# Probing Space-Time Geometry Using Young diagrams

Lwazi Nkumane

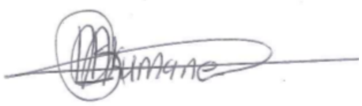
A thesis submitted to the University of the Witwatersrand, Faculty of Science in fulfillment of the academic requirements of the degree of Doctoral of Philosophy.

UNIVERSITY OF THE WITWATERSRAND
SCIENTIA ET LABORE
JOHANNESBURG

Johannesburg, 2018

## Declaration

I declare that the work contained in this thesis is my own work. Any work done previously by others or by myself has been acknowledged and put to references. This thesis is being submitted for the degree of Doctoral of Philosophy in the University of the Witwatersrand, Johannesburg and it has not been submitted before in any other tertiary institution.



Date: April, 2018

**Abstract**

Quantum field theories and theories of gravity play an essential role in understanding nature. A dramatic recent development has been the discovery that quantum field theories are equivalent or dual to theories of quantum gravity on negatively curved spacetime. This duality goes under the name of the AdS/CFT correspondence. Sometimes the computation of certain observables in field theory are more difficult than the computation of the same observables in the theory of gravity and the opposite is also true. This makes the correspondence a powerful tool, that might provide an approach to strong coupling dynamics.

We explore the AdS/CFT correspondence between type $IIB$ string theory on asymptotically $AdS_5 \times S^5$ backgrounds as our gravity theory and $\mathcal{N} = 4$ super Yang-Mills as our conformal field theory. We study BPS operators with bare dimension of order $N^2$ in the field theory and identify them with BPS geometries on the gravity side of the correspondence. The dynamics of 1/2 BPS geometries are identified with gauge invariant operators constructed using a single field in the field theory, while the dynamics of 1/4 BPS geometries are identified with gauge invariant operators constructed using two fields. We find a sector of the two matrix model defined by the $SU(2)$ sector in the field theory, that can be reduced to eigenvalue dynamics. The BPS operators in this sector are associated to solutions on the gravity side of the correspondence. We also identify the gauge invariant operators with bare dimension of order $N$, constructed using three fields, with 1/8 BPS giant graviton states. We count these gauge invariant operators constructed using three fields in the field theory and show that the counting of these operators is in agreement with the number of giant graviton states. We also demonstrate a correspondence between correlation functions of the field theory and the overlaps of the giant graviton wave functions.

By working in terms of the eigenvalues we have managed to go from the matrix, which contains $O(N^2)$ degrees of freedom, to the eigenvalues which are $O(N)$ degrees of freedom. Thus our work points to a significant simplification of the dynamics, something that deserves to be understood better. Another concrete result that we have achieved, is a proposal for some of the operators that are dual to the 1/4 BPS geometries. This is a genuine two matrix problem so it represents a novel extension of the understanding achieved by LLM of the 1/2 BPS geometries, constructed using a single matrix. The observables dual to new geometries have a bare dimension of $O(N^2)$. We have also considered operators with a bare dimension $O(N)$, which are dual to 1/4 BPS giant gravitons. In this case too, we demonstrate that the eigenvalue description is useful.

Almost all of the studies of the large $N$ limit of CFT have focused on the planar limit. Here, since the operator dimensions scale as we take $N \to \infty$, we are considering large $N$ but non-planar limits of the CFT. In these limits non-planar diagrams are not suppressed and the problem is considerably more difficult. The fact that we are able to explore this limit is concrete evidence for the power of the eigenvalue description and it suggests that a systematic treatment of large $N$ but non-planar limits is possible.

## Acknowledgements

I would like to thank my supervisor Professor Robert de Mello Koch for providing me with guidance and support during the course of this PhD. In addition to this, I would also like to thank him for dedicating his time to give me extra lectures. Furthermore, I would like to thank my family and friends for their support they gave me and I would also like to express my gratitude to my fellow colleague who contributed to my understanding of this field and not forgetting the following people who took their time to discussing and answering some of my questions: Mbavhalelo Mulokwe, James Mashiyane. It has been the grace of God that I made it this far and I am humble by that. This work was funded by National Research Foundation (NRF) through the South African Nuclear Human Asset and Research programme (SANHARP).

# Contents

# List of Figures

# Chapter 1

# Introduction

The AdS/CFT correspondence is a conjecture that was put forward by Maldacena in 1997 [1]. The conjecture asserts the existence of dualities between gauge theories at large $N$ and quantum gravity on asymptotically AdS spacetimes [1, 2, 3]. AdS stands for anti-de Sitter spacetime which is a spacetime that has a constant negative curvature, while CFT stands for conformal field theory. A conformal field theory is a quantum field theory that is invariant under conformal transformations. The CFT lives on the boundary of the AdS spacetime. The conjecture implies that if the quantum gravity theory lives in $d$ dimensions then the CFT side must live in $d-1$ dimensions. The great significance of this duality is that it allows us to study quantum gravity by considering quantum field theory. This is a significant new insight into the theory of quantum gravity. Indeed, our point of view is that the AdS/CFT correspondence provides a definition of quantum gravity.

The duality maps a formidable problem into a tractable problem. A common example where this duality is used is when one maps a theory which is strongly coupled (where calculations are hard to perform) to a weakly coupled theory (where weak coupling methods can be used). A typical example of this duality, and the setting that is relevant for this PhD, is the duality between $\mathcal{N}=4$ super Yang-Mills theory and type $IIB$ string theory on asymptotically $AdS_5 \times S^5$ spacetimes. For this example the field theory coupling $\lambda$ is related to the radius of curvature $R$ of the AdS space and the string length $l_s$ according to

$$\frac{R^4}{l_s^4} = \lambda.$$

When $\lambda$ is large the field theory is strongly coupled and perturbation theory can't be trusted. This is also the regime in which the curvature of the AdS space can be ignored and hence the string theory simplifies. When $\lambda$ is small the field theory is weakly coupled and perturbation theory can be used, but string curvature corrections can't be neglected, so that the string theory dynamics become extremely complicated.

We consider $\mathcal{N}=4$ super Yang-Mills theory on $S^3 \times R$. This theory has six spin-0 scalar fields $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$, one spin-1 gauge field and four spin-1/2 fermionic fields. We consider complex combinations of the scalar fields as follows

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6. \tag{1.1}$$

Operators of the $\mathcal{N}=4$ super Yang-Mills theory that are built from a single complex field $Z = \phi_1 + i\phi_2$, will enjoy an $SO(4)$ symmetry. This $SO(4)$ symmetry comes from the fact that the operator is independent of the scalar fields $\phi_3, \phi_4, \phi_5, \phi_6$ and hence invariant under $SO(4)$ rotations mixing them. Operators that are built from two complex fields $Z = \phi_1 + i\phi_2$ and $Y = \phi_3 + i\phi_4$ will have a $U(1)$ symmetry. This $U(1)$ symmetry comes from the fact that we are not using the field $X = \phi_5 + i\phi_6$ so there is an invariance under $U(1)$ transformations mixing $\phi_5, \phi_6$. Theories that are built from all three complex fields in the $\mathcal{N}=4$ super Yang-Mills theory do not have any of the symmetries just discussed because now all the scalar fields appear in our operator. On the gravity side of this duality we will consider geometries that are dual to states of a large dimension. When we talk about a "large dimension" we mean the classical dimension of the operator scales with a power of $N$ in the large $N$ limit.

As a consequence of the fact that two theories are equivalent, both must yield the same predictions for the values of all observables. The observables of CFT include the scaling dimension $\Delta$ which explicitly appears in the 2-point function, as follows

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}C}{|x_1-x_2|^{2\Delta_1}}, \tag{1.2}$$

where $C$ is a spacetime independent constant that is not observable and can be absorbed into the normalization of our fields. (1.2) is only true for 2-point functions of fields with a good scaling dimension. If two operators have different scaling dimensions their 2-point function vanishes. The AdS/CFT correspondence identifies the spectrum of anomalous dimensions of the CFT with the energy spectrum of the string theory. Studying the energy spectrum will enable us to explore the physics of spacetime excitations. Indeed, the energy spectrum of the hydrogen atom played an important role in the development of quantum mechanics, while the value of the Lamb shift played an important role in the development of quantum electrodynamics. Based on this experience, we expect the energy spectrum of spacetime excitations in quantum gravity will play a role in the development of the theory of quantum gravity.

One of the questions we aim to address in this PhD, concerns identifying the gravitational dual of specific BPS operators in the CFT. BPS operators are operators that are annihilated by some of the supercharges $Q^I_\alpha$ i.e.

$$[Q^I_\alpha, O_{BPS}] = 0.$$

To establish the equivalence of a CFT operator with a given gravitational dual, we need to compare computations performed in the CFT with computations performed in quantum gravity. This is a difficult task because, as we have discussed above, weak coupling gravity calculations must be compared to strongly coupled CFT. The reason we focus on BPS operators is that they enjoy certain non-renormalization theorems, which imply the computation can be performed at weak coupling and then extrapolated to strong coupling.

Correlation functions in quantum field theory are evaluated by summing Feynman diagrams. $\mathcal{N}=4$ super Yang-Mills theory is an example of a matrix model. The Feynman propagator for a matrix is a double line (to keep track of the row and column index) called a ribbon graph [4]. Thus computing correlation functions is the same as summing ribbon diagrams. Ribbon graphs are drawn on a 2-dimensional surface and they can be classified as either planar or non-planar diagrams. Planar diagrams are diagrams whose constituent ribbons do not cross when the diagram is drawn on a plane and non-planar diagrams are diagrams whose constituent ribbons do cross when drawn on the plane. Given a correlation function obtained as the sum of all the ribbon diagrams, there is a systematic approximation with leading term given by summing the planar diagrams in this pool of diagrams. This is called the planar limit. For the class of operators with a dimension that is held fixed as we take $N\to\infty$, the planar approximation gives an accurate approximation of the correlator. In this case the expansion in $1/N^2$ in the matrix model corresponds to the expansion in $\hbar$ in string theory. The planar diagrams in the matrix model correspond to tree level diagrams in string theory. In this case non-planar diagrams compute quantum corrections to the classical string theory.

If we consider operators with a dimension that grows parametrically with $N$, as $N^\alpha$ with $\alpha > \frac{1}{2}$, the planar approximation is no longer accurate [5]. In this case the gravity dual is no longer a weakly coupled string theory. For operators with a dimension of order $N$ the dual description is in terms of a quantum membrane, while for operators with a dimension of order $N^2$ the dual description involves a non-trivial deformation of the $AdS_5\times S^5$ spacetime. In this PhD thesis we focus on operators with a bare dimension of $O(N)$ or $O(N^2)$.

The computation of the anomalous dimensions in the planar limit can be carried out to all orders in $\hbar$ in the CFT. This is possible thanks to integrability. We say that a system is integrable when the number of conservation laws constraining the dynamics of the system is equal to the number of degrees of freedom of the system. In the case of the planar limit of $\mathcal{N} = 4$ SYM, integrability is demonstrated by showing that the dilatation operator is equal to the Hamiltonian of an integrable spin chain [6]. The dual string theory is also integrable at the classical level [7]. It is possible to argue, using integrability, that the planar limit of $\mathcal{N} = 4$ SYM is in exact agreement with classical string theory. Away from the planar limit, one does not in general expect integrability and further investigation of this case is needed. This is one of the motivations for the study presented in this thesis.

Within the framework of quantum field theory, the fundamental forces arise from the exchange of particles. In Einstein's non-linear theory of gravity, general relativity, the force of gravity is replaced by a curving of the spacetime geometry. Upon quantization we again expect particles to mediate the gravitational force. The fundamental particles that mediate the force of gravity are called gravitons. Gravitons are massless bosons that have spin two. In this thesis we will consider gravitons orbiting along a circle in $S^5$ which forms part of the $AdS_5 \times S^5$ background. In string theory, the graviton is only point like at low energy [12]. As the momentum of a graviton increases the graviton expands producing a sphere orbiting on the $S^5$. For this reason we talk about giant gravitons. Giant gravitons are described as branes in the internal sphere i.e in the $S^5$. We also have dual giant gravitons that are branes expanded in the $AdS_5$. The dual giant gravitons will play an essential role when we count 1/8 BPS operators in $\mathcal{N} = 4$ SYM.

The center of focus of this thesis will be to study spacetime geometries of type $IIB$ string theory on asymptotically $AdS_5 \times S^5$ backgrounds using BPS operators from the dual $\mathcal{N} = 4$ SYM. One of the questions we ask is how to identify operators that are dual to a specific spacetime geometry? We know that giant gravitons are constructed from operators with bare a dimension of $O(N)$. If we stack many of these giant gravitons together they back react on the geometry i.e they deform the original spacetime and this results in new geometries. Based on this observation we expect that by considering gauge invariant operators with a bare dimension of order $N^2$ in the dual theory we are studying new spacetime geometries.

The first set of operators we will use to study spacetime geometries are the 1/2 BPS operators. It has been shown that the 1/2 BPS sector has a field theory description in terms of free fermions [8, 9]. The goal is to study smooth geometries that correspond to these free fermion fields. For both sides of the duality we consider operators that are gauge invariant (i.e. operators that involve traces of the fields like $Tr(Z^n), Tr(Z)^n, etc$). States of $\mathcal{N} = 4$ SYM that are built from a single complex field have an $SO(4)$ symmetry. The 1/2 BPS operators are built using a single complex field and have a scaling dimension that is given by $\Delta = J_1$ where $J_1$ is a $U(1)$ charge in the $R$-symmetry group. The energy of the corresponding state in the dual gravity theory is $J_1$.

In the semi-classical limit, the 1/2 BPS states of supergravity are represented by droplets on a 2-dimensional plane embedded in the original 10-dimensional geometry that can be identified with states in the phase space of the fermions [10]. These 1/2 BPS geometries are dual to operators that are built from a single complex field $Z$ and so we can infer that the geometry has an isometry of $\mathbb{R} \times SO(4) \times SO(4)$. By increasing the energy of the graviton which corresponds to an increase of momentum ($E = J_1$), new spacetime excitations arise depending on how $J_1$ scales with $N$. Here are some examples of objects that arise as we increase $J_1$

i. For $J_1 \sim O(1)$, the single trace BPS states correspond to a point-like graviton.

ii. For $J_1 \sim O(\sqrt{N})$, the single trace BPS states correspond to a string.

iii. For $J_1 \sim O(N)$, the BPS states can be identified with giant gravitons.

iv. For $J_1 \sim O(N^2)$, the BPS states correspond to new spacetime geometries.

The configurations that are dual to arbitrary droplets of this phase space correspond to smooth geometries. We have two types of boundary conditions on this 2-dimensional plane in type $IIB$ string theory. The boundary conditions specify the geometries uniquely. The isometry of the 1/2 BPS geometry is $\mathbb{R} \times SO(4) \times SO(4)$, which implies that the metric will take the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{H+G} d\Omega_3^2 + e^{H-G} d\tilde{\Omega}_3^2.$$

This isometry, implies that the geometry has two 3-spheres. These spheres are represented by the $d\Omega_3^2$ and $d\tilde{\Omega}_3^2$ terms in the metric. By letting the radius of the first sphere and the second sphere be $r_1$ and $r_2$ respectively, we define the product of these radii by $y = e^H$ (i.e. $y = r_1 r_2$). We know the geometry will be smooth/regular if $y \neq 0$, but we are interested in studying geometries that include $y = 0$. We need to ask if it is possible to find regular solutions when $y = 0$? This question must be asked since we know that when one of the spheres has vanishing radius, the metric that describes the geometry might diverge (have a singularity). It turns out that it is possible to find regular (non-singular) solutions [10]. This is only true if we assign special boundary conditions on the two dimensional plane defined by $y = 0$. As we will discuss below, the metric is determined by a single function $z$. The boundary condition which ensures a regular geometry requires that the function $z$ takes the values $z = \pm\frac{1}{2}$ on the plane. This suggested that we identify the region $z = \frac{1}{2}$ as **occupied states** in the phase space and the region $z = -\frac{1}{2}$ as **unoccupied states** of the phase space in the free fermion description.

The second set of operators that we will use to study spacetime geometries are 1/4 BPS operators. 1/4 BPS operators are built from $Z$ and $Y$ fields and they have an isometry of $\mathbb{R} \times SO(4) \times U(1)$. The $\mathcal{N} = 4$ SYM operators that are built from the two complex fields $Z$ and $Y$ enjoy an $U(1)$ R-symmetry. The 1/4 BPS states have a scaling dimension which is given by $\Delta = J_1 + J_2$ where $J_1$ is the number of $Z$ fields in the operator and $J_2$ is the number of $Y$ fields in the operator. We consider BPS operators of the $SU(2)$ sector that include traces of products of both $Z$ and $Y$ matrices, which are genuine multi matrix observables. By performing explicit computations of correlation functions, we find evidence that there is a sector of the two matrix model defined by the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory, that can be reduced to eigenvalue dynamics. There is an interesting generalization of the usual Van der Monde determinant that plays a role. It is given by $\Delta(z, y) = \prod_{j>k}^{N} (z_j y_k - y_j z_k)$. The correlators in this sector that are computed using eigenvalue dynamics correctly reproduce the same answers as the correlators that are computed using the matrix model, provided that we consider the total number of complex fields (the sum of $Z$ and $Y$ fields) to be greater than or equal to $N$. It would be nice to give a first principles derivation of the generalized Van der monder determinant, which plays the same role as the Jacobian of the single matrix model when changing from the matrix to eigenvalue dynamics. This will open doors in the study of the $SU(3)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory. This problem can then be reduced eigenvalue dynamics which is a significant simplification.

The last set of operators we will consider are 1/8 BPS operators. The 1/8 BPS operators are constructed from $Z, Y$ and $X$ complex fields and they have an isometry of $\mathbb{R} \times SO(4)$. The 1/8 BPS operators are associated with a scaling dimension $\Delta = J_1 + J_2 + J_3$. In summary, the BPS geometries that we are interested in enjoy the following properties

| SUSY | QFT operators | Isometry |
|---|---|---|
| 1/2 BPS | $\Delta = J_1$ | $SO(4) \times SO(4) \times \mathbb{R}$ |
| 1/4 BPS | $\Delta = J_1 + J_2$ | $SO(4) \times SO(2) \times \mathbb{R}$ |
| 1/8 BPS | $\Delta = J_1 + J_2 + J_3$ | $SO(4) \times \mathbb{R}$ |

Table 1.1: Isometries of the BPS geometries dual to CFT operators.

Another type of comparison that one can use to check if two different theories are equivalent is to count the number of physical states (gauge invariant operators) that are dual to observables in the dual gravity theory and confirm if the counting matches. Focusing on the 1/8 BPS giant graviton states in type $IIB$ string theory on the $AdS_5 \times S^5$ background, the Hamiltonian of the system corresponds to the Hamiltonian of a 3-dimensional simple harmonic oscillator [64]. We represent the eigenstates of the single particle Hilbert space of this simple harmonic oscillator by

$$|n_1, n_2, n_3\rangle = \prod_{i=1,2,3} \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i}} |0\rangle. \tag{1.3}$$

Considering any number of particles preserves 1/8 of the supersymmetry. This system has a total energy of $E = J_1 + J_2 + J_3$. We can have more than one dual-giant with exactly the same quantum numbers. It turn out that we must treat the dual-giants as bosonic particles. The maximum number of dual-giants is $N$ and the total angular momenta is given by the sum of the individual dual-giant angular momenta as

$$J_i = \sum_{k=1}^{N} J_i^{(k)}. \tag{1.4}$$

The partition function that represents this system of dual-giant gravitons is then given by $N$ bosons in a 3-dimensional simple harmonic oscillator. Including the states with $J_i = 0$, allows us to count all states with a total of $N$ bosons. The counting of these dual-giants is done by first computing the grand partition function of this system. We will approach the study of the 1/8 BPS sector by identifying the operators in $\mathcal{N} = 4$ super Yang-Mills theory that are 1/8-BPS. We argue that the operators we construct are indeed dual to 1/8 BPS giant gravitons. Our evidence for the identification will come from counting these operators and showing agreement with independent counts of the number of giant graviton states, and by demonstrating a correspondence between correlation functions of the super Yang-Mills operators and overlaps of the giant graviton wave functions.

In conclusion, the central theme of this thesis is to use AdS/CFT to study spacetime geometries. The primary objective is to study BPS operators that are built from the complexified scalar fields of the $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times R$ and identify them with BPS geometries or non-perturbative states (giant gravitons) of type $IIB$ string theory on asymptotically $AdS_5 \times S^5$ spacetimes. If this goal is achieved it may allow a study of other outstanding problems, such as problems that involve black holes. Background relevant for the 1/2 and 1/4 BPS geometries will be covered in chapter 2 and the technical details that supplement this chapter are collected in appendix A. In chapter 3 we discuss the field theory relevant to this thesis. This includes the Schur polynomials, restricted Schur polynomials, state/operator correspondence of the Schur's and the non-interacting fermion wave functions and Gauss graphs. Chapter 4 covers the eigenvalue dynamics for multimatrix models and this is based on the paper [79] that was published during the course of my PhD. Chapter 5 covers the counting of Gauss graphs and this is based on the paper [80] that was recently published. In chapter 6, we present discussion and conclusions.

# Chapter 2

# BPS Geometries

## 2.1 1/2 BPS geometries.

In field theory, the $\mathcal{N} = 4$ SYM operators that are built from a single complex field $Z$ correspond to 1/2 BPS states. Since $\mathcal{N} = 4$ SYM is an example of a matrix model, the 1/ 2 BPS states are described by a one matrix model. As is well known, the dynamics of local gauge invariant operators in a one matrix model can be reduced to eigenvalue dynamics [15]. The eigenvalue dynamics can be mapped to the dynamics of free fermions. One fermion for each eigenvalue. The phase space of these non-interacting fermions is apparent in the dual supergravity description of the geometry as we now explain. The isometry of the 1/2 BPS geometries is $\mathbb{R}\times SO(4)\times SO(4)$. The $SO(4)\times SO(4)$ symmetry corresponds to the geometry of two 3-spheres. A very powerfull approach towards determining the 1/2 BPS supergravity solutions uses the Killing spinor equation. The Killing spinor equations are the equations that express the conditions for unbroken supersymmetry. Unlike the equations of motion which are second order equations, the Killing spinor equations are first order equations so they are much easier to work with. An interesting feature of the solutions obtained using the Killing spinor equation, is that there is a two dimensional plane on which one or both of the three spheres in the geometry shrink to zero size. To obtain a regular geometry, the boundary conditions must be chosen delicately: the plane is divided into black and white regions depending on which one of the spheres shrinks to zero size. This boundary condition can ultimately be identified with a state in the phase space of the fermions of the dual matrix model. A basis for the 1/2 BPS operators is provided by the Schur polynomials. In terms of the eigenvalues, the Schur polynomials define free fermion energy eigenfunctions. This allows us to identify each Schur polynomial with a phase space configuration and hence with a specific supergravity geometry. We will explore the link between 1/2 BPS geometries and the Schur polynomial in this chapter. We start by constructing the Killing spinor equation, and then discuss its solutions. We do this for both 1/2 BPS and 1/4 BPS geometries.

## 2.2 LLM solution

To obtain 1/2 BPS supergravity solutions we need to specify which supergravity fields are non-zero. The fields which participate, are the 10-dimensional metric, the frame field and the field strength. We want the variations of the boson and fermion fields to vanish under a supersymmetric transformation

$$\delta(\text{boson}) = 0, \qquad\qquad \delta(\text{fermion}) = 0.$$

This is what we mean by a BPS solution - the solution is invariant under a subset of the possible supersymmetry transformations. As we have explained, we look for solutions with only the metric and the five form field strength excited. Given the $\mathbb{R} \times SO(4) \times SO(4)$ isometry of the solution, we expect

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H_1}d\Omega_3^2 + e^{H}d\Omega_3^2 \\ F_{(5)} &= F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3, \end{aligned}$$

where $\mu$, $\nu$ = 0,1,2,3 and $H$ and $H_1$ are functions that depends on the coordinates of the metric $g_{\mu\nu}$. We now write the Killing spinor equation in the form

$$\nabla_M \eta + \frac{i}{480} \Gamma^{M_1 M_2 M_3 M_4 M_5} F^{(5)}_{M_1 M_2 M_3 M_4 M_5} \Gamma_M \eta = 0, \tag{2.1}$$

where indices range over $0, 1, \cdots 9$. Our goal is to review the solution of this Killing spinor equation [10].

### 2.2.1 Solution to the Killing spinor equation

We want to solve (2.1) which is the Killing spinor equation. This equation contains the covariant derivative $\nabla_M$ and the 10-dimensional gamma matrices $\Gamma^{M_1 M_2 M_3 M_4 M_5}$. In order to solve (2.1) we will have to explicitly work out the quantities that appear in the Killing spinor equation. We will break the 10-dimensional covariant derivative into three pieces. Two of the pieces will correspond to the covariant derivative of the two 3-spheres and the third piece will correspond to a 4-dimensional covariant derivative. We will simplify these covariant derivatives as much as we can so that we end up with a simple expression to evaluate. We will also simplify the 10-dimensional gamma matrices by breaking them into three pieces where two of these pieces will correspond to the two 3-spheres and the third piece will be the normal 4-dimensional gamma matrices denoted as $\gamma_\mu$. We will start with the analysis of the gamma matrices. We choose the basis of the gamma matrices to be

$$\gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3, \quad \Gamma_\mu = \gamma_\mu \otimes 1 \otimes 1 \otimes 1 \tag{2.2}$$

$$\Gamma_a = \gamma_5 \otimes \sigma_a \otimes 1 \otimes \hat{\sigma}_1, \quad \Gamma_{\tilde{a}} = \gamma_5 \otimes 1 \otimes \tilde{\sigma}_a \otimes \hat{\sigma}_2 \tag{2.3}$$

where $\mu = 0, \cdots, 3$ and the $\sigma$'s are the ordinary Pauli matrices. From this basis of the gamma matrices we define chirality matrices as follows

$$\Gamma^{(3)} = -\frac{i}{3!}\epsilon_{abc}\Gamma^{abc} = \gamma_5 \otimes 1 \otimes 1 \otimes \hat{\sigma}_1, \tag{2.4}$$

$$\Gamma^{(\tilde{3})} = -\frac{i}{3!}\epsilon_{\tilde{a}\tilde{b}\tilde{c}}\Gamma^{\tilde{a}\tilde{b}\tilde{c}} = \gamma_5 \otimes 1 \otimes 1 \otimes \hat{\sigma}_2, \tag{2.5}$$

$$\Gamma_{11} = \frac{1}{10!}\epsilon_{M_1 \cdots M_{10}}\Gamma^{M_1 \cdots M_{10}}, \tag{2.6}$$

$$\Gamma_{11} = \Gamma_0 \cdots \Gamma_3 \prod \Gamma_a \prod \Gamma_{\tilde{a}} = \gamma^5 \hat{\sigma}^3 \tag{2.7}$$

where $\Gamma$'s are the 10 dimensional gamma matrices. In the following, capital Latin indices run over $0, 1, \cdots, 9$, little Latin letters run over one sphere and the tilded little Latin letters run over the other sphere. Finaly, $\mu, \nu$ run over $0, 1, 2, 3$. The chirality of the spinor $\eta$ is given by

$$\Gamma_{11}\eta = \gamma^5 \hat{\sigma}_3 \eta = \eta. \tag{2.8}$$

The above discussion specifies how we construct the 10-dimensional gamma matrices. We will move on to the analysis of the covariant derivative. Consider the spinor $\chi$ on the unit radius sphere, that obeys the equation

$$\nabla_c \chi_a = a\frac{i}{2}\gamma_c \chi_a, \quad a = \pm 1. \tag{2.9}$$

Here the index $a$ on the spinor takes the value $\pm 1$ and it should not be confused with the spacetime index $a$ that runs over a sphere. The correct interpretation of the index should be clear from the context. Recall that when the covariant derivative acts on $V_\nu{}^a$ (see appendix A) it gives

$$\nabla_\mu V_\nu{}^a = \partial_\mu V_\nu{}^a + \omega_\mu{}^a{}_b V_\nu{}^b - \Gamma^\sigma{}_{\mu\nu} V_\sigma{}^a$$

where $\omega_\mu{}^a{}_b$ is the spin connection, written in terms of the vierbein (i.e. local frame) components and $\Gamma^\sigma_{\nu\mu}$ is the affine connection. We will now simplify the expression for the affine connection that determines the covariant derivative on the 4-dimensional spacetime and on the two 3-spheres. The affine connection of the 4-dimensional spacetime is

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}).$$

Since both 3-spheres are round, knowing the affine connection of one of the 3-spheres, we automatically know the affine connection of the other sphere. Consider the affine connection of the $e^{H+G}d\Omega_3^2$ term of the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\tilde{\Omega}_3^2, \tag{2.10}$$

where $H$ and $G$ are functions that depends on the direction of $x^\mu$. Decompose this affine connection into two parts as follows

$$\begin{aligned}\tilde{\Gamma}^{S}{}_{MN} &= \frac{1}{2}\tilde{g}^{SA}(\partial_M \tilde{g}_{AN} + \partial_N \tilde{g}_{AM} - \partial_A \tilde{g}_{MN}) \\ &= \frac{1}{4} g^{SA}\Big[\partial_M (H+G) g_{AN} + \partial_N (H+G) g_{AM} - \partial_A (H+G) g_{MN}\Big] + \Gamma^{S}{}_{MN}.\end{aligned}$$

$\tilde{\Gamma}^{S}{}_{MN}$ is the affine connection that appears when we take the covariant derivative on the 3-sphere $d\Omega_3$. Let $\nabla'_a$ denote the covariant derivative on the 3-sphere of unit radius. This covariant derivative contains the spin connection $\omega_\mu{}^a{}_b$

$$\nabla'_a V_\nu{}^a = \partial_a V_\nu{}^a + \omega_a{}^a{}_b V_\nu{}^b - \Gamma^\sigma_{a\nu} V_\sigma{}^a \tag{2.11}$$

and

$$\tilde{\Gamma}^{S}{}_{MN} = \bar{\Gamma}^{S}{}_{MN} + \Gamma^{S}{}_{MN}$$

where $\bar{\Gamma}^{S}{}_{MN}$ is given by

$$\bar{\Gamma}^{S}{}_{MN} = \frac{1}{4} g^{SA}\Big[\partial_M (H+G) g_{AN} + \partial_N (H+G) g_{AM} - \partial_A (H+G) g_{MN}\Big].$$

The second and the third terms cancel for all values of $M$. Consequently

$$\bar{\Gamma}^{S}{}_{MN} = \frac{1}{4} g^{SA}\Big[\partial_M (H+G) g_{AN}\Big].$$

We will make use of the following identities

$$\begin{aligned}\gamma_\nu \gamma^\nu &= D I_D, \\ \{\gamma^\mu, \gamma^\nu\} &= 2 g^{\mu\nu} I_D, \\ \gamma_\nu \gamma^\mu \gamma^\nu &= (2-D)\gamma^\mu, \\ \gamma_\mu &= g_{\mu\nu}\gamma^\nu.\end{aligned}$$

$D$ is the dimension of the manifold equipped with metric $g_{\mu\nu}$. For the 3-sphere $D = 3$. Notice that the covariant derivative defined in (2.11) acts on an object $V_\nu{}^a$, which can be thought of as a vector valued one form. When we factor the connection $\bar{\Gamma}^{S}{}_{MN}$ out on the left hand side, to obtain an expression for $\nabla'_a$, we will include a factor of $(I_D)^a{}_b = \delta^a{}_b$. A simple computation shows

$$\begin{aligned}\bar{\Gamma}^{S}{}_{MN} I_D &= \frac{1}{4} g^{SA}\Big[\partial_M (H+G)\Big] g_{AN} I_D \\ &= \frac{1}{16}\Big[4\gamma^S \gamma_N\Big]\Big[\partial_M (H+G)\Big] I_D.\end{aligned}$$

Since the notation of the gamma matrices $\gamma$ is reserved for 4 space-time dimensions, we denote the gamma matrices on the 3-sphere by $\Gamma$. Taking into consideration that the functions $H$ and $G$ depend on the coordinates $x^\mu$, the covariant derivatives of the two 3-spheres $a$ and $\tilde{a}$ are given by

$$\nabla_a = \nabla'_a - \frac{1}{4}\Gamma^\alpha\Gamma_a\partial_\alpha(H+G), \quad \nabla_{\tilde{a}} = \nabla'_{\tilde{a}} - \frac{1}{4}\Gamma^\alpha\Gamma_{\tilde{a}}\partial_\alpha(H-G). \tag{2.12}$$

Now that we have clarified the 10-dimensional gamma matrices and the covariant derivatives, the next natural thing to do is to consider the 10-dimensional spinor $\eta$. We decompose $\eta$ into a product of a 4-dimensional spinor $\epsilon$ and two 3-sphere spinors $\chi_a$ and $\chi_{\tilde{b}}$. This decomposition will allow each component of the decomposed covariant derivative to act on the rightful component of the spinor. The decomposition of $\eta$ is done as follows

$$\eta = \epsilon_{a,b} \otimes \chi_a \otimes \chi_{\tilde{b}} \tag{2.13}$$

where $\chi_a$ and $\chi_{\tilde{b}}$ satisfy (2.9) and $a, b = \pm 1$. This completes our discussion of the first term of (2.1). We will now consider the second term in (2.1). We denote this term by

$$M \equiv \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}. \tag{2.14}$$

The expression (2.14) contains the 10-dimensional gamma matrices and the five form field strength. We will now focus on how to decompose the five form field strength. In the expression of the five form field strength, we have wedge products of forms with the volume forms of the 3-spheres. Therefore, we will start the analysis of the five form field strength by considering the three form defined on the 3-sphere. The 3-sphere measure is given by

$$d\Omega_3 = \sin^2\theta_1 \sin\theta_2 d\theta_1 \wedge d\theta_2 \wedge d\theta_3.$$

From our metric we can define a new volume form for the 3-sphere $d\hat{\Omega}_3^{(3)}$ in terms of the old one $d\Omega_3^{(3)}$

$$d\hat{\Omega}_3^{(3)} = e^{\frac{1}{2}(H+G)}d\theta_1 \wedge e^{\frac{1}{2}(H+G)}\sin\theta_1 d\theta_2 \wedge e^{\frac{1}{2}(H+G)}\sin\theta_1\sin\theta_2 d\theta_3.$$

Therefore our new 3-sphere measures are

$$\begin{aligned} d\hat{\Omega}_3 &= e^{\frac{3}{2}(H+G)}d\Omega_3, \\ d\Omega_3 &= e^{-\frac{3}{2}(H+G)}d\hat{\Omega}_3. \end{aligned}$$

Using the same procedure, we find

$$d\tilde{\Omega}_3 = e^{-\frac{3}{2}(H-G)}d\hat{\tilde{\Omega}}_3.$$

The 5-form field strength can be expressed as

$$F_{(5)} = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge e^{-\frac{3}{2}(H+G)}d\hat{\Omega}_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge e^{-\frac{3}{2}(H-G)}d\hat{\tilde{\Omega}}_3.$$

Now, we can write (2.14) as

$$\begin{aligned} M &= \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5} \\ &= \frac{i}{480}\left(\Gamma^{\mu_1\nu_1\theta_1\theta_2\theta_3}F_{\mu_1\nu_1}e^{-\frac{3}{2}(H+G)}\epsilon_{\theta_1\theta_2\theta_3} + \Gamma^{\mu_1\nu_1\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_3}\tilde{F}_{\mu_1\nu_1}e^{-\frac{3}{2}(H-G)}\epsilon_{\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_3}\right). \end{aligned}$$

We can decompose $F^{(5)}$ into two components,

$$F^{(5)} = F_I^{(5)} - F_{II}^{(5)}.$$

$F_I^{(5)}$ and $F_{II}^{(5)}$ are non-zero on different spheres in the geometry. Consider the following manipulation of the first term (a similar analysis applies to the second term)

$$F_I^{(5)} = e^{-\frac{3}{2}(H+G)} F_{\mu_1\nu_1\theta_1\theta_2\theta_3}(dx^{\mu_1} \wedge dx^{\nu_1} \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3) \sin^2\theta_1 \sin\theta_2.$$

We define the volume form $dV = \sin^2\theta_1 \sin\theta_2 dx^{\mu_1} \wedge dx^{\nu_1} \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3$. Now, the expression for $M$ becomes

$$MdV = \frac{i}{480}\frac{5!}{3!2!}\left(e^{-\frac{3}{2}(H+G)}\Gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}\epsilon_{\theta_1\theta_2\theta_3}\Gamma^{\theta_1\theta_2\theta_3}dV - e^{-\frac{3}{2}(H-G)}\Gamma^{\mu_1\nu_1}\tilde{F}_{\mu_1\nu_1}\epsilon_{\tilde{\theta_1}\tilde{\theta_2}\tilde{\theta_3}}\Gamma^{\tilde{\theta_1}\tilde{\theta_2}\tilde{\theta_3}}dV\right). \tag{2.15}$$

Simplifying the factor up front and dividing by $dV$, we find

$$M = \frac{i}{48}\left(e^{-\frac{3}{2}(H+G)}\Gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}\epsilon_{\theta_1\theta_2\theta_3}\Gamma^{\theta_1\theta_2\theta_3} - e^{-\frac{3}{2}(H-G)}\Gamma^{\mu_1\nu_1}\tilde{F}_{\mu_1\nu_1}\epsilon_{\tilde{\theta_1}\tilde{\theta_2}\tilde{\theta_3}}\Gamma^{\tilde{\theta_1}\tilde{\theta_2}\tilde{\theta_3}}\right). \tag{2.16}$$

The two terms are related by the self-duality of the field strength as spelled out in (A.0.15). Further using (2.3), (2.4), this simplifies to

$$\begin{aligned}
M &= \frac{ie^{-\frac{3}{2}(H+G)}}{48}\left(2\Gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}\epsilon_{\theta_1\theta_2\theta_3}\Gamma^{\theta_1\theta_2\theta_3}\right) \\
&= \frac{ie^{-\frac{3}{2}(H+G)}}{48}\left(2\gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}(3!i)\Gamma^{(3)}\right) \\
&= \frac{-e^{-\frac{3}{2}(H+G)}}{4}\left(\gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}\Gamma^{(3)}\right) \\
&= \frac{-e^{-\frac{3}{2}(H+G)}}{4}\gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}\gamma_5 \otimes 1 \otimes 1 \otimes \hat{\sigma}_1 \\
&= \frac{-e^{-\frac{3}{2}(H+G)}}{4}\gamma^{\mu_1\nu_1}F_{\mu_1\nu_1}\gamma_5\hat{\sigma}_1.
\end{aligned}$$

This compact expression for $M$ will simplify our study of the Killing spinor equation. We will decompose (2.1) into a Killing spinor equation in 4-dimensions and Killing spinor equations for the two 3-spheres. The Killing spinor equation in 4-dimensions is given by

$$\nabla_\mu\epsilon + M\gamma_\mu\epsilon = 0 \tag{2.17}$$

where $\mu = 0, 1, 2, 3$. Recall that $\nabla_{S^3}$ acting on scalar functions $f(\theta_1, \theta_2, \theta_3)$ on the sphere gives

$$\nabla_{S^3} f = \frac{\partial f}{\partial\theta_1}\hat{\theta}_1 + \frac{1}{\sin\theta_1}\frac{\partial f}{\partial\theta_2}\hat{\theta}_2 + \frac{1}{\sin^2\theta_1 \sin\theta_2}\frac{\partial f}{\partial\theta_3}\hat{\theta}_3.$$

Using our metric we find that

$$\begin{aligned}
\nabla'_{S^3} f &= \frac{1}{e^{\frac{1}{2}(H+G)}}\frac{\partial f}{\partial\theta_1}\hat{\theta}_1 + \frac{1}{e^{\frac{1}{2}(H+G)}}\frac{1}{\sin\theta_1}\frac{\partial f}{\partial\theta_2}\hat{\theta}_2 + \frac{1}{e^{\frac{1}{2}(H+G)}}\frac{1}{\sin\theta_1^2 \sin\theta_2}\frac{\partial f}{\partial\theta_3}\hat{\theta}_3 \\
&= e^{-\frac{1}{2}(H+G)}\nabla_{S^3} f,
\end{aligned} \tag{2.18}$$

$$\tilde{\nabla}'_{\tilde{S}^3} f = e^{-\frac{1}{2}(H-G)}\tilde{\nabla}_{\tilde{S}^3} f. \tag{2.19}$$

Using (2.1), (2.12) and (2.9), the Killing spinor equations for the two 3-spheres are given by

$$\begin{aligned}
D_{S^3}\epsilon &= (iae^{-\frac{1}{2}(H+G)}\Gamma_{(3)} + \frac{1}{2}\gamma^\mu\partial_\mu(H+G))\epsilon + 2M\epsilon \\
&= (iae^{-\frac{1}{2}(H+G)}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\gamma^\mu\partial_\mu(H+G))\epsilon + 2M\epsilon = 0
\end{aligned} \tag{2.20}$$

and

$$D_{\tilde{S}^3}\epsilon = (ibe^{-\frac{1}{2}(H-G)}\Gamma_{(\tilde{3})} + \frac{1}{2}\gamma^\mu\partial_\mu(H-G))\epsilon - 2M\epsilon$$
$$= (ibe^{-\frac{1}{2}(H-G)}\gamma_5\hat{\sigma}_2 + \frac{1}{2}\gamma^\mu\partial_\mu(H-G))\epsilon - 2M\epsilon = 0. \quad (2.21)$$

We have managed to reduce the Killing spinor equation (2.1) to equations (2.17), (2.20) and (2.21). To extract useful information from these equations, we will make use of spinor bilinears.

### 2.2.2 Spinor bilinears

In order to make progress in solving equations (2.17), (2.20) and (2.21) we will introduce spinor bilinears that are constructed from the 4-dimensional spinor $\epsilon$. Spinor bilinears are functions that contain two spinors in each term. Here is a list of spinor bilinears that will be useful in our analysis

$$K_\mu = -\bar{\epsilon}\gamma_\mu\epsilon, \quad L_\mu = \bar{\epsilon}\gamma^5\gamma_\mu\epsilon, \quad \bar{\epsilon} = \epsilon^\dagger\Gamma^0$$
$$f_1 = i\bar{\epsilon}\hat{\sigma}_1\epsilon, \quad f_2 = i\bar{\epsilon}\hat{\sigma}_2\epsilon, \text{and} \quad Y_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\hat{\sigma}_1\epsilon. \quad (2.22)$$

Here are some useful identities that we will use below

$$[A,BC] = ABC - BCA$$
$$= ABC + BAC - BAC - BCA$$
$$= \{A,B\}C - B\{A,C\}, \quad (2.23)$$
$$\gamma_{\mu\nu} = \gamma_\mu\gamma_\nu \quad \text{If} \quad \mu \neq \nu, \quad (2.24)$$
$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad (2.25)$$
$$\gamma_{\mu\nu}\gamma_\lambda = 2g_{\lambda\nu}\gamma_\mu - 2g_{\lambda\mu}\gamma_\nu + \gamma_{\mu\nu\lambda}, \quad (2.26)$$
$$\gamma_\lambda\gamma_{\mu\nu} = -2g_{\lambda\nu}\gamma_\mu + 2g_{\lambda\mu}\gamma_\nu + \gamma_{\mu\nu\lambda}, \quad (2.27)$$
$$\gamma_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5, \quad (2.28)$$
$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \quad (2.29)$$
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \quad (2.30)$$
$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}. \quad (2.31)$$

We will evaluate derivatives of the spinor bilinears, something that will be used repeatedly when solving the Killing spinor equations. We start by taking the covariant derivative of $f_2$ and using (2.17) to get

$$\nabla_\mu f_2 = i(\nabla_\mu\bar{\epsilon})\hat{\sigma}_2\epsilon + i\bar{\epsilon}\hat{\sigma}_2(\nabla_\mu\epsilon)$$
$$= -iM\gamma_\mu\bar{\epsilon}\hat{\sigma}_2\epsilon - i\bar{\epsilon}\hat{\sigma}_2M\gamma_\mu\epsilon$$
$$= \frac{1}{4}e^{-\frac{3}{2}(H+G)}\bar{\epsilon}\bigg(\gamma_\mu\gamma_{\rho\lambda}\gamma_5\hat{\sigma_3} - \gamma_{\rho\lambda}\gamma_5\hat{\sigma_3}\gamma_\mu\bigg)\epsilon F^{\rho\lambda}$$
$$= e^{-\frac{3}{2}(H+G)}F_{\mu\sigma}K^\sigma. \quad (2.32)$$

Now consider $\nabla_\mu f_1$ which gives

$$\nabla_\mu f_1 = i(\nabla_\mu\bar{\epsilon})\hat{\sigma}_1\epsilon + i\bar{\epsilon}\hat{\sigma}_1(\nabla_\mu\epsilon)$$
$$= -iM\gamma_\mu\bar{\epsilon}\hat{\sigma}_1\epsilon - i\bar{\epsilon}\hat{\sigma}_1M\gamma_\mu\epsilon$$
$$= \frac{i}{4}e^{-\frac{3}{2}(H+G)}\bar{\epsilon}\bigg(\gamma_\mu\gamma_{\rho\lambda}F^{\rho\lambda}\gamma_5\hat{\sigma}_1\hat{\sigma}_1 + \gamma_{\rho\lambda}F^{\rho\lambda}\gamma_5\hat{\sigma}_1\hat{\sigma}_1\gamma_\mu\bigg)\epsilon$$
$$= \frac{i}{2}e^{-\frac{3}{2}(H+G)}\varepsilon_{\rho\lambda\mu\nu}K^\nu F^{\rho\lambda}. \quad (2.33)$$

For $\nabla_\nu K_\mu$ we find

$$\nabla_\nu K_\mu = -(\nabla_\nu\bar{\epsilon})\gamma_\mu\epsilon - i\bar{\epsilon}\gamma_\mu(\nabla_\nu\epsilon)$$
$$= M\gamma_\nu\bar{\epsilon}\gamma_\mu\epsilon + i\bar{\epsilon}\gamma_\mu M\gamma_\nu\epsilon.$$

Using the identities (2.23) - (2.31) we find

$$\nabla_\nu K_\mu = -e^{-\frac{3}{2}(H+G)}\bigg(F_{\mu\nu}f_2 - \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}f_1\bigg). \tag{2.34}$$

We also note that

$$\begin{aligned}
\nabla_\nu K_\mu + \nabla_\mu K_\nu &= -e^{-\frac{3}{2}(H+G)}\bigg(F_{\mu\nu}f_2 - \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}f_1\bigg) - e^{-\frac{3}{2}(H+G)}\bigg(F_{\nu\mu}f_2 - \frac{1}{2}\varepsilon_{\nu\mu\lambda\rho}F^{\lambda\rho}f_1\bigg)\\
&= -e^{-\frac{3}{2}(H+G)}\bigg(F_{\mu\nu}f_2 - \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}f_1\bigg) + e^{-\frac{3}{2}(H+G)}\bigg(F_{\mu\nu}f_2 - \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}f_1\bigg)\\
&= 0.
\end{aligned} \tag{2.35}$$

This result tells us that $K_\mu$ is a Killing vector. The computation of $\nabla_\nu L_\mu$ gives

$$\begin{aligned}
\nabla_\nu L_\mu &= (\nabla_\nu\bar{\epsilon})\gamma^5\gamma_\mu\epsilon + \bar{\epsilon}\gamma^5\gamma_\mu(\nabla_\nu\epsilon)\\
&= -M\gamma_\nu\bar{\epsilon}\gamma^5\gamma_\mu\epsilon - \bar{\epsilon}\gamma^5\gamma_\mu M\gamma_\nu\epsilon.
\end{aligned}$$

Again, using the identities (2.23) - (2.31), we obtain

$$\nabla_\nu L_\mu = e^{-\frac{3}{2}(H+G)}\bigg(\frac{1}{2}g_{\mu\nu}F_{\lambda\rho}Y^{\lambda\rho} - F_\mu{}^\rho Y_{\rho\nu} - F_\nu{}^\rho Y_{\rho\mu}\bigg). \tag{2.36}$$

Next, note that

$$\begin{aligned}
\nabla_\nu L_\mu - \nabla_\mu L_\nu &= e^{-\frac{3}{2}(H+G)}\bigg(\frac{1}{2}g_{\mu\nu}F_{\lambda\rho}Y^{\lambda\rho} - F_\mu{}^\rho Y_{\rho\nu} - F_\nu{}^\rho Y_{\rho\mu}\bigg) - e^{-\frac{3}{2}(H+G)}\bigg(\frac{1}{2}g_{\nu\mu}F_{\lambda\rho}Y^{\lambda\rho} - F_\nu{}^\rho Y_{\rho\mu} - F_\mu{}^\rho Y_{\rho\nu}\bigg).\\
&= 0.
\end{aligned} \tag{2.37}$$

This equation can be re-expressed as

$$\vec{\nabla}\times\vec{L} = 0.$$

Which tells us that

$$\int_{x_1}^{x_2}\vec{L}\cdot d\vec{x} = \text{ path independent.}$$

This implies that $dL = \vec{L}\cdot d\vec{x}$ is an exact differential. An important result that can be deduced from our analysis is

$$K\cdot L = 0. \tag{2.38}$$

Further, one can show, using (2.22) that

$$L^2 = -K^2. \tag{2.39}$$

Again, using (2.22), we can prove the relation

$$L^2 = -K^2 = f_1^2 + f_2^2. \tag{2.40}$$

An important spinor bilinear is the one-form

$$\omega_\mu = \epsilon^t\Gamma^2\gamma_\mu\epsilon \tag{2.41}$$

which is closed

$$d\omega = 0.$$

These relationships will prove useful when we determine the components of the 10-dimensional metric. Note that the ultimate goal behind solving the Killing spinor equation is to find the explicit form of this 10-dimensional metric.

### 2.2.3 Application of the spinor bilinears

We will now use the relations we have derived for spinor bilinears to determine the components of our 4-dimensional metric tensor. We will use these relations to solve for the spinor bilinears $f_1$ and $f_2$. Once we have these spinor bilinears we will be able to write down the components of the 10-dimensional metric tensor.

We start by recalling that $K^\mu$ is a Killing vector (it is actually a time-like Killing vector which corresponds to $\partial/\partial t$ ). Previously, we saw that the path integral of $L$ from $x_1$ to $x_2$ is path independent allowing us to conclude that $L$ represents a spatial coordinate. We choose this spatial coordinate to be the $y$ coordinate as follows

$$\Lambda dy = L_\mu dx^\mu, \quad \Lambda = \pm 1. \tag{2.42}$$

This choice is possible because the integral of $L_\mu dx^\mu$ between two points is path independent. Later we will identify $y$ as the product of the radii of $S^3$ and $\tilde{S}^3$. The sign of $\Lambda$ can be fixed later. Using the relation $K \cdot L = 0$, we can write

$$0 = K^\mu L_\mu = K^y L_y = \gamma K^y.$$

This tells us that the components of $K^\mu$ which are $K^\mu = (K^y, K^{x^1}, K^{x^2}, K^t)$ can be written as $K^\mu = (0, K^{x^1}, K^{x^2}, K^t)$. Taking into consideration that $K^\mu$ is a time-like Killing vector, we can set the other spatial components of $K^\mu$ to zero such that we end up with $K^\mu = (0, 0, 0, K^t)$. Now that we know $K^\mu$ has a non-zero component which is $K^t$, we choose this component to be $K^t = 1$. Thus the time-Killing vector is expressed as

$$\begin{aligned} K &= K^t \frac{\partial}{\partial x^t} \\ &= \frac{\partial}{\partial t}. \end{aligned}$$

We can now write the 4-dimensional metric in terms of these preferred coordinates as follows

$$ds^2 = h^2 dy^2 + \hat{g}_{\alpha\beta} dx^\alpha dx^\beta. \tag{2.43}$$

$x^\alpha$ contains two spatial dimensions and one time dimension. We can relate the $g_{tt}$ and the $g^{yy}$ coefficient of the metric, as we now explain. Note that

$$\begin{aligned} L^2 &= L_y^2 g^{yy} = g^{yy}, \\ K^2 &= (K^t)^2 g_{tt} = g_{tt}. \end{aligned}$$

Using equation (2.39), we find

$$g_{tt} = -g^{yy}. \tag{2.44}$$

Simplifying the metric we obtain

$$ds^2 = h^2(dy^2 + \tilde{h}_{ij} dx^i dx^j) - h^{-2}(dt + V_i dx^i)^2, \tag{2.45}$$

where $i, j = 1, 2$. We have pulled out the factor $h^2$ for later convenience.

