_0[J]}{dJ_{ij}} = \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right)+\mathsf{Tr}(JM)} M_{ji}. \tag{2.2.3}$$

The correlation functions are given by

$$\langle \dots \rangle_0 = \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right)} \dots \tag{2.2.4}$$

The subscript 0 indicates that we are studying the free theory. Combining (2.2.4) and (2.2.3), we have

$$\begin{aligned} \langle M_{ij} M_{kl} \dots M_{xy} \rangle_0 &= \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right)} M_{ij} M_{kl} \dots M_{xy} \\ &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \cdots \frac{d}{dJ_{yx}} Z_0[J]\big|_{J=0}. \end{aligned} \tag{2.2.5}$$

Here, we have fixed the normalization of the measure by

$$\langle 1 \rangle_0 = Z_0[J]\big|_{J=0} = \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right)} = 1. \tag{2.2.6}$$

2.2.1 Computation of correlators using the generating function. According to (2.2.5), one way to compute correlators is to determine $Z[J]$. The correlators are equal to derivatives of $Z_0[J]$ evaluated at $J=0$. Our first goal is to compute $Z_0[J]$.

We can perform the Gaussian integral in (2.2.3) by completing the square in the exponent. Towards this end, note that

$$\begin{aligned}
-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)+\mathrm{Tr}\left(JM\right) &= -\frac{1}{2}\left[\mathrm{Tr}\left(\omega M^2\right)-2\mathrm{Tr}\left(JM\right)\right] \\
&= -\frac{1}{2}\left[\mathrm{Tr}\left(\omega M^2 - JM - MJ\right)\right] \\
&= -\frac{\omega}{2}\left[\mathrm{Tr}\left(M^2-\frac{JM-MJ}{\omega}+\frac{J^2}{\omega}-\frac{J^2}{\omega}\right)\right] \\
&= -\frac{\omega}{2}\mathrm{Tr}\left(M^2-\frac{JM-MJ}{\omega}+\frac{J^2}{\omega^2}\right)+\mathrm{Tr}\left(\frac{J^2}{2\omega}\right) \\
&= -\frac{\omega}{2}\mathrm{Tr}\left(\left[M-\frac{J}{\omega}\right]^2\right)+\mathrm{Tr}\left(\frac{J^2}{2\omega}\right).
\end{aligned} \tag{2.2.7}$$

Therefore (in obtaining (2.2.8) we have used (2.2.6)))

$$\begin{aligned}
Z_0[J] &= \int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(\left[M-\frac{J}{\omega}\right]^2\right)+\mathrm{Tr}\left(\frac{J^2}{2\omega}\right)} \\
&= e^{\mathrm{Tr}\left(\frac{J^2}{2\omega}\right)}\int [dM]\, e^{-\frac{\omega}{2}\mathrm{Tr}\left(\left[M-\frac{J}{\omega}\right]^2\right)} \\
Z_0[J] &= e^{\mathrm{Tr}\left(\frac{J^2}{2\omega}\right)}.
\end{aligned} \tag{2.2.8}$$

When we differentiate the expression for $Z_0[J]$ in (2.2.8) with respect to J_{ij}, we obtain

$$\frac{dZ_0[J]}{dJ_{ij}} = \frac{1}{\omega}J_{ji}Z_0[J]. \tag{2.2.9}$$

Now, let us determine some correlators:

- $\langle M_{ij}M_{kl}\rangle_0$:

$$\begin{aligned}
\langle M_{ij}M_{kl}\rangle_0 &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}Z_0[J]\big|_{J=0} \\
&= \left[\frac{d}{dJ_{ji}}\left(\frac{1}{\omega}J_{kl}Z_0[J]\right)\right]_{J=0} \\
&= \frac{1}{\omega}\left[\left(\delta_{il}\delta_{jk}Z_0[J]+\frac{1}{\omega}J_{kl}J_{ij}Z_0[J]\right)\right]_{J=0}.
\end{aligned} \tag{2.2.10}$$

The second term in this expression vanishes when $J=0$. Therefore it follows that

$$\langle M_{ij}M_{kl}\rangle_0 = \frac{1}{\omega}\delta_{il}\delta_{jk}. \tag{2.2.11}$$

- $\langle M_{ij}M_{kl}M_{mn}\rangle_0$:

We use the result found in (2.2.10) to obtain

$$\begin{aligned}\langle M_{ij}M_{kl}M_{mn}\rangle_0 &= \frac{1}{\omega}\left[\frac{d}{dJ_{nm}}\left(\delta_{il}\delta_{jk}Z_0[J]+\frac{1}{\omega}J_{kl}J_{ij}Z_0[J]\right)\right]_{J=0}\\ &= \frac{1}{\omega^2}\left[\left(\delta_{il}\delta_{jk}J_{mn}+\delta_{nk}\delta_{ml}J_{ij}+J_{kl}\delta_{ni}\delta_{mj}+\frac{1}{\omega}J_{mn}J_{kl}J_{ij}\right)Z_0[J]\right]_{J=0}.\end{aligned} \tag{2.2.12}$$

When $J=0$, all terms in this expression vanish, so that

$$\langle M_{ij}M_{kl}M_{mn}\rangle_0 = 0. \tag{2.2.13}$$

- $\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle_0$:

  We use the result we obtained in (2.2.12) to find

$$\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle_0 = \frac{1}{\omega^2}\left[\frac{d}{dJ_{qp}}\left\{\left(\delta_{il}\delta_{jk}J_{mn}+\delta_{nk}\delta_{ml}J_{ij}+J_{kl}\delta_{ni}\delta_{mj}+\frac{1}{\omega}J_{mn}J_{kl}J_{ij}\right)Z_0[J]\right\}\right]_{J=0}. \tag{2.2.14}$$

  The terms which do not vanish when $J=0$ are the first three terms of this expression. The final result is

$$\langle M_{ij}M_{kl}M_{mn}M_{pq}\rangle_0 = \frac{1}{\omega^2}\left(\delta_{il}\delta_{jk}\delta_{qm}\delta_{pn}+\delta_{nk}\delta_{ml}\delta_{qi}\delta_{pj}+\delta_{qk}\delta_{pl}\delta_{ni}\delta_{mj}\right). \tag{2.2.15}$$

Using (2.2.11) and (2.2.15), we can compute $\langle \mathrm{Tr}\,(M^2)\rangle_0$, $\left\langle \mathrm{Tr}\,(M)^2\right\rangle_0$ and $\langle \mathrm{Tr}\,(M^4)\rangle_0$. The results are

- $\langle \mathrm{Tr}\,(M^2)\rangle_0$:

$$\begin{aligned}\left\langle \mathrm{Tr}\,(M)^2\right\rangle_0 &= \langle M_{ii}M_{jj}\rangle_0\\ &= \frac{1}{\omega}\delta_{ij}\delta_{ij}\\ &= \frac{1}{\omega}\delta_{jj}\\ \left\langle \mathrm{Tr}\,(M)^2\right\rangle_0 &= \frac{1}{\omega}N.\end{aligned} \tag{2.2.16}$$

- $\left\langle \mathrm{Tr}\,(M)^2\right\rangle_0$:

$$\begin{aligned}\left\langle \mathrm{Tr}\,(M)^2\right\rangle_0 &= \langle M_{ij}M_{ji}\rangle_0\\ &= \frac{1}{\omega}\delta_{ii}\delta_{jj}\\ \left\langle \mathrm{Tr}\,(M)^2\right\rangle_0 &= \frac{1}{\omega}N^2.\end{aligned} \tag{2.2.17}$$

- $\left\langle \text{Tr}\left(M^4\right)\right\rangle_0$:

$$\begin{aligned}\left\langle \text{Tr}\left(M^4\right)\right\rangle_0 &= \frac{1}{\omega^2}\left(\delta_{il}\delta_{jj}\delta_{il}\delta_{nn} + \delta_{nj}\delta_{ll}\delta_{ii}\delta_{nj} + \delta_{ij}\delta_{nl}\delta_{ni}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(\delta_{il}\delta_{il}N^2 + \delta_{nj}\delta_{nj}N^2 + \delta_{ij}\delta_{nl}\delta_{ni}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(N^3 + N^3 + \delta_{nl}\delta_{nj}\delta_{lj}\right)\\ &= \frac{1}{\omega^2}\left(2N^3 + \delta_{lj}\delta_{lj}\right)\\ \left\langle \text{Tr}\left(M^4\right)\right\rangle_0 &= \frac{1}{\omega^2}\left(2N^3 + N\right).\end{aligned} \tag{2.2.18}$$

2.2.2 Computation of correlators using the Schwinger-Dyson equation. The Schwinger-Dyson equation is obtained by using the fact that correlators are invariant under the change of variable $M_{ij} \to M_{ij} + \delta M_{ij}$ in the path integral. To derive the Schwinger-Dyson equation, let $\mathcal{O}$ be an arbitrarily observable, with correlator given by

$$\langle \mathcal{O} \rangle_0 = \int [dM]\, e^{-\frac{\omega}{2}\text{Tr}\left(M^2\right)}\mathcal{O} = \int [dM]\, F(M), \tag{2.2.19}$$

where

$$F(M) = e^{-\frac{\omega}{2}\text{Tr}\left(M^2\right)}\mathcal{O}. \tag{2.2.20}$$

Now, consider the transformation $M_{ij} \to M'_{ij} = M_{ij} + \delta M_{ij}$. Note that

$$M_{ij} = M'_{ij} - \delta M_{ij} \quad \text{and} \quad [dM] = \left[dM'\right], \tag{2.2.21}$$

so that

$$\langle \mathcal{O} \rangle_0 = \int \left[dM'\right] F(M' - \delta M). \tag{2.2.22}$$

In this expression, we can expand $F(M' - \delta M)$ as a Taylor series assuming that the δM_{ij}'s are small. The result is

$$\begin{aligned}\langle \mathcal{O} \rangle_0 &= \int \left[dM'\right]\left(F(M') - \delta M_{ij}\frac{\partial F(M')}{\partial M'_{ij}}\right)\\ &= \int \left[dM'\right] F(M') - \delta M_{ij}\int \left[dM'\right]\frac{\partial F(M')}{\partial M'_{ij}}\\ &= \langle \mathcal{O} \rangle_0 - \delta M_{ij}\int \left[dM'\right]\frac{\partial F(M')}{\partial M'_{ij}}.\end{aligned} \tag{2.2.23}$$

Since this last expression is true for any δM_{ij}, it follows that

$$\int [dM]\,\frac{\partial F(M)}{\partial M_{ij}} = 0, \tag{2.2.24}$$

which is the Schwinger-Dyson equation. This equation allows us to easily compute the correlators. For example

(a) $\left\langle \mathsf{Tr}\,(M)^2 \right\rangle_0 = N$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ii}} \left[\mathsf{Tr}\,(M)\, e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[\frac{\partial}{\partial M_{ii}} \mathsf{Tr}\,(M) - \frac{1}{2} \mathsf{Tr}\,(M)\, \frac{\partial}{\partial M_{ii}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM]\, [\delta_{ii} - \mathsf{Tr}\,(M)\, M_{ii}]\, e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle N - \mathsf{Tr}\,(M)^2 \right\rangle_0 .
\end{aligned} \tag{2.2.25}$$

(b) $\left\langle \mathsf{Tr}\,(M)^4 \right\rangle_0 = 3N^2$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ii}} \left[\mathsf{Tr}\,(M)^3\, e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[3 \left(\frac{\partial}{\partial M_{ii}} \mathsf{Tr}\,(M) \right) \mathsf{Tr}\,(M)^2 - \frac{1}{2} \mathsf{Tr}\,(M)^3\, \frac{\partial}{\partial M_{ii}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[3\delta_{ii} \mathsf{Tr}\,(M)^2 - \mathsf{Tr}\,(M)^3\, M_{ii} \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle 3N \mathsf{Tr}\,(M)^2 - \mathsf{Tr}\,(M)^4 \right\rangle_0 \\
&= \left\langle 3N^2 - \mathsf{Tr}\,(M)^4 \right\rangle_0 .
\end{aligned} \tag{2.2.26}$$

(c) $\left\langle \mathsf{Tr}\,(M)^6 \right\rangle_0 = 15N^3$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ii}} \left[\mathsf{Tr}\,(M)^5\, e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[5 \left(\frac{\partial}{\partial M_{ii}} \mathsf{Tr}\,(M) \right) \mathsf{Tr}\,(M)^4 - \frac{1}{2} \mathsf{Tr}\,(M)^5\, \frac{\partial}{\partial M_{ii}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[5\delta_{ii} \mathsf{Tr}\,(M)^4 - \mathsf{Tr}\,(M)^5\, M_{ii} \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle 5N \mathsf{Tr}\,(M)^4 - \mathsf{Tr}\,(M)^6 \right\rangle_0 \\
&= \left\langle 15N^3 - \mathsf{Tr}\,(M)^6 \right\rangle_0 .
\end{aligned} \tag{2.2.27}$$

(d) $\left\langle \mathsf{Tr}\,(M)^{2n+2} \right\rangle_0 = (2n+1)N \mathsf{Tr}\,(M)^{2n} = (2n+1)(2n-1)(2n-3)\cdots 1\, N^{n+1}$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ii}} \left[\mathsf{Tr}\,(M)^{2n+1}\, e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[(2n+1) \left(\frac{\partial}{\partial M_{ii}} \mathsf{Tr}\,(M) \right) \mathsf{Tr}\,(M)^{2n} - \frac{1}{2} \mathsf{Tr}\,(M)^{2n+1}\, \frac{\partial}{\partial M_{ii}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[(2n+1)\delta_{ii} \mathsf{Tr}\,(M)^4 - \mathsf{Tr}\,(M)^{2n+1}\, M_{ii} \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle (2n+1)N \mathsf{Tr}\,(M)^{2n} - \mathsf{Tr}\,(M)^{2n+2} \right\rangle_0 .
\end{aligned} \tag{2.2.28}$$

(e) $\left\langle \mathsf{Tr}\left(M^2\right)\right\rangle_0 = N^2$,

$$
\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ij}} \left[M_{ij} e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[\frac{\partial}{\partial M_{ij}} M_{ij} - \frac{1}{2} M_{ij} \frac{\partial}{\partial M_{ij}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[\delta_{ii}\delta_{jj} - M_{ij}M_{ji} \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle N^2 - \mathsf{Tr}\left(M^2\right) \right\rangle_0 .
\end{aligned} \tag{2.2.29}
$$

(f) $\left\langle \mathsf{Tr}\left(M^2\right)^2 \right\rangle_0 = (N^2+2)N^2$,

$$
\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ij}} \left[\mathsf{Tr}\left(M^2\right) M_{ij} e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[\left(\frac{\partial}{\partial M_{ij}} \mathsf{Tr}\left(M^2\right) \right) M_{ij} + \mathsf{Tr}\left(M^2\right) \frac{\partial}{\partial M_{ij}} M_{ij} - \frac{1}{2} \mathsf{Tr}\left(M^2\right) M_{ij} \frac{\partial}{\partial M_{ij}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[2M_{ij}M_{ji} + \delta_{ii}\delta_{jj}\mathsf{Tr}\left(M^2\right) - \mathsf{Tr}\left(M^2\right) M_{ij}M_{ji} \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle (2+N^2)\mathsf{Tr}\left(M^2\right) - \mathsf{Tr}\left(M^2\right)^2 \right\rangle_0 .
\end{aligned} \tag{2.2.30}
$$

(g) $\left\langle \mathsf{Tr}\left(M^2\right)^{n+1} \right\rangle_0 = (N^2+2n) \left\langle \mathsf{Tr}\left(M^2\right)^n \right\rangle_0$,

$$
\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ij}} \left[\mathsf{Tr}\left(M^2\right)^n M_{ij} e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \right] \\
&= \int [dM] \left[n\mathsf{Tr}\left(M^2\right)^{n-1} \left(\frac{\partial}{\partial M_{ij}} \mathsf{Tr}\left(M^2\right) \right) M_{ij} + \mathsf{Tr}\left(M^2\right)^n \frac{\partial}{\partial M_{ij}} M_{ij} \right. \\
&\quad \left. - \frac{1}{2} \mathsf{Tr}\left(M^2\right)^n M_{ij} \frac{\partial}{\partial M_{ij}} \mathsf{Tr}\left(M^2\right) \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[2n\mathsf{Tr}\left(M^2\right)^{n-1} M_{ij}M_{ji} + \delta_{ii}\delta_{jj}\mathsf{Tr}\left(M^2\right)^n - \mathsf{Tr}\left(M^2\right)^n M_{ij}M_{ji} \right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle (2n+N^2)\mathsf{Tr}\left(M^2\right)^n - \mathsf{Tr}\left(M^2\right)^{n+1} \right\rangle_0 .
\end{aligned} \tag{2.2.31}
$$

(h) $\left\langle \mathsf{Tr}\left(M^2\right)^3 \right\rangle_0 = (N^2+4)(N^2+2)N^2$,

(i) $\left\langle \mathsf{Tr}\left(M^2\right)^4 \right\rangle_0 = (N^2+6)(N^2+4)(N^2+2)N^2$,

We have used the recursion relation in (g) to find (h) and (i).

(j) $\left\langle \mathsf{Tr}\left(M^4\right)\right\rangle_0 = 2N^3 + N$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ij}} \left[(M^3)_{ij} e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)}\right] \\
&= \int [dM] \left[\frac{\partial}{\partial M_{ij}}(M^3)_{ij} - \frac{1}{2}(M^3)_{ij}\frac{\partial}{\partial M_{ij}}\mathsf{Tr}\left(M^2\right)\right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[\sum_{k=0}^{2} \mathsf{Tr}\left(M^k\right)\mathsf{Tr}\left(M^{2-k}\right) - (M^3)_{ij}M_{ji}\right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle 2N\mathsf{Tr}\left(M^2\right) + \mathsf{Tr}\left(M\right)^2 - \mathsf{Tr}\left(M^4\right)\right\rangle_0 .
\end{aligned} \tag{2.2.32}$$

(k) $\left\langle \mathsf{Tr}\left(M\right)\mathsf{Tr}\left(M^3\right)\right\rangle_0 = 3N^2$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ij}} \left[\mathsf{Tr}\left(M\right)(M^2)_{ij} e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)}\right] \\
&= \int [dM] \left[\left(\frac{\partial}{\partial M_{ij}}\mathsf{Tr}\left(M\right)\right)(M^2)_{ij} + \mathsf{Tr}\left(M\right)\frac{\partial}{\partial M_{ij}}(M^2)_{ij} - \frac{1}{2}\mathsf{Tr}\left(M\right)(M^2)_{ij}\frac{\partial}{\partial M_{ij}}\mathsf{Tr}\left(M^2\right)\right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[\delta_{ij}(M^2)_{ij} + 2N\mathsf{Tr}\left(M\right)\mathsf{Tr}\left(M\right) - \mathsf{Tr}\left(M\right)(M^2)_{ij}M_{ji}\right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle \mathsf{Tr}\left(M^2\right) + 2N\mathsf{Tr}\left(M\right)^2 - \mathsf{Tr}\left(M\right)\mathsf{Tr}\left(M^3\right)\right\rangle_0 .
\end{aligned} \tag{2.2.33}$$

(l) $\left\langle \mathsf{Tr}\left(M^6\right)\right\rangle_0 = 5N^4 + 10N^2$,

$$\begin{aligned}
0 &= \int [dM]\, \frac{\partial}{\partial M_{ij}} \left[(M^5)_{ij} e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)}\right] \\
&= \int [dM] \left[\frac{\partial}{\partial M_{ij}}(M^5)_{ij} - \frac{1}{2}(M^5)_{ij}\frac{\partial}{\partial M_{ij}}\mathsf{Tr}\left(M^2\right)\right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \int [dM] \left[\sum_{k=0}^{4} \mathsf{Tr}\left(M^k\right)\mathsf{Tr}\left(M^{4-k}\right) - (M^5)_{ij}M_{ji}\right] e^{-\frac{1}{2}\mathsf{Tr}\left(M^2\right)} \\
&= \left\langle 2N\mathsf{Tr}\left(M^4\right) + 2\mathsf{Tr}\left(M\right)\mathsf{Tr}\left(M^3\right) + \mathsf{Tr}\left(M^2\right)^2 - \mathsf{Tr}\left(M^6\right)\right\rangle_0 .
\end{aligned} \tag{2.2.34}$$

(m) $\left\langle \mathsf{Tr}\left(M\right)^2\mathsf{Tr}\left(M^4\right)\right\rangle_0 = 2N^5 + N^3 + 12N^2$,

(n) $\left\langle \mathsf{Tr}\left(M^2\right)\mathsf{Tr}\left(M^4\right)\right\rangle_0 = (N^2+4)(2N^3+N)$,

(o) $\left\langle \mathsf{Tr}\left(M^2\right)\mathsf{Tr}\left(M^6\right)\right\rangle_0 = (N^2+6)(5N^4+10N^2)$,

(p) $\left\langle \mathsf{Tr}\left(M^4\right)^2\right\rangle_0 = 4N^6 + 40N^4 + 61N^2$.

Here, the value of ω was taken to be 1.

2.2.3 Computation of correlators using ribbon graphs. Yang-Mills theories are theories with local gauge invariance. All physical observables must be invariant under the local gauge symmetry. For the matrix model we declare that the local gauge symmetry is $M \to UMU^\dagger$, where U is an unitary matrix, and we require that all physical observables are invariant under this transformation. Therefore, observables are given by products of traces. We will now introduce ribbon graphs, which are diagrams used to compute the correlators of observables in the matrix model, in the same way that we compute correlators in scalar quantum field theory using Feynman diagrams. The rules to draw ribbon diagrams are as follows:

- Each element M_{ij} of the matrix becomes a pair of dots labeled i and j.
- Place the labeled pairs of dots on a line and join pairs of pairs of dots with a pair of lines, a ribbon, without twisting the ribbon or crossing the line on which pairs live.
- Link dots which are labeled with same index by a solid line.

We state the following rules to compute the contribution from a particular ribbon graph

1. Each closed loop contributes a factor of N.
2. Each ribbon contributes a factor of $\frac{1}{\omega}$.

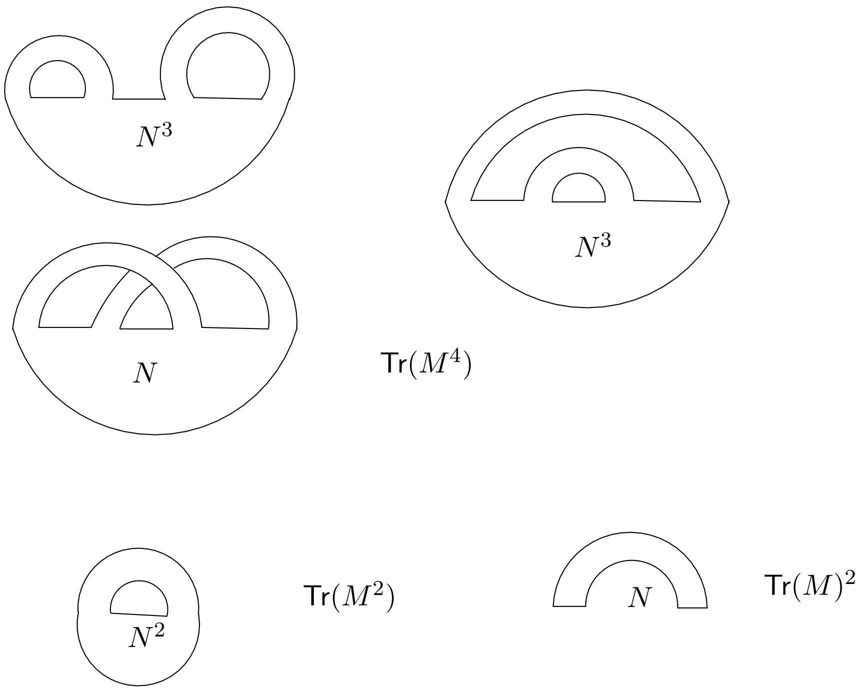


Figure 2.1: Ribbon graph diagrams used to compute $\left\langle \text{Tr}\,(M)^2 \right\rangle_0$, $\left\langle \text{Tr}\,(M^2) \right\rangle_0$ and $\left\langle \text{Tr}\,(M^4) \right\rangle_0$.

We remark that the number of ribbon graphs is equal to $(n-1)!!$ if n is the total number of matrices in the observables in the correlator. This rule is evident in the correlator computations illustrated in the Figures (2.1) and (2.2). Of course, the correlator is only non-zero if n is even.

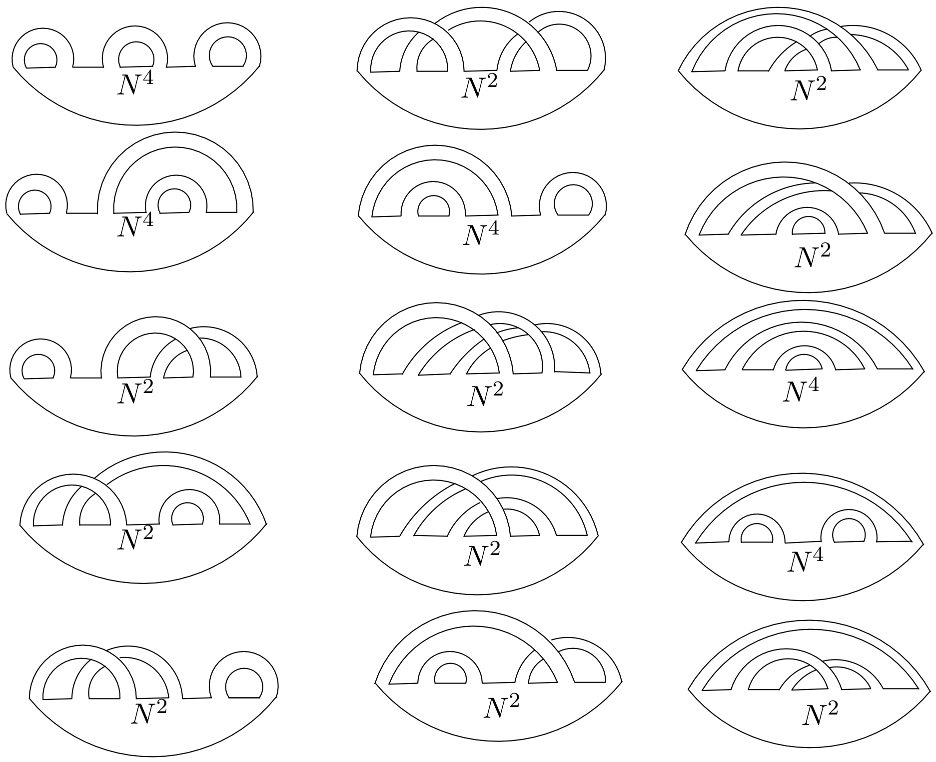


Figure 2.2: Ribbon graph diagrams used to compute $\left\langle \mathsf{Tr}\left(M^6\right)\right\rangle_0$.

2.3 Interacting model

Above we have developed the ribbon graph rules that can be used to compute correlators in the free (Gaussian) matrix model. In this section our goal is to give a set of rules that can be used to compute correlators even when interactions are turned on. To obtain an interacting matrix model we add a term quartic in M. The correlators are defined as follows

$$\langle \ldots \rangle = \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right) - g\mathsf{Tr}\left(M^4\right)} \ldots, \tag{2.3.1}$$

where g is the coupling constant. The generating function associated with this model takes the form

$$Z[J] = \int [dM]\, e^{-\frac{\omega}{2}\mathsf{Tr}\left(M^2\right) - g\mathsf{Tr}\left(M^4\right) + \mathsf{Tr}(JM)}. \tag{2.3.2}$$

The normalization of the generating function is fixed by requiring that $\langle 1 \rangle = 1$. To achieve this we introduce a new generating function called $\tilde{Z}[J]$ such that

$$\tilde{Z}[J] = \frac{Z[J]}{Z[0]}. \tag{2.3.3}$$

In this case, we have the normalized correlators defined as

$$\begin{aligned}\langle M_{ij}M_{kl}\ldots M_{xy}\rangle &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}\cdots\frac{d}{dJ_{yx}}\tilde{Z}[J]\big|_{J=0}\\ &= \frac{1}{Z[0]}\frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}\cdots\frac{d}{dJ_{yx}}Z[J]\big|_{J=0}\\ &= \frac{1}{Z[0]}\int[dM]\,e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)-g\mathrm{Tr}\left(M^4\right)}M_{ij}M_{kl}...M_{xy}. \end{aligned}\tag{2.3.4}$$

It is not possible to evaluate (2.3.4) exactly. We can however develop a perturbative approach by assuming that g is small and expanding (2.3.4) as a power series in g. Thus for any observable $\mathcal{O}$ we have

$$\begin{aligned}\langle\mathcal{O}\rangle &= \frac{1}{Z[0]}\int[dM]\,e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)-g\mathrm{Tr}\left(M^4\right)}\mathcal{O}\\ &= \frac{1}{Z[0]}\sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\int[dM]\,e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)}\left[\mathrm{Tr}\left(M^4\right)\right]^n\mathcal{O}\\ &= \frac{1}{Z[0]}\sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\left\langle\left[\mathrm{Tr}\left(M^4\right)\right]^n\mathcal{O}\right\rangle_0. \end{aligned}\tag{2.3.5}$$

Of course, the generating function itself can be expanded as

$$Z[J] = \sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\int[dM]\,e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)+\mathrm{Tr}(JM)}\left[\mathrm{Tr}\left(M^4\right)\right]^n,\tag{2.3.6}$$

which implies

$$\begin{aligned}Z[0] &= \sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\int[dM]\,e^{-\frac{\omega}{2}\mathrm{Tr}\left(M^2\right)}\left[\mathrm{Tr}\left(M^4\right)\right]^n\\ &= \sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\left\langle\left[\mathrm{Tr}\left(M^4\right)\right]^n\right\rangle_0. \end{aligned}\tag{2.3.7}$$

Finally, the correlator of any observable $\mathcal{O}$ can be written as

$$\langle\mathcal{O}\rangle = \frac{\sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\left\langle\left[\mathrm{Tr}\left(M^4\right)\right]^n\mathcal{O}\right\rangle_0}{\sum_{n=0}^{+\infty}\frac{(-g)^n}{n!}\left\langle\left[\mathrm{Tr}\left(M^4\right)\right]^n\right\rangle_0}.\tag{2.3.8}$$

Starting from (2.3.8), we can use perturbation theory to compute any correlators, to any order in g. For example at order g^2, we have

$$\begin{aligned}
\langle \mathcal{O} \rangle &= \frac{\langle \mathcal{O} \rangle_0 - g \left\langle \mathrm{Tr}\left(M^4\right) \mathcal{O} \right\rangle_0 + \frac{g^2}{2} \left\langle \left[\mathrm{Tr}\left(M^4\right)\right]^2 \mathcal{O} \right\rangle_0 + O(g^3)}{1 - g \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle_0 + \frac{g^2}{2} \left\langle \left[\mathrm{Tr}\left(M^4\right)\right]^2 \right\rangle_0 + O(g^3)} \\
&= \left[\langle \mathcal{O} \rangle_0 - g \left\langle \mathrm{Tr}\left(M^4\right) \mathcal{O} \right\rangle_0 + \frac{g^2}{2} \left\langle \left[\mathrm{Tr}\left(M^4\right)\right]^2 \mathcal{O} \right\rangle_0 + O(g^3) \right] \\
&\times \left[1 + g \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle_0 - \frac{g^2}{2} \left\langle \left[\mathrm{Tr}\left(M^4\right)\right]^2 \right\rangle_0 + O(g^3) \right] \\
&= \langle \mathcal{O} \rangle_0 - g \left\{ \left\langle \mathrm{Tr}\left(M^4\right) \mathcal{O} \right\rangle_0 - \langle \mathcal{O} \rangle_0 \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle_0 \right\} \\
&+ \frac{g^2}{2} \left\{ \left\langle \left[\mathrm{Tr}\left(M^4\right)\right]^2 \mathcal{O} \right\rangle_0 - 2 \left\langle \mathrm{Tr}\left(M^4\right) \mathcal{O} \right\rangle_0 \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle_0 - \langle \mathcal{O} \rangle_0 \left\langle \left[\mathrm{Tr}\left(M^4\right)\right]^2 \right\rangle_0 \right\} + O(g^3).
\end{aligned} \tag{2.3.9}$$

Since we have expressed everything in terms of correlators of the Gaussian matrix model, we can proceed as we did before. For $\mathcal{O} = \mathrm{Tr}\,(M)^2$, we have

$$\begin{aligned}
\left\langle \mathrm{Tr}\,(M)^2 \right\rangle &= \left\langle \mathrm{Tr}\left(M^2\right)\right\rangle_0 - g \left\{ \left\langle \mathrm{Tr}\left(M^4\right) \mathrm{Tr}\left(M^2\right)\right\rangle_0 - \left\langle \mathrm{Tr}\left(M^2\right)\right\rangle_0 \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle_0 \right\} + O(g^2) \\
&= N^2 - g(8N^3 + 4N) + O(g^2),
\end{aligned} \tag{2.3.10}$$

where we have used the results (e), (n) and (j) from Section (2.2.1).

2.3.1 Ribbon graphs rules in the interacting model. First, the following rules are added to the existing rules, to obtain the ribbon graph diagrams for the interacting theory

1. Each matrix element M_{ij} becomes a pair of dots. Indices that are summed are connected by a line.
2. To compute the order n contribution to the correlator, include n vertices that allow four ribbons to meet at a point (see Figure (2.3)).

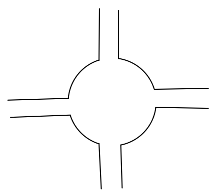


Figure 2.3: Figure showing a vertex.

3. Join pairs of dots, as well as the open ends of the vertices, with ribbons, without twisting the ribbon.

With these rules:

1. Each ribbon contributes a factor of $\frac{1}{\omega}$.
2. Each closed loop contributes a factor of N.

3. Each vertex contributes a factor of $-g$.
4. There is a factor of $\frac{1}{n!}$ for any diagram with n vertices.
5. Drop all terms which include vacuum diagrams.

All diagrams constructed just by joining vertices with ribbons are called vacuum diagrams. The normalization in (2.3.3) removes all vacuum diagrams as illustrated in (2.3.9).

We will give the ribbon graph diagrams for the order g-contribution to $O = \mathrm{Tr}\left(M^2\right)$ in the Figure below. Without normalization, we have the diagrams shown in Figure (2.4) when we add one 4-vertex.

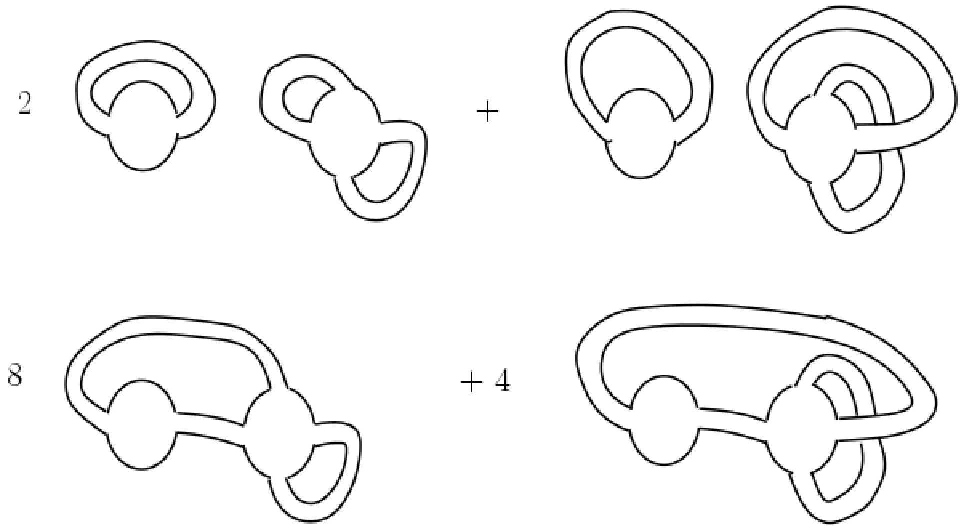


Figure 2.4: Ribbon graphs corresponding to $\left\langle \mathrm{Tr}\left(M^2\right)\right\rangle$ without the normalization.

We see that the vacuum diagrams are subtracted in (2.3.10) by the terms with coefficient $-g$.

2.3.2 Large N limit of the theory and 't Hooft expansion. The computation of correlators using a power series in g becomes problematic when N becomes large i.e when $N \to +\infty$. In fact, for each increasing power of g we have an increasing power of N in the expansion. Thus the leading term is not well determined. We can overcome this problem by setting $gN = \lambda$ which is held fixed when $N \to +\infty$ and $g \to 0$. This is known as the 't Hooft expansion. For every observable $\mathcal{O}$, the correlators take the following form

$$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} f_n(\lambda) N^{2-2n}. \tag{2.3.11}$$

We will now argue that the power of N in this expansion is related to the topology of some surface. This further demonstrates how powerful ribbon graph techniques are for the study of matrix models. It suggests that the summation is performed over all possible surfaces that we will identify as the worldsheet of a string's evolution. This gives an insight into the dynamics of the system modelled by our matrix model.

To determine the N dependence of the ribbon graph diagram for large N, it is convenient to set M to be equal to $\sqrt{N}M'$. The generating function is then given by

$$Z[J] = \int \left[dM'\right] e^{-\frac{N\omega}{2}\mathrm{Tr}\left(M'^2\right)+\lambda N \mathrm{Tr}\left(M'^4\right)+\mathrm{Tr}\left(J'M'\right)}. \tag{2.3.12}$$

From (2.3.12), we deduce that for each ribbon graph diagram:

1. Each ribbon contributes a factor of $\frac{1}{\omega N}$.
2. Each vertex contributes a factor of λN.
3. Each closed loop contributes a factor of N.

Therefore, if we denote by E the number of ribbons, by V the number of vertices and F the number of closed loops, we have

$$\frac{1}{(\omega N)^E}(\lambda N)^V N^F = \frac{1}{\omega^E} N^{F+V-E}\lambda^V, \tag{2.3.13}$$

in which $\chi = F + V - E$ is identified as the Euler characteristic of the topology of a surface. These surfaces are triangulated by ribbon graphs. If each closed loop is considered as the boundary of a piece of rubber then the surface is formed by gluing them together.

One can argue that smooth deformations leave χ invariant. Under the continuous operation of shrinking an edge to nothing, two distinct vertices have been joined to one vertex and the number of faces does not change as shown in Figure (2.5). The number of edges becomes $E' = E - 1$, the number of vertices becomes $V' = V - 1$ with the number of faces unchanged $F' = F$. This gives

$$F' + V' - E' = F + (V - 1) - (E - 1) = \chi. \tag{2.3.14}$$

Another continuous deformation is given by shrinking a face to nothing. If m is the number of edges bounding the face, after we shrink the number of edges will be $E' = E - m$. The number of vertices will be reduced by $m - 1$ since we have to join m vertices into one. Thus there are $V' = V - m + 1$ vertices. In this case, we have

$$F' + V' - E' = (F - 1) + (V - m + 1) - (E - m) = \chi. \tag{2.3.15}$$

An example of shrinking a face, with $m = 5$, is illustrated in Figure (2.6).

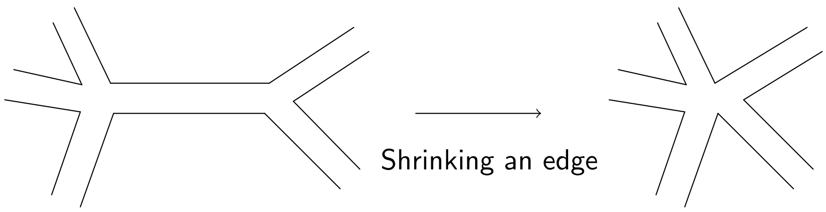


Figure 2.5: Shrinking an edge to nothing.

2.3.3 Planar limit as a classical limit. In the preceding section (2.2.2) we have only considered the free theory. In this case, it is easy to check that if N is large (i.e. $N \to +\infty$), the expectation value of a product is equal to the product of the expectation values

$$\left\langle \prod_i \mathrm{Tr}\,(M^{n_i}) \right\rangle_0 = \prod_i \langle \mathrm{Tr}\,(M^{n_i})\rangle_0 \qquad n_i = 2, 4, 6, \ldots, 2k. \tag{2.3.16}$$

This result holds for the large N limit of any matrix model. In this case, the only ribbon graph diagrams that contribute to the value of the correlators are ribbon graph diagrams that can be drawn on the

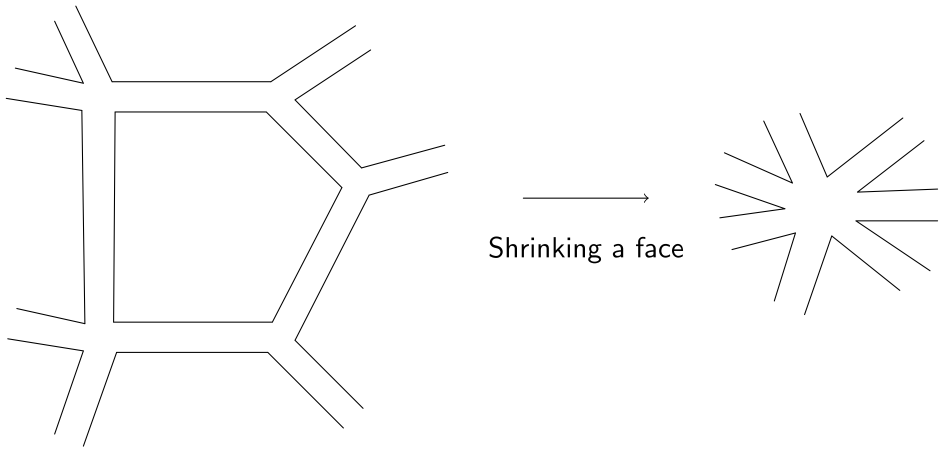


Figure 2.6: Shrinking a face to nothing

surface of a sphere. These diagrams are called planar diagrams. Consider the example of $\left\langle \text{Tr}\left(M^2\right)\right\rangle_0$, $\left\langle \text{Tr}\left(M^4\right)\right\rangle_0$ and $\left\langle \text{Tr}\left(M^6\right)\right\rangle_0$. We have

$$\left\langle \text{Tr}\left(M^2\right)\right\rangle_0 = N^2 \tag{2.3.17}$$
$$\left\langle \text{Tr}\left(M^4\right)\right\rangle_0 = 2N^3 + N \longrightarrow 2N^3 \tag{2.3.18}$$
$$\left\langle \text{Tr}\left(M^6\right)\right\rangle_0 = 5N^4 + 10N^2 \longrightarrow 5N^4. \tag{2.3.19}$$

Consequently we can verify that factorization holds for $\left\langle \text{Tr}\left(M^2\right) \text{Tr}\left(M^4\right)\right\rangle_0$ and $\left\langle \text{Tr}\left(M^2\right) \text{Tr}\left(M^6\right)\right\rangle_0$:

$$\left\langle \text{Tr}\left(M^2\right) \text{Tr}\left(M^4\right)\right\rangle_0 = (N^2+4)(2N^3+N) \longrightarrow \left\langle \text{Tr}\left(M^2\right)\right\rangle_0 \left\langle \text{Tr}\left(M^4\right)\right\rangle_0 = 2N^5 \tag{2.3.20}$$
$$\left\langle \text{Tr}\left(M^2\right) \text{Tr}\left(M^6\right)\right\rangle_0 = (N^2+6)(5N^4+10N^2) \longrightarrow \left\langle \text{Tr}\left(M^2\right)\right\rangle_0 \left\langle \text{Tr}\left(M^6\right)\right\rangle_0 = 5N^6. \tag{2.3.21}$$

Factorization is a strong indication that the system is in a classical-like limit. To explain why factorization is an indication that we are in a classical limit of the theory, we need to recall how correlators are computed. Let $\mathcal{O}$ be an observable which takes the value $\mathcal{O}(i)$ when the system is in a given state i, with a probability μ_i. Thus, the expectation value is given by

$$\langle \mathcal{O} \rangle = \sum_i \mu_i \mathcal{O}(i) \quad \text{with } \sum_i \mu_i = 1, \tag{2.3.22}$$

in which the sum is taken over all possible states of the system. Then, for any product of observables $\mathcal{O}_j$ with $j = 1, 2, \ldots n$, we have

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle = \sum_i \mu_i \mathcal{O}_1(i)\mathcal{O}_2(i)\ldots\mathcal{O}_n(i). \tag{2.3.23}$$

Now, if factorization holds, the above equation is equal to

$$\langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \ldots \langle \mathcal{O}_n \rangle = \sum_{i_1} \mu_{i_1} \mathcal{O}_1(i_1) \sum_{i_2} \mu_{i_2} \mathcal{O}_2(i_2) \cdots \sum_{i_n} \mu_{i_n} \mathcal{O}_n(i_n) \quad \forall n. \tag{2.3.24}$$

The equality of the Equations (2.3.23) and (2.3.24) implies that there exists only one state i' of the system such that

$$\mu_i = \begin{cases} 1 & \text{if } i = i' \\ 0 & \text{otherwise} \end{cases} , \tag{2.3.25}$$

which means that only one configuration contributes to the value of the correlators. This corresponds to the planar limit of our theory. Moreover, we have argued that corrections to the values of correlators come from ribbon graphs that triangulate higher genus surfaces in the large N limit. These corrections are due to fluctuations about the classical limit, so that we identify them with $\hbar$ corrections. In this way we see that the matrix model suggests that we identify $\hbar = \frac{1}{N^2}$.

In summary, we have started by defining the generating function Z for our matrix model, in order to compute correlators. There are different ways to evaluate correlators. For example, we can differentiate the generating function, or use Schwinger-Dyson equations. In addition, we are able to draw diagrams called ribbon graphs, to compute correlators. The description making use of ribbon graphs has suggested an interpretation of the computation of correlation functions as a sum over surfaces. From this, we have argued that our matrix model is related to a theory of strings with two parameters $\hbar = \frac{1}{N^2}$ and the 't Hooft coupling $\lambda = gN$ where g and N are respectively the coupling constant and the rank of the gauge group in the matrix model. The large N limit of the matrix model gives the classical limit of the dual string theory.

3. Group representation theory and Schur polynomials

In the previous chapter we have developed methods allowing us to compute correlation functions in a matrix model (Corley et al., 2002). We have found that in the large N limit, correlators are correctly reproduced by summing only the planar diagrams. In fact, this conclusion is only true when the number of matrices Δ in our observable is held fixed as we take $N \to \infty$. If we scale Δ with N, the sums over huge numbers of ribbon graphs implies that non-planar diagrams contribute (de Mello Koch et al., 2009). In this case, ribbon graph techniques are ineffective. Therefore, we have to develop new techniques to study this limit of the theory. These new techniques use some basic notions from group theory and representation theory.

3.1 Background group theory

3.1.1 Definition. A group is a set $\mathcal{G}$ together with a map $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ called group composition, which verifies the following axioms:

1. The group composition is closed

$$g_1 \cdot g_2 \in \mathcal{G}, \quad \forall g_1, g_2 \in \mathcal{G}. \tag{3.1.1}$$

2. There exists an element e called the identity of $\mathcal{G}$ such that

$$e \cdot g = g \cdot e = g, \quad \forall g \in \mathcal{G}. \tag{3.1.2}$$

3. For every $g \in \mathcal{G}$, there exists an element $g^{-1} \in \mathcal{G}$ called inverse of g such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e. \tag{3.1.3}$$

4. The group operation $\cdot$ is associative, which means

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3, \quad \forall g_1, g_2, g_3 \in \mathcal{G}. \tag{3.1.4}$$

3.1.2 Order of a group $\mathcal{G}$. The order of the group $\mathcal{G}$ is the number of elements of $\mathcal{G}$, denoted by $|\mathcal{G}|$. We say that the group $\mathcal{G}$ is finite if $|\mathcal{G}|$ is finite. Here are a few examples:

- S_n: The group of permutations of n objects is a finite group of order $|\mathcal{G}| = n!$.
- $GL(n, K)$: The group of all $n \times n$ invertible matrices whose matrix elements belong to a field K.
- $U(N)$: The group of $N \times N$ unitary matrices.

The last two groups listed are not finite.

3.1.3 Multiplication Table. For any finite group $\mathcal{G}$, we can construct a table which gives the composition of two elements belonging to the group.

3.1.4 Example. S_2: Group of permutations of 2 objects. In this case we have two group elements: we can swap the two objects or we do nothing. These two operations that we have performed correspond to the elements of S_2 which are the identity element, denote by $()$ and the swap operation by $(1,2)$. Thus, the group S_2 can be written as

$$S_2 = \{(), (1,2)\}. \tag{3.1.5}$$

Consider the multiplication table for S_2. It is obvious that if we perform the swap operation twice, we obtain the same configuration. Moreover, if we do nothing and then swap the result is the same as just swapping the elements. Following the same reasoning, we obtain the following table.

	()	(1,2)
()	()	(1,2)
(1,2)	(1,2)	()

Table 3.1: Multiplication table of S_2.

The notation adopted here is the cycle notation. This notation is used to define any element of any group of permutations of any number of objects. The notation $\sigma = (a, b, c)$ means b takes the position of a, c takes the position b and a takes the position of c and anything else does not change position. Then, we can write

$$\sigma(a) = b, \quad \sigma(b) = c, \quad \sigma(c) = a \text{ and } \quad \sigma(d) = d \quad \text{for } d \neq a, b, c. \tag{3.1.6}$$

3.1.5 Example. S_3: Group of permutations of 3 objects. Using the cycle notation to write the elements of the group S_3, we obtain

$$S_3 = \{(), (1,2), (1,2,3), (1,3,2), (2,3), (1,3)\}. \tag{3.1.7}$$

The multiplication table for this group is given by the following table.

	()	(1,2)	(1,2,3)	(1,3,2)	(2,3)	(1,3)
()	()	(1,2)	(1,2,3)	(1,3,2)	(2,3)	(1,3)
(1,2)	(1,2)	()	(2,3)	(1,3)	(1,2,3)	(1,3,2)
(1,2,3)	(1,2,3)	(1,3)	(1,3,2)	()	(1,2)	(2,3)
(1,3,2)	(1,3,2)	(2,3)	()	(1,2,3)	(1,3)	(1,2)
(2,3)	(2,3)	(1,3,2)	(1,3)	(1,2)	()	(1,2,3)
(1,3)	(1,3)	(1,2,3)	(1,2)	(2,3)	(1,3,2)	()

Table 3.2: Multiplication table of S_3.

3.2 Matrix representations of a group

We recall that $GL(n, K)$ is the set of all invertible $n \times n$ matrices whose entries are elements of a given field K.

3.2.1 Definition. A matrix representation of a group $\mathcal{G}$ is map $\Gamma(\cdot) : \mathcal{G} \to GL(n, K)$ such that:

$$\Gamma(g_1)\Gamma(g_2) = \Gamma(g_1 g_2), \quad \forall g_1, g_2 \in \mathcal{G}. \tag{3.2.1}$$

The product on the right hand side of this formula is the usual matrix multiplication. The product on the left had side is the composition law of the group $\mathcal{G}$.

3.2.2 Equivalent representations. We say that $\Gamma_R(\cdot)$ and $\Gamma_S(\cdot)$ are equivalent representations of the group $\mathcal{G}$ if there exists an invertible matrix M such that

$$\Gamma_R(g) = M\Gamma_S(g)M^{-1} \quad \forall g \in \mathcal{G}. \tag{3.2.2}$$

This implies that

$$\mathsf{Tr}\left(\Gamma_R(g)\right) = \mathsf{Tr}\left(\Gamma_S(g)\right) \quad \forall g \in \mathcal{G}. \tag{3.2.3}$$

3.2.3 Characters of group elements. The character of group element g in representation R is given by

$$\chi_R(g) = \mathsf{Tr}\left(\Gamma_R(g)\right). \tag{3.2.4}$$

We point out that two representations are equivalent if and only if their characters are equal. To prove that, let $A_1, A_2, ..., A_N$ stand for the matrices which represent the elements of a group $\mathcal{G}$ in representation R and $A'_1, A'_2, ..., A'_N$ stand for the matrices of representation R'. If R and R' are equivalent representation, we know that $A'_i = MA_iM^{-1}$ for $i = 1$ to N. We can see that the characters in representation R and R' are equal. To show that the equality of the characters in the two representations is enough to prove the representations are equivalent, we need to argue that equality of the characters is enough to prove that the trace of any product of matrices in representation R is equal to the trace of any product of matrices in the representation R'. Indeed, this last statement is true if and only if A_i and A'_i are related by $A'_i = MA_iM^{-1}$ for $i = 1$ to N. But the product of matrices in a given representation is the matrix which represents the product of the elements of the group. Thus this proves that two representations are equivalent if and only if their characters are equal.

3.2.4 Direct sum of two representations. Let $\Gamma_R(\cdot)$ and $\Gamma_S(\cdot)$ be two representations of the group $\mathcal{G}$. The direct sum of $\Gamma_R(\cdot)$ and $\Gamma_S(\cdot)$ is another representation $\Gamma(\cdot)$ of $\mathcal{G}$ defined by

$$\Gamma(g) = \begin{bmatrix} \Gamma_R(g) & O_{d_1 \times d_2} \\ O_{d_2 \times d_1} & \Gamma_S(g) \end{bmatrix}, \quad \forall g \in \mathcal{G}. \tag{3.2.5}$$

We denote the direct sum of $\Gamma_R(\cdot)$ and $\Gamma_S(\cdot)$ by

$$\Gamma(g) = \Gamma_R(g) \oplus \Gamma_S(g). \tag{3.2.6}$$

3.2.5 Reducible representation. Any representation that is equivalent to a block diagonal representation is called a reducible representation. Any representation that is not reducible is called irreducible.

Let V be a d-dimensional vector space and V_1 a subspace of V. We say that V_1 is an invariant subspace of $\mathcal{G}$ if

$$\forall \left|v\right\rangle \in V_1 \Rightarrow \Gamma(g)\left|v\right\rangle \in V_1, \quad \forall g \in \mathcal{G}. \tag{3.2.7}$$

This means that $\Gamma(\cdot)$ is block diagonal which implies that any irreducible representation has no invariant proper subspaces.

3.3 Fundamental orthogonality relation

The fundamental orthogonality relation will play an important role when we apply group representation theory to matrix models. Schur's Lemmas play an important part in the proof of the fundamental orthogonality relation. For that reason, we state (without proof) the two Lemmas in this section.

3.3.1 Schur lemma 1. Let R be an irreducible representation of $\mathcal{G}$. If we have some matrix A satisfying $\Gamma_R(g)A = A\Gamma_R(g), \forall g \in \mathcal{G}$, then $A = \lambda\mathbb{I}$ with $\lambda \in \mathbb{C}$.

3.3.2 Schur lemma 2. Let R and S be two inequivalent irreducible representations of $\mathcal{G}$. The only solution to $\Gamma_R(g)A = A\Gamma_S(g), \forall g \in \mathcal{G}$ is $A = 0$.

3.3.3 Fundamental orthogonality relation. To derive the Fundamental orthogonality relation, we start by studying the collection of matrices

$$[B(R,S,b,\alpha)]_{a\beta} = \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta}. \tag{3.3.1}$$

Choosing a specific value for R, S, b and α chooses a specific matrix from the collection. The row index of this matrix is a and the column index is β. First, let us multiply by $\Gamma_S(g_1)_{\beta\gamma}$ to obtain

$$\begin{aligned}
[B(R,S,b,\alpha)]_{\alpha\beta}\,\Gamma_S(g_1)_{\beta\gamma} &= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta}\Gamma_S(g_1)_{\beta\gamma} \\
&= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(gg_1)_{\alpha\gamma}.
\end{aligned} \tag{3.3.2}$$

Now, change the summation variable from g to $\bar{g} = gg_1$ or $g^{-1} = g_1\bar{g}^{-1}$. Thus

$$\begin{aligned}
[B(R,S,b,\alpha)]_{\alpha\beta}\,\Gamma_S(g_1)_{\beta\gamma} &= \sum_{\bar{g}\in\mathcal{G}} \Gamma_R(g_1\bar{g}^{-1})_{ab}\Gamma_S(\bar{g})_{\alpha\gamma} \\
&= \Gamma_R(g_1)_{ac} \sum_{\bar{g}\in\mathcal{G}} \Gamma_R(\bar{g}^{-1})_{cb}\Gamma_S(\bar{g})_{\alpha\gamma} \\
&= \Gamma_R(g_1)_{ac}\,[B(R,S,b,\alpha)]_{c\gamma}\,.
\end{aligned} \tag{3.3.3}$$

Using Schur's Lemmas, we conclude

$$[B(R,S,b,\alpha)]_{c\gamma} = \delta_{RS}\delta_{c\gamma}\lambda(b,\alpha,R) \tag{3.3.4}$$

where

$$\lambda(b,\alpha,R) = \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{\alpha b}. \tag{3.3.5}$$

This last value is obtained by evaluating the trace Tr $(B(R,S,b,\alpha))$ by using (3.3.1) and then by using (3.3.4). We now have the fundamental orthogonality relation

$$\sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta} = \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{a\beta}\delta_{\alpha b}. \tag{3.3.6}$$

The fundamental orthogonality relation could be used to derive an orthogonality relation for characters. To see this, we set $b = a$ and $\beta = \alpha$ and sum over α and a to obtain

$$\begin{aligned}\sum_{a\alpha}\sum_{g\in\mathcal{G}}\Gamma_R(g^{-1})_{aa}\Gamma_S(g)_{\alpha\alpha} &= \frac{|\mathcal{G}|}{d_R}\delta_{RS}\sum_{a\alpha}\delta_{a\alpha}\delta_{\alpha a} \\ &= \frac{|\mathcal{G}|}{d_R}\delta_{RS}\sum_{a}\delta_{aa}. \end{aligned} \tag{3.3.7}$$

By identifying

$$\sum_a \Gamma_R(g^{-1})_{aa} = \text{Tr}\left(\Gamma_R(g^{-1})\right) = \chi_R(g^{-1}) \tag{3.3.8}$$

and

$$\sum_\alpha \Gamma_S(g)_{\alpha\alpha} = \text{Tr}\left(\Gamma_S(g)\right) = \chi_S(g), \tag{3.3.9}$$

(3.3.7) becomes

$$\sum_{g\in\mathcal{G}}\chi_R(g^{-1})\chi_S(g) = \delta_{RS}|\mathcal{G}|. \tag{3.3.10}$$

This is the orthogonality relation for characters. We can specialize this relation to a unitary representation or to an orthogonal representation as follows

- For a unitary representation, $\text{Tr}\left(\Gamma_R(g^{-1})\right) = \text{Tr}\left(\Gamma_R(g)^{-1}\right) = \text{Tr}\left(\Gamma_R(g)^{\dagger}\right)$ so that

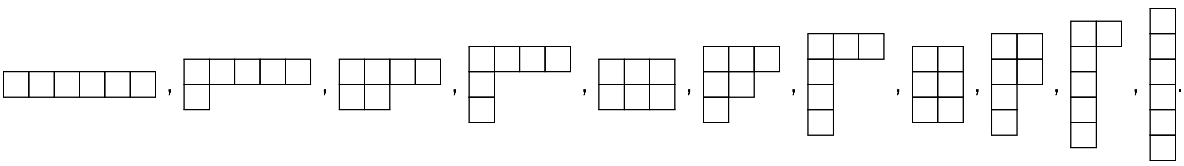
$$\sum_{g\in\mathcal{G}}\chi_R^{\dagger}(g)\chi_S(g) = \delta_{RS}|\mathcal{G}|. \tag{3.3.11}$$

- For an orthogonal representation, $\text{Tr}\left(\Gamma_R(g^{-1})\right) = \text{Tr}\left(\Gamma_R(g)^{-1}\right) = \text{Tr}\left(\Gamma_R(g)^{T}\right) = \text{Tr}\left(\Gamma_R(g)\right)$ so that

$$\sum_{g\in\mathcal{G}}\chi_R(g)\chi_S(g) = \delta_{RS}|\mathcal{G}|. \tag{3.3.12}$$

3.4 Matrix representations of the symmetric group S_n

3.4.1 Young diagrams. A Young diagram is a finite collection of boxes arranged in left-justified rows, with the row lengths weakly decreasing. Young diagrams are used to label the complete set of inequivalent irreducible representations of the symmetric group S_n, as well as the states in the carrier space of these representations. With $k = 6$ boxes, the following diagrams are all of the valid Young diagrams



We immediately see that S_6 has eleven inequivalent irreducible representations.

3.4.2 Hook lengths. The hook length of a box x in Young diagram R, denoted by hook(x), is the number of boxes that are in the same row to the right of x plus the number of boxes in the same column below x plus one (for x itself).

3.4.3 The dimensions of an irrep of S_n. The dimension of an irreducible representation R of the symmetric group S_n is given by

$$d_R = \frac{n!}{\prod_{x\in R} \mathsf{hook}(x)}. \tag{3.4.1}$$

3.4.4 Young-Yamanouchi states. The Young-Yamanouchi states are obtained by decorating the Young diagram. For a Young diagram with n boxes, the Young-Yamanouchi states are obtained by filling each box in the Young diagram with an integer from 1 to n such that for each box the integers in the boxes placed to the right and below it are less than the integer which labels the given box itself. These Young-Yamanouchi states are basis vectors of the vector space in which we can construct the representation of the symmetric group S_n. The Young-Yamanouchi states are a complete set, i.e. the number of Young-Yamanouchi states is equal to the dimension of the irreducible representation.

We will now study two examples which illustrate the labeling of Young-Yamanouchi states.

3.4.5 Example. Consider R $= \ydiagram{2,1}$ which is an irreducible representation of S_3. The dimension of this representation is

$$d_{\ydiagram{2,1}} = \frac{3!}{\ytableaushort{31,1}} = 2. \tag{3.4.2}$$

This matches the fact that there are two possible Young-Yamanouchi states, which are given by

$\left|\ytableaushort{31,2}\right\rangle, \left|\ytableaushort{32,1}\right\rangle.$

3.4.6 Example. Our second example considers R $= \ydiagram{3,2,1}$ which is an irreducible representation of S_6. The dimension of this representation is

$$d_{\ydiagram{3,2,1}} = \frac{6!}{\ytableaushort{531,31,1}} = 16. \tag{3.4.3}$$

There are indeed 16 possible Young-Yamanouchi states given by

$\left|\ytableaushort{631,52,4}\right\rangle, \left|\ytableaushort{641,52,3}\right\rangle, \left|\ytableaushort{651,42,3}\right\rangle, \left|\ytableaushort{641,53,2}\right\rangle, \left|\ytableaushort{651,43,2}\right\rangle, \left|\ytableaushort{632,51,4}\right\rangle, \left|\ytableaushort{642,51,3}\right\rangle, \left|\ytableaushort{652,41,3}\right\rangle, \left|\ytableaushort{643,51,2}\right\rangle, \left|\ytableaushort{653,41,2}\right\rangle,$
$\left|\ytableaushort{654,31,2}\right\rangle, \left|\ytableaushort{642,53,1}\right\rangle, \left|\ytableaushort{652,43,1}\right\rangle, \left|\ytableaushort{643,52,1}\right\rangle, \left|\ytableaushort{653,42,1}\right\rangle, \left|\ytableaushort{654,32,1}\right\rangle.$

3.4.7 Matrix representations of the adjacent 2-cycles of S_n. To build the matrix representation of the symmetric group S_n, we only need to compute the matrix representations of the adjacent 2-cycles $(i, i+1)$ for $i = 1, 2, \ldots (n-1)$ of S_n. This follows because any element of the group can be written as a product of adjacent 2-cycles. To build the matrix representations $\Gamma_R(.)$ we need the action of these matrices on each basis vector of the corresponding vector space. The action of $\Gamma_R((i, i+1))$ on the Young-Yamanouchi state $|YY\rangle$ is given by:

$$\Gamma_R((i,i+1))\,|YY\rangle = \frac{1}{c_i - c_{i+1}}\,|YY\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}\,|YY\rangle_{(i,i+1)} \tag{3.4.4}$$

where $|YY\rangle_{(i,i+1)}$ is the state obtained by permuting the labels i and $i+1$ in the state $|YY\rangle$ and c_i, c_{i+1} are respectively the content of the box labelled by i and the content of the box labelled by $i+1$. The content of each box is obtained as follows

- The content of the box in the left-top corner of the diagram is zero.
- The content of the box placed immediately on the right of a given box is equal to the content of that box plus one.
- The content of the box placed immediately below a given box is equal to the content of that box minus one.

3.4.8 Example. R $= \begin{array}{ll} \square & \square \\ \square & \end{array}$, a representation of S_3.

Basis: $\left| \begin{array}{ll} 3 & 1 \\ 2 & \end{array} \right\rangle$, $\left| \begin{array}{ll} 3 & 2 \\ 1 & \end{array} \right\rangle$.

Dimension of this space $d_R = 2$.

Matrix representation of $\sigma_i = \{(1,2),(2,3)\}$:

$$\Gamma_R((1,2)) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \; \Gamma_R((2,3)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We know that $(1,2)(1,2) = ()$ so that the above matrices must square to the identity. It is easy to see that this is indeed the case.

3.4.9 Example. R $= \begin{array}{ll} \square & \square \\ \square & \\ \square & \end{array}$, a representation of S_4.

Basis: $\left| \begin{array}{ll} 4 & 1 \\ 3 & \\ 2 & \end{array} \right\rangle$, $\left| \begin{array}{ll} 4 & 2 \\ 3 & \\ 1 & \end{array} \right\rangle$, $\left| \begin{array}{ll} 4 & 3 \\ 2 & \\ 1 & \end{array} \right\rangle$.

Dimension of this space $d_R = 3$.

Matrix representation of $\sigma_i = \{(1,2),(2,3),(3,4)\}$:

$$\Gamma_R((1,2)) = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; \Gamma_R((2,3)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \; \Gamma_R((3,4)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is again easy to see that the above matrices square to the identity.

3.5 Complex matrix model and Schur polynomials

In this section, we start by introducing a model with one complex matrix Z. We will see that non-vanishing correlators are of the form $\langle Z_{i_1 j_1} Z_{i_2 j_2} \cdots Z_{i_n j_n} Z^{\dagger}{}_{k_1 l_1} Z^{\dagger}{}_{k_2 l_2} \cdots Z^{\dagger}{}_{k_n l_n} \rangle$ i.e. the number of Z fields must equal the number of $Z^{\dagger}$ fields. We will demonstrate how group theory is implemented in the computation of correlators. We will first write the correlator $\langle Z_{i_1 j_1} Z_{i_2 j_2} \cdots Z_{i_n j_n} Z^{\dagger}{}_{k_1 l_1} Z^{\dagger}{}_{k_2 l_2} \cdots Z^{\dagger}{}_{k_n l_n} \rangle$ as a sum over all possible permutations of the symmetric group S_n. This is achieved by considering a tensor product of n copies of the vector space in which the matrix field Z is acting. In other words, we work on the tensor vector space $V_N^{\otimes n}$ if the matrix field Z is acting on V_N. Thereafter, we argue that any multi-trace structure can be written as a single trace on $V_N^{\otimes n}$. After that, we introduce a set of projectors on the irreducible subspaces of $V_N^{\otimes n}$ that are related to the construction of the Schur polynomials. Next, we define the Schur polynomials and argue that we can compute any correlators using their linear combinations.

3.5.1 Complex matrix model. Using two Hermitian matrices M_1 and M_2, we can construct a free complex matrix model. The model of two Hermitian matrices has the following generating function

$$Z_C[J_1, J_2] = \int dM_1\, dM_2 e^{-\frac{1}{2}\mathrm{Tr}\left(M_1^2\right)-\frac{1}{2}\mathrm{Tr}\left(M_2^2\right)+\mathrm{Tr}(J_1 M_1)+\mathrm{Tr}(J_2 M_2)}. \tag{3.5.1}$$

This generating function is, as usual, normalized so that

$$Z_C[J_1, J_2]\big|_{J_1=J_2=0} = 1. \tag{3.5.2}$$

One can evaluate the integrations with respect to M_1 and M_2 independently. The use of the Gaussian as we did in (2.2.8) gives

$$Z_C[J_1, J_2] = Z_0[J_1]Z_0[J_2], \tag{3.5.3}$$

where $Z_0[J]$ is the generating function for single matrix model. Since, we have

$$Z_C[J_1, 0] = Z_0[J_1] \tag{3.5.4}$$

$$Z_C[0, J_2] = Z_0[J_2] \tag{3.5.5}$$

one can reproduce the value of the following correlator

$$\begin{aligned}\langle (M_1)_{ij}(M_1)_{kl}\rangle &= \frac{d}{(dJ_1)_{ji}}\frac{d}{(dJ_1)_{lk}}Z_0[J_1]_{|_{J_1=0}}\\ &= \delta_{il}\delta_{jk}.\end{aligned} \tag{3.5.6}$$

Similarly for $\langle (M_2)_{ij}(M_2)_{kl}\rangle$, we find the same result

$$\begin{aligned}\langle (M_2)_{ij}(M_2)_{kl}\rangle &= \frac{d}{(dJ_2)_{ji}}\frac{d}{(dJ_2)_{lk}}Z_0[J_2]_{|_{J_2=0}}\\ &= \delta_{il}\delta_{jk}.\end{aligned} \tag{3.5.7}$$

From (3.5.3), we can show that correlators of any operator built from one matrix alone, either M_1 or M_2, agree with the correlators in the single matrix model.

Now consider operators involving the two matrices M_1 and M_2. First, we have

$$\begin{aligned}\langle (M_1)_{ij}(M_2)_{kl}\rangle &= \frac{d}{(dJ_1)_{ji}}\frac{d}{(dJ_2)_{lk}}Z_C[J_1, J_2]_{|_{J_1,J_2=0}}\\ &= \frac{d}{(dJ_1)_{ji}}Z_0[J_1]_{|_{J_1=0}}\frac{d}{(dJ_2)_{lk}}Z_0[J_2]_{|_{J_2=0}}\\ &= 0,\end{aligned} \tag{3.5.8}$$

which is expected. In fact, the correlator of any observable involving the two matrices is given by

$$\begin{aligned}&\langle (M_1)_{i_1j_1}(M_1)_{i_2j_2}\dots(M_1)_{i_nj_n}(M_2)_{k_1l_1}(M_2)_{k_2l_2}\dots(M_2)_{k_ml_m}\rangle\\ &= \langle (M_1)_{i_1j_1}(M_1)_{i_2j_2}\dots(M_1)_{i_nj_n}\rangle\,\langle (M_2)_{k_1l_1}(M_2)_{k_2l_2}\dots(M_2)_{k_ml_m}\rangle\,,\end{aligned} \tag{3.5.9}$$

for the free theory. Introduce the complex matrix field defined by

$$Z = \frac{1}{\sqrt{2}}(M_1 + iM_2), \tag{3.5.10}$$

with Hermitian conjugate given by

$$Z^\dagger = \frac{1}{\sqrt{2}}(M_1 - iM_2). \tag{3.5.11}$$

We will use (3.5.6), (3.5.8) and (3.5.7) to find

$$\begin{aligned} \langle Z_{ij} Z_{kl} \rangle &= \frac{1}{2} \left\langle (M_1 + iM_2)_{ij} (M_1 + iM_2)_{kl} \right\rangle \\ &= \frac{1}{2} \langle (M_1)_{ij} (M_1)_{kl} \rangle - \frac{1}{2} \langle (M_2)_{ij} (M_2)_{kl} \rangle \\ &= 0. \end{aligned} \tag{3.5.12}$$

Similarly, we have

$$\left\langle Z^\dagger_{ij} Z^\dagger_{kl} \right\rangle = 0 \tag{3.5.13}$$

$$\left\langle Z_{ij} Z^\dagger_{kl} \right\rangle = \delta_{il} \delta_{jk}. \tag{3.5.14}$$

The evaluation of correlators using Wick's theorem proves that

$$\left\langle Z_{i_1 j_1} Z_{i_2 j_2} \cdots Z_{i_n j_n} Z^\dagger{}_{k_1 l_1} Z^\dagger{}_{k_2 l_2} \cdots Z^\dagger{}_{k_m l_m} \right\rangle = 0 \quad \text{if} \quad n \neq m. \tag{3.5.15}$$

That is to say the number of Zs and $Z^\dagger$s appearing in the correlator should be equal since the contractions Z-Z or $Z^\dagger$-$Z^\dagger$ vanish. Therefore we need to compute correlators of the form

$$\left\langle Z_{i_1 j_1} Z_{i_2 j_2} \cdots Z_{i_n j_n} Z^\dagger{}_{k_1 l_1} Z^\dagger{}_{k_2 l_2} \cdots Z^\dagger{}_{k_n l_n} \right\rangle. \tag{3.5.16}$$

The number of ways of contracting n of Z to n of $Z^\dagger$ is equal to $n!$, which is exactly equal to the number of ways of permuting n distinct objects. This already gives some insight into how we use group theory in the computation of correlators. The next section will develop this idea to show how group theory is used.

Another way of computing the correlators is using the generating function. From the two-matrix model in (3.5.1), we obtain the generating function of the complex matrix model by changing fields from M_1 and M_2 to Z and $Z^\dagger$ according to (3.5.10) and (3.5.11). In fact, we have

$$dM_1 dM_2 = |\mathcal{J}| dZ dZ^\dagger \tag{3.5.17}$$

where $\mathcal{J}$ is the determinant of the Jacobian of the change of variables, which is a constant number. Further, the integrand in (3.5.1) becomes

$$e^{-\frac{1}{2}\mathrm{Tr}\left(M_1^2\right) - \frac{1}{2}\mathrm{Tr}\left(M_2^2\right) + \mathrm{Tr}(J_1 M_1) + \mathrm{Tr}(J_2 M_2)} = e^{-\mathrm{Tr}\left(Z Z^\dagger\right) + \mathrm{Tr}\left(J^\dagger Z\right) + \mathrm{Tr}\left(J Z^\dagger\right)} \tag{3.5.18}$$

where the sources J and $J^\dagger$ are given by

$$J = \frac{1}{\sqrt{2}}(J_1 + iJ_2) \tag{3.5.19}$$

$$J^\dagger = \frac{1}{\sqrt{2}}(J_1 - iJ_2). \tag{3.5.20}$$

The generating function (3.5.1), written in terms of Z and $Z^\dagger$ is

$$Z_C = |\mathcal{J}| \int dZ\, dZ^\dagger\, e^{-\mathsf{Tr}\left(ZZ^\dagger\right)+\mathsf{Tr}\left(J^\dagger Z\right)+\mathsf{Tr}\left(JZ^\dagger\right)}. \tag{3.5.21}$$

With this redefinition of fields, the correlators are given by

$$\langle \mathcal{O} \rangle = |\mathcal{J}| \int dZ\, dZ^\dagger e^{-\mathsf{Tr}\left(ZZ^\dagger\right)} \mathcal{O}. \tag{3.5.22}$$

These can be computed by taking appropriate derivatives of (3.5.21) with respect to J and $J^\dagger$ where

$$\frac{d}{dJ_{ij}} = \frac{1}{\sqrt{2}} \left(\frac{d}{(dJ_1)_{ji}} - i\frac{d}{(dJ_2)_{ji}} \right) \tag{3.5.23}$$

$$\frac{d}{dJ^\dagger_{kl}} = \frac{1}{\sqrt{2}} \left(\frac{d}{(dJ_1)_{kl}} + i\frac{d}{(dJ_2)_{kl}} \right). \tag{3.5.24}$$

As an example, the correlators in (3.5.12), (3.5.14) and (3.5.13) are given by

$$\begin{aligned}&\langle Z_{ij} Z_{kl} \rangle \\ &= \frac{d}{dJ^\dagger_{ij}} \frac{d}{dJ^\dagger_{kl}} Z_{C|_{J=0}} = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{d}{(dJ_1)_{ji}} + i\frac{d}{(dJ_2)_{ji}} \right) \left(\frac{d}{(dJ_1)_{kl}} + i\frac{d}{(dJ_2)_{kl}} \right) Z_C[J_1, J_2]_{J_1,J_2=0}\end{aligned} \tag{3.5.25}$$

$$\begin{aligned}&\left\langle {Z^\dagger}_{ij} {Z^\dagger}_{kl} \right\rangle \\ &= \frac{d}{dJ_{ij}} \frac{d}{dJ_{kl}} Z_{C|_{J=0}} = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{d}{(dJ_1)_{ji}} - i\frac{d}{(dJ_2)_{ji}} \right) \left(\frac{d}{(dJ_1)_{kl}} - i\frac{d}{(dJ_2)_{kl}} \right) Z_C[J_1, J_2]_{J_1,J_2=0}\end{aligned} \tag{3.5.26}$$

$$\begin{aligned}&\left\langle Z_{ij} {Z^\dagger}_{kl} \right\rangle \\ &= \frac{d}{dJ^\dagger_{ij}} \frac{d}{dJ_{kl}} Z_{C|_{J=0}} = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{d}{(dJ_1)_{ji}} + i\frac{d}{(dJ_2)_{ji}} \right) \left(\frac{d}{(dJ_1)_{kl}} - i\frac{d}{(dJ_2)_{kl}} \right) Z_C[J_1, J_2]_{J_1,J_2=0},\end{aligned} \tag{3.5.27}$$

where $Z_C[J_1, J_2]$ is the generating function for the two-matrix model in (3.5.1) and Z_C the generating function for the complex matrix model in (3.5.3). Since we know how to obtain $Z_C[J_1, J_2]$, any correlators can be computed. In view of the number of derivatives we have to perform, this is a tedious and long computation. Ribbon graphs can be used to obtain the correlators but it is also not practical when the number of fields in the correlators increase.

As a remark, the generalization to the multi-matrix model (Bhattacharyya et al., 2008a) can be achieved using a similar approach. From a model with $2p$ Hermitian matrices $M_1, M_2, N_1, N_2, \cdots$, we can define

p complex fields $Z, Y, \cdots$ such that

$$Z = \frac{1}{\sqrt{2}}\left(M_1 + M_2\right), \qquad Z^\dagger = \frac{1}{\sqrt{2}}\left(M_1 - M_2\right) \tag{3.5.28}$$

$$Y = \frac{1}{\sqrt{2}}\left(N_1 + N_2\right), \qquad Y^\dagger = \frac{1}{\sqrt{2}}\left(N_1 - N_2\right), \tag{3.5.29}$$

and so on. The generating function will be given by

$$Z_C = |\mathcal{J}| \int dZ\, dZ^\dagger\, e^{-\text{Tr}(ZZ^\dagger)-\text{Tr}(YY^\dagger)+\text{Tr}(J^\dagger Z)+\text{Tr}(JZ^\dagger)+\text{Tr}(R^\dagger Y)+\text{Tr}(RY^\dagger)+\cdots} \tag{3.5.30}$$

$$= Z_C[J_1, J_2, R_1, R_2, \cdots], \tag{3.5.31}$$

where $\mathcal{J}$ is the determinant of the Jacobian matrix from the change of variables and $Z_C[J_1, J_2, R_1, R_2, \cdots]$ is the generating function of the model with $2p$ Hermitian matrices. It is clear that

$$Z_C[J_1, J_2, R_1, R_2, \cdots] = Z_0[J_1]Z_0[J_2]Z_0[R_1]Z_0[R_2]\cdots \tag{3.5.32}$$

with $Z_0[J]$ the generating function of a free matrix model with sources $J_1, J_2, R_1, R_2, \cdots$ which are the sources for the fields $M_1, M_2, N_1, N_2, \cdots$. We can also introduce sources for the complex fields as follows

$$J = \frac{1}{\sqrt{2}}(J_1 + iJ_2), \qquad J^\dagger = \frac{1}{\sqrt{2}}(J_1 - iJ_2) \tag{3.5.33}$$

$$R = \frac{1}{\sqrt{2}}(R_1 + iR_2), \qquad R^\dagger = \frac{1}{\sqrt{2}}(R_1 - iR_2), \tag{3.5.34}$$

and similar equations for the other sources.

Using the identity (3.5.32) the computation of correlators is straightforward, by taking appropriate derivatives with respect to $J, J^\dagger, R, R^\dagger, \cdots$.

3.5.2 Correlation functions using group theory. We will compute correlators using group theory, following Corley et al. (2002). In the following, we use Z^i_j to denote the matrix element of Z in the ith row and jth column. Let us also introduce the following notation

$$(Z^{\otimes n})^I_J = Z^{i_1}_{j_1} \dots Z^{i_n}_{j_n} \tag{3.5.35}$$

$$\sigma^I_J = \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_n}_{j_{\sigma(n)}}, \qquad \sigma \in S_n. \tag{3.5.36}$$

Using the above notation, the correlator $\left\langle Z^{i_1}_{j_1} \dots Z^{i_n}_{j_n} Z^{\dagger k_1}_{l_1} \dots Z^{\dagger k_n}_{l_n} \right\rangle$ is expressed as a sum over all possible permutations of S_n

$$\left\langle Z^{i_1}_{j_1} \dots Z^{i_n}_{j_n} Z^{\dagger k_1}_{l_1} \dots Z^{\dagger k_n}_{l_n} \right\rangle = \left\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \right\rangle = \sum_{\sigma \in S_n} \sigma^I_L (\sigma^{-1})^K_J. \tag{3.5.37}$$

As an example, consider the case where $n = 3$. This computation uses the action of S_3 on the space

$V_N^{\otimes 3}$. Thus, for the correlators, we have

$$\begin{aligned}\left\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} Z^{\dagger k_1}_{l_1} Z^{\dagger k_2}_{l_2} Z^{\dagger k_3}_{l_3} \right\rangle &= \left\langle (Z^{\otimes 3})^I_J (Z^{\dagger \otimes 3})^K_L \right\rangle \\ &= \sum_{\sigma \in S_3} (\sigma)^I_L (\sigma^{-1})^K_J \\ &= (1)^I_L (1^{-1})^K_J + ((1,2))^I_L ((1,2)^{-1})^K_J + ((1,2,3))^I_L ((1,2,3)^{-1})^K_J \\ &+ ((1,3,2))^I_L ((1,3,2)^{-1})^K_J + ((2,3))^I_L ((2,3)^{-1})^K_J + ((1,3))^I_L ((1,3)^{-1})^K_J \\ &= (1)^I_L (1)^K_J + ((1,2))^I_L ((1,2))^K_J + ((1,2,3))^I_L ((1,3,2))^K_J \\ &+ ((1,3,2))^I_L ((1,2,3))^K_J + ((2,3))^I_L ((2,3))^K_J + ((1,3))^I_L ((1,3))^K_J ,\end{aligned} \tag{3.5.38}$$

where, by (3.5.36), we have

$$\begin{aligned}(1)^I_L (1)^K_J &= \delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{i_3}_{l_3} \delta^{k_1}_{j_1} \delta^{k_2}_{j_2} \delta^{k_3}_{j_3} \\ ((1,2))^I_L ((1,2))^K_J &= \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{i_3}_{l_3} \delta^{k_1}_{j_2} \delta^{k_2}_{j_1} \delta^{k_3}_{j_3} \\ ((1,2,3))^I_L ((1,3,2))^K_J &= \delta^{i_1}_{l_2} \delta^{i_2}_{l_3} \delta^{i_3}_{l_1} \delta^{k_1}_{j_3} \delta^{k_2}_{j_1} \delta^{k_3}_{j_2} \\ ((1,3,2))^I_L ((1,2,3))^K_J &= \delta^{i_1}_{l_3} \delta^{i_2}_{l_1} \delta^{i_3}_{l_2} \delta^{k_1}_{j_2} \delta^{k_2}_{j_3} \delta^{k_3}_{j_1} \\ ((2,3))^I_L ((2,3))^K_J &= \delta^{i_1}_{l_1} \delta^{i_2}_{l_3} \delta^{i_3}_{l_2} \delta^{k_1}_{j_1} \delta^{k_2}_{j_3} \delta^{k_3}_{j_2} \\ ((1,3))^I_L ((1,3))^K_J &= \delta^{i_1}_{l_3} \delta^{i_2}_{l_2} \delta^{i_3}_{l_1} \delta^{k_1}_{j_3} \delta^{k_2}_{j_2} \delta^{k_3}_{j_1} .\end{aligned}$$

This is an explicit demonstration that the sum over Wick contractions can be realized as a sum over permutations. Now we give the argument that any multi-trace structure can be written as a single trace on the space $V_N^{\otimes n}$. By allowing $\sigma \in S_n$ to permute the factors of V_N in $V_N^{\otimes n}$, we have

$$\begin{aligned}\mathsf{Tr}\left(\sigma Z^{\otimes n}\right) &= (\sigma)^I_J (Z^{\otimes n})^J_I \\ &= \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_n}_{j_{\sigma(n)}} Z^{j_1}_{i_1} \dots Z^{j_n}_{i_n} \\ &= Z^{j_1}_{j_{\sigma(1)}} \dots Z^{j_n}_{j_{\sigma(n)}} .\end{aligned} \tag{3.5.39}$$

Consider the case where $\sigma = (1,2) = (1,2)(3)$. We have

$$\begin{aligned}\mathsf{Tr}\left(\sigma Z^{\otimes 3}\right) &= \mathsf{Tr}\left(((1,2)(3)) Z^{\otimes 3}\right) \\ &= Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} \\ &= \mathsf{Tr}\left(Z^2\right) \mathsf{Tr}\left(Z\right) .\end{aligned} \tag{3.5.40}$$

In general, for any permutation which belongs to conjugacy class with cycle structure

$$\sigma = \sigma_{J_1} \sigma_{J_2} \cdots \sigma_{J_k}, \quad J_i \geq 1 \tag{3.5.41}$$

where σ_{J_i} is a J_i-cycle (a permutation of length J_i), we have

$$\mathsf{Tr}\left(\sigma Z^{\otimes n}\right) = \mathsf{Tr}\left(Z^{J_1}\right) \mathsf{Tr}\left(Z^{J_2}\right) \cdots \mathsf{Tr}\left(Z^{J_k}\right) . \tag{3.5.42}$$

Any multitrace structure can be written as a single trace in the tensor vector space $V_N^{\otimes n}$ (Corley et al., 2002).