#### Determining $f_1$ and $f_2$

$K^\mu$ has a component given by $K^t = 1$. Solving equation (2.32) we get

$$\begin{aligned} \nabla_\mu f_2 &= -e^{\frac{2}{3}(H+G)} F_{\mu\nu} K^\nu \\ &= -e^{\frac{2}{3}(H+G)} F_{\mu t} K^t \\ &= -e^{\frac{2}{3}(H+G)} \partial_\mu B_t \end{aligned}$$

where $\partial_\mu B_t$ can be expressed as

$$\partial_\mu B_t = F_{\mu\nu}K^\nu = -F_{\mu\nu}\bar{\epsilon}\gamma^\nu\epsilon = -\frac{1}{4}\bar{\epsilon}[\gamma_\mu, \not{F}]\epsilon.$$

We can write (2.20) and it adjoint as

$$\frac{1}{2}e^{-\frac{3}{2}(H+G)}\not{F}\epsilon = (iae^{\frac{1}{2}(H+G)} + \frac{1}{2}\gamma_5\not{\partial}(H+G)\hat{\sigma_1})\epsilon,$$
$$\frac{1}{2}e^{-\frac{3}{2}(H+G)}\bar{\epsilon}\not{F} = \bar{\epsilon}(iae^{\frac{1}{2}(H+G)} + \frac{1}{2}\gamma_5\not{\partial}(H+G)\hat{\sigma_1})$$

where $\not{F} = \gamma_{\lambda\sigma}F^{\lambda\sigma}$. We obtain $\partial_\mu B_t$ by manipulating the last three equations and using the properties of the Pauli matrices. The final expression is

$$\partial_\mu B_t = -e^{\frac{3}{2}(H+G)}\frac{1}{2}\partial_\mu(H+G)f_2. \tag{2.46}$$

Using (2.32) we find

$$\partial_\mu f_2 = \frac{1}{2}f_2\partial_\mu(H+G). \tag{2.47}$$

One can check that the solutions to these last two differential equations are

$$f_2 = 4\alpha e^{\frac{1}{2}(H+G)}, \quad B_t = -\alpha e^{2(H+G)} \tag{2.48}$$

where $\alpha$ is constant that can be obtained using suitable boundary conditions. We can also compute $f_1$ and $\tilde{B}_t$, following the same procedure. The results are the same as for $f_2$ and $B_t$ up to a minus sign in front of $G$ in the exponent. The expression of $f_1$ and $\tilde{B}_t$ are

$$f_1 = 4\beta e^{\frac{1}{2}(H-G)}, \quad \tilde{B}_t = -\beta e^{2(H-G)}. \tag{2.49}$$

By choosing an appropriate rescaling of the Killing spinor, we can set $4\beta = 1$. Using the properties of the Pauli matrices and equation (2.8), we re-write (2.20) and (2.21) as

$$\hat{\sigma_1}\not{\partial}H\epsilon = (-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)})\epsilon, \tag{2.50}$$
$$\bar{\epsilon}\hat{\sigma_1}\not{\partial}H = \bar{\epsilon}(-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)}). \tag{2.51}$$

Using

$$\partial_\mu H f_1 = i\partial_\mu H\bar{\epsilon}\hat{\sigma_1}\epsilon$$

in (2.50) and (2.51), we have

$$\begin{aligned}\partial_\mu H f_1 &= -iae^{-\frac{1}{2}(H+G)}\bar{\epsilon}\gamma_5\gamma_\mu\epsilon \\ &= -iae^{-\frac{1}{2}(H+G)}L_\mu.\end{aligned}$$

Using (2.49) and (2.42) we can solve this equation as follows

$$\begin{aligned}\partial_\mu He^{\frac{1}{2}(H-G)} &= -ae^{-\frac{1}{2}(H+G)}L_\mu \\ \Rightarrow \int dHe^H &= -adx^\mu L_\mu \\ e^H &= -a\Lambda dy.\end{aligned}$$

We want to identify the $y$ coordinate with $e^H$, which implies that

$$e^H = y, \quad \Lambda = -a. \tag{2.52}$$

From this result we see that we can determine $a$ by specifying $\Lambda$. Recall that $\Lambda$ takes two possible values, $\Lambda = \pm 1$. To fix $\alpha$ multiply (2.50) by $\bar{\epsilon}\gamma^5\hat{\sigma}^1$, to find

$$\begin{aligned}
\bar{\epsilon}\gamma^5\gamma^\mu\epsilon\partial_\mu H &= -iae^{-\frac{1}{2}(H+G)}\bar{\epsilon}\hat{\sigma}^1\epsilon + be^{-\frac{1}{2}(H-G)}\bar{\epsilon}\gamma^5\hat{\sigma}^1\epsilon \\
g^{yy}\bar{\epsilon}\gamma^5\gamma_y\epsilon\partial_y H &= -iae^{-\frac{1}{2}(H+G)}f_1 + be^{-\frac{1}{2}(H-G)}\bar{\epsilon}\gamma^5(-i\hat{\sigma}^2\hat{\sigma}^3)\epsilon \\
g^{yy}L_y\frac{\partial_y y}{e^H} &= -iae^{-\frac{1}{2}(H+G)}f_1 + be^{-\frac{1}{2}(H-G)}\bar{\epsilon}(-i\hat{\sigma}^2\hat{\sigma}^3)\gamma^5\epsilon \\
h^{-2}\Lambda &= -iae^{\frac{1}{2}(H-G)}f_1 - ibe^{\frac{1}{2}(H+G)}\bar{\epsilon}\hat{\sigma}^2\epsilon \\
h^{-2}\Lambda &= -iae^{\frac{1}{2}(H-G)}f_1 - be^{\frac{1}{2}(H+G)}f_2 \\
h^{-2}\Lambda &= -af_1^2 - \frac{b}{4\alpha}f_2^2.
\end{aligned}$$

To simplify this expression use the fact that $a = \pm 1$, so that $a^2 = 1$ and

$$\begin{aligned}
h^{-2}\Lambda &= -af_1^2 - \frac{ba^2}{4\alpha}f_2^2 \\
\Rightarrow -ah^{-2} &= -a(f_1^2 + \frac{ba}{4\alpha}f_2^2).
\end{aligned} \tag{2.53}$$

Using our normalization $K^t = 1$, $g_{tt} = h^{-2}$ and equation (2.40), we see that

$$h^{-2} = f_1^2 + f_2^2,$$

which, since $a, b = \pm 1$, implies that $4ab\alpha = 1$. Further more, we can choose $4\alpha = 4\beta = 1$, which forces the condition $a = b$. Through out this derivation we used properties of the Pauli matrices and the fact that $H$ depends only on $y$. Manipulating equation (2.50), we get

$$\begin{aligned}
\left(\hat{\sigma}_1\gamma^\mu\partial_\mu H + iae^{-\frac{1}{2}(H+G)}\gamma_5 - be^{-\frac{1}{2}(H-G)}\right)\epsilon = 0 \\
\left(\hat{\sigma}_1\gamma^y\partial_y \ln y + iae^{-\frac{1}{2}(H+G)}\gamma_5 - be^{-\frac{1}{2}(H-G)}\right)\epsilon = 0 \\
\left(\hat{\sigma}_1\gamma^y\frac{1}{y} + iae^{-\frac{1}{2}(H+G)}\gamma_5 - be^{-\frac{1}{2}(H-G)}\right)\epsilon = 0.
\end{aligned}$$

We can simplify this last equation to obtain

$$\left(\frac{1}{yh}\hat{\sigma}_1\Gamma^3 + iae^{-\frac{1}{2}(H+G)}\gamma_5 - be^{-\frac{1}{2}(H-G)}\right)\epsilon = 0. \tag{2.54}$$

Using (2.53), (2.48) and (2.49) and $4ab\alpha = 1$ as well as $a = b$ we learn that

$$(\sqrt{1 + e^{-2G}}\hat{\sigma}_1\Gamma^3 + aie^{-G}\gamma_5 - a)\epsilon = 0. \tag{2.55}$$

Using $\bar{\epsilon} = \epsilon^\dagger\gamma^0 = \epsilon^\dagger\Gamma^0$, $K^t = 1$ and the definition of $K$ from equation (2.22) we find the relation $\epsilon^\dagger\epsilon = 1$. Using the definition of $L_\mu$ from (2.22) and $L_y = -a$, we find

$$\epsilon^\dagger\Gamma^0\Gamma^5\Gamma^3\epsilon = -a.$$

This results implies that we have the projection condition

$$\left(1 + a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon = 0 \quad \text{or} \quad \left(1 + ia\Gamma_1\Gamma_2\right)\epsilon = 0. \tag{2.56}$$

Above we used the fact that $a^2 = 1$ and $\Gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$. To see that this is really a projector condition, we will check the property satisfied by any projector, namely if $P$ is a projector then $P^2 = P$. Let us check if this property holds for (2.56)

$$\begin{aligned}\left(1 + a\Gamma^0\Gamma^5\Gamma^3\right)\left(1 + a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon =&\left(1 + (a\Gamma^0\Gamma^5\Gamma^3)(a\Gamma^0\Gamma^5\Gamma^3) + 2a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon\\ =&\left(1 + a^2(\Gamma^0\Gamma^0)(\Gamma^5\Gamma^5)(\Gamma^3\Gamma^3) + 2a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon\\ =&\left(2 + 2a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon\\ =&2\left(1 + a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon.\end{aligned}$$

We see that (2.56) is a projector condition with the factor of two in the last expression, implying that the projector is not properly normalized. The two projectors (2.55) and (2.56) imply that the Killing spinor $\epsilon$ has the form

$$\epsilon = e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1}\epsilon_1, \quad \Gamma^3\hat{\sigma}^1\epsilon = a\epsilon_1, \quad \sinh 2\delta = ae^{-G}. \tag{2.57}$$

We further write the spinor $\epsilon_1$ in terms of a constant spinor $\epsilon_0$, by using the explicit expression for $f_2$, which implies

$$\epsilon_1 = e^{\frac{1}{4}(H+G)}\epsilon_0.$$

We are now in a position to compute $\omega_\mu$ given by equation (2.41). The non-vanishing components of $\omega_\mu$ are $\omega_1$ and $\omega_2$. A simple computation implies

$$\begin{aligned}\omega_{\hat{2}} &= \epsilon^t\Gamma^2\Gamma^2\epsilon\\ &= e^{\frac{1}{2}(H+G)}\epsilon_0^t(e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1})^t(e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1})\epsilon_0\\ &= e^{\frac{1}{2}(H+G)}\epsilon_0^t e^{i\delta a(\gamma^5)^t + i\delta\gamma^5 a}\epsilon_0\\ &= e^{\frac{1}{2}(H+G)}\epsilon_0^t e^{i\delta a[(\gamma^5)^t + \gamma^5]}\epsilon_0\\ &= e^{\frac{1}{2}(H+G)}\epsilon_0^t\left(1 + 0 + \frac{(2i\delta\gamma^5 a)^2}{2!} + 0 + \frac{(2i\delta\gamma^5 a)^4}{4!} + 0 + \frac{(2i\delta\gamma^5 a)^6}{6!} + 0 + \frac{(2i\delta\gamma^5 a)^8}{8!} + 0\cdots\right)\epsilon_0\\ &= e^{\frac{1}{2}(H+G)}\epsilon_0^t\left(1 + \frac{(2\delta)^2}{2!} + \frac{(2\delta)^4}{4!} + \frac{(2\delta)^6}{6!} + \cdots\right)\epsilon_0\\ &= e^{\frac{1}{2}(H+G)}\epsilon_0^t\epsilon_0\cosh 2\delta.\end{aligned}$$

To simplify this expression we have used $(\gamma^5)^2 = -1$, equation (2.57) and the relation $\gamma_\mu^t = -\Gamma^2\gamma_\mu\Gamma^2$ which follows from the Clifford algebra. Using (2.57) and the relation $h^{-2} = y(e^G + e^{-G})$, we get

$$\omega_{\hat{2}} = h^{-1}\epsilon_0^t\epsilon_0. \tag{2.58}$$

Following a very similar procedure, we find that $\omega_{\hat{1}}$ is given by

$$\omega_{\hat{1}} = -iah^{-1}\epsilon_0^t\epsilon_0. \tag{2.59}$$

$\omega$ is now given by

$$\begin{aligned}\omega &= \omega_{\hat{c}}e_\mu^{\hat{c}}dx^\mu\\ &= \omega_{\hat{1}}e_\mu^{\hat{1}}dx^\mu + \omega_{\hat{2}}e_\mu^{\hat{2}}dx^\mu\\ &= (-iah^{-1}e_i^{\hat{1}} + h^{-1}e_i^{\hat{2}})dx^i.\end{aligned}$$

In term of $e_i^{\hat{j}} = h\tilde{e}_i^{\hat{j}}$, this last expression becomes

$$\omega = (\text{constant})(\tilde{e}_i^{\hat{1}} + ia\tilde{e}_i^{\hat{2}})dx^i. \tag{2.60}$$

Therefore, we can write

$$\begin{aligned}\omega &= \omega_a dx^a \\ &= (idx^1 + dx^2).\end{aligned}$$

Next, note that

$$\begin{aligned}dx^a &= e^a{}_\mu dx^\mu \\ &= hdx^\mu.\end{aligned}$$

This tells us that the 2-dimensional metric tensor $\tilde{h}_{ij}$ in equation (2.45) is given by $\tilde{h}_{ij} = \delta_{ij}$. The 10-dimensional metric now takes the form

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \delta_{ij}dx^i dx^j) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2. \tag{2.61}$$

Our next task is to investigate the gauge fields. The self-duality condition obeyed by the 10-dimensional 5-form flux (see appendix) implies that we only have one independent gauge field in 4-dimensions. Using the definition of the field strength and its dual, we write the gauge field as follows

$$\begin{aligned}B &= B_t(dt + V) + \hat{B}, \\ d\hat{B} + B_t dV &= -h^2 e^{3G} *_3 d\tilde{B}_t, \qquad (2.62)\\ \tilde{B} &= \tilde{B}_t(dt + V) + \hat{\tilde{B}}, \\ d\hat{\tilde{B}} + \tilde{B}_t dV &= h^2 e^{-3G} *_3 dB_t. \qquad (2.63)\end{aligned}$$

$\hat{B}, \hat{\tilde{B}}$ have no components along the time direction and the 3 dimensional flat space epsilon $*_3$ is directed along $y, x_1, x_2$. We saw from (2.34) that $K$ is a Killing vector and $dK$ is given by $dK = 2\big(e^{-(H+G)}F + e^{-(H-G)}\tilde{F}\big)$. Using this expression of $dK$ and the above expressions of the gauge field, we can write the following relation

$$\frac{1}{2}dK = e^{-(H+G)}F + e^{-(H-G)}\tilde{F} = -\frac{1}{2}d[h^{-2}(dt + V)].$$

We find the expression for $V$ as follows

$$\begin{aligned}\frac{1}{2}h^{-2}dV &= e^{-(H+G)}(d\hat{B} + B_t dV) - e^{-(H-G)}d(d\hat{\tilde{B}} + \tilde{B}_t dV) \\ &= h^2(e^{-H+2G} *_3 d\tilde{B}_t - e^{-H-2G} *_3 dB_t).\end{aligned}$$

To move from the first line to the second line we used equations (2.62) and (2.63). Now, using the equations

$$\begin{aligned}\tilde{B}_t &= -\frac{1}{4}e^{2(H-G)}, \quad d\tilde{B}_t = -\frac{1}{2}e^{2(H-G)}(dH - dG) \\ B_t &= -\frac{1}{4}e^{2(H+G)}, \quad dB_t = -\frac{1}{2}e^{2(H+G)}(dH + dG)\end{aligned}$$

the expression for $dV$ becomes

$$\begin{aligned}dV &= 2h^4\left[e^{-H+2G} *_3 \left(-\frac{1}{2}e^{2(H-G)}(dH - dG)\right) + e^{-H-2G} *_3 \left(\frac{1}{2}e^{2(H+G)}(dH + dG)\right)\right] \\ &= 2h^4\left[\left(-\frac{1}{2}e^H *_3 (dH - dG)\right) + \left(\frac{1}{2}e^H *_3 (dH + dG)\right)\right] \\ &= 2h^4\left[\left(e^H *_3 dG\right)\right] \\ &= 2h^4 y *_3 dG.\end{aligned}$$

By defining the new variable $z \equiv \frac{1}{2}\tanh G$, we find

$$dV = \frac{1}{y} *_3 dz \tag{2.64}$$

where $dz = 2h^4 y^2 dG$ and we have used the relation $h^{-2} = y(e^G + e^{-G})$. The consistency condition $d(dV) = 0$ tells us that we can write this last equation as

$$\begin{aligned} &\partial_\alpha(\frac{1}{y} *_3 \partial_\alpha z) = 0 \\ \Rightarrow\ &\frac{1}{y}\partial_i^2 z + \partial_y(\frac{1}{y}\partial_y z) = 0 \end{aligned} \tag{2.65}$$

where $\alpha = x_1, x_2, y$ and $i = x_1, x_2$. From equation (2.62), (2.63) and (2.64) we can find the gauge field

$$\begin{aligned} d\hat{B} &= -h^2 e^{3G} *_3 d\tilde{B}_t - B_t dV \\ &= -h^2 e^{3G} *_3 (-\frac{1}{2}e^{2(H-G)}(dH - dG)) + \frac{1}{4}e^{2(H+G)}\frac{1}{y} *_3 dz \\ &= \left(\frac{ye^G}{2y(e^G + e^{-G})} - \frac{y(e^G + e^{-G})dze^G}{4}\right) *_3 + \frac{y}{4}e^{2G} *_3 dz \\ &= \left(\frac{1}{2(1+e^{-2G})} - \frac{ydz}{4}\right) *_3 . \end{aligned}$$

One can show that this equation can be written as

$$d\hat{B} = -\frac{1}{4}y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right). \tag{2.66}$$

Following the same procedure, we find $d\hat{\tilde{B}}$

$$d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right).$$

This brings us to the end of the derivation of the 1/2 BPS geometry. The important quantities defining this geometry are

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G}\tilde{\Omega}_3^2, \tag{2.67}$$

$$h^{-2} = 2y\cosh G, \tag{2.68}$$

$$z = \frac{1}{2}\tanh G. \tag{2.69}$$

Here we notice that after solving the Killing spinor equation we are able to relate the different functions that appear in our 10-dimensional metric, to the single function $z$. $z$ itself is obtained by solving the Laplace equation (2.65) together with suitable boundary conditions. The allowed boundary conditions follow by requiring that the geometry is not singular, as we explain in the next section.

## 2.3 Conditions for a smooth 1/2 BPS geometry

We will derive the necessary conditions for a regular 1/2 BPS geometry. We will then illustrate these conditions with an example. Recall that the $SO(4) \times SO(4)$ isometry of the 1/2 BPS geometry corresponds to two 3-spheres with radii $r$ and $\tilde{r}$. We have defined a new coordinate by the product of the radii of the two 3-spheres i.e. $y = r\tilde{r}$. We can identify the radii of the two 3-spheres of (2.67) as

$$r^2 = ye^G, \qquad \tilde{r}^2 = ye^{-G}, \qquad r^2\tilde{r}^2 = y^2. \tag{2.70}$$

A singularity might occur in this metric when $r$ or $\tilde{r}$ vanish. It turns out that there is a way of ensuring that this metric is non-singular at $y = 0$. We require that at $y = 0$ we have

$$G = \pm\infty, \qquad z = \pm\frac{1}{2}. \tag{2.71}$$

We will investigate the case where $z = \frac{1}{2}$. Let us consider a vicinity of a point where $y$ goes to zero, while $-b^2 = ye^G$ remains finite. From our definition of $z$ (i.e $z = \frac{1}{2}\tanh G$), we see that $z$ at this vicinity admits the following expansion

$$z \sim \frac{1}{2} - e^{-2G} = \frac{1}{2} + y^2 a(x^1, x^2) + \cdots \tag{2.72}$$

This relation implies that

$$ay^2 = -e^{-2G}.$$

Setting $a = -b^2$, we find the relation

$$\begin{aligned} y^2b^2 &= e^{-2G} \\ \Rightarrow yb &= e^{-G}. \end{aligned}$$

Now we can write the radii of the two 3-spheres as

$$\begin{aligned} r^2 &= ye^G = y\frac{1}{yb} = \frac{1}{b}, \\ \tilde{r}^2 &= ye^{-G} = y^2 b \end{aligned}$$

and we can re-express (2.68) as

$$\begin{aligned} h^{-2} &= 2y\cosh G = y(e^G + e^{-G}) \\ &= \frac{1}{b} + y^2 b \\ h^{-2}\Big|_{y=0} &= \frac{1}{b}. \end{aligned}$$

We learn that $h$ and the radius $r$ of the 3-sphere $d\Omega_3$ remain finite as we take $y = 0$. The metric in the $y$ direction and in the $d\tilde{\Omega}_3$ direction then becomes

$$h^2dy^2 + ye^{-G}d\tilde{\Omega}_3^2 \sim b(dy^2 + y^2 d\tilde{\Omega}_3) \tag{2.73}$$

which is the non-singular flat space metric. What we learn from this analysis is that it possible to find a non-singular solutions as long as we require $z = \pm\frac{1}{2}$ when $y = 0$.

## 2.4 1/2 BPS geometry from Schur polynomials

We now want to consider the dual description of the 1/2 BPS geometries. 1/2 BPS geometries correspond to configurations that are dual to arbitrary droplets (filled states) of the free fermions field in phase space. Free fermion fields can be discussed using the language of Schur polynomial i.e. there is a formalism that can be used to map free fermions states to Schur polynomials. The goal of this section is to use the Schur polynomials that are labelled by Young diagrams, to describe the 1/2 BPS geometry. In terms of Schur polynomials, the 1/2 BPS geometry correspond to black and white concentric rings as shown in the diagram below

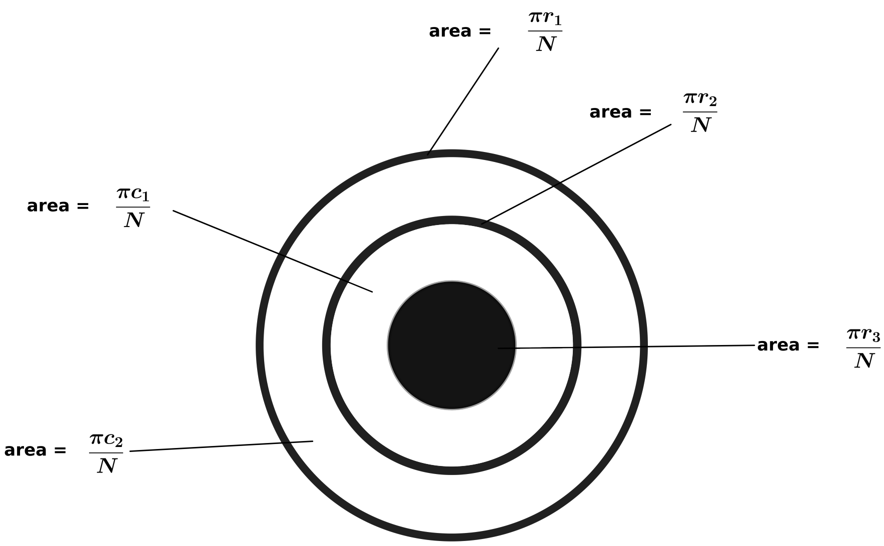



Figure 2.1: 1/2 BPS geometry from Schur polynomials.

The black regions correspond to the boundary condition $z = \dfrac{1}{2}$ and the white region correspond to the boundary condition $z = -\dfrac{1}{2}$. We identify this geometry with the Young diagram shown in the figure below

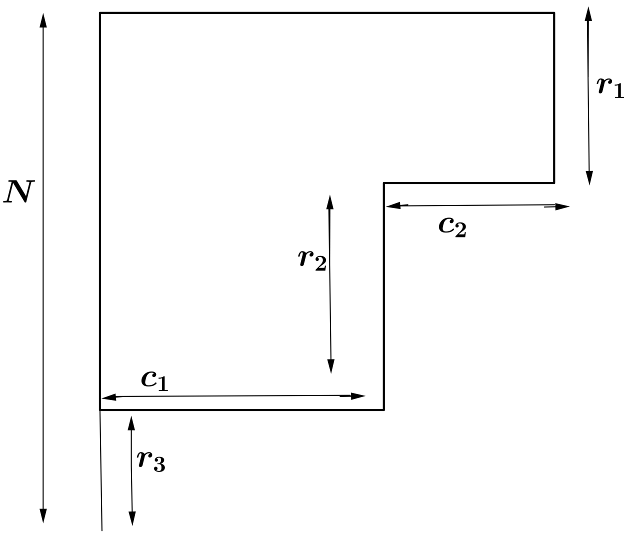



Figure 2.2: 1/2 BPS geometry from Young diagram.

The sides that are labelled by $r_1, r_2$ and $r_3$ on the Young diagram are identified with the black rings on the 1/2 BPS geometry and the sides that are labelled by $c_1$ and $c_2$ on the Young diagram are identified with the white rings on the 1/2 BPS geometry. Therefore it is possible to study the 1/2 BPS geometry using the Schur polynomials as an alternative to the free fermion description of the field theory. What we have done so far was to construct the 1/2 BPS geometry. We allow one of the two 3-spheres to shrink to zero on the $y = 0$ plane while keeping the other fixed and as a result we have managed to get a non-singular solution when $y = 0$ provided that we set $z = \pm\frac{1}{2}$. Now that we know how to construct smooth 1/2 BPS geometries, our next goal is to look at the 1/4 BPS geometries.

## 2.5 1/4 BPS geometries.

In field theory, some of the $\mathcal{N}=4$ states that are built from two complex fields $Z$ and $Y$ correspond to 1/4 BPS states. The isometry of 1/4 BPS geometries is $\mathbb{R}\times SO(4)\times U(1)$. The $SO(4)\times U(1)$ symmetry corresponds to the isomerties of a 3-sphere and a 1-sphere. To find the solution to the 1/4 BPS geometry we will solve the Killing spinor equations just like we did for the 1/2 BPS geometry. An interesting feature of the solutions obtained using the Killing spinor equation, is that there is a 4-dimensional plane on which one or both of the spheres (the 1-sphere and the 3-sphere) in the geometry shrink to zero size. To obtain a regular geometry, the boundary conditions must be chosen carefully. We represent the 4-dimensional plane by a complex space $\mathbb{C}^2$. This space come equipped with a Kähler metric. We will define a function $Z(z_a,\bar{z}_a)$ that specifies the boundary conditions for smooth 1/4 BPS geometries in terms of the Kähler potential. Once we know this function, we can turn the problem of finding 1/4 BPS geometries into the problem of solving a differential equation known as the Monge-Ampére equation. By solving the Monge-Ampére equation that appears in our analysis, we would determine the 1/4 BPS geometry. This section is a review of the beautiful papers [50] and [53].

### 2.5.1 The metric and the five-form of the 1/4 BPS geometries

The metric and the five-form of this geometry are given by [1]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{H-G}d\Omega_3^2 + e^{H+G}d\psi^2 \tag{2.74}$$

$$F_{(5)} = F_{(2)}\wedge d\Omega_3 + \tilde{F}_{(4)}\wedge d\psi \tag{2.75}$$

$$= F_{\mu_1\mu_2}dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3 + \tilde{F}_{\mu_3\mu_4\mu_5\mu_6}dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi \tag{2.76}$$

where $\mu_1,\cdots,\mu_6$ take values $=0,\cdots,5$ and $g_{\mu\nu}=diag(1,-1,-1,-1,-1,-1)$. Our first task is find the Hodge dual of the five form field strength. This is done as follows

$$\begin{aligned}
*F_{(5)} &= *(F_{(2)}\wedge d\Omega_3) + *(\tilde{F}_{(4)}\wedge d\psi)\\
&= *(F_{\mu_1\mu_2}dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3) + *(\tilde{F}_{\mu_3\mu_4\mu_5\mu_6}dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi)\\
&= \frac{1}{(6-2)!}F^{\mu_1\mu_2}\sqrt{|detg|}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi\\
&+ \frac{1}{(6-4)!}\sqrt{|det\bar{g}|}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\tilde{F}^{\mu_3\mu_4\mu_5\mu_6}dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3\\
&= \frac{1}{4!}F^{\mu_1\mu_2}e^{-H-2G}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi\\
&+ \frac{1}{2!}e^{H+2G}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\tilde{F}^{\mu_3\mu_4\mu_5\mu_6}dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3
\end{aligned}$$

where $\det g$ and $\det\bar{g}$ are given by

$$\sqrt{|detg|} = \frac{e^{\frac{1}{2}(H-G)}}{e^{\frac{3}{2}(H+G)}} = e^{-H-2G} \tag{2.77}$$

$$\sqrt{|det\bar{g}|} = \frac{e^{\frac{3}{2}(H+G)}}{e^{\frac{1}{2}(H-G)}} = e^{H+2G}. \tag{2.78}$$

We write the dual field strength as

$$\begin{aligned}
F_{(2)} &= 2e^{H+2G} *_6 \tilde{F}_{(4)}\\
\tilde{F}_{\mu_1\mu_2\mu_3\mu_4} &= -\frac{1}{2}e^{-H-2G}\varepsilon_{\mu_1\mu_2\mu_1\mu_2\mu_3\mu_4} *_6 F^{\mu_5\mu_6}.
\end{aligned}$$

[1]The full metric for this geometry should be given by $ds^2=g_{\mu\nu}dx^\mu dx^\nu+e^{H-G}d\Omega_3^2+e^{H+G}(d\psi+A)^2$, but the quantity $A$ is found to be zero and hence we will not include it in our analysis

Notice that $\varepsilon_{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4}$ is in Minkowski space. Raising the indices of this Levi-Civita symbol will cost us a minus sign, that is $\varepsilon_{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4} = -\varepsilon^{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4}$. Now lets check the double Hodge duality

$$\begin{aligned}
**F_{(5)} &= *\Bigg[\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\Bigg(\frac{1}{4!}e^{-H-2G}F^{\mu_1\mu_2}(dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi) \\
&+\frac{1}{2!}e^{H+2G}\tilde{F}^{\mu_3\mu_4\mu_5\mu_6}(dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3^2)\Bigg)\Bigg] \\
&= \varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}(-\varepsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6})\Bigg(\frac{1}{4!}\frac{1}{4!}\frac{e^{-H-2G}}{e^{-H-2G}}F_{\mu_1\mu_2}(dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3^2) \\
&+\frac{1}{2!}\frac{1}{2!}\frac{e^{H+2G}}{e^{H+2G}}\tilde{F}_{\mu_3\mu_4\mu_5\mu_6}(dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi)\Bigg) \\
&= -\Bigg(\frac{(6-2)!4!}{4!4!}F_{\mu_1\mu_2}(dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3^2)+\frac{(6-4)!2!}{2!2!}\tilde{F}_{\mu_3\mu_4\mu_5\mu_6}(dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi)\Bigg) \\
&= -\Bigg(F_{\mu_1\mu_2}(dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\Omega_3^2)+\tilde{F}_{\mu_3\mu_4\mu_5\mu_6}(dx^{\mu_3}\wedge dx^{\mu_4}\wedge dx^{\mu_5}\wedge dx^{\mu_6}\wedge d\psi)\Bigg) \\
&= -F_{(5)}.
\end{aligned}$$

We see that the double dual of this five-form is related to the original by a minus sign. In order to find the smooth geometries that preserve 1/4 BPS worth of symmetry, we need to write down the Killing spinor equation and solve it.

### 2.5.2 1/4 BPS Killing spinor equation

To determine the 1/4 BPS geometries, we need to solve the Killing spinor equation. The Killing spinor equation in this case is given by

$$0 = \nabla_M\eta + \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F_{M_1M_2M_3M_4M_5}\Gamma_M\eta \tag{2.79}$$

where $\eta$ is a 10-dimensional spinor. To solve this Killing spinor equation, choose the following basis for the gamma matrices with Lorentz indices as

$$\begin{aligned}
\Gamma_\mu &= \gamma_\mu\otimes\hat{\sigma}_1\otimes\mathbb{I}_2, \\
\Gamma_a &= \mathbb{I}_8\otimes\hat{\sigma}_2\otimes\sigma_a, \\
\Gamma_9 &= \gamma_7\otimes\hat{\sigma}_1\otimes\mathbb{I}_2
\end{aligned} \tag{2.80}$$

where $\mu = 0, 1, \cdots, 5$, $a = 6, 7, 8$ and $\gamma_7 = i\Gamma_0\cdots\Gamma_5$. The chirality condition is

$$\Gamma_{10}\eta = \Gamma_0\Gamma_1\cdots\Gamma_9\eta = \hat{\sigma}_3\eta = \eta. \tag{2.81}$$

We decompose the 10-dimensional Killing spinor $\eta$ as follows

$$\eta = \epsilon\otimes\chi_\alpha, \tag{2.82}$$

where $\epsilon$ is the 6-dimensional Killing spinor and $\chi_\alpha$ is the Killing spinor on a unit sphere. The Killing spinor on the sphere satisfies

$$\nabla_a\chi_\alpha = \frac{i\alpha}{2}\sigma_a\chi_\alpha, \quad a = \pm1. \tag{2.83}$$

The dependence on coordinates of the spinor is as follows

$$\varepsilon(x^\mu, \Omega_3, \psi) = e^{\frac{i}{2}n\psi}\varepsilon(x^\mu, \Omega_3).$$

This equation tells us that we must have the following relation

$$\partial_\psi \varepsilon = \frac{in}{2}\varepsilon. \tag{2.84}$$

Before, we derived the covariant derivative on $S^3$ to be

$$\begin{aligned}\nabla_a &= \nabla'_a - \frac{1}{4}\Gamma^\lambda \Gamma_a \partial_\lambda (H+G) \\ &= \nabla'_a - \frac{i}{4}\sigma_a \hat{\sigma}_3 \gamma^\lambda \partial_\lambda (H+G). \end{aligned} \tag{2.85}$$

The covariant derivative on $S^1$ is given by

$$\nabla_\psi = \partial_\psi + \frac{1}{4}\Gamma_\psi \Gamma^\lambda \partial_\lambda (H-G) \tag{2.86}$$

$$= \partial_\psi + \frac{1}{4}\gamma_7 \gamma^\lambda \partial_\lambda (H-G). \tag{2.87}$$

Denote the second term of (2.79) by $M$

$$M = \frac{i}{480}\Gamma^{M_1 M_2 M_3 M_4 M_5} F_{M_1 M_2 M_3 M_4 M_5}. \tag{2.88}$$

Using our results from the 1/2 BPS analysis we can write this as follows

$$\begin{aligned} M &= \frac{i}{480}\left(\frac{5!}{3!2!}\Gamma^{\mu_1\mu_2}\varepsilon_{\theta_1\theta_2\theta_3}F_{\mu_1\mu_2}\Gamma^{\theta_1\theta_2\theta_3} - \frac{5!}{4!1!}\Gamma^{\mu_3\mu_4\mu_5\mu_6}\Gamma^\psi e^{-\frac{1}{2}(H-G)}\tilde{F}_{\mu_3\mu_4\mu_5\mu_6}\right) \\ &= \frac{i}{480}\left(10\gamma^{\mu_1\mu_2}e^{-\frac{3}{2}(H+G)}F_{\mu_1\mu_2}(3!i)\hat{\sigma}_2 + \frac{5}{4}\Gamma_{\mu'_2\mu'_1}(\varepsilon^{\mu'_1\mu'_2\mu_3\mu_4\mu_5\mu_6}\gamma_7)\gamma_7\hat{\sigma}_1 e^{-\frac{3}{2}(H+G)}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}F^{\mu_1\mu_2}\right) \\ &= -\frac{1}{8}e^{-\frac{3}{2}(H+G)}\not{F}\left(\hat{\sigma}_2 + i\hat{\sigma}_1\right) \\ &= -\frac{1}{8}e^{-\frac{3}{2}(H+G)}\not{F}\hat{\sigma}_2\left(1+\hat{\sigma}_3\right) \end{aligned}$$

where we have used

$$\begin{aligned} \gamma_7 &\equiv \frac{1}{6!}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \\ \Gamma^{(7)} &= \frac{1}{6!}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} = \gamma_7 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \\ \gamma_\mu\gamma_\nu &= \gamma_{\mu\nu} \qquad \text{iff} \quad \mu < \nu. \end{aligned}$$

Now that we have managed to decompose the quantities that appear in the Killing spinor equation, we can decompose the Killing spinor equation into three parts. Using (2.79), (2.85) and (2.87), we find

$$\underline{\mathcal{D}_\mu \epsilon = 0}$$

$$\nabla_\mu \epsilon + M\Gamma_\mu \epsilon = 0$$

$$\Rightarrow \nabla_\mu \epsilon + N\hat{\sigma}_2\gamma_\mu\hat{\sigma}_1\epsilon = 0 \tag{2.89}$$

$$\Rightarrow \nabla_\mu \epsilon - iN\gamma_\mu\hat{\sigma}_3\epsilon = 0 \tag{2.90}$$

$$\therefore \nabla_\mu \epsilon - iN\gamma_\mu\epsilon = 0 \tag{2.91}$$

where $N$ is given by

$$N = -\frac{1}{4}\not{F}e^{-\frac{3}{2}(G+H)} \quad \text{and} \quad M = N\hat{\sigma}_2.$$

$$\underline{\mathcal{D}_{S^3}\epsilon = 0}$$

$$\begin{aligned}
0 &= \left(\frac{i\alpha}{2}\sigma_a e^{-\frac{1}{2}(H+G)} - \frac{i}{4}\sigma_a\hat{\sigma}_3\gamma^\lambda\partial_\lambda(H+G) + N\hat{\sigma}_2\Gamma_a\right)\epsilon \\
\Rightarrow 0 &= \left(\frac{i\alpha}{2}\sigma_a e^{-\frac{1}{2}(H+G)} - \frac{i}{4}\sigma_a\hat{\sigma}_3\gamma^\lambda\partial_\lambda(H+G) + N\hat{\sigma}_2\hat{\sigma}_2\sigma_a\right)\epsilon \\
\therefore 0 &= \left(\frac{i\alpha}{2}e^{-\frac{i}{2}(H+G)} - \frac{i}{4}\gamma^\lambda\partial_\lambda(H+G) + N\right)\epsilon. \qquad (2.92)
\end{aligned}$$

$$\underline{\mathcal{D}_{S^1}\epsilon = 0}$$

$$\begin{aligned}
&\left(\frac{in}{2}e^{-\frac{1}{2}(H-G)} + \frac{1}{4}\gamma_7\gamma^\lambda\partial_\lambda(H-G) + N\hat{\sigma}_2\Gamma_{10}\right)\epsilon = 0 \\
\Rightarrow &\left(\frac{in}{2}e^{-\frac{1}{2}(H-G)} + \frac{1}{4}\gamma_7\gamma^\lambda\partial_\lambda(H-G) + N\hat{\sigma}_2\gamma_7\hat{\sigma}_1\right)\epsilon = 0 \\
\Rightarrow &\left(\frac{in}{2}e^{-\frac{1}{2}(H-G)} + \frac{1}{4}\gamma_7\gamma^\lambda\partial_\lambda(H-G) - iN\gamma_7\hat{\sigma}_3\right)\epsilon = 0 \\
\therefore &\left(\frac{in}{2}e^{-\frac{1}{2}(H-G)} + \frac{1}{4}\gamma_7\gamma^\lambda\partial_\lambda(H-G) - i\gamma_7 N\right)\epsilon = 0. \qquad (2.93)
\end{aligned}$$

This reduces (2.79) to (2.91), (2.92) and (2.93). To solve the Killing spinor equation (2.79), we will again introduce spinor bilinears.

### 2.5.3 Spinor bilinears

To solve the equations (2.91), (2.92) and (2.93) we will introduce spinor bilinears constructed from the 6-dimensional spinor $\epsilon$. The list of spinor bilinear which we are interested in are

$$\begin{aligned}
&f_1 = \bar{\epsilon}\gamma_7\epsilon \quad f_2 = i\bar{\epsilon}\epsilon \quad K_\mu = \bar{\epsilon}\gamma_\mu\epsilon \quad L_\mu = \bar{\epsilon}\gamma_\mu\gamma_7\epsilon \\
&Y_{\mu\lambda} = i\bar{\epsilon}\gamma_{\mu\nu}\gamma_7\epsilon \quad V_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\epsilon \quad \Omega_{\mu\nu\lambda} = i\bar{\epsilon}\gamma_{\mu\nu\lambda}\epsilon. \qquad (2.94)
\end{aligned}$$

Our next step is to take the covariant derivatives of the first four spinor bilinears. Consider $f_1$

$$\begin{aligned}
\nabla_\mu f_1 &= (\nabla_\mu\bar{\epsilon})\gamma_7\epsilon + \bar{\epsilon}\gamma_7(\nabla_\mu\epsilon) \\
&= -\bar{\epsilon}M\gamma_\mu\gamma_7\epsilon - \bar{\epsilon}\gamma_7\gamma_\mu M\epsilon \\
&= \frac{i}{4}e^{-\frac{3}{2}(G+H)}\bar{\epsilon}(2\gamma_{\mu\kappa\lambda})\gamma_7 F^{\kappa\lambda}\epsilon \\
&= -\frac{1}{3!}e^{\frac{1}{2}(G-H)}\Omega^{\rho\sigma\tau}F_{\mu\rho\sigma\tau}
\end{aligned}$$

where we have used the self-dual relation

$$F_{\mu\rho\sigma\tau} = -\frac{1}{2}e^{-2G-H}\varepsilon_{\mu\rho\sigma\tau\kappa\lambda}F^{\kappa\lambda}. \qquad (2.95)$$

The remaining differential identities are derived in the same way as the corresponding formulas found for the 1/2 BPS geometry. Using equations (2.23)- (2.31), we find the following results

$$\nabla_\mu f_2 = -e^{-\frac{3}{2}(G+H)}F_{\mu\lambda}K^\lambda, \qquad (2.96)$$

$$\nabla_\mu K_\rho = e^{-\frac{3}{2}(G+H)} f_2 F_{\mu\rho} + \frac{1}{2} e^{\frac{1}{2}(G-H)} F_{\mu\rho\sigma\tau} Y^{\sigma\tau}, \tag{2.97}$$

$$\nabla_\mu L_\rho = e^{-\frac{3}{2}(G+H)} \left[ F_\mu{}^\lambda Y_{\lambda\rho} + F_\rho{}^\lambda Y_{\lambda\mu} + \frac{1}{2} g_{\mu\rho} F^{\kappa\lambda} Y_{\kappa\lambda} \right], \tag{2.98}$$

$$\nabla_\kappa \Omega_{\mu\nu\lambda} = \frac{1}{4} e^{-\frac{3}{2}(G+H)} F^{\pi\rho} \bar{\varepsilon}(\gamma_{\mu\nu\lambda}\gamma_{\pi\rho}\gamma_\kappa - \gamma_\kappa\gamma_{\pi\rho}\gamma_{\mu\nu\lambda})\varepsilon. \tag{2.99}$$

One can show that $K_\mu$ satisfies Killing's equation using equation (2.97) and hence $K_\mu$ is a Killing vector. Using equation (2.98) we can show that

$$\nabla_\mu L_\rho - \nabla_\rho L_\mu = 0$$

which implies that $dL = 0$. The following relations continue to hold for the 1/4 BPS analysis

$$L^2 = -K^2 = f_1^2 + f_2^2, \tag{2.100}$$

$$L \cdot K = 0. \tag{2.101}$$

By anti-symmetrizing (2.99) we find

$$d\Omega_{\kappa\lambda\mu\nu} = 4 f_1 e^{-\frac{1}{2}(H-G)} \tilde{F}_{\kappa\lambda\mu\nu}. \tag{2.102}$$

Note that most of our results match what we obtained for study of the 1/2 BPS geometry. We can again write $L = dy$ and $K^\mu = K^t = t$. Finally the six dimensional metric is

$$ds_6^2 = h^2 dy^2 + \hat{g}_{ij} dx^i dx^j. \tag{2.103}$$

We can relate $g_{tt}$ and $g^{yy}$ using $K^2 = -L^2$, so that our six dimensional metric becomes

$$ds_6^2 = -h^{-2}(dt + \omega)^2 + h^2 dy^2 + f_2^{-2} h_{ab} dx^a dx^b. \tag{2.104}$$

Notice that $\omega$ in this metric plays the same role as $V_i dx^i$ in the 1/2 BPS geometry metric. The factor $f_2^{-2}$ was included in front of the 4-dimensional metric tensor $h_{ab}$ for later convenient. To determine this metric we need to find the spinor bilinears $f_1$ and $f_2$.

### 2.5.4 Solving for $f_1$ and $f_2$

To find $f_1$ and $f_2$, we follow the same procedure as in the 1/2 BPS case, where we evaluated the covariant derivatives of these spinors bilinears. Take the conjugate of (2.92) and (2.93) to find

$$\bar{\varepsilon}\left( -\frac{i\alpha}{2} e^{-\frac{1}{2}(H+G)} - \frac{i}{4}\gamma^\lambda \partial_\lambda (H+G) - N \right) = 0, \tag{2.105}$$

$$\bar{\varepsilon}\left( -\frac{in}{2} e^{-\frac{1}{2}(H-G)} + \frac{i}{4}\gamma^\lambda \gamma_7 \partial_\lambda (H-G) + i\gamma_7 N \right) = 0. \tag{2.106}$$

To find $f_1$ and $f_2$ from these last two equations we need to evaluate $[\gamma_\mu, N]$ and $\{\gamma_\mu, N\}$. We find

$$\begin{aligned} [\gamma_\mu, N] &= \gamma_\mu N - N\gamma_\mu \\ &= -\frac{2}{4} e^{-\frac{3}{2}(G+H)} F^{\lambda\sigma}(g_{\mu\lambda}\gamma_\sigma - g_{\mu\sigma}\gamma_\lambda) \\ &= -e^{-\frac{3}{2}(G+H)} F_\mu{}^\sigma \gamma_\sigma \end{aligned}$$

and

$$\begin{aligned} \bar{\varepsilon}\{\gamma_\mu, N\}\gamma_7\varepsilon &= \bar{\varepsilon}\left[ \gamma_\mu N + N\gamma_\mu \right]\gamma_7\varepsilon \\ &= \bar{\varepsilon}\left[ -\frac{2}{4} e^{-\frac{3}{2}(G+H)} F^{\lambda\sigma}(\gamma_{\mu\lambda\sigma}) \right]\gamma_7\varepsilon \\ &= -i\partial_\mu f_1. \end{aligned}$$

Now, multiplying (2.92) by $\bar{\varepsilon}\gamma_\mu$ and (2.105) by $\gamma_\mu\varepsilon$ and adding the two equations, we get

$$0 = \bar{\varepsilon}\gamma_\mu\bigg(\frac{i\alpha}{2}e^{-\frac{1}{2}(H+G)} - \frac{i}{4}\gamma^\lambda\partial_\lambda(H+G) + N\bigg)\epsilon + \bar{\varepsilon}\bigg(-\frac{i\alpha}{2}e^{-\frac{1}{2}(H+G)} - \frac{i}{4}\gamma^\lambda\partial_\lambda(H+G) - N\bigg)\gamma_\mu\varepsilon$$
$$0 = -\frac{1}{2}f_2\partial_\mu(H+G) - e^{-\frac{3}{2}(G+H)}F_{\mu\sigma}K^\sigma$$
$$\Rightarrow \partial_\mu f_2 = \frac{1}{2}f_2\partial(H+G).$$

The solution to this differential equation is

$$f_2 = \kappa e^{\frac{1}{2}(H+G)} \tag{2.107}$$

where $\kappa$ is a constant of integration. To find $f_1$ we multiply (2.93) by $\bar{\varepsilon}\gamma_\mu$ and (2.106) by $\gamma_\mu\varepsilon$ and add the two equations to get

$$0 = \bar{\varepsilon}\gamma_\mu\bigg(\frac{in}{2}e^{-\frac{1}{2}(H-G)} + \frac{1}{4}\gamma_7\gamma^\lambda\partial_\lambda(H-G) - i\gamma_7 N\bigg)\epsilon + \bar{\varepsilon}\bigg(-\frac{in}{2}e^{-\frac{1}{2}(H-G)} + \frac{1}{4}\gamma^\lambda\gamma_7\partial_\lambda(H-G) + i\gamma_7 N\bigg)\gamma_\mu\varepsilon$$
$$0 = \bar{\varepsilon}\bigg(\frac{1}{2}\gamma_7\partial_\mu(H-G) - i\{\gamma_\mu, N\}\gamma_7\bigg)\epsilon$$
$$\Rightarrow \partial_\mu f_1 = \frac{1}{2}f_1\partial_\mu(H-G).$$

The solution to this differential equation is

$$f_1 = \lambda e^{\frac{1}{2}(H-G)}. \tag{2.108}$$

Now, combining (2.92), (2.93) and (2.108), we find

$$e^H = y. \tag{2.109}$$

With the correct choice of normalization, as for the 1/2 BPS case, $\lambda = \kappa = 1$, we can express $h$ in terms of $f_1$ and $f_2$ as follows

$$h^{-2} = f_1^2 + f_2^2 \tag{2.110}$$
$$= y(e^G + e^{-G}) \tag{2.111}$$
$$= 2y\cosh G. \tag{2.112}$$

Using these results, we can write the 10-dimensional metric of the 1/4 BPS geometry as

$$\begin{aligned} ds_{10}^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H-G}d\Omega_3^2 + e^{H+G}d\psi^2 \\ &= -h^{-2}(dt+\omega)^2 + f_2^{-2}h_{ab}dx^a dx^b + h^2dy^2 + y(e^{-G}d\Omega_3^2 + e^G d\psi^2) \\ &= -h^{-2}(dt+\omega)^2 + h^2[(e^{-2G}+1)h_{ab}dx^a dx^b + dy^2] + y(e^{-G}d\Omega_3^2 + e^G d\psi^2) \\ &= -h^{-2}(dt+\omega)^2 + h^2\left(\frac{1}{Z+\frac{1}{2}}h_{ab}dx^a dx^b + dy^2\right) + y(e^{-G}d\Omega_3^2 + e^G d\psi^2), \end{aligned} \tag{2.113}$$

where

$$Z = \frac{1}{2}\tanh G, \tag{2.114}$$
$$\frac{f_1}{f_2} = e^{-G}, \tag{2.115}$$
$$\frac{1}{1+e^{-2G}} = Z + \frac{1}{2}. \tag{2.116}$$

We learn that by solving the Killing spinor equation we are able to relate the different functions that appear in our 10-dimensional metric to the single function $Z(z_a, \bar{z}_a)$. At this point we have to ask how to determine the function $Z(z_a, \bar{z}_a)$? Towards this end, we now introduce the Kähler potential.

### 2.5.5 Kähler potential

A Kähler manifold is a complex manifold with a hermitian metric. The metric of the manifold is determined by a Kähler potential $K$. A Kähler manifold has three structures; a complex structure, a Riemannian structure and a symplectic structure. A symplectic form on a manifold $\mathcal{M}$ is a closed non-degenerate differential 2-form $\omega^{(2)}$ (i.e $d\omega = 0$). The Kähler potential is defined as a real valued function $K$ on a Kähler manifold for which the symplectic form $\omega$ can be written as $\omega = i\partial\bar{\partial}K$. Our goal here is to determine the function $Z$ using the Kähler potential. We will do this by using known facts about Kähler manifolds. From our 6-dimensional metric (2.104), we identify the 4-dimensional metric that lives on the Kähler manifold by $h_{ab}$. We can write this metric as

$$\begin{aligned} ds_4 &= h_{ab}dx^a dx^b \qquad (2.117)\\ &= \partial_a\bar{\partial}_b K dz^a d\bar{z}^b + \overline{\partial_a\bar{\partial}_b K dz^a d\bar{z}^b} \\ &= 2\partial_a\bar{\partial}_b K dz^a d\bar{z}^b \qquad (2.118) \end{aligned}$$

where $K(z_a, \bar{z}_a; y)$ is the Kähler potential. Finally, we can write the 10-dimensional metric of the 1/4 BPS geometry in terms of the Kähler potential as

$$ds_{10}^2 = -h^{-2}(dt+\omega)^2 + h^2\left(\frac{2}{Z+\frac{1}{2}}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b + dy^2\right) + y(e^{-G}d\Omega_3^2 + e^G d\psi^2). \qquad (2.119)$$

In the 1/2 BPS geometry, imposing the condition $d(dV) = 0$ implied a second order condition (2.65). In the 1/4 BPS analysis a second order condition again arises when we impose $d(d\omega) = 0$. This second order condition is solved by

$$Z = -\frac{1}{2}y\partial_y(y^{-1}\partial_y K). \qquad (2.120)$$

We have now achieved our goal by finding this relationship between $K$ and $Z$. We will now consider an example of computing the Kähler potential. This will give us a chance to illustrate the results we have just derived.

### 2.5.6 Example: The Kähler potential of $AdS_5 \times S^5$ as a 1/4 BPS geometry with SO(4) R-symmetry

The metric is given by

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho[\sin^2\alpha d\psi^2 + \cos^2\alpha d\tilde{\beta} + d\alpha^2] + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\Omega_3^2. \qquad (2.121)$$

Guided by the metric (2.119) we will extract expressions for $y, e^G$ and $h^{-2}$. Recall that $y$ is the product of the radii of $S^1$ and $S^3$ (i.e $y = r_\psi r_{\Omega_3}$). From this metric we see that

$$r_\psi = \sinh\rho\sin\alpha, \qquad r_{\Omega_3} = \sin\theta.$$

Comparing the factors in front of the 3-sphere and the 1-sphere of (2.119) and (2.121) we get the relations

$$e^G = \frac{\sin\theta}{\sinh\rho\sin\alpha}.$$

We can find $h^{-2}$ using the relation $h^{-2} = 2y\cosh G$, which implies

$$h^{-2} = \sinh^2\rho\sin^2\alpha + \sin^2\theta.$$

In summary, we have

$$\begin{aligned} y &= \sinh\rho\sin\alpha\sin\theta, \qquad (2.122)\\ e^G &= \frac{\sin\theta}{\sinh\rho\sin\alpha}, \qquad (2.123)\\ h^{-2} &= \sinh^2\rho\sin^2\alpha + \sin^2\theta. \qquad (2.124) \end{aligned}$$

Using the parametrization

$$x_1 = \cosh\rho\cos\theta, \qquad x_2 = \tanh\rho\cos\alpha,$$

we express $\cosh\rho, \cos\theta$ and $\sin\alpha$ in terms of plane coordinates as follows

$$\cosh\rho = \frac{1}{\sqrt{2(1-x_2^2)}}\left[1+r^2+y^2+\sqrt{(1+r^2+y^2)^2-4r^2}\right]^{\frac{1}{2}},$$

$$\cos\theta = \frac{1}{\sqrt{2}}\left[1+r^2+y^2-\sqrt{(1+r^2+y^2)^2-4r^2}\right]^{\frac{1}{2}},$$

$$\sin\alpha = \frac{\sqrt{1-x_2^2}}{\sqrt{2}}\left[\frac{-2y^2+x_2^2(r^2+y^2-1+\sqrt{(1+r^2+y^2)^2-4r^2})}{x_2^2(r^2+y^2-1)+x_2^4-y^2}\right]^{\frac{1}{2}}$$

where $r = x_1\sqrt{1-x_2^2}$. Using (2.69) we find

$$Z = -\frac{1}{2}\frac{r^2+y^2-1}{\sqrt{(r^2+y^2+1)^2-4r^2}}. \tag{2.125}$$

We can now find the Kähler potential using (2.120). The Kähler potential is given by

$$K = \frac{1}{4}\left[-R+(y^2+2)\log(1+r^2+y^2+R^2)-y^2\log\left(2\frac{(r^2-1)R+R^2-y^2(1+r^2+y^2)}{y(r^2-1)^2}\right)\right]K_0+y^2K_1 \tag{2.126}$$

where $R \equiv \sqrt{(1+r^2+y^2)^2-4r^2}$. This completes our discussion of the $AdS_5\times S^5$ geometry.

### 2.5.7 Monge-Ampére equation

Our goal here is derive the Monge-Ampére equation that governs the function $Z$ that determines the 1/4 BPS geometry. The Monge-Ampére equation is the differential equation that is solved by the Kähler potential. Since we know the relationship between the Kähler potential and $Z$, the Monge-Ampére equation fixes $Z$. Following the analysis [50, 51], we start by writing the expression for the complex structure $\mathcal{J}$, in terms of $L$ and $\omega$, as

$$d\mathcal{J} = e^H L\wedge d\omega. \tag{2.127}$$

We recall that $L$ is a closed form. We know that the non-zero component of $L$ is pointing in the $y$ direction and $\omega$ is the Kähler form. The equation (2.127) implies that the complex structure $\mathcal{J}$ is defined on the space perpendicular to the 4-dimensional space and the $y$ direction. Even though $\mathcal{J}$ live in the space that is perpendicular to the space of coordinates $(z_a,\bar{z}_a;y)$, it still has a $y, z_a$ and $\bar{z}_a$ dependence. Split the exterior derivative as

$$d = \tilde{d} + d_y + d_t$$

where $\tilde{d}$ takes into account the differentiation with respect to the coordinates $z_a$ and $z_b$ and their conjugates. The complex structure $\mathcal{J}$ is closed (i.e $\tilde{d}\mathcal{J}=0$). The $y$ dependence of (2.127) is

$$\partial_y\mathcal{J} = e^H\Lambda\tilde{d}\omega \tag{2.128}$$

where we have used (2.42) and $\omega$ is defined on the complex manifold where it derivative is denoted by $\tilde{d}$. Another usefully identity is the $y$ derivative of $\omega$ which is given by

$$\partial_y\omega = -\frac{1}{y}\mathcal{J}\cdot Z.$$

Using the consistency condition $d(d\omega) = 0$ and using equation (2.128), this last equation becomes

$$\begin{aligned} d(\partial_y \omega) &= d(-\frac{1}{y}\mathcal{J} \cdot Z) \\ y\partial_y \tilde{d}\omega &= -\tilde{d}(\mathcal{J} \cdot Z) \\ y\partial_y(\frac{\partial_y \mathcal{J}}{y}) &= -\tilde{d}(\mathcal{J} \cdot Z). \end{aligned}$$

The differential equation

$$y\partial_y(\frac{\partial_y \mathcal{J}}{y}) + \tilde{d}(\mathcal{J} \cdot Z) = 0 \tag{2.129}$$

is solved by (2.120). Another expression worth describing is the one-form of the Killing vector $dK$, given by

$$dK = \frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} dG \wedge K + dH \wedge K - (f_2^2 + f_1^2) d\omega$$

where the Killing vector $K$ is found by solving the Killing equation (2.97). This expression tells us that the one-form of the Killing vector $dK$ is pointing in a direction perpendicular to $y$ and $t$. The last term hints that this perpendicular direction is tangent to the Kähler manifold. Taking equation (2.95) and contracting with $L^\mu$, we get

$$L^\mu \partial_\mu f_1^2 = -\frac{1}{4} d\Omega_{\mu\rho\sigma\tau} L^\mu K^\rho I^{\sigma\tau}$$

where $\Omega$ is given by $\Omega = K \wedge I$. To make sure that the RHS is non-zero, we can set $\mu = y$ and $\rho = t$, to obtain

$$\begin{aligned} L^y f_1^2 \partial_y (H - G) &= -\frac{e^{\frac{1}{2}(H+G)} f_1}{4} d(K \wedge I)_{yt\sigma\tau} L^y K^t I^{\sigma\tau} \\ &= -\frac{e^{\frac{1}{2}(H+G)} f_1}{4} \left( \frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} dG \wedge K \wedge I + dH \wedge K \wedge I - (f_2^2 + f_1^2) d\omega \wedge I \right)_{yt\sigma\tau} L^y K^t I^{\sigma\tau} \\ &= -\frac{e^{\frac{1}{2}(H+G)} f_1}{4} \left( -(f_2^2 + f_1^2) d\omega \wedge I \right)_{yt\sigma\tau} L^y K^t I^{\sigma\tau}. \end{aligned}$$

This simplifies to

$$f_1 \Lambda \partial_y (H - G) = \frac{e^{\frac{1}{2}(H+G)}}{4} (f_2^2 + f_1^2) \tilde{d}\omega_{ab} I^{ab}. \tag{2.130}$$

The LHS tells us that the quantity on the RHS is pointing in the $y$ direction and the relationship between $\mathcal{J}$ and $I$ is given by

$$\mathcal{J} = e^{G+H} I.$$

We then have

$$\begin{aligned} f_1 \partial_y \Lambda (H - G) &= \frac{e^{\frac{1}{2}(H-G)}}{4} e^{-(G+H)} (f_2^2 + f_1^2) \tilde{d}\omega_{ab} \mathcal{J}^{ab} \\ 4e^{-\frac{1}{2}(H-G)} \frac{f_1}{f_2^2 + f_1^2} \partial_y (H - G) &= \tilde{d}\omega_{ab} \mathcal{J}^{ab}. \end{aligned}$$

Taking equation (2.128) and contracting with $\mathcal{J}$, we find

$$\begin{aligned} \mathcal{J}^{ab} \partial_y \mathcal{J}_{ab} &= \mathcal{J}^{ab} e^H \Lambda \tilde{d}\omega_{ab} \\ \mathcal{J}^{ab} \partial_y \mathcal{J}_{ab} &= 4e^H \Lambda e^{-\frac{1}{2}(H-G)} \frac{f_1}{f_2^2 + f_1^2} \partial_y (H - G). \end{aligned}$$

It is natural to associate the term $\mathcal{J}^{ab}\partial_y \mathcal{J}_{ab}$ with $\partial_y \log\det h_{ab}$ , that is

$$\mathcal{J}^{ab}\partial_y \mathcal{J}_{ab} \equiv \frac{1}{2}\partial_y \log\det h_{ab}.$$

We now have

$$\begin{aligned}\partial_y \log\det h_{ab} &= 2e^H e^{-\frac{1}{2}(H-G)}\frac{e^{\frac{1}{2}(H-G)}}{e^{(H+G)}+e^{(H-G)}}\partial_y H - 2e^H e^{-\frac{1}{2}(H-G)}\frac{e^{\frac{1}{2}(H-G)}}{e^{(H+G)}+e^{(H-G)}}\partial_y G\\ &= 2\frac{e^{-G}}{e^G+e^{-G}}\partial_y H - 2\frac{e^{-G}}{e^G+e^{-G}}\partial_y G.\end{aligned}$$

We simplify further by using the identities

$$\begin{aligned}\partial_y \log\left(Z+\frac{1}{2}\right) &= \frac{\operatorname{sech}^2 G\ \partial_y G}{2\left(\frac{1}{2}\tanh G+\frac{1}{2}\right)}\\ &= \frac{2e^{-G}}{e^G+e^{-G}}\partial_y G, &(2.131)\\ \frac{1}{2}-Z &= \frac{1}{1+e^{2G}}, &(2.132)\\ e^H &= y \Rightarrow \partial_y H = \frac{1}{y}. &(2.133)\end{aligned}$$

Then we have

$$\partial_y \log\det h_{ab} = \partial_y \log\left(Z+\frac{1}{2}\right) - \frac{2}{y}\left(\frac{1}{2}-Z\right). \tag{2.134}$$

Now, we solve this differential equation

$$\begin{aligned}\log\det h_{ab} &= \log\left(Z+\frac{1}{2}\right)+c_0-\log y + c_1 + 2\int dy\frac{1}{y}\left(-\frac{1}{2}y\partial_y(y^{-1}\partial_y K)\right) &(2.135)\\ &= \log\left(Z+\frac{1}{2}\right) - \log y - (y^{-1}\partial_y K)+c_3 &(2.136)\\ \det h_{ab} &= -y\left(Z+\frac{1}{2}\right)e^{-(y^{-1}\partial_y K)+c_3} &(2.137)\\ &= -y\frac{e^{-(y^{-1}\partial_y K)}}{2}\left(-y\partial_y(y^{-1}\partial_y K)+1\right)c_4 &(2.138)\end{aligned}$$

where the $c$'s are functions independent of $y$. To determine them we need to know the boundary conditions that govern the differential equation. In general $c_4$ is a function of $z$ and $\bar{z}$

$$\det h_{ab} = -y\frac{e^{-(y^{-1}\partial_y K)}}{2}\left(-y\partial_y(y^{-1}\partial_y K)+1\right)c_4(z,\bar{z}).$$

This is the Monge-Ampére equation that we were after.

In this section we have managed to solve the Killing spinor equation for the 1/4 BPS geometry. We find that in order to obtain smooth 1/4 BPS geometries we need to single out a 4-dimensional plane and impose boundary conditions on this plane. The boundary conditions force $Z(z_a,\bar{z}_a)=\pm\frac{1}{2}$. Thus finding 1/4 BPS geometries is replaced by the problem of solving a differential equation, the Monge-Ampére equation. This Monge-Ampére equation is solved by$Z(z_a,\bar{z}_a)$. In order to gain confidence in our analysis we will consider an example and test regularity of the 1/4 BPS geometry.

## 2.6 Checking regularity of a 1/4 BPS geometry

We would like to test our regularity condition for 1/4 BPS geometries. We will consider an example and test if we get a smooth geometry. We saw that $Z = \frac{1}{2}\tanh G$ admits the expansion (2.72). To solve for $K$, we equate (2.72) and (2.120) and choose $Z = -\frac{1}{2}$. This gives

$$\begin{aligned} -\frac{1}{2} + y^2 a_0 + \cdots &= -\frac{1}{2} y\partial_y(y^{-1}\partial_y K) \\ \frac{1}{y} + a_0' + \cdots &= \partial_y(y^{-1}\partial_y K) \\ \ln y + c_0 + \frac{1}{2}y^2 a_0' + \cdots &= y^{-1}\partial_y K \\ y\ln y + yc_0 + \frac{1}{2}y^3 a_0' + \cdots &= \partial_y K \\ \int y \ln y \; dy + \frac{1}{2}y^2 c_0 + \frac{1}{8}y^4 a_0' + c_1 + \cdots &= K. \end{aligned}$$

Up to order $y^2$ this expression simplifies to

$$K = \int y \ln y \; dy + K_0 + K_1 y^2 + O(y^4) \tag{2.139}$$

where $a_0', c_0, c_1, K_0$ and $K_1$ are functions of $z$ and $\bar{z}$. In the case where $Z = \frac{1}{2}$ we have

$$\frac{1}{2} + y^2 f_0 + \cdots = -\frac{1}{2} y\partial_y(y^{-1}\partial_y K)$$

and the Kähler potential is expressed as

$$K = -\int y \ln y \; dy + K_0 + K_1 y^2 + O(y^4). \tag{2.140}$$

Unlike in the 1/2 BPS case, here we find an additional constraint is imposed on the 4-dimensional metric $h_{a\bar{b}}$. This metric solves a Monge-Ampere type differential equation, given by

$$\log(\det h_{a\bar{b}}) = \log(Z + \frac{1}{2}) + \log y + \frac{1}{y}\partial_y K + D. \tag{2.141}$$

This equation simplifies to

$$\begin{aligned} \det h_{a\bar{b}} &= y(Z + \frac{1}{2}) e^{\frac{1}{y}\partial_y K} E & (2.142)\\ &= y(Z + \frac{1}{2}) e^{\frac{1}{y}\partial_y K} E, & (2.143) \end{aligned}$$

where $D$ and $E$ are functions that are independent of $y$. This last expression is approximately

$$\begin{aligned} \det h_{a\bar{b}} &= y(-\frac{1}{2} + y^2 f_0 + \frac{1}{2}) \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{1}{y}\partial_y \left( \int dy \; y \log y + K_0(z,\bar{z}) + y^2 K_1(z,\bar{z}) \right) \right]^n E \\ &= y^2(-\frac{1}{2} + y^2 f_0 + \frac{1}{2}) \sum_{n=0}^{\infty} \frac{1}{n!} \left[ K_1(z,\bar{z}) \right]^n E \\ &= y^4 f_0 \; e^{K_1(z,\bar{z})} E \\ &= y^4 g_0 \; e^{K_1(z,\bar{z})} \end{aligned}$$

where $g_0$ is a function of $z$ and $\bar{z}$ and both $f_0$ and $g_0$ are positive functions. To check for regularity, we will consider the metric (2.119) given by

$$ds_{10}^2 = -h^{-2}(dt+\omega)^2 + h^2 \left( \frac{2}{Z+\frac{1}{2}} \partial_a \bar{\partial}_b K dz^a d\bar{z}^b + dy^2 \right) + y(e^{-G} d\Omega_3^2 + e^G d\psi^2).$$

### 2.6.1 Regular geometry in the region $Z = \pm\frac{1}{2}$

**Region** $Z = -\frac{1}{2}$

Consider the case where $Z = -\frac{1}{2}$ which implies that $G = +\infty$. This represents the case where the $S^3$ shrinks to zero and $S^1$ remains finite size. We can use the following approximate forms for the functions in the metric

$$h^{-2} = 2y\cosh G \sim ye^{-G}, \tag{2.144}$$

$$ye^{-G}d\psi^2 = h^{-2}d\psi^2, \tag{2.145}$$

$$ye^{G}d\Omega_3^2 = y(yh^2)d\Omega_3^2 = y^2h^2\Omega_3^2, \tag{2.146}$$

$$\begin{aligned}\frac{2}{Z+\frac{1}{2}}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b &= \frac{2}{Z+\frac{1}{2}}h_{a\bar{b}}dz^a d\bar{z}^b \\ &= 2(-\frac{1}{2}+y^2a_0+\frac{1}{2})^{-1}h_{a\bar{b}}dz^a d\bar{z}^b \\ &= 2y^{-2}a_0^{-1}h_{a\bar{b}}dz^a d\bar{z}^b.\end{aligned} \tag{2.147}$$

The metric becomes

$$ds_{10}^2 = h^{-2}\Bigg[-(dt+\omega)^2 + d\psi^2\Bigg] + h^2\Bigg[2y^{-2}a_0^{-1}h_{a\bar{b}}dz^a d\bar{z}^b + dy^2 + y^2 d\Omega_3^2\Bigg]. \tag{2.148}$$

To demonstrate that this metric is regular as $y \to 0$, we need to focus on the term $2y^{-2}a_0^{-1}h_{a\bar{b}}dz^a d\bar{z}^b$ and make sure that it remains finite in this limit. As $y \to 0$ this term blows up unless all the components of $h_{a\bar{b}}$ vanish as $y^2$. Consider (2.147), in the limit $y = 0$, to learn

$$\begin{aligned}\det h_{a\bar{b}} &= 0 \\ \partial_1\bar{\partial}_1 K\partial_2\bar{\partial}_2 K - \partial_1\bar{\partial}_2 K\partial_2\bar{\partial}_1 K &= 0 \\ \partial_1\bar{\partial}_1 K_0\partial_2\bar{\partial}_2 K_0 - \partial_1\bar{\partial}_2 K_0\partial_2\bar{\partial}_1 K_0 &= 0.\end{aligned}$$

In the last line we used (2.140). This is the only restriction we find from equation (2.143). To make sure that we have a regular solution, we can can either require that $K_0 = 0$ so that

$$h_{a\bar{b}} = \partial_a\bar{\partial}_b K(z,\bar{z},y=0) = 0$$

or we can require that $K_0$ vanishes on the boundary of the region $Z = -\frac{1}{2}$ and remains finite inside the region.

**Region** $Z = \frac{1}{2}$

In the case $Z = \frac{1}{2}$, $G = -\infty$ and the circle $S^1$ shrinks to zero size while $S^3$ remains finite. In this case, we employ the following approximations

$$h^{-2} = 2y\cosh G \sim ye^{G}, \tag{2.149}$$

$$ye^{-G}d\psi^2 = y^2h^2d\psi^2, \tag{2.150}$$

$$ye^{G}d\Omega_3^2 = h^{-2}d\Omega_3^2, \tag{2.151}$$

$$\begin{aligned}\frac{2}{Z+\frac{1}{2}}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b &= 2(\frac{1}{2}+y^2a_0+\frac{1}{2})^{-1}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b \\ &= 2(1+y^2a_0)^{-1}\partial_a\bar{\partial}_b(-\int y\ln y\; dy + K_0 + K_1y^2 + O(y^4))dz^a d\bar{z}^b \\ &= 2\partial_a\bar{\partial}_b K_0 dz^a d\bar{z}^b.\end{aligned} \tag{2.152}$$

This last line is only true once we take $y = 0$. The metric then becomes

$$ds_{10}^2 = h^{-2}\bigg[-(dt+\omega)^2 + d\Omega_3^2\bigg] + h^2\bigg[2\partial_a\bar{\partial}_b K_0 dz^a d\bar{z}^b + dy^2 + y^2 d\psi^2\bigg]. \tag{2.153}$$

This metric automatically remains regular at $y = 0$ and $Z = \frac{1}{2}$.

### 2.6.2 Shape of the 1/4 BPS droplets

The boundaries of the droplets that determine the 1/2 BPS geometry, were allowed to take any arbitrary shape. We will now investigate the shape of the boundary of droplets of the 1/4 BPS description. To do that, we study the example of $AdS_5 \times S^5$ on $R \times S^3$. The metric of this space is given by

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \sin^2\theta d\psi^2 + d\theta^2 + \cos^2\theta[\cos^2\alpha d\phi_1^2 + \sin^2\alpha d\phi_2^2 + d\alpha^2]. \tag{2.154}$$

To introduce complex coordinates for this metric, we define

$$z_1 = r\cos\alpha e^{i(\phi_1+t)}, \qquad z_2 = r\sin\alpha e^{i(\phi_1+t)}, \qquad r = \cosh\rho\cos\theta. \tag{2.155}$$

Following the same procedure as in section (2.5.6), we find

$$y = \sinh\rho\sin\theta, \tag{2.156}$$

$$h^{-2} = \sinh^2\rho\sin^2\theta, \tag{2.157}$$

$$Z = \frac{h^2}{2}(r^2 + y^2 - 1) \tag{2.158}$$

and the Kähler potential is

$$K = \frac{1}{2}[\Psi - \log\Psi y^2 \log(\Psi - r^2) + y^2\log y - y^2] \tag{2.159}$$

where $\Psi \equiv \frac{1}{2}(r^2+y^2+1) + \sqrt{\frac{1}{2}(r^2+y^2+1)^2 + y^2}$. Since the product of the radii of $S^1$ and $S^3$ vanish when $y = 0$, we impose the following conditions on (2.156)

$$\rho = 0: \qquad r = \cos\theta \le 1, \quad Z = -\frac{1}{2}$$

$$\theta = 0: \qquad r = \cosh\rho \ge 1, \quad Z = \frac{1}{2}.$$

From the equation for $Z$ we see that the regions $Z = \frac{1}{2}$ and $Z = -\frac{1}{2}$ are separated by a 3-sphere with $r = 1$. In terms of the complex coordinates this correspond to

$$z_1\bar{z}_1 + z_2\bar{z}_2 = 1. \tag{2.160}$$

This boundary condition tells us that at each point on the 3-sphere, the space is locally flat. This can be seen explicitly by moving to new coordinates $v, R, \zeta$ such that

$$\sqrt{(r^2+y^2-1)+4y^2}, \qquad \cos^2\zeta = \frac{ye^G}{R}. \tag{2.161}$$

Useful relations between the old and the new coordinates are

$$ye^{-G} = R\sin^2\zeta, \quad y = \frac{R}{2}\sin 2\zeta, \quad Z = \frac{1}{2}\cos 2\zeta, \quad h^{-2} = R.$$

What we have to do now is to find the one-form $\omega$ from the Kähler potential. In general, $\omega$ is given by the equation

$$\omega = \frac{i}{2}\left[\frac{1}{y}\bar{\partial}\partial_y K - \frac{1}{y}\partial\partial_y K\right]. \tag{2.162}$$

To prove local flatness of our metric, we expand to order $O(R)$. To write the 4-dimensional metric using our new coordinates, first note that the Kähler potential from (2.159) depends on $v$ in the new coordinates, and we have

$$\begin{aligned} h_{ab} &= 2\partial_a\bar{\partial}_b K dz^a d\bar{z}^b \\ &= 2\partial_v K dz_a d\bar{z}_a + 2\partial_v^2 K|\bar{z}_a dz_a|^2. \end{aligned}$$

Now, taking the derivative of the Kähler potential, we get

$$\begin{aligned} \partial_v K &= \frac{1}{4(1+v)}[(v-y^2)+R] = \frac{R}{4}(\cos 2\zeta + 1) + O(R^2), \\ \partial_v^2 K &= \frac{1}{4}(\cos 2\zeta + 1) + O(R). \end{aligned}$$

The 4-dimensional metric then becomes

$$\begin{aligned} h_{ab} &= 2(\frac{R}{4}(\cos 2\zeta + 1) + O(R^2))dz^a d\bar{z}^b + 2(\frac{1}{4}(cos2\zeta + 1) + O(R))|\bar{z}_a dz_a|^2 \\ &= (Z + \frac{1}{2})[R dz^a d\bar{z}^b + |\bar{z}_a dz_a|^2] + \cdots \end{aligned}$$

Using (2.162) we find

$$\begin{aligned} \omega &= \frac{1}{8(1+v)}(-2 + \frac{4+v+y^2}{R})\eta \\ &= \left[\frac{1}{2R} + O(R)\right] \end{aligned}$$

where we have defined a new variable $\eta \equiv i(z_a d\bar{z}_a - \bar{z}_a dz_a)$. Plugging this to (2.119) we see that the subleading contribution to the 10-dimensional metric is given by

$$ds_{10}^2 \approx -dt\eta + dx_\perp dx_\perp + \frac{dR^2}{4R} + R(d\zeta^2 + \cos^2\zeta d\Omega_3^2 + \sin^2\zeta d\psi).$$

We see that this metric is regular at $R = 0$ and hence regularity is not affected by the subleading order. So, the boundary condition implies that each point on the 3-sphere is locally flat. Next consider the shape of the droplet of this geometry. Choosing suitable coordinates for this analysis, which happen to be spherical coordinates,

$$\tan\zeta = e^{-G}, \qquad R = \frac{2y}{\sin 2\zeta} \tag{2.163}$$

we find

$$Z = \frac{1}{2}\cos 2\zeta, \qquad h^{-2} = R. \tag{2.164}$$

Recall that $R$ in these coordinates measures the distance from the wall and we are interested in the leading term in a small $R$-expansion. The regularity condition requires that the coordinate $v$ which is defined by $v \equiv R\cos 2\zeta$ is independent of the $y$-coordinate, but it can depend on Kähler base. The metric that is obtained in the leading term of the R-expansion is a flat metric. It can be checked that the one-form $dv$ lies in the Kähler subspace, that is $\partial_y v = 0$ and this is true at leading order in $R$. Using (2.120) we express $Z$ as

$$\cos 2\zeta = -y\partial_y(y^{-1}\partial_y K) \tag{2.165}$$

where $\cos 2\zeta$ can also be expressed in terms of the coordinates $v$ and $y$ as $\cos 2\zeta = \dfrac{v}{\sqrt{v^2+4y^2}}$. This allows us to write the Kähler potential in terms of $v$ and $y$ using (2.159) as

$$K = \frac{v}{8}\sqrt{v^2+8y^2} + \frac{y^2}{2}\left(\log(v+\sqrt{v^2+4y^2}) - \log y\right) + K_0(z,\bar{z}) + y^2 K_1(z,\bar{z}). \tag{2.166}$$

The expression for $\omega$ is given by

$$\omega = \frac{i}{2y}\partial_y\partial_v K(\partial - \bar{\partial})v \tag{2.167}$$

$$= \frac{1}{y}\partial_y\partial_v K\eta \tag{2.168}$$

$$\sim \frac{1}{R}\eta \tag{2.169}$$

where we have used equation (2.162) and that $K$ is a function of $y$ and $v$, and we used $\eta \equiv \dfrac{i}{2}(\partial - \bar{\partial})v$. Plugging this result into the metric (2.119), the leading term is

$$ds^2 = \underbrace{-2R\omega dt - R\omega + \frac{1}{R\cos^2\zeta}\left(2\partial_\nu^2 K|\partial v|^2 + 2\partial_\nu K\partial\bar{\partial}v - \frac{1}{4}\cos^2\zeta dv^2\right)}_{I} + \underbrace{\frac{dR^2}{R} + R(d\zeta^2 + \cos^2\zeta d\Omega_3^2 + \sin^2\zeta d\phi^2)}_{II}.$$

From this metric we can easily tell that (II) gives a regular geometry while (I) has a possible singularity at $R\cos^2\zeta = 0$. To ensure $I$ is regular, we proceed as follows

$$\begin{aligned} ds_4^2 &= -2R\omega dt - R\omega + \frac{1}{R\cos^2\zeta}\left(2\partial_\nu^2 K|\partial v|^2 + 2\partial_\nu K\partial\bar{\partial}v - \frac{1}{4}\cos^2\zeta dv^2\right) \\ &= -2R\omega dt - \left(\frac{R}{y^2}(\partial_y\partial_v K)^2 - \frac{2\partial_v K}{R\cos^2\zeta}\right)\eta^2 + \frac{2\partial_\nu K}{R\cos^2\zeta}\partial\bar{\partial}v + \frac{1}{4R}\left(\frac{2\partial_v^2 K}{\cos^2\zeta} - 1\right)dv^2. \end{aligned}$$

To move from the first line to the second line we expanded $|\partial v|^2$ and used (2.168). Lets analyse each possible non-regular term of this metric, to leading order in $R$. Using (2.159) we have

$$\begin{aligned} \partial_v K &\sim R, \\ \partial_v^2 K &= \frac{1}{2}\cos^2\zeta + O(R), \\ \partial_y\partial_v K &= \frac{y}{R} + O(R). \end{aligned}$$

From this we conclude that the metric $ds_4^2$ is regular and we can write it as

$$\begin{aligned} ds_4^2 &= -2R\omega dt - \left(\frac{R}{y^2}(\partial_y\partial_v K)^2 - \frac{2\partial_v K}{R\cos^2\zeta}\right)\eta^2 + \frac{2\partial_v K}{R\cos^2\zeta}\partial\bar{\partial}v + \frac{1}{4R}\left(\frac{2\partial_v^2 K}{\cos^2\zeta} - 1\right)dv^2 \\ &= -2\eta dt + \partial\bar{\partial}v + \lambda_1 dv^2 + \lambda_2\eta^2. \end{aligned} \tag{2.170}$$

From this metric and the above approximation scheme we see that $\lambda_1$ and $\lambda_2$ should be independent of $v$ and $y$. To make sure that metric (2.170) stays regular we must require that $ds_4^2$ is determined by a holomorphic one-form $\xi$. The reason we do this is because we know that $\xi$ is continuous everywhere and it guarantees regularity. Therefore, we write the metric as

$$\partial\bar{\partial}v + \lambda_1 dv^2 + \lambda_2\eta^2 = \xi\bar{\xi} + O(v).$$

The RHS tells us that the anti-holomorphic terms on the LHS must sum to zero. This means all the terms of form $dz^a dz^b$ and $d\bar{z}^a d\bar{z}^b$ must cancel on the LHS. This tell us that we can split this last equation into two parts which are holomorphic and non-holomorphic as follows

$$\begin{aligned}\xi\bar{\xi} + O(v) &= \partial\bar{\partial}v + \lambda_1 dv^2 + \lambda_2 \left[\frac{i}{2}(\partial - \bar{\partial})v\right]\left[\frac{i}{2}(\partial - \bar{\partial})v\right] \\ &= \partial\bar{\partial}v + \lambda_1 dv^2 - \frac{\lambda_2}{4}\left[\partial v\partial v + \bar{\partial}v\bar{\partial}v - \partial v\bar{\partial}v - \bar{\partial}v\partial v\right]. \end{aligned} \tag{2.171}$$

The **anti-holomorphic** part is given by

$$\lambda_1 dv^2 - \frac{\lambda_2}{4}\left[\partial v\partial v + \bar{\partial}v\bar{\partial}v\right] = O(v) \tag{2.172}$$

and the **holomorphic** part is given by

$$\partial\bar{\partial}v + \frac{\lambda_2}{2}\partial v\bar{\partial}v = \xi\bar{\xi} + O(v). \tag{2.173}$$

To write a general equation of the type (2.173), first notice that on the surface of the regions $Z = \pm\frac{1}{2}$ where $v = 0$, the one-form $\xi$ describes a two dimensional space. This implies that by changing coordinates we can always write $\xi$ as $\xi = f dw$ were $dw$ is a one-form and $f$ is an arbitrary continuous function on this space. Making this kind of change of coordinate and requiring that $dv$ and $dw$ are independent, we can write the holomorphic equation (2.173) as

$$\partial_a\bar{\partial}_b v + \lambda\partial v_a\bar{\partial}_b v = g\partial_a w\bar{\partial}_b w + O(v). \tag{2.174}$$

We conclude that the shape of the droplet whose boundary condition is defined by $v(z_a, \bar{z}_b) = 0$ will give a smooth metric only if $v$ satisfies (2.174). Equation (2.174) put an extra constraint on $v$. In the 1/2 BPS analysis we saw that the droplet can take any shape, while here in the 1/4 BPS case the boundary condition of the droplet obeys a nontrivial contraint.

### 2.6.3 Condition for regular droplets

We see that an important equation to understand in the 1/4 BPS analysis is (2.174). By specifying $\lambda$ and the boundary condition of $v$, we can uniquely solve (2.174). To determine $\lambda$ we need to take the determinant of (2.174). Taking the determinant we obtain

$$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = O(v). \tag{2.175}$$

Since $dv$ and $dw$ are independent

$$\det(\partial_a\bar{\partial}_b v)\Big|_{v=0} \neq 0. \tag{2.176}$$

After finding $\lambda$, one can go back to the differential equation (2.174) and check the remaining conditions

$$\partial_a\bar{\partial}_b v + \lambda\partial v_a\bar{\partial}_b v = \xi\bar{\xi} + O(v), \qquad \xi_a dz^a = f dw + v\xi'. \tag{2.177}$$

Lets do some examples to test (2.174).

### 2.6.4 Example: Regular geometry from constrained droplets

We will study various examples of surfaces, and check if they obey (2.174). The procedure that we will follow in checking the singularity or the regularity of the surface is as follows

1. Write down the equation for the geometry in terms of $v$ such that it allows the condition $v = 0$ on the surface.

2. Check if the condition (2.176) is satisfied. If it is not satisfied then the surface defined by $v$ is not regular and we stop the calculation. If it is satisfied, we then proceed to the next step.

3. Use (2.175) to determine $\lambda$.

4. Once we find $\lambda$, we check if the conditions in (2.177) are satisfied, If they are we conclude that the surface defines a regular solution and if not the surface leads to a singular solution.

It is important to note that the conditions imposed by (2.174) are the same conditions given by (2.176) and (2.177), but in a more convenient way.

**Example:** $v = z_a\bar{z}_a - 1$

Following the procedure we outlined above, we have

1. $v = z_a\bar{z}_a - 1$ represents the surface of a sphere with radius 1.

2. $$\det(\partial_a\bar{\partial}_b v)\Big|_{v=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= 1.$$

   Since this determinant is not equal to zero, we proceed to step 3.

3. $$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = \det\begin{pmatrix} \partial_1\bar{\partial}_1 v + \lambda\partial_1 v\bar{\partial}_1 v & \partial_1\bar{\partial}_2 v + \lambda\partial_1 v\bar{\partial}_2 v \\ \partial_2\bar{\partial}_1 v + \lambda\partial_2 v\bar{\partial}_1 v & \partial_2\bar{\partial}_2 v + \lambda\partial_2 v\bar{\partial}_2 v \end{pmatrix}$$
$$= \det\begin{pmatrix} 1+\lambda z_1\bar{z}_1 & \lambda z_2\bar{z}_1 \\ \lambda z_1\bar{z}_2 & 1+\lambda z_2\bar{z}_2 \end{pmatrix}.$$

   Evaluating this at order $O(v)$, which correspond to $O(\lambda)$, we get

$$\begin{aligned}\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) &= 1 + \lambda z_a\bar{z}_a \\ &= 1 + \lambda(v+1) \\ &= 1 + \lambda + \lambda v.\end{aligned}$$

   To satisfy (2.175), $\lambda = -1$.

4. The one-form that corresponds to the solution of $\lambda$ in step 3 can be read from the equation

$$\begin{pmatrix} 1+\lambda z_1\bar{z}_1 & \lambda z_2\bar{z}_1 \\ \lambda z_1\bar{z}_2 & 1+\lambda z_2\bar{z}_2 \end{pmatrix} = \begin{pmatrix} \xi_1\bar{\xi}_1 & \xi_1\bar{\xi}_2 \\ \xi_2\bar{\xi}_1 & \xi_2\bar{\xi}_2 \end{pmatrix} + O(v)$$

   which implies that

$$\xi_1\bar{\xi}_2 = -z_2\bar{z}_1, \quad \xi_2\bar{\xi}_1 = -z_1\bar{z}_2.$$

   We can read off the values of the $\xi$'s. They are given by

$$\xi_1 = z_2, \quad \bar{\xi}_2 = -\bar{z}_1; \qquad \xi_2 = -z_1, \quad \bar{\xi}_1 = \bar{z}_2.$$

   Using the expansion in (2.177) we have

$$\begin{aligned}\xi &= \xi_1 dz_1 + \xi_2 dz_2 \\ &= z_2 dz_1 - z_1 dz_2.\end{aligned} \tag{2.178}$$

   This solution satisfies the differential conditions (2.177). We conclude that this droplet leads to a regular solution.

**Example:** $v = z_1\bar{z}_1 - z_2\bar{z}_2 - C$

1. $v = z_1\bar{z}_1 - z_2\bar{z}_2 - C$, represents the surface of a hyperboloid.

2. 
$$\det(\partial_a\bar{\partial}_b v)\Big|_{v=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ = -1.$$

   The determinant is not equal to zero so we proceed to the next step.

3. 
$$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = \det\begin{pmatrix} \partial_1\bar{\partial}_1 v + \lambda\partial_1 v\bar{\partial}_1 v & \partial_1\bar{\partial}_2 v + \lambda\partial_1 v\bar{\partial}_2 v \\ \partial_2\bar{\partial}_1 v + \lambda\partial_2 v\bar{\partial}_1 v & \partial_2\bar{\partial}_2 v + \lambda\partial_2 v\bar{\partial}_2 v \end{pmatrix} \\ = \det\begin{pmatrix} 1 + \lambda z_1\bar{z}_1 & -\lambda z_2\bar{z}_1 \\ -\lambda z_1\bar{z}_2 & -1 + \lambda z_2\bar{z}_2 \end{pmatrix}.$$

   At order $O(v)$, this determinant is
$$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = -1 + (\lambda z_2\bar{z}_2 - \lambda z_1\bar{z}_1) \\ = -(1 + \lambda v + \lambda C).$$

   To satisfy the (2.175), $\lambda = -\frac{1}{C}$.

4. The one-form $\xi$ is represented by
$$\xi = \frac{1}{\sqrt{C}}\Big( - z_2 dz_1 + z_1 dz_2\Big).$$

   This solution satisfies the differential conditions of (2.177). Therefore this droplet again leads to a regular solution.

**Example:** $v = z_a\bar{z}_a + a z_1 z_2 + a\bar{z}_1\bar{z}_2 - C$

1. $v = z_a\bar{z}_a + a z_1 z_2 + a\bar{z}_1\bar{z}_2 - C$.

2. 
$$\det(\partial_a\bar{\partial}_b v)\Big|_{v=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = 1.$$

   The determinant is not equal to zero so proceed to step 3.

3. 
$$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = \det\begin{pmatrix} \partial_1\bar{\partial}_1 v + \lambda\partial_1 v\bar{\partial}_1 v & \partial_1\bar{\partial}_2 v + \lambda\partial_1 v\bar{\partial}_2 v \\ \partial_2\bar{\partial}_1 v + \lambda\partial_2 v\bar{\partial}_1 v & \partial_2\bar{\partial}_2 v + \lambda\partial_2 v\bar{\partial}_2 v \end{pmatrix} \\ = \det\begin{pmatrix} 1 + \lambda(\bar{z}_1 + a z_2)(z_1 + a\bar{z}_2) & \lambda(\bar{z}_1 + a z_2)(z_2 + a\bar{z}_1) \\ \lambda(\bar{z}_2 + a z_1)(z_1 + a\bar{z}_2) & 1 + \lambda(z_2 + a\bar{z}_1)(\bar{z}_2 + a z_1) \end{pmatrix}.$$

   At order $O(v)$, this determinant is
$$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = 1 + \lambda(\bar{z}_1 + a z_2)(z_1 + a\bar{z}_2) + \lambda(z_2 + a\bar{z}_1)(\bar{z}_2 + a z_1) \\ = \lambda(\bar{z}_1 z_1 + a\bar{z}_1\bar{z}_2 + a z_2 z_1 + a^2 z_2\bar{z}_2) + \lambda(z_2\bar{z}_2 + a z_2 z_1 + a\bar{z}_1\bar{z}_2 + a^2\bar{z}_1 z_1) \\ = 1 + \lambda(v + C) + \lambda a^2(\bar{z}_a z_a) + \lambda(a(z_2 z_1 + \bar{z}_1\bar{z}_2)) \\ = 1 + \lambda(v + C) + \lambda a^2(v + C - a z_2 z_1 - a\bar{z}_1\bar{z}_2) + \lambda(a(z_2 z_1 + \bar{z}_1\bar{z}_2)) \\ = 1 + \lambda(v + C)(1 + a^2) + \lambda a(z_2 z_1 + \bar{z}_1\bar{z}_2)(1 - a^2).$$

Remember that we want to $\lambda$ to be a constant i.e. it must not depend on $z$ and $\bar{z}$. To satisfy all the necessary conditions we need

$$a(1-a^2)=0, \qquad \lambda = -\frac{1}{C(1+a^2)}. \tag{2.179}$$

There are three possible values of $a$, which are $a=0$ and $a=\pm 1$. We have already investigated the case where $a=0$ which leads to a regular solution. Now consider the case where $a=1$. The case $a=-1$ is very similar. When $a=1$, the surface is $v = z_a\bar{z}_a + z_1 z_2 + \bar{z}_1\bar{z}_2 - C$.

4. The one-form $\xi$ is written as

$$\xi = \frac{1}{\sqrt{2C}}\Big((\bar{z}_1+z_2)dz_1 - (z_1+\bar{z}_2)dz_2\Big).$$

Plugging this equation to (2.177), we see that there is an extra contribution which is of order $O(1)$ which now becomes the leading order. We conclude that this droplet leads to a solution that is **not regular** because $v$ fails to satisfy condition (2.177).

In summary, the geometries that arise by solving the 1/4 BPS Killing spinor equation are not all regular. For regular solutions we need to set boundary conditions on the plane $y=0$ which fix $Z=\pm\frac{1}{2}$. This still does not guarantee regularity of the metric, one need to also check each term on the resulting metric that it is regular and the best way to do that is to perform some expansion in one of the parameters of the metric. Once this is done, it is then very crucial that one need to check this new geometry that it satisfy (2.174). Once it satisfy (2.174) and (2.177) we conclude that the droplets of the geometry are regular.

# Chapter 3

# Field Theory

## 3.1 Introduction

In this chapter we turn to a study of conformal field theory (CFT). Computing the $n$-point correlation functions allows one to study physical properties of a given system. The hermitian operators representing physical quantities are called observables. The possible measured values of observables correspond to the eigenvalues of the hermitian operators. We want to find a basis for the local operators of the CFT that will diagonalize the 2-point function. One choice for the operators that does this job are the Schur and restricted Schur polynomials. To study these operators we will first consider the simplest case of a single matrix free field theory.

## 3.2 Single matrix free field theory: Schur polynomial

Schur polynomials diagonalize the free field theory 2-point function. They are functions of a single matrix $Z$. $Z$ is a complex $N \times N$ matrix made out of a pair of hermitian $N \times N$ matrix scalar fields

$$Z = \phi_1 + i\phi_2.$$

The Schur polynomials $\chi_R(Z)$ are expressed as [8]

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Tr(\sigma Z^{\otimes n}). \tag{3.1}$$

The different ingredients appearing in this expression are defined as follows

(i.) $S_n$

$S_n$ is the group of permutations of $n$ objects. Let $A$ be an ordered set of $n$ elements $A = \{1, 2, \cdots, n\}$ i.e. the order in which the elements appear in this set matters. Permutations will rearrange the members of this set in a total of $n!$ distinct ways. As an example, consider a set of three elements $A = \{1, 2, 3\}$. For this set the possible permutations are

$$P = [\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \{3,2,1\}].$$

These permutation can also be expressed in a cycle notation. Using cycle notation, the possible permutations are

$$\begin{aligned}
\sigma_1 &= (1)(2)(3) \\
\sigma_2 &= (23)(1) \\
\sigma_3 &= (12)(3) \\
\sigma_4 &= (123) \\
\sigma_5 &= (132) \\
\sigma_6 &= (13)(2).
\end{aligned} \tag{3.2}$$

When $\sigma$ acts on set $A$, it shuffles the position or the order of the elements in this set. As an example, consider the action of $\sigma_2$ and $\sigma_4$ on set $A$.

$$\begin{aligned}(23)(1)\{1,2,3\} &= \{1,3,2\},\\ (123)\{1,2,3\} &= \{2,3,1\}.\end{aligned}$$

The details of how we carry out the multiplication of permutations is important. There are two possible conventions for multiplying permutations. We think of permutations as a map $\sigma_\omega(i) = l$ where $\omega = 1, 2, \cdots, n!$ labels the permutation and $i = 1, 2, ..., n$. $l$ is the number in the $i^{th}$ position of the set after it has been permuted by $\sigma_\omega$. The two conventions for multiplying permutations are

$$\begin{aligned}\sigma\cdot\rho(i) &= \sigma(\rho(i)) \qquad (I)\\ &= \rho(\sigma(i)) \qquad (II)\end{aligned}$$

and both of these multiplication rules are consistent. As an example of this new point of view $\sigma_4(i)$ above is described as

$$\sigma_4(1) = 2,\ \sigma_4(2) = 3,\ \sigma_4(3) = 1.$$

The operation $\sigma_4 A$ then becomes $\sigma_4 A = \{\sigma_4(1), \sigma_4(2), \sigma_4(3)\} = \{2, 3, 1\}$. We want to study the two permutation multiplication rules in what follows.

**(I)**

This multiplication is the one which is more natural for us. Choose $\sigma = (12)$ and $\rho = (23)$, then

$$\sigma(\rho(1)) = 2,\ \sigma(\rho(2)) = 3,\ \sigma(\rho(3)) = 1.$$

This implies that

$$\begin{aligned}\sigma(\rho(i))A &= \{1,2,3\}\\ &= (123)A.\end{aligned}$$

Thus for $\sigma = (12)$ and $\rho = (23)$ we have $\sigma\cdot\rho(i) = \sigma(\rho(i)) = (123)$.

**(II)**

In what follows we will also realize permutations as a matrix acting in the vector space $V_N^{\otimes n}$. In this space, permutations are realized as

$$(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\cdots\delta^{i_n}_{j_{\sigma(n)}}.$$

As an example $\sigma_4 = (123)$ acting on $V_N^{\otimes 3}$ gives

$$\begin{aligned}(123)^I_K V^K &= \delta^{i_1}_{k_{\sigma(1)}}\delta^{i_2}_{k_{\sigma(2)}}\delta^{i_3}_{k_{\sigma(3)}} V_1^{k_1}V_2^{k_2}V_3^{k_3}\\ &= \delta^{i_1}_{k_2}\delta^{i_2}_{k_3}\delta^{i_3}_{k_1} V_1^{k_1}V_2^{k_2}V_3^{k_3}\\ &= V_1^{i_3}V_2^{i_1}V_3^{i_2} = (V_2V_3V_1)^I.\end{aligned}\tag{3.3}$$

Now, act on the same vector space $V_N^{\otimes 3}$ with the product of permutations given by (12)(23). The result is

$$\begin{aligned}(12)^I_J(23)^J_K V^K &= \delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\delta^{i_3}_{j_{\sigma(3)}}\delta^{j_1}_{k_{\sigma(1)}}\delta^{j_2}_{k_{\sigma(2)}}\delta^{j_3}_{k_{\sigma(3)}} V_1^{k_1}V_2^{k_2}V_3^{k_3}\\ &= \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}\delta^{j_1}_{k_1}\delta^{j_2}_{k_3}\delta^{j_3}_{k_2} V_1^{k_1}V_2^{k_2}V_3^{k_3}\\ &= V_1^{i_2}V_2^{i_3}V_3^{i_1} = (V_3V_1V_2)^I.\end{aligned}\tag{3.4}$$

We see from (3.3) and (3.4) that we get two different answers. We have found that

$$(12)^I_J(23)^J_K = (132)^I_K.$$

This corresponds to the rule $\sigma \cdot \rho(i) = \rho(\sigma(i))$. The cycle notation is cyclic. For example the 3-cycles below are the same permutation

$$(123) = (231) = (312).$$

In addition, the cycle made of $n$ 1-cycles

$$(1)(2)\cdots(n)$$

is the identity (when acting on a set, the identity leaves the set unchanged). From now on we will not indicate the 1-cycles. The total number of elements that belongs to $S_n$ which is called the order of $S_n$ is $n!$.
In group theory, $S_n$ is known as the symmetric group which is a group of permutations. A representation of the symmetric group is a mapping $\Gamma_R(\sigma)$ of $S_n$ onto a set of matrices which respects the following rules:

(I.) $\Gamma_R(e) = 1$

(II.) $\Gamma_R(\sigma_1)\Gamma_R(\sigma_2) = \Gamma_R(\sigma_1\sigma_2)$

where $\Gamma_R(e)$ is the identity operator (matrix) and (II) tells us that the group composition law of the symmetric group can be expressed as matrix multiplication. A complete set of irreducible inequivalent representations of the symmetric group $S_n$ are labelled by all possible valid Young diagrams with $n$ boxes.

(ii.) **Young diagram** $R$

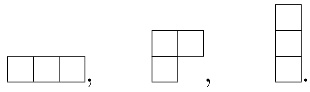
A Young diagram is a finite collection of $n$ boxes stacked together in a manner that the row lengths are weakly decreasing i.e. each row has the same or shorter length than its predecessor. A Young diagram contains the same information as an integer partition. For example consider $S_3$. We can partition 3 as follows:

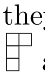
$$n = 3: \quad 3, \quad 2+1, \quad 1+1+1.$$

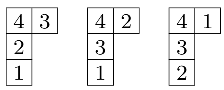
Associate $R$ with the partitions of 3 to get

$$n = 3: \qquad 3, \quad 2+1, \quad 1+1+1$$

$R \vdash 3$ :



We refer to $R$ as an irreducible representation. We see that $S_3$ has 3 inequivalent irreducible representations. The basis of an irreducible representation is labelled by the so-called Young-Yamanouchi symbols. The Young-Yamanouchi symbols are obtained by decorating the Young diagrams with integers $1, 2, 3, \cdots, n$. We assign each box a unique integer. The integers that label the boxes should range from 1 to $n$, where $n$ corresponds to the total number of boxes of the Young diagram. The integer entries in each row decrease as we move to the right and in each column they decrease as we move down. As an example, the valid Young-Yamanouchi symbols for $R =$  are



Any other symbol constructed from this $R$ will not be valid. This includes diagrams such as

$$\begin{array}{|c|c|}\hline 3&4\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 4&2\\\hline 1\\\cline{1-1} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\;.$$

Each Young-Yamanouchi symbol labels a basis vector of a vector space $V_R^{S_n}$. We will use the notation $|R\rangle$ for the Young-Yamanouchi states belonging to irreducible representation $R$. We say that the dimension of this vector space is the dimension of $R$. Thus the dimension of the irrep $\yng(2,1,1)$ is 3. We can compute the dimensions of the irreducible representations of the symmetric group using the Young diagrams labelling the irreducible representations. To do this we need to introduce the hooks of a Young diagram. The hook of a box $Q$ of the Young diagram is equal to the number of boxes to the right of $Q$ plus the number of boxes below $Q$ plus the box itself. This is to say, each box $Q$ has an elbow which is formed by a horizontal line moving to right of the box $Q$ and a vertical line moving from $Q$ to the bottom of the Young diagram. The hook length associated to the box is the number of boxes that the arm covers.