3.5.3 Remark. For any $\sigma, \tau \in S_n$, we have

$$\sigma^I_J \tau^J_K = (\tau\sigma)^I_K. \tag{3.5.43}$$

To prove this, note that

$$\begin{aligned}\sigma^I_J \tau^J_K &= \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \dots \delta^{i_n}_{j_{\sigma(n)}} \delta^{j_1}_{k_{\tau(1)}} \delta^{j_2}_{k_{\tau(2)}} \dots \delta^{j_n}_{k_{\tau(n)}} \\ &= \prod_{q=1}^{n} \delta^{i_q}_{j_{\sigma(q)}} \prod_{p=1}^{n} \delta^{j_p}_{k_{\tau(p)}}.\end{aligned} \tag{3.5.44}$$

Now, let $p = \sigma(q)$. Then, we can write

$$\begin{aligned}\sigma^I_J \tau^J_K &= \prod_{q=1}^{n} \delta^{i_q}_{j_{\sigma(q)}} \delta^{j_{\sigma(q)}}_{k_{\tau(\sigma(q))}} \\ &= \prod_{q=1}^{n} \delta^{i_q}_{k_{\tau(\sigma(q))}} \\ &= \delta^{i_1}_{k_{\tau\sigma(1)}} \delta^{i_2}_{k_{\tau\sigma(2)}} \dots \delta^{i_n}_{k_{\tau\sigma(n)}} \\ &= (\tau\sigma)^I_K.\end{aligned} \tag{3.5.45}$$

3.5.4 Projection operators. Previously, by enlarging the space we work in from V_N to $V_N^{\otimes n}$, any multi-trace structure could be written as a single trace structure. In addition, the sum over Wick contractions can be written in terms of the action of the symmetric group S_n on $V_N^{\otimes n}$. Now, we introduce a suitable set of projection operators on the space $V_N^{\otimes n}$ so that we can construct an orthogonal basis of operators, known as the Schur polynomials, which can be used to compute the correlators of any observables. We will give some properties of these projection operators. We will then compute their trace on $V_N^{\otimes n}$.

The projection operators are given by

$$(P_R)^I_J = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \sigma^I_J, \tag{3.5.46}$$

and they satisfy the general properties of any complete set of projectors

$$P_R P_S = \delta_{RS} P_S \tag{3.5.47}$$

and

$$\sum_R P_R = \mathbb{I} \tag{3.5.48}$$

in which the summation is taken over all possible inequivalent irreducible representations.

Besides the properties mentioned above, we also have the following the commutation relation

$$P_R \psi = \psi P_R, \quad \forall \psi \in S_n. \tag{3.5.49}$$

These facts are proved as follows

$$(P_R)^I_J(P_S)^J_K = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma^I_J\frac{d_S}{n!}\sum_{\tau\in S_n}\chi_S(\tau)\tau^J_K$$
$$= \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma,\tau\in S_n}\chi_R(\sigma)\chi_S(\tau)\sigma^I_J\tau^J_K. \tag{3.5.50}$$

Using (3.5.43), we have

$$(P_R)^I_J(P_S)^J_K = \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma,\tau\in S_n}\chi_R(\sigma)\chi_S(\tau)(\tau\sigma)^I_K. \tag{3.5.51}$$

Now, change variables from σ, τ to $\sigma, \psi = \tau\sigma$. We find $\tau = \psi\sigma^{-1}$. Then, we have

$$(P_R)^I_J(P_S)^J_K = \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma,\tau\in S_n}\chi_R(\sigma)\chi_S(\psi\sigma^{-1})(\psi)^I_K$$
$$= \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma,\psi\in S_n}\Gamma_R(\sigma)_{ii}\Gamma_S(\psi)_{jk}\Gamma_S(\sigma^{-1})_{kj}(\psi)^I_K. \tag{3.5.52}$$

By the Fundamental Orthogonality Relation, we have

$$\sum_{\sigma\in S_n}\Gamma_R(\sigma)_{ii}\Gamma_S(\sigma^{-1})_{kj} = \frac{n!}{d_R}\delta_{RS}\delta_{ij}\delta_{ik}$$
$$= \frac{n!}{d_R}\delta_{RS}\delta_{jk}. \tag{3.5.53}$$

Therefore, we obtain

$$(P_R)^I_J(P_S)^J_K = \frac{d_S}{n!}\sum_{\psi\in S_n}\Gamma_S(\psi)_{jk}\delta_{RS}\delta_{jk}(\psi)^I_K$$
$$= \frac{d_S}{n!}\sum_{\psi\in S_n}\Gamma_S(\psi)_{kk}\delta_{RS}(\psi)^I_K$$
$$= \delta_{RS}\frac{d_S}{n!}\sum_{\psi\in S_n}\Gamma_S(\psi)_{kk}(\psi)^I_K$$
$$= \delta_{RS}P_S. \tag{3.5.54}$$

The commutation relation $[P_R, \psi]$ is proved as follows

$$(P_R)^I_J(\psi)^J_K = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma^I_J(\psi)^J_K. \tag{3.5.55}$$

Now, let us change the summation variable from σ to τ where

$$\sigma^I_J\psi^J_K = \psi^I_J\tau^J_K \tag{3.5.56}$$

or,

$$\begin{aligned}\sigma^I_J &= \psi^I_L \tau^L_K (\psi^{-1})^K_J \\ &= \psi^I_L (\psi^{-1}\tau)^L_J \\ &= (\psi^{-1}\tau\psi)^I_J. \end{aligned} \tag{3.5.57}$$

Then

$$\begin{aligned}(P_R)^I_J(\psi)^J_K &= \frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)\psi^I_J\tau^J_K \\ &= \frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\tau)\psi^I_J\tau^J_K \\ &= \psi^I_J\frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\tau)\tau^J_K \\ &= \psi^I_J(P_R)^J_K. \end{aligned} \tag{3.5.58}$$

In the above equations, we have used the fact that two conjugate elements have the same character.

The trace of the operator P_R is given by:

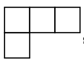
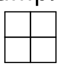
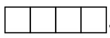
$$\text{Tr}\,(P_R) = d_R \text{Dim}_R \tag{3.5.59}$$

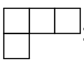
where

$$\text{Dim}_R = \frac{\text{factor}_R}{\text{Hooks}_R}. \tag{3.5.60}$$

To prove the result (3.5.59), we are using the fact that there are the commuting actions of the Symmetric group S_n and the Unitary group $U(N)$ on $V_N^{\otimes n}$. Thus we can simultaneously diagonalize the actions of S_n and $U(N)$ in the space $V_N^{\otimes n}$. Moreover, the space $V_N^{\otimes n}$ decomposes into subspaces labelled by Young diagrams R with at most N rows, such that each subspace has basis vectors labelled by a symmetric group label (i.e the Young-Yamanouchi symbols YY) and a unitary group label (i.e the Gelfand-Tsetlin pattern GT). Thus each basis vector has the form $|YY, GT\rangle$. In the subspace labelled by Young diagram R, the total number of Young-Yamanouchi symbols is d_R, while the total number of Gelfand-Tsetlin patterns is Dim_R. Consequently, the dimension of the subspace labelled by R is $d_R\text{Dim}_R$ and the number of states in $V_N^{\otimes n}$ is

$$N_{\text{states}} = \sum_R d_R \text{Dim}_R, \tag{3.5.61}$$

which is equal to N^n, the dimension of the whole space $V_N^{\otimes n}$. Secondly, the projection operator P_R is projecting into the subspace labelled by R of $V_N^{\otimes n}$ whose dimension is $d_R\text{Dim}_R$. Therefore, the trace to P_R is equal to $d_R\text{Dim}_R$. As an example, consider the actions of S_4 and $U(2)$ on $V_2^{\otimes 4}$. Then, the subspaces are labelled by R = ,  and .

- For R= :

$$\text{factor}_R = \begin{array}{|c|c|c|}\hline N & N+1 & N+2 \\ \hline N-1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \text{and } \text{Hooks}_R = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},$$

$$\begin{aligned}\text{Dim}_R &= \frac{1}{8}(N+2)(N+1)(N-1)N = \frac{1}{8}(2+2)(2+1)(2-1)2 = 3, \\ d_R &= 3.\end{aligned} \tag{3.5.62}$$

- For R= $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}$:

$$\text{factor}_R = \begin{array}{|c|c|}\hline N & N+1 \\ \hline N-1 & N \\ \hline\end{array} \quad \text{and } \text{Hooks}_R = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array},$$

$$\begin{aligned}\text{Dim}_R &= \frac{1}{12}(N+1)(N-1)N^2 = \frac{1}{12}(2+1)(2-1)2^2 = 1, \\ d_R &= 2.\end{aligned} \tag{3.5.63}$$

- For R= $\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline\end{array}$:

$$\text{factor}_R = \begin{array}{|c|c|c|c|}\hline N & N+1 & N+2 & N+3 \\ \hline\end{array} \quad \text{and } \text{Hooks}_R = \begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline\end{array},$$

$$\begin{aligned}\text{Dim}_R &= \frac{1}{24}(N+3)(N+2)(N+1)N = \frac{1}{24}(2+3)(2+2)(2+1)2 = 5, \\ d_R &= 1.\end{aligned} \tag{3.5.64}$$

As a result, we have

$$\sum_R d_R \text{Dim}_R = 3\times 3 + 2\times 1 + 1\times 5 = 16, \tag{3.5.65}$$

which is equal to $N^n = 2^4 = 16$.

3.5.5 Schur polynomials. We will define the Schur polynomials (Corley et al., 2002). We will then compute their correlators. After that we will argue that any multi-trace structure can be written as a linear combination of Schur polynomials.

For a single complex matrix model, the Schur polynomial is defined by

$$\begin{aligned}\chi_R(Z) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J(Z^{\otimes n})^J_I \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\text{Tr}\left(\sigma Z^{\otimes n}\right).\end{aligned} \tag{3.5.66}$$

Using (3.5.46), the Schur polynomial can be written as a single trace

$$\chi_R(Z) = \frac{1}{d_R}\mathsf{Tr}\left(P_R Z^{\otimes n}\right). \tag{3.5.67}$$

Now, we can compute the following correlators

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle &= \left\langle \frac{1}{d_R}\mathsf{Tr}\left(P_R Z^{\otimes n}\right)\frac{1}{d_S}\mathsf{Tr}\left(P_S Z^{\dagger\otimes n}\right)\right\rangle \\ &= \frac{1}{d_R}\frac{1}{d_S}\left\langle \mathsf{Tr}\left(P_R Z^{\otimes n}\right)\mathsf{Tr}\left(P_S Z^{\dagger\otimes n}\right)\right\rangle \\ &= \frac{1}{d_R}\frac{1}{d_S}\left\langle (P_R)^I_J (Z^{\otimes n})^J_I (P_S)^K_L (Z^{\dagger\otimes n})^L_K\right\rangle \\ &= \frac{1}{d_R}\frac{1}{d_S}(P_R)^I_J (P_S)^K_L\left\langle (Z^{\otimes n})^J_I (Z^{\dagger\otimes n})^L_K\right\rangle.\end{aligned} \tag{3.5.68}$$

Using (3.5.37), we have

$$\left\langle (Z^{\otimes n})^J_I (Z^{\dagger\otimes n})^L_K\right\rangle = \sum_{\sigma\in S_n}\sigma^J_K(\sigma^{-1})^L_I, \tag{3.5.69}$$

so that (3.5.68) becomes

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle &= \frac{1}{d_R}\frac{1}{d_S}\sum_{\sigma\in S_n}(P_R)^I_J(P_S)^K_L\sigma^J_K(\sigma^{-1})^L_I \\ &= \frac{1}{d_R}\frac{1}{d_S}\sum_{\sigma\in S_n}\mathsf{Tr}\left(P_R\sigma P_S\sigma^{-1}\right).\end{aligned} \tag{3.5.70}$$

Now, we use the fact that P_S commutes with σ to find

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle &= \frac{1}{d_R}\frac{1}{d_S}\sum_{\sigma\in S_n}\mathsf{Tr}\left(P_R P_S\right) \\ &= \frac{n!}{d_R d_S}\mathsf{Tr}\left(P_R P_S\right).\end{aligned} \tag{3.5.71}$$

Finally, (3.5.47) allows us to write

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle &= \frac{n!}{d_R d_S}\delta_{RS}\mathsf{Tr}\left(P_S\right) \\ &= \frac{n!}{d_R d_S}\delta_{RS}d_S\mathsf{Dim}_S,\end{aligned} \tag{3.5.72}$$

where Dim_R is given by (3.5.60). Thus, we obtain

$$\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle = \frac{n!}{d_R}\delta_{RS}\frac{\mathsf{factor}_S}{\mathsf{Hooks}_S}. \tag{3.5.73}$$

Now, using $d_R = \frac{n!}{\mathsf{Hooks}_R}$, (3.5.73) becomes

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle &= \delta_{RS}\mathsf{factor}_S\frac{\mathsf{Hooks}_R}{\mathsf{Hooks}_S} \\ &= \delta_{RS}\mathsf{factor}_S.\end{aligned} \tag{3.5.74}$$

The following will show that any multi-trace operator can be written in terms of a linear combination of Schur polynomials. This will use the orthogonality relation in group theory and the definition of the Schur polynomials. We have

$$\begin{aligned}\sum_{R}\sum_{\sigma\in S_n}\chi_R(\sigma)\chi_S(\sigma)\chi_R(Z) &= \sum_{R}\sum_{\sigma\in S_n}\Gamma_R(\sigma)_{ii}\Gamma_S(\sigma)_{jj}\chi_R(Z)\\ &= \sum_R \frac{n!}{d_R}\delta_{RS}\delta_{ij}\delta_{ij}\chi_R(Z)\\ &= n!\chi_S(Z)\\ &= \sum_{\sigma\in S_n}\chi_S(\sigma)\mathsf{Tr}\left(\sigma Z^{\otimes n}\right).\end{aligned} \tag{3.5.75}$$

This last identity shows that

$$\mathsf{Tr}\left(\sigma Z^{\otimes n}\right) = \sum_R \chi_R(\sigma)\chi_R(Z), \tag{3.5.76}$$

which is combined with (3.5.42) to complete the argument that any multi-trace structure can be written as a linear combination of Schur polynomials.

3.5.6 Example. To illustrate the results obtained above, consider the symmetric group S_3 and the gauge group $U(3)$. The symmetric group S_3 has three conjugacy classes whose representatives are $(), (1,2)$ and $(1,2,3)$. There are three inequivalent irreducible representations associated to the groups S_3 and $U(3)$. They are labelled by R $= \yng(1,1,1)$, $\yng(3)$ and $\yng(2,1)$. The correlators of the Schur polynomials in these representation are

$$\left\langle \chi_{\yng(1,1,1)}(Z)\chi^{\dagger}_{\yng(1,1,1)}(Z)\right\rangle = N(N-1)(N-2) \tag{3.5.77}$$

$$\left\langle \chi_{\yng(3)}(Z)\chi^{\dagger}_{\yng(3)}(Z)\right\rangle = N(N+1)(N+2) \tag{3.5.78}$$

$$\left\langle \chi_{\yng(2,1)}(Z)\chi^{\dagger}_{\yng(2,1)}(Z)\right\rangle = N(N+1)(N-1). \tag{3.5.79}$$

The table of characters is given below

R	$\{()\}$	$\{(1,2),(2,3),(1,3)\}$	$\{(1,3,2),(1,2,3)\}$
$\yng(1,1,1)$	1	-1	1
$\yng(3)$	1	1	1
$\yng(2,1)$	2	0	-1

Table 3.3: Table of characters for S_3 in a representation R

Using the character table, the Schur polynomials are given by

$$\chi_{\yng(1,1,1)}(Z) = \frac{1}{6}\left(\text{Tr}\left(Z^3\right) - 3\text{Tr}\left(Z\right)\text{Tr}\left(Z^2\right) + 2\text{Tr}\left(Z^3\right)\right) \tag{3.5.80}$$

$$\chi_{\yng(3)}(Z) = \frac{1}{6}\left(\text{Tr}\left(Z^3\right) + 3\text{Tr}\left(Z\right)\text{Tr}\left(Z^2\right) + 2\text{Tr}\left(Z^3\right)\right) \tag{3.5.81}$$

$$\chi_{\yng(2,1)}(Z) = \frac{1}{6}\left(2\text{Tr}\left(Z^3\right) - 2\text{Tr}\left(Z^3\right)\right). \tag{3.5.82}$$

The different multi-trace structures written as linear combinations of Schur polynomials are given by

$$\begin{aligned}\text{Tr}\left(Z\right)^3 &= \text{Tr}\left(\left(\right) Z^{\otimes 3}\right)\\ &= \chi_{\yng(1,1,1)}(Z) + \chi_{\yng(3)}(Z) + 2\chi_{\yng(2,1)}(Z)\end{aligned} \tag{3.5.83}$$

$$\begin{aligned}\text{Tr}\left(Z\right)\text{Tr}\left(Z^2\right) &= \text{Tr}\left(\left(1,2\right) Z^{\otimes 3}\right)\\ &= -\chi_{\yng(1,1,1)}(Z) + \chi_{\yng(3)}(Z)\end{aligned} \tag{3.5.84}$$

$$\begin{aligned}\text{Tr}\left(Z^3\right) &= \text{Tr}\left(\left(1,2,3\right) Z^{\otimes 3}\right)\\ &= \chi_{\yng(1,1,1)}(Z) + \chi_{\yng(3)}(Z) - \chi_{\yng(2,1)}(Z).\end{aligned} \tag{3.5.85}$$

With these linear combinations and the use of the orthogonality of the Schur polynomials, we find the correlators of the different multi-trace structures given in the table below.

$\mathcal{O}_1\mathcal{O}_2^\dagger$	$\text{Tr}\left(Z\right)^3$	$\text{Tr}\left(Z\right)\text{Tr}\left(Z^2\right)$	$\text{Tr}\left(Z^3\right)$
$\text{Tr}\left(Z\right)^3$	$6\,N^3$	$6\,N^2$	$6\,N$
$\text{Tr}\left(Z\right)\text{Tr}\left(Z^2\right)$	$6\,N^2$	$2\left(N^2+2\right)N$	$6\,N^2$
$\text{Tr}\left(Z^3\right)$	$6\,N$	$6\,N^2$	$3\left(N^2+1\right)N$

Table 3.4: Correlators of multi-trace structures under the gauge group $U(N=3)$.

The results in this table are in agreement with the correlators of the Schur polynomials given in (3.5.79). We can also find the results

$$\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle = JN^J\left(1+O\left(\frac{1}{N^2}\right)\right) \tag{3.5.86}$$

$$\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^K\right)\text{Tr}\left(Z^{\dagger J+K}\right)\right\rangle = JK(J+K)N^{J+K-1}\left(1+O\left(\frac{1}{N^2}\right)\right) \tag{3.5.87}$$

even though, we have not considered N to be large. The general results for these correlators are stated in Chapter (5). They will be used to compute correlators involving giant gravitons. It is also easy to check that

$$\left\langle \text{Tr}\left(Z\right)^J\text{Tr}\left(Z^\dagger\right)^J\right\rangle = J!\,N^J, \tag{3.5.88}$$

as shown in Table (3.4).

3.6 Two-matrix model and restricted Schur polynomials

In this section, we add a new matrix field Y, which is acting on the same vector space V_N as the previous matrix field Z (Balasubramanian et al., 2005). We then consider observables of the form

$$\text{Tr}\left(Z^{n_1}Y^{m_1}Z^{n_2}Y^{m_2}\dots Z^{n_k}Y^{m_k}\right). \tag{3.6.1}$$

Let us study the case with n matrix fields Z and m matrix fields Y. For that, we work on $V_N^{\otimes n+m}$ and we introduce the following notation:

$$(Z^{\otimes n}Y^{\otimes m})^I_J = Z^{i_1}_{j_1}Z^{i_2}_{j_2}\dots Z^{i_n}_{j_n}Y^{i_{n+1}}_{j_{n+1}}Y^{i_{n+2}}_{j_{n+2}}\dots Y^{i_{n+m}}_{j_{n+m}}. \tag{3.6.2}$$

We have also

$$\text{Tr}\left(\rho Z^{\otimes n}Y^{\otimes m}\right) = Z^{i_1}_{i_{\rho(1)}}Z^{i_2}_{i_{\rho(2)}}\dots Z^{i_n}_{i_{\rho(n)}}Y^{i_{n+1}}_{i_{\rho(n+1)}}Y^{i_{n+2}}_{i_{\rho(n+2)}}\dots Y^{i_{n+m}}_{i_{\rho(n+m)}}, \tag{3.6.3}$$

where $\rho \in S_{n+m}$.

The observable above is invariant under the action of $S_n \times S_m$ which follows because the Ys and Zs are bosonic fields. Thus permutations of the Zs among each other and the Ys among each other is a symmetry of the theory. We have

$$(\sigma)^I_J(Z^{\otimes n}Y^{\otimes m})^J_K(\sigma^{-1})^K_L = (Z^{\otimes n}Y^{\otimes m})^I_L, \quad \sigma \in S_n \times S_m \subset S_{n+m}. \tag{3.6.4}$$

So

$$\text{Tr}\left(\rho Z^{\otimes n}Y^{\otimes m}\right) = \text{Tr}\left(\rho\sigma(Z^{\otimes n}Y^{\otimes m})\sigma^{-1}\right) \tag{3.6.5}$$
$$= \text{Tr}\left(\sigma^{-1}\rho\sigma(Z^{\otimes n}Y^{\otimes m})\right). \tag{3.6.6}$$

This argument shows that ρ and $\sigma^{-1}\rho\sigma$ with $\sigma \in S_n \times S_m$, give rise to the same observable. In the following, we say that g and h are restricted conjugate if

$$g = \sigma^{-1}h\sigma \quad \text{with } g, h \in S_{n+m}, \sigma \in S_n \times S_m. \tag{3.6.7}$$

Consequently, the number of observables will equal the number of restricted conjugacy classes.

We know that after restricting to a subgroup, a given irreducible representation will decompose into a number of irreducible representations of the subgroup in which more than one copy of a representation of the subgroup may appear. We distinguish identical copies by adding a new label α, called the multiplicity label. Moreover, to label the representation of S_{n+m} we need a Young diagrams R with $n+m$ boxes, another Young diagram r with n boxes to label a representation of S_n and another Young diagram s with m boxes to label the representation of S_m. Thus the representation of $S_n \times S_m$ is labelled by $(r,s)\alpha$. Consequently, we denote states in the carrier space $(r,s)\alpha$ of the representation of $S_n \times S_m$ by $|R,(r,s)\alpha,i\rangle$ where the label i runs from 1 to $d_r d_s$ in which d_r and d_s are the dimensions of the representations labelled by s and r.

Now, we can define the restricted character of the group elements of S_{n+m} by (de Mello Koch et al., 2007a)

$$\chi_{R,(r,s)\alpha\beta}(\sigma) = \sum_{i=1}^{d_r d_s} \langle R,(r,s)\alpha;i|\,\Gamma_R(\sigma)\,|R,(r,s)\beta;i\rangle, \quad \sigma \in S_{n+m}, \tag{3.6.8}$$

in which we restrict the trace by only summing over the indices that belong to the specific irreducible representation of the subgroup $S_n \times S_m$.

Next, we can define operators, the so called restricted Schur polynomials, which are single traces on $V^{\otimes n+m}$, by (Bhattacharyya et al., 2008a)

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \text{Tr}\left(\sigma Z^{\otimes n} Y^{\otimes m}\right) \tag{3.6.9}$$

$$= \text{Tr}\left(P_{R,(r,s)\alpha\beta} Z^{\otimes n} Y^{\otimes m}\right), \tag{3.6.10}$$

where

$$P_{R,(r,s)\alpha\beta} = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma). \tag{3.6.11}$$

The operators $P_{R,(r,s)\alpha\beta}$ satisfy

$$\left[\sigma, P_{R,(r,s)\alpha\beta}\right] = 0, \quad \sigma \in S_n \times S_m \tag{3.6.12}$$

$$P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta} = \delta_{RS}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\lambda P_{R,,(r,s)\alpha\delta}, \quad \lambda \in K = \mathbb{R}, \mathbb{C} \tag{3.6.13}$$

$$\Gamma_{(r,s)\alpha}(\sigma) P_{R,(r,s)\alpha\beta} = P_{R,(r,s)\alpha\beta}\Gamma_{(r,s)\beta}(\sigma), \quad \sigma \in S_n \times S_m. \tag{3.6.14}$$

Here, the correlation function $\langle (Z^{\otimes n} Y^{\otimes m})^I_J (Z^{\dagger \otimes n} Y^{\dagger \otimes m})^K_L \rangle$ can be written in terms of permutations as

$$\left\langle (Z^{\otimes n} Y^{\otimes m})^I_J (Z^{\dagger \otimes n} Y^{\dagger \otimes m})^K_L \right\rangle = \sum_{\sigma \in S_n \times S_m} (\sigma^{-1})^I_L (\sigma)^K_J. \tag{3.6.15}$$

So, by setting $A = R, (r,s)\alpha\beta$ and $B = T, (t,u)\gamma\delta$, we have

$$\left\langle \chi_A(Z,Y) \chi_B(Z,Y)^\dagger \right\rangle = \sum_{\sigma \in S_n \times S_m} \text{Tr}\left(P_A \sigma P_B \sigma^{-1}\right) \tag{3.6.16}$$

$$= n!m! \text{Tr}\left(P_A P_B\right). \tag{3.6.17}$$

Then using (3.6.13), we see that $\chi_A(Z,Y)$ and $\chi_B(Z,Y)$ are orthogonal. Thus the computation of $\langle \chi_A(Z,Y) \chi_B(Z,Y)^\dagger \rangle$ is easily achieved.

In conclusion, correlators involving operators constructed from order N fields were studied. In this chapter we found that the correlation functions of any gauge theory operator can be computed exactly using Schur polynomials. The orthogonality of these operators simplifies the computations. Further, we can sum lots more than just the planar diagrams by using the new group theory methods.

4. Holographic computation of scalar field correlation functions

In this chapter, we compute correlators of scalar fields using holography. The tools that are necessary for these computations are introduced in the first section. We then consider two and three-point correlation functions.

4.1 Holographic principle and the AdS/CFT correspondence

The AdS_{d+1}/CFT_d correspondence (Maldacena, 1999) is the correspondence between the theory of gravity in $(d+1)$-dimensional AdS space with conformal field theory on the d-dimensional space that is the boundary of the AdS spacetime. The AdS_{d+1}/CFT_d correspondence is a realization of the holographic principle which postulates the exact equality of a theory of gravity in a $(d+1)$-dimensional bulk spacetime and quantum field theory on the d-dimensional boundary. Using the AdS_{d+1}/CFT_d correspondence we can evaluate the correlation functions of the CFT side as follows:

a- Determine the bulk field ϕ dual to the operator $\mathcal{O}$ of dimension Δ. The boundary condition ϕ_0 for ϕ plays the role of a source for $\mathcal{O}$. Indeed, the generating function Z_{AdS} in the quantum gravity and Z_{CFT} in the CFT are related by (Witten, 1998)

$$Z_{CFT}[\mathcal{O}] = \left\langle \exp\left(i\int d^dx\,\phi_0\mathcal{O}\right)\right\rangle_{CFT} = Z_{AdS}(\phi_0), \tag{4.1.1}$$

where ϕ_0 is the field configuration at the boundary of the AdS spacetime.

b- Minimize the action on the AdS side to obtain the equations of motion.

c- Solve the equations of motion for ϕ with boundary condition ϕ_0.

d- Insert the solution into the action on the AdS side.

e- Take variational derivatives with respect to the boundary configuration ϕ_0, which plays the role of a source for the operator $\mathcal{O}$.

Now, we may perform a Wick rotation and use the saddle point approximation to evaluate Z_{AdS} so that

$$Z_{AdS}(\phi_0) \approx \exp\left(-S_{EAdS}(\phi_0)\right), \tag{4.1.2}$$

where $S_{EAdS}(\phi_0)$ is the Euclidean action on the AdS side evaluated at the solution of the equation of motion for ϕ with boundary configuration ϕ_0. This approximation amounts to ignoring quantum gravity corrections. In the dual CFT this corresponds to studying the $N\to\infty$ limit. The correlation functions of the CFT side are finally obtained as follows

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_{CFT} = \frac{\delta}{\delta\phi_0(x_1)}\frac{\delta}{\delta\phi_0(x_2)}\cdots\frac{\delta}{\delta\phi_0(x_n)}\exp\left(-S_{EAdS}(\phi_0)\right)\Big|_{\phi_0=0}. \tag{4.1.3}$$

4.2 Two-point correlation functions from the AdS side

According to the recipe outlined above, to compute correlation functions from the gravity side using the AdS_{d+1}/CFT_d correspondence, we need the explicit expression for the action on the AdS side. Using the action, we should derive the equations of motion and solve them. It will be useful to first consider the metric on the AdS space. As a toy model, we couple one massive scalar field to gravity in the $(d+1)$-dimensional AdS background.

4.2.1 The metric of AdS space. (Witten, 1998; Freedman et al., 1999; D'Hoker and Freedman, 2002) The space in which we work is the Euclidean continuation of AdS$_{d+1}$, which can be written as the surface

$$-(Y_{-1})^2 + (Y_0)^2 + \sum_{i=1}^{d}(Y_i)^2 = -\frac{1}{a^2}, \quad Y_{-1} > 0 \tag{4.2.1}$$

in a $(d+2)$-dimensional embedding space with metric

$$ds^2 = -dY_{-1}^2 + dY_0^2 + \sum_{i=1}^{d}(dY)_i^2. \tag{4.2.2}$$

This space has negative curvature

$$R = -d(d+1)a^2. \tag{4.2.3}$$

Now we change coordinates from Y_i to z_i as follows

$$z_i = \frac{Y_i}{a(Y_{-1}+Y_0)} \tag{4.2.4}$$

$$z_0 = \frac{1}{a^2(Y_{-1}+Y_0)}, \tag{4.2.5}$$

or, equivalently

$$\frac{z_i}{z_0} = aY_i \quad \text{for } i \neq 0 \tag{4.2.6}$$

$$z_0 = \frac{1}{a^2(Y_{-1}+Y_0)}. \tag{4.2.7}$$

Differentiate to find

$$adY_i = \frac{z_0 dz_i - z_i dz_0}{z_0^2}, \tag{4.2.8}$$

and

$$dz_0 = -\frac{dY_0 + dY_i}{a^2(Y_{-1}+Y_0)^2}. \tag{4.2.9}$$

Then

$$\begin{aligned}(dY_i)^2 &= \frac{1}{a^2 z_0^4}\left[z_0^2(dz_i)^2 + z_i^2(dz_i)^2 - 2z_0 z_i dz_0 dz_i\right] \\ &= \frac{1}{a^2 z_0^4}\left[z_0^2(dz_i)^2 + z_i^2(dz_i)^2 - z_0 dz_0 d(z_i^2)\right],\end{aligned} \tag{4.2.10}$$

and

$$(dz_0)^2 = \frac{(dY_{-1}+dY_0)^2}{a^4(Y_{-1}+Y_0)^4} = a^4 z_0^4 (dY_{-1}+dY_0)^2. \tag{4.2.11}$$

Thus, it follows that

$$\sum_{i=1}^{d}(dY)_i^2 = \frac{1}{a^2 z_0^4}\left[z_0^2\sum_{i=1}^{d} dz_i^2 + (dz_i)^2\sum_{i=1}^{d} z_i^2 - z_0 dz_0 d\left(\sum_{i=1}^{d} z_i^2\right)\right]. \tag{4.2.12}$$

Using

$$z_i = a z_0 Y_i, \tag{4.2.13}$$

we obtain

$$\sum_{i=1}^{d}(dY_i)^2 = \frac{1}{a^2 z_0^4}\left[z_0^2\sum_{i=1}^{d} dz_i^2 + a^2 z_0^2 (dz_i)^2\sum_{i=1}^{d} Y_i^2 - a^2 z_0^3 dz_0 d\left(\sum_{i=1}^{d} Y_i^2\right)\right]. \tag{4.2.14}$$

Now, from (4.2.1) it follows that

$$\sum_{i=1}^{d}(Y_i)^2 = -\frac{1}{a^2} + (Y_{-1})^2 - (Y_0)^2 \tag{4.2.15}$$

$$d\left(\sum_{i=1}^{d}(Y_i)^2\right) = 2Y_{-1}dY_{-1} - 2Y_0 dY_0. \tag{4.2.16}$$

And hence, (4.2.14) becomes

$$\begin{aligned}\sum_{i=1}^{d}(dY_i)^2 &= \frac{1}{a^2 z_0^4}\left[z_0^2\sum_{i=1}^{d} dz_i^2 + a^2 z_0^2 (dz_i)^2\left(-\frac{1}{a^2} + (Y_{-1})^2 - (Y_0)^2\right) - 2a^2 z_0^3 dz_0\left(Y_{-1}dY_{-1} - Y_0 dY_0\right)\right] \\ &= \frac{1}{a^2 z_0^2}\left[\sum_{i=1}^{d}(dz_i)^2 + (dz_0)^2\right] + \frac{1}{z_0^2}\left[(dz_i)^2\left((Y_{-1})^2 - (Y_0)^2\right) - 2z_0 dz_0\left(Y_{-1}dY_{-1} - Y_0 dY_0\right)\right].\end{aligned} \tag{4.2.17}$$

Finally replace z_0 and dz_0 in the second term of the RHS of this equation with (4.2.5) and (4.2.9) to obtain

$$\sum_{i=1}^{d}(dY_i)^2 = \frac{1}{a^2 z_0^2}\left[\sum_{i=1}^{d}(dz_i)^2 + (dz_0)^2\right] - dY_0^2 + dY_{-1}^2, \tag{4.2.18}$$

which gives the induced metric in the form of the Lobaschevsky upper half-space:

$$ds^2 = \frac{1}{a^2 z_0^2}\left(\sum_{\mu=0}^{d} dz_\mu^2\right). \tag{4.2.19}$$

4.2.2 The wave equation. For a free scalar field, of mass m, in flat $(d+1)$-dimensional Minkowski spacetime, the action is given by

$$S = \int d^{d+1}x\mathcal{L}, \tag{4.2.20}$$

where the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2, \tag{4.2.21}$$

with $d^{d+1}x = dx_0dx_1\cdots dx_d$ the measure and $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1,-1,...,-1)$ corresponds to the Lorentzian metric tensor.

For any theory in a curved space, we use the invariant measure $d^{d+1}x\sqrt{g} = d^{d+1}x\sqrt{\det(g_{\mu\nu})}$, and we replace the tensor metric $\eta_{\mu\nu}$ of the flat space by the metric tensor $g_{\mu\nu}$ of the curved space. We have to replace usual derivatives ∂_μ by covariant derivatives ∇_μ. Here, the usual derivative ∂_μ coincides with ∇_μ because we are considering a scalar field. Therefore, after a Wick rotation, we have the Euclidean action for a massive scalar field given by (Freedman et al., 1999)

$$S_E[\phi] = \int d^{d+1}x\sqrt{g}\mathcal{L}_E = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]. \tag{4.2.22}$$

We derive the equations of motion from the Euler-Lagrange equation which is

$$\partial_\rho\left[\frac{\partial(\sqrt{g}\mathcal{L}_E)}{\partial(\partial_\rho\phi)}\right] - \frac{\partial(\sqrt{g}\mathcal{L}_E)}{\partial\phi} = 0, \tag{4.2.23}$$

with

$$\begin{aligned}
\partial_\rho\left[\frac{\partial(\sqrt{g}\mathcal{L}_E)}{\partial(\partial_\rho\phi)}\right] &= \partial_\rho\left[\frac{\partial}{\partial(\partial_\rho\phi)}\left(\frac{\sqrt{g}}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]\right)\right] \\
&= \partial_\rho\left(\frac{\sqrt{g}}{2}g^{\mu\nu}\left[\delta_{\mu\rho}\partial_\nu\phi + \delta_{\nu\rho}\partial_\mu\phi\right]\right) \\
&= \partial_\rho\left(\frac{\sqrt{g}}{2}\left[g^{\rho\nu}\partial_\nu\phi + g^{\rho\mu}\partial_\mu\phi\right]\right) \\
&= \partial_\rho\left[\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right] && (4.2.24) \\
\frac{\partial(\sqrt{g}\mathcal{L}_E)}{\partial\phi} &= \frac{\partial}{\partial\phi}\left(\frac{\sqrt{g}}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]\right) \\
&= \sqrt{g}m^2\phi. && (4.2.25)
\end{aligned}$$

In the curved space, the equation of motion is

$$\frac{1}{\sqrt{g}}\partial_\mu\left[\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right] - m^2\phi = 0. \tag{4.2.26}$$

Now, use the metric of the AdS spacetime,

$$\begin{cases} g_{\mu\nu} &= z_0^{-2}\delta_{\mu\nu} \\ g^{\mu\nu} &= z_0^2\delta_{\mu\nu} \end{cases}, \quad \mu,\nu = 0,1,\cdots,d \tag{4.2.27}$$

to find

$$g = \det\left(g_{\mu\nu}\right) = \det\left(z_0^{-2}\delta_{\mu\nu}\right) = \prod_{\mu=0}^{d}\left(z_0^{-2}\delta_{\nu\nu}\right) = z_0^{-2(d+1)}. \tag{4.2.28}$$

The equation of motion becomes

$$z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{z})\right] + z_0^2\frac{\partial^2}{\partial \vec{z}^2}\phi(z_0,\vec{z}) - m^2\phi(z_0,\vec{z}) = 0, \tag{4.2.29}$$

or, equivalently

$$z_0^2\frac{\partial^2}{\partial z_0^2}\phi(z_0,\vec{z}) + (-d+1)z_0\frac{\partial}{\partial z_0}\phi(z_0,\vec{z}) + z_0^2\frac{\partial^2}{\partial \vec{z}^2}\phi(z_0,\vec{z}) - m^2\phi(z_0,\vec{z}) = 0 \tag{4.2.30}$$

and the Euclidean action S_E is

$$S_E = \frac{1}{2}\int \frac{d^d z\, dz_0}{z_0^{d+1}}\left[\partial_\mu\phi z_0^2\partial_\mu\phi + m^2\phi^2\right]. \tag{4.2.31}$$

4.2.3 The solution of the wave equation. The bulk-to-boundary Green's function associated to the equation of motion is given by (Witten, 1998)

$$K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right)^{\Delta}, \tag{4.2.32}$$

where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4}+m^2} \qquad m^2 = \Delta(\Delta-d). \tag{4.2.33}$$

We will now verify that the above bulk-to-boundary Green's function exhibits the necessary singular behaviour

$$z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x}) \underset{z_0\to 0}{\longrightarrow} \delta(\vec{z}-\vec{x}) = \begin{cases} \infty & \text{if } \vec{z}=\vec{x} \\ 0 & \text{otherwise} \end{cases}. \tag{4.2.34}$$

Consider two cases

- For $\vec{z}\neq\vec{x}$:

$$z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{z_0^{2\Delta-d}}{[z_0^2+(\vec{z}-\vec{x})^2]^{\Delta}} \underset{z_0\to 0}{\longrightarrow} 0. \tag{4.2.35}$$

- For $\vec{z}=\vec{x}$:

$$z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{1}{z_0^d} \underset{z_0\to 0}{\longrightarrow} \infty. \tag{4.2.36}$$

Moreover, we have

$$\int d^d z\, z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\int d^d z\, \frac{z_0^{2\Delta-d}}{\left(z_0^2+(\vec{z}-\vec{x})^2\right)^\Delta}$$
$$= C_\Delta \int d^d z\, \frac{z_0^{2\Delta-d}}{\left(z_0^2+(\vec{z})^2\right)^\Delta}, \tag{4.2.37}$$

with

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}\right)}. \tag{4.2.38}$$

We obtain the last line by employing the change of variables $\vec{z}' = \vec{z}-\vec{x}$, which leaves the measure invariant. Now change to d-dimensional spherical coordinates such that

$$d^d z = dz_1 \dots dz_d = r^{d-1} dr\, d\Omega_{d-1} \tag{4.2.39}$$
$$\vec{z}^2 = z_1^2+\cdots+z_d^2 = r^2, \tag{4.2.40}$$

where r is a radial coordinate and $d\Omega_{d-1}$ is the measure for the angular variables. The integral over the angular variables gives

$$\int d\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{4.2.41}$$

Thus, we have

$$\int d^d z\, z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty dr \frac{z_0^{2\Delta-d}}{\left(z_0^2+r^2\right)^\Delta}. \tag{4.2.42}$$

Now change coordinate, from r to $u = \frac{r^2}{z_0^2+r^2}$ to find

$$\int d^d z\, z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^1 du u^{\frac{d}{2}-1}(1-u)^{\Delta-\frac{d}{2}-1}. \tag{4.2.43}$$

Finally, using the identity

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 du u^{p-1}(1-u)^{q-1}, \tag{4.2.44}$$

we find that

$$\int d^d z\, z_0^{\Delta-d} K_\Delta(z_0,\vec{z},\vec{x}) = 1, \tag{4.2.45}$$

which complete the proof that $z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x})$ tends to the delta function $\delta(\vec{z}-\vec{x})$ as $z_0 \to 0$.