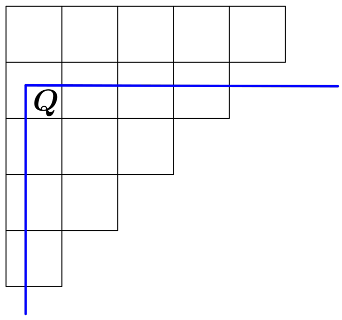



Figure 3.1: The hooks of the box $Q$ is 7.

The hook lengths of the above Young diagram, are given by

$$\text{Hook lengths} = \begin{array}{|c|c|c|c|c|}\hline 9&7&5&3&1\\\hline 7&5&3&1\\\cline{1-4} 5&3&1\\\cline{1-3} 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

and the dimension of the irreducible representation $R$ is given by

$$\begin{aligned} d_R &= \frac{n!}{\prod_{x\in R} hooks(x)} \\ &\equiv \frac{n!}{hooks_R}. \end{aligned} \tag{3.5}$$

As an example, for $R = \yng(5,4,3,2,1)$ we have the hook lengths $= \begin{array}{|c|c|c|c|c|}\hline 9&7&5&3&1\\\hline 7&5&3&1\\\cline{1-4} 5&3&1\\\cline{1-3} 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$ and hence

$$d_{\yng(5,4,3,2,1)} = \frac{15!}{9\cdot 7\cdot 5\cdot 3\cdot 1\cdot 7\cdot 5\cdot 3\cdot 1\cdot 5\cdot 3\cdot 1\cdot 3\cdot 1\cdot 1} = 292864.$$

Thus, this tells us that there are 292864 valid Young-Yamanouchi symbols for this Young diagram. Each inequivalent irreducible representation of the symmetric group is given by a set of matrices $\Gamma_R((\sigma))$ which act on a vector space whose basis is labelled by the Young-Yamanouchi symbols. The matrix elements of $\Gamma_R((\sigma))$ are specified by

$$\Gamma((k,k+1))|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}\,|R_{(k,k+1)}\rangle, \tag{3.6}$$

where $c_i$ is the content of a box of the Young diagram $R$. The content of a box of a Young diagram $R$ in row $a$ and column $b$ is given by $b-a$. The numbers $k$ and $k+1$ in (3.6) refer to boxes labelled $k$ and $k+1$ in the Young-Yamanouchi symbol for state $R$. Below is an example of a Young diagram with the content of each box displayed

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| −1 | 0 | 1 | 2 | |
| −2 | −1 | 0 | | |
| −3 | −2 | | | |
| −4 | | | | |

$|R\rangle$ are the Young-Yamanouchi states constructed from the Young diagram $R$. $|R_{(k,k+1)}\rangle$ is obtained from the Young-Yamanouchi state $|R\rangle$ by exchanging the position of the box labelled $k$ with the box labelled $k+1$, as illustrated below

$$|R\rangle = \begin{array}{|c|c|c|}\hline \phantom{k} & & \\ \hline & k & \\ \cline{1-2} k{+}1 & & \\ \cline{1-1}\end{array}, \qquad |R_{(k,k+1)}\rangle = \begin{array}{|c|c|c|}\hline \phantom{k} & & \\ \hline & k{+}1 & \\ \cline{1-2} k & & \\ \cline{1-1}\end{array}.$$

To illustrate how to compute $\Gamma_R((\sigma))$, choose $\sigma = (23)$ and $R = \yng(3,2)$. Then

$$\Gamma_{\yng(3,2)}((23))\left|\begin{smallmatrix}5&4&3\\2&1\end{smallmatrix}\right\rangle = -\frac{1}{3}\left|\begin{smallmatrix}5&4&3\\2&1\end{smallmatrix}\right\rangle + \frac{\sqrt{8}}{3}\left|\begin{smallmatrix}5&4&2\\3&1\end{smallmatrix}\right\rangle$$
$$\Gamma_{\yng(3,2)}((23))\left|\begin{smallmatrix}5&4&2\\3&1\end{smallmatrix}\right\rangle = \frac{1}{3}\left|\begin{smallmatrix}5&4&2\\3&1\end{smallmatrix}\right\rangle + \frac{\sqrt{8}}{3}\left|\begin{smallmatrix}5&4&3\\2&1\end{smallmatrix}\right\rangle$$
$$\Gamma_{\yng(3,2)}((23))\left|\begin{smallmatrix}5&3&1\\4&2\end{smallmatrix}\right\rangle = -\left|\begin{smallmatrix}5&3&1\\4&2\end{smallmatrix}\right\rangle$$
$$\Gamma_{\yng(3,2)}((23))\left|\begin{smallmatrix}5&3&2\\4&1\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}5&3&2\\4&1\end{smallmatrix}\right\rangle$$
$$\Gamma_{\yng(3,2)}((23))\left|\begin{smallmatrix}5&4&1\\3&2\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}5&4&1\\3&2\end{smallmatrix}\right\rangle.$$

Choosing the following basis

$$\left|\begin{smallmatrix}5&4&3\\2&1\end{smallmatrix}\right\rangle = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \left|\begin{smallmatrix}5&4&2\\3&1\end{smallmatrix}\right\rangle = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \left|\begin{smallmatrix}5&3&1\\4&2\end{smallmatrix}\right\rangle = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}$$
$$\left|\begin{smallmatrix}5&3&2\\4&1\end{smallmatrix}\right\rangle = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \left|\begin{smallmatrix}5&4&1\\3&2\end{smallmatrix}\right\rangle = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

we obtain

$$\Gamma_{\tiny\yng(3,2)}((23)) = \begin{bmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 & 0 & 0 \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is an important identity obeyed by the matrix elements of the matrix representation of any finite group, called the **fundamental orthogonality relation**, and it is given by

$$\sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{\alpha\beta} = \frac{|\mathcal{G}|}{d_R}\delta_{RS}\delta_{a\beta}\delta_{b\alpha} \tag{3.7}$$

where $|\mathcal{G}|$ is the order of the group $\mathcal{G}$.

Now let us turn our attention to the $U(N)$ group. We have been using Young diagrams to label the irreducible representations of the symmetric group. How do we label the irreducible representations of the unitary group? Thanks to Schur-Weyl duality, we can also label the irreducible representations of the unitary group using Young diagrams. We can compute the dimensions of $R$ of a $U(N)$ irreducible representation using an equation similar to (3.5), obtained by replacing $n!$ with $f_R$

$$Dim_R = \frac{f_R}{\prod_{x\in R} hooks(x)}.$$

$f_R$ represents the product of *factors* of the Young diagram $R$. The *factors* of $U(N)$ are given by the $S_n$ content plus $N$ for each box of the Young diagram. As an example, consider the following Young diagram with its *factors* displayed

| | | | | |
|---|---|---|---|---|
| $N{+}0$ | $N{+}1$ | $N{+}2$ | $N{+}3$ | $N{+}4$ |
| $N{-}1$ | $N{+}0$ | $N{+}1$ | $N{+}2$ | |
| $N{-}2$ | $N{-}1$ | $N{+}0$ | | |
| $N{-}3$ | $N{-}2$ | | | |
| $N{-}4$ | | | | |

(iii.) **Character** $\chi_R(\sigma)$

A character $\chi_R(\sigma)$ is the trace of the matrix representing group element $\sigma$ in irreducible representation $R$ of the symmetric group

$$\chi_R(\sigma) = Tr(\Gamma_R(\sigma)).$$

A very useful identity satisfied by the characters of any finite group is the **character orthogonality relation**

$$\sum_{g\in\mathcal{G}} \chi_R(\sigma)\chi_S(\sigma) = |\mathcal{G}|\delta_{RS}. \tag{3.8}$$

This can be proved by taking suitable traces of the fundamental orthogonality relation (3.7).

(iv.) $Z^{\otimes n}$ **and** $\sigma Z^{\otimes n}$

$Z^{\otimes n}$ is the tensor product of $n$ copies of matrix $Z$ and $\sigma Z^{\otimes n}$ simply swaps or permutes indices of the $n$ copies of $Z$ as follows

$$(Z^{\otimes n})^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} \cdots Z^{i_n}_{j_n}$$
$$(\sigma Z^{\otimes n})^I_J = Z^{i_1}_{j_{\sigma(1)}} Z^{i_2}_{j_{\sigma(2)}} Z^{i_3}_{j_{\sigma(3)}} \cdots Z^{i_n}_{j_{\sigma(n)}}.$$

Taking a trace of these operators ensures that all the indices are contracted, i.e.

$$Tr(\sigma Z^{\otimes n}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}}.$$

(v.) **Projectors**
Operators that project onto one of the subspaces corresponding to an irreducible representation of $S_n$, are given by

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma$$

where $R$ labels the irreducible representation. In terms of this projection operator, the Schur polynomial can be written as

$$\chi_R(Z) = Tr(P_R Z^{\otimes n})$$

where the above trace is over $V_N^{\otimes n}$.

We have explored the Schur polynomials in some detail. We will now compute correlation functions that involve the Schur polynomials.

### 3.2.1 The 2-point function of Schur polynomials

As stated before, the Schur polynomials diagonalize the 2-point function. The 2-point function of the Schur polynomial is given by [8]

$$\langle \chi_R(Z) \chi_T(Z^\dagger) \rangle = \delta_{RT} f_R. \tag{3.9}$$

To understand this formula we will compute one example which has a Young diagram with 3 boxes. The possible Young diagrams are

$$\yng(3) \qquad \yng(2,1) \qquad \yng(1,1,1) \tag{3.10}$$

and the possible 2-point functions that we can compute from these Young diagrams are

$$\begin{array}{ccc} \langle \chi_{\tiny\yng(3)}(Z)\chi_{\tiny\yng(3)}(Z^\dagger)\rangle & \langle \chi_{\tiny\yng(2,1)}(Z)\chi_{\tiny\yng(2,1)}(Z^\dagger)\rangle & \langle \chi_{\tiny\yng(1,1,1)}(Z)\chi_{\tiny\yng(1,1,1)}(Z^\dagger)\rangle \\ \langle \chi_{\tiny\yng(3)}(Z)\chi_{\tiny\yng(2,1)}(Z^\dagger)\rangle & \langle \chi_{\tiny\yng(3)}(Z)\chi_{\tiny\yng(1,1,1)}(Z^\dagger)\rangle & \langle \chi_{\tiny\yng(2,1)}(Z)\chi_{\tiny\yng(1,1,1)}(Z^\dagger)\rangle. \end{array} \tag{3.11}$$

Using the definition of the Schur polynomials, we can write

$$\chi_{\tiny\yng(1,1,1)}(Z) = \frac{1}{3!}\Big(Tr(Z)^3 + 3Tr(Z)Tr(Z^2) + 2Tr(Z^3)\Big), \tag{3.12}$$

$$\chi_{\tiny\yng(2,1)}(Z) = \frac{1}{3!}\Big(2Tr(Z)^3 - 2Tr(Z^3)\Big), \tag{3.13}$$

$$\chi_{\tiny\yng(3)}(Z) = \frac{1}{3!}\Big(Tr(Z)^3 - 3Tr(Z)Tr(Z^2) + 2Tr(Z^3)\Big). \tag{3.14}$$

For complex matrix fields, the correlators are computed by Wick contracting $Z$ fields with $Z^\dagger$ fields. The basic contraction is

$$\langle Z_{ij} Z^\dagger_{kl} \rangle = \delta_{il}\delta_{jk}. \tag{3.15}$$

Using this and Wick's theorem, we easily find, for example

$$\begin{aligned} \langle Tr(Z)Tr(Z^\dagger)\rangle &= \langle Z_{ii} Z^\dagger_{jj}\rangle \\ &= \delta_{ij}\delta_{ij} \\ &= N \end{aligned} \tag{3.16}$$

and

$$\langle Tr(Z)Tr(Z)\rangle = 0, \qquad \langle Tr(Z^\dagger)Tr(Z^\dagger)\rangle = 0. \tag{3.17}$$

We will now state the result of the correlators that will be relevant in computing the 2-point functions in (3.11). The results are

$$\langle Tr(Z)^3 Tr(Z^\dagger)^3\rangle = 6N^3 \tag{3.18}$$
$$\langle Tr(Z)^3 Tr(Z^{3\dagger})\rangle = 6N \tag{3.19}$$
$$\langle Tr(Z)^3 Tr(Z^{2\dagger})Tr(Z^\dagger)\rangle = 6N^2 \tag{3.20}$$
$$\langle Tr(Z^{3\dagger})Tr(Z^2)Tr(Z)\rangle = 6N^2 \tag{3.21}$$
$$\langle Tr(Z^{3\dagger})Tr(Z^3)\rangle = 3N + 3N^3 \tag{3.22}$$
$$\langle Tr(Z^{2\dagger})Tr(Z^\dagger)Tr(Z^2)Tr(Z)\rangle = 4N + 2N^3. \tag{3.23}$$

Using these results, we find

$$\langle \chi_{\yng(3)}(Z)\chi_{\yng(3)}(Z^\dagger)\rangle = N(N+1)(N+2) = f_{\yng(3)} \tag{3.24}$$
$$\langle \chi_{\yng(2,1)}(Z)\chi_{\yng(2,1)}(Z^\dagger)\rangle = N(N+1)(N-1) = f_{\yng(2,1)} \tag{3.25}$$
$$\langle \chi_{\yng(1,1,1)}(Z)\chi_{\yng(1,1,1)}(Z^\dagger)\rangle = N(N-1)(N-2) = f_{\yng(1,1,1)} \tag{3.26}$$
$$\langle \chi_{\yng(3)}(Z)\chi_{\yng(2,1)}(Z^\dagger)\rangle = 0 \tag{3.27}$$
$$\langle \chi_{\yng(3)}(Z)\chi_{\yng(1,1,1)}(Z^\dagger)\rangle = 0 \tag{3.28}$$
$$\langle \chi_{\yng(2,1)}(Z)\chi_{\yng(1,1,1)}(Z^\dagger)\rangle = 0. \tag{3.29}$$

These results confirm (3.9). Higher point correlation functions such as 3-point, 4-point, $\cdots$, of the Schur polynomials are easily evaluated using

$$\chi_R(Z)\chi_S(Z) = \sum_T f_{RST}\chi_T(Z) \tag{3.30}$$

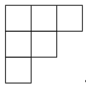
where $f_{RST}$ is the Littlewood-Richardson coefficient. This equation tells us that we can always resolve a product of two Schur polynomials into a sum of Schur polynomials weighted by the factor $f_{RST}$. This completes our discussion of the Schur polynomials. The generalization to more than one complex matrix field leads naturally to the restricted Schur polynomials.

## 3.3 Multi-matrix free field theory: Restricted Schur polynomials.

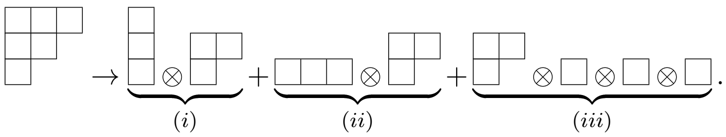
The restricted Schur polynomials $\chi_{R,(r,s)}(Z,Y)$ are built using two complex matrix fields $Y$and $Z$. They also diagonalize the two point function. They are given by [31]

$$\chi_{R,(r,s)_{\alpha\beta}}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{m+n}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma)Tr(\sigma Z^{\otimes n}\otimes Y^{\otimes m}). \tag{3.31}$$

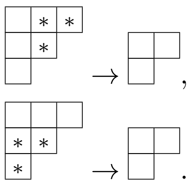
$R$ labels an irreducible representation of $S_{n+m}$, since the Young diagram $R$ has $n+m$ boxes. $r$ and $s$ label an irreducible representation of $S_n \times S_m$ which is a subgroup of $S_{n+m}$. $r$ is a Young diagram with $n$ boxes which we can think of as organizing the $Z$ fields. $s$ is a Young diagram with $m$ boxes which we can think of as organizing the $Y$ fields. The Young diagram $R$ tells us how the $Z$ and $Y$ fields are assembled. When the irreducible representation $S_{n+m}$ subduces a specific irreducible representation of $S_n \times S_m$ more than once, we distinguish these copies by introducing the multiplicity indices $\alpha$ and $\beta$ which label the different copies. To see how these multiplicities come about, we will consider an example. Consider the irreducible representation



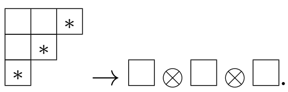
We restrict this irreducible representation from $S_6$ to $S_3 \times S_3$ by pulling the boxes off as illustrated below



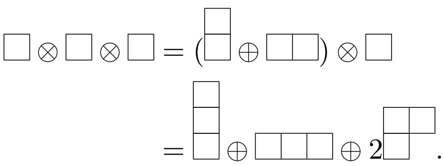

The rule for restricting $S_6$ to $S_3 \times S_3$, is that we need to pull off 3 boxes and when assembling the removed boxes into an irreducible representation of $S_3$ we must respect any shared sides. If we look at (*i*) and (*ii*) above and put stars in the boxes that are removed, the rule we use is demonstrated as follows



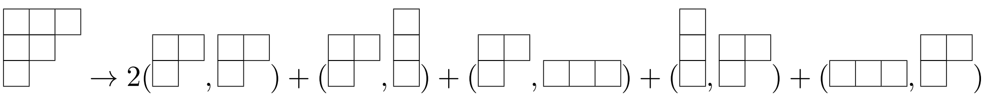
For term (*iii*), there are no common sides between the removed boxes, so that

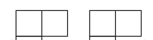


To form an irreducible representation of $S_3$, $\square\otimes\square\otimes\square$ can be combined as follows



Therefore, we have



Notice that irreducible representation  appears twice. The role of the multiplicity index is to distinguish these two copies. Here, we will consider a simple case where the subspace $S_n \times S_m$

of $S_{m+n}$ has no multiplicities. In general, when we consider a Young diagram $R$ with $p = 2$, where $p$ is the number of rows of the Young diagram, we will not have multiplicities when we restrict the irreducible representation of $S_{n+m}$ to $S_n \times S_m$. Another special case that does not lead to multiplicities arises when we restrict $S_{n+m}$ to $S_n \times S_m$ with $m = 1$ or $m = 2$. From now on we will consider $m = 2$ and we will drop the multiplicity indices. Our restricted Schur polynomial then becomes [31]

$$\chi_{R,(r,s)}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{m+n}} \chi_{R,(r,s)}(\sigma) Tr(\sigma Z^{\otimes n} \otimes \sigma Y^{\otimes m}). \tag{3.32}$$

The restricted character $\chi_{R,(r,s)}(\sigma)$ is defined as [60]

$$\chi_{R,(r,s)}(\sigma) = Tr_{R,(r,s)} \Gamma((\sigma)) \tag{3.33}$$

$$= \sum_A \langle R,(r,s);A|\Gamma_R(\sigma)|R,(r,s);A\rangle \tag{3.34}$$

where $A$ labels different Young-Yamanouchi states and the trace of $\sigma Z^{\otimes n} \otimes Y^{\otimes m}$ is given by

$$Tr(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}} Y^{i_{n+3}}_{i_{\sigma(n+3)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}.$$

The 2-point function of the restricted Schur polynomial is given by [31]

$$\langle \chi_{R,(r,s)}(Z,Y) \chi_{T,(t,u)}(Z,Y)^\dagger \rangle = \frac{hook_R}{hooks_r hooks_s} f_R \delta_{RT} \delta_{rt} \delta_{su}.$$

This complete our discussion of the restricted Schur polynomials. To further understand our formula for the restricted Schur polynomial we will find it useful to introduce certain projectors.

## 3.4 Projectors.

The projectors we study in this section will be used to evaluate the restricted characters we need to construct restricted Schur polynomials. The projectors project to the subspace used to define the restricted character so that we can write

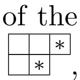
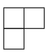
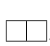
$$\chi_{R,(r,s)}(\sigma) = Tr(P_{R,(r,s)} \Gamma_R(\sigma))$$

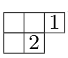
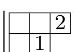
where $P_{R,(r,s)}$ is the projector. The projector $P_{R,(r,s)}$ projects from an irreducible representation of $S_{n+m}$ to an irreducible representation of $S_n \times S_m$ and is given by

$$P_{R,(r,s)} = \frac{d_s d_r}{m!n!} \sum_{\sigma \in S_{n \times m}} \chi_{(r,s)}(\sigma) \Gamma_R(\sigma) \tag{3.35}$$

where $\sigma = \sigma_1 \circ \sigma_2$, $\sigma_1 \in S_n$ and $\sigma_2 \in S_m$. The character of $\sigma = \sigma_1 \circ \sigma_2 \in S_n \times S_m$ with $\sigma_1 \in S_n$ and $\sigma_2 \in S_m$ can be written as $\chi_{(r,s)}(\sigma) = \chi_r(\sigma_1)\chi_s(\sigma_2)$. We want to argue that the projector can be expressed as

$$P_{R,(r,s)} = \frac{d_s}{m!} \sum_{\sigma \in S_m} \chi_s(\sigma) \Gamma_R(\sigma) \otimes \mathbb{I}_r, \tag{3.36}$$

if we restrict the projector to a carefully chosen subspace. The subspace respects a partition of the blocks of $R$ into blocks associated to the $Z$s and the blocks associated to the $Y$s. Consider $R =$ , $r =$  and $s =$ . We can form two Young-Yamanouchi states by interchanging the position of the starred boxes

$|\psi, i\rangle = \alpha \Big|$  $\Big\rangle + \beta \Big|$  $\Big\rangle$.

The projector (3.36) will only act on the space with states labelled by $\young(531,42)$, $\young(532,41)$, $\young(541,32)$ and $\young(542,31)$ which is a proper subspace of the carrier space of $R$. We represent the Young-Yamanouchi symbols by $|\psi, i\rangle$ as follows

$$\begin{aligned} |\psi,1\rangle &= \alpha \left|\young(531,42)\right\rangle + \beta \left|\young(532,41)\right\rangle, \\ |\psi,2\rangle &= \alpha \left|\young(541,32)\right\rangle + \beta \left|\young(542,31)\right\rangle. \end{aligned}$$

The subspace we consider respects the partition of $R$ into $n$ and $m$ boxes. Let's evaluate the action of

$$\frac{d_r}{n!} \sum_{\sigma_1 \in S_n} \chi_r(\sigma_1)\Gamma_R(\sigma_1)$$

on the vector $\left|\young(54*,3*)\right\rangle$. With the help of the character table we find

$$\begin{aligned} \chi_{\yng(2,1)}(\mathbb{I}) &= 2 \\ \chi_{\yng(2,1)}((\bullet,\bullet)) &= 0 \\ \chi_{\yng(2,1)}((\bullet,\bullet,\bullet)) &= -1 \end{aligned}$$

where $\mathbb{I}$ is the identity, $(\bullet,\bullet)$ is a 2-cycle and $(\bullet,\bullet,\bullet)$ is a 3-cycle. There are two possible distinct 3-cycles that can be formed from $\sigma_1$ and they are (345) and (354). $\Gamma_R((345))\left|\young(54*,3*)\right\rangle$ is given by

$$\begin{aligned} (345)\left|\young(54*,3*)\right\rangle &= (34)(45)\left|\young(54*,3*)\right\rangle \\ &= -\frac{1}{2}\left|\young(54*,3*)\right\rangle + \frac{\sqrt{3}}{2}\left|\young(53*,4*)\right\rangle \end{aligned}$$

and $\Gamma_R((354))\left|\young(54*,3*)\right\rangle$ is given by

$$\begin{aligned} (354)\left|\young(54*,3*)\right\rangle &= (54)(34)\left|\young(54*,3*)\right\rangle \\ &= -\frac{1}{2}\left|\young(54*,3*)\right\rangle - \frac{\sqrt{3}}{2}\left|\young(53*,4*)\right\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d_r}{n!} \sum_{\sigma_1 \in S_n} \chi_r(\sigma_1)\Gamma_R(\sigma_1)\left|\young(54*,3*)\right\rangle &= \frac{2}{3!}\Big(2\cdot\Gamma_R(\mathbb{I}) + 0[\Gamma_R(12)+\Gamma_R(13)+\Gamma_R(23)] - [\Gamma_R(345)+\Gamma_R(354)]\Big)\left|\young(54*,3*)\right\rangle \\ &= \frac{1}{3}\Big((2+\frac{1}{2}+\frac{1}{2})\left|\young(54*,3*)\right\rangle + (\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})\left|\young(53*,4*)\right\rangle\Big) \\ &= \left|\young(54*,3*)\right\rangle. \end{aligned}$$

Therefore this implies that

$$\frac{d_r}{n!} \sum_{\sigma_1 \in S_n} \chi_r(\sigma_1)\Gamma_R(\sigma_1) = \mathbb{I}_r.$$

So, we have shown that, after restricting to a carefully chosen subspace, we have

$$\left(\frac{d_s}{m!}\sum_{\sigma_2\in S_m}\chi_s(\sigma_2)\Gamma_R(\sigma_2)\right)\left(\frac{d_r}{n!}\sum_{\sigma_1\in S_n}\chi_r(\sigma_1)\Gamma_R(\sigma_1)\right)=\left(\frac{d_s}{m!}\sum_{\sigma_2\in S_m}\chi_s(\sigma_2)\Gamma_R(\sigma_2)\right)\otimes\mathbb{I}_r.$$

From now on we will always assume we have restricted ourselves to this subspace so that we can simply write

$$P_{R,(r,s)}=\frac{d_s}{m!}\sum_{\sigma\in S_m}\chi_s(\sigma)\Gamma_R(\sigma). \tag{3.37}$$

Projectors have the property that they square to themselves and their trace is equal to the dimension of the subspace they project to. For the projectors we have introduced we can show that

1. $P_{R,(r',t)}\cdot P_{R,(r,s)}=\delta_{st}\delta_{rr'}P_{R,(r',t)}$.
2. $Tr(P_{R,(r,s)})=d_sd_r$.

Taking the product of $P_{R,(r,s)}$ and $P_{R,(r',t)}$, we get

$$\begin{aligned}
P_{R,(r,s)}\cdot P_{R,(r',t)}&=\frac{d_s}{m!}\sum_{\sigma\in S_m}\chi_s(\sigma)\Gamma_R(\sigma)\cdot\frac{d_t}{m!}\sum_{\psi\in S_m}\chi_t(\psi)\Gamma_R(\psi)\\
&=\frac{d_sd_t}{m!m!}\sum_{\sigma\in S_m}\sum_{\psi\in S_m}\chi_s(\sigma)\Gamma_R(\sigma)\chi_t(\psi)\Gamma_R(\psi)\\
&=\frac{d_sd_t}{m!m!}\sum_{\sigma\in S_m}\sum_{\psi\in S_m}\chi_s(\sigma)\chi_t(\psi)\Gamma_R(\sigma\psi) \qquad \text{set } \tau=\sigma\psi\\
&=\frac{d_sd_t}{m!m!}\sum_{\sigma\in S_m}\sum_{\tau\in S_m}\chi_s(\sigma)\chi_t(\sigma^{-1}\tau)\Gamma_R(\tau)\\
&=\frac{d_sd_t}{m!m!}\sum_{\sigma\in S_m}\sum_{\tau\in S_m}\Gamma_s(\sigma)_{aa}\Gamma_t(\sigma^{-1})_{bc}\Gamma_t(\tau)_{cb}\Gamma_R(\tau) \qquad \text{sum over } \sigma \text{ and use (3.7)}\\
&=\frac{d_sd_t}{m!m!}\sum_{\tau\in S_m}\Gamma_t(\tau)_{cb}\Gamma_R(\tau)\delta_{ab}\delta_{ac}\delta_{st}\frac{m!}{d_s}\\
&=\frac{d_t}{m!}\sum_{\tau\in S_m}Tr(\Gamma_t(\tau))\Gamma_R(\tau)\delta_{st}\\
&=\frac{d_t}{m!}\sum_{\tau\in S_m}\chi_t(\tau)\Gamma_R(\tau)\delta_{st},
\end{aligned}$$

which proves the first statement. Taking the trace of $P_{R,(r,s)}$ we have

$$Tr(P_{R,(r,s)})=Tr\left(\frac{d_s}{m!}\frac{d_r}{n!}\sum_{\sigma_2\in S_m}\sum_{\sigma_1\in S_n}\chi_s(\sigma_2)\chi_r(\sigma_1)\Gamma_R(\sigma_2\circ\sigma_1)\right).$$

By summing over all of the irreducible representation $(a,b)$ that can be subduced by $R$ we can rewrite

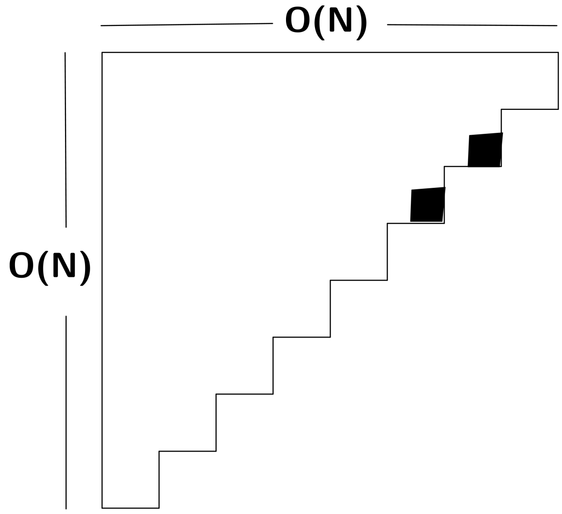
$\Gamma_R(\sigma_2 \circ \sigma_1) = \sum_{a \vdash m, b \vdash n} \Gamma_{a,b}(\sigma_2 \circ \sigma_1)$. This leads to

$$\begin{aligned}
Tr(P_{R,(r,s)}) &= \frac{d_s}{m!}\frac{d_r}{n!} \sum_{\sigma_2 \in S_m} \sum_{\sigma_1 \in S_n} \chi_s(\sigma_2)\chi_r(\sigma_1) \sum_{a \vdash m, b \vdash n} \chi_{a,b}(\sigma_2 \circ \sigma_1) \\
&= \frac{d_s}{m!}\frac{d_r}{n!} \sum_{\sigma_2 \in S_m} \sum_{\sigma_1 \in S_n} \chi_s(\sigma_2)\chi_r(\sigma_1) \sum_a \chi_a(\sigma_2) \sum_b \chi_b(\sigma_1) \\
&= \frac{d_s}{m!}\frac{d_r}{n!} \sum_{\sigma_2 \in S_m} \sum_a \chi_s(\sigma_2)\chi_a(\sigma_2) \sum_{\sigma_1 \in S_n} \sum_b \chi_r(\sigma_1)\chi_b(\sigma_1) \qquad \text{using (3.8)} \\
&= \frac{d_s}{m!}\frac{d_r}{n!} m!n!\delta_{as}\delta_{br} \\
&= d_s d_r
\end{aligned}$$

where $d_r d_s$ are the dimensions of $r$ and $s$. This proves the second property. Now that we have seen how the projectors are used, lets construct them.

### 3.4.1 Construction of the projectors.

Consider the Young diagram




The number of boxes in this Young diagram is order $O(N^2)$. We want to construct arbitrary projectors given that the shaded boxes are associated with $Y$ fields and the unshaded boxes with $Z$ fields [1]. The transparent boxes are associated to $S_n$ and the shaded boxes to $S_m$. Given a Young diagram $R$, we can divide it into $r$ and $s$ boxes where $s$ are the impurities and $r$ is what remains of $R$ after removing the impurities. To construct the projectors $P_{R,(r,s)}$, we will need to know the possible irreducible representations that are allowed. This is done by taking the tensor product of the shaded boxes

$$s = \square \otimes \square$$

$$s = \square\square \quad \text{or} \quad s = \begin{smallmatrix}\square \\ \square\end{smallmatrix}.$$

[1] From now on we will refer to shaded boxes as impurities.

Having this result, we can construct the projectors following equation (3.37)

$$
\begin{aligned}
P_{R,(r,s)} &= \frac{d_s}{m!} \sum_{\sigma \in S_m} \chi_s(\sigma) \Gamma_R(\sigma) \\
&= \frac{d_s}{m!} \sum_{\sigma \in S_m} \langle s|\Gamma_s(\sigma)|s\rangle \Gamma_R(\sigma).
\end{aligned}
$$

We identify two possible Young-Yamanouchi states $|1\rangle$ and $|2\rangle$ below

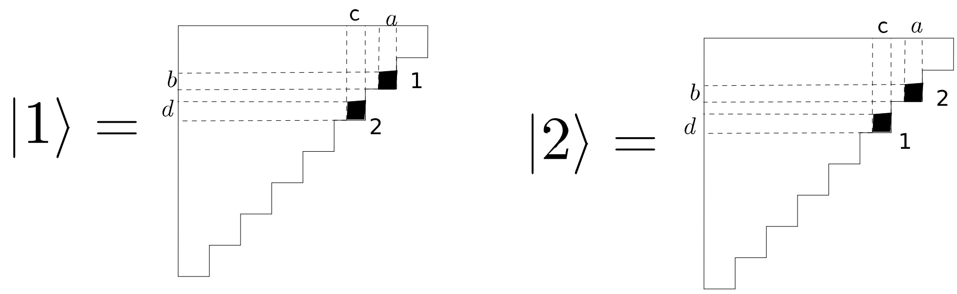



Figure 3.2: The possible Young-Yamanouchi states for operators with two impurities. The labels on the diagonal of the Young-diagram are the Young-Yamanouchi numberings.

The positions of the boxes are specified by the column and row the box belongs to. We use $c_{ab}$ to denote the box in column $a$ and row $b$. We always take $a > c$ and $b < d$ such that $c_{ab} > c_{cd}$. To evaluate the relevant projectors, we will need to know the explicit form of $\Gamma_R((1))$ and $\Gamma_R((12))$. Projectors with two impurities ($m = 2$) are given by

$$
\begin{aligned}
P_{\square\square} &= \frac{d_s}{2}\Big[\langle \square\square|\Gamma_{\square\square}((1))|\square\square\rangle \Gamma_R((1)) + \langle \square\square|\Gamma_{\square\square}((12))|\square\square\rangle \Gamma_R((12))\Big], \\
P_{\substack{\square\\\square}} &= \frac{d_s}{2}\Big[\langle \substack{\square\\\square}|\Gamma_{\substack{\square\\\square}}((1))|\substack{\square\\\square}\rangle \Gamma_R((1)) + \langle \substack{\square\\\square}|\Gamma_{\substack{\square\\\square}}((12))|\substack{\square\\\square}\rangle \Gamma_R((12))\Big].
\end{aligned}
$$

Above we have adopted the notation $P_{R,(r,\square\square)} = P_{\square\square}$ and $P_{R,(r,\substack{\square\\\square})} = P_{\substack{\square\\\square}}$. The action of $\Gamma_R(12)$ is

$$
\begin{aligned}
\Gamma_R((12))|1\rangle &= \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle, \\
\Gamma_R((12))|2\rangle &= \frac{1}{c_{(cd)} - c_{(ab)}}|2\rangle + \sqrt{1 - \frac{1}{(c_{(cd)} - c_{(ab)})^2}}|1\rangle.
\end{aligned}
$$

Solving these two equation, we get

$$
\begin{aligned}
\Gamma_R((12)) = {} & \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle\langle 1| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle\langle 1| \\
& - \frac{1}{c_{(ab)} - c_{(cd)}}|2\rangle\langle 2| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|1\rangle\langle 2|, \qquad (3.38) \\
\Gamma_R((1)) = {} & |1\rangle\langle 1| + |2\rangle\langle 2|. \qquad (3.39)
\end{aligned}
$$

Using the fact that the states are orthogonal i.e $\langle \square\square | \square\square \rangle = \langle \substack{\square\\\square} | \substack{\square\\\square} \rangle = 1$, we obtain

$$
\begin{aligned}
P_{\square\square} = \frac{d_s}{2}\Bigg[ & |1\rangle\langle 1| + |2\rangle\langle 2| + \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle\langle 1| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle\langle 1| \\
& - \frac{1}{c_{(ab)} - c_{(cd)}}|2\rangle\langle 2| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|1\rangle\langle 2|\Bigg]
\end{aligned}
\tag{3.40}
$$

and

$$
\begin{aligned}
P_{\substack{\square\\\square}} = \frac{d_s}{2}\Bigg[ & |1\rangle\langle 1| + |2\rangle\langle 2| - \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle\langle 1| - \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle\langle 1| \\
& + \frac{1}{c_{(ab)} - c_{(cd)}}|2\rangle\langle 2| - \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|1\rangle\langle 2|\Bigg].
\end{aligned}
\tag{3.41}
$$

It is simple to verify that [2]

$$P_{\square\square} \cdot P_{\square\square} = P_{\square\square}$$

and

$$P_{\square\square} \cdot P_{\substack{\square\\\square}} = 0.$$

Further

$$
\begin{aligned}
Tr(P_{\square\square}) &= \frac{1}{2}Tr\Bigg[|1\rangle\langle 1| + |2\rangle\langle 2| + \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle\langle 1| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle\langle 1| \\
&\quad - \frac{1}{c_{(ab)} - c_{(cd)}}|2\rangle\langle 2| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|1\rangle\langle 2|\Bigg] \\
&= \frac{1}{2}Tr\Bigg[\langle 1|1\rangle + \langle 2|2\rangle + \frac{1}{c_{(ab)} - c_{(cd)}}\langle 1|1\rangle + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}\langle 1|2\rangle \\
&\quad - \frac{1}{c_{(ab)} - c_{(cd)}}\langle 2|2\rangle + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}\langle 2|1\rangle\Bigg] \\
&= \frac{1}{2}Tr\Bigg[1 + 1 + \frac{1}{c_{(ab)} - c_{(cd)}} - \frac{1}{c_{(ab)} - c_{(cd)}}\Bigg] \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
Tr(P_{\substack{\square\\\square}}) &= \frac{1}{2}Tr\Bigg[\langle 1|1\rangle + \langle 2|2\rangle - \frac{1}{c_{(ab)} - c_{(cd)}}\langle 1|1\rangle - \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}\langle 1|2\rangle \\
&\quad + \frac{1}{c_{(ab)} - c_{(cd)}}\langle 2|2\rangle - \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}\langle 2|1\rangle\Bigg] \\
&= \frac{1}{2}Tr\Bigg[1 + 1 - \frac{1}{c_{(ab)} - c_{(cd)}} + \frac{1}{c_{(ab)} - c_{(cd)}}\Bigg] \\
&= 1.
\end{aligned}
$$

Again, these are the expected results.

[2] See appendix B for detail computations.

## 3.5 Non-Interacting Fermions

In this section we will study a system that will allow us to make the connection between non-interacting fermions and the Schur polynomials. The system that we will consider is of a charged particle subjected to an external magnetic field. The fermions in this system do not interact with each other. These fermions feel the potential of a harmonic oscillator. To introduce this subject, we will first study a single particle in a magnetic field.

### 3.5.1 Particle in an external magnetic field

The Hamiltonian of the one particle system that we will study is given by

$$H = \frac{1}{2}((P_x + y)^2 + (P_y - x)^2). \tag{3.42}$$

This is the Hamiltonian for an electron that is coupled to an external magnetic field in units where the charge $e$, mass $m$, speed of light $c$ and the magnetic field $B$ are set to unity. This can clearly be seen from the equation

$$H = \frac{1}{2m}\left(P_x + \frac{eBy}{c}\right)^2 + \frac{1}{2m}\left(P_y - \frac{eBx}{c}\right)^2.$$

To check if this Hamiltonian is coupled to external magnetic field, we will need to show that the equations of motion reproduce the Lorentz force. Starting from Hamilton's equations

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}}, \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} \tag{3.43}$$

we find

$$\begin{aligned} \dot{x} &= P_x + y, \qquad \dot{y} = P_y - x \\ \dot{P}_x &= P_y - x, \qquad \dot{P}_y = -P_x + y. \end{aligned}$$

Taking the second derivative of the positions and use the above results, we find

$$\ddot{x} = 2P_y - 2x = 2\dot{y}, \qquad \ddot{y} = -2P_x - 2y = -2\dot{x}. \tag{3.44}$$

This is the Lorentz force, it might not look like it at first glance but once we restore $e, m$ and $B$ this last expression is indeed the Lorentz force. Now let's introducing the coordinates

$$z = x + iy \quad \bar{z} = x - iy.$$

We know that the momentum operator in position space is given by $P_x = -i\frac{\partial}{\partial x}$ (where we have set $\hbar = 1$). We would like to write the linear combination of $P_x$ and $P_y$ in terms of our new coordinates $z$ and $\bar{z}$. Doing so, we find

$$P_x + iP_y = -2i\frac{\partial}{\partial \bar{z}} \equiv P_{\bar{z}}, \tag{3.45}$$

$$P_x - iP_y = -2i\frac{\partial}{\partial z} \equiv P_z. \tag{3.46}$$

We can check the commutation relation $[P_z, z]$ using the test function $f(z)$

$$\begin{aligned} [P_z, z]f(z) &= [-2i\frac{\partial}{\partial z}, z]f(z) \\ &= (-2i)\left(\frac{\partial}{\partial z}(zf(z)) - z\frac{\partial}{\partial z}f(z)\right) \\ &= (-2i)\left(f(z) + z\frac{\partial f(z)}{\partial z} - z\frac{\partial f(z)}{\partial z}\right) \\ &= -2if(z). \end{aligned}$$

Therefore this tells us that

$$[P_z, z] = -2i.$$

The commutation relation $[P_z, \bar{z}]$ is given by

$$\begin{aligned}
[P_z, \bar{z}]g(z,\bar{z}) &= [-2i\frac{\partial}{\partial z}, \bar{z}]g(z,\bar{z}) \\
&= (-2i)\left(\frac{\partial}{\partial z}(\bar{z}g(z,\bar{z})) - \bar{z}\frac{\partial}{\partial z}g(z,\bar{z})\right) \\
&= (-2i)\left(0 + \bar{z}\frac{\partial g(z,\bar{z})}{\partial z} - \bar{z}\frac{\partial g(z,\bar{z})}{\partial z}g\right) \\
&= 0. \qquad (3.47)
\end{aligned}$$

Using the same procedure we find that

$$[P_{\bar{z}}, \bar{z}] = -2i, \qquad [P_{\bar{z}}, z] = 0. \qquad (3.48)$$

We therefore have the following commutation relation

$$[P_z, z] = 2i, \qquad [P_z, \bar{z}] = 0 \qquad (3.49)$$
$$[P_{\bar{z}}, \bar{z}] = -2i, \qquad [P_{\bar{z}}, z] = 0. \qquad (3.50)$$

Since we are dealing with harmonic oscillators, we can always write the Hamiltonian in terms of creation and annihilation operators (ladder operators). The Hamiltonian given by (3.42) can be written as

$$H = \frac{1}{4}(a^\dagger a + a a^\dagger). \qquad (3.51)$$

One can check that the creation and annihilation operators are given by

$$a = P_x + y + i(P_y - x) = P_{\bar{z}} - iz, \qquad (3.52)$$
$$a^\dagger = P_x + y - i(P_y - x) = P_z - i\bar{z}. \qquad (3.53)$$

To check if this is correct, we plug these ladder operators into (3.51)

$$\begin{aligned}
H &= \frac{1}{4}\Big[(P_x + y - i(P_y - x))(P_x + y + i(P_y - x)) + (P_x + y + i(P_y - x))(P_x + y - i(P_y - x))\Big] \\
&= \frac{1}{4}\Big[(P_x + y)^2 + (P_y - x)^2 + (P_x + y)^2 + (P_y - x)^2\Big] \\
&= \frac{1}{2}\Big[(P_x + y)^2 + (P_y - x)^2\Big]
\end{aligned}$$

and we see that this reproduces (3.42). Now consider the commutator of the ladder operators $a$ and $a^\dagger$

$$\begin{aligned}
[a, a^\dagger] &= (P_{\bar{z}} - iz)(P_z + i\bar{z}) - (P_z + i\bar{z})(P_{\bar{z}} - iz) \\
&= \Big[P_{\bar{z}}P_z + z\bar{z} + iP_{\bar{z}}\bar{z} - izP_z\Big] - \Big[P_z P_{\bar{z}} + \bar{z}z - P_z z + i\bar{z}P_{\bar{z}}\Big] \\
&= [P_{\bar{z}}, P_z] + [z, \bar{z}] + i[P_{\bar{z}}, \bar{z}] + [P_z, z] \\
&= 0 + 0 + i(-2i) + i(-2i) \\
&= 2 + 2 \\
&= 4.
\end{aligned}$$

Therefore, the commutator of $a$ and $a^\dagger$ is

$$[a, a^\dagger] = 4. \tag{3.54}$$

Our next task is to determine the ground state wave function. This is done by taking the annihilation operator and letting it act on the vacuum state

$$a|0\rangle = 0.$$

Writing the ground state in position space as $\langle z, \bar{z}|0\rangle = \psi_0(z, \bar{z})$, we find

$$\langle z, \bar{z}|a|0\rangle = 0$$
$$\Big( -2i \cdot \frac{\partial}{\partial \bar{z}} - iz \Big) \psi_0(z, \bar{z}) = 0. \tag{3.55}$$

This differential equation can easily be solved and the solution is given by

$$\psi_0(z, \bar{z}) = f(z) e^{-\frac{z\bar{z}}{2}}. \tag{3.56}$$

It is convenient to choose the basis of the (degenerate) solutions as follows

$$\psi_l(z, \bar{z}) = \mathcal{N} z^l e^{-\frac{z\bar{z}}{2}} \tag{3.57}$$

where $l$ represents the state of angular momentum. The angular momentum operator is defined as

$$
\begin{aligned}
L_z &= xP_y - yP_x && (3.58)\\
&= x\Big(\frac{-i\partial}{\partial y}\Big) - y\Big(\frac{-i\partial}{\partial x}\Big) && (3.59)\\
&= -i\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big) && (3.60)\\
&= -i\Big[\frac{z+\bar{z}}{2}\Big(\frac{P_{\bar{z}} - P_z}{2}\Big) - \frac{z-\bar{z}}{2i}\Big(\frac{P_{\bar{z}} + P_z}{-2i}\Big)\Big]. && (3.61)
\end{aligned}
$$

Let this operator act on the wave function $\psi_l(z, \bar{z})$

$$
\begin{aligned}
L_z\psi_l(z, \bar{z}) &= -i\Big[\frac{z+\bar{z}}{2}\Big(\frac{P_{\bar{z}} - P_z}{2}\Big) - \frac{z-\bar{z}}{2i}\Big(\frac{P_{\bar{z}} + P_z}{-2i}\Big)\Big]\psi_l(z, \bar{z}) && (3.62)\\
&= \frac{-i}{4}\Big[2\bar{z}P_{\bar{z}} - 2zP_z\Big]\psi_l(z, \bar{z}) && (3.63)\\
&= \frac{-i}{2}\Big[\bar{z}P_{\bar{z}} - zP_z\Big]\mathcal{N} z^l e^{-\frac{z\bar{z}}{2}} && (3.64)\\
&= \frac{-i}{2}\Big[\bar{z}(-2i)\frac{\partial}{\partial \bar{z}} - z(-2i)\frac{\partial}{\partial z}\Big]\mathcal{N} z^l e^{-\frac{z\bar{z}}{2}} && (3.65)\\
&= -\Big[\bar{z}z^l(-\frac{1}{2}z) - zlz^{l-1} - zz^l(-\frac{1}{2}\bar{z})\Big]\mathcal{N} e^{-\frac{z\bar{z}}{2}} && (3.66)\\
&= lz^l\mathcal{N} e^{-\frac{z\bar{z}}{2}} && (3.67)\\
&= l\psi_l(z, \bar{z}). && (3.68)
\end{aligned}
$$

This confirms that $l$ represents the eigenvalue of angular momentum. Now, lets determine the normalization factor $\mathcal{N}$. $\mathcal{N}$ is determine by using the fact that the wave function is normalized, that is

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dz d\bar{z}. \tag{3.69}$$

This calculation is easily done by moving to polar coordinates where $z = re^{i\theta}$.

$$1 = \mathcal{N}^2 \int_0^{2\pi} \int_0^{\infty} r dr d\theta \sin\theta r^{2l} e^{-r^2} \tag{3.70}$$

$$= \mathcal{N}^2 (2\pi) \int_0^{\infty} r dr d\theta \sin\theta r^{2l} e^{-r^2} \tag{3.71}$$

$$= \mathcal{N}^2 \frac{2\pi}{2} l! \left( 1^{2l+2} \right). \tag{3.72}$$

To move from the second line to the third line we have used the identity

$$\int_0^{\infty} x^{2n+1} e^{\frac{-x^2}{a^2}} = \frac{n!}{2} a^{2n+2}. \tag{3.73}$$

The normalization factor then becomes

$$\mathcal{N} = \left( \frac{1}{\pi l!} \right)^{\frac{1}{2}} \tag{3.74}$$

and the wave function is given by

$$\psi_l(z, \bar{z}) = \frac{1}{\sqrt{\pi l!}} z^l e^{\frac{-z\bar{z}}{2}}. \tag{3.75}$$

Now that we have understood the single particle in an external magnetic field, we will now study $N$ particles.