The solution to the equation of motion can be written using the bulk-to-boundary Green's functions as follows

$$\phi(z_0,\vec{z}) = \int d^d x K_\Delta(z_0,\vec{z},\vec{x})\phi_0(\vec{x})$$
$$= \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\int d^d x \left(\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right)^\Delta \phi_0(\vec{x}). \tag{4.2.46}$$

To demonstrate that this is a solution, we plug this expression into the wave equation. We need to compute

$$\frac{\partial}{\partial z_0}K_\Delta(z_0,\vec{z},\vec{x}) = K_\Delta(z_0,\vec{z},\vec{x})\left[\Delta\frac{1}{z_0}\frac{(\vec{z}-\vec{x})^2-z_0^2}{z_0^2+(\vec{z}-\vec{x})^2}\right] \tag{4.2.47}$$

$$\frac{\partial^2}{\partial z_0^2}K_\Delta(z_0,\vec{z},\vec{x}) = K_\Delta(z_0,\vec{z},\vec{x})\left\{\Delta\frac{\partial}{\partial z_0}\left[\frac{1}{z_0}\frac{(\vec{z}-\vec{x})^2-z_0^2}{z_0^2+(\vec{z}-\vec{x})^2}\right]+\left[\Delta\frac{1}{z_0}\frac{(\vec{z}-\vec{x})^2-z_0^2}{z_0^2+(\vec{z}-\vec{x})^2}\right]^2\right\} \tag{4.2.48}$$

$$\frac{\partial}{\partial z_i}K_\Delta(z_0,\vec{z},\vec{x}) = K_\Delta(z_0,\vec{z},\vec{x})\left[\Delta\frac{-2(z_i-x_i)}{z_0^2+(\vec{z}-\vec{x})^2}\right] \tag{4.2.49}$$

$$\frac{\partial^2}{\partial z_i^2}K_\Delta(z_0,\vec{z},\vec{x}) = K_\Delta(z_0,\vec{z},\vec{x})\left\{\Delta\frac{\partial}{\partial z_i}\left[\frac{-2(z_i-x_i)}{z_0^2+(\vec{z}-\vec{x})^2}\right]+\left[\Delta\frac{2(z_i-x_i)}{z_0^2+(\vec{z}-\vec{x})^2}\right]^2\right\}. \tag{4.2.50}$$

After evaluating each derivative, we have

$$z_0^2\frac{\partial^2}{\partial z_0^2}K_\Delta = K_\Delta\left\{\frac{\Delta\left[z_0^4-4z_0^2(\vec{z}-\vec{x})^2-(\vec{z}-\vec{x})^4\right]+\Delta^2\left[z_0^4-2z_0^2(\vec{z}-\vec{x})^2+(\vec{z}-\vec{x})^4\right]^2}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right\} \tag{4.2.51}$$

$$(-d+1)z_0\frac{\partial}{\partial z_0}K_\Delta = K_\Delta\left\{\Delta(-d+1)\frac{(\vec{z}-\vec{x})^4-z_0^4}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right\} \tag{4.2.52}$$

$$z_0^2\sum_{i=1}^{d}\frac{\partial^2}{\partial z_i^2}K_\Delta = K_\Delta\left\{\frac{\Delta\left[-2d(z_0^4+z_0^2(\vec{z}-\vec{x})^2)+4z_0^2(\vec{z}-\vec{x})^2\right]+4\Delta^2z_0^2(\vec{z}-\vec{x})^2}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right\} \tag{4.2.53}$$

$$-m^2K_\Delta = K_\Delta\left\{-m^2\frac{\left[z_0^4+2z_0^2(\vec{z}-\vec{x})^2+(\vec{z}-\vec{x})^4\right]}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\right\}. \tag{4.2.54}$$

Combining these terms we have

$$\begin{aligned}
&(-d+1)z_0\frac{\partial}{\partial z_0}K_\Delta + z_0^2\frac{\partial^2}{\partial z_0^2}K_\Delta + z_0^2\sum_{i=1}^{d}\frac{\partial^2}{\partial z_i^2}K_\Delta - m^2K_\Delta\\
&= \frac{K_\Delta}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^2}\left\{z_0^4\Big[\Delta+\Delta^2-\Delta(-d+1)-2d\Delta-m^2\Big]\right.\\
&+ z_0^2(\vec{z}-\vec{x})^2\Big[-4\Delta-2\Delta^2-2d\Delta+4\Delta+4\Delta^2-2m^2\Big]\\
&\left.+ (\vec{z}-\vec{x})^4\Big[-\Delta+\Delta^2+\Delta(-d+1)-m^2\Big]\right\}.
\end{aligned} \tag{4.2.55}$$

The RHS of this equation vanishes after we replace m^2 by $\Delta^2-d\Delta$ as dictated by (4.2.33). Therefore K_Δ is a solution to (4.2.29).

4.2.4 The two-point correlation function in position space. Given the classical solution we can evaluate the action and, thanks to (4.1.1) and (4.1.2), compute correlation functions of the CFT.

They are given by taking derivatives of the generating function Z_{AdS}. Thus, we now have

$$\left\langle \mathcal{O}(\vec{X})\mathcal{O}(\vec{Y})\right\rangle = \frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})} Z_{AdS}(\phi_0)\big|_{\phi_0=0} \tag{4.2.56}$$

$$= \frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})} \exp\left(-S_{EAdS}(\phi_0)\right)\big|_{\phi_0=0} \tag{4.2.57}$$

$$= -\frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})} S_E(\phi_0)\big|_{\phi_0=0}. \tag{4.2.58}$$

To evaluate this last expression, plug the solution $\phi(z_0,\vec{z}) = \int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})$, where $z=(z_0,\vec{z})$ denotes the coordinates in the AdS_{d+1}, into the classical action to obtain

$$\begin{aligned} S_E =& \frac{1}{2}\int \frac{d^dz\, dz_0}{z_0^{d+1}}\Big[\partial_\mu\left(\int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\right) z_0^2\partial_\mu\left(\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y})\right) \\ &+ m^2 \int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x}) \int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y})\Big]. \end{aligned} \tag{4.2.59}$$

We then need

$$\begin{aligned} \frac{\delta}{\delta\phi_0(\vec{Y})} S_E = \frac{1}{2}\int \frac{d^dz\, dz_0}{z_0^{d+1}}\Big[&\partial_\mu K_\Delta(z,\vec{Y})\left(z_0^2\partial_\mu\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y}) + z_0^2\partial_\mu\int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\right) \\ &+ m^2 K_\Delta(z,\vec{Y})\left(\int d^dy K_\Delta(z,\vec{y})\phi_0(\vec{y}) + \int d^dx K_\Delta(z,\vec{x})\phi_0(\vec{x})\right)\Big] \end{aligned}$$

and

$$\frac{\delta}{\delta\phi_0(\vec{X})}\frac{\delta}{\delta\phi_0(\vec{Y})} S_E = \frac{1}{2}\int \frac{d^dz\, dz_0}{z_0^{d+1}}\left[2\partial_\mu K_\Delta(z,\vec{X}) z_0^2 \partial_\mu K_\Delta(z,\vec{Y}) + 2m^2 K_\Delta(z,\vec{X})K_\Delta(z,\vec{Y})\right].$$

Finally, the 2-point correlator is given by

$$\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = -\int \frac{d^dz\, dz_0}{z_0^{d+1}}\left(\partial_\mu K_\Delta(z,\vec{x}) z_0^2\partial_\mu K_\Delta(z,\vec{y}) + m^2 K_\Delta(z,\vec{x})K_\Delta(z,\vec{y})\right). \tag{4.2.60}$$

In this equation the variables $\vec{X}$ and $\vec{Y}$ are replaced by $\vec{x}$ and $\vec{y}$. This can be rewritten as

$$\begin{aligned} \langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = -\int d^dz\, dz_0 z_0^{-d-1}\Big(&\partial_0 K_\Delta(z,\vec{x}) z_0^2\partial_0 K_\Delta(z,\vec{y}) \\ &+ \sum_{i=1}^{d}\partial_i K_\Delta(z,\vec{x}) z_0^2\partial_i K_\Delta(z,\vec{y}) + m^2 K_\Delta(z,\vec{x})K_\Delta(z,\vec{y})\Big). \end{aligned} \tag{4.2.61}$$

To evaluate the above expression, it proves useful to perform an integration by parts, to obtain

$$\begin{aligned} \int d^dz\, dz_0 z_0^{-d-1}\partial_0 K_\Delta(z,\vec{x}) z_0^2\partial_0 K_\Delta(z,\vec{y}) &= \int d^dz\, dz_0 \partial_0 K_\Delta(z,\vec{x}) z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y}) \\ &= \int d^dz \left[K_\Delta(z,\vec{x}) z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y})\right]_{z_0=\epsilon\to 0}^{\infty} \\ &- \int d^dz\, dz_0 K_\Delta(z,\vec{x})\partial_0\left[z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y})\right], \end{aligned} \tag{4.2.62}$$

and

$$\begin{aligned}\int d^d z\, dz_0 z_0^{-d-1}\partial_i K_\Delta(z,\vec{x}) z_0^2 \partial_i K_\Delta(z,\vec{y}) &= \int d^d z\, dz_0 \partial_i K_\Delta(z,\vec{x}) z_0^{-d+1}\partial_i K_\Delta(z,\vec{y}) \\ &= \int d^{d-1} z\, dz_0 \left[K_\Delta(z,\vec{x}) z_0^{-d+1}\partial_i K_\Delta(z,\vec{y})\right]_{z_i=-\infty}^{z_i=+\infty} \\ &- \int d^d z\, dz_0 z_0^{-d+1} K_\Delta(z,\vec{x})\partial_i\partial_i K_\Delta(z,\vec{y}) \\ &= -\int d^d z\, dz_0 K_\Delta(z,\vec{x}) z_0^{-d+1}\partial_i\partial_i K_\Delta(z,\vec{y}). \end{aligned} \quad (4.2.63)$$

Using these results the correlators become

$$\begin{aligned}\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= -\int d^d z \left[K_\Delta(z,\vec{x}) z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y})\right]_{z_0=\epsilon\to 0}^{\infty} \\ &+ \int d^d z\, dz_0 K_\Delta(z,\vec{x}) \left(-\partial_0\left[z_0^{-d+1}\partial_0 K_\Delta(z,\vec{y})\right] - \sum_{i=1}^{d} z_0^{-d+1}\partial_i\partial_i K_\Delta(z,\vec{y}) + m^2 z_0^{-d-1} K_\Delta(z,\vec{y})\right) \\ &= \lim_{\epsilon\to 0}\int d^d z \epsilon^{1-d} K_\Delta(\epsilon,\vec{z},\vec{x})\frac{\partial}{\partial z_0} K_\Delta(z_0,\vec{z},\vec{y})\big|_{z_0=\epsilon} \\ &+ \int d^d z\, dz_0\, z_0^{-d-1} K_\Delta(z,\vec{x})\left(-z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}K_\Delta(z,\vec{y})\right] - z_0^2\frac{\partial^2}{\partial \vec{z}^2}K_\Delta(z,\vec{y}) + m^2 K_\Delta(z,\vec{y})\right). \end{aligned} \quad (4.2.64)$$

Using the free equation of motion for K_Δ, we have

$$\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = \lim_{\epsilon\to 0}\int d^d z \epsilon^{1-d} K_\Delta(\epsilon,\vec{z},\vec{x})\frac{\partial}{\partial z_0} K_\Delta(z_0,\vec{z},\vec{y})\big|_{z_0=\epsilon}. \quad (4.2.65)$$

Given

$$\begin{aligned}\frac{\partial}{\partial z_0} K_\Delta(z_0,\vec{z},\vec{y}) &= \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{\partial}{\partial z_0}\left\{\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right)^{\Delta}\right\} \\ &= \frac{\Gamma(\Delta)\Delta}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{\partial}{\partial z_0}\left\{\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right\}\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right)^{\Delta-1} \\ &= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\left(\frac{(\vec{z}-\vec{y})^2 - z_0^2}{[z_0^2+(\vec{z}-\vec{y})^2]^2}\right)\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right)^{\Delta-1}, \end{aligned} \quad (4.2.66)$$

we can write

$$\begin{aligned}\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\lim_{\epsilon\to 0}\int d^d z \epsilon^{1-d} K_\Delta(\epsilon,\vec{z},\vec{x})\left(\frac{(\vec{z}-\vec{y})^2-\epsilon^2}{[\epsilon^2+(\vec{z}-\vec{y})^2]^2}\right)\left(\frac{\epsilon}{\epsilon^2+(\vec{z}-\vec{y})^2}\right)^{\Delta-1} \\ &= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\lim_{\epsilon\to 0}\int d^d z \epsilon^{\Delta-d} K_\Delta(\epsilon,\vec{z},\vec{x})\left(\frac{(\vec{z}-\vec{y})^2-\epsilon^2}{[\epsilon^2+(\vec{z}-\vec{y})^2]^2}\right)\left(\frac{1}{\epsilon^2+(\vec{z}-\vec{y})^2}\right)^{\Delta-1}. \end{aligned} \quad (4.2.67)$$

Now, we use the fact that $z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x}) \to \delta(\vec{z}-\vec{x})$ when $z_0 \to 0$. We finally obtain

$$\begin{aligned}
\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\lim_{\epsilon\to 0}\int d^dz\,\delta(\vec{z}-\vec{x})\left(\frac{(\vec{z}-\vec{y})^2-\epsilon^2}{[\epsilon^2+(\vec{z}-\vec{y})^2]^2}\right)\left(\frac{1}{\epsilon^2+(\vec{z}-\vec{y})^2}\right)^{\Delta-1}\\
&= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\lim_{\epsilon\to 0}\left(\frac{(\vec{x}-\vec{y})^2-\epsilon^2}{[\epsilon^2+(\vec{x}-\vec{y})^2]^2}\right)\left(\frac{1}{\epsilon^2+(\vec{x}-\vec{y})^2}\right)^{\Delta-1}\\
&= \frac{\Gamma(\Delta+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}. \qquad (4.2.68)
\end{aligned}$$

This is precisely the form expected for the two point function of a scalar primary operators of dimension Δ in CFT, that we have reproduced from a supergravity computation.

4.2.5 The two-point correlation function in momentum space. First, Fourier transform the variable $\vec{z}$ in (4.2.29), as follows (Witten, 1998)

$$\phi(z_0,\vec{z}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k}). \qquad (4.2.69)$$

This gives

$$\begin{aligned}
&z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k})\right] + z_0^2\frac{\partial^2}{\partial\vec{z}^2}\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k})\\
&- m^2\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k}) = 0\\
&z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k})\right] + z_0^2\frac{1}{(2\pi)^{\frac{d}{2}}}\sum_{j=1}^{d}\frac{\partial^2}{\partial z_j^2}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k})\\
&- m^2\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\phi(z_0,\vec{k}) = 0\\
&\frac{1}{(2\pi)^{\frac{d}{2}}}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{z}}\left[z_0^{d+1}\frac{\partial}{\partial z_0}z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}) - z_0^2k^2\phi(z_0,\vec{k}) - m^2\phi(z_0,\vec{k})\right] = 0, \qquad (4.2.70)
\end{aligned}$$

which implies

$$z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})\right] - (z_0^2k^2+m^2)\phi(z_0,\vec{k}) = 0. \qquad (4.2.71)$$

The solution of this equation is given by

$$\phi(z_0,\vec{k}) = z_0^{\frac{d}{2}}F_\nu(ik\,z_0), \qquad (4.2.72)$$

where F_ν is the solution of the Bessel equation of index

$$\nu = \sqrt{\frac{d^2}{4}+m^2} = \Delta - \frac{d}{2}. \qquad (4.2.73)$$

To see this, begin by rewriting the Fourier space equation of motion (4.2.71) as

$$z_0^{d+1}\left[(-d+1)z_0^{-d}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}) + z_0^{-d+1}\frac{\partial^2}{\partial z_0^2}\phi(z_0,\vec{k})\right] - (z_0^2k^2+m^2)\phi(z_0,\vec{k}) = 0. \qquad (4.2.74)$$

Using the explicit form of the solution given in (4.2.72), we have

$$\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}) = \frac{d}{2} z_0^{\frac{d}{2}-1} F_\nu(ik\,z_0) + ik\,z_0^{\frac{d}{2}} \frac{\partial}{\partial(ikz_0)} F_\nu(ik\,z_0) \tag{4.2.75}$$

$$\begin{aligned}\frac{\partial^2}{\partial(ikz_0)^2}\phi(z_0,\vec{k}) &= \frac{d}{2}\left(\frac{d}{2}-1\right) z_0^{\frac{d}{2}-2} F_\nu(ik\,z_0) + 2ik\frac{d}{2} z_0^{\frac{d}{2}-1} \frac{\partial}{\partial(ikz_0)} F_\nu(ik\,z_0)\\ &+ (ik)^2\, z_0^{\frac{d}{2}} \frac{\partial^2}{\partial(ikz_0)^2} F_\nu(ik\,z_0).\end{aligned} \tag{4.2.76}$$

The Fourier space equation of motion becomes

$$\begin{aligned}&\frac{d}{2}(-d+1) z_0^{\frac{d}{2}} F_\nu(ik\,z_0) + ik\,(-d+1) z_0^{\frac{d}{2}+1} \frac{\partial}{\partial(ikz_0)} F_\nu(ik\,z_0)\\ &+ \left[\frac{d}{2}\left(\frac{d}{2}-1\right) z_0^{\frac{d}{2}} F_\nu(ik\,z_0) + ik\,d\,z_0^{\frac{d}{2}+1} \frac{\partial}{\partial(ikz_0)} F_\nu(ik\,z_0)\right.\\ &\left.+ (ik)^2\, z_0^{\frac{d}{2}+2} \frac{\partial^2}{\partial(ikz_0)^2} F_\nu(ik\,z_0)\right]\\ &- (z_0^2 k^2 + m^2) z_0^{\frac{d}{2}} F_\nu(ik\,z_0) = 0.\end{aligned} \tag{4.2.77}$$

Finally, regroup terms, and factor out $z_0^{\frac{d}{2}}$ to get

$$(ikz_0)^2 \frac{\partial^2}{\partial(ikz_0)^2} F_\nu(ik\,z_0) + (ikz_0)\frac{\partial}{\partial(ikz_0)} F_\nu(ik\,z_0) + \left[(ikz_0)^2 - \left(\frac{d^2}{4} + m^2\right)\right] F_\nu(ik\,z_0) = 0, \tag{4.2.78}$$

which is the Bessel equation of index $\nu = \sqrt{\frac{d^2}{4} + m^2} = \Delta - \frac{d}{2}$. For what follows it will be useful to write the action in terms of Fourier components. The Fourier space action is

$$S = \frac{1}{2}\int dz_0\, d\vec{k}\, d\vec{k}'\, z_0^{-d+1} \delta(\vec{k}+\vec{k}') \left[\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}') + \left(k^2 + \frac{m^2}{z_0^2}\right)\phi(z_0,\vec{k})\phi(z_0,\vec{k}')\right]. \tag{4.2.79}$$

Integrate by parts with respect to z_0 and use the solution of the wave equation to get

$$S = \frac{1}{2}\int d\vec{k}\, d\vec{k}'\, \delta(\vec{k}+\vec{k}') \lim_{z_0\to\epsilon} z_0^{-d+1} \left[\phi(z_0,\vec{k})\frac{\partial}{\partial z_0}\phi(z_0,\vec{k}')\right]. \tag{4.2.80}$$

The solution to the classical equation of motion can be written as

$$\phi(z_0,\vec{k}) = K^\epsilon(z_0,\vec{k})\phi_b(\vec{k}), \tag{4.2.81}$$

with

$$\phi_b(\vec{k}) = \phi(\epsilon,\vec{k}) \tag{4.2.82}$$

and

$$K^\epsilon(z_0,\vec{k}) = \left(\frac{z_0}{\epsilon}\right)^{\frac{d}{2}} \frac{\mathcal{K}_\nu(kz_0)}{\mathcal{K}_\nu(k\epsilon)}. \tag{4.2.83}$$

The function $\mathcal{K}_\nu$ is the modified Bessel function which vanishes as $z_0 \to 0$. The boundary conditions for the field imply that

$$\begin{cases} K^\epsilon(z_0,\vec{k}) \underset{z_0\to\epsilon}{\longrightarrow} 1 \\ K^\epsilon(z_0,\vec{k}) \underset{z_0\to\infty}{\longrightarrow} 0 \end{cases} . \tag{4.2.84}$$

The two-point correlator in momentum space becomes

$$\left\langle \mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\right\rangle = -\epsilon^{-d+1}\delta(\vec{k}+\vec{k}') \lim_{z_0\to\epsilon} \frac{\partial}{\partial z_0} K^\epsilon(z_0,\vec{k}). \tag{4.2.85}$$

To evaluate this limit, we need the expansion of $\mathcal{K}_\nu$ near-the boundary, which is

$$\mathcal{K}_\nu(u) = u^{-\nu}\big(a_0 + a_2 u^2 + \dots\big) + u^\nu\big(b_0 + b_2 u^2 + \dots\big) \tag{4.2.86}$$

$$\mathcal{K}'_\nu(u) = u^{-\nu-1}\left[-\nu\, a_0 + (2-\nu)a_2 u^2 + \dots\right] + u^{\nu-1}\left[\nu\, b_0 + (2+\nu)b_2 u^2 + \dots\right]. \tag{4.2.87}$$

In this expansion, we only need a_0 and b_0, which are

$$a_0 = 2^{\nu-1}\Gamma(\nu) \tag{4.2.88}$$

$$b_0 = -2^{-\nu-1}\frac{\Gamma(1-\nu)}{\nu}. \tag{4.2.89}$$

We also need the expansion, valid for small z_0

$$\frac{\partial}{\partial z_0}\left[z_0^{\frac{d}{2}}\mathcal{K}_\nu(kz_0)\right] = z_0^{\frac{d}{2}-1}\left[\frac{d}{2}\mathcal{K}_\nu(kz_0) + kz_0\mathcal{K}'_\nu(kz_0)\right] \tag{4.2.90}$$

$$\begin{aligned} &= z_0^{\frac{d}{2}-1}\Bigg\{(kz_0)^{-\nu}\left[\left(\frac{d}{2}-\nu\right)a_0 + \left(2+\frac{d}{2}-\nu\right)a_2(kz_0)^2 + \dots\right] \\ &+ (kz_0)^\nu\left[\left(\frac{d}{2}+\nu\right)b_0 + \left(2+\frac{d}{2}+\nu\right)b_2(kz_0)^2 + \dots\right]\Bigg\}. \end{aligned} \tag{4.2.91}$$

Using the above expansion we find

$$\epsilon^{-d+1}\lim_{z_0\to\epsilon}\frac{\partial}{\partial z_0}K^\epsilon(z_0,\vec{k}) = \epsilon^{-d}\left(\left[\frac{d}{2}-\nu+c_2(k\epsilon)^2+\dots\right] + (k\epsilon)^{2\nu}\frac{b_0}{a_0}\left[2\nu + d_2(k\epsilon)^2+\cdots\right]\right), \tag{4.2.92}$$

where c_i and d_i are constants. The final expression for the correlator in the momentum space is obtained by dropping terms of integer power of k^2 since they will be proportional to the delta function $\delta(\vec{x}-\vec{y})$ and its derivatives in position space. These contact terms will not contribute to our correlator since we only consider fields at non-coincident points. Only the first terms of the series of non-integer power of k determine the physical correlator. Keeping only the leading term as $\epsilon \to 0$,

$$\begin{aligned} \epsilon^{-d+1}\lim_{z_0\to\epsilon}\frac{\partial}{\partial z_0}K^\epsilon(z_0,\vec{k}) &= \epsilon^{-d}(k\epsilon)^{2\nu}\frac{b_0}{a_0}2\nu \\ &= -\epsilon^{2(\Delta-d)}2\nu\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}. \end{aligned} \tag{4.2.93}$$

Thus the correlator is

$$\left\langle \mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\right\rangle = \delta(\vec{k}+\vec{k}')\epsilon^{2(\Delta-d)}2\nu\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}, \tag{4.2.94}$$

which can be written in position space as

$$\begin{aligned}\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= \frac{1}{(2\pi)^d}\int d^dk\, d^dk'\, e^{i\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{y}}\left\langle\mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\right\rangle\\ &= \epsilon^{2(\Delta-d)}\frac{1}{(2\pi)^d}\int d^dk\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}2\nu\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\\ &= \frac{1}{\pi^{\frac{d}{2}}}\frac{(2\Delta-d)\Gamma(\Delta)}{\Gamma(\Delta-\frac{d}{2})}\frac{1}{(\vec{x}-\vec{y})^{2\Delta}}.\end{aligned} \tag{4.2.95}$$

To obtain this result we have used the inverse Fourier transform identity

$$\frac{1}{(2\pi)^d}\int d^dk\, e^{i\vec{k}\cdot\vec{X}}k^{2\nu} = \frac{2^{2\nu}}{\pi^{\frac{d}{2}}}\frac{\Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma(-\nu)}\frac{1}{|\vec{X}|^{2\nu+d}}, \tag{4.2.96}$$

and we have replaced ν by $\Delta-\frac{d}{2}$. Comparing the two results obtained in (4.2.68) and (4.2.95), they differ by a factor

$$\frac{\Delta}{2\Delta-d}. \tag{4.2.97}$$

Different results are also obtained if we evaluate the limit $\epsilon\to 0$ of the expression of the propagator in (4.2.83) before we evaluate the derivative in (4.2.85). In this case we have

$$K^{\epsilon}(z_0,\vec{k}) = \left(\frac{z_0}{\epsilon}\right)^{\frac{d}{2}}\frac{\mathcal{K}_\nu(kz_0)}{a_0(k\epsilon)^{-\nu}}. \tag{4.2.98}$$

Then the correlator in the momentum space takes the form

$$\left\langle\mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\right\rangle = \delta(\vec{k}+\vec{k}')\epsilon^{2(\Delta-d)}\left(\nu+\frac{d}{2}\right)\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}. \tag{4.2.99}$$

This result again differs by a factor of $\frac{\nu+\frac{d}{2}}{2\nu}=\frac{\Delta}{2\Delta-d}$ when compared to (4.2.95).

In summary, we have computed the two-point function in two different ways. The results differ by a factor of $\frac{\Delta}{2\Delta-d}$. This difference is explained by the way we drop terms when we evaluate the limit $\epsilon\to 0$. This is a convincing demonstration that the computation of the two point function is indeed subtle. Subtle means here we have to be careful on doing the computations. In fact, we have extracted the value of two-point function from the subtraction of two divergent integrals.

4.2.6 The correct value of the two-point correlation function. It can be argued that the correct value of the two point correlation function is given by (Freedman et al., 1999)

$$\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = \frac{1}{\pi^{\frac{d}{2}}}\frac{(2\Delta-d)\Gamma(\Delta)}{\Gamma(\Delta-\frac{d}{2})}\frac{1}{(\vec{x}-\vec{y})^{2\Delta}}. \tag{4.2.100}$$

Computations in Fourier space and in position space gave a different answer. We want to better understand the origin of this subtlety when we evaluate the relevant integrals.

First, we can see that the computation of the two-point function in position space required the introduction of a cutoff (4.2.62) to regulate the divergence of z_0^{-d+1} at the boundary. In addition, the bulk-to-boundary Green's function K_Δ has a singular behaviour as described in (4.2.34). The computation of the two-point correlation function in momentum space, is also subtle, due to the cutoff.

One can extract the two-point function from the Ward Identity for the three-point correlation function of the conserved current $\mathcal{J}_i(\vec{z})$ with the scalar operators $\mathcal{O}(\vec{x})$ and $\mathcal{O}(\vec{y})$ both of which are operators of scale dimension Δ. The computation of the three-point function is not divergent and so can be carried out without ever introducing a cut off. The Ward identity relating the three and two point correlation functions $\langle \mathcal{J}_i(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle$ and $\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle$ confirms the result (4.2.100).

The three-point correlation function $\langle \mathcal{J}_i(\vec{z})\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle$ will be computed from the AdS supergravity side. The sources for the conserved flavor currents $\mathcal{J}_i^a(\vec{x})$ are the boundary values $A_i^a(\vec{x})$ of the gauge potentials $A_\mu^a(x_0, \vec{x})$. After coupling to the background gauge field, the action for the scalar becomes

$$S[\phi^I, A_\mu^a] = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left[g^{\mu\nu}\nabla_\mu\phi^I\nabla_\nu\phi^I + m^2\phi^I\phi^I\right], \tag{4.2.101}$$

with

$$\nabla_\mu\phi^I = \partial_\mu\phi^I - iA_\mu^a(T^a)^{IJ}\phi^J. \tag{4.2.102}$$

In the equation above, $(T^a)^{IJ}$ are the generators in a real representation of the $SO(6)$ flavor group. They are imaginary antisymmetric matrices.

In order to evaluate correlators from the gravity side, we again have to use the bulk-to-boundary Green's function K_Δ in (4.2.32) for the two scalar operators $\mathcal{O}^I(\vec{x})$ and $\mathcal{O}^J(\vec{y})$. We also need the Green's functions $G_{\mu i}(z, \vec{x})$ which are the bulk-to-boundary Green's functions for the gauge field. Using these bulk-to-boundary Green's functions, we will be able to write down gauge field solutions to the equation of motion with boundary values $A_i^a(\vec{x})$. They are given by (Freedman et al., 1999)

$$\begin{aligned} G_{\mu i}(z,\vec{x}) &= C_d \frac{z_0^{d-2}}{\left[z_0^2 + (\vec{z}-\vec{y})^2\right]^{d-1}} J_{\mu i}(z-\vec{x}) \\ &= C_d\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{y})^2}\right)^{d-2}\partial_\mu\left(\frac{(z-\vec{x})_i}{z_0^2+(\vec{z}-\vec{x})^2}\right), \end{aligned} \tag{4.2.103}$$

with

$$C_d = \frac{\Gamma(d)}{2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}. \tag{4.2.104}$$

Using the AdS_{d+1}/CFT_d correspondence, we have

$$\begin{aligned} \left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle &= \frac{\delta}{\delta A_i^a(\vec{z})}\frac{\delta}{\delta\phi_0^I(\vec{x})}\frac{\delta}{\delta\phi_0^J(\vec{y})}\exp\left(-S\right)\Big|_{\phi_0^I=\phi_0^J=A_i^a=0} \\ &= -\frac{\delta}{\delta A_i^a(\vec{z})}\frac{\delta}{\delta\phi_0^I(\vec{x})}\frac{\delta}{\delta\phi_0^J(\vec{y})}S\Big|_{\phi_0^I=\phi_0^J=A_i^a=0}. \end{aligned} \tag{4.2.105}$$

Expanding $\nabla_\mu\phi^K\nabla_\nu\phi^K$ we have

$$\begin{aligned} \nabla_\mu\phi^K\nabla_\nu\phi^K &= \left(\partial_\mu\phi^K - iA_\mu^b(T^b)^{KL}\phi^L\right)\left(\partial_\nu\phi^K - iA_\nu^c(T^c)^{KM}\phi^M\right) \\ &= \partial_\mu\phi^K\partial_\nu\phi^K - i\partial_\mu\phi^K A_\nu^c(T^c)^{KM}\phi^M \\ &\quad - iA_\mu^b(T^b)^{KL}\phi^L\partial_\nu\phi^K - A_\mu^b(T^b)^{KL}\phi^L A_\nu^c(T^c)^{KM}\phi^M. \end{aligned} \tag{4.2.106}$$

The only terms which contribute to (4.2.105) are the cubic vertices in (4.2.106), so that we obtain

$$\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle = (T^a)^{IJ}\int \frac{d^dw\, dw_0}{w_0^{d+1}} G_{\mu i}(w,\vec{z})w_0^2\Big[K_\Delta(w,\vec{x})\frac{\partial}{\partial w_\mu}K_\Delta(w,\vec{y}) - K_\Delta(w,\vec{y})\frac{\partial}{\partial w_\mu}K_\Delta(w,\vec{x})\Big]. \quad (4.2.107)$$

To evaluate this integral we set first $\vec{z}=0$ and perform an inversion. The result is

$$\begin{aligned}
&\int \frac{d^dw\, dw_0}{w_0^{d+1}} G_{\mu i}(w,0)w_0^2\Big[K_\Delta(w,\vec{x})\frac{\partial}{\partial w_\mu}K_\Delta(w,\vec{y}) - K_\Delta(w,\vec{y})\frac{\partial}{\partial w_\mu}K_\Delta(w,\vec{x})\Big]\\
&= \int \frac{d^dw\, dw_0}{w_0^{d+1}} C_d \frac{w_0^{d-2}}{\left[w_0^2+\vec{w}^2\right]^{d-1}} J_{\mu i}(w)w_0^2\Big[K_\Delta(w,\vec{x})\frac{\partial}{\partial w_\mu}K_\Delta(w,\vec{y}) - K_\Delta(w,\vec{y})\frac{\partial}{\partial w_\mu}K_\Delta(w,\vec{x})\Big]\\
&= C_d\int \frac{d^dw'\, dw'_0}{w_0'^{d+1}} w_0'^{d-1}\frac{w'_0}{w^2}\vec{x}'^{2\Delta}\vec{y}'^{2\Delta}\Big[K_\Delta(w',\vec{x}')\frac{\partial w'_\rho}{\partial w_\mu}\frac{\partial}{\partial w'_\rho}K_\Delta(w,\vec{y}) - K_\Delta(w',\vec{y}')\frac{\partial w'_\rho}{\partial w_\mu}\frac{\partial}{\partial w'_\rho}K_\Delta(w',\vec{x}')\Big]\\
&= C_d\vec{x}'^{2\Delta}\vec{y}'^{2\Delta}\int \frac{d^dw'\, dw'_0}{w'_0} J_{\mu i}(w)J_{\rho\mu}(w)\Big[K_\Delta(w',\vec{x}')\frac{\partial}{\partial w'_\rho}K_\Delta(w,\vec{y}) - K_\Delta(w',\vec{y}')\frac{\partial}{\partial w'_\rho}K_\Delta(w',\vec{x}')\Big]\\
&= \frac{C_d}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}}\int \frac{d^dw'\, dw'_0}{w'_0}\delta_{\rho i}\Big[K_\Delta(w',\vec{x}')\frac{\partial}{\partial w'_\rho}K_\Delta(w,\vec{y}) - K_\Delta(w',\vec{y}')\frac{\partial}{\partial w'_\rho}K_\Delta(w',\vec{x}')\Big]\\
&= \frac{C_d}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}}\int \frac{d^dw'\, dw'_0}{w'_0}\Big[K_\Delta(w',\vec{x}')\frac{\partial}{\partial w'_i}K_\Delta(w,\vec{y}) - K_\Delta(w',\vec{y}')\frac{\partial}{\partial w'_i}K_\Delta(w',\vec{x}')\Big].
\end{aligned} \quad (4.2.108)$$

We now integrate the first term of the integral above by parts with respect to w'_i to obtain

$$\begin{aligned}
\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle &= -2(T^a)^{IJ}\frac{C_d}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}}\int \frac{d^dw'\, dw'_0}{w'_0}K_\Delta(w',\vec{y}')\frac{\partial}{\partial w'_i}K_\Delta(w',\vec{x}')\\
&= 2(T^a)^{IJ}\frac{C_d}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}}\int \frac{d^dw'\, dw'_0}{w'_0}K_\Delta(w',\vec{y}')\frac{\partial}{\partial x'_i}K_\Delta(w',\vec{x}')\\
&= 2(T^a)^{IJ}\frac{C_d}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}}\frac{\partial}{\partial x'_i}\int \frac{d^dw'\, dw'_0}{w'_0}K_\Delta(w',\vec{y}')K_\Delta(w',\vec{x}'),
\end{aligned} \quad (4.2.109)$$

where we have used the following identity

$$\frac{\partial}{\partial w'_i}K_\Delta(w',\vec{x}') = -\frac{\partial}{\partial x'_i}K_\Delta(w',\vec{x}'). \quad (4.2.110)$$

The above integral can be evaluated using Feynman parameter methods. For example (Freedman et al., 1999)

$$\int d^dz dz_0 \frac{z_0^a}{\left[z_0^2+(\vec{z}-\vec{x})^2\right]^b\left[z_0^2+(\vec{z}-\vec{y})^2\right]^c} = I(a,b,c,d)\,|\vec{x}-\vec{y}|^{1+a+d-2b-2c}, \quad (4.2.111)$$

with

$$I(a,b,c,d) = \frac{\pi^{\frac{d}{2}}}{2}\frac{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(b+c-\frac{d}{2}-\frac{a}{2}-\frac{1}{2})}{\Gamma(b)\Gamma(c)}\frac{\Gamma(\frac{a}{2}+\frac{1}{2}+\frac{d}{2}-b)\Gamma(\frac{a}{2}+\frac{1}{2}+\frac{d}{2}-c)}{\Gamma(a+1+d-b-c)}. \quad (4.2.112)$$

Therefore, according to (4.2.111), the integral in (4.2.109) becomes

$$\begin{aligned}
\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle &= 2(T^a)^{IJ}\frac{C_d C_\Delta^2}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}}\frac{\partial}{\partial x_i'}\int d^d w'\, dw_0' \frac{w_0'^{2\Delta-1}}{\left[w_0'^2+(\vec{w}'-\vec{x}')^2\right]^\Delta\left[w_0'^2+(\vec{w}'-\vec{y}')^2\right]^\Delta}\\
&= 2(T^a)^{IJ}\frac{C_d C_\Delta^2}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}} I\big(2\Delta-1,\Delta,\Delta,d\big)\frac{\partial}{\partial x_i'}\left|\vec{x}'-\vec{y}'\right|^{d-2\Delta}\\
&= -2(T^a)^{IJ}\frac{C_d C_\Delta^2}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}} I\big(2\Delta-1,\Delta,\Delta,d\big)(2\Delta-d)(x_i'-y_i')\left|\vec{x}'-\vec{y}'\right|^{d-2\Delta-2}\\
&= -2(T^a)^{IJ}\frac{C_d C_\Delta^2}{\vec{x}^{2\Delta}\vec{y}^{2\Delta}} I\big(2\Delta-1,\Delta,\Delta,d\big)(2\Delta-d)\left(\frac{x_i}{\vec{x}^2}-\frac{y_i}{\vec{y}}\right)\frac{\vec{x}^{d-2\Delta-2}\vec{y}^{d-2\Delta-2}}{|\vec{x}-\vec{y}|^{d-2\Delta-2}}.
\end{aligned} \tag{4.2.113}$$

The last line is obtained by performing an inversion using the identity

$$\frac{1}{(\vec{x}'-\vec{y}')^2} = \frac{\vec{x}^2\vec{y}^2}{(\vec{x}-\vec{y})^2}. \tag{4.2.114}$$

Now, perform a translation such that $\vec{x}\to\vec{x}-\vec{z}$ and $\vec{y}\to\vec{y}-\vec{z}$ to find

$$\begin{aligned}
\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle &= -2(T^a)^{IJ}\frac{(2\Delta-d)C_d C_\Delta^2 I\big(2\Delta-1,\Delta,\Delta,d\big)}{(\vec{x}-\vec{z})^{d-2}(\vec{y}-\vec{z})^{d-2}(\vec{x}-\vec{y})^{-d+2\Delta+2}}\left(\frac{x_i-z_i}{(\vec{x}-\vec{z})^2}-\frac{y_i-z_i}{(\vec{y}-\vec{z})^2}\right)\\
&= -(T^a)^{IJ}\frac{\Gamma(\Delta)\Gamma\left(\frac{d}{2}\right)}{\pi^d\Gamma(\Delta-\frac{d}{2})}\left(\Delta-\frac{d}{2}\right)\\
&\quad\times\frac{1}{(\vec{x}-\vec{z})^{d-2}(\vec{y}-\vec{z})^{d-2}(\vec{x}-\vec{y})^{-d+2\Delta+2}}\left(\frac{x_i-z_i}{(\vec{x}-\vec{z})^2}-\frac{y_i-z_i}{(\vec{y}-\vec{z})^2}\right),
\end{aligned} \tag{4.2.115}$$

where we have evaluated $I\big(2\Delta-1,\Delta,\Delta,d\big)$ using (4.2.112). Compare this result with the computation from the CFT side, which is given by

$$\begin{aligned}
&\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle = -\epsilon(S_i^a)^{IJ}(\vec{z},\vec{x},\vec{y})\\
&= -\epsilon(d-2)(T^a)^{IJ}\frac{1}{(\vec{x}-\vec{z})^{d-2}(\vec{y}-\vec{z})^{d-2}(\vec{x}-\vec{y})^{-d+2\Delta+2}}\left(\frac{x_i-z_i}{(\vec{x}-\vec{z})^2}-\frac{y_i-z_i}{(\vec{y}-\vec{z})^2}\right).
\end{aligned} \tag{4.2.116}$$

We find

$$\epsilon = \frac{\Gamma(\Delta)\Gamma\left(\frac{d}{2}\right)}{\pi^d(d-2)\Gamma(\Delta-\frac{d}{2})}\left(\Delta-\frac{d}{2}\right). \tag{4.2.117}$$

Now, use the Ward identity relating $\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle$ and $\left\langle \mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle$ which is

$$\begin{aligned}
&\frac{\partial}{\partial z_i}\left\langle \mathcal{J}_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle\\
&= \delta(\vec{x}-\vec{z})(T^a)^{IK}\left\langle \mathcal{O}^K(\vec{x})\mathcal{O}^J(\vec{y})\right\rangle + \delta(\vec{y}-\vec{z})(T^a)^{JK}\left\langle \mathcal{O}^I(\vec{x})\mathcal{O}^K(\vec{y})\right\rangle\\
&= \epsilon\frac{(d-2)2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}(T^a)^{IJ}\left[\delta(\vec{x}-\vec{z})-\delta(\vec{y}-\vec{z})\right]\frac{1}{(\vec{x}-\vec{y})^{2\Delta}}.
\end{aligned} \tag{4.2.118}$$

This Ward identity is satisfied if the two-point correlator is given by (4.2.100) with the same constant ϵ given in (4.2.117).

To derive the Ward identity above, consider

$$\int [\mathcal{D}\phi] \exp(iS[\phi])\phi^I(x)\phi^J(y) - \int [\mathcal{D}\phi'] \exp(iS[\phi'])\phi'^I(x)\phi'^J(y) = 0 \qquad (4.2.119)$$

where in the second term above we have made a change of variables

$$\phi^I \to \phi'^I = \phi^I + \delta\phi^I. \qquad (4.2.120)$$

Assume that the path integral measure is invariant under this change of variables. The change in the action is

$$S[\phi'] = S[\phi] + \delta S[\phi]. \qquad (4.2.121)$$

Thus we can write

$$\int [\mathcal{D}\phi] \exp(iS[\phi]) \left\{\phi^I(x)\phi^J(y) - (1 + i\delta S[\phi])[\phi^I(x) + \delta\phi^I(x)][\phi^J(y) + \delta\phi^J(y)]\right\} = 0. \qquad (4.2.122)$$

From the first order variation, we have

$$\int [\mathcal{D}\phi] \exp(iS[\phi]) \left\{i\delta S[\phi]\phi^I(x)\phi^J(y) + \delta\phi^I(x)\phi^J(y) + \phi^I(x)\delta\phi^J(y)\right\} = 0. \qquad (4.2.123)$$

For the transformation we consider, $\delta\phi^I$ is generated using the $SU(4)$ R-symmetry of the CFT. Thus we have

$$\delta S[\phi] = \int d^d w \partial_i \alpha^a \mathcal{J}_i^a = -\int d^d w \alpha^a \partial_i \mathcal{J}_i^a, \qquad (4.2.124)$$

and

$$\delta\phi^I(x) = i\alpha^a(x)(T^a)^{IK}\phi^K(x), \qquad (4.2.125)$$

where $(T^a)^{IK}$ are the generators of the Lie algebra of $SU(4)$. They are antisymmetric. Therefore, (4.2.123) becomes

$$i\left\langle \int d^d w \alpha^a(w)\partial_i \mathcal{J}_i^a(w)\phi^I(x)\phi^J(y)\right\rangle = i(T^a)^{IK}\left\langle \alpha^a(x)\phi^K(x)\phi^J(y)\right\rangle + i(T^a)^{JK}\left\langle \phi^I(x)\alpha^a(y)\phi^K(y)\right\rangle. \qquad (4.2.126)$$

We finally choose α^a to be

$$\alpha^a(x) = \beta^a\delta(x - z), \qquad (4.2.127)$$

where β^a is a small constant parameter. We find then the following Ward identity

$$\left\langle \partial_i \mathcal{J}_i^a(z)\phi^I(x)\phi^J(y)\right\rangle = (T^a)^{IK}\delta(x - z)\left\langle \phi^K(x)\phi^J(y)\right\rangle + (T^a)^{JK}\delta(y - z)\left\langle \phi^I(x)\phi^K(y)\right\rangle. \qquad (4.2.128)$$

To obtain (4.2.118), we use the antisymmetric property of the generators $(T^a)^{IJ}$ and the conformal form of the two-point correlators.

Note that we determined the correct normalization for the current $\mathcal{J}_i^I$ using the fact that the associated charge Q^I generates the symmetry transformation

$$\phi^I(x) \to e^{iQ^I}\phi^I(x)e^{-iQ^I}. \qquad (4.2.129)$$

The variation of the field ϕ^I is given by

$$\delta\phi^I(x) = i[Q^I, \phi^I(x)] \qquad (4.2.130)$$

for an infinitesimal transformation.

4.3 Three-point correlation functions from the AdS side

Consider three scalar fields ϕ_I, $I = 1, 2, 3$ in the supergravity theory with mass m_I and interaction vertices of the form $\mathcal{L}_1 = \phi_1\phi_2\phi_3$ and $\mathcal{L}_2 = \phi_1 g^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_3$. The corresponding three-point amplitudes are

$$A_1(\vec{x},\vec{y},\vec{z}) = -\int \frac{d^d w\, dw_0}{w_0^{d+1}} K_{\Delta_1}(w,\vec{x})K_{\Delta_2}(w,\vec{y})K_{\Delta_3}(w,\vec{z}) \tag{4.3.1}$$

$$A_2(\vec{x},\vec{y},\vec{z}) = -\int \frac{d^d w\, dw_0}{w_0^{d+1}} K_{\Delta_1}(w,\vec{x})\partial_\mu K_{\Delta_2}(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,\vec{z}), \tag{4.3.2}$$

where K_{Δ_I} are the Green's functions given in (4.2.32). The above correlators are conformally covariant and are of the form required by conformal symmetry

$$A_i(\vec{x},\vec{y},\vec{z}) = \frac{a_i}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{z}-\vec{x}|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{4.3.3}$$

with

$$a_1 = -\frac{\Gamma\left(\frac{1}{2}[\Delta_1+\Delta_2-\Delta_3]\right)\Gamma\left(\frac{1}{2}[\Delta_2+\Delta_3-\Delta_1]\right)\Gamma\left(\frac{1}{2}[\Delta_3+\Delta_1-\Delta_2]\right)}{2\pi^d\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})}\Gamma\left(\frac{1}{2}[\Delta_1+\Delta_2+\Delta_3-d]\right) \tag{4.3.4}$$

$$a_2 = a_1\left[\Delta_2\Delta_3 + \frac{1}{2}(d-\Delta_1-\Delta_2-\Delta_3)(\Delta_2+\Delta_3-\Delta_1)\right]. \tag{4.3.5}$$

Use the translation symmetry to set one boundary point to 0, say $\vec{z}=0$. Then use inversion symmetry, $z'_\mu = \frac{z_\mu}{z^2}$, which is a discrete symmetry, to determine the three-point amplitudes. The measure in the integral is invariant under translation and inversion. The bulk-to-boundary propagators transform as

$$K_{\Delta_1}(w,\vec{x}) = K_{\Delta_1}(w',\vec{x}')|\vec{x}'|^{\Delta_1} \tag{4.3.6}$$

$$K_{\Delta_2}(w,\vec{y}) = K_{\Delta_2}(w',\vec{y}')|\vec{y}'|^{\Delta_2} \tag{4.3.7}$$

$$K_{\Delta_3}(w,0) = C_{\Delta_3}w_0'^{\Delta_3}. \tag{4.3.8}$$

Thus we obtain

$$\begin{aligned} A_1(\vec{x},\vec{y},0) &= -|\vec{x}'|^{\Delta_1}|\vec{y}'|^{\Delta_2}C_{\Delta_3}\int\frac{d^d w'\,dw_0'}{w_0'^{d+1}}K_{\Delta_1}(w',\vec{x}')K_{\Delta_2}(w',\vec{y}')w_0'^{\Delta_3} \\ &= -|\vec{x}'|^{\Delta_1}|\vec{y}'|^{\Delta_2}C_{\Delta_3}C_{\Delta_1}C_{\Delta_2}\int d^d w'\,dw_0'\frac{w_0'^{\Delta_3+\Delta_1+\Delta_2-d-1}}{\left[w_0'^2+(\vec{w}'-\vec{x}')^2\right]^{\Delta_1}\left[w_0'^2+(\vec{w}'-\vec{y}')^2\right]^{\Delta_2}} \\ &= -|\vec{x}'|^{\Delta_1}|\vec{y}'|^{\Delta_2}C_{\Delta_3}C_{\Delta_1}C_{\Delta_2}I(\Delta_3-\Delta_1-\Delta_2-d-1,\Delta_1,\Delta_2,d)|\vec{x}'-\vec{y}'|^{\Delta_3-\Delta_1-\Delta_2}. \end{aligned} \tag{4.3.9}$$

Use the expression for $I(\Delta_3-\Delta_1-\Delta_2-d-1,\Delta_1,\Delta_2,d)$ given in (4.2.112) and for C_{Δ_i} given in (4.2.38) and perform an inversion such that (D'Hoker and Freedman, 2002)

$$\frac{1}{|\vec{x}'-\vec{y}'|^2} = \frac{|\vec{x}|^2|\vec{y}|^2}{|\vec{x}-\vec{y}|^2} \tag{4.3.10}$$

$$|\vec{x}'|^2 = \frac{1}{|\vec{x}|^2} \tag{4.3.11}$$

$$|\vec{y}'|^2 = \frac{1}{|\vec{y}|^2} \tag{4.3.12}$$

After a translation to bring the boundary point back to $\vec{z}$, we obtain the three-point amplitude A_1. To compute A_2 we can proceed similarly by knowing that $\partial_\mu K_{\Delta_2}(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,\vec{z})$ is an invariant contraction under inversion so that

$$\begin{aligned}
&\partial_\mu K_{\Delta_2}(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,0)\\
&=|\vec{y}'|^{2\Delta_2}\partial'_\mu K_{\Delta_2}(w',\vec{y}')w_0'^2\partial'_\mu K_{\Delta_3}(w',0)\\
&=|\vec{y}'|^{2\Delta_2}C_{\Delta_2}C_{\Delta_3}\frac{\partial}{\partial w_0'}\left[\frac{w_0'}{w_0'^2+(\vec{w}'-\vec{y}')^2}\right]^{\Delta_2}w_0'^2\frac{\partial}{\partial w_0'}w_0'^{\Delta_3}\\
&=\Delta_2\Delta_3|\vec{y}'|^{2\Delta_2}C_{\Delta_2}C_{\Delta_3}w_0'^{\Delta+\Delta_3}\left[\frac{1}{[w_0'^2+(\vec{w}'-\vec{y}')^2]^{\Delta_2}}-\frac{2w_0'}{[w_0'^2+(\vec{w}'-\vec{y}')^2]^{\Delta_2+1}}\right].
\end{aligned} \tag{4.3.13}$$

After using (4.2.112) we obtain the quoted result for A_2.

In conclusion, the holographic principle is used to compute correlation functions in the strong coupling limit of the CFT, by using supergravity on $AdS_5\times S^5$. The computation of two-point functions from the gravity side is subtle due to divergences. The answer that was obtained is ambiguous. Using a Ward Identity, we decided on the correct result. The origin of these divergences is still to be understood with the hope that we can provide a well motivated regularization procedure. Indeed, we have already seen that these divergences are boundary effects that require a cutoff near the boundary of the AdS space to regulate the divergences. The result of the computation depends on the way this cutoff is manipulated.

5. Correlation functions of the giant gravitons

Previously, a divergence was found in the two-point function of operators with dimension of order 1. In this chapter, we will see that there are also divergences in correlators of operators with dimension of order N. Thus, these divergences are a rather general feature of the theory. Here we are considering three-point function of two giant gravitons, both on the five-sphere and in the 5-dimensional Anti-de Sitter space, and one point-like graviton. We use the $D3$-brane Dirac-Born-Infeld (DBI) actions on the gravity side, and Schur polynomials and a single trace chiral primary in the gauge theory computation (Bissi et al., 2011). The gauge theory and the string theory results in Bissi et al. (2011) do not match since there were divergences as Lin (2012) pointed out in his paper. The extremal correlators of Schur polynomials and single trace operators match exactly with string theory computations using analytic continuation from non-extremal correlators to extremal correlators (Lin, 2012).

5.1 Correlation functions of giant gravitons from the CFT side

5.1.1 Giant gravitons and Schur polynomials. As we know, Schur polynomial operators, introduced in Chapter (3), are specific combinations of traces of the $\mathcal{N}=4$ SYM scalars. Their correlation functions are given by

$$\left\langle \chi_R(Z)\chi_R(Z^\dagger)\right\rangle = \delta_{RS}\prod_{i,j\in R}(N-i+j) = \delta_{RS} f_R \tag{5.1.1}$$

$$\left\langle \chi_R(Z)\chi_S(Z)\chi_T(Z^\dagger)\right\rangle = g(R,S,T)\prod_{i,j\in T}(N-i+j) = g(R,S,T) f_T, \tag{5.1.2}$$

where $g(R,S,T)$ is the Littlewood-Richardson coefficient which counts the multiplicity with which the representation T appears in the tensor product of the representation R and S (Corley et al., 2002). The product $\prod_{i,j\in R}$ goes over all boxes of the Young tableau of the representation R with i denoting the row number and j the column number. This product is defined as the product of the factors, denoted by f_R, of the Young tableau of the representation R.

The S^5 giant gravitons of dimension $k \leq N$ are mapped to Schur polynomials in the representations labelled by Young diagrams with one column of length k, known as the antisymmetric representation with k boxes, while the dual AdS$_5$ giants map to Schur polynomials in the representations labelled by Young diagrams with one row of length k, known as the symmetric representation with k boxes (Corley et al., 2002; Caputa et al., 2012). It is useful to note the following results obtained directly from (5.1.1) and (5.1.2),

$$\left\langle \chi_k(Z)\chi_k(Z^\dagger)\right\rangle = \prod_{j=1}^{k}(N-1+j) \tag{5.1.3}$$

$$\left\langle \chi_{1^k}(Z)\chi_{1^k}(Z^\dagger)\right\rangle = \prod_{i=1}^{k}(N-i+1) \tag{5.1.4}$$

$$\left\langle \chi_J(Z)\chi_{k-J}(Z)\chi_k(Z^\dagger)\right\rangle = \prod_{j=1}^{k}(N-1+j) \tag{5.1.5}$$

$$\left\langle \chi_{1^J}(Z)\chi_{1^{k-J}}(Z)\chi_{1^k}(Z^\dagger)\right\rangle = \prod_{i=1}^{k}(N-i+1). \tag{5.1.6}$$

5.1.2 Single trace chiral primaries. Single-trace operators built from a single complex scalar field Z are dual to point-like strings moving along an equator of S^5 with angular momentum J. The simplest example of a chiral primary operator is

$$\mathcal{O}_J(x) = \mathrm{Tr}\left(Z^J(x)\right). \tag{5.1.7}$$

In zero-dimensions, their two-point and three-point functions are given by

$$\begin{aligned}\left\langle \mathrm{Tr}\left(Z^J\right)\mathrm{Tr}\left(Z^{\dagger J}\right)\right\rangle &= \frac{1}{J+1}\left[\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right]\\ &= JN^J\left[1+\begin{pmatrix} J+1 \\ 4\end{pmatrix}\frac{1}{N^2}+\dots\right],\end{aligned} \tag{5.1.8}$$

and

$$\begin{aligned}&\left\langle \mathrm{Tr}\left(Z^J\right)\mathrm{Tr}\left(Z^K\right)\mathrm{Tr}\left(Z^{\dagger J+K}\right)\right\rangle\\ &= \frac{1}{J+K+1}\left[\frac{\Gamma(N+J+K+1)}{\Gamma(N)} - \frac{\Gamma(N+J+1)}{\Gamma(N-K)}\right.\\ &\left. + \frac{\Gamma(N+1)}{\Gamma(N-J-K)} - \frac{\Gamma(N+K+1)}{\Gamma(N-J)}\right]\\ &= JK(J+K)N^{J+K-1}\left\{1+\begin{pmatrix} J+K-1 \\ 2\end{pmatrix}\left[\begin{pmatrix} J \\ 2\end{pmatrix}+\begin{pmatrix} K \\ 2\end{pmatrix}-1\right]\frac{1}{3!N^2}+\dots\right\}.\end{aligned} \tag{5.1.9}$$

The structure constant is given by

$$\begin{aligned}C_{J,K,K+J} &= \frac{\langle \mathcal{O}^J\mathcal{O}^K\mathcal{O}^{\dagger J+K}\rangle}{\sqrt{\langle \mathcal{O}^J\mathcal{O}^{\dagger J}\rangle\langle \mathcal{O}^K\mathcal{O}^{\dagger K}\rangle\langle \mathcal{O}^{J+K}\mathcal{O}^{\dagger J+K}\rangle}}\\ &= \frac{\sqrt{JK(J+K)}}{N}\left[1+O\left(\frac{1}{N^2}\right)\right]\end{aligned} \tag{5.1.10}$$

To compute these correlators, it is helpful to note that

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) = \sum_{R\vdash n}\chi_R(\sigma)\chi_R(Z). \tag{5.1.11}$$

We take $\sigma \in S_J$ as an element of the conjugacy class of J-cycles so that

$$\text{Tr}\left(\sigma Z^{\otimes J}\right) = \text{Tr}\left(Z^J\right). \tag{5.1.12}$$

Therefore

$$\begin{aligned}\left\langle \text{Tr}\left(Z^J\right) \text{Tr}\left(Z^{\dagger J}\right)\right\rangle &= \sum_{R\vdash J} \chi_R(\sigma) \sum_{S\vdash J} \chi_S(\sigma) \left\langle \chi_R(Z)\chi_S(Z^\dagger)\right\rangle \\ &= \sum_{R\vdash J}\sum_{S\vdash J} \chi_R(\sigma)\chi_S(\sigma)\delta_{RS} f_R \\ &= \sum_{R\vdash J} [\chi_R(\sigma)]^2 f_R. \end{aligned} \tag{5.1.13}$$

For a J-cycle $\sigma \in S_J$, the character $\chi_R(\sigma)$ is only non-zero for the representations $R \vdash J$ corresponding to Young tableaux with row length k and column length $J-k$ for $k = 1$ to J. These are called hook representations. The character of a J-cycle is either plus one or negative one for any hook representation

$$\chi_R(\sigma) = \chi_R(\sigma^{-1}) = \pm 1, \tag{5.1.14}$$

and the corresponding product of factors f_R for this representation is given by

$$f_R = \prod_{j=1}^{k}(N+j-1) \prod_{i=1}^{J-k}(N-i)(-1)^{J-K}. \tag{5.1.15}$$

We finally deduce that the correlator $\left\langle \text{Tr}\left(Z^J\right) \text{Tr}\left(Z^{\dagger J}\right)\right\rangle$ is equal to the sum of products of factors f_R over all representations R corresponding to Young tableaux with row length k and column length $J-k$ for $k = 1$ to J. Thus,

$$\left\langle \text{Tr}\left(Z^J\right) \text{Tr}\left(Z^{\dagger J}\right)\right\rangle = \sum_{k=1}^{J}\prod_{j=1}^{k}(N+j-1) \prod_{i=1}^{J-k}(N-i), \tag{5.1.16}$$

which gives (5.1.8) after evaluating the summation over the index k. The three-point correlation function in (5.1.9) is obtained with very similar manipulations.

5.1.3 The three-point correlation functions with two giant gravitons. The structure constant corresponding to a three-point function involving two giant gravitons moving on S^5 with angular momenta $k-J$ and k wrapping an $S^3 \subset \text{AdS}_5$, and one light string dual to a chiral primary operator of the type $\text{Tr}\left(Z^J\right)$ is given by

$$\left\langle O^{J,J}\right\rangle_S^{gauge} = \frac{\left\langle \text{Tr}\left(Z^J\right) \chi_{k-J}(Z)\chi_k(Z^\dagger)\right\rangle}{\sqrt{\left\langle \chi_k(Z)\chi_k(Z^\dagger)\right\rangle \left\langle \chi_{k-J}(Z)\chi_{k-J}(Z^\dagger)\right\rangle \left\langle \text{Tr}\left(Z^J\right) \text{Tr}\left(Z^{\dagger J}\right)\right\rangle}}. \tag{5.1.17}$$

Similarly, the structure constant corresponding to a three-point function involving two giant gravitons moving on S^5 with angular momenta $k-J$ and k wrapping an $S^3 \subset S^5$, and one light string dual to a chiral primary operator of the type $\text{Tr}\left(Z^J\right)$ is given by

$$\left\langle O^{J,J}\right\rangle_A^{gauge} = \frac{\left\langle \text{Tr}\left(Z^J\right) \chi_{1^{k-J}}(Z)\chi_{1^k}(Z^\dagger)\right\rangle}{\sqrt{\left\langle \chi_{1^k}(Z)\chi_{1^k}(Z^\dagger)\right\rangle \left\langle \chi_{1^{k-J}}(Z)\chi_{1^{k-J}}(Z^\dagger)\right\rangle \left\langle \text{Tr}\left(Z^J\right) \text{Tr}\left(Z^{\dagger J}\right)\right\rangle}}. \tag{5.1.18}$$

In order to compute these structure constants, we expand $\text{Tr}\left(Z^J\right)$ in the basis of Schur polynomials using

$$\text{Tr}\left(Z^J\right) = \text{Tr}\left(\sigma Z^{\otimes J}\right) = \sum_{R\vdash J} \chi_R(\sigma)\chi_R(Z), \tag{5.1.19}$$

where σ is a J-cycle permutation. Then, we have

$$\left\langle \text{Tr}\left(Z^J\right)\chi_{k-J}(Z)\chi_k(Z^\dagger)\right\rangle = \sum_{R\vdash J}\chi_R(\sigma)\left\langle \chi_R(Z)\chi_{k-J}(Z)\chi_k(Z^\dagger)\right\rangle \tag{5.1.20}$$

$$\left\langle \text{Tr}\left(Z^J\right)\chi_{1^{k-J}}(Z)\chi_{1^k}(Z^\dagger)\right\rangle = \sum_{R\vdash J}\chi_R(\sigma)\left\langle \chi_R(Z)\chi_{1^{k-J}}(Z)\chi_{1^k}(Z^\dagger)\right\rangle. \tag{5.1.21}$$

In these two equations above, only the completely symmetric representation (Young diagram with one row of length J) and the completely antisymmetric representation (Young diagram of one column of length J) contribute respectively in the first correlation function and the second one. With

$$\chi_J(\sigma) = 1 \tag{5.1.22}$$

$$\chi_{1^J}(\sigma) = (-1)^{J-1}, \tag{5.1.23}$$

we obtain

$$\left\langle \text{Tr}\left(Z^J\right)\chi_{k-J}(Z)\chi_k(Z^\dagger)\right\rangle = \prod_{j=1}^{k}(N-1+j) \tag{5.1.24}$$

$$\left\langle \text{Tr}\left(Z^J\right)\chi_{1^{k-J}}(Z)\chi_{1^k}(Z^\dagger)\right\rangle = (-1)^{J-1}\prod_{i=1}^{k}(N-i+1). \tag{5.1.25}$$

Using these results, we have

$$\begin{aligned}\langle O^{J,J}\rangle_S^{gauge} &= \frac{\prod_{j=1}^{k}(N-1+j)}{\sqrt{\prod_{j=1}^{k}(N-1+j)\prod_{j=1}^{k-J}(N-1+j)\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle}} \\ &= \sqrt{\frac{\prod_{j=1}^{k}(N-1+j)}{\prod_{j=1}^{k-J}(N-1+j)\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle}} \\ &= \sqrt{\frac{\prod_{j=k-J+1}^{k}(N-1+j)}{\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle}},\end{aligned} \tag{5.1.26}$$

and

$$\begin{aligned}\langle O^{J,J}\rangle_A^{gauge} &= \frac{\prod_{i=1}^{k}(N-i+1)}{\sqrt{\prod_{i=1}^{k}(N-i+1)\prod_{i=1}^{k-J+1}(N-i+1)\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle}} \\ &= \sqrt{\frac{\prod_{i=1}^{k}(N-i+1)}{\prod_{i=1}^{k-J+1}(N-i+1)\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle}} \\ &= \sqrt{\frac{\prod_{i=k-J+1}^{k}(N-i+1)}{\left\langle \text{Tr}\left(Z^J\right)\text{Tr}\left(Z^{\dagger J}\right)\right\rangle}}.\end{aligned} \tag{5.1.27}$$

We finally obtain the structure constants as

$$\langle O^{J,J}\rangle_S^{gauge} = \frac{1}{\sqrt{J}}\left(1+\frac{k}{N}\right)^{\frac{J}{2}} \tag{5.1.28}$$

$$\langle O^{J,J}\rangle_A^{gauge} = (-1)^{J-1}\frac{1}{\sqrt{J}}\left(1-\frac{k}{N}\right)^{\frac{J}{2}}, \tag{5.1.29}$$

where we have used the following limits

$$N\to\infty \qquad k\to\infty \qquad \frac{k}{N}\ \text{finite} \qquad J\ll k. \tag{5.1.30}$$

These limits correspond to large Young tableaux and a small chiral primary operator.

5.2 Correlation functions of giant gravitons from the string theory side

5.2.1 Giant graviton on the five-sphere S^5. The two-point function of a giant graviton can be computed holographically at large N, by using a classical D-brane solution in string theory. We will review the relevant solution in this section.

On $AdS_5\times S^5$, the metric $g_{\mu\nu}$ is

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\widetilde{\Omega}_3^2 + d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta d\Omega_3^2. \tag{5.2.1}$$

The action for the $D3$-brane is

$$S_{D3} = -\frac{N}{2\pi^2}\int d^4\sigma\big(\sqrt{-g} - P[C_4]\big), \tag{5.2.2}$$

where $\sigma^a = (\sigma^0,\sigma^1,\sigma^2,\sigma^3)$ are the worldvolume coordinates and $P[C_4]$ is the pull back of the 4-form potential of the $D3$-brane (Grisaru et al., 2000). If we denote by x^μ for $\mu=0$ to 9 the coordinates on the target spacetime, we have

$$\begin{aligned} g &= \det\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\right), \quad a,b=0,\dots,3 \\ &= \det\begin{bmatrix} g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^0}\frac{\partial x^\nu}{\partial\sigma^0} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^0}\frac{\partial x^\nu}{\partial\sigma^1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^0}\frac{\partial x^\nu}{\partial\sigma^2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^0}\frac{\partial x^\nu}{\partial\sigma^3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^0} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^0} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^0} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^3} \end{bmatrix}. \end{aligned} \tag{5.2.3}$$

The giant graviton on S^5 has a worldvolume evolving on $(\mathbb{R}\subset AdS_5)\times(S^3\subset S^5)$. The wordvolume coordinates σ^a are chosen as follows (Grisaru et al., 2000)

$$\sigma^0 = t, \quad \sigma^i = \chi_i, i=1,2,3 \tag{5.2.4}$$

where χ_i are angles covering the $S^3\subset S^5$. We also have

$$\rho = 0, \quad \phi = \phi(t). \tag{5.2.5}$$

The 4-form potential C_4 is proportional to the volume element Vol(Ω_3)

$$C_{\phi\chi_1\chi_2\chi_3} = \cos^4\theta \text{Vol}(\Omega_3). \tag{5.2.6}$$

In to order to write the action, we need the expression for g which is obtained by considering (5.2.4) and (5.2.5). With

$$d\Omega_3^2 = d\chi_1^2 + \sin^2\chi_1\left(d\chi_2^2 + \sin^2\chi_2 d\chi_3^2\right) \tag{5.2.7}$$

as a metric on S^3 where

$$\chi_1, \chi_2 \in [0, \pi], \quad \chi_3 \in [0, 2\pi], \tag{5.2.8}$$

we obtain

$$\begin{aligned} g &= \det\begin{bmatrix} g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial \chi_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial \chi_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial \chi_3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_1}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_1}\frac{\partial x^\nu}{\partial \chi_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_1}\frac{\partial x^\nu}{\partial \chi_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_1}\frac{\partial x^\nu}{\partial \chi_3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_2}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_2}\frac{\partial x^\nu}{\partial \chi_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_2}\frac{\partial x^\nu}{\partial \chi_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_2}\frac{\partial x^\nu}{\partial \chi_3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_3}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_3}\frac{\partial x^\nu}{\partial \chi_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_3}\frac{\partial x^\nu}{\partial \chi_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \chi_3}\frac{\partial x^\nu}{\partial \chi_3} \end{bmatrix} \\ &= \det\begin{bmatrix} -1 + \dot\phi^2\sin^2\theta & 0 & 0 & 0 \\ 0 & \cos^2\theta & 0 & 0 \\ 0 & 0 & \cos^2\theta\sin^2\chi_1 & 0 \\ 0 & 0 & 0 & \cos^2\theta\sin^2\chi_1\sin^2\chi_2 \end{bmatrix} \\ &= -(1 - \dot\phi^2\sin^2\theta)\cos^6\theta\sin^4\chi_1\sin^2\chi_2. \end{aligned} \tag{5.2.9}$$

The action is then

$$\begin{aligned} S &= -\frac{N}{2\pi^2}\int_{\mathbb{R}\times S^3} dt\, d\chi_1\, d\chi_2\, d\chi_3\left(\cos^3\theta\sin^2\chi_1\sin\chi_2\sqrt{1 - \dot\phi^2\sin^2\theta} - \dot\phi\cos^4\theta\sin^2\chi_1\sin\chi_2\right) \\ &= -N\int dt\left[\cos^3\theta\sqrt{1 - \dot\phi^2\sin^2\theta} - \dot\phi\cos^4\theta\right]. \end{aligned} \tag{5.2.10}$$

The conserved angular momentum is

$$k = \frac{\delta L}{\delta\dot\phi} = N\left[\frac{\dot\phi\cos^3\theta\sin^2\theta}{\sqrt{1 - \dot\phi^2\sin^2\theta}} + \cos^4\theta\right]. \tag{5.2.11}$$

We want to rewrite the action in terms of $l = \frac{k}{N}$. Towards this end, we need to express $\dot\phi$ as a function of l. From (5.2.11), we have

$$\left(l - \cos^4\theta\right)^2 = \frac{\dot\phi^2\cos^6\theta\sin^4\theta}{1 - \dot\phi^2\sin^2\theta} \tag{5.2.12}$$

which gives

$$\dot\phi = \frac{l - \cos^4\theta}{\sin\theta\sqrt{\left(l - \cos^4\theta\right)^2 + \sin^2\theta\cos^6\theta}}. \tag{5.2.13}$$

Therefore the action becomes

$$S = \int dt \frac{\cos^4\theta}{\sin\theta} \frac{l - \cos^2\theta}{\sqrt{\left(l - \cos^4\theta\right)^2 + \sin^2\theta\cos^6\theta}}. \tag{5.2.14}$$

The energy is defined as

$$E = \dot{\phi}k - L. \tag{5.2.15}$$

We replace $\dot{\phi}$, k and L by their expressions as functions of l, to obtain

$$E = \frac{N}{\sin\theta}\sqrt{\left(l - \cos^4\theta\right)^2 + \sin^2\theta\cos^6\theta}. \tag{5.2.16}$$

We can see that the energy is minimized for

$$\cos^2\theta = l \tag{5.2.17}$$

and that

$$E_{\text{min}} = k, \quad S_{\text{min}} = 0. \tag{5.2.18}$$

Further, we have

$$\dot{\phi} = 1. \tag{5.2.19}$$

5.2.2 Giant graviton on AdS_5. The giant graviton on AdS_5 has worldvolume on $\mathbb{R}\times S^3$ embedded in the AdS_5. The worldvolume coordinates σ^a are choosen as follows

$$\sigma^0 = t, \quad \sigma^i = \widetilde{\chi}_i, \quad i = 1,2,3 \tag{5.2.20}$$

where $\widetilde{\chi}_i$ are angles covering the $S^3 \subset AdS_5$. We also have

$$\rho = \text{constant}, \quad \phi = \phi(t). \tag{5.2.21}$$

The action of the anti-$D3$-brane is given by

$$S_{\bar{D}3} = -\frac{N}{2\pi^2}\int d^4\sigma\left(\sqrt{-g} + P[C_4]\right). \tag{5.2.22}$$

The 4-form potential C_4 is proportional to the volume element Vol$(\widetilde{\Omega}_3)$ on S^3 embedded in the AdS, such that

$$C_{\phi\widetilde{\chi}_1,\widetilde{\chi}_2,\widetilde{\chi}_3} = -\sinh^4\rho\,\text{Vol}(\widetilde{\Omega}_3). \tag{5.2.23}$$

The metric on the three-sphere S^3 is the same as in (5.2.7) after replacing the χ_i by $\widetilde{\chi}_i$. In this case, we obtain

$$\begin{aligned}
g &= \det\begin{bmatrix} g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial \widetilde{\chi}_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial \widetilde{\chi}_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial \widetilde{\chi}_3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_1}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_1}\frac{\partial x^\nu}{\partial \widetilde{\chi}_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_1}\frac{\partial x^\nu}{\partial \widetilde{\chi}_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_1}\frac{\partial x^\nu}{\partial \widetilde{\chi}_3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_2}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_2}\frac{\partial x^\nu}{\partial \widetilde{\chi}_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_2}\frac{\partial x^\nu}{\partial \widetilde{\chi}_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_2}\frac{\partial x^\nu}{\partial \widetilde{\chi}_3} \\ g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_3}\frac{\partial x^\nu}{\partial t} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_3}\frac{\partial x^\nu}{\partial \widetilde{\chi}_1} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_3}\frac{\partial x^\nu}{\partial \widetilde{\chi}_2} & g_{\mu\nu}\frac{\partial x^\mu}{\partial \widetilde{\chi}_3}\frac{\partial x^\nu}{\partial \widetilde{\chi}_3} \end{bmatrix} \\
&= \det\begin{bmatrix} -\cosh^2\rho + \dot{\phi}^2 & 0 & 0 & 0 \\ 0 & \sinh^2\rho & 0 & 0 \\ 0 & 0 & \sinh^2\rho\sin^2\widetilde{\chi}_1 & 0 \\ 0 & 0 & 0 & \sinh^2\rho\sin^2\widetilde{\chi}_1\sin^2\widetilde{\chi}_2 \end{bmatrix} \\
&= -\left(\cosh^2\rho - \dot{\phi}^2\right)\sinh^6\rho\sin^4\widetilde{\chi}_1\sin^2\widetilde{\chi}_2.
\end{aligned} \tag{5.2.24}$$

The action becomes

$$S = -\frac{N}{2\pi^2}\int_{\mathbb{R}\times S^3} dt\, d\widetilde{\chi}_1\, d\widetilde{\chi}_2\, d\widetilde{\chi}_3\left(\sinh^3\rho\sin^2\widetilde{\chi}_1\sin\widetilde{\chi}_2\sqrt{\cosh^2\rho-\dot{\phi}^2}-\sinh^4\rho\sin^2\widetilde{\chi}_1\sin\widetilde{\chi}_2\right)$$
$$= -N\int dt\left[\sinh^3\rho\sqrt{\cosh^2\rho-\dot{\phi}^2}-\sinh^4\rho\right]. \tag{5.2.25}$$

The conserved angular momentum is

$$\widetilde{k} = \frac{\delta L}{\delta\dot{\phi}} = N\frac{\dot{\phi}\sinh^3\rho}{\sqrt{\cosh^2\rho-\dot{\phi}^2}}. \tag{5.2.26}$$

Following our treatment of the giant graviton on S^5, we rewrite the action in terms of $\widetilde{l} = \frac{\widetilde{k}}{N}$, by expressing $\dot{\phi}$ as a function of $\widetilde{l}$. From (5.2.26), we have

$$\dot{\phi} = \frac{\widetilde{l}\cosh\rho}{\sqrt{\sinh^6\rho+\widetilde{l}^2}}. \tag{5.2.27}$$

The action is now

$$S = -N\int dt\,\sinh^4\rho\left[\frac{\sinh^2\rho\cosh\rho}{\sqrt{\sinh^6\rho+\widetilde{l}^2}}-1\right] \tag{5.2.28}$$

with the energy defined as

$$E = \dot{\phi}\widetilde{k} - L = N\left[\cosh\rho\sqrt{\sinh^6\rho+\widetilde{l}^2}-\sinh^4\rho\right]. \tag{5.2.29}$$

The minimum of the energy is realized with

$$\sinh^2\rho = \widetilde{l} \tag{5.2.30}$$

such that

$$E_{min} = \widetilde{k}, \quad S_{min} = 0. \tag{5.2.31}$$

Finally, we plug (5.2.30) into (5.2.27) to have

$$\dot{\phi} = \frac{\widetilde{l}\cosh\rho}{\sqrt{\widetilde{l}^2(\sinh^2\rho+1)}} = 1. \tag{5.2.32}$$

5.2.3 Structure constant of the antisymmetric giant graviton. The DBI part of the Euclidean supergravity action is given by

$$S_{DBI} = \frac{N}{2\pi^2}\int d^4\sigma\sqrt{g}. \tag{5.2.33}$$

We will need the variation

$$\delta\sqrt{g} = -\frac{1}{\sqrt{g}}\delta g = \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab}, \tag{5.2.34}$$

where we have used the identity

$$\delta g = \delta \det g_{ab} = g\, g^{ab}\delta g_{ab}. \tag{5.2.35}$$

With

$$\delta g_{ab} = \delta g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}, \quad a,b = 0,1,2,3 \quad \mu,\nu = 0,1,\dots,9 \tag{5.2.36}$$

we find

$$\begin{aligned}\delta S_{DBI} &= \frac{N}{2\pi^2}\int d^4\sigma \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab}\\ &= \frac{N}{2\pi^2}\int d^4\sigma \frac{1}{2}\sqrt{g}g^{ab}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\delta g_{\mu\nu}.\end{aligned} \tag{5.2.37}$$

Since the induced metric g_{ab} is diagonal, the inverse g^{ab} will also be diagonal. Thus we have

$$\delta S_{DBI} = \frac{N}{2\pi^2}\int d^4\sigma \frac{1}{2}\sqrt{g}g^{aa}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^a}\delta g_{\mu\nu} \tag{5.2.38}$$

with

$$g^{00}\frac{\partial x^\mu}{\partial\sigma^0}\frac{\partial x^\nu}{\partial\sigma^0}\delta g_{\mu\nu} = g^{00}\left[\delta g_{tt} + \left(\frac{\partial\phi}{\partial t}\right)^2\delta g_{\phi\phi}\right] = g^{00}\big(\delta g_{tt} - \delta g_{\phi\phi}\big), \tag{5.2.39}$$

$$g^{11}\frac{\partial x^\mu}{\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^1}\delta g_{\mu\nu} = g^{11}\delta g_{\chi_1\chi_1}, \tag{5.2.40}$$

$$g^{22}\frac{\partial x^\mu}{\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^2}\delta g_{\mu\nu} = g^{22}\delta g_{\chi_2\chi_2}, \tag{5.2.41}$$

$$g^{33}\frac{\partial x^\mu}{\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^3}\delta g_{\mu\nu} = g^{33}\delta g_{\chi_3\chi_3}. \tag{5.2.42}$$

Here, after the Wick rotation into Euclidean AdS_5, we have

$$g_{tt}dt^2 \to -g_{tt}dt_E^2 = g_{t_Et_E}dt_E^2 \tag{5.2.43}$$

$$\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t} \to -\frac{\partial\phi}{\partial t_E}\frac{\partial\phi}{\partial t_E}. \tag{5.2.44}$$

Thus, with $\frac{\partial\phi}{\partial t} = \dot\phi = 1$ and $\rho = 0$, one finds

$$g_{ab} = \text{diag}\big(\cos^2\theta, cos^2\theta, \cos^2\theta\sin^2\chi_1, \cos^2\theta\sin^2\chi_1\sin^2\chi_2\big) \tag{5.2.45}$$

$$g^{ab} = \text{diag}\left(\frac{1}{\cos^2\theta}, \frac{1}{cos^2\theta}, \frac{1}{\cos^2\theta\sin^2\chi_1}, \frac{1}{\cos^2\theta\sin^2\chi_1\sin^2\chi_2}\right). \tag{5.2.46}$$

The fluctuation of the metric and the 4-form potential are given by (Lee et al., 1998; Arutyunov and Frolov, 2000; Zarembo, 2010; Bissi et al., 2011)

$$\delta g_{\mu\nu} = \left[-\frac{6\Delta}{5}g_{\mu\nu} + \frac{4}{\Delta+1}\nabla_{(\mu}\nabla_{\nu)}\right]s^\Delta(X)Y_\Delta(\Omega) \tag{5.2.47}$$

$$\delta g_{\alpha\beta} = 2\Delta g_{\alpha\beta}s^\Delta(X)Y_\Delta(\Omega) \tag{5.2.48}$$

$$\delta C_{\mu_1\mu_2\mu_3\mu_4} = -4\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5}s^\Delta(X)Y_\Delta(\Omega) \tag{5.2.49}$$

$$\delta C_{\alpha_1\alpha_2\alpha_3\alpha_4} = 4\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}s^\Delta(X)\nabla^{\alpha_5}Y_\Delta(\Omega) \tag{5.2.50}$$

where the indices μ, ν are AdS_5 and α, β are S^5 indices. $s^\Delta(X)$ is a scalar field propagating on AdS_5 with mass squared equal to $\Delta(\Delta-4)$ and $Y_\Delta(\Omega)$ are spherical harmonics on the five-sphere S^5. The traceless symmetric double covariant derivative is defined by

$$\nabla_{(\mu}\nabla_{\nu)} = \frac{1}{2}\big(\nabla_\mu\nabla_\nu + \nabla_\mu\nabla_\nu\big) - \frac{1}{5}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla_\sigma. \tag{5.2.51}$$