### 3.5.2 $N$ particles in an external magnetic field

Focus on $N = 4$ and recall that we are dealing with fermions, which have distinct states. The ground state has one particle in $l = 0$, one in $l = 1$ , one in $l = 2$ , one in $l = 3$. Working in $z$ coordinates, we write the anti-symmetric wave function between the 4 particles as

$$\psi_{0,1,2,3}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3, z_4, \bar{z}_4) = \begin{vmatrix} \psi_0(z_1) & \psi_1(z_1) & \psi_2(z_1) & \psi_3(z_1) \\ \psi_0(z_2) & \psi_1(z_2) & \psi_2(z_2) & \psi_3(z_2) \\ \psi_0(z_3) & \psi_1(z_3) & \psi_2(z_3) & \psi_3(z_3) \\ \psi_0(z_4) & \psi_1(z_4) & \psi_2(z_4) & \psi_3(z_4) \end{vmatrix}.$$

Using (3.75), this reduces to

$$\begin{aligned} \psi_{0,1,2,3}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3, z_4, \bar{z}_4) &= \begin{vmatrix} 1 & z_1 & z_1^2 & z_1^3 \\ 1 & z_2 & z_2^2 & z_2^3 \\ 1 & z_3 & z_3^2 & z_3^3 \\ 1 & z_4 & z_4^2 & z_4^3 \end{vmatrix} e^{\frac{z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + z_4\bar{z}_4}{2}} \\ &= (z_2 - z_1)(z_3 - z_1)(z_3 - z_2)(z_4 - z_1)(z_4 - z_2)(z_4 - z_3) e^{\frac{z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + z_4\bar{z}_4}{2}}. \end{aligned}$$

Now lets excite each particle by one. This will give

$$\begin{aligned} \psi_{1,2,3,4}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3, z_4, \bar{z}_4) &= \begin{vmatrix} \psi_1(z_1) & \psi_2(z_1) & \psi_3(z_1) & \psi_4(z_1) \\ \psi_1(z_2) & \psi_2(z_2) & \psi_3(z_2) & \psi_4(z_2) \\ \psi_1(z_3) & \psi_2(z_3) & \psi_3(z_3) & \psi_4(z_3) \\ \psi_1(z_4) & \psi_2(z_4) & \psi_3(z_4) & \psi_4(z_4) \end{vmatrix} \\ &= \begin{vmatrix} z_1 & z_1^2 & z_1^3 & z_1^4 \\ z_2 & z_2^2 & z_2^3 & z_2^4 \\ z_3 & z_3^2 & z_3^3 & z_3^4 \\ z_4 & z_4^2 & z_4^3 & z_4^4 \end{vmatrix} e^{\frac{z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + z_4\bar{z}_4}{2}} \\ &= (z_2 - z_1)(z_3 - z_1)(z_3 - z_2)(z_4 - z_1)(z_4 - z_2)(z_4 - z_3) z_1 z_2 z_3 z_4 e^{\frac{z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + z_4\bar{z}_4}{2}}. \end{aligned}$$

We want to compare these wave functions to Schur polynomials. We choose the basis where $Z$ is diagonal

$$Z = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{bmatrix}. \tag{3.76}$$

Exciting each particle corresponds to a Schur polynomial labelled by a Young diagram with one box in each row of the Young diagram, $R = \yng(1,1,1,1)$. The Schur polynomial $\chi_{\yng(1,1,1,1)}(Z)$ is given by

$$\begin{aligned}
\chi_{\yng(1,1,1,1)}(Z) &= \frac{1}{4!}\Big(Tr(Z)^4 - 6Tr(Z)^2Tr(Z^2) + 8Tr(Z)Tr(Z^3) + 3Tr(Z^2)Tr(Z^2) - Tr(Z^4)\Big) \\
&= z_1z_2z_3z_4.
\end{aligned}$$

In general there is clear relationship between the wave function of the fermions and the Schur polynomial. The relationship for the example we study is

$$\psi_{1,2,3,4}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3,z_4,\bar{z}_4) = \chi_{\yng(1,1,1,1)}(Z)\psi_{0,1,2,3}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_1,\bar{z}_1,z_1,\bar{z}_1).$$

As a second example consider the case where we excite only the first 3 particles. Our wave function becomes

$$\begin{aligned}
\psi_{0,2,3,4}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3,z_4,\bar{z}_4) &= \begin{vmatrix} \psi_0(z_1) & \psi_2(z_1) & \psi_3(z_1) & \psi_4(z_1) \\ \psi_0(z_2) & \psi_2(z_2) & \psi_3(z_2) & \psi_4(z_2) \\ \psi_0(z_3) & \psi_2(z_3) & \psi_3(z_3) & \psi_4(z_3) \\ \psi_0(z_4) & \psi_2(z_4) & \psi_3(z_4) & \psi_4(z_4) \end{vmatrix} \\
&= \begin{vmatrix} 1 & z_1^2 & z_1^3 & z_1^4 \\ 1 & z_2^2 & z_2^3 & z_2^4 \\ 1 & z_3^2 & z_3^3 & z_3^4 \\ 1 & z_4^2 & z_4^3 & z_4^4 \end{vmatrix} e^{\frac{z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3+z_4\bar{z}_4}{2}} \\
&= (z_2-z_1)(z_3-z_1)(z_3-z_2)(z_4-z_1)(z_4-z_2)(z_4-z_3) \\
&\quad \times (z_2z_3z_4 + z_1z_3z_4 + z_1z_2z_4 + z_1z_2z_3)e^{\frac{z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3+z_4\bar{z}_4}{2}}.
\end{aligned}$$

Since we excite 3 particles the Schur polynomial becomes

$$\chi_{\yng(1,1,1)}(Z) = \frac{1}{3!}\Big(Tr(Z)^3 - 3Tr(Z)Tr(Z^2) + 2Tr(Z^3)\Big) \tag{3.77}$$

$$= (z_2z_3z_4 + z_1z_3z_4 + z_1z_2z_4 + z_1z_2z_3). \tag{3.78}$$

Therefore

$$\psi_{0,2,3,4}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3,z_4,\bar{z}_4) = \chi_{\yng(1,1,1)}(Z)\psi_{0,1,2,3}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3,z_4,\bar{z}_4). \tag{3.79}$$

If we denote the row length of $r$ by $r_i$, the general rule is

$$\psi_{0+r_4,1+r_3,2+r_2,3+r_1}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3,z_4,\bar{z}_4) = \chi_r(Z)\psi_{0,1,2,3}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3,z_4,\bar{z}_4). \tag{3.80}$$

### 3.5.3 State operator correspondence

We compute correlators using the formula

$$\langle \cdots \rangle = \int [dZdZ^\dagger] e^{-Tr(ZZ^\dagger)} \ldots \tag{3.81}$$

where $Z$ is an $N \times N$ matrix. Using a change of coordinates we can write this integral in terms of the eigenvalues of $Z$ as

$$\int [dZdZ^\dagger] e^{-Tr(ZZ^\dagger)} \cdots \rightarrow \int \prod_{i=1}^{N} dz_i d\bar{z}_i J(z, \bar{z}) e^{-\sum_j z_j \bar{z}_j} \cdots$$

where the Jacobian above can be expressed as [3]

$$J = \Delta(z)\Delta(\bar{z}), \quad \text{and} \quad \Delta(\lambda) \equiv \prod_{i<j} (\lambda_i - \lambda_j). \tag{3.82}$$

Now we want to show that the two point function of Schur polynomials is equal to the overlap of fermion wave functions. The connection is

$$\begin{aligned} \langle \chi_R(Z) \chi_S(Z^\dagger) \rangle &= \int [dZdZ^\dagger] e^{-Tr(ZZ^\dagger} \chi_R(Z) \chi_S(Z^\dagger) \\ &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i \Delta(z) \Delta(\bar{z}) e^{-\sum_j z_j \bar{z}_j} \chi_R(Z) \chi_S(Z^\dagger). \end{aligned} \tag{3.83}$$

The fermion wave function in (3.80) can be written as

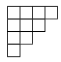
$$\psi_{0+R_N, 1+R_{N-1}, 2+R_{N-2}, 3+R_{N-3}, \cdots, N-1+R_1}(z_1, \bar{z}_1, z_2, \bar{z}_2, \cdots, z_N, \bar{z}_N) = \chi_R(Z) \Delta(z) e^{-\frac{1}{2} \sum_j z_j \bar{z}_j}. \tag{3.84}$$

Then we find

$$\langle \chi_R(Z) \chi_S(Z^\dagger) \rangle = \int \prod_{i=1}^{N} dz_i d\bar{z}_i \psi_{0+R_N, 1+R_{N-1}, 2+R_{N-2}, 3+R_{N-3}, \cdots, N-1+R_1} \psi^*_{0+S_N, 1+S_{N-1}, 2+S_{N-2}, 3+S_{N-3}, \cdots, N-1+S_1}. \tag{3.85}$$

To illustrate this last equation, we will consider an example.

**Example: correlators and overlaps**

Here we will compute the two point function of $\langle \chi_R(Z^\dagger) \chi_R(Z) \rangle$ where $R =$  of (3.85). First, let write the ground state wave function as

$$\tilde{\psi}_{gs}(z_1, \bar{z}_1, \cdots, z_N, \bar{z}_N) = \epsilon^{a_1 a_2 \cdots a_n} \tilde{\psi}_0(z_{a_1}, \bar{z}_{a_1}) \tilde{\psi}_1(z_{a_2}, \bar{z}_{a_2}) \cdots \tilde{\psi}_{N-1}(z_{a_N}, \bar{z}_{a_N}) \tag{3.86}$$

where $\tilde{\psi}_l(z, \bar{z}) = z^l e^{-\frac{z\bar{z}}{2}}$. Lets first compute the normalization of this ground state

$$1 = \mathcal{N}^2 \int \prod_{i=1}^{N} dz_i d\bar{z}_i \tilde{\psi}_{gs}(z_1, \bar{z}_1, \cdots, z_N, \bar{z}_N) \tilde{\psi}^*_{gs}(z_1, \bar{z}_1, \cdots, z_N, \bar{z}_N) \tag{3.87}$$

$$= \mathcal{N}^2 N! \prod_{i=1}^{N} dz_i d\bar{z}_i \prod_{j=1}^{N} |\bar{\psi}_{j-1}(z_j, \bar{z}_j)|^2 \tag{3.88}$$

$$= \mathcal{N}^2 N! \prod_{j=0}^{N} j! \tag{3.89}$$

which tells us that the normalization is given by

$$\mathcal{N} = \frac{1}{\sqrt{N!}} \frac{1}{\prod_{j=0}^{N} j!}. \tag{3.90}$$

[3] See appendix C for the derivation of this Jacobian.

Now that we have the normalization, we can compute the overlaps of our choice. The wave function that will correspond to an arbitrary Young diagram $R$ is given by

$$\tilde{\psi}_R(z_1, \cdots, \bar{z}_N) = \epsilon^{a_1 a_2 \cdots a_n} \tilde{\psi}_{0+R_N}(z_{a_1}, \bar{z}_{a_1}) \tilde{\psi}_{1+R_{N-1}}(z_{a_2}, \bar{z}_{a_2}) \cdots \tilde{\psi}_{N-1+R_1}(z_{a_N}, \bar{z}_{a_N}). \tag{3.91}$$

The correlation function of Schur polynomials is given by

$$\langle \tilde{\psi}_R | \tilde{\psi}_R \rangle = \frac{1}{\prod_{j=0}^N j!} \int \prod_{i=1}^N dz_i d\bar{z}_i |\tilde{\psi}_R(z_1, \cdots, z_N)|^2 \tag{3.92}$$

$$= \frac{1}{\prod_{j=0}^N j!} \int \prod_{i=1}^N dz_i d\bar{z}_i |\tilde{\psi}_{0+R_N}(z_{a_1}, \bar{z}_{a_1})|^2 |\tilde{\psi}_{1+R_{N-1}}(z_{a_2}, \bar{z}_{a_2})|^2 \cdots |\tilde{\psi}_{N-1+R_1}(z_{a_N}, \bar{z}_{a_N})|^2 \tag{3.93}$$

$$= \prod_{j=0}^{N-1} \frac{(j + R_{N-j})!}{j!}. \tag{3.94}$$

Specialize to our original example where $R = \yng(4,3,2,1)$ and take $N = 7$. From this Young diagram, we have the row length $R_1 = 4$, which correspond to the number of boxes in the first row of $R$. The other row lengths are given by $R_2 = 3, R_3 = 2, R_4 = 1, R_5 = 0, R_6 = 0, R_7 = 0$. This correlator is evaluated to be

$$\begin{aligned} \langle \chi_R(Z^\dagger) \chi_R(Z) \rangle &= \prod_{j=0}^{6} \frac{(j + R_{7-j})!}{j!} \\ &= \frac{(0 + R_7)!}{0!} \frac{(1 + R_6)!}{1!} \frac{(2 + R_5)!}{2!} \frac{(3 + R_4)!}{3!} \frac{(4 + R_3)!}{4!} \frac{(5 + R_2)!}{5!} \frac{(6 + R_1)!}{6!} \\ &= \frac{4!}{3!} \frac{6!}{4!} \frac{8!}{5!} \frac{10!}{6!} \\ &= (4) \cdot (6 \cdot 5) \cdot (8 \cdot 7 \cdot 6) \cdot (10 \cdot 9 \cdot 8 \cdot 7). \end{aligned} \tag{3.95}$$

Recall that the 2-point function of the Schur polynomials is given by

$$\langle \chi_R(Z^\dagger) \chi_R(Z) \rangle = f_R \tag{3.96}$$

$$\Rightarrow \langle \chi_{\yng(4,3,2,1)} \chi^\dagger_{\yng(4,3,2,1)} \rangle = N(N+1)(N+2)(N+3) \cdot (N-1)N(N+1) \cdot (N-2)(N-1) \cdot (N-3) \tag{3.97}$$

$$= (7 \cdot 8 \cdot 9 \cdot 10) \cdot (6 \cdot 7 \cdot 8) \cdot (5 \cdot 6) \cdot 4. \tag{3.98}$$

To move from the second line to the third line we set $N = 7$. We see that (3.95) and (3.98) are the same and this verifies that (3.85) holds for this example. This results tells us that for every operator i.e Schur polynomial, there is a corresponding state (fermion wave function). Computing correlators of operators maps into computing overlaps of the states.

### 3.5.4 Correspondence between row lengths and the eigenvalues

Recall that the matrix $Z$ has a total of $N$ eigenvalues and the Schur polynomial has a total of $N$ rows. Our goal here is map the eigenvalues into row lengths of the Young diagram which labels the Schur polynomial. It is known that the row length determine the angular momenta of the fermions according to

$$l_i = N - i + r_i \tag{3.99}$$

and each eigenvalue is the coordinate of the particle. In the large $N$ limit the variance of the eigenvalues vanishes, so each eigenvalue takes on a definite value. Using the rules of quantum mechanics we can compute this value of the eigenvalue by computing the expected value of $z\bar{z}$. This is done by using

$$\psi_l(z,\bar{z}) = \frac{1}{\sqrt{\pi l!}} z^l e^{-\frac{z\bar{z}}{2}} = \langle z,\bar{z}|\psi_l\rangle. \tag{3.100}$$

Computing $\langle\psi_l|z\bar{z}|\psi_l\rangle$ we get

$$\begin{aligned}
\langle\psi_l|z\bar{z}|\psi_l\rangle &= \int_{\infty}^{\infty} dz d\bar{z} \bar{\psi}_l z\bar{z}\psi_l \\
&= \frac{1}{\pi l!}\int_{\infty}^{\infty} dz d\bar{z} z^{2l} e^{-z\bar{z}}|z|^2 \\
&= \frac{1}{\pi l!}\int_0^{2\pi}\int_0^r r\sin\theta d\theta dr r^{2l} r^2 e^{-r^2} \\
&= \frac{2\pi}{\pi l!}\int_0^r dr r^{2l+3} e^{-r^2} \\
&= \frac{2}{l!}\left(\frac{1}{2}\Gamma(2+l)\right) \\
&= \frac{2}{l!}\left(\frac{1}{2}(l+1)!\right) \\
&= l+1.
\end{aligned} \tag{3.101}$$

The Schur polynomial corresponds to some definite state of the large $N$ theory. The large $N$ state will have a definite eigenvalue distribution associated with it. The eigenvalues and the row lengths give the same information.

## 3.6 Generalizing to 2 matrices

Our goal here is to go beyond one matrix which will allow us to go beyond the 1/2 BPS sector. We approach this problem on the field theory side by using the restricted Schur polynomial operators.

### 3.6.1 BPS Operator for two matrices

We consider a local gauge invariant operator that is constructed using the $m$ $Y$ fields and $n$ $Z$ fields. Any such operator can be written as a sum of terms of the form

$$Tr(\sigma Y^{\otimes m} Y^{\otimes n}) = Y^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}. \tag{3.102}$$

Recall that the restricted Schur polynomial is

$$\chi_{R,(r,s)_{\alpha\beta}}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) Tr(\sigma Y^{\otimes m} Z^{\otimes n}) \tag{3.103}$$

and that $Tr(\sigma Z^{\otimes n} Y^{\otimes m})$ can be written as

$$Tr(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_{T,(t,u)_{\alpha\beta}} \frac{d_T n! m!}{d_t d_u (n+m)!} \chi_{T,(t,u)_{\alpha\beta}}(\sigma^{-1}) \chi_{T,(t,u)_{\alpha\beta}}(Z,Y) \tag{3.104}$$

where $\chi_{T,(t,u)_{\alpha\beta}}(\sigma^{-1})$ is the restricted character, which can be written as

$$\chi_{R,(r,s)_{\alpha\beta}}(\sigma) = Tr(P_{R,(r,s)_{\alpha\beta}} \Gamma^{(R)}(\sigma)) \tag{3.105}$$

where the intertwining map is

$$P_{R,(r,s)_{\alpha\beta}} = 1_r \otimes \sum_a |s,a;\alpha\rangle\langle s,a;\beta|. \tag{3.106}$$

BPS operators can be represented in terms of Gauss graphs and they correspond to the Gauss graphs with open strings that loop back to the same giant. Using the language of restricted Schur polynomials labelled by Young diagram, the Gauss graph operator is given by

$$O_{R,r}^{\vec{m}}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s \vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s} \Gamma_{jk}^{(s)}(\sigma) B_{j\mu_1}^{s\to 1_H} B_{k\mu_2}^{s\to 1_H} O_{R,(r,s)_{\mu_1\mu_2}} \tag{3.107}$$

where $|H|$ is given by

$$H = S_{m_1} \times S_{m_2} \times \cdots S_{m_p}$$

and

$$\frac{1}{|H|} \sum_{\gamma\in H} \Gamma_{jk}^{(s)}(\gamma) = \sum_{\mu} B_{j\mu}^{s\to 1_H} B_{k\mu}^{s\to 1_H}. \tag{3.108}$$

Notice that the LHS looks like a projector for character $\chi(\gamma) = 1$, that is

$$\begin{aligned} \frac{1}{|H|} \sum_{\gamma\in H} \chi(\gamma)\Gamma_{jk}^{(s)}(\gamma) &= \frac{1}{|H|} \sum_{\gamma\in H} 1 \cdot \Gamma_{jk}^{(s)}(\gamma) \\ &= P_{jk}. \end{aligned}$$

We will work out an example that will allow us to illustrate this last equation.

**Example:**$B_{j\mu}^{s\to 1_H}$ **for two rows with** $m_1 = 2$ **and** $m_2 = 1$

Consider the case where we have a Young diagram with two impurities in the first row and one impurity in the second row i.e $m_1 = 2$ and $m_2 = 1$.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ & * & * \\ \hline \ & \ & \ & \ & * \\ \cline{1-5} \end{array} \tag{3.109}$$

For this case $|H|$ is given by

$$\begin{aligned} H &= S_{m_1} \times S_{m_2} \\ &= S_2 \times S_1 \\ &= \{\mathbb{I}, (12)\} \times \{\mathbb{I}\} \\ &= \{\mathbb{I}, (12)\}. \end{aligned} \tag{3.110}$$

Picking $s = \tiny\yng(2,1)$ this Young diagram has two Young-Yamanouchi states which are

$$|1\rangle = \tiny\young(31,2), \qquad |2\rangle = \tiny\young(32,1).$$

Now the LHS of (3.108) gives

$$\begin{aligned} \frac{1}{2}\left[\Gamma^{\tiny\yng(2,1)}(\mathbb{I}) + \Gamma^{\tiny\yng(2,1)}(12)\right] &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\right] \\ &= \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix}. \end{aligned}$$

This final expression is the projector, it satisfy both the projector properties which are

$$P^2 = P, \qquad Tr(P) = 1.$$

To find the vector $B_{j\mu}^{s\to 1_H}$, we will solve

$$P|v\rangle = |v\rangle.$$

Choosing the vector to be $\begin{bmatrix} a \\ b \end{bmatrix}$, we have

$$\frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solving this we get

$$a = \frac{\sqrt{3}b}{3}$$

and the vector becomes $\begin{bmatrix} \frac{\sqrt{3}b}{3} \\ b \end{bmatrix}$. Using the fact that this vector is normalized we have

$$\frac{3b^2}{9} + b^2 = 1$$
$$b = \frac{\sqrt{3}}{2}$$

and

$$|v\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}.$$

This vector is the branching coefficient

$$B_{j\mu}^{s\rightarrow 1_H} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}. \tag{3.111}$$

One can easily check that

$$|v\rangle\langle v| = P.$$

The BPS Gauss graph operators correspond to $\sigma = 1$. Using this we can write (3.107) as

$$\begin{aligned} O^{BPS}_{R,r,\vec{m}}(\sigma) &= O^{\vec{m}}_{R,r}(1) \\ &= \frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}\Gamma^s_{jk}(1)B_{j\mu_1}^{s\rightarrow 1_H}B_{k\mu_2}^{s\rightarrow 1_H}O_{R,(r,s)_{\mu_1\mu_2}} \\ &= \frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}\delta_{jk}B_{j\mu_1}^{s\rightarrow 1_H}B_{k\mu_2}^{s\rightarrow 1_H}O_{R,(r,s)_{\mu_1\mu_2}} \\ &= \frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}B_{k\mu_1}^{s\rightarrow 1_H}B_{k\mu_2}^{s\rightarrow 1_H}O_{R,(r,s)_{\mu_1\mu_2}} \\ &= \frac{|H|}{\sqrt{m!}}\sum_{s\vdash m}\sum_{\mu}\sqrt{d_s}O_{R,(r,s)_{\mu\mu}}. \end{aligned}$$

On the third line we used

$$B_{k\mu_1}^{s\rightarrow 1_H}B_{k\mu_2}^{s\rightarrow 1_H} = \delta_{\mu_1\mu_2}.$$

This completes our discussion of the BPS operators. In the next chapter we will motivate a description of these BPS operators in terms of the eigenvalues of the $Z$ and $Y$ matrices.

# Chapter 4

# Eigenvalue Dynamics for Multimatrix Models

## 4.1 Motivation

The large $N$ expansion continues to be a promising approach towards the strong coupling dynamics of quantum field theories. For example, 't Hooft's proposal that the large $N$ expansions of Yang-Mills theories are equivalent to the usual perturbation expansion in terms of topologies of worldsheets in string theory[4] has been realized concretely in the AdS/CFT correspondence[1]. Besides the usual planar limit where classical operator dimensions are held fixed as we take $N \to \infty$, there are non-planar large $N$ limits of the theory [11] defined by considering operators with a bare dimension that is allowed to scale with $N$ as we take $N \to \infty$. These limits are also relevant for the AdS/CFT correspondence. Indeed, operators with a dimension that scales as $N$ include operators relevant for the description of giant graviton branes[12, 13, 14] while operators with a dimension of order $N^2$ include operators that correspond to new geometries in supergravity[10, 8, 9]. These convincing motivations have motivated sustained study of large $N$ field theory. Despite this, carrying out the large $N$ expansion for most matrix models is still beyond our current capabilities.

One class of models for which the large $N$ expansion can be computed are the singlet sector of matrix quantum mechanics of a single hermitian matrix[15]. We can also consider a complex matrix model as long as we restrict ourselves to potentials that are analytic in $Z$ (summed with the dagger of this which needs to be added to get a real potential) and observables constructed out of traces of a product of $Z$s or out of a product of $Z^\dagger$s[16]. In these situations we can reduce the problem to eigenvalue dynamics. This is a huge reduction in degrees of freedom since we have reduced from $O(N^2)$ degrees of freedom, associated to the matrix itself, to $O(N)$ eigenvalue degrees of freedom. Studying saddle points of the original matrix action does not reproduce the large $N$ values of observables. This is a consequence of the large number of degrees of freedom: we expect fluctuations to be suppressed by $1/N^2$ so that if $N^2$ variables in total are fluctuating, then we can have fluctuations of size $1/N^2 \times N^2 \sim 1$ which are not suppressed as $N \to \infty$. In terms of eigenvalues there are only $N$ variables fluctuating so that fluctuations are bounded by $N \times 1/N^2 \sim 1/N$ which vanishes as $N \to \infty$. Thus, classical eigenvalue dynamics captures the large $N$ limit. For example, one can formulate the physics of the planar limit by using the density of eigenvalues as a dynamical variable. The resulting collective field theory defines a field theory that explicitly has $1/N$ as the loop expansion parameter[17, 18]. It has found both application in the context of the $c = 1$ string[19, 20, 21] and in descriptions of the LLM geometries[22].

Standard arguments show that eigenvalue dynamics corresponds to a familiar system: non-interacting fermions in an external potential[15]. This makes the description extremely convenient because the fermion dynamics is rather simple. This eigenvalue dynamics is also a natural description of the large $N$ but non-planar limits discussed above. Giant graviton branes which have expanded into the $AdS_5$ of the spacetime correspond to highly excited fermions or, equivalently, to single highly excited eigenvalues: the giant graviton is an eigenvalue[13, 9]. Giant graviton branes which have expanded into the $S^5$ of the spacetime correspond to holes in the Fermi sea, and hence to collective excitations of the eigenvalues where many eigenvalues are excited[9]. Half-BPS geometries also have a natural interpretation in terms of the eigenvalue dynamics: every fermion state can be identified with a particular supergravity geometry[8, 9]. The map between the two descriptions was discovered by Lin, Lunin and Maldacena in [10]. The fermion state can be specified by stating which states in phase space are occupied by a fermion, so we can divide phase space up into occupied and unoccupied states. By requiring regularity of the corresponding supergravity solution exactly the same structure arises: the complete set of regular solutions are specified by boundary conditions obtained by dividing a certain plane into black (identified with occupied states in the fermion phase space) and white (unoccupied states) regions. See [10] for the details.

Our main goal in this chapter is to ask if a similar eigenvalue description can be constructed for a two matrix model. Further, if such a construction exists, does it have a natural AdS/CFT interpretation? Work with a similar motivation but focusing on a different set of questions has appeared in[23, 24, 25, 26, 27]. We will consider the dynamics of two complex matrices, corresponding to the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory. Further, we consider the theory on $R\times S^3$ and expand all fields in spherical harmonics of the $S^3$. We will consider only the lowest $s$-wave components of these expansions so that the matrices are constant on the $S^3$. The reduction to the $s$-wave will be motivated below. In this way we find a matrix model quantum mechanics of two complex matrices. Expectation values are computed as follows

$$\langle\cdots\rangle=\int[dZdZ^\dagger dYdY^\dagger]e^{-S}\cdots \tag{4.1}$$

At first sight it appears that any attempts to reduce (4.1) to an eigenvalue description are doomed to fail: the integral in (4.1) runs over two independent complex matrices $Z$ and $Y$ which will almost never be simultaneously diagonalizable. However, perhaps there is a class of questions, generalizing the singlet sector of a single hermitian matrix model, that can be studied using eigenvalue dynamics. To explore this possibility, let's review the arguments that lead to eigenvalue dynamics for a single complex matrix $Z$. We can use the Schur decomposition[16, 28, 29],

$$Z=U^\dagger DU \tag{4.2}$$

with $U$ a unitary matrix and $D$ an upper triangular matrix, to explicitly change variables. Since we only consider observables that depend on the eigenvalues (the diagonal elements of $D$) we can integrate $U$ and the off diagonal elements of $D$ out of the model, leaving only the eigenvalues. The result of the integrations over $U$ and the off diagonal elements of $D$ is a non trivial Jacobian. Denoting the eigenvalues of $Z$ by $z_i$, those of $Z^\dagger$ are given by complex conjugation, $\bar{z}_i$. The resulting Jacobian [1] is[16]

$$J=\Delta(z)\Delta(\bar{z}) \tag{4.3}$$

where

$$\begin{aligned}\Delta(z) &= \begin{vmatrix} 1 & 1 & \cdots & 1\\ z_1 & z_2 & \cdots & z_N\\ \vdots & \vdots & \vdots\vdots\vdots & \vdots\\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1}\end{vmatrix}\\ &= \prod_{j>k}^{N}(z_j-z_k)\end{aligned} \tag{4.4}$$

is the usual Van der Monde determinant. A standard argument now maps this into non-interacting fermion dynamics[15]. Trying to apply a very direct change of variables argument to the two matrix

[1]The alternative derivation of this Jacobian is found in appendix C

model problem appears difficult. There is however an approach which both agrees with the above non-interacting fermion dynamics and can be generalized to the two matrix model. The idea is to construct a basis of operators that diagonalizes the inner product of the free theory. The construction of an orthogonal basis, given by the Schur polynomials, was achieved in [8]. Each Schur polynomial $\chi_R(Z)$ is labelled by a Young diagram $R$ with no more than $N$ rows. In [8] the exact (to all orders in $1/N$) two point function of Schur polynomials was constructed. The result is

$$\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle = f_R\delta_{RS} \tag{4.5}$$

where all spacetime dependence in the correlator has been suppressed. This dependence is trivial as it is completely determined by conformal invariance. The notation $f_R$ denotes the product of the factors of Young diagram $R$. Remarkably there is an immediate and direct connection to non-interacting fermions: the fermion wave function can be written as

$$\psi_R(\{z_i,\bar{z}_i\}) = \chi_R(Z)\Delta(z)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}. \tag{4.6}$$

This relation can be understood as a combination of the state operator correspondence (we associate a Schur polynomial operator on $R^4$ to a wave function on $R\times S^3$) and the reduction to eigenvalues (which is responsible for the $\Delta(z)$ factor)[9]. In this map the number of boxes in each row of $R$ determines the amount by which each fermion is excited. In this way, each row in the Young diagram corresponds to a fermion and hence to an eigenvalue. Having one very long row corresponds to exciting a single fermion by a large amount, which corresponds to a single large (highly excited) eigenvalue. In the dual AdS gravity, a single long row is a giant graviton brane that has expanded in the $AdS_5$ spacetime. Having one very long column corresponds to exciting many fermions by a single quantum, which corresponds to many eigenvalues excited by a small amount. In the dual AdS gravity, a single long column is a giant graviton brane that has expanded in the $S^5$ space.

The first questions we should tackle when approaching the two matrix problem should involve operators built using many $Z$ fields and only a few $Y$ fields. In this case at least a rough outline of the one matrix physics should be visible, and experience with the one matrix model will prove to be valuable.

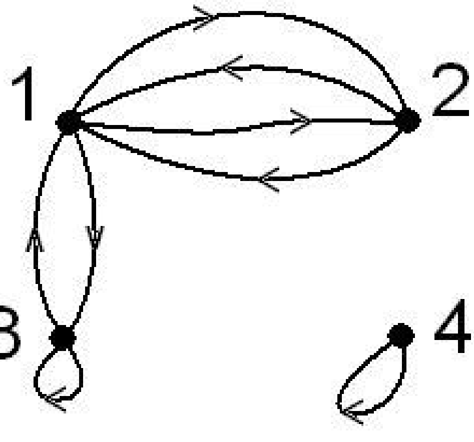



Figure 4.1: An example of a graph labelling an operator with a definite scaling dimension. Each node corresponds to an eigenvalue. Edges connect the different nodes so that the eigenvalues are interacting.

For the case of two matrices we can again construct a basis of operators that diagonalize the free field two point function. These operators $\chi_{R,(r,s)ab}(Z,Y)$ are a generalization of the Schur polynomials, called restricted Schur polynomials[30, 31, 32]. They are labelled by three Young diagrams $(R,r,s)$ and two multiplicity labels $(a,b)$. For an operator constructed using $n$ $Z$s and $m$ $Y$s, $R\vdash n+m$, $r\vdash n$ and $s\vdash m$. The multiplicity labels distinguish between different copies of the $(r,s)$ irreducible representation of $S_n\times S_m$ that arise when we restrict the irreducible representation $R$ of $S_{n+m}$ to the $S_n\times S_m$ subgroup. The two point function is

$$\langle \chi_{R,(r,s)ab}(Z,Y)\chi_{T,(t,u)cd}(Z^\dagger,Y^\dagger)\rangle = f_R\frac{\text{hooks}_R}{\text{hooks}_r\text{hooks}_s}\delta_{RT}\delta_{rt}\delta_{su}\delta_{ac}\delta_{bd} \tag{4.7}$$

where $f_R$ was defined after (4.5) and $\text{hooks}_a$ denotes the product of the hook lengths associated to Young diagram $a$. These operators do not have a definite dimension. However, they only mix weakly under the action of the dilatation operator and they form a convenient basis in which to study the spectrum of anomalous dimensions[33]. This action has been diagonalized in a limit in which $R$ has order 1 rows (or columns), $m \ll n$ and $n$ is of order $N$. Operators of a definite dimension are labelled by graphs composed of nodes that are traversed by oriented edges[34, 35]. There is one node for each row, so that each node corresponds to an eigenvalue. The directed edges start and end on the nodes. There is one edge for each $Y$ field and the number of oriented edges ending on a node must equal the number of oriented edges emanating from a node. See figure 1 for an example of a graph labelling an operator. This picture, derived in the Yang-Mills theory, has an immediate and compelling interpretation in the dual gravity: each node corresponds to a giant graviton brane and the directed edges are open string excitations of these branes. The constraint that the number of edges ending on a node equals the number of edges emanating from the node is simply encoding the Gauss law on the brane world volume, which is topologically an $S^3$. For this reason the graphs labelling the operators are called Gauss graphs. If we are to obtain a system of non-interacting eigenvalues, we should only consider Gauss graphs that have no directed edges stretching between nodes. See figure 2 for an example. In fact, these all correspond to BPS operators. We thus arrive at a very concrete proposal:

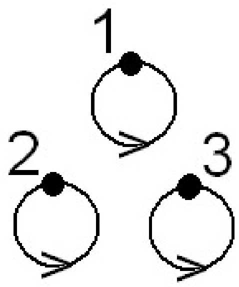



Figure 4.2: An example of a graph labelling a BPS operator. Each node corresponds to an eigenvalue. There are no edges connecting the different nodes so that these eigenvalues are not interacting.

**If there is a free fermion description arising from the eigenvalue dynamics of the two matrix model, it will describe the BPS operators of the $SU(2)$ sector.**

The BPS operators are associated to supergravity solutions of string theory. Indeed, the only one-particle states saturating the BPS bound in gravity are associated to massless particles and lie in the supergravity multiplet. Thus, eigenvalue dynamics will reproduce the supergravity dynamics of the gravity dual.

The BPS operators are all constructed from the $s$-wave of the spherical harmonic expansion on $S^3$[9]. This is our motivation for only considering operators constructed using the $s$-wave of the fields $Y$ and $Z$. One further comment is that it is usually not consistent to simply restrict to a subset of the dynamical degrees of freedom. Indeed, this is only possible if the subset of degrees of freedom dynamically decouples from the rest of the theory. In the case that we are considering this is guaranteed to be the case, in the large $N$ limit, because the Chan-Paton indices of the directed edges are frozen at large $N$ [34].

We should mention that eigenvalue dynamics as dual to supergravity has also been advocated by Berenstein and his collaborators[36, 37, 38, 39, 40, 41, 42]. See also [43, 44, 45, 46] for related studies. Using a combination of numerical and physical arguments, which are rather different to the route we have followed, compelling evidence for this proposal has already been found. The basic idea is that at strong coupling the commutator squared term in the action forces the Higgs fields to commute and hence, at strong coupling, the Higgs fields of the theory should be simultaneously diagonalizable. In this case, an eigenvalue description is possible. Notice that our argument is a weak coupling large $N$ argument, based on diagonalization of the one loop dilatation operator, that comes to precisely the same conclusion. In this chapter we will make some exact analytic statements that agree with and, in our opinion, refine some of the physical picture of the above studies. For example, we will start to make precise statements about what eigenvalue dynamics does and does not correctly reproduce.

## 4.2 Eigenvalue Dynamics for $\mathbf{AdS_5 \times S^5}$

To motivate our proposal for eigenvalue dynamics, we will review the $\frac{1}{2}$-BPS sector stressing the logic that we will subsequently use. The way in which a direct change of variables is used to derive the eigenvalue dynamics can be motivated by considering correlation functions of arbitrary observables $\cdots$ that are functions only of the eigenvalues. Because we are considering BPS operators, correlators computed in the free field theory agree with the same computations at strong coupling[47], so that we now work in the free field theory. Performing the change of variables we find

$$\begin{aligned} \langle \cdots \rangle &= \int [dZ dZ^\dagger] e^{-\text{Tr} Z Z^\dagger} \cdots \\ &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_k \bar{z}_k} \Delta(z) \Delta(\bar{z}) \cdots \\ &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i |\psi_{\text{gs}}(\{z_i, \bar{z}_i\})|^2 \cdots \end{aligned}$$

where the groundstate wave function is given by

$$\psi_{\text{gs}}(\{z_i, \bar{z}_i\}) = \Delta(z) e^{-\frac{1}{2}\sum_i z_i \bar{z}_i}. \tag{4.8}$$

We will shortly qualify the adjective "groundstate". Under the state-operator correspondence, this wave function is the state corresponding to the identity operator. The above transformation is equivalent to the identification

$$[dZ] e^{-\frac{1}{2}\text{Tr}(Z Z^\dagger)} \leftrightarrow \prod_{i=1}^{N} dz_i \, \psi_{\text{gs}}(\{z_i, \bar{z}_i\}). \tag{4.9}$$

The role of each of the elements of the wave function is now clear:

1. Under the state operator correspondence, dimensions of operators map to energies of states. The dimensions of BPS operators are not corrected, i.e. they take their free field values. This implies an evenly spaced spectrum and hence a harmonic oscillator wave function. This explains the $e^{-\frac{1}{2}\sum_i z_i \bar{z}_i}$ factor. It also suggests that the wavefunction will be a polynomial times this Gaussian factor.

2. There is a gauge symmetry $Z \to U Z U^\dagger$ that is able to permute the eigenvalues. Consequently we are discussing identical particles. Two matrices drawn at random from the complex Gaussian ensemble will not have degenerate eigenvalues, so we choose the particles to be fermions. This matches the fact that the wave function is a Slater determinant.

The wave function (4.8) satisfies these properties. Further, if we require that the wavefunction is a polynomial in the eigenvalues $z_i$ times the exponential $e^{-\frac{1}{2}\sum_i z_i \bar{z}_i}$, then (4.8) is the state of lowest energy (we did not write down a Hamiltonian, but any other wave function has more nodes and hence a higher energy) so it deserves to be called the ground state. The wave function (4.8) is the state corresponding to the $AdS_5 \times S^5$ spacetime in the $\frac{1}{2}$-BPS sector.

The above discussion can be generalized to write down a wave function corresponding to the $AdS_5 \times S^5$ spacetime in the $SU(2)$ sector. The equation (4.9) is generalized to

$$[dZ dY] e^{-\frac{1}{2}\text{Tr}(Z Z^\dagger) - \frac{1}{2}\text{Tr}(Y Y^\dagger)} \to \prod_{i=1}^{N} dz_i dy_i \, \Psi_{\text{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}). \tag{4.10}$$

The wave function must obey the following properties:

1. Our wave functions again describe states that correspond to BPS operators. The dimensions of the BPS operators take their free field values, implying an evenly spaced spectrum and hence a harmonic oscillator wave function. This suggests the wave function is a polynomial times the Gaussian $e^{-\frac{1}{2}\sum_i z_i\bar{z}_i-\frac{1}{2}\sum_i y_i\bar{y}_i}$ factor.

2. There is a gauge symmetry $Z \to UZU^\dagger$ and $Y \to UYU^\dagger$ that is able to permute the eigenvalues. Consequently we are discussing $N$ identical particles. Matrices drawn at random will not have degenerate eigenvalues, so we choose the particles to be fermions. Thus we expect the wave function is a Slater determinant.

We are working within the AdS/CFT correspondence. Our main goal is to understand how geometry in the dual gravity theory emerges. We expect a smooth geometry with small curvature emerges in the strongly coupled limit of the CFT. Correlators of operators belonging to the BPS sector of $\mathcal{N}=4$ SYM take their free field values even in the strong coupling limit[47]. Thus, although we study the free field theory our intuition should come from the dual gravity. In the free field theory the eigenvalue density is expected to have a U(1)×U(1) symmetry (as in (4.18)). This follows simply by integrating over the non-eigenvalue degrees of freedom in the Gaussian two matrix model. The strong coupling answer, where we again integrate over the non-eigenvalue degrees of freedom of the two matrices, but now in the strong coupling limit, will not match this free matrix model. It will match the dual gravity. In the $\text{AdS}_5\times\text{S}^5$ geometry we have an $SO(6)$ isometry of the $S^5$, which acts in the dual field theory as $SO(6)$ rotations of the six adjoint scalars of $\mathcal{N}=4$ SYM (see (4.69)). These are $\mathcal{R}$ symmetry rotations. When we restrict to the eigenvalues of $Z$ and $Y$, we reduce this to an $SO(4)$ symmetry. Since the geometry should emerge from the eigenvalues[36], this symmetry should manifest in the single eigenvalue probability density. This leads us to the last property we impose on our theory:

3. The probability density associated to a single particle $\rho_{\text{gs}}(z_1,\bar{z}_1,y_1,\bar{y}_1)$ must have an $SO(4)$ symmetry, i.e. it should be a function of $|z_i|^2+|y_i|^2$.

The single particle probability density referred to in point 3 above is given, for any state $\Psi(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})$ as usual, by

$$\rho(z_1,\bar{z}_1,y_1,\bar{y}_1)=\int\prod_{i=2}^{N}dz_id\bar{z}_idy_id\bar{y}_i|\Psi(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2. \tag{4.11}$$

There is a good reason why the single particle probability density is an interesting quantity to look at: at short distances the eigenvalues feel a repulsion from the Slater determinant, which vanishes when two eigenvalues are equal. At long distances the confining harmonic oscillator potential dominates, ensuring the eigenvalues are clumped together in some finite region and do not wander off to infinity. In the end we expect that at large $N$ the locus where the eigenvalues lie defines a specific surface, generalizing the idea of a density of eigenvalues for the single matrix model. This large $N$ surface is captured by $\rho(z_1,\bar{z}_1,y_1,\bar{y}_1)$. We will make this connection more explicit in a later section.

There appears to be a unique wave function singled out by the above requirements. It is given by

$$\Psi_{\text{gs}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})=\mathcal{N}\Delta(z,y)e^{-\frac{1}{2}\sum_k z_k\bar{z}_k-\frac{1}{2}\sum_k y_k\bar{y}_k} \tag{4.12}$$

where

$$\Delta(z,y) = \begin{vmatrix} y_1^{N-1} & y_2^{N-1} & \cdots & y_N^{N-1} \\ z_1y_1^{N-2} & z_2y_2^{N-2} & \cdots & z_Ny_N^{N-2} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ z_1^{N-2}y_1 & z_2^{N-2}y_2 & \cdots & z_N^{N-2}y_N \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix}$$

$$= \prod_{j>k}^{N} (z_j y_k - y_j z_k) \tag{4.13}$$

generalizes the usual Van der Monde determinant and $\mathcal{N}$ is fixed by normalizing the wave function. Normalizing the wave function in the state picture corresponds to choosing a normalization in the original matrix model so that the expectation value of 1 is 1. In the next section we will discuss the proposal (4.12) with a special emphasis on the symmetries realized by this wavefunction. As we will review, a wave function given as a product of Van der Monde determinants is also a natural guess. We will argue that (4.12) realizes more symmetries than a product of Van der Monde determinants does. We will then use the wave function to compute correlators. Surprisingly, for a large class of correlators the wave function (4.12) gives the exact answer.

## 4.3 Symmetries of the AdS$_5$×S$^5$ Wavefunction

The original two (complex) matrix model enjoys an $SO(4) \simeq SU(2)_L \times SU(2)_R$ symmetry. Indeed, the generators

$$\begin{aligned}
J_3^R &= Z_{ij}\frac{\partial}{\partial Z_{ij}} - Z_{ij}^\dagger \frac{\partial}{\partial Z_{ij}^\dagger} + Y_{ij}\frac{\partial}{\partial Y_{ij}} - Y_{ij}^\dagger \frac{\partial}{\partial Y_{ij}^\dagger}, \\
J_+^R &= Y_{ij}\frac{\partial}{\partial Z_{ij}^\dagger} - Z_{ij}\frac{\partial}{\partial Y\dagger_{ij}} \qquad J_-^R = Z_{ij}^\dagger \frac{\partial}{\partial Y_{ij}} - Y_{ij}^\dagger \frac{\partial}{\partial Z_{ij}}, \\
J_3^L &= Z_{ij}\frac{\partial}{\partial Z_{ij}} - Z_{ij}^\dagger \frac{\partial}{\partial Z_{ij}^\dagger} - Y_{ij}\frac{\partial}{\partial Y_{ij}} + Y_{ij}^\dagger \frac{\partial}{\partial Y_{ij}^\dagger}, \\
J_+^L &= Y_{ij}^\dagger \frac{\partial}{\partial Z_{ij}^\dagger} - Z_{ij}\frac{\partial}{\partial Y_{ij}} \qquad J_-^L = Z_{ij}^\dagger \frac{\partial}{\partial Y_{ij}^\dagger} - Y_{ij}\frac{\partial}{\partial Z_{ij}},
\end{aligned} \tag{4.14}$$

annihilate $\mathrm{Tr}(ZZ^\dagger) + \mathrm{Tr}(YY^\dagger)$. The above $SO(4)$ symmetry can also be realized at the level of the eigenvalues. In this case, the generators are

$$\begin{aligned}
J_3^R &= z_i\frac{\partial}{\partial z_i} - \bar{z}_i\frac{\partial}{\partial \bar{z}_i} + y_i\frac{\partial}{\partial y_i} - \bar{y}_i\frac{\partial}{\partial \bar{y}_i}, \\
J_+^R &= y_i\frac{\partial}{\partial \bar{z}_i} - z_i\frac{\partial}{\partial \bar{y}_i} \qquad J_-^R = \bar{z}_i\frac{\partial}{\partial y_i} - \bar{y}_i\frac{\partial}{\partial z_i}, \\
J_3^L &= z_i\frac{\partial}{\partial z_i} - \bar{z}_i\frac{\partial}{\partial \bar{z}_i} - y_i\frac{\partial}{\partial y_i} + \bar{y}_i\frac{\partial}{\partial \bar{y}_i}, \\
J_+^L &= \bar{y}_i\frac{\partial}{\partial \bar{z}_i} - z_i\frac{\partial}{\partial y_i} \qquad J_-^L = \bar{z}_i\frac{\partial}{\partial \bar{y}_i} - y_i\frac{\partial}{\partial z_i}.
\end{aligned} \tag{4.15}$$

It is simple to verify that

$$J_3^L \Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = J_+^L \Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = J_-^L \Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = 0 \tag{4.16}$$

so that the wave function is manifestly invariant under $SU(2)_L$. Further, since

$$J_3^R \Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = N(N-1)\, \Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) \tag{4.17}$$

it transforms covariantly under $U(1) \subset SU(2)_R$ generated by $J_3^R$. Thus, in summary, out of the original $SO(4)$ symmetry, the wave function is invariant under $SU(2)_L$ and covariant under a $U(1) \subset SU(2)_R$. Since we will restrict to the subset of BPS operators that are holomorphic in $Y$ and $Z$, this is the biggest symmetry we should expect.

A few comments are in order. If the interaction is switched off, the system is invariant under separate $U(N)$ actions on $Z$ and $Y$. Thus, in this case, the model has a $U(N) \times U(N)$ symmetry. If we restrict ourselves to correlators of operators that never have $Y$s and $Z$s in the same trace, the wave function

$$\Psi_{\rm VdM} = \mathcal{N}\Delta(z)\Delta(y) e^{-\frac{1}{2}\sum_j (z_j\bar{z}_j + y_j\bar{y}_j)} \tag{4.18}$$

will reproduce the exact values for all correlators. Notice that this wave function is covariant under $U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$ generated by $J_3^L$ and $J_3^R$, i.e. it has less symmetry than (4.12). Further, if we consider correlators of operators that include products of $Z$ and $Y$ matrices the symmetry is broken to $U(N)$. The integration over the non-eigenvalue degrees of freedom is nontrivial, but the result will again be a polynomial in the eigenvalues. The precise form of the polynomial will depend on the choice of operators in the correlator and we will not get a simple rule for translating a specific operator. In the next section we will show that using (4.12), we will in fact obtain a simple rule for translating a specific operator into the eigenvalue language and the translation will not depend on the choice of the other operators in the correlator. For these reasons, we do not discuss $\Psi_{\rm VdM}$ further.

To end this section we consider the location of the zeros of (4.12). For each eigenvalue we have a vector with coordinates $(z_i, y_i)$ on $\mathbb{C}^2$. Physically we expect that the wave function must vanish whenever $n > 1$ eigenvalues coincide, leading to an enhanced symmetry of the joint eigenvalue configuration[36]. The wave function vanishes whenever the vectors associated to two distinct eigenvalues are parallel, i.e. whenever $(z_i, y_i) = \lambda(z_j, y_j)$. If $\lambda \neq 1$ the eigenvalues are not coincident, there is no enhanced symmetry of the joint eigenvalue configuration and physically there is no reason why such an eigenvalue configuration should be weighted with zero. Thus, there are more zeroes than what we expect. Clearly then (4.12) will get various things wrong, but given that it realizes more symmetries that $\Psi_{\rm VdM}$, it may be good enough for some computations. We will confirm this in the next section by showing that this wave function reproduces the correct exact answer for a large class of matrix model correlators.

Finally, note that it is useful to think of the wave function as a function of two points in $\mathbb{C}P^1 \times \mathbb{C}^*$, with $(z_i, y_i)$ simultaneously the coordinates of a point and the affine coordinates of the projective sphere base. With this interpretation, the singularities are associated with points coinciding in the base which is physically more sensible.

## 4.4 Correlators

In this section we will provide detailed tests of this wave function by computing correlators with the wave function and comparing them to the exact results from the matrix model. The comparison is accomplished by using the equation

$$\int [dY dZ dY^\dagger dZ^\dagger] e^{-{\rm Tr}(ZZ^\dagger)-{\rm Tr}(YY^\dagger)} \cdots = \int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 \cdots \tag{4.19}$$

to compute correlators of observables (denoted by $\cdots$ above) that depend only on the eigenvalues. We have already argued above that we expect that the observables that are correctly computed using eigenvalue dynamics are the BPS operators of the CFT. As a first example, consider correlators of traces $O_J = {\rm Tr}(Z^J)$. These can be computed exactly in the matrix model, using a variety of different techniques - see for example [16, 48, 28]. The result is

$$\langle {\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle = \frac{1}{J+1}\Big[\frac{(J+N)!}{(N-1)!} - \frac{N!}{(N-J-1)!}\Big] \tag{4.20}$$

if $J < N$ and

$$\langle {\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!} \tag{4.21}$$

if $J \geq N$. These expressions could easily be expanded to generate the $1/N$ expansion if we wanted to do that. We would now like to consider the eigenvalue computation. It is useful to write the wave function as

$$\Psi_{\rm gs}(\{z_i,\bar z_i,y_i,\bar y_i\})=\frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1a_2\cdots a_N}\frac{z^0_{a_1}y^{N-1}_{a_1}}{\sqrt{0!(N-1)!}}\cdots\frac{z^{k-1}_{a_k}y^{N-k}_{a_k}}{\sqrt{(k-1)!(N-k)!}}\cdots$$
$$\cdots\frac{z^{N-1}_{a_N}y^0_{a_N}}{\sqrt{(N-1)!0!}}e^{-\frac12\sum_q z_q\bar z_q-\frac12\sum_q y_q\bar y_q}. \quad (4.22)$$

The gauge invariant observable in this case is given by

$$\mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})=\sum_{i=1}^N z_i^J\sum_{j=1}^N \bar z_j^J. \quad (4.23)$$

It is now straightforward to find

$$\int\prod_{i=1}^N dz_id\bar z_idy_id\bar y_i|\Psi_{\rm gs}(\{z_i,\bar z_i,y_i,\bar y_i\})|^2\sum_{i=1}^N z_i^J\sum_{j=1}^N\bar z_j^J=\frac{1}{J+1}\frac{(J+N)!}{(N-1)!}. \quad (4.24)$$

When evaluating the above integral, only the terms with $i=j$ contribute. From this result we see that we have not reproduced traces with $J<N$ correctly - we don't even get the leading large $N$ behaviour right. We have, however, correctly reproduced the exact answer (to all orders in $1/N$) of the two point function for all single traces of dimension $N$ or greater. For $J>N$ there are trace relations of the form

$$\mathrm{Tr}(Z^J)=\sum_{i,j,\dots,k}c_{ij\dots k}\mathrm{Tr}(Z^i)\mathrm{Tr}(Z^j)\cdots\mathrm{Tr}(Z^k) \quad (4.25)$$

$i,j,...,k\le N$ and $i+j+\cdots+k=J$. The fact that we reproduce two point correlators of traces with $J>N$ exactly implies that we also start to reproduce sums of products of traces of less than $N$ fields. This suggests that the important thing is not the trace structure of the operator, but rather the dimension of the state.

The fact that we only reproduce observables that have a large enough dimension is not too surprising. Indeed, supergravity can't be expected to correctly describe the back reaction of a single graviton or a single string. To produce a state in the CFT dual to a geometry that is different from the AdS vacuum one needs to allow a number of giant gravitons (eigenvalues) to condense. The eigenvalue dynamics is correctly reproducing the two point function of traces when their energy is greater than that required to blow up into a giant graviton.

With a very simple extension of the above argument we can argue that we also correctly reproduce the correlator $\langle\mathrm{Tr}(Y^J)\mathrm{Tr}(Y^{\dagger J})\rangle$ with $J\ge N$. A much more interesting class of observables to consider are mixed traces, which contain both $Y$ and $Z$ fields. To build BPS operators using both $Y$ and $Z$ fields we need to construct symmetrized traces. A very convenient way to perform this construction is as follows

$$\mathcal{O}_{J,K}=\frac{J!}{(J+K)!}\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K\mathrm{Tr}(Z^{J+K}). \quad (4.26)$$

The normalization up front is just the inverse of the number of terms that appear. With this normalization, the translation between the matrix model observable and an eigenvalue observable is

$$\mathcal{O}_{J,K}\leftrightarrow\sum_i z_i^Jy_i^K. \quad (4.27)$$

Since we could not find this computation in the literature, we will now explain how to evaluate the matrix model two point function exactly, in the free field theory limit. Since the dimension of BPS operators are not corrected, this answer is in fact exact. To start, perform the contraction over the $Y, Y^\dagger$ fields

$$\begin{aligned}\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle &= \left(\frac{J!}{(J+K)!}\right)^2\langle \mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \mathrm{Tr}(Z^{J+K})\mathrm{Tr}\left(Y^\dagger\frac{\partial}{\partial Z^\dagger}\right)^K \mathrm{Tr}(Z^{\dagger\, J+K})\rangle \\ &= \left(\frac{J!}{(J+K)!}\right)^2 K!\langle \mathrm{Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right)^K \mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger\, J+K})\rangle. \end{aligned} \quad (4.28)$$

Given the form of the matrix model two point function

$$\langle Z_{ij}Z^\dagger_{kl}\rangle = \delta_{il}\delta_{jk} \quad (4.29)$$

we know that we can write any free field theory correlator as

$$\langle\cdots\rangle = e^{\mathrm{Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right)}\cdots\Big|_{Z=Z^\dagger=0}. \quad (4.30)$$

Using this identity we now find

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \left(\frac{J!}{(J+K)!}\right)^2 K!\frac{(J+K)!}{J!}\langle \mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger\, J+K})\rangle. \quad (4.31)$$

Thus, the result of the matrix model computation is

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\left[\frac{(J+K+N)!}{(N-1)!} - \frac{N!}{(N-J-K-1)!}\right] \quad (4.32)$$

if $J+K<N$ and

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!} \quad (4.33)$$

if $J+K\geq N$. Notice that for these two matrix observables we again get a change in the form of the correlator as the dimension of the trace exceeds $N$.