The variation of the DBI part of the Euclidean action is

$$\begin{aligned}\delta S_{DBI} = \frac{N}{2\pi^2}\int_{\mathbb{R}\times S^3} & dt\, d\chi_1\, d\chi_2\, d\chi_3 \cos^2\theta \sin^2\chi_1 \sin\chi_2 Y_\Delta(\Omega) \\ \times & \left[\left(-\frac{6\Delta}{5}g_{tt} + \frac{4}{\Delta+1}\nabla_{(t}\nabla_{t)} - 2\Delta\sin^2\theta\right) + 6\Delta\cos^2\theta\right] s^\Delta(X),\end{aligned} \tag{5.2.52}$$

where t is now the time coordinate in the Euclidean AdS_5. To compute the double traceless symmetric covariant derivative $\nabla_{(t}\nabla_{t)}$, we use the metric of the Euclidean AdS_5 given by

$$ds^2_{EAdS_5} = \frac{1}{z^2}\big[(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dz^2)^2\big]. \tag{5.2.53}$$

From (5.2.51), we have

$$\nabla_{(t}\nabla_{t)}s^\Delta(X) = \nabla_t\nabla_t s^\Delta(X) - \frac{1}{5}g_{tt}g^{\mu\mu}\nabla_\mu\nabla_\mu s^\Delta(X), \tag{5.2.54}$$

with

$$g^{x^ix^i}\nabla_{x_i}\nabla_{x_i}s^\Delta(X) = g^{x^ix^i}\big(\partial^2_{x^i} - \Gamma^z_{x^ix^i}\partial_z\big)s^\Delta(X) = z^2\left(\partial^2_{x^i} - \frac{1}{z}\partial_z\right)s^\Delta(X), \quad i=0,1,2,3 \tag{5.2.55}$$

$$g^{zz}\nabla_z\nabla_z s^\Delta(X) = g^{zz}\big(\partial^2_{zz} - \Gamma^z_{zz}\partial_z\big)s^\Delta(X) = z^2\left(\partial^2_z + \frac{1}{z}\partial_z\right)s^\Delta(X). \tag{5.2.56}$$

In addition,

$$\partial_{x^i}s^\Delta(X) = 0, \tag{5.2.57}$$

since the bulk-to-boundary propagator $s^\Delta(X)$ is function only of z. It is given by

$$s^\Delta(X) = \frac{\Delta+1}{N\Delta^{\frac{1}{2}}2^{2-\frac{\Delta}{2}}}\frac{z^\Delta}{x_B^{2\Delta}} = \frac{a_\Delta z^\Delta}{x_B^{2\Delta}}, \quad a_\Delta = \frac{\Delta+1}{N\Delta^{\frac{1}{2}}2^{2-\frac{\Delta}{2}}}. \tag{5.2.58}$$

It follows from (5.2.55) and (5.2.56) that

$$\nabla_{(t}\nabla_{t)}s^\Delta(X) = \left[\partial_t^2 - \frac{1}{5}g_{tt}\big(-3\Delta + \Delta(\Delta-1)\big)\right]s^\Delta(X), \quad g_{tt}=1. \tag{5.2.59}$$

Therefore, (5.2.52) becomes

$$\begin{aligned}\delta S_{DBI} &= \frac{N}{2}\cos^2\theta\int dt\, Y_\Delta(\Omega)\left(\frac{4}{\Delta+1}\partial_t^2 - \frac{2\Delta(\Delta-1)}{\Delta+1} - 8\Delta\sin^2\theta + 6\Delta\right)s^\Delta(X) \\ &= N\cos^2\theta\int dt\, Y_\Delta(\Omega)\left(\frac{2}{\Delta+1}\big(\partial_t^2 - \Delta^2\big) + 4\Delta\cos^2\theta\right)s^\Delta(X)\end{aligned} \tag{5.2.60}$$

with

$$Y_\Delta(\Omega) = \frac{\sin^\Delta\theta\, e^{i\Delta\phi}}{2^{\frac{\Delta}{2}}} = \widetilde{Y}_\Delta e^{\Delta t}, \qquad \widetilde{Y}_\Delta = \frac{\sin^\Delta\theta}{2^{\frac{\Delta}{2}}}. \tag{5.2.61}$$

We have $z = \frac{R}{\cosh t}$. Thus we find

$$\partial_t s^\Delta(X) = -\Delta\frac{\sinh t}{\cosh t}s^\Delta(X) \tag{5.2.62}$$

$$\partial_t^2 s^\Delta(X) = \left[\Delta^2 - \Delta(\Delta+1)\frac{1}{\cosh^2 t}\right] s^\Delta(X). \tag{5.2.63}$$

After evaluating the second derivatives of $s^\Delta(X)$ with respect to t and substituting the expressions for $Y_\Delta(\Omega)$ and $s^\Delta(X)$ into (5.2.60), we obtain

$$\delta S_{DBI} = a_\Delta \cos^2\theta \int dt\, \left(4\cos^2\theta \widetilde{Y}_\Delta \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^\Delta t} - 2\widetilde{Y}_\Delta\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t}\right). \tag{5.2.64}$$

The variation of the WZ part in the action is

$$\begin{aligned}
\delta S_{WZ} &= -i\frac{N}{2\pi^2}\int d^4\sigma P[\delta C_4] \\
&= -i\frac{N}{2\pi^2}\int dt d\chi_1 d\chi_2 d\chi_3 \left(\frac{\partial\phi}{\partial t}4s^\Delta(X)\partial_\theta Y_\Delta(\Omega)\right)\sin\theta\cos^3\theta\sin^2\chi_1\sin\chi_2 \\
&= -N\cos^2\theta\int dt\, \Big(4\sin\theta\cos\theta\,\partial_\theta Y_\Delta(\Omega)\Big)s^\Delta(X) \\
&= -a_\Delta N\cos^2\theta\int dt \Big(4\sin\theta\cos\theta\,\partial_\theta\widetilde{Y}_\Delta\Big)\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^\Delta t}
\end{aligned} \tag{5.2.65}$$

where the fluctuation of the 4-form potential is given in (5.2.50). The total variation in the action is

$$\begin{aligned}
\delta S &= \delta S_{DBI} + \delta S_{WZ} \\
&= 4a_\Delta\cos^2\theta\int dt\, \left(\cos^2\theta\widetilde{Y}_\Delta - \sin\theta\cos\theta\,\partial_\theta\widetilde{Y}_\Delta\right)\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^\Delta t} - 2a_\Delta\cos^2\theta\,\widetilde{Y}_\Delta\int dt\,\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t}.
\end{aligned} \tag{5.2.66}$$

In Bissi et al. (2011), the following terms was dropped

$$\delta S_{div} = 4a_\Delta\cos^2\theta\int dt\, \left(\cos^2\theta\widetilde{Y}_\Delta - \sin\theta\cos\theta\,\partial_\theta\widetilde{Y}_\Delta\right)\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^\Delta t} \tag{5.2.67}$$

since

$$\left(\sin\theta\cos\theta\partial_\theta - \Delta\cos^2\theta\right)\widetilde{Y}_\Delta = 0, \tag{5.2.68}$$

for $\widetilde{Y}_\Delta$ given in (5.2.61). They keep

$$\begin{aligned}
\delta S &= -\frac{\cos^2\theta\,\sin^\Delta\theta(\Delta+1)\sqrt{\Delta}}{2}\int dt\,\frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t} \\
&= -(2\mathcal{R})^\Delta\cos^2\theta\,\sin^\Delta\theta\sqrt{\Delta}
\end{aligned} \tag{5.2.69}$$

which is computed using the integral

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, \frac{e^{\Delta t}}{\cosh^{\Delta+2l} t} &= 2^{\Delta+2l-1} \int_{-\infty}^{\infty} dt\, \left(\frac{e^{2t}}{1+e^{2t}}\right)^{\Delta+2l} (e^{-2t})^l \\ &= 2^{\Delta+2l-1} \int_0^1 du\, u^{\Delta+l-1}(1-u)^{l-1} \\ &= 2^{\Delta+2l-1} B(\Delta+l,l) \\ &= 2^{\Delta+2l-1} \frac{\Gamma(\Delta+l)\Gamma(l)}{\Gamma(\Delta+2l)}. \end{aligned} \tag{5.2.70}$$

The second line of this equation is obtained by performing the change of variable from t to $u = \frac{e^{2t}}{1+e^{2t}}$. Therefore Bissi et al. (2011) found the structure constant of the three-point function equal to

$$\langle O^{J,J} \rangle_A^{string} = \sqrt{J} \frac{k}{N} \left(1 - \frac{k}{N}\right)^{\frac{J}{2}} \tag{5.2.71}$$

which does not match with the computation from the gauge theory side. We will see in the following section that this mismatch is due to the subtleties in the computation of the extremal correlators. There are divergences that need to be regularized carefully.

5.2.4 Regularization procedure for the extremal correlators . Lin (2012) noticed that the integral appearing in the variation of the action is subtle. In fact, the integral in (5.2.70) is divergent when the limit $l \to 0$ is taken, since there is the factor $\Gamma(l)$. Since the term $\left(\sin\theta\cos\theta\partial_\theta - \Delta\cos^2\theta\right)\widetilde{Y}_\Delta$ vanishes in (5.2.68), it is clear that the total variation of the action $\delta S = \delta S_{DBI} + \delta S_{WZ}$ contains a term of the form $0 \cdot \infty$.

Lin (2012) computed the extremal correlators as the limit of the non-extremal correlators (Buchbinder and Tseytlin, 2012). Instead of Y_Δ, he has used spherical harmonics (Appendix of Skenderis and Taylor (2007)) that are expressed in terms of the hypergeometric function ${}_2F_1$,

$$\begin{aligned} Y_{\Delta,j} &= c_{\Delta,j} \sin^j\theta\, e^{ij\phi} {}_2F_1(-l, j+l+2, j+1; \sin^2\theta) \\ &= c_{\Delta,j} e^{ij\phi} F_{\Delta,j}, \qquad F_{\Delta,j} = \sin^j\theta\, {}_2F_1(-l, j+l+2, j+1; \sin^2\theta) \end{aligned} \tag{5.2.72}$$

where the constant of normalization is given by

$$c_{\Delta,j} = \frac{\Gamma(j+l+1)\sqrt{(j+l+1)(l+1)}2^{-\frac{j}{2}}}{\Gamma(l+2)\Gamma(j+1)\sqrt{j+2l+1}2^l} \tag{5.2.73}$$

with $l = 0,1,2,3,\ldots$ and $j = \Delta - 2l$. In this case, we notice that $Y_{\Delta,j}$ tends to the spherical harmonic given in (5.2.61) as $l \to 0$ and

$$\begin{aligned} &\left(\Delta\cos^2\theta - \sin\theta\cos\theta\partial_\theta\right) F_{\Delta,j} \\ &= 2l\cos^2\theta \left[F_{\Delta,j} + \sin^{j+2}\theta \frac{j+l+2}{j+1} {}_2F_1(-l+1, j+l+3, j+2; \sin^2\theta)\right], \end{aligned} \tag{5.2.74}$$

where the following property of the hypergeometric function has been used

$$\frac{d}{dx} {}_2F_1(a,b,c;x) = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; x). \tag{5.2.75}$$

The integral in (5.2.67) that has been dropped in the calculation of Bissi et al. (2011) contributes

$$\delta S_{div} = 4a_\Delta c_{\Delta,j}\cos^2\theta\left(\Delta\cos^2\theta - \sin\theta\cos\theta\,\partial_\theta\right)F_{\Delta,j}\int dt\,\frac{\mathcal{R}^\Delta e^{jt}}{\cosh^{j+2l}t}\Big|_{l\to 0}$$
$$= (2\mathcal{R})^\Delta\frac{1}{\sqrt{j}}\left(1+j\frac{k}{N}\right)\left(1-\frac{k}{N}\right)^{\frac{j}{2}}, \tag{5.2.76}$$

where we have used

$${}_2F_1(-l, j+l+2, j+1; \sin^2\theta)\big|_{l\to 0} = 1 \tag{5.2.77}$$

$${}_2F_1(-l+1, j+l+3, j+2; \sin^2\theta)\big|_{l\to 0} = \frac{1}{j+2}\left[\frac{j+1}{\cos^2\theta}+\frac{1}{\cos^4\theta}\right] \tag{5.2.78}$$

$$l\,\Gamma(l)\big|_{l\to 0} = 1 \tag{5.2.79}$$

with

$$\left[F_{\Delta,j} + \sin^{j+2}\theta\frac{j+l+2}{j+1}{}_2F_1(-l+1, j+l+3, j+2; \sin^2\theta)\right]_{l\to 0} = \frac{\sin^j\theta\,(1+j\cos^2\theta)}{\cos^4\theta(j+1)}. \tag{5.2.80}$$

Adding (5.2.76) to (5.2.71), Lin (2012) found the structure constant

$$\langle O^{J,J}\rangle_A^{string} = \frac{1}{\sqrt{J}}\left(1-\frac{k}{N}\right)^{\frac{J}{2}} \tag{5.2.81}$$

which is in perfect agreement with the gauge theory computation using Schur polynomials.

5.2.5 Structure constant of the symmetric giant graviton. For the symmetric giant graviton, the variation of the DBI part of the action is again given by (5.2.60) but with the induced metric given by

$$g_{ab} = \text{diag}\big(\sinh^2\rho, \sinh^2\rho, \sinh^2\rho\cos^2\vartheta, \sinh^2\rho\sin^2\vartheta\big). \tag{5.2.82}$$

We are working on the Euclidean $AdS_5\times S^5$ space with metric

$$ds^2_{EAdS_5} = \cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\big(d\vartheta^2+\cos^2\vartheta d\phi_1^2+\sin^2\vartheta d\phi_2^2\big). \tag{5.2.83}$$

The coordinates on the worldvolume of the $D3$-brane are

$$\sigma^0 = t,\quad \sigma^1=\vartheta,\quad \sigma^2=\phi_1,\quad \sigma^3=\phi_2. \tag{5.2.84}$$

We also mention that

$$\left(\frac{\partial\phi}{\partial t}\right)^2 = -1 \tag{5.2.85}$$

because we have performed a Wick rotation and we must obey the constraint on the motion of the giant graviton given in (5.2.32). With these results in hand, we find

$$\sqrt{g} = \sinh^4\rho\cos\vartheta\sin\vartheta, \tag{5.2.86}$$

and

$$g^{00}\frac{\partial x^\mu}{\partial\sigma^0}\frac{\partial x^\nu}{\partial\sigma^0}\delta g_{\mu\nu} = g^{00}\big(\delta g_{tt} - \delta g_{\phi\phi}\big), \tag{5.2.87}$$

$$g^{11}\frac{\partial x^\mu}{\partial\sigma^1}\frac{\partial x^\nu}{\partial\sigma^1}\delta g_{\mu\nu} = g^{11}\delta g_{\vartheta\vartheta}, \tag{5.2.88}$$

$$g^{22}\frac{\partial x^\mu}{\partial\sigma^2}\frac{\partial x^\nu}{\partial\sigma^2}\delta g_{\mu\nu} = g^{22}\delta g_{\phi_1\phi_1}, \tag{5.2.89}$$

$$g^{33}\frac{\partial x^\mu}{\partial\sigma^3}\frac{\partial x^\nu}{\partial\sigma^3}\delta g_{\mu\nu} = g^{33}\delta g_{\phi_2\phi_2}. \tag{5.2.90}$$

The fluctuation of the metric is given in (5.2.47) and (5.2.48) such that

$$\begin{aligned}\delta g_{tt} &= \left[-\frac{6}{7}g_{tt} + \frac{4}{\Delta+1}\nabla_{(t}\nabla_{t)}\right]s\\ &= \frac{2}{\Delta+1}\left[2\nabla_t\nabla_t - \Delta(\Delta-1)g_{tt}\right]s = h_{tt},\end{aligned} \tag{5.2.91}$$

where we have used the definition of the traceless symmetric double covariant derivative in (5.2.51) in which we compute the trace on the $EAdS_5$ using the Poincare coordinate in (5.2.53) so that

$$g^{\rho\sigma}\nabla_\rho\nabla_\sigma s = \big[\Delta(\Delta-1) - 3\Delta\big]s, \quad s = s^\Delta(X)Y_\Delta(\Omega). \tag{5.2.92}$$

Using the same approach as above, we have

$$\delta g_{\vartheta\vartheta} = \frac{2}{\Delta+1}\left[2\nabla_\vartheta\nabla_\vartheta - \Delta(\Delta-1)g_{\vartheta\vartheta}\right]s = h_{\vartheta\vartheta}, \tag{5.2.93}$$

$$\delta g_{\phi_1\phi_1} = \frac{2}{\Delta+1}\left[2\nabla_{\phi_1}\nabla_{\phi_1} - \Delta(\Delta-1)g_{\phi_1\phi_1}\right]s = h_{\phi_1\phi_1}, \tag{5.2.94}$$

$$\delta g_{\phi_2\phi_2} = \frac{2}{\Delta+1}\left[2\nabla_{\phi_2}\nabla_{\phi_2} - \Delta(\Delta-1)g_{\phi_2\phi_2}\right]s = h_{\phi_2\phi_2}. \tag{5.2.95}$$

Therefore the variation of the DBI part of the action is

$$\begin{aligned}\delta S_{DBI} &= \frac{N}{2\pi^2}\int d^4\sigma\frac{1}{2}\sqrt{g}g^{aa}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^a}\delta g_{\mu\nu}\\ &= \frac{N}{2\pi^2}\int d^4\sigma\frac{1}{2}\sinh^4\rho\cos\vartheta\sin\vartheta\left[\frac{1}{\sinh^2\rho}\left[h_{tt} - 2\Delta s\right]\right.\\ &\left. + \frac{1}{\sinh^2\rho}h_{\vartheta\vartheta} + \frac{1}{\sinh^2\rho\cos^2\vartheta}h_{\phi_1\phi_1} + \frac{1}{\sinh^2\rho\sin^2\vartheta}h_{\phi_2\phi_2}\right]\\ &= \frac{N}{4\pi^2}\int_{-\infty}^{\infty}dt\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2\int_0^{\pi}d\vartheta\sinh^2\rho\cos\vartheta\sin\vartheta\left[-2\Delta s + h_{tt} + h_{\vartheta\vartheta} + \frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} + \frac{h_{\phi_2\phi_2}}{\sin^2\vartheta}\right]\end{aligned} \tag{5.2.96}$$

where we have set

$$h_{\mu\nu} = \frac{2}{\Delta+1}\left[2\nabla_\mu\nabla_\nu - \Delta(\Delta-1)g_{\mu\nu}\right]s. \tag{5.2.97}$$

To compute the double covariant derivative we use the metric in (5.2.83) to determine the Christoffel connection and we obtain

$$\nabla_t\nabla_t s = (\partial_t^2 + \cosh\rho\sinh\rho\partial_\rho)s, \tag{5.2.98}$$
$$\nabla_\vartheta\nabla_\vartheta s = (\partial_\vartheta^2 + \cosh\rho\sinh\rho\partial_\rho)s, \tag{5.2.99}$$
$$\nabla_{\phi_1}\nabla_{\phi_1} s = (\partial_{\phi_1}^2 + \cos^2\vartheta\cosh\rho\sinh\rho\partial_\rho - \cos\vartheta\sin\vartheta\partial_\vartheta)s, \tag{5.2.100}$$
$$\nabla_{\phi_2}\nabla_{\phi_2} s = (\partial_{\phi_2}^2 + \sin^2\vartheta\cosh\rho\sinh\rho\partial_\rho + \cos\vartheta\sin\vartheta\partial_\vartheta)s. \tag{5.2.101}$$

Here we have

$$Y_\Delta(\Omega) = \frac{e^{\Delta t}}{2^{\frac{\Delta}{2}}} \tag{5.2.102}$$
$$s^\Delta(X) = \frac{\Delta+1}{N\Delta^{\frac{1}{2}}2^{2-\frac{\Delta}{2}}}\frac{\mathcal{R}^\Delta}{\big(\cosh\rho\cosh t - \cos\vartheta\sin\phi_1\sinh\rho\big)^\Delta}, \quad \mathcal{R}^\Delta = \left(\frac{R}{x_B^2}\right)^\Delta \tag{5.2.103}$$
$$s = s^\Delta(X)Y_\Delta(\Omega) = \frac{\Delta+1}{2^2\sqrt{\Delta}N}\frac{\mathcal{R}^\Delta e^{\Delta t}}{\big(\cosh\rho\cosh t - \cos\vartheta\sin\phi_1\sinh\rho\big)^\Delta}. \tag{5.2.104}$$

Using the fluctuation of the 4-form potential in (5.2.49), the variation of the WZ part of the action is given by

$$\begin{aligned}\delta S_{WZ} &= -i\frac{N}{2\pi^2}\int d^4\sigma P[\delta C_4]\\ &= -i\frac{N}{2\pi^2}\int_{-\infty}^{\infty} dt\int_0^{2\pi} d\phi_1\int_0^{2\pi} d\phi_2\int_0^{\pi} d\vartheta(-i4\nabla^\rho s^\Delta(X)Y_\Delta(\Omega))\cosh\rho\sinh^3\rho\cos\vartheta\sin\vartheta\\ &= -\frac{2N}{\pi^2}\int_{-\infty}^{\infty} dt\int_0^{2\pi} d\phi_1\int_0^{2\pi} d\phi_2\int_0^{\pi} d\vartheta\cosh\rho\sinh^3\rho\cos\vartheta\sin\vartheta\partial_\rho s.\end{aligned} \tag{5.2.105}$$

Now we want to obtain the total variation. For this we need

$$\partial_t\, s^\Delta = -\Delta s^\Delta\cosh\rho\sin t z, \tag{5.2.106}$$
$$\partial_t^2\, s^\Delta = \Delta s^\Delta\left((\Delta+1)(\cosh\rho\sinh t\ z)^2 - \cosh\rho\cosh t\ z\right), \tag{5.2.107}$$
$$\partial_\vartheta\, s^\Delta = -\Delta s^\Delta\sin\vartheta\sin\phi_1\sinh\rho\ z, \tag{5.2.108}$$
$$\partial_\vartheta^2\, s^\Delta = \Delta s^\Delta\left((\Delta+1)(\sin\vartheta\sin\phi_1\sinh\rho\ z)^2 - \cos\vartheta\sin\phi_1\sinh\rho\ z\right), \tag{5.2.109}$$
$$\partial_\rho\, s^\Delta = \Delta s^\Delta(\cos\vartheta\sinh\phi_1\cosh\rho\ z - \sinh\rho\cosh t\ z), \tag{5.2.110}$$
$$\partial_{\phi_1}\, s^\Delta = \Delta s^\Delta\cos\vartheta\cos\phi_1\sinh\rho\ z, \tag{5.2.111}$$
$$\partial_{\phi_1}^2\, s^\Delta = \Delta s^\Delta\left((\Delta+1)(\cos\vartheta\cos\phi_1\sinh\rho\ z)^2 - \cos\vartheta\sin\phi_1\sinh\rho\ z\right), \tag{5.2.112}$$
$$\partial_{\phi_2}^2\, s^\Delta = 0. \tag{5.2.113}$$

Then we obtain

$$h_{tt} = \frac{2}{\Delta+1}\Big\{2\Delta\big[(\Delta+1)(\cosh\rho\cosh t\ z)^2 - (\Delta+1)(\cosh\rho\ z)^2 - \cosh\rho\cosh t\ z\big]$$
$$+ 2\cosh\rho\sinh\rho\ \partial_\rho - \Delta(\Delta-1)\cosh^2\rho\Big\}s \tag{5.2.114}$$

$$h_{\vartheta\vartheta} = \frac{2}{\Delta+1}\Big\{2\Delta\big[(\Delta+1)(\sin\phi_1\sinh\rho\ z)^2 - (\Delta+1)(\cos\vartheta\sin\phi_1\sinh\rho\ z)^2 - \cos\vartheta\sin\phi_1\sinh\rho\ z\big]$$
$$+ 2\cosh\rho\sinh\rho\ \partial_\rho - \Delta(\Delta-1)\sinh^2\rho\Big\}s \tag{5.2.115}$$

$$\frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} = \frac{2}{\Delta+1}\Big\{2\Delta\left[(\Delta+1)(\cos\phi_1\sinh\rho\ z)^2 - \frac{\sin\phi_1\sinh\rho\ z}{\cos\vartheta}\right] + 2\frac{\sin\vartheta}{\cos\vartheta}\ \partial_\vartheta$$
$$+ 2\cosh\rho\sinh\rho\ \partial_\rho - \Delta(\Delta-1)\sinh^2\rho\Big\}s \tag{5.2.116}$$

$$\frac{h_{\phi_2\phi_2}}{\sin^2\vartheta} = \frac{2}{\Delta+1}\Big\{2\cosh\rho\sinh\rho\ \partial_\rho + 2\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta - \Delta(\Delta-1)\sinh\rho^2\Big\}s.$$

Using these results we now easily find

$$-2\Delta s + h_{tt} + h_{\vartheta\vartheta} + \frac{h_{\phi_1\phi_1}}{\cos^2\vartheta} + \frac{h_{\phi_2\phi_2}}{\sin^2\vartheta}$$
$$= -2\Delta s + \frac{2}{\Delta+1}\Big\{2\Delta\Big[(\Delta+1)(\cosh\rho\cosh t\ z)^2 - (\Delta+1)(\cos\vartheta\sin\phi_1\sinh\rho\ z)^2 - (\Delta+1)(\cosh\rho\ z)^2$$
$$+ (\Delta+1)(\cos\phi_1\sinh\rho\ z)^2 - \cosh\rho\cosh t\ z - \cos\vartheta\sin\phi_1\sinh\rho\ z - \frac{\sin\phi_1\sinh\rho\ z}{\cos\vartheta}\Big]$$
$$-2\left(\frac{\sin^2\vartheta - \cos^2\vartheta}{sin\vartheta\cos\vartheta}\right)\partial_\vartheta + 8\cosh\rho\sinh\rho\ \partial_\rho - \Delta(\Delta-1)\big[\cosh^2\rho + 3\sinh^\rho\big]\Big\}s$$
$$= -2\Delta s + \frac{2}{\Delta+1}\Big\{2\Delta\Big[(\Delta+1)\cosh\rho\cosh t\ z + (\Delta+1)\cos\vartheta\sin\phi_1\sinh\rho\ z - (\Delta+1)z^2 - \frac{\sin\phi_1\sinh\rho\ z}{\cos\vartheta}\Big]$$
$$+ 2\Delta\left(\frac{1}{\cos\vartheta} - 2\cos\vartheta\right)\sin\phi_1\sinh\rho\ z + 8\cosh\rho\sinh\rho\ \partial_\rho - \Delta(\Delta-1)\big[\cosh^2\rho + 3\sinh^\rho\big]\Big\}s$$
$$= -2\Delta s + \frac{2}{\Delta+1}\Big\{2\Delta\big[\Delta + 2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z - (\Delta+1)z^2\big]$$
$$+ 8\cosh\rho\sinh\rho\partial_\rho - \Delta(\Delta-1)\big[1 + 4\sinh^2\rho\big]\Big\}s$$
$$= \frac{2}{\Delta+1}\Big\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z - (\Delta+1)z^2\big]$$
$$+ 8\cosh\rho\sinh\rho\partial_\rho - 4\Delta(\Delta-1)\sinh^2\rho\Big\}s.$$

The sum of the variation of the DBI part and the WZ part is

$$
\begin{aligned}
\delta L =& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\left[\frac{2}{\Delta+1}\left\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\right.\right.\\
&\left.\left.+\,8\cosh\rho\sinh\rho\partial_\rho-4\Delta(\Delta-1)\sinh^2\rho\right\}s-8\cosh\rho\sinh\rho\partial_\rho s\right]\\
=& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{2}{\Delta+1}\left\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\right.\\
&\left.-\,4(\Delta-1)\cosh\rho\sinh\rho\partial_\rho-4\Delta(\Delta-1)\sinh^2\rho\right\}s\\
=& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{2}{\Delta+1}\left\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\right.\\
&\left.-\,4\Delta(\Delta-1)\big[\cosh^2\rho\cos\vartheta\sinh\phi_1\sinh\rho z-\cosh\rho\sinh^2\rho\cosh tz\big]-4\Delta(\Delta-1)\sinh^2\rho\right\}s\\
=& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{2}{\Delta+1}\left\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\right.\\
&\left.-\,4(\Delta-1)\cosh\rho\sinh\rho\partial_\rho-4\Delta(\Delta-1)\sinh^2\rho\right\}s\\
=& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{2}{\Delta+1}\left\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\right.\\
&\left.-\,4\Delta(\Delta-1)\big[-\cosh^2\rho+\cosh\rho\cosh tz\big]-4\Delta(\Delta-1)\sinh^2\rho\right\}s\\
=& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{2}{\Delta+1}\left\{2\Delta\big[2(\Delta-2)\cos\vartheta\sin\phi_1\sinh\rho z-(\Delta+1)z^2\big]\right.\\
&\left.-\,4(\Delta-1)\cosh\rho\sinh\rho\partial_\rho-4\Delta(\Delta-1)\sinh^2\rho\right\}s\\
=& \frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{2}{\Delta+1}\left\{2\Delta\big[-2(\Delta-2)-(\Delta+1)z^2\big]\right.\\
&\left.+\,4\Delta(\Delta-1)\cosh^2\rho-4\Delta(\Delta-1)\sinh^2\rho\right\}s\\
=& -\frac{N}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\;4\Delta z^2 s\\
=& -\frac{\sqrt{\Delta}(\Delta+1)}{4\pi^2}\sinh^2\rho\cos\vartheta\sin\vartheta\frac{\mathcal{R}^\Delta e^{\Delta t}}{\big(\cosh\rho\cosh t-\cos\vartheta\sin\phi_1\sinh\rho\big)^{\Delta+2}}.
\end{aligned}
\tag{5.2.117}
$$

Therefore the variation of the action becomes

$$
\begin{aligned}
\delta S = &-\int_{-\infty}^{\infty} dt \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{\pi} d\vartheta \\
&\times \frac{(\Delta+1)\sqrt{\Delta}}{4\pi^2} \cos\vartheta \sin\vartheta \sinh^2\rho \frac{\mathcal{R}^\Delta e^{\Delta t}}{\big(\cosh\rho\cosh t - \cos\vartheta\sin\phi_1\sinh\rho\big)^{\Delta+2}} \\
= &-\frac{(\Delta+1)\sqrt{\Delta}}{2\pi} \sinh^2\rho \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\phi_1 \int_0^1 d(\cos\vartheta) \\
&\times \cos\vartheta \frac{\mathcal{R}^\Delta e^{\Delta t}}{\big(\cosh\rho\cosh t - \cos\vartheta\sin\phi_1\sinh\rho\big)^{\Delta+2}} \\
= &-\frac{(\Delta+1)\sqrt{\Delta}}{2\pi} \frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho} \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\phi_1 \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t} \int_0^1 d\lambda \frac{\lambda}{\Big(1-\lambda\frac{\sin\phi_1\tanh\rho}{\cosh t}\Big)^{\Delta+2}}.
\end{aligned} \tag{5.2.118}
$$

Now, we use the fact that

$$
\frac{1}{(1-u)^\alpha} = \sum_{k=0}^{\infty} \alpha(\alpha+1)\dots(\alpha+k-1)\frac{u^k}{k!} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(k+1)\Gamma(\alpha)} u^k \tag{5.2.119}
$$

to find

$$
\begin{aligned}
\delta S = &-\frac{(\Delta+1)\sqrt{\Delta}}{2\pi} \frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho} \int_{-\infty}^{\infty} dt \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t} \int_0^{2\pi} d\phi_1 \int_0^1 d\lambda \\
&\times \sum_{k=0}^{\infty} \lambda^{k+1} \frac{\Gamma(\Delta+k+2)}{\Gamma(k+1)\Gamma(\Delta+2)} \left(\frac{\sin\phi_1\tanh\rho}{\cosh t}\right)^k \\
= &-\frac{(\Delta+1)\sqrt{\Delta}}{2\pi} \frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho} \int_{-\infty}^{\infty} dt \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t} \int_0^{2\pi} d\phi_1 \\
&\times \sum_{k=0}^{\infty} \frac{1}{k+2} \frac{\Gamma(\Delta+k+2)}{\Gamma(k+1)\Gamma(\Delta+2)} \left(\frac{\sin\phi_1\tanh\rho}{\cosh t}\right)^k.
\end{aligned} \tag{5.2.120}
$$

Using the integral

$$
\int_0^{2\pi} dx \sin^k x = \begin{cases} 0 & k = 2p+1 \\ \frac{(2p)!}{2^{2p}(p!)^2}\pi & k = 2p, \end{cases} \quad p \in \mathbb{N}. \tag{5.2.121}
$$

we obtain

$$
\begin{aligned}
\delta S = &-\frac{(\Delta+1)\sqrt{\Delta}}{2} \frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho} \int_{-\infty}^{\infty} dt \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2} t} \\
&\times \sum_{k=0}^{\infty} \frac{1}{2k+2} \frac{\Gamma(\Delta+2k+2)}{\Gamma(2k+1)\Gamma(\Delta+2)} \frac{\Gamma(2k+1)}{2^{2k}\Gamma(k+1)^2} \left(\frac{\tanh\rho}{\cosh t}\right)^{2k} \\
= &-\frac{(\Delta+1)\sqrt{\Delta}}{2} \frac{\sinh^2\rho}{\cosh^{\Delta+2}\rho} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} dt \frac{\mathcal{R}^\Delta e^{\Delta t}}{\cosh^{\Delta+2k+2} t} \\
&\times \frac{1}{2^{2k}} \frac{\Gamma(\Delta+2k+2)}{\Gamma(\Delta+2)\Gamma(k+2)\Gamma(k+1)} \tanh^{2k}\rho.
\end{aligned} \tag{5.2.122}
$$

Now we need to evaluate

$$\int_{-\infty}^{\infty} dt \, \frac{e^{\Delta t}}{\cosh^{\Delta+2k+2} t} = 2^{\Delta+2k+2} \int_{-\infty}^{\infty} dt \, \left(\frac{e^{2t}}{1+e^{2t}}\right)^{\Delta+2k+2} (e^{-2t})^{k+1}$$
$$= 2^{\Delta+2k+1} \int_0^1 du \, u^{\Delta+k} (1-u)^k$$
$$= 2^{\Delta+2k+1} B(\Delta+k+1, k+1)$$
$$= 2^{\Delta+2k+1} \frac{\Gamma(\Delta+k+1)\Gamma(k+1)}{\Gamma(\Delta+2k+2)} \tag{5.2.123}$$

in which we have again used the change of variable $u = \frac{e^{2t}}{1+e^{2t}}$. The variation of the action becomes

$$\delta S = -\frac{(\Delta+1)\sqrt{\Delta}}{2} \frac{\sinh^2 \rho}{\cosh^{\Delta+2} \rho} \sum_{k=0}^{\infty} \mathcal{R}^{\Delta} 2^{\Delta+2k+1} \frac{\Gamma(\Delta+k+1)\Gamma(k+1)}{\Gamma(\Delta+2k+2)}$$
$$\times \frac{1}{2^{2k}} \frac{\Gamma(\Delta+2k+2)}{\Gamma(\Delta+2)\Gamma(k+2)\Gamma(k+1)} \tanh^{2k} \rho$$
$$= -(2\mathcal{R})^{\Delta} \frac{1}{\sqrt{\Delta}\cosh^{\Delta} \rho} \sum_{k=0}^{\infty} \frac{\Gamma(\Delta+k+1)}{\Gamma(\Delta)\Gamma(k+2)} \tanh^{2k+2} \rho. \tag{5.2.124}$$

Using the identity

$$\cos^{2\Delta} x = \left(\frac{1}{1-\tanh^2 x}\right)^{\Delta} = \sum_{k=0}^{\infty} \frac{\Gamma(\Delta+k)}{\Gamma(\Delta)\Gamma(k+1)} \tanh^{2k} x$$
$$= 1 + \sum_{k=1}^{\infty} \frac{\Gamma(\Delta+k)}{\Gamma(\Delta)\Gamma(k+1)} \tanh^{2k} x$$
$$= 1 + \sum_{k=0}^{\infty} \frac{\Gamma(\Delta+k+1)}{\Gamma(\Delta)\Gamma(k+2)} \tanh^{2k+2} x, \tag{5.2.125}$$

we obtain

$$\delta S = -\frac{(2\mathcal{R})^{\Delta}}{\sqrt{\Delta}} \big(\cosh^{\Delta} \rho - \cosh^{-\Delta} \rho \big). \tag{5.2.126}$$

The structure constant of the symmetric giant graviton is (Bissi et al., 2011)

$$\langle O^{J,J} \rangle_S^{string} = \frac{1}{\sqrt{J}} \left[\left(1+\frac{k}{N}\right)^{\frac{J}{2}} - \left(1+\frac{k}{N}\right)^{-\frac{J}{2}} \right]. \tag{5.2.127}$$

This result perfectly matches the gauge theory computations in (5.1.29) when $\frac{k}{N} \to \infty$. For $\frac{k}{N} \to 0$, the result tends to (5.1.10).

In conclusion, divergences are also found in extremal correlators of giant gravitons on AdS_5. Thus divergences appear in correlators of operators of scale dimensions Δ of the order of the gauge parameter N. Thus, the divergences appear to be a rather general feature of correlators of the theory. The regularization that was chosen is the analytic continuation of the non-extremal correlators to the extremal cases. This regularization allows to us extract the finite parts from the divergent integrals and with this prescription the gauge theory and gravity side computations are in agreement. However, this is not yet satisfactory since we do not have an independent way of fixing the regularization.

6. Correlation functions of Kaluza-Klein gravitons

In this chapter, we will find divergences in higher point functions which again suggests that these divergences are a general feature of the theory. A harmonic expansion in supergravity to determine the Kaluza-Klein modes will be performed. Using these modes, we find that the calculation of extremal correlators in supergravity is subject to the same subtlety of regularization.

6.1 Supergravity computation of $\langle \mathcal{O}_{t,s}^{k_1} \mathcal{O}_{\phi}^{k_2} \mathcal{O}_{\phi}^{k_3} \rangle$

Starting from a theory of gravity in $D = 10$ dimensions, we compactify five of the dimensions on a five-sphere to obtain a theory of gravity in 5 dimensions, such that the background metric is that of AdS space. This dimensional reduction is achieved using a spherical harmonic decomposition of the fluctuations of the background fields. We obtain the scalar Kaluza-Klein modes ϕ^k of the dilaton ϕ and other scalar Kaluza-Klein modes t^k and s^k arising from the 4-form and the graviton with indices on the sphere. Thereafter, we compute the three-point correlation functions $\langle \mathcal{O}_{t}^{k_1} \mathcal{O}_{\phi}^{k_2} \mathcal{O}_{\phi}^{k_3} \rangle$ and $\langle \mathcal{O}_{s}^{k_1} \mathcal{O}_{\phi}^{k_2} \mathcal{O}_{\phi}^{k_3} \rangle$, using the AdS_5/CFT_4 correspondence, where $\mathcal{O}_t^k, \mathcal{O}_s^k, \mathcal{O}_\phi^{k_3}$ are the SYM operators respectively dual to the supergravity scalar fields ϕ^k, t^k and s^k. The calculation of $\langle \mathcal{O}_{t}^{k_1} \mathcal{O}_{\phi}^{k_2} \mathcal{O}_{\phi}^{k_3} \rangle$, for the non-extremal case $k_1 < k_2 + k_3$ and the extremal case $k_1 = k_2 + k_3$ are performed following D'Hoker et al. (1999).