Next, consider the eigenvalue computation. We need to perform the integral

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\mathrm{gs}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{k=1}^N z_k^J y_k^K\sum_{j=1}^N \bar{z}_j^J\bar{y}_j^K. \quad (4.34)$$

After some straightforward manipulations we have

$$\begin{aligned}\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \pi^{-2N}\int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \frac{|z_1|^0|y_1|^{2N-2}}{0!(N-1)!}\cdots\frac{|z_k|^{2k-2}|y_k|^{2N-2k}}{(k-1)!(N-k)!}\cdots \\ \frac{|z_N|^{2N-2}|y_N|^0}{(N-1)!0!}\times e^{-\sum_q z_q\bar{z}_q-\sum_q y_q\bar{y}_q}\sum_{k,j=1}^N z_k^J y_k^K\bar{z}_j^J\bar{y}_j^K. \end{aligned} \quad (4.35)$$

Only terms with $k=j$ contribute so that

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \sum_{k=1}^N\frac{(N-k+K)!}{(N-k)!}\frac{(J+k-1)!}{(k-1)!} = \frac{K!J!}{(K+J+1)!}\frac{(J+K+N)!}{(N-1)!}. \quad (4.36)$$

Thus, we again correctly reproduce the exact (to all orders in $1/N$) answer for the two point function of single trace operators of dimension $N$ or greater. Inspecting (4.14) we notice that we have obtained $\mathcal{O}_{J,K}$ from $\mathcal{O}_{J+K}$ by applying $J^L_-$, that is, by applying an $SU(2)_L$ rotation. Since both the original matrix description and the eigenvalue description enjoy $SU(2)_L$ symmetry, the agreement of the $\langle\mathcal{O}^\dagger_{J,K}\mathcal{O}_{J,K}\rangle$ correlator is not independent of the agreement of the $\langle\mathcal{O}^\dagger_{J+K}\mathcal{O}_{J+K}\rangle$ correlator.

It is also interesting to consider multi trace correlators. We will start with the correlator between a double trace and a single trace and we will again start with the matrix model computation

$$\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}\times$$
$$\langle \mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^{K_1}\mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^{K_2}\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}\left(Y^\dagger\frac{\partial}{\partial Z^\dagger}\right)^{K_1+K_2}\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle. \tag{4.37}$$

We could easily set $K_1 = K_2 = 0$ and obtain traces involving only a single matrix. Begin by contracting all $Y, Y^\dagger$ fields to obtain

$$\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}(K_1+K_2)!\times$$
$$\langle\frac{\partial}{\partial Z_{i_1j_1}}\cdots\frac{\partial}{\partial Z_{i_{K_1}j_{K_1}}}\mathrm{Tr}(Z^{J_1+K_1})\frac{\partial}{\partial Z_{i_{K_1+1}j_{K_1+1}}}\cdots\frac{\partial}{\partial Z_{i_{K_1+K_2}j_{K_1+K_2}}}\mathrm{Tr}(Z^{J_2+K_2})$$
$$\frac{\partial}{\partial Z^\dagger_{j_1i_1}}\cdots\frac{\partial}{\partial Z^\dagger_{j_{K_1+K_2}i_{K_1+K_2}}}\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle. \tag{4.38}$$

It is now useful to integrate by parts with respect to $Z^\dagger$, using the identity

$$\langle\frac{\partial}{\partial Z_{ij}}f(Z)\,g(Z)\,\frac{\partial}{\partial Z^\dagger_{ji}}h(Z^\dagger)\rangle = n_f\langle f(Z)\,g(Z)\,h(Z^\dagger)\rangle \tag{4.39}$$

where $f(Z)$ is of degree $n_f$ in $Z$. Repeatedly using this identity, we find

$$\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}(K_1+K_2)!\times$$
$$\frac{(J_1+K_1)!}{J_1!}\frac{(J_2+K_2)!}{J_2!}\langle\mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle$$
$$= \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2)!}\langle\mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle. \tag{4.40}$$

This last correlator is easily computed. For example, if $J_1 + K_1 < N$ and $J_2 + K_2 < N$ we have

$$\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\Big[\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}$$
$$+\frac{N!}{(N-J_1-K_1-J_2-K_2-1)!} - \frac{(N+J_1+K_1)!}{(N-J_2-K_2-1)!}$$
$$- \frac{(N+J_2+K_2)!}{(N-J_1-K_1-1)!}\Big] \tag{4.41}$$

and if $J_1 + K_1 \geq N$ and $J_2 + K_2 \geq N$ we have

$$\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}. \tag{4.42}$$

It is a simple exercise to check that, in terms of eigenvalues, we have

$$\begin{aligned}
\langle \mathcal{O}_{J_1,K_1}\mathcal{O}_{J_2,K_2}\mathcal{O}^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\\
&\times\sum_{k=1}^N z_k^{J_1}y_k^{K_1}\sum_{l=1}^N z_l^{J_2}y_l^{K_2}\sum_{j=1}^N \bar{z}_j^{J_1+J_2}\bar{y}_j^{K_1+K_2}\\
&= \frac{(J_1+J_2)!(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}
\end{aligned} \tag{4.43}$$

so that once again we have reproduced the exact answer as long as the dimension of each trace is not less than $N$. The agreement that we have observed for multi trace correlators continues as follows: as long as the dimension of each trace is greater than $N-1$ the matrix model and the eigenvalue descriptions agree and both give

$$\langle O_{J_1,K_1}O_{J_2,K_2}\cdots O_{J_n,K_n}O^\dagger_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!}\delta_{J_1+\cdots+J_n,J}\delta_{K_1+\cdots+K_n,K} \tag{4.44}$$

for the exact value of this correlator. We have limited ourselves to a single daggered observable in the above expression for purely technical reasons: it is only in this case that we can compute the matrix model correlator using the identity (4.39). It would be interesting to develop analytic methods that allow more general computations.

Finally, we can also test multi trace correlators with a dimension of order $N^2$. A particularly simple operator is the Schur polynomial labelled by a Young diagram $R$ with $N$ rows and $M$ columns (i.e $R$ is rectangular). For this $R$ we have

$$\chi_R(Z) = (\det Z)^M = z_1^M z_2^M\cdots z_N^M, \tag{4.45}$$

$$\chi_R(Z^\dagger) = (\det Z^\dagger)^M = \bar{z}_1^M\bar{z}_2^M\cdots\bar{z}_N^M. \tag{4.46}$$

The dual LLM geometry is labelled by an annulus boundary condition that has an inner radius of $\sqrt{M}$ and an outer radius of $\sqrt{M+N}$. The two point correlator of this Schur polynomial is

$$\begin{aligned}
\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \chi_R(Z)\chi_R(Z^\dagger)|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\\
&= \pi^{-2N}\int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \frac{|z_1|^{0+2M}|y_1|^{2N-2}}{0!(N-1)!}\cdots\frac{|z_k|^{2k-2+2M}|y_k|^{2N-2k}}{(k-1)!(N-k)!}\\
&\times\ \cdots\frac{|z_N|^{2N-2+2M}|y_N|^0}{(N-1)!0!}\times e^{-\sum_q z_q\bar{z}_q-\sum_q y_q\bar{y}_q}\\
&= \prod_{i=1}^N\frac{(i-1+M)!}{(i-1)!}
\end{aligned} \tag{4.47}$$

which is again the exact answer for this correlator.

After this warm up example we will now make a few comments that are relevant for the general case. The details are much more messy, so we will not manage to make very precise statements. We have however included this discussion as it does provide a guide as to when eigenvalue dynamics is applicable. A Schur polynomial labelled with a Young diagram $R$ that has row lengths $r_i$ is given in terms of eigenvalues as (our labelling of the rows is defined by $r_1\ge r_2\ge\cdots\ge r_N$)

$$\chi_R(Z) = \frac{\epsilon_{a_1a_2\cdots a_N}z_{a_1}^{N-1+r_1}z_{a_2}^{N-2+r_2}\cdots z_{a_N}^{r_N}}{\epsilon_{b_1b_2\cdots b_N}z_{b_1}^{N-1}z_{b_2}^{N-2}\cdots z_{b_{N-1}}}. \tag{4.48}$$

Using this expression, we can easily write the exact two point function as follows

$$\begin{aligned}
\langle \chi_R(Z)\chi_R(Z^\dagger)\rangle &= \frac{1}{N!\pi^N}\prod_{j=0}^{N-1}\frac{1}{j!}\int\prod_{i=1}^{N} dz_i d\bar{z}_i \epsilon_{a_1a_2\cdots a_N} z_{a_1}^{N-1+r_1} z_{a_2}^{N-2+r_2}\cdots z_{a_N}^{r_N} \\
&\quad \times \epsilon_{b_1b_2\cdots b_N}\bar{z}_{b_1}^{N-1+r_1}\bar{z}_{b_2}^{N-2+r_2}\cdots \bar{z}_{b_N}^{r_N} e^{-\sum_k z_k\bar{z}_k} \\
&= \prod_{j=0}^{N-1}\frac{(j+r_{N-j})!}{j!} = f_R. \qquad (4.49)
\end{aligned}$$

Using our wave function we can compute the two point function of Schur polynomials. The result is

$$\begin{aligned}
\langle \chi_R(Z)\chi_R(Z^\dagger)\rangle &= \int\prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i \chi_R(Z)\chi_R(Z^\dagger)|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \\
&= \frac{1}{\pi^N}\prod_{j=0}^{N-1}\frac{1}{j!}\int\prod_{i=1}^{N} dz_i d\bar{z}_i \; |z_{a_1}|^{2N-2}|z_{a_2}|^{2N-4}\cdots|z_{a_{N-1}}|^2 \\
&\quad \times \frac{\epsilon_{b_1b_2\cdots b_N} z_{b_1}^{N-1+r_1} z_{b_2}^{N-2+r_2}\cdots z_{b_N}^{r_N}}{\epsilon_{c_1c_2\cdots c_N} z_{c_1}^{N-1} z_{c_2}^{N-2}\cdots z_{c_{N-1}}} \\
&\quad \times \frac{\epsilon_{d_1d_2\cdots d_N} \bar{z}_{d_1}^{N-1+r_1} \bar{z}_{d_2}^{N-2+r_2}\cdots \bar{z}_{d_N}^{r_N}}{\epsilon_{e_1e_2\cdots e_N} \bar{z}_{e_1}^{N-1} \bar{z}_{e_2}^{N-2}\cdots \bar{z}_{e_{N-1}}} e^{-\sum_k z_k\bar{z}_k}. \qquad (4.50)
\end{aligned}$$

When the integration over the angles $\theta_i$ associated to $z_i = r_i e^{i\theta_i}$ are performed, a non-zero result is only obtained if powers of the $z_i$ match the powers of the $\bar{z}_i$. The difference between the above expression and the exact answer is simply that in the eigenvalue expression these powers are separately set to be equal in the measure and in the product of Schur polynomials - there are two matchings, while in the exact answer the power of $z_i$ arising from the product of the measure and the product of Schur polynomials is matched to the power of $\bar{z}_i$ from the product of the measure and the product of Schur polynomials - there is a single matching happening. Thus, the eigenvalue computation may miss some terms that are present in the exact answer[2]. For Young diagrams with a few corners and $O(N^2)$ boxes (the annulus above is a good example) the eigenvalues clump into groupings, with each grouping collecting eigenvalues of a similar size corresponding to rows with a similar row length[46]. This happens because the product of the Gaussian fall off $e^{-z\bar{z}}$ and a polynomial of fixed degree $|z^2|^n$ is sharply peaked at $|z| = n$. Thus, for example if $r_i \approx M_1$ for $i = 1, 2, \cdots, \frac{N}{2}$ and $r_i \approx M_2$ for $i = 1 + \frac{N}{2}, 2 + \frac{N}{2}, \cdots, N$ with $M_1$ and $M_2$ well separated ($M_1 - M_2 \geq O(N)$), under the integral we can replace

$$\frac{\epsilon_{b_1b_2\cdots b_N} z_{b_1}^{N-1+r_1} z_{b_2}^{N-2+r_2}\cdots z_{b_N}^{r_N}}{\epsilon_{c_1c_2\cdots c_N} z_{c_1}^{N-1} z_{c_2}^{N-2}\cdots z_{c_{N-1}}} \to \prod_{i=1}^{\frac{N}{2}} z_{a_i}^{M_1} z_{a_{i+\frac{N}{2}}}^{M_2}. \qquad (4.51)$$

After making a replacement of this type, we recover the exact answer. This replacement is not exact - we need to appeal to large $N$ to justify it. It would be very interesting to explore this point further and to quantify in general (if possible) what the corrections to the above replacement are. For Young diagrams with many corners, row lengths are not well separated and there is no similar grouping that occurs, so that the eigenvalue description will not agree with the exact result, even at large $N$. A good example of a geometry with many corners is the superstar[49]. The corresponding LLM boundary condition is a number of very thin concentric annuli, so that we effectively obtain a gray disk, signaling a singular supergravity geometry. It is then perhaps not surprising that the eigenvalue dynamics does not correctly reproduce this two point correlator.

Having discussed the two point function of Schur polynomials in detail, the product rule

$$\chi_R(Z)\chi_S(Z) = \sum_T f_{RST}\chi_T(Z) \qquad (4.52)$$

[2]This is the reason why (4.24) only captures one of the terms present in the two point function for $J < N$.

with $f_{RST}$ a Littlewood-Richardson coefficient, implies that there is no need to consider correlation functions of products of Schur polynomials.

## 4.5 Other backgrounds

In the $\frac{1}{2}$ BPS sector there is a wave function corresponding to every LLM geometry. The (not normalized) wave function has already been given in (4.6). In this section we consider the problem of writing eigenvalue wave functions that correspond to geometries other than $\mathrm{AdS}_5\times\mathrm{S}^5$. The simplest geometry we can consider is the annulus geometry considered in the previous section, where we argued that the eigenvalue dynamics reproduces the exact correlator of the Schur polynomials dual to this geometry. Our proposal for the state that corresponds to this LLM spacetime is

$$\Psi_{\mathrm{LLM}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1a_2\cdots a_N}\frac{z_{a_1}^M y_{a_1}^{N-1}}{\sqrt{M!(N-1)!}}\cdots\frac{z_{a_k}^{k-1+M}y_{a_k}^{N-k}}{\sqrt{(k-1+M)!(N-k)!}} \cdots\frac{z_{a_N}^{N-1+M}y_{a_N}^0}{\sqrt{(N-1+M)!0!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q-\frac{1}{2}\sum_q y_q\bar{y}_q}. \tag{4.53}$$

This is simply obtained by multiplying the ground state wave function by the relevant Schur polynomial and normalizing the resulting state. The connection between matrix model correlators and expectation values computed using the above wave function is the following[3]

$$\begin{aligned}\langle\,\cdots\,\rangle_{\mathrm{LLM}} &= \frac{\langle\,\cdots\,\chi_R(Z)\chi_R(Z^\dagger)\rangle}{\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle}\\ &= \int\prod_{i=1}^N dz_id\bar{z}_idy_id\bar{y}_i|\Psi_{\mathrm{LLM}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\,\cdots\end{aligned} \tag{4.54}$$

We can use this wave function to compute correlators that we are interested in. Traces involving only $Z$s for example lead to

$$\begin{aligned}\langle\mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})\rangle_{\mathrm{LLM}} &= \int\prod_{i=1}^N dz_id\bar{z}_idy_id\bar{y}_i|\Psi_{\mathrm{LLM}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{k=1}^N z_k^J\sum_{l=1}^N\bar{z}_l^J\\ &= \sum_{k=0}^{N-1}\frac{(J+k+M)!}{(k+M)!}\\ &= \frac{1}{J+1}\left[\frac{(J+M+N)!}{(M+N-1)!}-\frac{(J+M)!}{(M-1)!}\right]\end{aligned} \tag{4.55}$$

which agrees with the exact result, as long as $J > N-1$. Thus, in this background, eigenvalue dynamics is correctly reproducing the same set of correlators as in the original $\mathrm{AdS}_5\times\mathrm{S}^5$ background. Traces involving only $Y$ fields are also correctly reproduced

$$\begin{aligned}\langle\mathrm{Tr}(Y^J)\mathrm{Tr}(Y^{\dagger J})\rangle_{\mathrm{LLM}} = \int\prod_{i=1}^N dz_id\bar{z}_idy_id\bar{y}_i|\Psi_{\mathrm{LLM}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{k=1}^N y_k^J\sum_{l=1}^N\bar{y}_l^J\\ = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!}\end{aligned} \tag{4.56}$$

where $J\geq N$. Notice that these results are again exact, i.e. we reproduce the matrix model correlators to all orders in $1/N$. Finally, let's consider the most interesting case of traces involving both matrices. The LLM wave function we have proposed does not reproduce the exact matrix model computation. The matrix model computation gives

[3]The new normalization for matrix model correlators is needed to ensure that the identity operator has expectation value 1. This matches the normalization adopted in the eigenvalue description.

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle_{\rm LLM} &= \left(\frac{J!}{(J+K)!}\right)^2 \langle {\rm Tr}\left(Y\frac{\partial}{\partial Z}\right)^K {\rm Tr}(Z^{J+K}){\rm Tr}\left(Y^\dagger\frac{\partial}{\partial Z^\dagger}\right)^K {\rm Tr}(Z^{\dagger\, J+K})\rangle_{\rm LLM} \\
&= \left(\frac{J!}{(J+K)!}\right)^2 K!\langle {\rm Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right)^K {\rm Tr}(Z^{J+K}){\rm Tr}(Z^{\dagger\, J+K})\rangle_{\rm LLM} \\
&= \left(\frac{J!}{(J+K)!}\right)^2 K!\frac{(J+K)!}{J!}\langle {\rm Tr}(Z^{J+K}){\rm Tr}(Z^{\dagger\, J+K})\rangle_{\rm LLM} \\
&= \frac{J!K!}{(J+K+1)!}\left[\frac{(J+K+M+N)!}{(M+N-1)!}-\frac{(J+K+M)!}{(M-1)!}\right] \qquad (4.57)
\end{aligned}$$

if $J+K\geq N$. Next, consider the eigenvalue computation. We need to perform the integral

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle_{\rm LLM,eigen} &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^N z_k^J y_k^K \sum_{j=1}^N \bar{z}_j^J \bar{y}_j^K \\
&= \sum_{k=1}^N \frac{(N-k+K)!}{(N-k)!}\frac{(J+M+k-1)!}{(M+k-1)!}. \qquad (4.58)
\end{aligned}$$

It is not completely trivial to compare (4.57) and (4.58), but it is already clear that they do not reproduce exactly the same answer. To simplify the discussion, let's consider the case that $M = O(\sqrt{N})$. In this case, in the large $N$ limit, we can drop the second term in (4.57) to obtain

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle_{\rm LLM} = \frac{J!K!}{(J+K+1)!}\frac{(J+K+M+N)!}{(M+N-1)!}(1+\cdots) \qquad (4.59)$$

where $\cdots$ stand for terms that vanish as $N\to\infty$. In the sum appearing in (4.58), change variables from $k$ to $k'-M$ and again appeal to large $N$ to write

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle_{\rm LLM,eigen} &= \sum_{k'=M+1}^{M+N} \frac{(N+M-k'+K)!}{(N+M-k')!}\frac{(J+k'-1)!}{(k'-1)!} \\
&= \sum_{k'=1}^{M+N} \frac{(N+M-k'+K)!}{(N+M-k')!}\frac{(J+k'-1)!}{(k'-1)!}(1+\cdots) \\
&= \frac{J!K!}{(J+K+1)!}\frac{(J+K+M+N)!}{(M+N-1)!}(1+\cdots). \qquad (4.60)
\end{aligned}$$

In the last two lines above $\cdots$ again stands for terms that vanish as $N\to\infty$. Thus, we find agreement between (4.57) and (4.58). It is again convincing to see genuine multi matrix observables reproduced by the eigenvalue dynamics. Notice that in this case the agreement is not exact, but rather is realized in the large $N$ limit. This is what we expect for the generic situation - the $\mathrm{AdS}_5\times\mathrm{S}^5$ case is highly symmetric and the fact that eigenvalue dynamics reproduces so many observables exactly is a consequence of this symmetry. We only expect eigenvalue dynamics to reproduce classical gravity, which should emerge from the CFT at $N=\infty$.

Much of our intuition came from thinking about the Gauss graph operators constructed in [34, 35]. It is natural to ask if we can write down wave functions dual to the Gauss graph operators. The simplest possibility is to consider a Gauss graph operator obtained by exciting a single eigenvalue by $J$ levels, and then attaching a total of $K$ $Y$ strings to it. The extreme simplicity of this case follows because we can write the (normalized) Gauss graph operator in terms of a familiar Schur polynomial as

$$\hat{O} = \sqrt{\frac{J!}{K!(J+K)!}\frac{(N-1)!}{(N+J+K-1)!}}{\rm Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \chi_{(J+K)}(Z) \qquad (4.61)$$

where we have used the notation $(n)$ to denote a Young diagram with a single row of $n$ boxes. Consider the correlator

$$\begin{aligned}\langle \hat{O}\mathrm{Tr}(Y^\dagger)^K \mathrm{Tr}(Z^{\dagger J})\rangle &= \langle \mathrm{Tr}\left(\frac{\partial}{\partial Y}\right)^K \hat{O}\mathrm{Tr}(Z^{\dagger J})\rangle \\ &= \sqrt{\frac{J!K!}{(J+K)!}\frac{(N+J+K-1)!}{(N-1)!}}. \end{aligned} \tag{4.62}$$

This answer is exact, in the free field theory. In what limit should we compare this answer to eigenvalue dynamics? Our intuition is coming from the $\frac{1}{2}$- BPS sector where we know that rows of Schur polynomials correspond to eigenvalues and we know exactly how to write the corresponding wave function. If we only want small perturbations of this picture, we should keep $K \ll J$. In this case we should simplify

$$\begin{aligned}\frac{J!}{(J+K)!} &\to \frac{1}{J^K} \\ \frac{(N+J+K-1)!}{(N-1)!} &= \frac{(N+J+K-1)!}{(N+J-1)!}\frac{(N+J-1)!}{(N-1)!} \\ &\to (N+J-1)^K \frac{(N+J-1)!}{(N-1)!}. \end{aligned} \tag{4.63}$$

How should we scale $J$ as we take $N \to \infty$? The Schur polynomials are a sum over all possible matrix trace structures. We want these sums to be dominated by traces with a large number of matrices ($N$ or more) in each trace. To accomplish this we will scale $J = O(N^{1+\epsilon})$ with $\epsilon > 0$. In this case, at large $N$, we can replace

$$\frac{1}{J^K}(N+J-1)^K \to 1 \tag{4.64}$$

and hence, the result that should be reproduced by the eigenvalue dynamics is given by

$$\langle \hat{O}\mathrm{Tr}(Y^\dagger)^K \mathrm{Tr}(Z^{\dagger J})\rangle = \sqrt{K!\frac{(N+J-1)!}{(N-1)!}}. \tag{4.65}$$

In the eigenvalue computation, we will use the wave function of the ground state and the wave function of the Gauss graph operator ($\Psi_{\mathrm{GG}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$) to compute the amplitude

$$\int \prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i \Psi^*_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})(\sum_i \bar{y}_i)^K \sum_j \bar{z}_j^J \Psi_{\mathrm{GG}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}). \tag{4.66}$$

We expect the amplitude (4.66) to reproduce (4.65). Our proposal for the wave function corresponding to the above Gauss graph operator is

$$\begin{aligned}\Psi_{GG}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = &\frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2 \cdots a_N}\frac{z_{a_1}^0 y_{a_1}^{N-1}}{\sqrt{0!(N-1)!}}\cdots\frac{z_{a_k}^{k-1}y_{a_k}^{N-k}}{\sqrt{(k-1)!(N-k)!}}\cdots \\ &\cdots\frac{z_{a_{N-1}}^{N-2}y_{a_{N-1}}}{\sqrt{(N-2)!1!}}\frac{z_{a_N}^{J+N-1}y_{a_N}^K}{\sqrt{(J+N-1)!K!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q - \frac{1}{2}\sum_q y_q \bar{y}_q}. \end{aligned} \tag{4.67}$$

The eigenvalue with the largest power of $z$ (i.e. $z_{a_N}$) was the fermion at the very top of the Fermi sea. It has been excited by $J$ powers of $z$ and $K$ powers of $y$. It is now trivial to verify that (4.66) does indeed reproduce (4.65).

Finally, the state with three eigenvalues excited by $J_1 > J_2 > J_3$ and with $K_1 > K_2 > K_3$ strings attached to each eigenvalue is given by

$$\Psi_{GG}(\{z_i,\bar z_i,y_i,\bar y_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1a_2\cdots a_N}\frac{z^0_{a_1}y^{N-1}_{a_1}}{\sqrt{0!(N-1)!}}\cdots\frac{z^{k-1}_{a_k}y^{N-k}_{a_k}}{\sqrt{(k-1)!(N-k)!}}\cdots$$
$$\cdots\frac{z^{N-4}_{a_{N-3}}y^3_{a_{N-3}}}{\sqrt{(N-4)!3!}}\frac{z^{J_3+N-3}_{a_{N-2}}y^{2+K_3}_{a_{N-2}}}{\sqrt{(J_3+N-3)!(2+K_3)!}}\frac{z^{J_2+N-2}_{a_{N-1}}y^{K_2+1}_{a_{N-1}}}{\sqrt{(J_2+N-2)!(K_2+1)!}}$$
$$\times\frac{z^{J_1+N-1}_{a_N}y^{K_1}_{a_N}}{\sqrt{(J_1+N-1)!K_1!}}e^{-\frac12\sum_q z_q\bar z_q-\frac12\sum_q y_q\bar y_q}. \tag{4.68}$$

The generalization to any Gauss graph operator is now clear.

## 4.6 Connection to Supergravity

In this section we would like to explore the possibility that the eigenvalue dynamics of the $SU(2)$ sector has a natural interpretation in supergravity. The relevant supergravity solutions have been considered in [50, 51, 52, 53].

There are 6 adjoint scalars in the $\mathcal{N}=4$ super Yang-Mills theory that can be assembled into the following three complex combinations

$$Z=\phi^1+i\phi^2,\qquad Y=\phi^3+i\phi^4,\qquad X=\phi^5+i\phi^6. \tag{4.69}$$

The operators we consider are constructed using only $Z$ and $Y$ so that they are invariant under the $U(1)$ which rotates $\phi^5$ and $\phi^6$. Further, since our operators are BPS they are built only from the $s$-wave spherical harmonic components of $Y$ and $Z$, so that they are invariant under the $SO(4)$ symmetry which acts on the $S^3$ of the $R\times S^3$ spacetime on which the CFT is defined. Local supersymmetric geometries with $SO(4)\times U(1)$ isometries have the form[50, 53]

$$ds^2_{10}=-h^{-2}(dt+\omega)^2+h^2\Big[\frac{2}{Z+\frac12}\partial_a\bar\partial_bK dz^ad\bar z^b+dy^2\Big]+y(e^Gd\Omega^2_3+e^{-G}d\psi^2) \tag{4.70}$$

$$d\omega=\frac{i}{y}\Big(\partial_a\bar\partial_b\partial_yK dz^ad\bar z^b-\partial_aZdz^ady+\bar\partial_aZd\bar z^ady\Big). \tag{4.71}$$

Here $z^1$ and $z^2$ is a pair of complex coordinates and $K$ is a Kahler potential which may depend on $y$, $z^a$ and $\bar z^a$. $y^2$ is the product of warp factors for $S^3$ and $S^1$. Thus we must be careful and impose the correct boundary conditions at the $y=0$ hypersurface if we are to avoid singularities. The $y=0$ hypersurface includes the four dimensional space with coordinates given by the $z^a$. These boundary conditions require that when the $S^3$ contracts to zero, we need $Z=-\frac12$ and when the $\psi$-circle collapses we need $Z=\frac12$[50, 53]. There is a surface separating these two regions, and hence, defining the supergravity solution. So far the discussion given closely matches what is found for the $\frac12$-BPS supergravity solutions. In that case the $y=0$ hypersurface includes a two dimensional space which is similarly divided into two regions, giving the black droplets on a white plane. The edges of the droplets are completely arbitrary, which is an important difference from the case we are considering. The surface defining local supersymmetric geometries with $SO(4)\times U(1)$ isometries is not completely arbitrary - it too has to satisfy some additional constraints as spelled out in [53]. It is natural to ask if the surface defining the supergravity solution is visible in the eigenvalue dynamics?

To answer this question we will now review how the surface defining the local supersymmetric geometries with $SO(4)\times U(1)$ isometries corresponding to the $\frac14$-BPS LLM geometries is constructed. According to [53], the boundary condition for these geometries have walls between the two boundary conditions determined by the equation[4]

$$z^2\bar z^2=e^{-2\hat D(z^1,\bar z^1)} \tag{4.72}$$

[4] This next equation is (6.35) of [53]. We will relate $z^1$ and $z^2$ to $z_i$ (the eigenvalues of $Z$) and $y_i$ (the eigenvalues of $Y$) when we make the correspondence to eigenvalues.

where $\hat{D}(z^1,\bar{z}^1)$ is determined by expanding the function $D$ as follows (it is the $y$ coordinate that we set to zero to get the LLM plane)

$$D = \log(y) + \hat{D}(z,\bar{z}) + O(y). \tag{4.73}$$

The function $D$ is determined by the equations

$$y\partial_y D = \frac{1}{2} - Z, \qquad V = -i(dz\partial_z - d\bar{z}\partial_{\bar{z}})D \tag{4.74}$$

where $Z(y,z^1,\bar{z}^1)$ is the function obeying Laplace's equation that determines the LLM solution and $V(y,z^1,\bar{z}^1)$ is the one form appearing in the combination $(dt+V)^2$ in the LLM metric.

Consider an annulus that has an outer edge at radius $M+N$ and an inner edge at a radius $M$. This solution has (these solutions were constructed in the original LLM paper [10])

$$\begin{aligned}Z(y,z^1,\bar{z}^1) &= -\frac{1}{2}\left(\frac{|z^1|^2+y^2-M}{\sqrt{(|z^1|^2+y^2+M)^2-4|z^1|^2M}}\right.\\ &\left.+\frac{|z^1|^2+y^2-M-N}{\sqrt{(|z^1|^2+y^2+M+N)^2-4|z^1|^2(M+N)}}\right),\\ V(y,z^1,\bar{z}^1) &= \frac{d\phi}{2}\left(\frac{|z^1|^2+y^2+M}{\sqrt{(|z^1|^2+y^2+M)^2-4|z^1|^2M-}}\right.\\ &\left.+\frac{|z^1|^2+y^2+M+N}{\sqrt{(|z^1|^2+y^2+M+N)^2-4|z^1|^2(M+N)}}\right).\end{aligned}$$

Evaluating at $y=0$, the second of (4.74) says

$$V = -i(dz\partial_z - d\bar{z}\partial_{\bar{z}})\hat{D}. \tag{4.75}$$

Setting $z^1 = re^{-i\phi}$ and assuming that $\hat{D}$ depends only on $r$ we find

$$r\frac{\partial\hat{D}}{\partial r} = -\frac{M+N}{r^2-M-N} + \frac{M}{r^2-M} \tag{4.76}$$

which is solved by

$$\hat{D} = \frac{1}{2}\log\frac{|z^1\bar{z}^1-M|}{|z^1\bar{z}^1-M-N|}. \tag{4.77}$$

Thus, the wall between the two boundary conditions is given by

$$|z^2|^2 = \frac{M+N-z^1\bar{z}^1}{z^1\bar{z}^1-M}. \tag{4.78}$$

The same analysis applied to the $\mathrm{AdS}_5\times\mathrm{S}^5$ solution gives

$$|z^1|^2 + |z^2|^2 = N. \tag{4.79}$$

For the pair of geometries described above, we know the wave function in the eigenvalue description. We will now return to the eigenvalue description and see how these surfaces are related to the eigenvalue wave functions.

At large $N$, since fluctuations are controlled by $1/N^2$, we expect a definite eigenvalue distribution. These eigenvalues will trace out a surface specified by the support of the single fermion probability density

$$\rho(z_1,\bar{z}_1,y_1,\bar{y}_1) = \int\prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2. \tag{4.80}$$

Denote the points lying on this surface using coordinates $z, y$.

Using the wave function $\Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ corresponding to the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime, the probability density for a single eigenvalue is

$$\rho(z,\bar{z},y,\bar{y}) = \frac{1}{N\pi^2}\sum_{i=0}^{N-1}\frac{(z\bar{z})^i}{i!}\frac{(y\bar{y})^{N-i-1}}{(N-i-1)!}e^{-z\bar{z}-y\bar{y}}$$
$$= \frac{(z\bar{z}+y\bar{y})^{N-1}}{N\pi^2(N-1)!}e^{-z\bar{z}-y\bar{y}}. \tag{4.81}$$

Which is maximised at

$$z\bar{z}+y\bar{y} = N-1. \tag{4.82}$$

Thus, if we identify the points $z, y$ with the supergravity coordinates $z^1, z^2$ as follows

$$z^2 = y, \qquad z^1 = z \tag{4.83}$$

we find

$$|z^1|^2 + |z^2|^2 = N \tag{4.84}$$

at large $N$, so that the eigenvalues condense on the surface that defines the wall between the two boundary conditions.

Let's now compute the positions of our eigenvalues, using $\Psi_{\rm LLM}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$. The probability density for a single eigenvalue is easily obtained by computing the following integral

$$\rho(z_1,\bar{z}_1,y_1,\bar{y}_1) = \int\prod_{i=2}^{N}dz_i d\bar{z}_i dy_i d\bar{y}_i|\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2$$
$$= \frac{1}{N\pi^2}\sum_{i=0}^{N-1}\frac{(z_1\bar{z}_1)^{M+i}}{(M+i)!}\frac{(y_1\bar{y}_1)^{N-i-1}}{(N-i-1)!}e^{-z_1\bar{z}_1-y_1\bar{y}_1}. \tag{4.85}$$

Following the analysis we performed above, we find that the probability density is maximised when the following relations are satisfied

$$\sum_{i=0}^{N-1}\left[\frac{(y\bar{y})^{N-i-1}}{(N-i-1)!}\left(\frac{(z\bar{z})^{M+i-1}}{(M+i-1)!}-\frac{(z\bar{z})^{M+i}}{(M+i)!}\right)\right] = 0 \tag{4.86}$$

$$\sum_{i=0}^{N-1}\left[\frac{(z\bar{z})^{M+i}}{(M+i)!}\left(\frac{(y\bar{y})^{N-i-2}}{(N-i-2)!}-\frac{(y\bar{y})^{N-i-1}}{(N-i-1)!}\right)\right] = 0. \tag{4.87}$$

The above holds only if each term in each sum is zero. We find

$$z\bar{z} = M+i, \qquad y\bar{y} = N-i-1, \qquad i = 1,2,...,N-1. \tag{4.88}$$

Thus, if we identify the points $z, y$ and the supergravity coordinate $z^1, z^2$ as follows

$$z^2 = \frac{y}{\sqrt{|z|^2-M}}, \qquad z^1 = z \tag{4.89}$$

we find that (4.78) gives

$$\frac{|y|^2}{i} = \frac{M+N-|z|^2}{|z|^2-M} \tag{4.90}$$

in complete agreement with where our wave function is localized. This again shows that the eigenvalues are collecting on the surface that defines the wall between the two boundary conditions. Although these examples are rather simple, they teach us something important: the map between the eigenvalues and the supergravity coordinates depends on the specific geometry we consider.

The fact that eigenvalues condense on the surface that defines the wall between the two boundary conditions is something that was already anticipated by Berenstein and Cotta in [38]. The proposal of [38] identifies the support of the eigenvalue distribution with the degeneration locus of the three sphere in the full ten dimensional metric. Our results appear to be in perfect accord with this proposal.

## 4.7 Outlook

There are a number of definite conclusions resulting from our study. One of our key results is that we have found substantial evidence for the proposal that there is a sector of the two matrix model that is described (sometimes exactly) by eigenvalue dynamics. This is rather non-trivial since, as we have already noted, it is simply not true that the two matrices can be simultaneously diagonalized. The fact that we have reproduced correlators of operators that involve products of both matrices in a single trace is convincing evidence that we are reproducing genuine two matrix observables. The observables we can reproduce correspond to BPS operators. In the dual gravity these operators map to supergravity states corresponding to classical geometries. The local supersymmetric geometries with $SO(4) \times U(1)$ isometries are determined by a surface that defines the boundary conditions needed to obtain a non-singular supergravity solution. At large $N$ where we expect classical geometry, the eigenvalues condense on this surface. In this way the supergravity boundary conditions appear to match the large $N$ eigenvalue description perfectly.

The eigenvalue dynamics appears to provide some sort of a coarse grained description. Correlators of operators dual to states with a very small energy are not reproduced correctly: for example the energy of states dual to single traces has to be above some threshold ($N$) before they are correctly reproduced. For complicated operators with a detailed multi trace structure we would thus expect to get the gross features correct, but we may miss certain finer details - see the discussion after (4.50). Developing this point of view, perhaps using the ideas outlined in [43], may provide a deeper understanding of the eigenvalue wave functions.

The eigenvalue description we have developed here is explicit enough that we could formulate the dynamics in terms of the density of eigenvalues. This would provide a field theory that has $1/N$ appearing explicitly as a coupling. It would be very interesting to work out, for example, what the generalization of the Das-Jevicki Hamiltonian[54] is.

The picture of eigenvalue dynamics that we are finding here is almost identical to the proposal discussed by Berenstein and his collaborators[36, 37, 38, 39, 40, 41, 42], developed using numerical methods and clever heuristic arguments. The idea of these works is that the eigenvalues represent microscopic degrees of freedom. At large $N$ one can move to collective degrees of freedom that represent the 10 dimensional geometry of the dual gravitational description. This is indeed what we are seeing. They have also considered cases with reduced supersymmetry and orbifold geometries[55, 56, 57]. These are natural examples to consider using the ideas and methods we have developed in this chapter. Developing other examples of eigenvalue dynamics will allow us to further test the proposals for wave functions and the large $N$ distributions of eigenvalues that we have put forward in this chapter.

An important question that should be tackled is to ask how one could derive (and not guess) the wave functions we have described. Progress with this question is likely to give some insights into how it is even possible to have a consistent eigenvalue dynamics. One would like to know when an eigenvalue description is relevant and to what classes of observables it is applicable.

Another important question is to consider the extension to more matrices, including gauge and fermion degrees of freedom. The Gauss graph labelling of operators continues to work when we include gauge fields and fermions[58, 59], so that our argument goes through without modification and we again expect that eigenvalue dynamics in these more general settings will be an effective approach to compute these more general correlators of BPS operators. Another important extension is to consider the eigenvalue dynamics, perturbed by off diagonal elements, which should allow one to start including stringy degrees of freedom. Can this be done in a controlled systematic fashion? In this context, the studies carried out in [60, 61, 62], will be relevant.

# Chapter 5

# From Giants to Gauss graphs

## 5.1 Chapter introduction

The AdS/CFT correspondence[1] provides a beautiful realization of 't Hooft's proposal that the large $N$ limit of Yang-Mills theories are equivalent to string theory[4]. Most studies of the correspondence have focused on the planar limit, which holds classical operator dimensions fixed as we take $N \to \infty$. There are non-planar large $N$ limits of the theory [11], which are defined by considering operators with a bare dimension that is allowed to scale with $N$ as we take $N \to \infty$. These limits are relevant for the AdS/CFT correspondence. The limit on which we will focus in this study considers operators with a dimension that scales as $N$. Our focus is on operators relevant for the description of giant graviton branes[12, 13, 14].

The worldvolume of the most general $\frac{1}{8}$-BPS giant graviton can be described as the intersection of a holomorphic complex surface in $\mathbb{C}^3$ with the five sphere $S^5$ of the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime[63]. It is possible to quantize these giant graviton configurations and then to count them[64]. Remarkably, this quantization leads to the Hilbert Space of $N$ noninteracting Bose particles in a 3d harmonic oscillator potential, a result conjectured in [65]. In [66] $\frac{1}{8}$-BPS states which carry three independent angular momenta on $S^5$ were counted. This counting problem can again be mapped to counting energy eigenstates of a system of $N$ bosons in a 3-dimensional harmonic oscillator. Both of these analysis [64, 66] make use of a world volume description of the branes. Finally, an index to count single trace BPS operators operators has been constructed [67, 68]. The index has been computed both at weak coupling (using the gauge theory) and at strong coupling (as a sum over the spectrum of free massless particles in $\mathrm{AdS}_5\times\mathrm{S}^5$) and the results again agree with [64, 66].

Given the AdS/CFT correspondence, this counting should also arise in the dual $\mathcal{N}=4$ super Yang-Mills theory, when the operators of a bare dimension of order $N$ and vanishing anomalous dimension are considered. One of our goals in this study is to demonstrate this.

A crucial ingredient in the study of operators with a bare dimension of order $N$, has been the construction of bases of operators developed in [8, 30, 69, 70, 71, 31, 32, 72]. These bases diagonalize the free field theory two point function to all order in $1/N$ and mix weakly when the Yang-Mills coupling is switched on. Using these bases as a starting point, the spectrum of anomalous dimensions for a class operators of bare dimension of order $N$ has been computed in [73, 34, 35]. The operators are constructed using the three complex adjoint scalars $Z$, $Y$ and $X$. We use $n$ $Z$s, $m$ $Y$s and $p$ $X$s, fixing $n\sim N$ and $m,p\ll n$. This implies that we are focusing on small deformations of $\frac{1}{2}$-BPS giant gravitons. The operators of a definite scaling dimension are labeled by a permutation $\sigma\in S_m\times S_p$ and a triple of Young diagrams $R\vdash n+m+p$ and $r\vdash n$. The explicit form of these operators is

$$O^{\vec{m},\vec{p}}_{R,r}(\sigma) = \frac{|H_X\times H_Y|}{\sqrt{p!m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}_1,\vec{\mu}_2}\sqrt{d_s d_t}\Gamma^{(s,t)}_{jk}(\sigma) \times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}_1} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\mu}_2} O_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}. \tag{5.1}$$

The Young diagrams $R$ and $r$ both have $q$ rows for operators dual to a state of $q$ giant gravitons. Each box in $R$ is associated with one of the complex fields, so that we can talk of a box as being a

$Z$ box, a $Y$ box or an $X$ box. $r$ collects all of the $Z$ boxes. The difference in the row length of the $q$th row in $R$ and $q$th row in $r$ is equal to the number of $X$s ($= p_q$) and $Y$s ($= m_q$) in row $q$, so that $R_q - r_q = m_q + p_q$. The right most boxes are $X$ boxes, the left most boxes $Z$ boxes and the $Y$ boxes are sandwiched in the middle. The $q$ dimensional vector $\vec{m}$ collects the $m_i$, while $\vec{p}$ collects the $p_i$. The branching coefficients $B_{j\vec{\mu}}^{(s,t)\to 1_{H_Y\times H_X}}$ resolve the operator that projects from $(s,t)$, with $s\vdash m$, $t\vdash p$, an irreducible representation of $S_m\times S_p$, to the trivial (identity) representation of the product group $H_Y\times H_X$ with $H_Y = S_{m_1}\times S_{m_2}\times\cdots S_{m_q}$ and $H_X = S_{p_1}\times S_{p_2}\times\cdots S_{p_q}$, i.e.

$$\frac{1}{|H_X\times H_Y|}\sum_{\gamma\in H_X\times H_Y}\Gamma_{ik}^{(s,t)}(\gamma) = \sum_{\vec{\mu}} B_{i\vec{\mu}}^{(s,t)\to 1_{H_X\times H_Y}} B_{k\vec{\mu}}^{(s,t)\to 1_{H_X\times H_Y}}. \tag{5.2}$$

The operators $O_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}$ are normalized versions of the restricted Schur polynomials [31]

$$\chi_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}(Z,X,Y) = \frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\chi_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}Y^{\otimes m}X^{\otimes p}), \tag{5.3}$$

which themselves provide a basis for the gauge invariant operators of the theory. The restricted characters $\chi_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}(\sigma)$ are defined by tracing the matrix representing group element $\sigma$ in representation $R$ over the subspace giving an irreducible representation $(r,s,t)$ of the subgroup $S_n\times S_m\times S_p$. There is more than one choice for this subspace and the multiplicity labels $\vec{\mu}_1\vec{\mu}_2$ resolve this ambiguity. The operators $O_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}$ given by

$$O_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2} = \sqrt{\frac{\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{\mathrm{hooks}_R f_R}}\,\chi_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2} \tag{5.4}$$

have unit two point function. Although the definition of the Gauss graph operators $O_{R,r}(\sigma)$ is technically rather involved, they have a very natural and simple interpretation in terms of the dual giant graviton branes plus open string excitations. A Gauss graph operator that is labeled by a Young diagram $R$ that has $q$ rows corresponds to a system of $q$ giant gravitons. The $Y$ and $X$ fields describe the open string excitations of the giants. Each such field corresponds to a directed edge, an open string, which can end on any two (not necessarily distinct) of the $q$ branes. The permutation $\sigma\in S_m\times S_p$ is a label which tells us precisely how the $m$ $Y$'s and the $p$ $X$'s are draped between the $q$ giant gravitons. The picture of directed edges stretched between $q$ dots is highly suggestive of a brane plus open string system, as reflected in our language. This interpretation is further supported by that fact that the only configurations that appear have the same number of strings starting or terminating on any given giant. This nicely implements the Gauss Law of the brane world volume theory implied by the fact that the giant graviton has a compact world volume. The Gauss graph operators which correspond to BPS states have all open strings described by loops that start at a given giant and loop back to the same giant, i.e. no open strings stretch between giants. In this case, we simply need to specify which brane the open string belongs to and this is most conveniently done by partially labeling Young diagram $R$: in each box we place a $z$, an $x$ or a $y$. Each row in the operator consists mainly of $Z$ fields, corresponding to the fact that the unexcited giant graviton is dual to a half-BPS operator built only from $Z$s. The number of $x$ and $y$ boxes in a given row tell us how many $X$ and $Y$ strings attach to the corresponding giant.

In the next section we will show the counting of these BPS states agrees with the counting of [64, 66]. Motivated by this observation, we explore the link between the $N$ particle description employing the 3d harmonic oscillator and the super Yang-Mills operators in section 5.3. Our results shed light on the attractive possibility of an $N$ particle description of multi matrix models, suggesting that there maybe an extension of the famous free fermion/eigenvalue description of single matrix models [15]. Finally, we refer the reader to [74] and [75] for further related background.

## 5.2 Counting

As discussed in the introduction, our description of $\frac{1}{8}$-BPS operators is in terms of a Young diagram $R$ with partially labeled boxes. When the boxes corresponding to $Y$ and $X$ fields are removed from the rows of $R$, we are left with the valid Young diagram $r$. An example of a valid $\frac{1}{8}$-BPS operator is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline z & z & z & z & z & z & z & z & y & y & x \\ \hline z & z & z & z & z & z & z & y & x \\ \cline{1-9} z & z & z & y & y \\ \cline{1-5}\end{array} \tag{5.5}$$

The boxes with label $z$ belong to the Young diagram $r$ and the boxes with label $y$ or $x$ are the ones that are removed from the Young diagram $R$ to obtain $r$. The operator labeled by the Young diagram shown in (5.5) corresponds to a system of 3 giant gravitons, with 2 $Y$ strings and an $X$ string attached to the first giant, a $Y$ and an $X$ string attached to the second giant and 2 $Y$ strings attached to the third giant. This description in terms of Gauss graph operators is valid in the case where $n$ the total number of boxes of the Young diagram $r$ and $m+p$ the total number of the boxes that are added to the Young diagram $r$ to form $R$, are both large and of order $N \gg 1$. In addition, $m+p \ll n$ and the number of rows of the Young diagram $R$ is of order $1 = N^0$. Finally, the length of any row of $R$ is of order $N$, as is the difference between the length of any two consecutive rows.

Let us first start by fixing our notation. We will denote by $R_i$ the number of boxes in the $i$th row of $R$, and we will denote by $m_i$ and $p_i$ the number of $Y$ and $X$ boxes to be removed from the $i$th row of $R$ to obtain $r$. Furthermore, $q$ will stand for the number of rows of $R$, $n$ will stand for the total number of boxes of $r$, $m = \sum_{i=1}^{q} m_i$ and $p = \sum_{i=1}^{q} p_i$. Hence, the total number of boxes of $R$ is then $n+m+p$. If we denote by $r_i$ the number of boxes in the $i$th row of $r$, then we have

$$r_i = R_i - m_i - p_i.$$

In our conventions, we start the numbering of rows from top to bottom. As already mentioned above, this description of $\frac{1}{8}$-BPS states is proved to work[34] in the cases that

$$R_i \sim N, \qquad R_{i+1} - R_i \gg m + p \sim N, \qquad q \sim N^0. \tag{5.6}$$

We call this the displaced corners approximation because the neighboring corners of $R$ are separated by a huge number of columns. Outside this regime, things are more complicated and it is not even known if partially labeled Young diagrams can be used to describe these $\frac{1}{8}$-BPS states. The number of $\frac{1}{8}$-BPS operators is the same as the number of possible pairs $(R; r)$ counted with multiplicity equal to the number of ways of assigning a valid vector $\vec{m} = (m_1, m_2, \ldots, m_q)$. Note that once the pair $(R; r)$ and the vector $\vec{m}$ are given, the vector $\vec{p} = (p_1, p_2, \ldots, p_q)$ is determined. The first step towards counting the number of Gauss graph operators entails writing a generating function for the number of pairs $(R; r)$. Our starting point is the observation that the Young diagrams are in one to one correspondence with partitions of integers. The generating function of the latter is given by

$$Z = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{k=0}^{\infty} D_k q^k \tag{5.7}$$

where $D_k$ is the number of possible ways to partition an integer $k$. This counting is too coarse for us to reach our goals: we need to track the number of parts in the partition which corresponds to the number of rows in the Young diagram. Indeed, we must encode the information about $q$ the number of rows of $R$, as well as the information about the different possible $m_i$'s and $p_i$'s in such a partition, to ensure that we are counting states in the regime in which the Gauss graph operators provide a trustworthy description. Both modifications are easy to take into account in our case of interest where $m_i + p_i + r_i \ll m_{i+1} + p_{i+1} + r_{i+1}$ for all values of $i = 1, 2, \ldots, q$. The number of ways to partition an integer $k$ is given by the number of solutions to the equation

$$k = \sum_i \chi_i n_i, \qquad n_1 \geq n_2 \geq \cdots > 0, \qquad \chi_i \geq 0. \tag{5.8}$$

Notice that the term $\chi_i n_i$ in the above equation is associated to the term $(q^{n_i})^{\chi_i}$ in the expansion of $Z$. This term appears in the expansion for the term $(1 - q^{n_i})^{-1}$. Clearly then, to keep track of contributions from different rows $\chi_i$ we just need to multiply $q^n$ by an extra parameter $\chi$ and track the power of $\chi$. So, we consider the following modification of the partition function $Z$

$$Z = \prod_{i=1}^{\infty} \frac{1}{1-\chi q^n} = \sum_{k,d=0}^{\infty} D_{k;d} \chi^d q^k \tag{5.9}$$

where $D_{k;d}$ counts the number of Young diagrams with $k$ boxes and $d$ rows. Next consider the information associated to the $m_i$'s and $p_i$'s. There is a potential complication because we want both $R$ and $r$ to be Young diagrams. However, in the displaced corners limit, we can ensure that this is not an issue. Indeed, by taking $m, p \ll |r_{i+1} - r_i|$ for all $i$, we ensure that we can never pile enough $Y$ and $X$ boxes onto a row to make it longer than the row above it. Thus, we may treat the $m_i$'s and $p_i$'s as independent, except for the requirement that $\sum_{i=1}^{q} m_i = m$ and $\sum_{i=1}^{q} p_i = p$. In terms of the partition function $Z$, this is equivalent to associating to each term $q^{\chi_i n_i}$, a term $p^{b_i m_i} r^{c_i p_i}$ , where $b_i \neq \chi_i$ and $c_i \neq \chi_i$ in general. The latter condition is equivalent to associating the term $p^l r^m$, with $l, m = 0, 1, \ldots$ for each term $q^n$ in the product form of $Z$ in equation (5.9). Thus, we finally obtain the generating function

$$Z = \prod_{l=0}^{\infty} \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \frac{1}{1-\chi p^l r^m q^n} = \sum_{d,m,p,n} D_{m,p,n;d} \chi^d p^m r^p q^n \tag{5.10}$$

where $D_{m,p,n;d}$ counts the number of diagrams $R$ with $(n+m+p)$ boxes and $d$ rows, that is the result of adding $m+p$ boxes that are randomly distributed over the $d$ rows of the Young diagram $r$ with $n$ boxes. Our construction of the Gauss graph operators only holds when the displaced corners approximation holds. Thus, we trust $D_{m,p,n;d}$ to count the number of Gauss graph operators for a system of $d \sim N^0$ giant gravitons when $n, m, p \sim N$ and $n \gg m + p$. This is the main result of this section.

We want to compare this to the counting of $\frac{1}{8}$-BPS giant gravitons. As we discussed in the introduction, this counting problem can be mapped to counting energy eigenstates of a system of $N$ bosons in a 3-dimensional harmonic oscillator. The grand canonical partition function for bosons in a 3-dimensional simple harmonic oscillator is given by

$$Z(\zeta, q_1, q_2, q_3) = \prod_{n_1=0}^{\infty} \prod_{n_2=0}^{\infty} \prod_{n_3=0}^{\infty} \frac{1}{1-\zeta q_1^{n_1} q_2^{n_2} q_3^{n_3}} \tag{5.11}$$

with the fugacity $\zeta$ being dual to particle number[66]. Notice that (5.10) exactly matches the grand canonical partition function (5.11) for bosons in a harmonic oscillator potential with $\chi$ playing the role of the fugacity. This is in harmony with the fact that the number of rows matches the number of giant gravitons. In the 3-dimensional harmonic oscillator we have 3 types of excitations, counted by $q_1$, $q_2$ and $q_3$. These map into the three types of boxes ($X$, $Y$ or $Z$ boxes) appearing in $R$, counted by $p$, $q$ and $r$. Thus, long rows in $R$ map to highly excited particles. This proves our first claim: the counting of the Gauss graph operators matches the counting of $\frac{1}{8}$ BPS giant gravitons.

## 5.3 Matching States to Operators

The fact that the number of Gauss graph operators matches the number of energy eigenstates states of a system of bosons in a 3-dimensional harmonic oscillator potential, motivates us to look for a correspondence between the two. To start we will consider operators $O_{R,r}^{\vec{m},\vec{p}}(\sigma)$ labeled by Young diagrams that have a single row. In this case we don't need to encode a complicated shape for $R$, so we will simply list the number of $Z$s, $Y$s and $X$s in the operator as $O_{n,m,p}$. Since this row has $O(N)$ boxes, we have a system of $N$ bosons and one of them is highly excited. The idea is that since we have one highly excited particle, we can use a single particle description and overlaps of the single particle wave functions will match correlation functions of Gauss graph operators in the CFT. We focus on $R$'s with a single row because the computations are so simple to carry out in this case that we can compute many quantities exactly. There is a simple formula for the Gauss graph operators we consider, in terms of the Schur polynomials

$$O_{n,m,p}(Z,Y,X)=\mathcal{N}\mathrm{Tr}\left(Y\frac{d}{dZ}\right)^m\mathrm{Tr}\left(X\frac{d}{dZ}\right)^p\chi_{(n+m+p)}(Z) \tag{5.12}$$

where

$$\mathcal{N}=\sqrt{\frac{n!(N-1)!}{m!p!(n+m+p)!(N+n+m+p-1)!}}. \tag{5.13}$$

We are using the notation $(k)$ to denote a Young diagram that has a single row of $k$ boxes. There are a number of natural operators that act on the Gauss graphs. For example, we have

$$\mathrm{Tr}(Y\frac{d}{dZ})^{k_1}\mathrm{Tr}(X\frac{d}{dZ})^{k_2}O_{n,m,p}(Z,Y,X)\ \propto\ O_{n-k_1-k_2,m+k_1,p+k_2}(Z,Y,X). \tag{5.14}$$

Thus, a natural correlator to consider is given by

$$\langle O^\dagger_{n-k_1-k_2,m+k_1,p+k_2}\mathrm{Tr}\left(Y\frac{d}{dZ}\right)^{k_1}\mathrm{Tr}\left(X\frac{d}{dZ}\right)^{k_2}O_{n,m,p}\rangle=\sqrt{\frac{(m+k_1)!(p+k_2)!n!}{m!p!(n-k_1-k_2)!}}. \tag{5.15}$$

To describe a single particle in a 3d harmonic oscillator, we need three sets of creation and annihilation operators

$$\left[a_z,a_z^\dagger\right]=\left[a_y,a_y^\dagger\right]=\left[a_x,a_x^\dagger\right]=1. \tag{5.16}$$

Using the above oscillators we can create a state with an arbitrary number of $z$ quanta, $y$ quanta or $x$ quanta. We suggest that the correspondence between Gauss graph operators and particle states is as follows

$$O_{n,m,p}\qquad\leftrightarrow\qquad|O_{n,m,p}\rangle=\frac{1}{\sqrt{n!m!p!}}(a_x^\dagger)^p(a_y^\dagger)^m(a_z^\dagger)^n|0\rangle. \tag{5.17}$$

The correspondence identifies the number of $z$, $y$ or $x$ quanta in the particle state with the number of $Z$s, $Y$s or $X$s in the Gauss graph operator. There is a natural extension to include operators, suggested by this identification. For example

$$\mathrm{Tr}(Y\frac{d}{dZ})^{k_1}\mathrm{Tr}(X\frac{d}{dZ})^{k_2}\qquad\leftrightarrow\qquad(a_y^\dagger)^{k_1}(a_x^\dagger)^{k_2}(a_z)^{k_1+k_2}. \tag{5.18}$$

As a test of the proposed correspondence, note that

$$\begin{aligned}&\langle O_{n-k_1-k_2,m+k_1,p+k_2}|(a_y^\dagger)^{k_1}(a_x^\dagger)^{k_2}(a_z)^{k_1+k_2}|O_{n,m,p}\rangle\\ =\ &\frac{\langle 0|(a_z)^n(a_y)^{m+k_1}(a_x)^{p+k_2}(a_z^\dagger)^n(a_y^\dagger)^{m+k_1}(a_x^\dagger)^{p+k_2}|0\rangle}{\sqrt{n!m!p!(n-k_1-k_2)!(m+k_1)!(p+k_2)!}}\\ =\ &\sqrt{\frac{n!(m+k_1)!(p+k_2)!}{(n-k_1-k_2)!m!p!}}\end{aligned} \tag{5.19}$$

which is in complete agreement with (5.15). Very similar computations comparing, for example

$$\langle O^\dagger_{n-k_1,m-k_2,p-k_3}\left(\mathrm{Tr}\frac{d}{dZ}\right)^{k_1}\left(\mathrm{Tr}\frac{d}{dY}\right)^{k_2}\left(\mathrm{Tr}\frac{d}{dX}\right)^{k_3}O_{n,m,p}\rangle \tag{5.20}$$

and

$$\langle O_{n-k_1,m-k_2,p-k_3}|(a_z)^{k_1}(a_y)^{k_2}(a_x)^{k_3}|O_{n,m,p}\rangle \tag{5.21}$$

show that we should identify

$$\begin{aligned}
a_x &\leftrightarrow \sqrt{\frac{m+n+p}{N+m+n+p}}\,\mathrm{Tr}\left(\frac{d}{dX}\right),\\
a_y &\leftrightarrow \sqrt{\frac{m+n+p}{N+m+n+p}}\,\mathrm{Tr}\left(\frac{d}{dY}\right),\\
a_z &\leftrightarrow \sqrt{\frac{m+n+p}{N+m+n+p}}\,\mathrm{Tr}\left(\frac{d}{dZ}\right).
\end{aligned} \tag{5.22}$$

These computations make use of the reduction rule of [76, 60].

We now want to argue that the identifications we have developed above have a natural extension which identifies Gauss graph operators with $q$ rows with a $q$ particle system. Towards this end, we first point out a dramatic simplification in the formula for the Gauss graph operators, arising when we specialize to BPS operators. As discussed in the introduction, in this case we set the permutation $\sigma$ appearing in (5.1) to the identity. Using the orthogonality of the branching coefficients we then find

$$\begin{aligned}
\sum_{j,k}\Gamma^{(s,t)}_{jk}(\mathbf{1})B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec\mu_1}B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec\mu_2} &= \sum_{j,k}\delta_{jk}B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec\mu_1}B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec\mu_2}\\
&= \delta_{\vec\mu_1\vec\mu_2}.
\end{aligned} \tag{5.23}$$

This leads to the following formula (the operators below are normalized to have a unit two point function; they differ from the operators in (5.1) that are not normalized, by a factor of $\sqrt{|H_X\times H_Y|}$)

$$O^{\vec m,\vec p}_{R,r}(X,Y,Z) = \frac{1}{n!m!p!}\sqrt{\frac{|H_X\times H_Y|\mathrm{hooks}_r}{\mathrm{hooks}_R f_R}}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}\left(P_{R,r}\Gamma_R(\sigma)\right)\mathrm{Tr}(\sigma X^{\otimes p}Y^{\otimes m}Z^{\otimes n}). \tag{5.24}$$

$P_{R,r}$ is a projector on the carrier space of $R$. It projects to the subspace of Young-Yammonouchi states that have $1,2,...,m+p$ distributed in the boxes that belong to $R$ but not $r$ and $m+p+1,...,m+p+n$ distributed in the boxes that belong to $R$ and $r$. Using this formula, it is straight forward to prove that

$$\mathrm{Tr}\left(\frac{d}{dX}\right)O^{\vec m,\vec p}_{R,r}(X,Y,Z) = \sum_{i=1}^{q}\sqrt{\frac{p_i\ c_{RR^{(1)}_i}}{n_i+m_i+p_i}}O^{\vec m,\vec p^{(1)}_i}_{R^{(1)}_i,r}(X,Y,Z), \tag{5.25}$$

$$\mathrm{Tr}\left(\frac{d}{dY}\right)O^{\vec m,\vec p}_{R,r}(X,Y,Z) = \sum_{i=1}^{q}\sqrt{\frac{m_i\ c_{RR^{(1)}_i}}{n_i+m_i+p_i}}O^{\vec m^{(1)}_i,\vec p}_{R^{(1)}_i,r}(X,Y,Z), \tag{5.26}$$

$$\mathrm{Tr}\left(\frac{d}{dZ}\right)O^{\vec m,\vec p}_{R,r}(X,Y,Z) = \sum_{i=1}^{q}\sqrt{\frac{n_i\ c_{RR^{(1)}_i}}{n_i+m_i+p_i}}O^{\vec m,\vec p}_{R^{(1)}_i,r^{(1)}_i}(X,Y,Z). \tag{5.27}$$

The first formula above is exact. The last two hold only in the large $N$ limit. We have introduced some new notation: the Young diagram $R^{(n)}_i$ is obtained from $R$ by dropping $n$ boxes from row $i$ of $R$. Further, $\vec p^{(n)}_i$ is obtained from vector $\vec p$ by replacing $p_i\to p_i-n$ and similarly for $\vec m^{(n)}_i$. Finally, $c_{RR^{(1)}_i}$ is the factor of the box that belongs to $R$ but not to $R^{(1)}_i$. Recall that a box in row $i$ and column $j$ has factor $N-i+j$. For the proof of these formulas, we use the notation

$$\mathcal{N} = \frac{1}{n!m!p!}\sqrt{\frac{|H_X\times H_Y|\mathrm{hooks}_r}{\mathrm{hooks}_R f_R}}$$

and

$$\mathrm{Tr}(\sigma\cdot X^{\otimes p}\otimes Y^{\otimes m}\otimes Z^{\otimes n}) = X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.$$

We will now prove (5.25). A simple computation shows

$$\begin{aligned}
\frac{dO^{\vec{m},\vec{p}}_{R,r}}{dX^i_i} &= p\mathcal{N}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}(P_{R,r}\Gamma^{(R)}(\sigma))\mathrm{Tr}(\sigma\,\cdot\,1\otimes\,X^{\otimes p-1}\otimes Y^{\otimes\,m}\otimes Z^{\otimes\,n})\\
&= p\mathcal{N}\sum_{\sigma\in S_{n+m+p-1}}\sum_{i=1}^{n+m}\mathrm{Tr}(P_{R,r}\Gamma^{(R)}(\sigma\,(i,1))\mathrm{Tr}(\sigma(i,1)\,\cdot\,1\otimes\,X^{\otimes p-1}\otimes Y^{\otimes\,m}\otimes Z^{\otimes\,n})\\
&= p\mathcal{N}\sum_{\sigma\in S_{n+m+p-1}}\mathrm{Tr}(P_{R,r}\Gamma^{(R)}(\sigma)[N+\sum_{i=2}^{n+m}(i,1)])\mathrm{Tr}(\sigma\,\cdot\,1\otimes\,X^{\otimes p-1}\otimes Y^{\otimes\,m}\otimes Z^{\otimes\,n}).
\end{aligned}$$

Since we are summing over elements of the subgroup $S_{n+m+p-1}\subset S_{n+m+p}$ we can decompose the trace over the irreducible representation of $S_{n+m+p}$ as a sum of traces over irreducible representation $R^{(1)}_i$ of the subgroup $S_{n+m+p-1}$. Now use the fact that $N+\sum_{i=2}^{n+m}(i,1)$ gives $c_{RR^{(1)}_i}$ = the factor of the box dropped from $R$ when acting on any state in the carrier space of $R$ that also belongs to the $R^{(1)}_i$ subspace. We find

$$\frac{dO^{\vec{m},\vec{p}}_{R,r}}{dX^j_j}=\sum_{i=1}^{q}f^{(i)}_{\mathcal{N}}\;c_{RR^{(1)}_i}O^{\vec{m},\vec{p}^{(1)}_i}_{R^{(1)}_i,r} \tag{5.28}$$

where the factor

$$f^{(i)}_{\mathcal{N}}=\sqrt{\frac{p_i}{(n_i+m_i+p_i)c_{RR^{(1)}_i}}} \tag{5.29}$$

accounts for the change in the normalization factor $\mathcal{N}$ of the operator. This is an exact formula - it does not depend on large $N$ or on the displaced corners approximation. Next consider the proof of (5.26) and (5.27). Consider

$$\begin{aligned}
\frac{dO^{\vec{m},\vec{p}}_{R,r}}{dY^i_i} &= m\mathcal{N}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}(P_{R,r}\Gamma^{(R)}(\sigma))\mathrm{Tr}(\sigma\,\cdot\,X^{\otimes p}\otimes 1\,\otimes Y^{\otimes\,m-1}\otimes Z^{\otimes\,n})\\
&= m\mathcal{N}\sum_{\sigma\in S_{n+m+p-1}}\sum_{i=1}^{n+m}\mathrm{Tr}(P_{R,r}\Gamma^{(R)}(\sigma)\mathrm{Tr}((p+1,1)\sigma(p+1,1)\,\cdot\,1\,\otimes\,X^{\otimes p}\otimes Y^{\otimes\,m-1}\otimes Z^{\otimes\,n})\\
&= m\mathcal{N}\sum_{\sigma\in S_{n+m+p-1}}\mathrm{Tr}(P_{R,r}\Gamma^{(R)}((1,p+1)\sigma)(1,p+1))\mathrm{Tr}(\sigma\,\cdot\,1\,\otimes\,X^{\otimes p}\otimes Y^{\otimes\,m-1}\otimes Z^{\otimes\,n}).
\end{aligned}$$

The new feature in the above derivation is the presence of the $(1,p+1)\in S_{n+m+p}$ factors needed to swap the removed $Y$ box to the end of the row so that it can be removed, using the same manipulations as above. The evaluation of the action of these factors is most easily performed using Young's orthogonal representation, which gives a rule for the action of adjacent permutations (i.e. permutations of the form $(i,i+1)$) on Young-Yamanouchi (hereafter abbreviated YY) states. Let $|Y\rangle$ denote a YY state, and let $|Y(i\leftrightarrow i+1)\rangle$ denote the YY state obtained by swapping boxes $i$ and $i+1$. A box in row $a$ and column $b$ has content given by $b-a$. Denote the content of the box in $|Y\rangle$ filled with $j$ by $c_j$. The rule is

$$(i,i+1)|Y\rangle=\frac{1}{c_i-c_{i+1}}|Y\rangle+\sqrt{1-\frac{1}{(c_i-c_{i+1})^2}}|Y(i\leftrightarrow i+1)\rangle. \tag{5.30}$$

This rule simplifies dramatically in the displaced corners limit, at large $N$. If the two boxes belong to the same row we find $(i,i+1)|Y\rangle=|Y\rangle$ and if not $(i,i+1)|Y\rangle=|Y(i\leftrightarrow i+1)\rangle$. This is all that is needed to complete the proof of (5.26) and (5.27) and it proceeds exactly as for the first rule proved above. Note that because we used simplifications of the large $N$ limit, (5.26) and (5.27) are not exact statements but hold only at large $N$. The three statements derived above admit some natural generalizations. For example, we can consider tracing over a product of derivatives to obtain

$$\mathrm{Tr}\left(\frac{d^k}{dX^k}\right) O^{\vec{m}\vec{p}}_{R,r}(X,Y,Z) = \sum_{i=1}^{q}\left(\frac{c_{RR_i^{(1)}}}{n_i+m_i+p_i}\right)^{\frac{k}{2}} \prod_{a=0}^{k-1}\sqrt{p_i-a}\ O^{\vec{p}_i^{(k)}\vec{m}}_{R_i^{(k)},r}(X,Y,Z). \tag{5.31}$$

There are obvious generalization when we have a product of $Y$ or $Z$ derivatives. We could also allow more than one type of derivative in a given trace, for example (in what follows $k=k_1+k_2$)

$$\mathrm{Tr}\left(\frac{d^{k_1}}{dX^{k_1}}\frac{d^{k_2}}{dY^{k_2}}\right) O^{\vec{m}\vec{p}}_{R,r}(X,Y,Z)$$
$$= \sum_{i=1}^{q}\left(\frac{c_{RR_i^{(1)}}}{n_i+m_i+p_i}\right)^{\frac{k}{2}} \prod_{a=0}^{k_1-1}\sqrt{p_i-a}\prod_{b=0}^{k_2-1}\sqrt{m_i-b}\ O^{\vec{m}_i^{(k_2)}\vec{p}_i^{(k_1)}}_{R_i^{(k)},r}(X,Y,Z). \tag{5.32}$$

By using these formulas for each trace successively, we can also easily evaluate expressions of this form

$$\mathrm{Tr}\left(\frac{d^{k_1}}{dX^{k_1}}\frac{d^{k_2}}{dY^{k_2}}\right)\cdots\mathrm{Tr}\left(\frac{d^{k_3}}{dZ^{k_3}}\right) O^{\vec{m}\vec{p}}_{R,r}(X,Y,Z). \tag{5.33}$$

To compare to a multi particle system of $q$ noninteracting particles, again in a 3-dimensional harmonic oscillator potential, we need to introduce $q$ copies of the oscillators $(I,J=1,...,q)$

$$\left[a_z^{(I)},a_z^{(J)\dagger}\right]=\left[a_y^{(I)},a_y^{(J)\dagger}\right]=\left[a_x^{(I)},a_x^{(J)\dagger}\right]=\delta^{IJ} \tag{5.34}$$

one copy for each particle. Each Gauss graph operator $O^{\vec{m}\vec{p}}_{R,r}$ is specified by giving the number of $Z$ boxes $(r_i)$, $Y$ boxes $(m_i)$ and $X$ boxes $(p_i)$ in the $i$th row for $i=1,...,q$. The corresponding multi particle state is

$$|O^{\vec{m}\vec{p}}_{R,r}\rangle = \prod_{I=1}^{q}\frac{(a_z^{(I)\dagger})^{r_I}}{\sqrt{r_I!}}\frac{(a_y^{(I)\dagger})^{m_I}}{\sqrt{m_I!}}\frac{(a_x^{(I)\dagger})^{p_I}}{\sqrt{p_I!}}|0\rangle. \tag{5.35}$$

Using these formulas we can compare (for example) the matrix elements

$$\langle O^{\vec{m}\vec{p}}_{R_q^{(k)},r_q^{(k)}}|(a_z^{(I)})^k|O^{\vec{m}\vec{p}}_{R,r}\rangle \tag{5.36}$$

to the correlation functions

$$\langle O^{\vec{m}\vec{p}\ \dagger}_{R_q^{(k)},r_q^{(k)}}\ \mathrm{Tr}\left(\frac{d^k}{dZ^k}\right)\ O^{\vec{m}\vec{p}}_{R,r}\rangle \tag{5.37}$$

to learn that we should identify

$$\mathrm{Tr}\left(\frac{d^k}{dZ^k}\right)\leftrightarrow\sum_{I=1}^{q}\left(\sqrt{\frac{N+m_I+n_I+p_I}{m_I+n_I+p_I}}\right)^k (a_z^{(I)})^k. \tag{5.38}$$

In the above formula $n_I$ is the number of $Z$ boxes in row $I$, $m_I$ the number of $Y$ boxes and $p_I$ the number of $X$ boxes. The general rule is $(k=k_1+k_2+k_3)$

$$\mathrm{Tr}\left(\frac{d^{k_1}}{dX^{k_1}}\frac{d^{k_2}}{dY^{k_2}}\frac{d^{k_3}}{dZ^{k_3}}\right)\leftrightarrow\sum_{I=1}^{q}\left(\sqrt{\frac{N+m_I+n_I+p_I}{m_I+n_I+p_I}}\right)^k (a_x^{(I)})^{k_1}(a_y^{(I)})^{k_2}(a_z^{(I)})^{k_3}. \tag{5.39}$$

It is easy to check that the ordering of operators inside the trace on the left hand side above does not matter, when acting on the operators we consider, at large $N$. Multi trace formulas use the above identification for each trace separately. For example

$$
\begin{aligned}
&\mathrm{Tr}\left(\frac{d^{k_1}}{dX^{k_1}}\frac{d^{k_2}}{dY^{k_2}}\frac{d^{k_3}}{dZ^{k_3}}\right)\mathrm{Tr}\left(\frac{d^{k_4}}{dX^{k_4}}\right)\\
&\leftrightarrow \quad \sum_{I=1}^{q}\left(\sqrt{\frac{N+m_I+n_I+p_I}{m_I+n_I+p_I}}\right)^{k}(a_x^{(I)})^{k_1}(a_y^{(I)})^{k_2}(a_z^{(I)})^{k_3}\\
&\qquad \times\sum_{J=1}^{q}\left(\sqrt{\frac{N+m_J+n_J+p_J}{m_J+n_J+p_J}}\right)^{k_4}(a_x^{(I)})^{k_4}. \qquad (5.40)
\end{aligned}
$$

By comparing overlaps between states with polynomials of creation and annihilation operators sandwiched in between and correlators of Gauss graph operators with traces of polynomials of the matrices and derivatives with respect to the matrices acting on the Gauss graph operators as in the examples we studied above, we can build any entry in the dictionary between the $q$ particle system and Gauss graph operators with $q$ rows.

## 5.4 Outlook

The description of giant gravitons, constructed using a world volume analysis, allows one to count the set of all $\frac{1}{8}$-BPS giant gravitons. This counting matches $N$ bosons in a 3-dimensional harmonic oscillator. It is also possible to define an index to count single trace BPS operators, and it can be computed both at weak coupling (using the gauge theory) and at strong coupling (as a sum over the spectrum of free massless particles in $\mathrm{AdS}_5\times\mathrm{S}^5$). The results of these different computations are in complete accord. One can compute the spectrum of anomalous dimensions, for operators with a bare dimension of order $N$, in the $\mathcal{N}=4$ super Yang-Mills theory[73, 34, 35]. In this study we have demonstrated that exactly the same counting (i.e. $N$ bosons in a 3-dimensional harmonic oscillator) results from counting operators of vanishing anomalous dimension in this spectrum. Motivated by this agreement, we have looked for a relation between multi particle wave functions and Gauss graph operators. Our basic result is that a map between particle wave functions for particles in a 3-dimensional harmonic oscillator and Gauss graph operators is easily constructed by comparing overlaps of wave functions of the particle system with correlators of Gauss graph operators. The correlator computations have made use of significant simplifications that arise for the BPS Gauss graph operators. The number of particles match the number of rows in the Young diagram labeling the Gauss graph operator. In our opinion, these results provide concrete evidence that the Gauss graph operators are indeed the operators dual to the $\frac{1}{8}$-BPS giant gravitons. To interpret the link between the particle system and the Gauss graph operators, recall the link between giant gravitons and an eigenvalue description of the multi matrix dynamics, which has been pursued in [9, 77]. Thus, the fact that the matrix model computations appear to be related to the dynamics of non-interacting particles gives hints as to how matrix model dynamics may simplify, along the line of the proposals of [36, 46, 42, 79].

# Chapter 6

# Discussion and Conclusion

We have studied the AdS/CFT correspondence between type $IIB$ string theory on asymptotically $AdS_5 \times S^5$ backgrounds and $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$. The boundary of $AdS_5 \times S^5$ in global coordinate is $R \times S^3$. We considered BPS operators in the field theory that are dual to BPS geometries on the gravity side of the correspondence. To confirm the equality of these dual theories, we computed observables such as correlation functions and we also counted gauge invariant operators in the field theory that correspond to giant gravitons in the gravity theory.

We showed that there is a dictionary that can be used to map the free fermion field (that is dual to an LLM geometry) to a Schur polynomial. We needed to know the ground state wave function of the free fermion field. This picture was tested by using the operator state correspondence where correlation functions of the Schur polynomial were compared to the overlaps of the wave function of the non-interacting fermion field theory. The results were in complete agreement.

We discussed the 1/2 BPS geometries that are dual to free fermions in phase space. The 1/2 BPS geometries are smooth geometries determined by a single function $z(x_i, y)$ which satisfies the Laplace equation. The fermions that are dual to 1/2 BPS geometries form an incompressible fluid in phase space. Each configuration of droplets in this phase space was assigned a geometry in the dual gravity. In order to obtain smooth geometries, the function $z(x_1, x_2, y = 0)$ must only take the values $z(x_1, x_2, y = 0) = \pm 1/2$.

We also looked at 1/4 BPS geometries, which are also smooth geometries. The 1/4 BPS geometries were determined by a single function $Z(z_a, \bar{z}_a, y)$ which satisfies the Monge-Ampere equation. The Monge-Ampere equation is a collection of coupled non-linear partial differential equations which are hard to solve.

An important difference between the analysis of the 1/2 and 1/4 BPS geometries is that the droplets of the 1/2 BPS geometries can take any arbitrary shape while the droplets of the 1/4 BPS geometry are constrained to take certain shapes. The shapes of the droplets of the 1/4 BPS geometries are described by the equation $v(z_a, \bar{z}_a) = 0$ and the function $v$ is governed by the differential equation (2.174). Up to this point, the thesis was a review of existing results.

A picture that links 1/4 BPS geometries with free fermion fields was developed. The proposed ground state wave function was used to compute several BPS correlators which were tested against the multi matrix model correlators with $Z$ and $Y$ fields. This wave function was constructed using eigenvalues of 2 complex matrix fields, $Z$ and $Y$. The computation of correlation functions using eigenvalues and then using the matrix model were in agreement only for correlation functions that have at least $N$ fields in each trace. We checked that the surface defining the supergravity solution is visible in the eigenvalues dynamics. This was achieved by looking at the surface where the eigenvalues condense and identifying it with the wall defining the boundary conditions in supergravity [79].

We also perfomed the counting of 1/8 BPS giant gravitons by deriving the grand partition function of $N$ bosons in a 3-dimensional harmonic oscillator. We find that the counting of the 1/8 BPS giant gravitons agrees with the counting of BPS Gauss graphs with $Z$, $Y$ and $X$ fields. We also confirm that the computation of correlation functions of these Gauss graphs match the overlaps of the proposed dual multi-particle states [80].

Our results have added yet more evidence to the validity of the AdS/CFT correspondence and it sheds light on how to study BPS geometries and giant gravitons in the CFT.

For future work, it would be nice to devise a systematic way of deriving the Van der Monde determinant for two matrices, which will confirm our proposal. It is interesting to extend the study of two matrices of the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills to a study of three matrices of the $SU(3)$ sector of $\mathcal{N}=4$ super Yang-Mills and to reduce it to eigenvalue dynamics. In our study of 1/8 BPS giant gravitons we did not consider 1/8 BPS operators corresponding to new geometries. It would be nice to extend our analysis and include the 1/8 BPS geometries.

# Appendix A

# Supergravity Background

In this chapter, we study the BPS geometries in type $IIB$ string theory. These are smooth geometries that live in 10-dimensions and are described by a metric. They are all asymptotically $AdS_5 \times S^5$. The 1/2 BPS metric that we are interested in has $R \times SO(4) \times SO(4)$ isometry. The 1/4 BPS metric enjoys $R \times SO(4) \times SO(2)$ isometry. Both the 1/2 BPS and the 1/4 BPS geometries have an $SO(4)$ isometry. This corresponds to a 3-sphere. To start, we will review the metric of a 3-sphere.

### A.0.1 Metric of a round 3-sphere of radius $R$

The equation for the 3-sphere embedded in $\mathbb{R}^4$ is

$$x^2 + y^2 + z^2 + w^2 = R^2. \tag{A.0.1}$$

In spherical coordinates, this equation is solved by

$$\begin{aligned} x &= R \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ y &= R \sin\theta_1 \sin\theta_2 \sin\theta_3, \\ z &= R \sin\theta_1 \cos\theta_2, \\ w &= R \cos\theta_1. \end{aligned}$$

$\theta_1, \theta_2$ and $\theta_3$ are the coordinates of the 3-sphere. The range of $\theta_1$ and $\theta_2$ is $[0, \pi]$ and $\theta_3$ ranges from 0 to $2\pi$. Any distance between two points in the 3-sphere can be measured by the metric of $\mathbb{R}^4$. Therefore we can induce the metric of a 3-sphere from the metric in $\mathbb{R}^4$. The distance between two points in $\mathbb{R}^4$ is measured by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 + dz^2 + dw^2. \tag{A.0.2}$$

Consider the change coordinates from $x$ to $\tilde{x}$. Writing $x = x(\tilde{x})$, $dx^\mu$ and $dx^\nu$ transform as $dx^m = \frac{\partial x^m}{\partial \tilde{x}^q} d\tilde{x}^q$. The metric transforms as follows

$$ds^2 = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\sigma} \frac{\partial x^\nu}{\partial \tilde{x}^\lambda} d\tilde{x}^\sigma d\tilde{x}^\lambda \tag{A.0.3}$$

and the new metric tensor is given by $\tilde{g}_{\sigma\lambda} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\sigma} \frac{\partial x^\nu}{\partial \tilde{x}^\lambda}$. The metric tensor induced on a manifold that is embedded in a larger manifold is called an induced metric. The induced metric of a 3-sphere embedded in $\mathbb{R}^4$ is given by

$$g_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^a} \frac{\partial x^\nu}{\partial \tilde{x}^b} \tag{A.0.4}$$

where $a, b = \theta_1, \theta_2$ and $\theta_3$. We will now use (A.0.4) to compute the components of the metric tensor of the 3-sphere. The non-vanishing terms are

$$\begin{aligned}
g_{\mu\nu}\frac{\partial x^\mu}{\partial \theta^1}\frac{\partial x^\nu}{\partial \theta^1}d\theta_1^2 =& \quad R^2 d\theta_1^2\bigg( \cos^2\theta_1 \sin^2\theta_2 \cos^2\theta_3 + \cos^2\theta_1 \sin^2\theta_2 \sin^2\theta_3 + \cos^2\theta_1 \cos^2\theta_2 + \sin^2\theta_1\bigg) \\
=& R^2 d\theta_1^2. \\
g_{\mu\nu}\frac{\partial x^\mu}{\partial \theta^2}\frac{\partial x^\nu}{\partial \theta^2}d\theta_2^2 =& \quad R^2 d\theta_2^2\bigg( \sin^2\theta_1 \cos^2\theta_2 \cos^2\theta_3 + \sin^2\theta_1 \cos^2\theta_2 \sin^2\theta_3 + \sin^2\theta_1 \sin^2\theta_2\bigg) \\
=& R^2 \sin^2\theta_1 d\theta_2^2. \\
g_{\mu\nu}\frac{\partial x^\mu}{\partial \theta^3}\frac{\partial x^\nu}{\partial \theta^3}d\theta_3^2 =& \quad R^2 d\theta_3^2\bigg( \sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_3 + \sin^2\theta_1 \sin^2\theta_2 \cos^2\theta_3\bigg) \\
=& R^2 \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2.
\end{aligned}$$

Therefore the metric of the round 3-sphere is given by

$$d\Omega_3^2 = R^2 d\theta_1^2 + R^2 \sin^2\theta_1 d\theta_2^2 + R^2 \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2. \tag{A.0.5}$$

### A.0.2 Hodge dual

Among the fields of type II $B$ gravity there is a self dual five form field strength. To check if forms are self dual or anti self dual, we need to use the Hodge dual. The Hodge dual is an operation that maps $k$-forms to the space of $(n-k)$-forms. This operation is represented by a star $*$. To understand the operation of the Hodge dual on forms, we will first have to understand how the Levi-Civita symbol $\epsilon$ changes in different coordinates.

#### $\epsilon^{MNP}$ in spherical coordinates

The map between Cartesian coordinates and spherical coordinates can be written as

$$\begin{aligned}
x &= r\cos\theta\sin\phi, \\
y &= r\sin\theta\sin\phi, \\
z &= r\cos\phi.
\end{aligned}$$

Consequently

$$\begin{aligned}
dx &= \cos\theta\sin\phi dr - r\sin\theta\sin\phi d\theta + r\cos\theta\cos\phi d\phi \\
dy &= \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi \\
dz &= \cos\phi dr - r\sin\phi d\phi.
\end{aligned}$$

We want to transform the Levi-Civita symbol to spherical coordinates starting from its expression in Cartesian coordinates. There is a one-to-one correspondence between vectors and directional vectors, that is

$$V = V^m \frac{\partial}{\partial x^m}. \tag{A.0.6}$$

We have basis vectors that are given by $\hat{e}_{(m)} = \dfrac{\partial}{\partial x^m}$. In other coordinate systems, these basis vectors are written as

$$\frac{\partial}{\partial \tilde{x}^n} = \frac{\partial x^m}{\partial \tilde{x}^n}\frac{\partial}{\partial x^m}.$$

Using this last expression and requiring that (A.0.6) remains the same under a change of basis, we find the following relation

$$\begin{aligned} V = V^m \frac{\partial}{\partial x^m} &= \tilde{V}^n \frac{\partial}{\partial \tilde{x}^n} \\ \Rightarrow V^m \frac{\partial}{\partial x^m} &= \tilde{V}^n \frac{\partial x^m}{\partial \tilde{x}^n} \frac{\partial}{\partial x^m}. \end{aligned}$$

From this result we conclude that the vector transformation law is given by

$$V^m = \tilde{V}^n \frac{\partial x^m}{\partial \tilde{x}^n}. \tag{A.0.7}$$

The Levi-Civita symbol is totally anti-symmetric. It transforms as follows

$$\epsilon^{MNP} = \frac{\partial x^M}{\partial \tilde{x}^Q} \frac{\partial x^N}{\partial \tilde{x}^R} \frac{\partial x^P}{\partial \tilde{x}^S} \tilde{\epsilon}^{QRS}$$

where $\epsilon^{123} = 1$ and $x^1 = x, x^2 = y$ and $x^3 = z$. Computing $\tilde{\epsilon}^{r\phi\theta}$, we find

$$\begin{aligned} \epsilon^{123} &= \frac{\partial x^1}{\partial \tilde{x}^r} \frac{\partial x^2}{\partial \tilde{x}^\theta} \frac{\partial x^3}{\partial \tilde{x}^\phi} \tilde{\epsilon}^{r\theta\phi} + \frac{\partial x^1}{\partial \tilde{x}^\phi} \frac{\partial x^2}{\partial \tilde{x}^\theta} \frac{\partial x^3}{\partial \tilde{x}^r} \tilde{\epsilon}^{\phi\theta r} + \frac{\partial x^1}{\partial \tilde{x}^\theta} \frac{\partial x^2}{\partial \tilde{x}^r} \frac{\partial x^3}{\partial \tilde{x}^\phi} \tilde{\epsilon}^{\theta r\phi} + \frac{\partial x^1}{\partial \tilde{x}^\theta} \frac{\partial x^2}{\partial \tilde{x}^\phi} \frac{\partial x^3}{\partial \tilde{x}^r} \tilde{\epsilon}^{\theta\phi r} \\ 1 &= r^2 \sin\phi \tilde{\epsilon}^{r\phi\theta} \\ \tilde{\epsilon}^{r\phi\theta} &= \frac{1}{r^2 \sin\phi}. \end{aligned} \tag{A.0.8}$$

We see that the term $r^2 \sin\phi$ is the same as $\sqrt{|det g_{\alpha\beta}|}$, where $g_{\alpha\beta}$ is the metric written using spherical coordinates. It is given by

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2\phi & 0 \\ 0 & 0 & r^2 \end{pmatrix}.$$

The Levi-Civita symbol in Cartesian coordinates $\epsilon^{123} = 1$ is $\tilde{\epsilon}^{r\phi\theta} = \dfrac{1}{\sqrt{|det g_{\alpha\beta}|}}$ in spherical coordinates. This last formula is general. To see this, consider the determinant of the $3 \times 3$ matrix $A$

$$\begin{aligned} det(A) &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\ &= A_{11}A_{22}A_{33} + A_{31}A_{12}A_{23} + A_{21}A_{32}A_{13} - A_{21}A_{12}A_{33} - A_{11}A_{32}A_{23} - A_{31}A_{22}A_{13}. \end{aligned}$$

This determinant can be re-written as

$$det(A) = \epsilon^{i_1 i_2 i_3} A_{i_1 1} A_{i_2 2} A_{i_3 3}. \tag{A.0.9}$$

Now, consider the Jacobian

$$|detJ| = \begin{vmatrix} \frac{\partial x^1}{\partial \tilde{x}^{i_1}} & \frac{\partial x^2}{\partial \tilde{x}^{i_1}} & \frac{\partial x^3}{\partial \tilde{x}^{i_1}} \\ \frac{\partial x^1}{\partial \tilde{x}^{i_2}} & \frac{\partial x^2}{\partial \tilde{x}^{i_2}} & \frac{\partial x^3}{\partial \tilde{x}^{i_2}} \\ \frac{\partial x^1}{\partial \tilde{x}^{i_3}} & \frac{\partial x^2}{\partial \tilde{x}^{i_3}} & \frac{\partial x^3}{\partial \tilde{x}^{i_3}} \end{vmatrix}.$$

We can write this Jacobian as

$$|detJ| = \epsilon^{i_1 i_2 i_3} \frac{\partial x^1}{\partial \tilde{x}^{i_1}} \frac{\partial x^2}{\partial \tilde{x}^{i_2}} \frac{\partial x^3}{\partial \tilde{x}^{i_3}}. \tag{A.0.10}$$

The right hand side of (A.0.10) looks exactly like the right hand of side of (A.0.8). Requiring that the inner product remain invariant under the change in coordinate ($dx_\mu dx^\mu = d\tilde{x}_\mu d\tilde{x}^\mu$), the metric transforms as

$$\tilde{g}_{kl} = g_{ij} \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial x^j}{\partial \tilde{x}^l}.$$

Taking the determinant of the metric tensor

$$\begin{aligned} det(g) &= \frac{1}{3!} \epsilon_{klm} \epsilon_{ijq} g_{ki} g_{lj} g_{mq} \\ &= \frac{1}{3!} (\epsilon_{klm} \frac{\partial x^a}{\partial \tilde{x}^k} \frac{\partial x^b}{\partial \tilde{x}^l} \frac{\partial x^c}{\partial \tilde{x}^m})(\epsilon_{ijq} \frac{\partial x^a}{\partial \tilde{x}^i} \frac{\partial x^b}{\partial \tilde{x}^j} \frac{\partial x^c}{\partial \tilde{x}^q}) g_{aa} g_{bb} g_{cc}. \end{aligned}$$

Define

$$\epsilon_{klm} \frac{\partial x^a}{\partial \tilde{x}^k} \frac{\partial x^b}{\partial \tilde{x}^l} \frac{\partial x^c}{\partial \tilde{x}^m} = T^{abc}$$

where $T^{abc}$ is totally anti-symmetric that is $T^{abc} = -T^{bac}$. Looking at equation (A.0.10) we see that $T^{123} = |detJ|$, therefore

$$\begin{aligned} det(g) &= \frac{6}{3!} (\epsilon_{klm} \frac{\partial x^1}{\partial \tilde{x}^k} \frac{\partial x^2}{\partial \tilde{x}^l} \frac{\partial x^3}{\partial \tilde{x}^m})(\epsilon_{ijq} \frac{\partial x^1}{\partial \tilde{x}^i} \frac{\partial x^2}{\partial \tilde{x}^j} \frac{\partial x^3}{\partial \tilde{x}^q}) \\ &= (T^{123})^2 \\ \Rightarrow T^{123} &= \sqrt{det(g)} \end{aligned}$$

where on the first line we used the fact that $T^{abc}T^{abc} = 6(T^{123})^2$. Therefore

$$|detJ| = \sqrt{|detg|}. \tag{A.0.11}$$

This is a general statement. We conclude that

$$\sqrt{|detg|} = \epsilon^{i_1 i_2 i_3} \frac{\partial x^1}{\partial \tilde{x}^{i_1}} \frac{\partial x^2}{\partial \tilde{x}^{i_2}} \frac{\partial x^3}{\partial \tilde{x}^{i_3}}.$$

#### $\epsilon_{MNP}$ in spherical coordinates

Following the reasoning used to derive (A.0.6), one can show that $V_M$ transforms as

$$V_M = \frac{\partial \tilde{x}^Q}{\partial x^M} \tilde{V}_Q.$$

We can derive the Levi-Civita symbol in spherical coordinates with lower indices, where $\epsilon_{123} = 1$ as follows

$$\begin{aligned} \epsilon_{123} &= \frac{\partial \tilde{x}^r}{\partial x^1} \frac{\partial \tilde{x}^\theta}{\partial x^2} \frac{\partial \tilde{x}^\phi}{\partial x^3} \tilde{\epsilon}_{r\theta\phi} + \frac{\partial \tilde{x}^\phi}{\partial x^1} \frac{\partial \tilde{x}^\theta}{\partial x^2} \frac{\partial \tilde{x}^r}{\partial x^3} \tilde{\epsilon}_{\phi\theta r} + \frac{\partial \tilde{x}^\theta}{\partial x^1} \frac{\partial \tilde{x}^r}{\partial x^2} \frac{\partial \tilde{x}^\phi}{\partial x^3} \tilde{\epsilon}_{\theta r\phi} + \frac{\partial \tilde{x}^\theta}{\partial x^1} \frac{\partial \tilde{x}^\phi}{\partial x^2} \frac{\partial \tilde{x}^r}{\partial x^3} \tilde{\epsilon}_{\theta\phi r} \\ \tilde{\epsilon}_{r\phi\theta} &= r^2 \sin\phi. \end{aligned}$$

Again, the Cartesian coordinate Levi-Civita symbol $\epsilon_{123} = 1$ is related to the spherical coordinate Levi-Civita symbol $\tilde{\epsilon}_{r\phi\theta} = \sqrt{|detg_{\alpha\beta}|}$. Now that we know how to compute the Levi-Civita symbol in different coordinates, we would like to investigate products of Levi-Civita symbols.

**The products of the Levi-Civita symbol with an anti-symmetric tensor in $d$ dimensional flat space:** $\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_n\nu_{n+1}\cdots\nu_d}A^{\nu_{n+1}\cdots\nu_d}$

Let $A^{\nu_{n+1}\cdots\nu_d}$ be a completely antisymmetric tensor. For the computation of $\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_n\nu_{n+1}\cdots\nu_d}A^{\nu_{n+1}\cdots\nu_d}$, we start with simple cases until we manage to motivate the general formula. We will use the formula

$$\begin{aligned}\epsilon^{i_1i_2\cdots i_n}\epsilon_{i_1i_2\cdots i_n} &= \delta^{i_1i_2\cdots i_n}_{i_1i_2\cdots i_n}\\ &= n!.\end{aligned}$$

To illustrate this formula, consider $\epsilon^{ijk}\epsilon_{pqr}$ where all indices runs from 1 to 3 ($i,j,k,p,q,r = 1,2,3$). Now consider

$$\epsilon^{ijk}\epsilon_{pqr} = \delta^i_p\delta^j_q\delta^k_r - \delta^i_p\delta^j_r\delta^k_q - \delta^i_q\delta^j_p\delta^k_r + \delta^i_q\delta^j_r\delta^k_p + \delta^i_r\delta^j_p\delta^k_q - \delta^i_r\delta^j_q\delta^k_p.$$

Setting $k = r$, $j = q$ and $i = p$, we find

$$\epsilon^{ijk}\epsilon_{ijk} = 3!.$$

Proceeding with $\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_n\nu_{n+1}\cdots\nu_d}A^{\nu_{n+1}\cdots\nu_d}$, we will consider different values for $d$ and $n$, until we understand the pattern.