The metric G_{mn} of the whole 10-dimensional manifold, with a background metric g_{mn} and a fluctuation h_{mn}, is given by (Kim et al., 1985; Lee et al., 1998; D'Hoker et al., 1999)

$$G_{mn} = g_{mn} + h_{mn}\,, \tag{6.1.1}$$

where

$$h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{h_2}{5} g_{\alpha\beta}, \qquad g^{\alpha\beta} h_{(\alpha\beta)} = 0, \tag{6.1.2}$$

$$h_{\mu\nu} = h'_{\mu\nu} - \frac{h_2}{3} g_{\mu\nu}, \qquad h'_{\mu\nu} = h'_{(\mu\nu)} + \frac{h'}{5} g_{\mu\nu}, \qquad g^{\mu\nu} h'_{(\mu\nu)} = 0, \tag{6.1.3}$$

$$F = \bar{F} + \delta F, \qquad \delta F_{ijklm} = 5 \nabla_{[i} a_{jklm]}. \tag{6.1.4}$$

Here the Latin indices $i, j, k, \cdots$ are used for the 10-dimensional manifold. Indices α, β, γ are S^5 indices, while μ, ν, λ are AdS_5 indices, $\bar{F}$ is the background value of the F-field. The S^5 and AdS_5 scales are set to 1. The choice of gauge is

$$\nabla^\alpha h_{\alpha\beta} = \nabla^\alpha h_{\mu\alpha} = \nabla^\alpha a_{\alpha\mu_1 m_2 m_3 m_4} = 0. \tag{6.1.5}$$

The expansions in terms of spherical harmonics of the fluctuations are

$$h'_{\mu\nu} = \sum Y^k h'^k_{\mu\nu}, \tag{6.1.6}$$

$$h_2 = \sum Y^k h'^k_2, \tag{6.1.7}$$

$$a_{\alpha_1\alpha_2\alpha_3\alpha_4} = \sum \nabla^\alpha Y^k \epsilon_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4} b^k, \tag{6.1.8}$$

$$\phi = \sum Y^k \phi^k. \tag{6.1.9}$$

The spherical harmonics obey

$$\nabla_\alpha \nabla^\alpha Y^k = -k(k+4)Y^k. \tag{6.1.10}$$

The modes h_2^k and b^k have coupled linear equations of motion (Kim et al., 1985; Lee et al., 1998; D'Hoker et al., 1999),

$$\left[\nabla_m \nabla^m b^k + \left(\frac{1}{2}h'^k - \frac{4}{3}h_2^k\right)\right] Y^k = 0, \tag{6.1.11}$$

$$\left[\left(\nabla_m \nabla^m - 32\right) h_2^k + 80\nabla_\alpha \nabla^\alpha b^k + \nabla_\alpha \nabla^\alpha \left(h'^k - \frac{16}{15}h_2^k\right)\right] Y^k = 0, \tag{6.1.12}$$

with three constraint equations

$$\left(\frac{1}{2}h'^k - \frac{8}{15}h_2^k\right)\nabla_{(\alpha}\nabla_{\beta)} Y^k = 0, \tag{6.1.13}$$

$$\left[\nabla_\mu h'^{k\mu\nu} - \nabla^\nu \left(h'^k - \frac{8}{15}h_2^k + 8b^k\right) - \frac{8}{4!}\epsilon^{\nu\mu_1\mu_2\mu_3\mu_4} a^k_{\mu_1\mu_2\mu_3\mu_4}\right]\nabla_\alpha Y^k = 0, \tag{6.1.14}$$

$$\left[a^k_{\mu_1\mu_2\mu_3\mu_4} + \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}\nabla^{\mu_5} b^k\right]\nabla_\alpha Y^k = 0. \tag{6.1.15}$$

The constraint between h_2^k and h'^k in (6.1.13) can also be rewritten as

$$h'^k = \frac{16}{15}h_2^k, \tag{6.1.16}$$

and so we find the following equations of motions,

$$\nabla_m \nabla^m b - \frac{4}{5}h_2 = 0, \tag{6.1.17}$$

$$\left(\nabla_m \nabla^m b - 32\right) h_2 + 80\nabla_\alpha \nabla^\alpha b = 0. \tag{6.1.18}$$

The fields h_2 and $h'_{\mu\nu}$ are not independent. The trace part of $h'_{\mu\nu}$ and h_2 are related by the equation given in (6.1.16). The traceless part of $h'_{\mu\nu}$ is related to h_2 by

$$h'_{(\mu\nu)} = \nabla_{(\mu}\nabla_{\nu)}\left(\frac{2}{5(k+1)(k+3)}(h_2^k - 30b^k)\right). \tag{6.1.19}$$

Using the property of spherical harmonics Y^k in (6.1.10), the equations of motions for h_2^k and b^k take the following forms

$$\nabla_\mu \nabla^\mu b^k = k(k+4)b^k + \frac{4}{5}h_2^k, \tag{6.1.20}$$

$$\nabla_\mu \nabla^\mu h_2^k = k(k+4)\left[80b^k + h_2^k\right] + 32h_2^k. \tag{6.1.21}$$

The diagonal combinations are

$$s^k = \frac{1}{20(k+2)}[h_2^k - 10(k+4)b^k], \tag{6.1.22}$$

$$t^k = \frac{1}{20(k+2)}[h_2^k + 10kb^k], \tag{6.1.23}$$

which satisfy

$$\nabla_\mu \nabla^\mu s^k = k(k-4) s^k, \tag{6.1.24}$$
$$\nabla_\mu \nabla^\mu t^k = (k+4)(k+8) t^k. \tag{6.1.25}$$

To prove these equations, we compute $\nabla_\mu \nabla^\mu s^k$ and $\nabla_\mu \nabla^\mu b^k$ using (6.1.21) and (6.1.20). We have

$$\begin{aligned}
\nabla_\mu \nabla^\mu s^k &= \frac{1}{20(k+2)} [\nabla_\mu \nabla^\mu h_2^k - 10(k+4) \nabla_\mu \nabla^\mu b^k] \\
&= \frac{1}{20(k+2)} \left\{ \left[k(k+4)[80 b^k + h_2^k] + 32 h_2^k \right] - 10(k+4) \left[k(k+4) b^k + \frac{4}{5} h_2^k \right] \right\} \\
&= k(k-4) \frac{1}{20(k+2)} \left\{ -10k(k+4) b^k + h_2^k \right\} \\
&= k(k-4) s^k,
\end{aligned} \tag{6.1.26}$$

and

$$\begin{aligned}
\nabla_\mu \nabla^\mu t^k &= \frac{1}{20(k+2)} [\nabla_\mu \nabla^\mu h_2^k + 10k \nabla_\mu \nabla^\mu b^k] \\
&= \frac{1}{20(k+2)} \left\{ \left[k(k+4)[80 b^k + h_2^k] + 32 h_2^k \right] + 10k \left[k(k+4) b^k + \frac{4}{5} h_2^k \right] \right\} \\
&= (k+4)(k+8) \frac{1}{20(k+2)} \left\{ h_2^k + 10k b^k \right\} \\
&= (k+4)(k+8) t^k.
\end{aligned} \tag{6.1.27}$$

6.1.1 Cubic Action. The cubic vertices needed for the computation of the three-point function come from the kinetic term for the dilaton in the 10-dimensional action, which is given by

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10} x \sqrt{G} \frac{1}{2} G^{mn} \partial_m \phi \partial_n \phi \,. \tag{6.1.28}$$

The excitations related to the two-form fields and the axion can be ignored by setting

$$h_{\mu\alpha} \equiv 0 \qquad h_{(\alpha\beta)} \equiv 0. \tag{6.1.29}$$

Let us now expand this part of the action in terms of the fluctuations. The determinant of the metric is

$$G = \det(G_{mn}) = \det(g_{mn} + h_{mn}) = \det(g_{mn}) \det(1 + g^{mn} h_{mn}). \tag{6.1.30}$$

Using the identity

$$\ln(\det M) = \mathrm{Tr}(\ln M), \tag{6.1.31}$$

it follows that

$$\begin{aligned}
\sqrt{G} &= \sqrt{\det g_{mn}}\, e^{\frac{1}{2} \ln \det(1 + g^{mn} h_{mn})} \\
&= \sqrt{g}\ \ e^{\frac{1}{2} \mathrm{Tr}(\ln(1 + g^{mn} h_{mn}))} \\
&= \sqrt{g}\, e^{\frac{1}{2} \mathrm{Tr}\left(\left(g^{mn} h_{mn} - \frac{(g^{mn} h_{mn})^2}{2} + \cdots \right) \right)} \\
&= \sqrt{g} \mathrm{Tr} \left(1 + \frac{1}{2} g^{mn} h_{mn} + \cdots \right).
\end{aligned} \tag{6.1.32}$$

Now use $g^{mn}h_{mn} = g^{\mu\nu}h_{\mu\nu} + g^{\alpha\beta}h_{\alpha\beta}$ and $\sqrt{g} = \sqrt{g_1}\sqrt{g_2}$ to find

$$\begin{aligned}\sqrt{G} &= \sqrt{g_1}\sqrt{g_2}\,\mathrm{Tr}\left(1 + \frac{1}{2}g^{\mu\nu}h_{\mu\nu} + \frac{1}{2}g^{\alpha\beta}h_{\alpha\beta} + \cdots\right)\\ &= \sqrt{g_1}\sqrt{g_2}\,\mathrm{Tr}\left(1 + \frac{1}{2}g^{\mu\nu}(h'_{\mu\nu} - \frac{h_2}{3}g_{\mu\nu}) + \frac{1}{2}g^{\alpha\beta}(h_{(\alpha\beta)} + \frac{h_2}{5}g_{\alpha\beta}) + \cdots\right)\\ &= \sqrt{g_1}\sqrt{g_2}\Big[1 - \frac{1}{3}h_2 + \frac{1}{2}h' + \cdots\Big]\end{aligned} \tag{6.1.33}$$

where g_1, g_2 indicate the determinant of the background metric on AdS_5 and S^5 respectively. Here the previous choice $h_{(\alpha\beta)} = 0$ is being used. The action can now be written as

$$\begin{aligned}S &= \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{g}\frac{1}{2}\Big(1 - \frac{1}{3}h_2 + \frac{1}{2}h' + \cdots\Big)\Big(g_{mn} - h_{mn} + \cdots\Big)\partial^m\phi\partial^n\phi\\ &= \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{g}\frac{1}{2}\Big(1 - \frac{1}{3}h_2 + \frac{1}{2}h' + \cdots\Big)\Big((g_{\mu\nu} - h_{\mu\nu})\partial^\mu\phi\partial^\nu\phi + (g_{\alpha\beta} - h_{\alpha\beta})\partial^\alpha\phi\partial^\beta\phi + \cdots\Big)\\ &= \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{g}\frac{1}{2}\Big(1 - \frac{1}{3}h_2 + \frac{1}{2}h' + \cdots\Big)\Big(\partial_\mu\phi\partial^\mu\phi + \partial_\alpha\phi\partial^\alpha\phi - h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - h_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi + \cdots\Big).\end{aligned} \tag{6.1.34}$$

Next, use the decomposition given in (6.1.2) and (6.1.3) to write

$$\begin{aligned}S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{g}\frac{1}{2}\Big(1 - \frac{1}{3}h_2 + \frac{1}{2}h' + \cdots\Big)\Big(&\partial_\mu\phi\partial^\mu\phi + \partial_\alpha\phi\partial^\alpha\phi\\ &- h'_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + \frac{h_2}{3}\partial_\mu\phi\partial^\nu\phi - \frac{h_2}{5}\partial_\alpha\phi\partial^\alpha\phi + \cdots\Big)\end{aligned} \tag{6.1.35}$$

which implies

$$\begin{aligned}S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{g}\Big[&\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi\\ &+ \left(\frac{1}{4}h' - \frac{4}{15}h_2\right)\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{4}h'\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}h'_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi + \cdots\Big].\end{aligned} \tag{6.1.36}$$

6.1.2 Dimensional Reduction. The spherical harmonics (Appendix B.4.2) are normalized as

$$\int Y^{k_1}Y^{k_2} = z(k)\delta^{k_1k_2}\,, \tag{6.1.37}$$

$$\int Y^{k_1}Y^{k_2}Y^{k_3} = a(k_1,k_2,k_3)\langle C^{k_1}C^{k_2}C^{k_3}\rangle\,, \tag{6.1.38}$$

where

$$z(k) = \frac{1}{2^{k-1}(k+1)(k+2)}, \tag{6.1.39}$$

$$a(k_1,k_2,k_3) = \frac{\omega_5}{(\Sigma+2)!2^{\Sigma-1}\alpha_1!\alpha_2!\alpha_3!}. \tag{6.1.40}$$

Here

$$\alpha_1 = \frac{1}{2}(k_2+k_3-k_1), \tag{6.1.41}$$
$$\alpha_2 = \frac{1}{2}(k_1+k_3-k_2), \tag{6.1.42}$$
$$\alpha_3 = \frac{1}{2}(k_1+k_2-k_3), \tag{6.1.43}$$
$$\Sigma = \frac{1}{2}(k_1+k_2-k_3), \tag{6.1.44}$$

and $\omega_5 = \pi^3$ is the area of a unit five-sphere.

To compute the 3–point functions $\langle \mathcal{O}^{k_1}_{t}\mathcal{O}^{k_2}_{\phi}\mathcal{O}^{k_3}_{\phi}\rangle$ we need to consider excitations for which the fields s^k are zero. From the definition of s^k and t^k given in (6.1.23), and the decomposition in (6.1.2), we find

$$h^k_2 = 10(k+4)b^k, \tag{6.1.45}$$
$$h^k_2 = 10(k+4)t^k. \tag{6.1.46}$$

The field $h'^k_{\mu\nu}$ can then be written as

$$\begin{aligned}
h'^k_{\mu\nu} &= h'^k_{(\mu\nu)} + \frac{h'^k}{5}g_{\mu\nu} \\
&= \nabla_{(\mu}\nabla_{\nu)}\Big[\frac{2}{5(k+1)(k+3)}(h^k_2 - 30b^k)\Big] + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\
&= \frac{2}{5(k+1)(k+3)}\nabla_{(\mu}\nabla_{\nu)}\Big(h^k_2 - \frac{30}{10(k+4)}h^k_2\Big) + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\
&= \frac{2}{5(k+3)(k+4)}\nabla_{(\mu}\nabla_{\nu)}h^k_2 + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu}.
\end{aligned} \tag{6.1.47}$$

By definition, the symmetric traceless covariant derivative of h^k_2 is given by

$$\nabla_{(\mu}\nabla_{\nu)}h^k_2 = \nabla_\mu\nabla_\nu h^k_2 - \frac{1}{5}g_{\mu\nu}\nabla_\rho\nabla^\rho h^k_2, \tag{6.1.48}$$

which gives

$$\begin{aligned}
h'^k_{\mu\nu} &= \frac{2}{5(k+3)(k+4)}\Big[\nabla_\mu\nabla_\nu h^k_2 - \frac{1}{5}g_{\mu\nu}10(k+4)\nabla_\rho\nabla^\rho t^k\Big] + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\
&= \frac{2}{5(k+3)(k+4)}\Big[\nabla_\mu\nabla_\nu h^k_2 - \frac{1}{5}g_{\mu\nu}10(k+4)(k+4)(k+8)t^k\Big] + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\
&= \frac{2}{5(k+3)(k+4)}\nabla_\mu\nabla_\nu h^k_2 - \frac{2(k+8)}{25(k+3)}g_{\mu\nu}h^k_2 + \frac{16}{15}\frac{h^k_2}{5}g_{\mu\nu} \\
&= \frac{2}{5(k+3)(k+4)}\nabla_\mu\nabla_\nu h^k_2 + \frac{2}{15}\frac{k}{k+3}h^k_2 g_{\mu\nu}.
\end{aligned} \tag{6.1.49}$$

Here, (6.1.46) and (6.1.25) were used.

Next evaluate the integral over the five-sphere in (6.1.36) to reduce the theory to 5-dimensions, using the normalization of the spherical harmonics introduced earlier.

The first term in (6.1.36) is

$$\begin{aligned}\int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{2}\sum_{k_1k_2} Y^{k_1}Y^{k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2}\\ &= \int \mathrm{d}^5x\sqrt{g_1}\frac{1}{2}\sum_{k_1k_2} z(k)\delta_{k_1k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2}\\ &= \int \mathrm{d}^5x\sqrt{g_1}\frac{z(k)}{2}\nabla_\mu\phi^k\nabla^\mu\phi^k. \end{aligned} \tag{6.1.50}$$

The second term in (6.1.36) is

$$\begin{aligned}\int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{2}\sum_{k_1k_2} \nabla_\alpha Y^{k_1}\nabla^\alpha Y^{k_2}\phi^{k_1}\phi^{k_2}\\ &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{2}\sum_{k_1k_2} k_2(k_2+4)Y^{k_1}Y^{k_2}\phi^{k_1}\phi^{k_2}\\ &= \int \mathrm{d}^5x\sqrt{g_1}\frac{1}{2}\sum_{k_1k_2} k_2(k_2+4)z(k)\delta_{k_1k_2}\phi^{k_1}\phi^{k_2}\\ &= \int \mathrm{d}^5x\sqrt{g_1}\frac{z(k)}{2}k_2(k_2+4)\phi^k\phi^k\,. \end{aligned} \tag{6.1.51}$$

In this computation, we performed an integration by parts of $\nabla_\alpha Y^{k_1}\nabla^\alpha Y^{k_2}$ as follows

$$\int_{S^5}\nabla_\alpha Y^{k_1}\nabla^\alpha Y^{k_2} = -\int_{S^5} Y^{k_1}\nabla_\alpha\nabla^\alpha Y^{k_2} = \int_{S^5} k_2(k_2+4)Y^{k_1}Y^{k_2} \tag{6.1.52}$$

where the identity given in (6.1.10) has been used. The third term in (6.1.36) is

$$\begin{aligned}\int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{4}h'\nabla_\mu\phi\nabla^\mu\phi &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{1}{4}\sum_{k_1}\frac{16}{15}Y^{k_1}h_2^{k_1}\sum_{k_2}Y^{k_2}\nabla_\mu\phi^{k_2}\sum_{k_3}Y^{k_3}\nabla^\mu\phi^{k_3}\\ &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\frac{4}{15}\sum_{k_1,k_2,k_3} Y^{k_1}Y^{k_2}Y^{k_3}10(k_1+4)t^{k_1}\nabla_\mu\phi^{k_2}\nabla^\mu\phi^{k_3}\\ &= a(k_1,k_2,k_3)\int \mathrm{d}^5x\sqrt{g_1}\frac{8}{3}(k_1+4)t^{k_1}\nabla_\mu\phi^{k_2}\nabla^\mu\phi^{k_3}\,. \end{aligned} \tag{6.1.53}$$

The fourth term in (6.1.36) is

$$\begin{aligned}&\int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\left(-\frac{1}{2}\right)h'_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi\\ &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\left(-\frac{1}{2}\right)\sum_{k_1,k_2,k_3}Y^{k_1}\left[\frac{2\nabla_\mu\nabla_\nu h_2^{k_1}}{5(k_1+3)(k_1+4)}+\frac{2}{15}\frac{k_1}{k_1+3}h_2^{k_1}g_{\mu\nu}\right]Y^{k_2}\nabla^\mu\phi^{k_2}Y^{k_3}\nabla^\nu\phi^{k_3}\\ &= \int \mathrm{d}^{10}x\sqrt{g_1}\sqrt{g_2}\left(-\frac{1}{2}\right)\sum_{k_1,k_2,k_3}\left[\frac{4}{(k_1+3)}\nabla_\mu\nabla_\nu t^{k_1}+\frac{2}{15}\frac{k_1}{k_1+3}10(k_1+4)t^{k_1}g_{\mu\nu}\right]Y^{k_1}Y^{k_2}Y^{k_3}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3}\\ &= -a(k_1,k_2,k_3)\int \mathrm{d}^5x\sqrt{g_1}\left[\frac{2}{(k_1+3)}\nabla_\mu\nabla_\nu t^{k_1}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3}+\frac{2}{3}\frac{k_1}{k_1+3}(k_1+4)t^{k_1}\nabla^\mu\phi^{k_2}\nabla_\mu\phi^{k_3}\right]. \end{aligned} \tag{6.1.54}$$

Combining these terms, the dimensionally reduced form of (6.1.36) is

$$S = \frac{1}{2\kappa_5^2} \int \mathrm{d}^5 x \sqrt{g_1} \Bigg[\frac{z(k)}{2} \left(\nabla_\mu \phi^k \nabla^\mu \phi^k - k(k+4) \phi^k \phi^k \right) + a(k_1, k_2, k_3) \left(2 \frac{(k_1+4)^2}{(k_1+3)} t^{k_1} \nabla^\mu \phi^{k_2} \nabla^\mu \phi^{k_3} - \frac{2}{k_1+3} \nabla_\mu \nabla_\nu t^{k_1} \nabla^\mu \phi^{k_2} \nabla^\nu \phi^{k_3} \right) \Bigg]. \tag{6.1.55}$$

The gravitational coupling constant κ_5^2 is related to the SYM parameter N by (D'Hoker and Freedman, 2002)

$$2\kappa_5^2 = \frac{8\pi^2}{N^2}. \tag{6.1.56}$$

6.1.3 Evaluation of the Action. The first cubic term can be manipulated as follows

$$\begin{aligned}
&\int_{AdS_5} t^{k_1} \nabla_\mu \phi^{k_2} \nabla^\mu \phi^{k_3} \\
&= \int_{AdS_5} t^{k_1} \frac{1}{2} \Big[\nabla^\mu \nabla_\mu (\phi^{k_2} \phi^{k_3}) - \nabla^\mu \nabla_\mu (\phi^{k_2}) \phi^{k_3} - \nabla^\mu \nabla_\mu (\phi^{k_3}) \phi^{k_3} \Big] \\
&= \int_{AdS_5} t^{k_1} \frac{1}{2} \nabla^\mu \nabla_\mu (\phi^{k_2} \phi^{k_3}) - \int_{AdS_5} \frac{1}{2} \Big(m_\phi^2(k_2) + m_\phi^2(k_3) \Big) t^{k_1} \phi^{k_2} \phi^{k_3} \\
&= \int_{AdS_5} \frac{1}{2} \Big[\nabla^\mu \nabla_\mu t^{k_1} \phi^{k_2} \phi^{k_3} + \nabla^\mu \big[t^{k_1} \nabla_\mu (\phi^{k_2} \phi^{k_3}) \big] - \nabla^\mu \big[\nabla_\mu t^{k_1} \phi^{k_2} \phi^{k_3} \big] - \Big(m_\phi^2(k_2) + m_\phi^2(k_3) \Big) t^{k_1} \phi^{k_2} \phi^{k_3} \Big] \\
&= \int_{AdS_5} \frac{1}{2} \Big(m_t^2(k_1) - m_\phi^2(k_2) - m_\phi^2(k_3) \Big) t^{k_1} \phi^{k_2} \phi^{k_3} + \frac{1}{2} \int_{\partial(AdS_5)} \Big(t^{k_1} D_n (\phi^{k_2} \phi^{k_3}) - \phi^{k_2} \phi^{k_3} D_n t^{k_1} \Big)
\end{aligned} \tag{6.1.57}$$

where D_n indicates the outward normal derivative to the boundary and $m_\phi^2(k)$ and $m_t^2(k)$ denote the masses of the fields ϕ^k and t^k, given by

$$m_\phi^2(k) \equiv k(k+4) \qquad m_t^2(k) \equiv (k+4)(k+8). \tag{6.1.58}$$

These masses appear from the equations of motion of ϕ^k and t^k. Using the divergence theorem, integrals of a total derivative are written as boundary integrals. Following the argument given in D'Hoker et al. (1999), the boundary integrals found in (6.1.57) cannot contributes to the three-point function for three points that are disjoint. Consequently, these integrals can always be dropped in our computations.

The second terms in (6.1.55) is computed as follows. Defining the quantity

$$P_{\mu\nu} = \frac{1}{2} \left(\nabla_\mu \phi^{k_2} \nabla_\nu \phi^{k_3} + \nabla_\nu \phi^{k_2} \nabla_\mu \phi^{k_3} \right) - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi^{k_2} \nabla_\lambda \phi^{k_3} \tag{6.1.59}$$

that satisfies the relation

$$\nabla^\mu P_{\mu\nu} = \frac{1}{2} \left(m_\phi^2(k_3) \phi^{k_3} \nabla_\nu \phi^{k_2} + m_\phi^2(k_2) \phi^{k_2} \nabla_\nu \phi^{k_3} \right), \tag{6.1.60}$$

we can write

$$\begin{aligned}
&\int_{AdS_5}\nabla_\mu\nabla_\nu t^{k_1}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3}=\int_{AdS_5}\nabla^\mu\nabla^\nu t^{k_1}\left[P_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi^{k_2}\nabla_\lambda\phi^{k_3}\right]\\
&=\int_{AdS_5}\left[\nabla^\nu\left[\nabla^\mu t^{k_1}P_{\mu\nu}\right]-\nabla^\mu t^{k_1}\nabla^\nu P_{\mu\nu}+\frac{1}{2}\nabla^\mu\nabla_\mu t^{k_1}\nabla^\lambda\phi^{k_2}\nabla_\lambda\phi^{k_3}\right]\\
&=\frac{1}{2}\int_{AdS_5}\left[m_t^2(k_1)t^{k_1}\nabla^\lambda\phi^{k_2}\nabla_\lambda\phi^{k_3}-\Big(m_\phi^2(k_3)\nabla^\mu t^{k_1}\phi^{k_3}\nabla_\mu\phi^{k_2}+m_\phi^2(k_2)\nabla^\mu t^{k_1}\phi^{k_2}\nabla_\mu\phi^{k_3}\Big)\right]\\
&+\int_{\partial(AdS_5)}\nabla^\mu t^{k_1}P_{\mu n}.
\end{aligned}\tag{6.1.61}$$

The first term in the integral is computed in (6.1.57), with the result

$$\frac{1}{4}\int_{AdS_5}m_t^2(k_1)\Big(m_t^2(k_1)-m_\phi^2(k_2)-m_\phi^2(k_3)\Big)t^{k_1}\phi^{k_2}\phi^{k_3}\tag{6.1.62}$$

The terms in parentheses can be written as

$$\begin{aligned}
&m_\phi^2(k_3)\nabla^\mu t^{k_1}\phi^{k_3}\nabla_\mu\phi^{k_2}+m_\phi^2(k_2)\nabla^\mu t^{k_1}\phi^{k_2}\nabla_\mu\phi^{k_3}\\
&=\nabla^\mu\left[m_\phi^2(k_3)t^{k_1}\phi^{k_3}\nabla_\mu\phi^{k_2}+m_\phi^2(k_2)t^{k_1}\phi^{k_2}\nabla_\mu\phi^{k_3}\right]-\Big(m_\phi^2(k_2)+m_\phi^2(k_3)\Big)t^{k_1}\nabla^\mu\phi^{k_2}\nabla_\mu\phi^{k_3}\\
&-t^{k_1}m_\phi^2(k_2)\phi^{k_2}\nabla^\mu\nabla_\mu\phi^{k_3}-t^{k_1}m_\phi^2(k_3)\nabla^\mu\nabla_\mu\phi^{k_2}\phi^{k_3}.
\end{aligned}\tag{6.1.63}$$

Using the equations of motion for ϕ^k and the divergence theorem, the contribution from the terms in parentheses is

$$\begin{aligned}
&\frac{1}{4}\int_{AdS_5}\Big[-\Big(m_\phi^2(k_2)+m_\phi^2(k_3)\Big)\Big(m_t^2(k_1)-m_\phi^2(k_2)-m_\phi^2(k_3)\Big)\\
&-4m_\phi^2(k_2)m_\phi^2(k_3)\Big]t^{k_1}\phi^{k_2}\phi^{k_3}.
\end{aligned}\tag{6.1.64}$$

Thus, combining (6.1.62) and (6.1.64), (6.1.61) yields

$$\begin{aligned}
&\int_{AdS_5}\nabla_\mu\nabla_\nu t^{k_1}\nabla^\mu\phi^{k_2}\nabla^\nu\phi^{k_3}\\
&=-\frac{1}{4}\int_{AdS_5}\Big(\big(m_\phi^2(k_2)-m_\phi^2(k_3)\big)^2-m_t^4(k_1)\Big)t^{k_1}\phi^{k_2}\phi^{k_3}\\
&-\frac{1}{2}\int_{\partial(AdS_5)}\Big(-D_n\phi^{k_3}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2}-D_n\phi^{k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_3}+D_n\phi^{k_1}\nabla_\lambda\phi^{k_2}\nabla^\lambda\phi^{k_3}\Big).
\end{aligned}\tag{6.1.65}$$

The total contribution from the action in (6.1.55) is

$$\begin{aligned}
&2\kappa_5^2S_{cubic}=\int_{AdS_5}a(k_1,k_2,k_3)t^{k_1}\phi^{k_2}\phi^{k_3}\\
&\times\left[\frac{(k_1+4)^2}{k_1+3}\Big(m_t^2(k_1)-m_\phi^2(k_2)-m_\phi^2(k_3)\Big)+\frac{1}{2(k_1+3)}\Big(\big(m_\phi^2(k_2)-m_\phi^2(k_3)\big)^2-m_t^4(k_1)\Big)\right]\\
&+\int_{\partial(AdS_5)}\frac{a(k_1,k_2,k_3)}{k_1+3}\Big(-D_n\phi^{k_3}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2}-D_n\phi^{k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_3}+D_n\phi^{k_1}\nabla_\lambda\phi^{k_2}\nabla^\lambda\phi^{k_3}\Big).
\end{aligned}\tag{6.1.66}$$

Using the expressions for the masses, the action is

$$2\kappa_5^2 S_{cubic} = -8\frac{(\Sigma+4)\alpha_1(\alpha_2+2)(\alpha_3+2)}{k_1+3}\int_{AdS_5} a(k_1,k_2,k_3)t^{k_1}\phi^{k_2}\phi^{k_3}+$$
$$\int_{\partial(AdS_5)}\frac{a(k_1,k_2,k_3)}{k_1+3}\Big(-D_n\phi^{k_3}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_2}-D_n\phi^{k_2}\nabla_\mu\phi^{k_1}\nabla^\mu\phi^{k_3}+D_n\phi^{k_1}\nabla_\lambda\phi^{k_2}\nabla^\lambda\phi^{k_3}\Big). \tag{6.1.67}$$

6.1.4 The three–point function for $k_1 < k_2 + k_3$. In this case, the three-point function for the boundary CFT comes from the bulk integral contributing to (6.1.67). The contribution of this term to the value of the three-point function is performed in (4.3.3) and is determined in terms of the conformal dimensions of the fields. From (4.2.33) and (6.1.58), the conformal dimensions Δ_i of t^{k_1} , ϕ^{k_2} and ϕ^{k_3} are found to be

$$\Delta_1 = k_1+8 \quad \Delta_2 = k_2+4 \quad \Delta_3 = k_3+4. \tag{6.1.68}$$

Substituting the Δ_i's into (4.3.3), we find (D'Hoker et al., 1999)

$$\Big\langle \mathcal{O}_t^{k_1}(x_1)\mathcal{O}_\phi^{k_2}(x_2)\mathcal{O}_\phi^{k_3}(x_3)\Big\rangle = \frac{1}{2\kappa_5^2}\frac{4}{\pi^4}\frac{a(k_1,k_2,k_3)}{x_{12}^{8+2\alpha_3}x_{13}^{8+2\alpha_2}x_{23}^{2\alpha_1}}$$
$$\times\frac{(\Sigma+4)(\alpha_2+2)(\alpha_3+2)}{k_1+3}\frac{\Gamma(\alpha_1+1)\Gamma(\alpha_3+4)\Gamma(\alpha_2+4)\Gamma(\Sigma+6)}{\Gamma(k_1+6)\Gamma(k_2+6)\Gamma(k_3+2)}. \tag{6.1.69}$$

There is a smooth limit as $k_1 \to k_2+k_3$,

$$\lim_{k_1\longrightarrow k_2+k_3}\Big\langle \mathcal{O}_t^{k_1}(x_1)\mathcal{O}_\phi^{k_2}(x_2)\mathcal{O}_\phi^{k_3}(x_3)\Big\rangle = \frac{1}{2\kappa_5^2}\frac{4}{\pi^4}\frac{a(k_2+k_3,k_2,k_3)}{x_{12}^{8+2k_2}x_{13}^{8+2k_3}}$$
$$\times\frac{(k_2+3)(k_2+2)^2(k_3+3)(k_3+2)^2(k_2+k_3+4)}{(k_2+k_3+3)}. \tag{6.1.70}$$

6.1.5 The extremal case $k_1 = k_2 + k_3$. For this case, the contribution to the three-point correlation function comes from the boundary term of (6.1.67). The regulation of the boundary integral is achieved by introducing a cutoff $z_0 = \epsilon$ near the boundary. The bulk-to-boundary propagator $K_\Delta^\epsilon(z,x)$, satisfying the Dirichlet boundary value problem, gives the solution for each field in the form

$$t^k(z) = \int \mathrm{d}^4x K_\Delta^\epsilon(z,x)\bar{t}^k(x) \tag{6.1.71}$$

where $\bar{t}^k(x)$ is the boundary source for $t^k(z)$. A similar equation holds for $\phi^k(z)$ but with different conformal dimensions. The three-point correlation function will be obtained by plugging these solutions into (6.1.67) and then using the holographic dictionary.

The three-point correlation function is computed using the Fourier transform of $K_\Delta^\epsilon(z,x)$, which is given by

$$K_\Delta^\epsilon(p) = \frac{z_0^{\frac{d}{2}}\mathcal{K}_{\Delta-\frac{d}{2}}(pz_0)}{\epsilon^{\frac{d}{2}}\mathcal{K}_{\Delta-\frac{d}{2}}(p\epsilon)}, \tag{6.1.72}$$

where $\mathcal{K}_\nu$ is the modified Bessel function of index ν, and d is the dimension of the boundary of AdS_{d+1}. We notice that ν is an integer for this case. The power series expansion of the modified Bessel function of index $\nu = n \in \mathbb{N}$ is given by

$$\mathcal{K}_n(x) = (-1)^{n+1} \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(n+m+1)!} \ln \frac{x}{2} + \frac{1}{2} \sum_{m=0}^{n-1} \left(\frac{x}{2}\right)^{-n+2m} \frac{(n-m-1)!}{m!}$$
$$+ (-1)^{n+1} \frac{1}{2} \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(n+m)!} \left[2C - \sum_{k=1}^{m+n} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right]. \tag{6.1.73}$$

Thus, we have

$$\mathcal{K}_{\Delta-\frac{d}{2}}(pz_0) = (-1)^{\Delta-\frac{d}{2}+1} \sum_{m=0}^{\infty} \frac{(pz_0/2)^{2m+\Delta-\frac{d}{2}}}{m!(\Delta-\frac{d}{2}+m+1)!} \ln \frac{pz_0}{2} + \frac{1}{2} \sum_{m=0}^{\Delta-\frac{d}{2}-1} \left(\frac{pz_0}{2}\right)^{-\Delta+\frac{d}{2}+2m} \frac{(\Delta-\frac{d}{2}-m-1)!}{m!}$$
$$+ (-1)^{\Delta-\frac{d}{2}+1} \frac{1}{2} \sum_{m=0}^{\infty} \frac{(pz_0/2)^{\Delta-\frac{d}{2}+2m}}{m!(\Delta-\frac{d}{2}+m)!} \left[2C - \sum_{k=1}^{m+\Delta-\frac{d}{2}} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right], \tag{6.1.74}$$

which implies

$$z_0 \frac{\partial}{\partial z_0} K^{\epsilon}_{\Delta}(p)\Big|_{z_0=\epsilon} = \frac{\frac{d}{2} z_0^{\frac{d}{2}} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0) + z_0^{\frac{d}{2}} (pz_0) \frac{\partial}{\partial(pz_0)} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0)}{\epsilon^{\frac{d}{2}} \mathcal{K}_{\Delta-\frac{d}{2}}(p\epsilon)}\Bigg|_{z_0=\epsilon}$$
$$= \frac{d}{2} + 2 \frac{\frac{pz_0}{2} \frac{\partial}{\partial(pz_0)} \mathcal{K}_{\Delta-\frac{d}{2}}(pz_0)}{\mathcal{K}_{\Delta-\frac{d}{2}}(p\epsilon)}\Bigg|_{z_0=\epsilon}. \tag{6.1.75}$$

Introducing

$$A_m = \left[m + \frac{1}{2}\Big(\Delta - \frac{d}{2}\Big) \right] \frac{(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m+1)!} \ln \frac{p\epsilon}{2}$$
$$+ \left[m + \frac{1}{2}\Big(\Delta - \frac{d}{2}\Big) \right] \frac{1}{2} \frac{(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m)!} \left[2C - \sum_{k=1}^{m+\Delta-\frac{d}{2}} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right] + \frac{2(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m+1)!},$$
$$B_m = \left[m - \frac{1}{2}\Big(\Delta - \frac{d}{2}\Big) \right] \frac{1}{2} \frac{(\Delta-\frac{d}{2}-m-1)!}{m!},$$
$$C_m = \left(\frac{p\epsilon}{2}\right)^{2\left(\Delta-\frac{d}{2}\right)} \left[\frac{(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m+1)!} \ln \frac{p\epsilon}{2} + \frac{1}{2} \frac{(-1)^{\Delta-\frac{d}{2}+1}}{m!(\Delta-\frac{d}{2}+m)!} \left[2C - \sum_{k=1}^{m+\Delta-\frac{d}{2}} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right] \right],$$
$$D_m = \frac{1}{2} \frac{(\Delta-\frac{d}{2}-m-1)!}{m!} \tag{6.1.76}$$

we can rewrite (6.1.75) as

$$
\begin{aligned}
&z_0\frac{\partial}{\partial z_0}K^{\epsilon}_{\Delta}(p)\Big|_{z_0=\epsilon}\\
&\quad=\frac{d}{2}+2\left[\frac{\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\left(A_0+A_1\left(\frac{p\epsilon}{2}\right)^2+\cdots\right)+\left(B_0+B_1\left(\frac{p\epsilon}{2}\right)^2+\cdots\right)}{\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\left(C_0+C_1\left(\frac{p\epsilon}{2}\right)^2+\cdots\right)+\left(D_0+D_1\left(\frac{p\epsilon}{2}\right)^2+\cdots\right)}\right]\\
&\quad=\frac{d}{2}+2\frac{B_0}{D_0}\left[\frac{\left(1+\frac{B_1}{B_0}\left(\frac{p\epsilon}{2}\right)+\cdots\right)+\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\left(\frac{A_0}{B_0}+\frac{A_1}{B_0}\left(\frac{p\epsilon}{2}\right)+\cdots\right)}{\left(1+\frac{D_1}{D_0}\left(\frac{p\epsilon}{2}\right)+\cdots\right)+\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\left(\frac{C_0}{D_0}+\frac{C_1}{D_0}\left(\frac{p\epsilon}{2}\right)+\cdots\right)}\right]\\
&\quad=\frac{d}{2}+2\frac{B_0}{D_0}\left[1+\frac{B_1}{B_0}\left(\frac{p\epsilon}{2}\right)+\cdots+\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\left(\frac{A_0}{B_0}+\frac{A_1}{B_0}\left(\frac{p\epsilon}{2}\right)+\cdots\right)\right]\\
&\qquad\left[1-\frac{D_1}{D_0}\left(\frac{p\epsilon}{2}\right)-\cdots-\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\left(\frac{C_0}{D_0}+\frac{C_1}{D_0}\left(\frac{p\epsilon}{2}\right)+\cdots\right)\right]. \qquad (6.1.77)
\end{aligned}
$$

We will not consider the terms with a positive integer power of p or the terms containing $\ln(p\epsilon)$ times higher powers of p. These term are either contact terms or are subleading. Thus, we only consider

$$
\begin{aligned}
z_0\frac{\partial}{\partial z_0}K^{\epsilon}_{\Delta}(p)\Big|_{z_0=\epsilon}&=\frac{d}{2}+2\frac{B_0}{D_0}+\cdots+2\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\frac{A_0}{D_0}+\cdots\\
&=\frac{d}{2}+(-\Delta+\frac{d}{2})+\cdots-2\left(\frac{p\epsilon}{2}\right)^{2\left(\Delta+\frac{d}{2}\right)}\frac{(-1)^{\Delta-\frac{d}{2}}\ln\frac{p\epsilon}{2}}{\frac{1}{2}\Gamma\left(\Delta-\frac{d}{2}\right)^2}+\cdots\\
&=(d-\Delta)+\cdots+\frac{(-1)^{\Delta-\frac{d}{2}+1}}{2^{2\left(\Delta-\frac{d}{2}\right)-2}\Gamma\left(\Delta-\frac{d}{2}\right)^2}(p\epsilon)^{2\left(\Delta-\frac{d}{2}\right)}\ln(p\epsilon)+\cdots\\
&=(d-\Delta)+\cdots a_{\Delta}(p\epsilon)^{2\left(\Delta-\frac{d}{2}\right)}\ln(p)+\cdots \qquad (6.1.78)
\end{aligned}
$$

Here $a_{\Delta}(p\epsilon)^{2(\Delta-\frac{d}{2})}\ln(p)$ leads to the value of the two-point amplitude given in (4.2.95), namely

$$
\frac{1}{(x-y)^{2\Delta}}\frac{(2\Delta-d)\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}. \qquad (6.1.79)
$$

The derivatives (6.1.67) wich are parallel to the boundary vanish faster than those in the normal direction when $\epsilon\to 0$ (D'Hoker et al., 1999). Therefore the relevant term in the product of three propagators is given by

$$
\begin{aligned}
&D_nK^{\epsilon}_{\Delta_2+\Delta_3}(p_1)D_nK^{\epsilon}_{\Delta_2}(p_2)D_nK^{\epsilon}_{\Delta_3}(p_3)\\
&=\left[(d-\Delta_2-\Delta_3)+\cdots a_{\Delta_2+\Delta_3}(p_1\epsilon)^{2(\Delta_2+\Delta_3-\frac{d}{2})}\ln(p_1)+\cdots\right]\\
&\quad\times\left[(d-\Delta_2)+\cdots a_{\Delta_2}(p_2\epsilon)^{2(\Delta_2-\frac{d}{2})}\ln(p_2)+\cdots\right]\\
&\quad\times\left[(d-\Delta_3)+\cdots a_{\Delta_3}(p_3\epsilon)^{2(\Delta_3-\frac{d}{2})}\ln(p_3)+\cdots\right]\\
&=\cdots+\epsilon^{2\Delta_2+2\Delta_3-2d}(d-\Delta_2-\Delta_3)a_{\Delta_2}a_{\Delta_3}p_2^{2(\Delta_2-\frac{d}{2})}\ln(p_2)p_3^{2(\Delta_3-\frac{d}{2})}\ln(p_3)+\cdots \qquad (6.1.80)
\end{aligned}
$$

We finally find, for the three-point function

$$\left\langle \mathcal{O}_t^{k_1}(x_1)\mathcal{O}_\phi^{k_2}(x_2)\mathcal{O}_\phi^{k_3}(x_3)\right\rangle = -\frac{1}{2\kappa_5^2}\frac{a(k_1,k_2,k_3)}{k_1+3}(d-\Delta_2-\Delta_3)$$
$$\left[\frac{1}{|\vec{x}_1-\vec{x}_2|^{2\Delta_2}}\frac{(2\Delta_2-d)\Gamma(\Delta_2)}{\pi^{\frac{d}{2}}\Gamma(\Delta_2-\frac{d}{2})}\right]\left[\frac{1}{|\vec{x}_1-\vec{x}_3|^{2\Delta_3}}\frac{(2\Delta_3-d)\Gamma(\Delta_3)}{\pi^{\frac{d}{2}}\Gamma(\Delta_3-\frac{d}{2})}\right]. \tag{6.1.81}$$

Now, substitute k_1 by k_2+k_3 and use the expressions (6.1.68) for the Δ_i. Thus with $d=4$, we obtain

$$\left\langle \mathcal{O}_t^{k_2+k_3}(x_1)\mathcal{O}_\phi^{k_2}(x_2)\mathcal{O}_\phi^{k_3}(x_3)\right\rangle$$
$$= \frac{1}{2\kappa_5^2}\frac{4}{\pi^4}\frac{a(k_2+k_3,k_2,k_3)}{x_{12}^{8+2k_2}x_{13}^{8+2k_3}}\frac{(k_2+3)(k_2+2)^2(k_3+3)(k_3+2)^2(k_2+k_3+4)}{(k_2+k_3+3)}. \tag{6.1.82}$$

This result is in agreement with the value of the three-point function obtained by analytic continuation, given in (6.1.70). In D'Hoker et al. (1999), it is mentioned that there is a regularization that may give a different result. This confirms the subtlety in the computation of the extremal correlators. D'Hoker et al. (1999) show that the Ward identity of a current and an extremal combination of scalar operators agrees with the extremal three-point correlation function.

6.2 Correlation functions from the gauge theory side

In this section, the corresponding correlation functions in the gauge theory are computed.

6.2.1 Chiral primary operators. The chiral primary operators (CPO) of SYM_4 are operators of the form

$$\mathcal{O}^I = C^I_{i_1\cdots i_k}\,\mathrm{Tr}\left(\phi^{i_1}\cdots\phi^{i_k}\right) \tag{6.2.1}$$

where $i_1,\cdots i_k$ are $SO(6)$ vector indices, ϕ^i are six $N\times N$ matrices transforming in the adjoint of $U(N)$, and $C^I_{i_1\cdots i_k}$ are totally symmetric traceless rank k tensors of $SO(6)$. Here the trace is taken over the $U(N)$ indices.

6.2.2 The propagator. The action is

$$S = \int d^4x\,\frac{1}{2g_{YM}^2}\mathrm{Tr}\left(F^2\right)+\cdots = \int d^4x\,\frac{1}{4g_{YM}^2}F^a_{\mu\nu}F^{a\mu\nu}+\cdots. \tag{6.2.2}$$

The Yang-Mills coupling g_{YM} and the string coupling g_s are related by

$$g_{YM}^2 = 4\pi g_s. \tag{6.2.3}$$

In this theory, the propagator is given by

$$\left\langle \phi_a^i(x_1)\phi_b^j(x_2)\right\rangle = \frac{g_{YM}^2\delta_{ab}\delta^{ij}}{(2\pi)^2x_{12}^2}, \tag{6.2.4}$$

where $a,b,\cdots$ are $U(N)$ color indices.

6.2.3 Correlators at large N. The correlators are computed using Wick's theorem. In the large N limit we only need to sum the planar diagrams. For the correlator of a product of two traces, we have

$$\begin{aligned}&\left\langle \text{Tr}\left(\phi^{i_1}(x_1)\cdots\phi^{i_k}(x_1)\right)\text{Tr}\left(\phi^{j_1}(x_2)\cdots\phi^{j_k}(x_2)\right)\right\rangle \\ &= \frac{N^k g_{YM}^{2k}\left(\delta^{i_1j_1}\delta^{i_2j_2}\ldots\delta^{i_kj_k}+\text{cyclic}\right)}{(2\pi)^{2k}x_{12}^{2k}}.\end{aligned} \tag{6.2.5}$$

The complete set of planar diagrams follow by contracting the i's and j's in the same cyclic order in which they appear in the traces.

Let $\mathcal{O}^{I_1}$ and $\mathcal{O}^{I_2}$ be two CPO specified by the tensors $C^{I_1}_{i_1\cdots i_k}$ and $C^{I_2}_{j_1\cdots j_k}$. The correlator of the product of these two operators is zero if their scaling dimensions are not equal. For $k_1 = k_2 = k$, we have

$$\begin{aligned}&\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\right\rangle \\ &= C^{I_1}_{i_1\cdots i_k}C^{I_2}_{j_1\cdots j_k}\left\langle \text{Tr}\left(\phi^{i_1}(x_1)\cdots\phi^{i_k}(x_1)\right)\text{Tr}\left(\phi^{j_1}(x_2)\cdots\phi^{j_k}(x_2)\right)\right\rangle \\ &= C^{I_1}_{i_1\cdots i_k}C^{I_2}_{j_1\cdots j_k}\frac{N^k g_{YM}^{2k}\left(\delta^{i_1j_1}\delta^{i_2j_2}\ldots\delta^{i_kj_k}+\text{cyclic}\right)}{(2\pi)^{2k}x_{12}^{2k}} \\ &= C^{I_1}_{i_1\cdots i_k}C^{I_2}_{i_1\cdots i_k}\frac{N^k g_{YM}^{2k}\,k}{(2\pi)^{2k}x_{12}^{2k}} \\ &= \left\langle C^{I_1}C^{I_2}\right\rangle\frac{\lambda^k\,k}{(2\pi)^{2k}x_{12}^{2k}}\end{aligned} \tag{6.2.6}$$

where the t'Hooft coupling is given by

$$\lambda = N g_{YM}^2 \tag{6.2.7}$$

and

$$\left\langle C^{I_1}C^{I_2}\right\rangle = C^{I_1}_{i_1\cdots i_k}C^{I_2}_{i_1\cdots i_k}. \tag{6.2.8}$$

There are k possible cyclic permutations with k pairs of indices which is why we obtain the factor k in the correlator above. The correlator of the 3-point function of CPOs is specified by three tensors of rank k_1, k_2 and k_3, as follows

$$\begin{aligned}&\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_2}(x_3)\right\rangle \\ &= C^{I_1}_{i_1\cdots i_{k_1}}C^{I_2}_{j_1\cdots j_{k_2}}C^{I_3}_{l_1\cdots l_{k_3}}\left\langle \text{Tr}\left(\phi^{i_1}(x_1)\cdots\phi^{i_{k_1}}(x_1)\right)\text{Tr}\left(\phi^{j_1}(x_2)\cdots\phi^{j_{k_2}}(x_2)\right)\text{Tr}\left(\phi^{l_1}(x_3)\cdots\phi^{l_{k_3}}(x_3)\right)\right\rangle \\ &= \left\langle C^{I_1}C^{I_2}C^{I_3}\right\rangle\frac{\lambda^{\Sigma}\,k_1k_2k_3}{N(2\pi)^{2\Sigma}x_{12}^{2\alpha_3}x_{23}^{2\alpha_1}x_{31}^{2\alpha_2}}\end{aligned} \tag{6.2.9}$$

where the contraction of the indices of the three symmetric traceless tensors is considered in Appendix A,

$$\left\langle C^{I_1}C^{I_2}C^{I_3}\right\rangle = C^{I_1}_{i_1\cdots i_{\alpha_3}j_1\cdots j_{\alpha_2}}C^{I_2}_{i_1\cdots i_{\alpha_3}l_1\cdots l_{\alpha_1}}C^{I_3}_{j_1\cdots j_{\alpha_2}l_1\cdots l_{\alpha_1}} \tag{6.2.10}$$

and

$$\Sigma = \frac{1}{2}(k_1+k_2+k_3). \tag{6.2.11}$$

The number of planar diagram is $k_1k_2k_3$.

Rescaling the CPOs

$$\mathcal{O}^I = \mathcal{O}^I \frac{(2\pi)^k}{\sqrt{\lambda\, k}} \tag{6.2.12}$$

gives a normalized 2-point function,

$$\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\right\rangle = \left\langle C^{I_1}C^{I_2}\right\rangle \frac{1}{x_{12}^{2k}}, \tag{6.2.13}$$

and the 3-point function becomes

$$\left\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\mathcal{O}^{I_2}(x_3)\right\rangle = \left\langle C^{I_1}C^{I_2}C^{I_3}\right\rangle \frac{\sqrt{k_1k_2k_3}}{N x_{12}^{2\alpha_3} x_{23}^{2\alpha_1} x_{31}^{2\alpha_2}}. \tag{6.2.14}$$

Notice that we can choose the C^I such that

$$\left\langle C^{I_1}C^{I_2}\right\rangle = \delta^{I_1 I_2}. \tag{6.2.15}$$

The CPOs become orthonormal operators. The large N counting gives us

$$\left\langle \mathcal{O}^{I_1}\mathcal{O}^{I_2}\mathcal{O}^{I_2}\right\rangle \begin{cases} = 0 & \text{if } k_1 \geq k_2 + k_3 \\ \approx \frac{1}{N} & \text{if } k_1 \leq k_2 + k_3 \end{cases}. \tag{6.2.16}$$

In summary, the computation of extremal correlators is subtle and divergences appear in higher point correlation functions.

7. Extremal correlators

In this chapter, we want to explore the subtlety in the computation of the extremal correlators. We will suggest that there maybe a connection to divergences in collinear amplitudes in quantum field theory. The comments we make in this chapter are simply suggestive. We have not had time to pursue them to their logical conclusion.

7.1 Extremal three- and n-point functions

Let $\mathcal{O}_\Delta$ and $\mathcal{O}_{\Delta_i}$, $i = 1, 2$ to n, be half-BPS chiral primary operators (CPOs) with dimension Δ and Δ_i. The extremal three-point function is of the form

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle \tag{7.1.1}$$

with $\Delta_1 = \Delta_2 + \Delta_3$. The supergravity coupling constant for the three-point interaction vanishes as $\mathcal{G}(\Delta_1; \Delta_2, \Delta_3) \approx \Delta_1 - \Delta_2 - \Delta_3$ and the AdS_5 bulk integral has a "pole" at the extremal dimension (D'Hoker et al., 1999; D'Hoker and Freedman, 2002),

$$\int_{AdS_5} \frac{d^5 z}{z_0^5} \prod_{i=1}^{3} \frac{z^{\Delta_i}}{\left(z_0^2 + (\vec{z} - \vec{x}_i)^2\right)^{\Delta_i}} \approx \frac{1}{\Delta_1 - \Delta_2 - \Delta_3}. \tag{7.1.2}$$

The generalization of the extremal three-point function is of the form

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle \tag{7.1.3}$$

with $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_n$. There is a conjecture (D'Hoker et al., 1999; D'Hoker and Freedman, 2002) that the associated supergravity bulk coupling must vanish as

$$\mathcal{G}(\Delta_1; \Delta_2, \cdots, \Delta_n) \approx \Delta - \Delta_1 - \Delta_2 - \cdots - \Delta_n \tag{7.1.4}$$

and the bulk integral exhibits a "pole"

$$\int_{AdS_5} \frac{d^5 z}{z_0^5} \frac{z_0^\Delta}{\left(z_0^2 + (\vec{z} - \vec{x})^3\right)^\Delta} \prod_{i=1}^{n} \frac{z^{\Delta_i}}{\left(z_0^2 + (\vec{z} - \vec{x}_i)^2\right)^{\Delta_i}} \approx \frac{1}{\Delta - \Delta_1 - \Delta_2 - \cdots - \Delta_n}. \tag{7.1.5}$$

The pole in the z-integration and the zero of the supergravity coupling compensate one another and consequently the analytic continuation in the dimension can be used to derive the value of the extremal correlators. A careful analysis establishes that while the bulk contribution vanishes, there remains a boundary contribution, as shown in the previous chapter 6. This provides a second way to compute extremal correlators from the gravity side.

7.2 Mapping between R-charge and angular momentum

This section will explain why we expect that extremal correlators are related to collinear amplitudes. This idea is based on a basic feature of the AdS/CFT correspondence which states that global symmetries in CFT are matched to isometries in the dual gravity. For example, the $SO(4,2)$ conformal symmetry maps to the isometry group of AdS_5; the $SU(4)$ R-symmetry maps to the isometry group of S^5. In fact, the $SU(4)$ R-symmetry group is a double cover of the special orthogonal group $SO(6)$, which is an isometry of S^5. This means that $SU(4)$ also has spinor representations. Thus, when there are fermions, we should use $SU(4)$ rather than $SO(6)$. The conserved charge associated with the isometry of the sphere is angular momentum. This leads to the conclusion that R-charge of an operator in the CFT maps to angular momentum of a state in the string theory. Using this correspondence, we can put the modes arising from an expansion in spherical harmonics on the S^5 into one-to-one correspondence with operators of definite R-charge. Of course, if there are degeneracies this will complicate things. As an example, chiral primary operators on the gauge theory side correspond to spherical harmonics obtained from Klein-Kaluza reduction (appendix B of Semenoff and Young (2006)). This correspondence between chiral primary operators and spherical harmonics was used (Chapters 5 and 6) where we compute correlators involving giant gravitons and Klein-Kaluza gravitons.

With this picture in mind, we can now explain why we think extremal correlators are linked to the scattering of particle states with parallel momenta. First, we can define three complex matrices using the six scalars ϕ_i, with $i = 1$ to 6, of the SYM theory, transforming in the vector representation of the $SO(6)$ R symmetry group. These matrices, from which chiral primary operators can be formed, are given by

$$Z = \phi_1 + i\phi_2, \quad Z^\dagger = \phi_1 - i\phi_2 \tag{7.2.1}$$

$$Y = \phi_3 + i\phi_4, \quad Y^\dagger = \phi_3 - i\phi_4 \tag{7.2.2}$$

$$X = \phi_5 + i\phi_6, \quad X^\dagger = \phi_5 - i\phi_6 \tag{7.2.3}$$

We can label the generators (=angular momenta) of $SO(6)$ as L_{ij} with $i, j = 1, 2, \ldots, 6$. The generator $L_{ij} = -L_{ji}$ perform an infinitesimal rotation in the ij-plane. These generators close the $SO(6)$ Lie algebra

$$[L_{ij}, L_{kl}] = i\delta_{jk}L_{il} - i\delta_{ik}L_{jl} - i\delta_{jl}L_{ik} + i\delta_{il}L_{jk}. \tag{7.2.4}$$

There is a basis of states in which L_{12}, L_{34} and L_{56} are simultaneously diagonal. Indeed, this follows because

$$[L_{12}, L_{34}] = 0 = [L_{12}, L_{56}] = [L_{34}, L_{56}]. \tag{7.2.5}$$

Collect these three quantum numbers into a vector $\vec{L} \equiv (L_{12}, L_{34}, L_{56})$. We will now compute the $\vec{L}$ quantum numbers of X, Y and Z. Under a rotation in the 12-plane

$$\begin{bmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \\ \phi_4' \\ \phi_5' \\ \phi_6' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix}. \tag{7.2.6}$$

Thus

$$\begin{aligned} Z' &= \phi_1' + i\phi_2' \\ &= \cos\theta\phi_1 + \sin\theta\phi_2 + i\big(\cos\theta\phi_2 - \sin\theta\phi_1\big) \\ &= e^{-i\theta}\big(\phi_1 + i\phi_2\big) = e^{-i\theta}Z \end{aligned} \tag{7.2.7}$$

$$Y' = \phi_3' + i\phi_4' = \phi_3 + i\phi_4 = Y \tag{7.2.8}$$

$$X' = \phi_5' + i\phi_6' = \phi_5 + i\phi_6 = X \tag{7.2.9}$$

Similarly, under a rotation in the 34-plane we have

$$Z' = Z, \quad Y' = e^{-i\theta} Y, \quad X' = X \tag{7.2.10}$$

and under a rotation in the 56-plane we have

$$Z' = Z, \quad Y' = Y, \quad X' = e^{-i\theta} X. \tag{7.2.11}$$

For this we read off

$$\vec{L} = (1,0,0) \qquad \text{for } Z \tag{7.2.12}$$

$$\vec{L} = (0,1,0) \qquad \text{for } Y \tag{7.2.13}$$

$$\vec{L} = (0,0,1) \qquad \text{for } X \tag{7.2.14}$$

An operator such as $\text{Tr}\,(Z^{n_1})\,\text{Tr}\,(Y^{n_2})\,\text{Tr}\,(X^{n_3})$ is mapped to a three-particle state with the momenta of the three particles given by $(n_1, 0, 0)$, $(0, n_2, 0)$ and $(0, 0, n_3)$. The three particles are moving in different directions. The multi-trace operator of the form $\text{Tr}\,(Z^{n_1})\,\text{Tr}\,(Z^{n_2})\,\text{Tr}\,(Z^{n_3})$, which participates in extremal correlators, corresponds to a three-particle state with momenta, $(n_1, 0, 0)$, $(n_2, 0, 0)$ and $(n_3, 0, 0)$. Thus, the particles in this state are moving collinearly. Clearly then, extremal correlators, for example

$$\left\langle \text{Tr}\,(Z^{n_1})\,\text{Tr}\,(Z^{n_2})\,\text{Tr}\,(Z^{n_3})\,\text{Tr}\left(Z^{\dagger n_1+n_2+n_3}\right)\right\rangle$$

map to processes with collinear particles. Non-extremal correlators

$$\left\langle \text{Tr}\,(Z^{n_1})\,\text{Tr}\,(Y^{n_2})\,\text{Tr}\,(X^{n_3})\,\text{Tr}\left(Z^{\dagger n_1} Y^{\dagger n_2} X^{\dagger n_3}\right)\right\rangle$$

are not related to states with collinear amplitudes. Having establish this connection, we will now explore properties of collinear amplitudes in QFT.

7.3 Collinear and soft divergences

We have just argued that extremal correlators are closely related to collinear amplitudes. These amplitudes are associated to processes where the momentum of the particles that are being described are parallel. In fact, it is well known that there are divergences associated with collinear amplitudes. This suggests the very attractive possibility that perhaps the divergences associated with extremal correlators can be interpreted as collinear singularities.

Collinear singularities are usually accompanied by soft or IR divergences. In this section our goal is to give a brief description of these divergences, how they are interpreted and how they are cancelled out. We hope that a similar approach to extremal correlators will resolve the subtleties we are studying in this dissertation.

Feynman diagrams are known to produce various divergent (infinite) expressions. What do they mean? Here, it is important to note that there are different sources of divergences and their meaning is very different for the different sources.

"Infrared divergences" is the name for the infinities that emerge because we have to integrate over arbitrarily long-wavelength (or low-energy) virtual particles (or quanta). They are produced when we send the minimum allowed momentum or energy of virtual particles to zero. In this case, loop diagrams are infinite. What does it mean? To answer this question it is helpful to start by recalling that ultraviolet (short-distance or high-energy) divergences usually imply that a quantum field theory is incomplete and should be thought of as a limit of a more accurate theory. Infrared divergences are quite different. The asymmetry between the two kinds of divergences arises because physics at long distances is derived from the physics at short distances.

How should we interpret infrared divergences? It is important to know that quantum field theory has two kinds of questions (with a whole continuum in between): questions that are are directly linked to the results of measurements, that are easily interpreted experimentally as well as questions that are natural and simple from a theoretical viewpoint, questions that are connected with fundamental concepts and quantities in the theory. It is the second type of question that suggests calculations that may produce infrared divergences.

A classic example in which IR divergences enter is in Quantum Electrodynamics, when we calculate the cross section for the scattering of two charged particles. Perturbative quantum field theory will produce a cross section that is a Taylor expansion in the fine-structure constant (or the electric charge). The first term in this expansion comes from a tree diagram. The particles simply exchange one virtual photon and it reproduces the predictions you could make using classical physics. Loop corrections provide quantum corrections to the classical prediction. Already the one-loop corrections suffers from infrared divergences. The amplitude includes a term proportional to $\ln(E_{min})$ - where E_{min} is the minimum allowed energy of a virtual photon in the loop. We should set this limit, E_{min}, to zero which produces a divergence. Remarkably, this divergences is canceled if we do the computation carefully. The quantity that will be compared with experiment is the cross section of an observable process, which is obtained by squaring the amplitude the Feynman diagram computes. The squared amplitude, $[\text{finite} + \ln(E_{min})]^2$, will produce terms like $(\text{finite}^2 + 2 \text{ finite } \ln(E_{min}) + ...)$. The dots contain higher powers of the fine-structure constant. To see how the term proportional to $\ln(E_{min})$ be canceled, note that a real experiment can't observe photons of arbitrarily low energies. We must actually compute the inclusive cross section in which we allow an arbitrary number of low energy photons, that are invisible to the experiment's detectors. These photons have such a low energy that they can never be observed by an detector. Thus, we must include diagrams with extra external low-energy (soft) photons. It's so soft that your device cannot see it. This extra diagram is also infrared divergent but, remarkably the sum is finite. This method of dealing with soft divergences is called the Bloch and Nordsieck Theorem (Bloch and Nordsieck, 1937). It says that as long as you sum over degenerate final states (the final state plus other versions of it in which soft photons are included), the answer you get for any physical quantity is free of infrared divergences.

The collinear divergences are a new effect, in which infinities are produced as a consequence of the fact that the momenta of particles in the amplitude are parallel. The treatment of these divergences is very similar to the treatment of soft divergences. Again by summing the correct classes of Feynman diagrams one obtains a result that is free of any divergences. The theorem stating the result is due to Kinoshita (1962), Lee and Nauenberg (1964). The theorem proves that if you sum over both degenerate final and initial states, the answer you get for any physical quantity is free of infrared divergences.

To summarize the discussion, all the divergences ultimately cancel as long as you properly calculate quantities that can be observed. In the cases we considered here, there are many degenerate states that can't be resolved and so to get a physical quantity you must sum over these degenerate possibilities.

7.4 Extremal correlators and collinear divergences

To summarize our discussion in this chapter, we have established a close correspondence between collinear amplitudes and extremal correlators. Further, we have reviewed the fact that collinear amplitudes display extra divergences, intimately related to the fact that the particles participating are collinear. This immediately suggests that the singularities present in extremal correlators may be related to collinear singularities. Assuming this is the case, what do we learn about the divergences in extremal correlators?

The divergences in collinear amplitudes are eliminated once one sums over degenerate initial and final states. One possible approach to the divergences in extremal correlators would entail exploring precisely what degenerate states are relevant for the correlator, and then exploring the sum over these. What effect would these sums have? An aspect that one would need to explore here would be the existence or nonexistence of threshold bound states, which would definitely have implications for the precise nature of the sets of degenerate states. Sadly, due to a lack of time, these interesting questions must be left for the future

8. Conclusions

We have motivated the AdS/CFT correspondence by exploring the planar limit of matrix models in Chapter 2. In Chapter 3 we have extended this analysis to explore the correlation functions of operators with a dimension of order N in the matrix model. This was achieved by employing techniques that exploit group representation theory, effectively allowing a study of finite N effects in the CFT. In Chapter 4 holographic methods of computing correlation functions have been introduced. These techniques allow a study of CFT correlators in the strong coupling and large N limit of the CFT. There are divergences that appear in the holographic evaluation of correlators, for the case of two point functions. For extremal correlators the divergences are more severe, as has been review in Chapter 5 and 6. As we have reviewed in these chapters, the correct values for the extremal correlators are obtained by performing an analytic continuation of the non-extremal correlators to the extremal case. In this way it is possible to obtain a perfect match of the gauge theory and gravity results.

This is however, far from a satisfactory understanding of the physics that is involved in these divergences. We have not understood the origin of these divergences and without this, it is difficult to motivate the analytic continuation that has been used. The key question we have explored in this MSc is an attempt to develop an understanding of these divergences in order that we can provide a complete understanding of these divergences.

In Chapter 7, exploiting the identification of R-charge in the CFT with angular momentum in the string theory, we have suggested that extremal correlators are mapped to amplitudes involving particles with parallel momenta. It is well known that collinear particles give rise to divergences so it is somewhat natural to identify these divergences with the divergences in extremal correlators. The Kinoshita-Lee-Nauenberg Theorem (Kinoshita, 1962; Lee and Nauenberg, 1964) states that all collinear divergences are removed by summing over degenerate initial and final states. Our identification suggests that, perhaps, by summing over degenerate initial and final states we can remove the divergences that appear in the extremal correlators. Future work should explore these preliminary ideas and establish a rigorous correspondence between the divergences that appear in extremal correlators and collinear divergences.

Appendices

AppendixA. Contractions of indices of three symmetric traceless tensors

Consider the problem of contracting an arbitrary number of symmetric traceless tensors. Let n_{pq} be the number of indices of the tensor C^{I_p} of rank k_p contracted with the indices of C^{I_q} of rank k_q. These numbers satisfy the relations

$$n_{pq} = n_{qp}, \tag{A.0.1}$$
$$n_{pp} = 0, \tag{A.0.2}$$
$$n_{p1} + n_{p2} + \cdots = k_p. \tag{A.0.3}$$

Specializing to three tensors, we have the constraints

$$n_{12} + n_{13} = k_1, \tag{A.0.4}$$
$$n_{12} + n_{23} = k_2, \tag{A.0.5}$$
$$n_{13} + n_{23} = k_3, \tag{A.0.6}$$

which have a unique solution given by (Lee et al., 1998)

$$n_{12} = n_{21} = \frac{1}{2}(k_1 + k_2 - k_3) = \alpha_3, \tag{A.0.7}$$
$$n_{13} = n_{31} = \frac{1}{2}(k_3 + k_1 - k_2) = \alpha_2, \tag{A.0.8}$$
$$n_{23} = n_{32} = \frac{1}{2}(k_2 + k_3 - k_1) = \alpha_1. \tag{A.0.9}$$

We will denote this contraction of three tensors by $\left\langle C^{I_1} C^{I_1} C^{I_1} \right\rangle$ with

$$\left\langle C^{I_1} C^{I_2} C^{I_3} \right\rangle = C^{I_1}_{i_1 \cdots i_{\alpha_3} j_1 \cdots j_{\alpha_2}} C^{I_2}_{i_1 \cdots i_{\alpha_3} l_1 \cdots l_{\alpha_1}} C^{I_3}_{j_1 \cdots j_{\alpha_2} l_1 \cdots l_{\alpha_1}}. \tag{A.0.10}$$

Here any repeated index is summed from 1 to 6.

AppendixB. Five-dimensional sphere and spherical harmonics

B.1 System of coordinates and symmetries on the n-sphere

A parametrisation of the n-dimensional sphere is given by (Semenoff and Young, 2006)

$$x^1 = \cos\theta_1, \tag{B.1.1}$$
$$x^2 = \sin\theta_1\cos\theta_2, \tag{B.1.2}$$
$$x^3 = \sin\theta_1\sin\theta_2\cos\theta_3, \tag{B.1.3}$$
$$x^4 = \sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4, \tag{B.1.4}$$
$$x^5 = \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos\theta_5, \tag{B.1.5}$$
$$x^6 = \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\cos\theta_6, \tag{B.1.6}$$
$$\cdots = \cdots \tag{B.1.7}$$
$$x^n = \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\cdots\sin\theta_{n-1}\cos\theta_n, \tag{B.1.8}$$
$$x^{n+1} = \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\cdots\sin\theta_{n-1}\sin\theta_n, \tag{B.1.9}$$

with $\theta_1, \cdots, \theta_{n-1} \in [0,\pi]$ and $\theta_n \in [0, 2\pi]$. These coordinates satisfy

$$\sum_{k=1}^{n+1}(x^k)^2 = 1 \tag{B.1.10}$$

The metric on the n-sphere is obtained from the metric of the $(n+1)$-dimensional Euclidean space. It is given by

$$ds^2_{S^n} = \frac{\partial x^\mu}{\partial\theta_i}\frac{\partial x^\mu}{\partial\theta_j}d\theta_i d\theta_j = g_{ij}d\theta_i d\theta_j \tag{B.1.11}$$

where

$$g_{S^n} = g_{ij} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \sin^2\theta_1 & 0 & \cdots & 0 \\ 0 & 0 & \sin^2\theta_1\sin^2\theta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{n-1} \end{bmatrix}. \tag{B.1.12}$$

Rotations are symmetries of the sphere. Further we can reparametrise the sphere using $\theta_i + \theta_{i0}$, for $i = 1$ to n, where θ_{i0} are constant. For example, the parametrisation obtained by changing θ_n into $\theta_n + \pi$ changes the coordinate x^{n+1} into $-x^{n+1}$. The measure $d\Omega_n = d\theta_1\cdots d\theta_n\sqrt{\det g_{S^n}}$ is invariant under all of these transformations. Any permutation of the coordinates $(x^1, \cdots, x^{n+1})$ leaves $d\Omega_n$ invariant which follows immediately from the definition of the metric on the sphere.

B.2 Volume and Area of an n-dimensional sphere

The volume V_n and the area S_n of an n-dimensional sphere are given by

$$S_n = \frac{2\pi^{\frac{n+1}{2}} R^n}{\Gamma(\frac{n+1}{2})}, \tag{B.2.1}$$

$$V_n = \frac{\pi^{\frac{n+1}{2}} R^{n+1}}{\Gamma(\frac{n+3}{2})}, \tag{B.2.2}$$

where R is the radius of the n-sphere.

n	S_n	V_n
0	$S_0 = 2$	$V_0 = 2\,R$
1	$S_1 = 2\,\pi R$	$V_1 = \pi R^2$
2	$S_2 = 4\,\pi R^2$	$V_2 = \frac{4}{3}\,\pi R^3$
3	$S_3 = 2\,\pi^2 R^3$	$V_3 = \frac{1}{2}\,\pi^2 R^4$
4	$S_4 = \frac{8}{3}\,\pi^2 R^4$	$V_4 = \frac{8}{15}\,\pi^2 R^5$
5	$S_5 = \pi^3 R^5$	$V_5 = \frac{1}{6}\,\pi^3 R^6$
6	$S_6 = \frac{16}{15}\,\pi^3 R^6$	$V_6 = \frac{16}{105}\,\pi^3 R^7$
7	$S_7 = \frac{1}{3}\,\pi^4 R^7$	$V_7 = \frac{1}{24}\,\pi^4 R^8$

B.3 Integrals of polynomials on the five-sphere

Let $x = (x^1, \cdots, x^6)$ be coordinates on $\mathbb{R}^6$. Let $d\Omega_5$ be the measure on the 5-sphere. The area of the unit 5-sphere is

$$\int_{S^5} d\Omega_5 = \Omega_5 = \pi^3. \tag{B.3.1}$$

We also have (Lee et al., 1998)

$$\frac{1}{\Omega_5}\int_{S^5} d\Omega_5 x^{i_1}\cdots x^{i_{2m}} = \frac{1}{2^{m-1}(m+2)!}[\text{all possible contractions}]. \tag{B.3.2}$$

This integral can be evaluated by using

$$\int_{S^5} d\Omega_5 x^{i_1}\cdots x^{i_{2m}} = \frac{\delta^{2m}}{\delta J_{i_1}\cdots\delta J_{i_{2m}}}\int_{S^5} d\Omega_5\, e^{J\cdot x}. \tag{B.3.3}$$

We will prove (B.3.2) by recursion. First notice that

$$\int_{S^5} d\Omega_5 x^{i_1}\cdots x^{i_{2m+1}} = A_{2m+1} = 0. \tag{B.3.4}$$

This is true because the measure on the S^5 is invariant under the transformation P_i (see section (B.1)) which maps $x = (\cdots, x^i, \cdots)$ into $x' = (\cdots, -x^i, \cdots)$. Therefore we have

$$A_1 = \int_{S^5} d\Omega_5 x^i = -\int_{S^5} d\Omega_5 x^i \tag{B.3.5}$$

which implies $A_1 = 0$. Similarly, A_{2m+1} vanishes since the product $x^{i_1} \cdot \cdots \cdot x^{i_{2m+1}}$ will be mapped to $-x^{i_1} \cdot \cdots \cdot x^{i_{2m+1}}$ under the transformation P_i which maps $x = (\cdots, x^i, \cdots)$ into $x' = (\cdots, -x^i, \cdots)$ such that x^i appears in the integrand an odd number of times.

We want to use the same reasoning for the evaluation of $\int_{S^5} d\Omega_5 x^{i_1} x^{i_2}$. We see that if $i_1 \neq i_2$ we can perform a transformation P_{i_1} so that $x^{i_1} x^{i_2}$ is mapped to $-x^{i_1} x^{i_2}$. Therefore the integral vanishes. For $i_1 = i_2$, using the symmetry properties of the sphere, we have

$$\int_{S^5} d\Omega_5 \, (x^{i_1})^2 = A_2 = \text{constant}, \tag{B.3.6}$$

with $i_1 = 1, 2, \cdots, 6$. This constant can be computed using the fact that

$$\sum_{i_k=1}^{6} (x^{i_k})^2 = 1.$$

Thus, we obtain

$$A_2 = \frac{1}{6} \sum_{i_k=1}^{6} \int_{S^5} d\Omega_5 \, (x^{i_k})^2 = \frac{1}{6} \Omega_5 = \frac{1}{2^{1-1}(1+2)!} \Omega_5. \tag{B.3.7}$$

Therefore we can write

$$\int_{S^5} d\Omega_5 x^{i_1} x^{i_2} = A_2 \delta^{i_1 i_2}. \tag{B.3.8}$$

Let us now consider the integral $\int_{S^5} d\Omega_5 x^{i_1} x^{i_2} x^{i_3} x^{i_4}$. This integral will be proportional to the following contraction of indices.

$$\int_{S^5} d\Omega_5 x^{i_1} x^{i_2} x^{i_3} x^{i_4} = A_4 \big(\delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3} \big). \tag{B.3.9}$$

For $i_1 = i_2 = i_3 = i_4 = k$, we have

$$\int_{S^5} d\Omega_5 x^k x^k x^k x^k = 3A_4. \tag{B.3.10}$$

For $i_1 = i_2 = k \neq i_3 = i_4 = l$, we have

$$\int_{S^5} d\Omega_5 x^k x^k x^l x^l = A_4 \tag{B.3.11}$$

These two equations are true for any k. Thus we can combine them to find

$$\sum_{k=1}^{6} \int_{S^5} d\Omega_5 x^k x^k x^l x^l = 5A_4 + 3A_4 = 8A_4. \tag{B.3.12}$$

Using the defining equations of the sphere we now find

$$8A_4 = \int_{S^5} d\Omega_5 x^l x^l \tag{B.3.13}$$

$$= A_2, \tag{B.3.14}$$

which implies

$$A_4 = \frac{A_2}{8} = \frac{1}{2^{2-1}(2+2)!}\Omega_5. \tag{B.3.15}$$

To proceed further we will derive a recursion relation between A_{2m} and A_{2m+2}. These quantities are given by

$$\int_{S^5} d\Omega_5 x^{i_1}\cdots x^{i_{2m-1}}x^{i_{2m}} = A_{2m}\big[\text{Contraction of the indices } i_1,\cdots,i_{2m}\big], \tag{B.3.16}$$

$$\int_{S^5} d\Omega_5 x^{i_1}\cdots x^{i_{2m+1}}x^{i_{2m+2}} = A_{2m+2}\big[\text{Contraction of the indices } i_1,\cdots,i_{2m+2}\big]. \tag{B.3.17}$$

For $i_1 = \cdots = i_{2m+2} = k$, the last integral equals to $A_{2m+2}(2m+2)!!$. If $i_1 = \cdots = i_{2m} = l \neq i_{2m+1} = i_{2m+2} = k$ it will be equal to $A_{2m+2}(2m)!!$. Thus, we obtain

$$\sum_{k=1}^{6}\int_{S^5} d\Omega_5 (x^l)^{2m}(x^k)^2 = 5A_{2m+2}(2m)!! + A_{2m+2}(2m+2)!!. \tag{B.3.18}$$

Again using the defining equation for the sphere we obtain

$$2(m+3)A_{2m+2}(2m)!! = \int_{S^5} d\Omega_5 (x^l)^{2m} = A_{2m}(2m)!!, \tag{B.3.19}$$

which implies

$$A_{2m+2} = \frac{A_{2m}}{2(m+3)}. \tag{B.3.20}$$

Solving this recursion relation, we find

$$A_{2m} = \frac{1}{2^{m-1}(m+2)!}\Omega_5. \tag{B.3.21}$$

This completes the proof of (B.3.2).

B.4 Spherical harmonics on the five sphere

B.4.1 Definitions. By construction, the spherical harmonics on the n-sphere are homogeneous harmonic polynomials of the $(n+1)$-dimensional Euclidean space restricted to the n-sphere. Any polynomial of the form

$$Y^{I,k} = C^I_{i_1\cdots i_k} x^{i1}\cdots x^{i_k} \tag{B.4.1}$$

where $C^I_{i_1\cdots i_k}$ is a traceless symmetric tensor of rank k, defines a spherical harmonic on the n-sphere. There are

$$d_k = C^k_{n+k} - C^k_{n+k-2}, \qquad C^k_n = \frac{n!}{k!(n-k)!} \tag{B.4.2}$$

linearly independent spherical harmonics Y^I defined in this way (Lee et al., 1998). Here C^k_n is the binomial coefficient. The label I is used to distinguish the different harmonics. The dimension of the subspace spanned by these spherical harmonics is exactly the number of linearly independent homogeneous harmonic polynomial of degree k restricted to the n-sphere. They are eigenfunctions of the Laplacian operator $\nabla^2_{S^n}$ on the n-sphere

$$\nabla^2_{S^n} Y^{I,k} = \nabla^\alpha \nabla_\alpha Y^{I,k} = -k(k+n-1)Y^{I,k}. \tag{B.4.3}$$

The set of spherical harmonics is an infinite dimensional space of continuous functions since any continuous function can be approximated by polynomials. This fact allows us to perform the decomposition of any continuous function into an infinite sum of spherical harmonics.

B.4.2 Integrals of Spherical harmonics . The integral of two spherical harmonics is given by

$$\int_{S^5} d\Omega_5 \; Y^{I_1,k_1} Y^{I_2,k_2} = C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} \int_{S^5} d\Omega_5 \; x^{i_1}\cdots x^{i_{k_1}} x^{j_1}\cdots x^{j_{k_1}}. \tag{B.4.4}$$

We perfom this integral using (B.3.2). We obtain zero if the two tensors are not of the same rank i.e if $k_1 \neq k_2$. In fact, the contraction of two indices belonging to the same tensor will vanish because our tensors are traceless. Thus, we only consider the case where $k_1 = k_2 = k$. In this case we have

$$\begin{aligned}
\int_{S^5} d\Omega_5 \; Y^{I_1,k} Y^{I_2,k} &= C^{I_1}_{i_1\cdots i_k} C^{I_2}_{j_1\cdots j_k} \int_{S^5} d\Omega_5 \; x^{i_1}\cdots x^{i_k} x^{j_1}\cdots x^{j_k} \\
&= C^{I_1}_{i_1\cdots i_k} C^{I_2}_{j_1\cdots j_k} \frac{1}{2^{k-1}(k+2)!}\Omega_5 \big[\text{all possible contractions of } i_1,\cdots,i_k,j_1,\cdots,j_k\big] \\
&= C^{I_1}_{i_1\cdots i_k} C^{I_2}_{j_1\cdots j_k} \frac{1}{2^{k-1}(k+2)!}\Omega_5 \big[\text{all possible contractions of } i_1,\cdots,i_k \text{ with } j_1,\cdots,j_k\big] \\
&= C^{I_1}_{i_1\cdots i_k} C^{I_2}_{1_1\cdots i_k} \frac{1}{2^{k-1}(k+2)!} k!\Omega_5 \\
\int_{S^5} d\Omega_5 \; Y^{I_1,k} Y^{I_2,k} &= \left\langle C^{I_1} C^{I_2} \right\rangle \frac{1}{2^{k-1}(k+1)(k+2)}\Omega_5,
\end{aligned} \tag{B.4.5}$$

where we have used the fact that the tensors C^{I_1} and C^{I_2} are symmetric and traceless. We have introduced the notation

$$\left\langle C^{I_1} C^{I_2} \right\rangle = C^{I_1}_{i_1\cdots i_k} C^{I_2}_{i_1\cdots i_k}. \tag{B.4.6}$$

In the above expressions, there are sums over $i_1,\cdots i_k$ from 1 to 6.

The integral of three spherical harmonics Y^{I_1,k_1}, Y^{I_2,k_2} and Y^{I_3,k_3} is performed in a similar way. We find

$$\int_{S^5} d\Omega_5 \; Y^{I_1,k_1} Y^{I_2,k_2} Y^{I_3,k_3} = C^{I_1}_{i_1\cdots i_{k_2}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \int_{S^5} d\Omega_5 \; x^{i_1}\cdots x^{i_{k_1}} x^{j_1}\cdots x^{j_{k_2}} x^{l_1}\cdots x^{l_{k_3}}. \tag{B.4.7}$$

Use (B.3.2) with m defined by $\Sigma = k_1 + k_2 + k_3 = 2m$ we obtain

$$\begin{aligned}
&\int_{S^5} d\Omega_5 \; Y^{I_1,k_1} Y^{I_2,k_2} Y^{I_3,k_3} \\
&= C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \frac{1}{2^{\frac{1}{2}\Sigma-1}\left(\frac{1}{2}\Sigma+2\right)!} \big[\text{all possible constractions of } i_1\cdots i_{k_2} j_1\cdots j_{k_2} j_l\cdots l_{k_3}\big].
\end{aligned} \tag{B.4.8}$$

Making use of Appendix A we have

$$
\begin{aligned}
&C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \big[\text{all possible constractions of } i_1\cdots i_{k_2} j_1\cdots j_{k_2} j_l\cdots l_{k_3}\big] \\
&= C^{I_1}_{i_1\cdots i_{k_1}} C^{I_2}_{j_1\cdots j_{k_2}} C^{I_3}_{l_1\cdots l_{k_3}} \big[\text{all possible ways by contracting } \alpha_1 \text{ indices between } C^{I_2} \text{ and } C^{I_3}, \\
&\quad \alpha_2 \text{ indices between } C^{I_3} \text{ and } C^{I_1},\ \alpha_3 \text{ indices between } C^{I_1} \text{ and } C^{I_2}\big] \\
&= \big\langle C^{I_1} C^{I_1} C^{I_1} \big\rangle \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!}.
\end{aligned}
\tag{B.4.9}
$$

Therefore, the integral of three spherical harmonics is given by

$$
\int_{S^5} d\Omega_5 \, Y^{I_1,k_1} Y^{I_2,k_2} Y^{I_3,k_3} = \big\langle C^{I_1} C^{I_1} C^{I_1} \big\rangle \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!} \frac{1}{2^{\frac{1}{2}\Sigma - 1} \left(\frac{1}{2}\Sigma + 2\right)!} \Omega_5. \tag{B.4.10}
$$

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