**Case 1:** $d - n = 1$

(I) $d = 2$ and $d - n = 1$:

$$\begin{aligned}\epsilon^{\mu_1\mu_2}\epsilon_{\mu_1\nu_2}A^{\nu_2} &= (\delta^{\mu_1}_{\mu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\mu_1})A^{\nu_2}\\ &= (2\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_2})A^{\nu_2}\\ &= \delta^{\mu_2}_{\nu_2}A^{\nu_2}\\ &= A^{\mu_2}.\end{aligned}$$

(II) $d = 3$ and $d - n = 1$:

$$\begin{aligned}\epsilon^{\mu_1\mu_2\mu_3}\epsilon_{\mu_1\mu_2\nu_3}A^{\nu_3} &= (\delta^{\mu_1}_{\mu_1}\delta^{\mu_2}_{\mu_2}\delta^{\mu_3}_{\nu_3} - \delta^{\mu_1}_{\mu_1}\delta^{\mu_2}_{\nu_3}\delta^{\mu_3}_{\mu_2} - \delta^{\mu_1}_{\mu_2}\delta^{\mu_2}_{\mu_1}\delta^{\mu_3}_{\nu_3} + \delta^{\mu_1}_{\mu_2}\delta^{\mu_2}_{\nu_3}\delta^{\mu_3}_{\mu_1} + \delta^{\mu_1}_{\nu_3}\delta^{\mu_2}_{\mu_1}\delta^{\mu_3}_{\mu_2} - \delta^{\mu_1}_{\nu_3}\delta^{\mu_2}_{\mu_2}\delta^{\mu_3}_{\mu_1})A^{\nu_3}\\ &= (3\cdot 3\delta^{\mu_3}_{\nu_3} - 3\delta^{\mu_3}_{\nu_3} - 3\delta^{\mu_3}_{\nu_3} + \delta^{\mu_3}_{\nu_3} + \delta^{\mu_3}_{\nu_3} - 3\delta^{\mu_3}_{\nu_3})A^{\nu_3}\\ &= 2\delta^{\mu_3}_{\nu_3}A^{\nu_3}\\ &= 2A^{\mu_3}.\end{aligned}$$

(III) $d = d$ and $d - n = 1$:
Following the same procedure as above, we find

$$\begin{aligned}\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_{d-1}\nu_d}A^{\nu_d} &= (d-1)!\delta^{\mu_d}_{\nu_d}A^{\nu_d}\\ &= (d-1)!A^{\mu_d}.\end{aligned}$$

**Case 2:** $d - n = 2$

(IV) $d = 3$ and $d - n = 2$ :

$$\begin{aligned}\epsilon^{\mu_1\mu_2\mu_3}\epsilon_{\mu_1\nu_2\nu_3}A^{\nu_2\nu_3} &= (\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_3} - \delta^{\mu_2}_{\nu_3}\delta^{\mu_3}_{\nu_2})A^{\nu_2\nu_3}\\ &= 2A^{\mu_2\mu_3}.\end{aligned}$$

(V) $d = d$ and $d - n = 2$ :
Following the same calculation as above we notice the following pattern

$$\begin{aligned}\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_{d-2}\nu_{d-1}\nu_d}A^{\nu_{d-1}\nu_d} &= (d-2)!(\delta^{\mu_{d-1}}_{\nu_{d-1}}\delta^{\mu_d}_{\nu_d} - \delta^{\mu_{d-1}}_{\nu_d}\delta^{\mu_d}_{\nu_{d-1}})A^{\nu_{d-1}\nu_d} \\ &= 2(d-2)!A^{\mu_{d-1}\mu_d}\end{aligned}$$

**Case 3:** $d - n$

(VI) $\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_n\nu_{n+1}\cdots\nu_d}A^{\nu_{n+1}\nu_{n+2}\cdots\nu_d}$ :
We are now in a position to compute this expression. With experience from the above examples we conclude that the general formula is given by

$$\epsilon^{\mu_1\mu_2\cdots\mu_d}\epsilon_{\mu_1\mu_2\cdots\mu_n\nu_{n+1}\cdots\nu_d}A^{\nu_{n+1}\nu_{n+2}\cdots\nu_d} = n!(d-n)!A^{\mu_{n+1}\mu_{n+2}\cdots\mu_d}. \tag{A.0.12}$$

### A.0.3 Self-dual operators.

Using our knowledge of the Hodge dual, we will compute $*F$ and show that it is self-dual. We will use the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\tilde{\Omega}_3^2 \tag{A.0.13}$$

and we define the 5-form $F_{(5)}$ to be

$$\begin{aligned}F_{(5)} &= F_{\mu\nu\theta_1\theta_2\theta_3}dx^\mu \wedge dx^\nu \wedge \sin^2\theta_1 \sin\theta_2 d\theta_1 \wedge d\theta_2 \wedge \theta_3 + \tilde{F}_{\mu\nu\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_3}dx^\mu \wedge dx^\nu \wedge \sin^2\tilde{\theta}_1 \sin\tilde{\theta}_2 d\tilde{\theta}_1 \wedge d\tilde{\theta}_2 \wedge \tilde{\theta}_3 \\ &= F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3\end{aligned}$$

where

$$d\Omega_3 = \sin^2\theta_1 \sin\theta_2 d\theta_1 \wedge d\theta_2 \wedge d\theta_3$$

and

$$d\tilde{\Omega}_3 = \sin^2\tilde{\theta}_1 \sin\tilde{\theta}_2 d\tilde{\theta}_1 \wedge d\tilde{\theta}_2 \wedge d\tilde{\theta}_3$$

are the two 3-spheres with a unit radius. The integral of $d\Omega_3$ is

$$\begin{aligned}\int d\Omega_3 &= \int_0^\pi \sin^2\theta_1 d\theta_1 \int_0^\pi \sin\theta_2 d\theta_1 \int_0^{2\pi} d\theta_3 \\ &= 2\pi^2.\end{aligned}$$

Using

$$\begin{aligned}\alpha^{(k)} &= \alpha_{i_1\cdots i_k}dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ *\alpha^{(k)} &= \frac{1}{(n-k)!}\epsilon_{i_1,\cdots,i_n}\sqrt{|det(g)|}\alpha^{i_1,i_2,\cdots,i_k}dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \cdots \wedge dx^{i_n},\end{aligned}$$

we can re-write $F_{(5)}$ as

$$\begin{aligned}*F_{(5)} &= \frac{1}{(4-2)!}F^{\mu\nu}\sqrt{|det(g)|}\;\epsilon_{\mu\nu\rho\lambda}(dx^\rho \wedge dx^\lambda \wedge d\tilde{\Omega}_3) \\ &+ \frac{1}{(4-2)!}\tilde{F}^{\mu\nu}\sqrt{|det(\tilde{g})|}\epsilon_{\mu\nu\rho\lambda}(dx^\rho \wedge dx^\lambda \wedge d\Omega_3)\end{aligned}$$

where $\sqrt{|det(g)|}$ is given by

$$\begin{aligned}\sqrt{|det(g)|} &= \sqrt{|det(g_\theta)|} \\ &= e^{-3G} det(R(\theta)) \\ &= e^{-3G}.\end{aligned}$$

Likewise

$$\begin{aligned}\sqrt{|det(\tilde{g})|} &= \sqrt{|det(g_{\tilde{\theta}})|} \\ &= e^{3G}.\end{aligned}$$

Therefore this reduces to

$$*F_{(5)} = \frac{1}{2!}\epsilon_{\mu\nu\rho\lambda}\Big(F^{\mu\nu}e^{-3G}(dx^\rho \wedge dx^\lambda \wedge d\tilde{\Omega}_3) + \tilde{F}^{\mu\nu}e^{3G}(dx^\rho \wedge dx^\lambda \wedge d\Omega_3)\Big).$$

Comparing this expression to the original 5-form we conclude that the field strength in 4-dimensions is

$$\begin{aligned}F &= e^{3G}\tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \\ \tilde{F} &= \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda}e^{-3G}F^{\sigma\lambda}dx^\mu \wedge dx^\nu \\ &= (\pm)e^{-3G} *_4 \tilde{F}.\end{aligned} \tag{A.0.14}$$

Note that since $\epsilon_{\mu\nu\sigma\lambda}$ is in Minkowski space, raising the indices of the Levi-Civita symbol will cost us a minus sign $\epsilon_{\mu\nu\sigma\lambda} = -\epsilon^{\mu\nu\sigma\lambda}$

$$\tilde{F} = - *_4 e^{-3G}F. \tag{A.0.15}$$

Taking the double Hodge dual, we obtain

$$\begin{aligned}**F &= *\Big[\frac{1}{2!}\epsilon_{\mu\nu\rho\lambda}\Big(F^{\mu\nu}(dx^\rho \wedge dx^\lambda \wedge d\tilde{\Omega}_3)) + \tilde{F}^{\mu\nu}(dx^\rho \wedge dx^\lambda \wedge d\Omega_3)\Big)\Big] \\ &= \frac{1}{2!}\frac{1}{2!}\epsilon_{\mu\nu\rho\lambda}(-\epsilon^{\mu\nu\rho\lambda})\Big(F_{\mu\nu}(dx^\mu \wedge dx^\nu \wedge d\Omega_3) + \tilde{F}_{\mu\nu}(dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3)\Big) \\ &= -\frac{2!(4-2)!}{2!2!}\Big(F_{\mu\nu}(dx^\mu \wedge dx^\nu \wedge d\Omega_3) + \tilde{F}_{\mu\nu}(dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3)\Big) \\ &= -F.\end{aligned}$$

So, we see that by taking the double Hodge dual we get back our original 5-form with an extra negative sign. From our study of electromagnetism, we know that the field strength is defined to be

$$F = dA$$

where $A$ is a 1-form, given by

$$A^{(1)} = A_i dx^i.$$

Then

$$\begin{aligned}F &= dA^{(1)} \\ &= \partial_j A_i dx^j \wedge dx^i.\end{aligned}$$

We know that we can break a tensor into a symmetric part and an anti-symmetric part, that is

$$\partial_j A_i = \frac{\partial_j A_i + \partial_i A_j}{2} + \frac{\partial_j A_i - \partial_i A_j}{2}.$$

Using the fact that $dx^j \wedge dx^i = -dx^i \wedge dx^j$, we can write $F$ as

$$\begin{aligned} F &= \frac{1}{2}(\partial_j A_i - \partial_i A_j) dx^j \wedge dx^i \\ &= \frac{1}{2} F_{ji} dx^j \wedge dx^i. \end{aligned} \tag{A.0.16}$$

The Hodge operator on this field strength in 4 dimensions is given by

$$*F = \frac{1}{2}\frac{1}{(4-2)!} F^{ij} \epsilon_{ijkl} dx^k \wedge dx^l. \tag{A.0.17}$$

Recall the expression from our $F_{(5)}$

$$F = e^{3G} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

Comparing this expression with (A.0.16), we conclude that from our $F_{(5)}$ we can also write

$$F = dB.$$

Again, recall the expression

$$\tilde{F} = \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda} e^{3G} F^{\sigma\lambda} dx^\mu \wedge dx^\nu.$$

Comparing this expression with (A.0.17), we learn that

$$\tilde{F} = d\tilde{B}.$$

### A.0.4 Geometry and Tensors

Geometry is a mathematical field that focus on the study of shape, size (such as lengths, areas, and volumes), relative position of figures, and the properties of space. The geometries of spaces are independent of the coordinates used to describe points in the space. Geometries are best described by mathematical objects called tensors. Tensors, just like scalars and vectors, can be used to describe physical properties. In fact tensors are a generalization of scalars and vectors. When we compute observables in the theory of gravity, we require that the physical quantities should be coordinate independent. It turns out that tensors are the right language to use when studying the theory of gravity because they are coordinate independent. To prove that tensor equations are coordinate independent, consider the tensor equation $T^{\mu\nu}{}_{\alpha\beta}(x) = 0$ . Under a coordinate transformation $x \to x'$, we expect that

$$T^{\mu\nu}{}_{\alpha\beta}(x) \to T^{\mu'\nu'}{}_{\alpha'\beta'}(x') = 0.$$

We can write $T^{\mu'\nu'}{}_{\alpha'\beta'}(x')$ in terms of $T^{\mu\nu}{}_{\alpha\beta}(x)$. Using the coordinate transformation laws for tensors, we have

$$T^{\mu'\nu'}{}_{\alpha'\beta'}(x') = \frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{\partial x'^{\nu'}}{\partial x^\nu}\frac{\partial x^\alpha}{\partial x'^\alpha}\frac{\partial x^\beta}{\partial x'^\beta} T^{\mu\nu}{}_{\alpha\beta}(x).$$

Notice that $\frac{\partial x'^{\mu'}}{\partial x^\mu}$, $\frac{\partial x'^{\nu'}}{\partial x^\nu}$, $\frac{\partial x^\alpha}{\partial x'^\alpha}$ and $\frac{\partial x^\beta}{\partial x'^\beta}$ can be represented by matrices and these matrices are always invertible since their determinant is non-zero. Therefore $T^{\mu'\nu'}{}_{\alpha'\beta'}(x') = 0$ implies that $T^{\mu\nu}{}_{\alpha\beta}(x) = 0$.

So we conclude that tensor equations are coordinate independent. Not every object is a tensors. Here are examples of objects which are tensors and objects which are not

$$\frac{\partial \phi}{\partial x^\nu} \qquad \text{tensor,}$$
$$\frac{\partial V^\mu}{\partial x^\nu} \qquad \text{not a tensor.}$$

However, it is possible that a tensor is composed of objects that are not tensors. An example of such a tensor is the covariant derivative

$$D_\nu V^\mu = \frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\alpha} V^\alpha, \tag{A.0.18}$$

$$D_\nu V_\mu = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\alpha{}_{\nu\mu} V_\alpha. \tag{A.0.19}$$

In the above expression, the first term on the right hand side is not a tensor but the second term on the right hand side corects it. This behaviour can be understood in the following way. Consider a function that depends on two variables $f(x, y)$. If we apply a change in this function $df(x, y)$ we must make sure that we consider the change in both the $x$ and $y$ direction

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

If we only take the change in one direction, this would not have represented the full change of the function. The same thing happened in the covariant derivative. The first term alone does not mean anything but the full change is the covariant derivative. The second term in the covariant derivative contains the affine connection which is known as the Christoffel symbol $\Gamma^\mu{}_{\nu\alpha}$.

### A.0.5 Christoffel symbols

The Christoffel symbols belong to a larger class of objects called connections. The affine connection connects nearby tangent space on a smooth manifold. It allows tangent vector fields to be differentiated as functions on the manifold with values in a fixed vector space. Given a metric, one can calculate the Christoffel symbol using the formula

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} (\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}). \tag{A.0.20}$$

This method of computing the Christoffel symbol is long and tedious. Fortunately, there is another simpler way that one can use to compute the Christoffel symbols. This is the geodesic method.

#### Geodesic equation

The geodesic equation is derived using the variational principle. This is done as follows: consider a Lorentz invariant action of a free massive particle

$$S = -m \int d\tau$$

where $d\tau^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$. We can re-write this action as

$$S = -m \int \sqrt{g_{\mu\nu}(x) dx^\mu dx^\nu}.$$

To perform the variation ($S \to S' = S + \delta S$), we introduce parameter $\lambda$ such that $d\tau$ becomes

$$d\tau = \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda. \tag{A.0.21}$$

Then we have

$$\int d\tau = \int \left(\frac{d\tau}{d\lambda}\right) d\lambda$$
$$= \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda.$$

Now, we vary the path $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$, and only focus on $\delta S$. We will set this equal to zero

$$\begin{aligned}
\delta \int d\tau &= \int \delta\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda \\
&= \int \frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-\frac{1}{2}} \delta\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right) d\lambda \\
&= \int \frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-\frac{1}{2}} \left(\delta g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\right) d\lambda \\
&= \int \frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-\frac{1}{2}} \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta x^\alpha \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right) d\lambda.
\end{aligned}$$

We have used the fact that $g_{\mu\nu}$ is a function of $x$ and that the metric is symmetric in $\mu$ and $\nu$. Integrating by parts the term $2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$, we have

$$2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{d}{d\lambda}\left(2g_{\mu\nu}\delta x^\mu \frac{dx^\nu}{d\lambda}\right) - 2\delta x^\mu \frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right).$$

When performing the integral the first term will vanish because a total derivative integrates to zero. Therefore

$$\begin{aligned}
2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} &= -2\delta x^\mu \frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right) \\
&= -2\delta x^\mu \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2}
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta \int d\tau &= \int \frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-\frac{1}{2}} \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta x^\alpha \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2\delta x^\mu \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2}\right) d\lambda \\
&= \int \frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-\frac{1}{2}} \delta x^\mu \left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} - 2\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} - 2g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2}\right) d\lambda.
\end{aligned}$$

Using (A.0.21) we can re-write the right hand side in terms of $d\tau$

$$\delta \int d\tau = \int \frac{1}{2}\delta x^\mu \left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - 2\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right) d\tau.$$

Rewrite

$$\begin{aligned}
2\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} &= \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} + \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} \\
&= \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}.
\end{aligned}$$

In the second term on the last line we used the symmetry of the indices of $\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}$. Now, denote the $\tau$-derivative by $\dot{x}^\nu$ and set $\delta\int d\tau = \delta S = 0$ . Then we have

$$\begin{aligned}
\delta \int d\tau &= \int \frac{1}{2}\delta x^\mu \left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau} - 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right) d\tau \\
0 &= \int \frac{1}{2}\delta x^\mu \left(\partial_\mu g_{\alpha\nu}\dot{x}^\alpha\dot{x}^\nu - \partial_\alpha g_{\mu\nu}\dot{x}^\alpha\dot{x}^\nu - \partial_\nu g_{\mu\alpha}\dot{x}^\nu\dot{x}^\alpha - 2g_{\mu\nu}\ddot{x}^\nu\right) d\tau \\
&= -\int \delta x^\mu \left[\frac{1}{2}\left(-\partial_\mu g_{\alpha\nu}\dot{x}^\alpha\dot{x}^\nu + \partial_\alpha g_{\mu\nu}\dot{x}^\alpha\dot{x}^\nu + \partial_\nu g_{\mu\alpha}\dot{x}^\nu\dot{x}^\alpha\right) + g_{\mu\nu}\ddot{x}^\nu\right] d\tau.
\end{aligned}$$

For the right hand side to be zero, the integrand must be zero. This implies that

$$
\begin{aligned}
g_{\mu\nu}\ddot{x}^{\nu} &= \frac{1}{2}\Big(\partial_\mu g_{\alpha\nu}\dot{x}^{\alpha}\dot{x}^{\nu} - \partial_\alpha g_{\mu\nu}\dot{x}^{\alpha}\dot{x}^{\nu} - \partial_\nu g_{\mu\alpha}\dot{x}^{\nu}\dot{x}^{\alpha}\Big) \\
\Rightarrow g^{\mu\rho}g_{\mu\nu}\ddot{x}^{\nu} &= \frac{1}{2}g^{\mu\rho}\Big(\partial_\mu g_{\alpha\nu}\dot{x}^{\alpha}\dot{x}^{\nu} - \partial_\alpha g_{\mu\nu}\dot{x}^{\alpha}\dot{x}^{\nu} - \partial_\nu g_{\mu\alpha}\dot{x}^{\nu}\dot{x}^{\alpha}\Big) \\
\Rightarrow 0 &= \ddot{x}^{\rho} + \frac{1}{2}g^{\mu\rho}\Big(-\partial_\mu g_{\alpha\nu} + \partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha}\Big)\dot{x}^{\alpha}\dot{x}^{\nu}.
\end{aligned}
$$

If we compare the term in the bracket with (A.0.20), then this last expression can be written as

$$
0 = \ddot{x}^{\rho} + \Gamma^{\rho}_{\alpha\nu}\dot{x}^{\alpha}\dot{x}^{\nu}. \tag{A.0.22}
$$

This is the geodesic equation in terms of the Chrisoffel symbol. Let's do an example that will allow us to illustrate how to compute the Christoffels symbol.

**Example: Computation of the Christoffel symbols using the geodesic equation**

Consider a 2-dimension Euclidean metric in polar coordinates given by

$$
ds^2 = r^2 d\theta + dr^2, \tag{A.0.23}
$$

where $g_{rr} = 1$ and $g_{\theta\theta} = r^2$. Our goal is to use the geodesic equation to compute the Christoffel symbols. Starting from

$$
S = \int \sqrt{r^2\ddot{\theta}^2 + \dot{r}^2}\; d\tau \equiv \int L d\tau. \tag{A.0.24}
$$

The Euler-Lagrange equations are

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\
\Rightarrow \ddot{r} &= r\dot{\theta}^2 \\
\Rightarrow {\Gamma^{r}}_{\theta\theta} &= -r,
\end{aligned} \tag{A.0.25}
$$

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \\
\Rightarrow \frac{d}{d\tau}\left(r^2\dot{\theta}\right) &= r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \\
\Rightarrow \ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} \\
\Rightarrow {\Gamma^{\theta}}_{r\theta} + {\Gamma^{\theta}}_{\theta r} &= \frac{2}{r}.
\end{aligned} \tag{A.0.26}
$$

These Christoffel symbols are symmetric under the interchange of the lower indices ${\Gamma^{\theta}}_{\theta r} = {\Gamma^{\theta}}_{r\theta}$, so we conclude that

$$
{\Gamma^{\theta}}_{\theta r} = \frac{1}{r} = {\Gamma^{\theta}}_{r\theta}. \tag{A.0.27}
$$

Therefore the Christoffel symbols of the 2-dimensional flat metric in polar coordinates are

$$
{\Gamma^{r}}_{\theta\theta} = -r, \quad \text{and} \quad {\Gamma^{\theta}}_{\theta r} = \frac{1}{r} = {\Gamma^{\theta}}_{r\theta}. \tag{A.0.28}
$$

The property of the Christoffel symbol that allows us to interchange the two lower indices $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$ is called the **no torsion condition**. In a space where there is no torsion, we have

$$
\begin{aligned}
D_\mu V_\nu - D_\nu V_\mu &= \partial_\mu V_\nu - \Gamma^{\alpha}_{\mu\nu}V_\alpha - \partial_\nu V_\mu + \Gamma^{\alpha}_{\nu\mu}V_\alpha \\
&= \partial_\mu V_\nu - \partial_\nu V_\mu.
\end{aligned}
$$

### A.0.6 Killing vector equation

Symmetries simplify theories a lot. By knowing the symmetries of the theory, one can often tell how the theory behaves. When studying theories of gravity, symmetries are encoded in the Killing vectors. A map that preserves the metric when the metric has been mapped from one Riemannian manifold to another Riemannian manifold is called an isometry. Suppose we have a point $P$ at position $x^\mu$ and we infinitesimaly move this point in a certain direction to a new position $x^\mu + \zeta^\mu(x)$ at point $P'$. By requiring that the metric of this point is the same we get

$$g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} dx^\mu dx^\nu.$$

This tells us that

$$\tilde{g}_{\mu\nu}(x+\zeta) = g_{\mu\nu}(x)$$

$$\begin{aligned}
\Rightarrow \tilde{g}_{\mu\nu}(x+\zeta) &= g_{\alpha\beta}(x+\zeta)\left(\frac{\partial(x^\alpha+\zeta^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta+\zeta^\beta)}{\partial x^\nu}\right) = g_{\mu\nu}(x) \\
\Rightarrow g_{\mu\nu}(x) &= \left(g_{\alpha\beta}(x) + \zeta^\lambda \frac{\partial g_{\alpha\beta}(x)}{\partial x^\lambda} + O(\zeta^2)\right)\left(\delta^\alpha_\mu + \frac{\partial \zeta^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial \zeta^\beta}{\partial x^\nu}\right) \\
&= g_{\alpha\beta}(x)\left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\mu \frac{\partial \zeta^\beta}{\partial x^\nu} + \frac{\partial \zeta^\alpha}{\partial x^\mu}\delta^\beta_\nu + O(\zeta^2)\right) + \zeta^\lambda \frac{\partial g_{\alpha\beta}(x)}{\partial x^\lambda}\delta^\alpha_\mu \delta^\beta_\nu + O(\zeta) \\
\Rightarrow g_{\mu\nu}(x) &= g_{\mu\nu}(x) + g_{\mu\beta}(x)\frac{\partial \zeta^\beta}{\partial x^\nu} + g_{\alpha\nu}(x)\frac{\partial \zeta^\alpha}{\partial x^\mu} + \zeta^\lambda \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} + O(\zeta^2).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0 &= g_{\mu\beta}(x)\frac{\partial \zeta^\beta}{\partial x^\nu} + g_{\nu\alpha}(x)\frac{\partial \zeta^\alpha}{\partial x^\mu} + \zeta^\lambda \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} \\
&= \frac{\partial}{\partial x^\nu}(g_{\mu\beta}(x)\zeta^\beta) - \frac{\partial g_{\mu\beta}(x)}{\partial x^\nu}\zeta^\beta + \frac{\partial}{\partial x^\mu}(g_{\nu\alpha}(x)\zeta^\alpha) - \frac{\partial g_{\nu\alpha}(x)}{\partial x^\mu}\zeta^\alpha + \zeta^\lambda \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} \\
&= \partial_\nu \zeta_\mu + \partial_\mu \zeta_\nu - \left(\partial_\nu g_{\mu\beta}(x) + \partial_\mu g_{\nu\beta}(x) - \partial_\beta g_{\mu\nu}(x)\right) g^{\beta\rho}\zeta_\rho.
\end{aligned}$$

Using equation (A.0.20), this expression is reduced to

$$\begin{aligned}
0 &= \partial_\nu \zeta_\mu - \Gamma^\rho_{\nu\mu}\zeta_\rho + \partial_\mu \zeta_\nu - \Gamma^\rho_{\mu\nu}\zeta_\rho \\
\Rightarrow 0 &= D_\nu \zeta_\mu + D_\mu \zeta_\nu.
\end{aligned} \tag{A.0.29}$$

This is the Killing equation. Symmetries are encoded in the **Killing vectors**. A Killing vector is a vector that satisfies the Killing equation (A.0.29). Symmetries are extracted from the Killing vector. To understand how the Killing equation works and how to extract symmetries, we will do two examples

**Example 1: Computing the Killing vectors for the metric $ds^2 = dx^2 + dy^2$**

The Christoffel symbols of the metric

$$ds^2 = dx^2 + dy^2 \tag{A.0.30}$$

are all zero. Therefore the Killing equation becomes

$$D_i \zeta_j + D_j \zeta_i = \partial_i \zeta_j + \partial_j \zeta_i.$$

For $i = j = x$

$$\begin{aligned} \partial_x \zeta_x + \partial_x \zeta_x &= 0 \\ \partial_x \zeta_x &= 0. \end{aligned}$$

The solution to this differential equation is

$$\zeta_x = f(y). \tag{A.0.31}$$

Doing the same thing with the $y$ component we get

$$\zeta_y = g(x). \tag{A.0.32}$$

Now setting $i = x$ and $j = 1$

$$\begin{aligned} \partial_x g(x) + \partial_y f(y) &= 0 \\ \partial_x g(x) &= -\partial_y f(y). \end{aligned}$$

We see that the left hand side depends on $x$ while the right hand side depends on $y$, for the LHS to be equal to the RHS these two functions must be equal to a constant $k$. This implies

$$\partial_x g(x) = k.$$

The solution to this equation is

$$g(x) = kx + A \tag{A.0.33}$$

and

$$\begin{aligned} -\partial_y f(y) &= k \\ f(y) &= -ky + B. \end{aligned} \tag{A.0.34}$$

Now, the Killing vector is given by

$$\begin{aligned} \zeta &= \zeta^i \partial_i \\ &= \zeta^y \partial_y + \zeta^x \partial_x \\ &= \Big(kx + A\Big)\partial_y + \Big(-ky + B\Big)\partial_x \\ &= k(x\partial_y - y\partial_x) + B\partial_x + A\partial_y \\ &= kL_z + BP_x + AP_y \end{aligned}$$

where

$$P_x = \partial_x, \quad P_y = \partial_y \text{ and } L_z = (x\partial_y - y\partial_x).$$

This result tells us that there are two Killing vectors corresponding to translation ($BP_x$ and $AP_y$) in two linearly independent directions and the third Killing vector $L_z$ corresponding to rotation about the $z$ axis.

**Example 2: Computing the Killing vectors for the metric** $ds^2 = r^2 d\theta + dr^2$

As we saw above, the non-zero Christoffel symbols of the metric

$$ds^2 = r^2 d\theta + dr^2$$

are

$$\Gamma^r{}_{\theta\theta} = -r, \quad \text{and} \quad \Gamma^\theta{}_{\theta r} = \frac{1}{r} = \Gamma^\theta{}_{r\theta}.$$

The Killing equation is

$$\begin{aligned}
0 &= \partial_\nu \zeta_\mu - \Gamma^\rho_{\nu\mu}\zeta_\rho + \partial_\mu \zeta_\nu - \Gamma^\rho_{\mu\nu}\zeta_\rho \\
0 &= D_\nu \zeta_\mu + D_\mu \zeta_\nu
\end{aligned}$$

where $\mu$ and $\nu$ take the values $\theta$ and $r$. Setting $\mu = r$ and $\nu = r$, we obtain

$$\begin{aligned}
D_r\zeta_r + D_r\zeta_r &= 0, \\
D_r\zeta_r &= 0.
\end{aligned}$$

Expressing this equation in terms of the Christoffel symbols we use

$$\begin{aligned}
D_r\zeta_r &= \partial_r\zeta_r - \Gamma^\alpha{}_{rr}\zeta_\alpha \\
&= \partial_r\zeta_r - \Gamma^r{}_{rr}\zeta_r - \Gamma^\theta{}_{rr}\zeta_\theta \\
&= \partial_r\zeta_r.
\end{aligned}$$

Therefore we have

$$\partial_r\zeta_r = 0$$

and the solution to this equation is

$$\zeta_r = f(\theta). \tag{A.0.35}$$

Setting $\mu = \theta$ and $\nu = \theta$, we get

$$\begin{aligned}
D_\theta\zeta_\theta + D_\theta\zeta_\theta &= 0 \\
D_\theta\zeta_\theta &= 0.
\end{aligned}$$

Expressing this equation in terms of the Christoffel symbols

$$\begin{aligned}
D_\theta\zeta_\theta &= \partial_\theta\zeta_\theta - \Gamma^\alpha{}_{\theta\theta}\zeta_\alpha \\
&= \partial_\theta\zeta_\theta - \Gamma^r{}_{\theta\theta}\zeta_r - \Gamma^\theta{}_{\theta\theta}\zeta_\theta \\
&= \partial_\theta\zeta_\theta - \Gamma^r{}_{\theta\theta}\zeta_r - 0 \\
&= \partial_\theta\zeta_\theta + r\zeta_r.
\end{aligned}$$

Then, this simplifies to

$$\begin{aligned}
\partial_\theta\zeta_\theta + r\zeta_r &= 0 \\
\partial_\theta\zeta_\theta + rf(\theta) &= 0.
\end{aligned}$$

The solution to this differential equation is

$$\zeta_\theta = -r\int d\theta' f(\theta') + \tilde{f}(r). \tag{A.0.36}$$

Now setting $\mu = r$ and $\nu = \theta$, the Killing equation becomes

$$D_r\zeta_\theta + D_\theta\zeta_r = 0.$$

In terms of the Christoffel symbols this equation simplifies to

$$\begin{aligned}
\partial_r\zeta_\theta + \partial_\theta\zeta_r - \frac{2}{r}\zeta_\theta &= 0 \\
\partial_r\Big(-r\int d\theta' f(\theta') + \tilde{f}(r)\Big) + \partial_\theta f(\theta) - \frac{2}{r}\Big(-r\int d\theta' f(\theta') + \tilde{f}(r)\Big) &= 0 \\
-\int d\theta' f(\theta') + \partial_r\tilde{f}(r) + \partial_\theta f(\theta) + 2\int d\theta' f(\theta') - \frac{2}{r}\tilde{f}(r) &= 0 \\
-\int d\theta' f(\theta') - \partial_\theta f(\theta) &= \partial_r\tilde{f}(r) - \frac{2}{r}\tilde{f}(r).
\end{aligned} \tag{A.0.37}$$

Again, we see that the LHS and the RHS are equal to a constant $k$. Lets first equate the LHS to $k$ and multiply by $\partial_\theta$, to get

$$\int d\theta' f(\theta') + \partial_\theta f(\theta) = -k$$
$$f(\theta) + \partial_\theta^2 f(\theta) = 0.$$

The solution to this differential equation is

$$f(\theta) = A\cos\theta + B\sin\theta.$$

Substitute the solution back into the differential equation to find $k$

$$\int d\theta'(A\cos\theta' + B\sin\theta') + \partial_\theta(A\cos\theta' + B\sin\theta') = -k$$
$$A\sin\theta - B\cos\theta - A\sin\theta + B\cos\theta = k$$
$$\Rightarrow k = 0.$$

Evaluating the RHS of (A.0.37), we find

$$\partial_r f(r) - \frac{2}{r} f(r) = k$$
$$\partial_r f(r) - \frac{2}{r} f(r) = 0.$$

The solution to this differential equation is

$$f(r) = Cr^2. \tag{A.0.38}$$

Therefore, we have

$$\zeta_r = A\cos\theta + B\sin\theta$$
$$\zeta_\theta = -r(A\sin\theta - B\cos\theta) + Cr^2.$$

Using our metric we have

$$\zeta_r = \zeta^r$$
$$\zeta_\theta = r^2\zeta^\theta.$$

The Killing vector is then given by

$$\begin{aligned}
\zeta &= \zeta^\mu \partial_\mu \\
&= \zeta^r \partial_r + \zeta^\theta \partial_\theta \\
&= (A\cos\theta + B\sin\theta)\partial_r + \frac{1}{r^2}(-r(A\sin\theta - B\cos\theta) + Cr^2)\partial_\theta \\
&= A(\cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta) + B(\sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta) + C\partial_\theta \\
&= AL_x + BL_y + CP_z.
\end{aligned}$$

These Killing vectors correspond to two rotations and one translation. Thus we reproduce the results of example 1 as we must.

**Conserved quantity from the Killing vectors**

We now know that symmetries of a geometrical theory are hidden in the Killing vectors. What are the conserved quantities associated to these symmetries? We can find these conserved quantities by using the geodesic equation, since these quantities are conserved along the geodesic. The conserved quantity is given by the equation

$$\frac{d}{d\tau}\left(\zeta_\mu \frac{dx^\mu}{d\tau}\right) = 0. \tag{A.0.39}$$

We can show that this equation is true. Use the geodesic equation as follows

$$\frac{d}{d\tau}\left(\zeta_\mu \frac{dx^\mu}{d\tau}\right) = \frac{d\zeta_\mu}{d\tau}\frac{dx^\mu}{d\tau} + \zeta_\mu \ddot{x}^\mu$$

where we have adopted the notation $\frac{dx^\mu}{d\tau} = \dot{x}^\mu$. Using the geodesic equation (A.0.22) and the fact that $\zeta$ is a function of $x^\mu$ which is parametrized by $\tau$ i.e. $\zeta_\mu = \zeta_\mu(x(\tau))$, we obtain

$$\begin{aligned}
\frac{d\zeta_\mu}{d\tau}\frac{dx^\mu}{d\tau} + \zeta_\mu \ddot{x}^\mu &= \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{d\zeta_\mu}{dx^\nu} - \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \zeta_\mu \\
&= \dot{x}^\alpha \dot{x}^\mu \partial_\alpha \zeta_\mu - \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \zeta_\mu \\
&= \dot{x}^\alpha \dot{x}^\mu \left(\partial_\alpha \zeta_\mu - \Gamma^\rho_{\alpha\mu}\zeta_\rho\right) \\
&= \dot{x}^\alpha \dot{x}^\mu D_\alpha \zeta_\mu \\
&= \frac{1}{2}\left(\dot{x}^\alpha \dot{x}^\mu D_\alpha \zeta_\mu + \dot{x}^\mu \dot{x}^\alpha D_\mu \zeta_\alpha\right) \\
&= \frac{1}{2}\dot{x}^\alpha \dot{x}^\mu \left(D_\alpha \zeta_\mu + D_\mu \zeta_\alpha\right) \\
&= 0.
\end{aligned}$$

where in the second last line we used the Killing equation. This conserved quantity is the metric product between the Killing vector and the geodesic tangent vector. Using the form of geodesic equation

$$0 = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) + \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{A.0.40}$$

we see that when the second term in the above equation is zero i.e.

$$0 = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) + 0 \tag{A.0.41}$$

then this equation is equivalent to (A.0.39). This tells us that the conserved quantities calculated from (A.0.39) are the same conserved quantities that will be calculated from (A.0.41). We will compute two examples of conserved quntities associated to the Killing vector. For these examples we will choose the metrics

$$ds^2 = dr^2 + r^2 d\theta$$

and

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

**Example: Computing the conserved quantities from the Killing vectors of the metric** $ds^2 = dr^2 + r^2 d\theta$

Recall that the non-zero Christoffel symbols for this metric were ${\Gamma^r}_{\theta\theta}$, ${\Gamma^\theta}_{r\theta}$ and ${\Gamma^\theta}_{\theta r}$. Since this metric is diagonal, we will use the non zero component $g_{rr} = 1$ and $g_{\theta\theta} = r^2$ to compute the conserved quantities. Setting $\mu = \nu = \theta$, in (A.0.40) we have

$$0 = \frac{d}{d\tau}\left(r^2 \frac{d\theta}{d\tau}\right).$$

We know that the conserved quantity is the one inside the bracket. Solving this we get

$$const = r^2 \frac{d\theta}{d\tau}.$$

Denoting $\frac{d\theta}{d\tau} = \omega$ and calling the constant $l$, we learn that

$$l = r^2 \omega. \tag{A.0.42}$$

This is the equation for the angular momentum per unit mass ($L = \frac{l}{m}$). So, this last equation tells us that the **angular momentum is the conserved quantity**.

**Example: Computing the conserved quantities from the Killing vectors of the Minkowski metric**

Using the Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

we get

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\right) + \frac{1}{2}\partial_\mu \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\ &= \frac{d}{d\tau}\left(\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\right). \end{aligned}$$

We will compute the conserved quantities by breaking the calculation into four parts.

1. **Setting** $\mu = \nu = t$

   We have

   $$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\eta_{tt}\frac{dt}{d\tau}\right) \\ const &= \frac{dt}{d\tau}. \end{aligned}$$

   We set this constant to be equal to $e$ which is interpreted as the relativistic energy per unit mass ($e = \frac{E}{m}$). The conserved quantity is the **energy per unit mass**.

2. **Setting** $\mu = \nu = x$

   We have

   $$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\eta_{xx}\frac{dx}{d\tau}\right) \\ const &= \frac{dx}{d\tau}. \end{aligned}$$

   We denote this constant by $p_x$ and it tells us that the **momentum per unit mass along the $x$-axis is conserved**.

3. **Setting** $\mu = \nu = y$

   Following the same procedure as above we find that

   $$p_y = \frac{dy}{d\tau} \tag{A.0.43}$$

   and this tells us that the **momentum per unit mass along the $y$ axis is conserved**

4. **Setting $\mu = \nu = z$**

   Again applying the same procedure we find that

$$p_z = \frac{dz}{d\tau} \tag{A.0.44}$$

   and this implies that the **momentum per unit mass along the $z$ axis is conserved.**

A big lesson that has been learned here is that searching for symmetries of the metric reduces to solving the Killing equation. The Killing vectors can be used to find conserved quantities.

### A.0.7 Spin connection

We have seen that the covariant derivative of a vector $V_\mu$ is given by

$$D_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\alpha{}_{\nu\mu} V_\alpha.$$

The first term represents the normal derivative while the second term describes the way in which basis $e_\mu$ changes along the manifold. This was easily read from the fact that the upper index $\alpha$ on the Christoffel was contracted with the index of the vector which describes the direction of this vector. Suppose we now want to take a covariant derivative of $V_\mu^a$. The covariant derivative of $V_\mu^a$ will have an additional term which should tell us how the vector changes in the direction represented by index $a$. Therefore the covariant derivative of $V_\mu^a$ is given by

$$D_\nu V_\mu^a = \partial_\nu V_\mu^a - \Gamma^\alpha{}_{\nu\mu} V_\alpha^a + w_\nu{}^a{}_b V_\mu^b$$

where $w_\nu{}^a{}_b$ is called the spin connection. We would like to express this spin connection in terms of quantities we know.

#### The vierbein

The vierbein (also known as the tetrad) are orthonormal vector fields that are defined on a Lorentzian manifold. The vierbein basis is an orthonormal basis independent of the coordinates. Therefore we can choose an orthonormal basis that is independent of the choice of coordinates. From a local perspective, any vector can be expressed as a linear combination of the fixed tetrad basis vectors at that point. We can write any coordinate basis in terms of the tetrad as the following linear combination

$$\hat{e}_{(\mu)}(x) = e_\mu{}^a(x)\hat{e}_{(a)} \tag{A.0.45}$$

$$\hat{e}_{(a)}(x) = e_\mu{}^a(x)\hat{e}_{(\mu)} \tag{A.0.46}$$

where $e_\mu{}^a$ is the vierbein and the components of this vierbein form a $4 \times 4$ invertible matrix. The inverse of $e_\mu{}^a$ is given by $e^\mu{}_a$ and they satisfy the following relations

$$e_\mu{}^a e^\mu{}_b = \delta_b^a \quad \text{and} \quad e_\mu{}^a e^\nu{}_a = \delta_\mu^\nu. \tag{A.0.47}$$

The vierbein are related to the metric in the following way

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_{a\nu}(x), \tag{A.0.48}$$

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x)\eta_{ab}, \tag{A.0.49}$$

$$\eta^{ab} = e_\mu{}^a(x) e^{b\mu}(x), \tag{A.0.50}$$

$$\eta_{ab} = g_{\mu\nu} e_a{}^\mu(x) e_b{}^\nu(x). \tag{A.0.51}$$

We will see that the vierbein plays a huge role in simplifying our calculations. It is a good place to start investigating how the vierbein transforms.

**Vierbein transformation**

Recall that we can write any vector $\vec{V}$ in terms of component of coordinate and non-coordinate orthonormal basis

$$\vec{V} = V^{\mu}\hat{e}_{(\mu)} = V^{a}\hat{e}_{(a)}.$$

So, the components of the vector are related by the vierbein field transformation as follows

$$V^{a} = e_{\mu}{}^{a}V^{\mu}$$
$$V^{\mu} = e^{\mu}{}_{a}V^{a}.$$

A tensor that has more than one index transforms as follows

$$\begin{aligned} V^{a}{}_{b} &= e_{\mu}{}^{a}V^{\mu}{}_{b} \\ &= e^{\nu}{}_{b}V_{\nu}{}^{a} \\ &= e_{\mu}{}^{a}e^{\nu}{}_{b}V^{\mu}{}_{\nu}. \end{aligned}$$

We transform the vierbein to inverse vierbein using the metric tensors in the following way

$$e^{\mu}{}_{a} = g^{\mu\nu}\eta_{ab}e_{\nu}{}^{b}.$$

The Lorentz transformation for $e_{(a)}$ is given by

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda^{a}_{a'}\hat{e}_{(a)}.$$

So far, we have been exploring the vierbein and it properties. To convince our selves that we understand them, we will do an example that will allow us to evaluate them.

**Example: Compute the vierbein of the spherical metric.**

The metric in spherical coordinates is given by

$$ds^2 = dr^2 + r^2 d\theta + r^2 \sin\theta d\phi. \tag{A.0.52}$$

The coordinate basis vectors for this metric are

$$e_r = \partial_r, \quad e_\theta = \partial_\theta \;\text{ and }\; e_\phi = \partial_\phi. \tag{A.0.53}$$

Not all of these basis vectors have unit length and also they do not have the same dimensions. These basis vectors satisfy the relation

$$e_\mu \cdot e_\nu = g_{\mu\nu}. \tag{A.0.54}$$

Now, let us compute the length of these vectors. Notice that the non-zero components of the metric tensor (A.0.52) are

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2 \;\text{ and }\; g_{\phi\phi} = r^2\sin^2\theta.$$

The length of these vectors are

$$\begin{aligned} e_r \cdot e_r &= 1 \quad \Rightarrow |e_r| = \sqrt{g_{rr}} = 1, \\ e_\theta \cdot e_\theta &= r^2 \quad \Rightarrow |e_\theta| = \sqrt{g_{\theta\theta}} = r, \\ e_\phi \cdot e_\phi &= r^2\sin^2\theta \quad \Rightarrow |e_\phi| = \sqrt{g_{\phi\phi}} = r\sin\theta. \end{aligned}$$

Lets denote the orthonormal basis by $e_{\hat{\mu}}$. The basis vectors satisfy

$$e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \eta_{\mu\nu}.$$

In spherical coordinates this basis becomes

$$e_{\hat{r}} = \partial_r, \quad e_{\hat{\theta}} = \frac{1}{r}\partial_\theta \text{ and } e_{\hat{\phi}} = \frac{1}{r\sin\theta}\partial_\phi.$$

Now we are in a position to compute the vierbein by using the equation

$$e_{\hat{a}} = e_{\hat{a}}{}^{b} e_b.$$

Choosing $\hat{a} = r$

$$\begin{aligned}
e_{\hat{r}} &= e_{\hat{r}}{}^{b} e_b \\
e_{\hat{r}} &= e_{\hat{r}}{}^{r} e_r + e_{\hat{r}}{}^{\theta} e_\theta + e_{\hat{r}}{}^{\phi} e_\phi \\
\partial_r &= e_{\hat{r}}{}^{r} \partial_r + e_{\hat{r}}{}^{\theta} \partial_\theta + e_{\hat{r}}{}^{\phi} \partial_\phi.
\end{aligned}$$

Reading off the values of the vierbein we find

$$e_{\hat{r}}{}^{r} = 1, \quad e_{\hat{r}}{}^{\theta} = e_{\hat{r}}{}^{\phi} = 0.$$

Choosing $\hat{a} = \theta$

$$\begin{aligned}
e_{\hat{\theta}} &= e_{\hat{\theta}}{}^{b} e_b \\
e_{\hat{\theta}} &= e_{\hat{\theta}}{}^{r} e_r + e_{\hat{\theta}}{}^{\theta} e_\theta + e_{\hat{\theta}}{}^{\phi} e_\phi \\
\frac{1}{r}\partial_\theta &= e_{\hat{\theta}}{}^{r} \partial_r + e_{\hat{\theta}}{}^{\theta} \partial_\theta + e_{\hat{\theta}}{}^{\phi} \partial_\phi.
\end{aligned}$$

Reading off the values of the vierbein, we have

$$e_{\hat{\theta}}{}^{\theta} = \frac{1}{r}, \quad e_{\hat{\theta}}^{r} = e_{\hat{\theta}}{}^{\phi} = 0.$$

Finally, choose $\hat{a} = \phi$

$$\begin{aligned}
e_{\hat{\phi}} &= e_{\hat{\phi}}{}^{b} e_b \\
e_{\hat{\phi}} &= e_{\hat{\phi}}{}^{r} e_r + e_{\hat{\phi}}{}^{\theta} e_\theta + e_{\hat{\phi}}{}^{\phi} e_\phi \\
\frac{1}{r\sin\theta}\partial_\phi &= e_{\hat{\phi}}{}^{r} \partial_r + e_{\hat{\phi}}{}^{\theta} \partial_\theta + e_{\hat{\phi}}{}^{\phi} \partial_\phi
\end{aligned}$$

Reading off the vierbein values, we get

$$e_{\hat{\phi}}{}^{\phi} = \frac{1}{r\sin\theta}, \quad e_{\hat{\phi}}{}^{r} = e_{\hat{\phi}}{}^{\phi} = 0.$$

We finally have

$$e_{\hat{a}}{}^{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}. \tag{A.0.55}$$

**Expression for the spin connection**

The expression for the spin connection $w_{\nu}{}^{a}{}_{b}$ can be derived by first expressing the metric $g_{\mu\nu}$ in terms of the vierbein

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}.$$

We know that the covariant derivative of the metric must be zero because the affine connection is chosen to ensure the covariant derivative of the metric vanishes

$$D_\alpha g_{\mu\nu} = 0.$$

We have

$$\begin{aligned}
D_\alpha g_{\mu\nu} &= D_\alpha(e^a_\mu e^b_\nu \eta_{ab}) \\
&= (D_\alpha e^a_\mu)e^b_\nu \eta_{ab} + e^a_\mu(D_\alpha e^b_\nu)\eta_{ab} + e^a_\mu e^b_\nu(D_\alpha \eta_{ab}) \\
&= (D_\alpha e^a_\mu)e^b_\nu \eta_{ab} + e^a_\mu(D_\alpha e^b_\nu)\eta_{ab}, \qquad \text{re-label } \mu \leftrightarrow \nu \text{ and } a \leftrightarrow b \\
&= 2(D_\alpha e^a_\mu)e^b_\nu \eta_{ab}.
\end{aligned}$$

This implies that

$$D_\alpha e^a_\mu = 0.$$

The left hand side of this equation can be expanded using the relevant connections, as follows

$$\begin{aligned}
D_\alpha e^a_\mu = \partial_\alpha e^a_\mu - \Gamma^\sigma_{\alpha\mu} e^a_\sigma + \omega_\alpha{}^a{}_b e^b_\mu &= 0 \\
\omega_\alpha{}^a{}_b e^b_\mu e^\mu_{b'} &= -\partial_\alpha e^a_\mu e^\mu_{b'} + \Gamma^\sigma_{\alpha\mu} e^a_\sigma e^\mu_{b'} \\
\omega_\alpha{}^a{}_{b'} &= e^\mu{}_{b'}\bigg( - \partial_\alpha e^a_\mu + \Gamma^\sigma_{\mu\alpha} e^a_\sigma\bigg).
\end{aligned} \tag{A.0.56}$$

This determines of the spin connection, telling us that we can compute them by using the Christoffel symbols and the tetrad (or vierbein). What happens when we take the covariant derivative of the Minkowski metric $\eta_{ab}$? We know that $D_\mu \eta_{ab} = 0$ and that all the Christoffel symbols of this metric are *zero*. This implies that

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} + \omega_\mu{}^{ac}\eta_{cb} + \omega_\mu{}^{bc}\eta_{ac} = 0.$$

Notice that we have two spin connections in this equation, this is because the $\eta_{ab}$ carries two Roman indices and each index must transform. Simplifying this we get

$$\begin{aligned}
0 &= \omega_\mu{}^{ac}\eta_{cb} + \omega_\mu{}^{bc}\eta_{ac} \\
\omega_\mu{}^{ab} &= -\omega_\mu{}^{ba}.
\end{aligned}$$

This tells us that the connection $\omega_\mu{}^{ab}$ is anti symmetric in the Lorentz indices.

**Spin connection transformation**

So far we have been exploring the properties of the spin connection but we haven't mentioned anything about how this spin connection transforms. To determine how the spin connection transforms, consider a vector that carries a Lorentz index $V^a$. Taking its covariant derivative, gives

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b.$$

Now require that the covariant derivative of this equation is Lorentz invariant, that is

$$\begin{aligned}
D_\mu V^{a'} &\to D_\mu(\Lambda^{a'}{}_a V^a) \\
&= (D_\mu \Lambda^{a'}{}_a)V^a + \Lambda^{a'}{}_a(D_\mu V^a).
\end{aligned}$$

This covariant derivative is Lorentz invariant

$$D_\mu V^{a'} = \Lambda^{a'}{}_a(D_\mu V^a)$$

if and only if the covariant derivative of the Lorentz transformation $\Lambda^{a'}{}_a$ is zero

$$D_\mu \Lambda^{a'}{}_a = 0. \tag{A.0.57}$$

(A.0.57) is a constraint equation that allows us to determine how the spin connection behaves under a Lorentz transformation. Using (A.0.57)

$$D_\mu \Lambda^{a'}{}_b = \partial_\mu \Lambda^{a'}{}_b + \omega_\mu{}^{a'}{}_c \Lambda^c{}_b - \omega_\mu{}^c{}_b \Lambda^{a'}{}_c = 0.$$

Multiply by $\Lambda^b{}_{b'}$, to get

$$\begin{aligned}\Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b + \Lambda^b{}_{b'}\omega_\mu{}^{a'}{}_c\Lambda^c{}_b - \Lambda^b{}_{b'}\omega_\mu{}^c{}_b\Lambda^{a'}{}_c = 0 \\ \omega_\mu{}^{a'}{}_c\delta^c_{b'} = \Lambda^b{}_{b'}\omega_\mu{}^c{}_b\Lambda^{a'}{}_c - \Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b.\end{aligned}$$

Therefore the spin connection transforms as

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^b{}_{b'}\omega_\mu{}^c{}_b\Lambda^{a'}{}_c - \Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b. \tag{A.0.58}$$

Now that we know how the spin connection transforms, we would like to interpret our answer. We will see that it is possible to identify spin connection with a gauge field for the local Lorentz group. This is motivated by comparing how the gauge fields transform. Consider the gauge field of a $U(1)$ gauge theory (QED) which is given by $A_\mu$. This gauge field transforms in the following way

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi.$$

Using the fact that

$$\begin{aligned}1 &= UU^\dagger \\ 0 &= \partial_\mu(UU^\dagger) \\ 0 &= (\partial_\mu U)U^\dagger + U\partial_\mu U^\dagger \\ \Rightarrow (\partial_\mu U)U^\dagger &= U\partial_\mu U^\dagger.\end{aligned}$$

Setting $e^{i\chi} = U$ we see that the $U(1)$ gauge field transforms as

$$\begin{aligned}A_\mu &= UA_\mu U^\dagger - (\partial_\mu U)U^\dagger \\ &= UA_\mu U^\dagger + U\partial_\mu U^\dagger.\end{aligned} \tag{A.0.59}$$

Comparing (A.0.59) with (A.0.58) which is given by

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^b{}_{b'}\omega_\mu{}^c{}_b\Lambda^{a'}{}_c - \Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b$$

we see that the spin connection is indeed the gauge field for the local Lorentz group: it transforms in the same way as the gauge field of the $U(1)$ group does. To ensure we understand the spin connection, we will consider an example in which we compute the spin connection of a given metric.

**Example: Spin connection**

Consider the metric

$$ds^2 = dr^2 + r^2 d\theta^2.$$

The veirbein of this metric is

$$e^\mu_a = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r}\end{pmatrix}. \tag{A.0.60}$$

We will compute the spin connection using equation (A.0.56)

$$\omega_\alpha{}^a{}_b = e^\mu{}_b\bigg( - \partial_\alpha e^a_\mu + \Gamma^\sigma_{\mu\alpha}e^a_\sigma\bigg).$$

We will break the calculation into four pieces according to the value of $a$ and $b$

(I) $a = r$ **and** $b = r$

To compute the non-zero spin connection we need $e^{\mu}_{\ b'}$ non-zero, which implies $\mu = r$

$$\begin{aligned}\omega_{\alpha\ r}^{\ r} &= e^{r}_{\ r}\Big( - \partial_\alpha e^r_r + \Gamma^{\sigma}_{r\alpha} e^r_\sigma \Big)\\ &= e^{r}_{\ r}\Big( - \partial_\alpha e^r_r + \Gamma^{r}_{\alpha r} e^r_r + \Gamma^{\theta}_{\alpha r} e^r_\theta \Big)\\ &= 0.\end{aligned}$$

Therefore we have

$$\omega_{r\ r}^{\ r} = \omega_{\theta\ r}^{\ r} = 0.$$

(II) $a = \theta$ **and** $b = \theta$
For a non-zero spin connection $\mu = \theta$

$$\begin{aligned}\omega_{\alpha\ \theta}^{\ \theta} &= e^{\theta}_{\ \theta}\Big( - \partial_\alpha e^\theta_\theta + \Gamma^{\sigma}_{\theta\alpha} e^\theta_\sigma \Big)\\ \omega_{\alpha\ \theta}^{\ \theta} &= e^{\theta}_{\ \theta}\Big( - \partial_\alpha e^\theta_\theta + \Gamma^{\theta}_{\theta\alpha} e^\theta_\theta + \Gamma^{r}_{\theta\alpha} e^\theta_r \Big).\end{aligned}$$

Therefore we have

$$\begin{aligned}\omega_{\theta\ \theta}^{\ \theta} &= 0\\ \omega_{r\ \theta}^{\ \theta} &= \frac{1}{r}\Big( - \partial_r r + \frac{1}{r} r\Big)\\ &= \frac{1}{r}\Big( -1 + 1\Big)\\ &= 0.\end{aligned}$$

(III) $a = r$ **and** $b = \theta$

In this case $\mu = \theta$

$$\omega_{\alpha\ \theta}^{\ r} = e^{\theta}_{\ \theta}\Big( - \partial_\alpha e^r_\theta + \Gamma^{r}_{\theta\alpha} e^r_r + \Gamma^{\theta}_{\theta\alpha} e^r_\theta \Big).$$

Therefore

$$\begin{aligned}\omega_{r\ \theta}^{\ r} &= 0\\ \omega_{\theta\ \theta}^{\ r} &= \frac{1}{r}(-r)\\ &= -1.\end{aligned}$$

(III) $a = \theta$ **and** $b = r$

For this case $\mu = r$

$$\omega_{\alpha\ r}^{\ \theta} = e^{r}_{\ r}\Big( - \partial_\alpha e^\theta_r + \Gamma^{r}_{r\alpha} e^\theta_r + \Gamma^{\theta}_{r\alpha} e^\theta_\theta \Big).$$

Therefore we have

$$\begin{aligned}\omega_{r\ r}^{\ \theta} &= 0\\ \omega_{\theta\ r}^{\ \theta} &= (0 + 0 + \Gamma^{\theta}_{r\theta} e^\theta_\theta)\\ &= 1.\end{aligned}$$

The spin connections are

$$\omega_{r\ r}^{\ r} = \omega_{\theta\ r}^{\ r} = 0, \quad \omega_{\theta\ \theta}^{\ \theta} = \omega_{r\ \theta}^{\ \theta} = 0, \quad \omega_{r\ \theta}^{\ r} = 0, \quad \omega_{\theta\ \theta}^{\ r} = -1, \quad \omega_{r\ r}^{\ \theta} = 0 \ \text{ and } \ \omega_{\theta\ r}^{\ \theta} = 1.$$

### A.0.8 Spinors

A spinor field $\Psi_\alpha$ is an irreducible representation under boosts and rotations (under Lorentz transformations). Earlier, we saw that the covariant derivative of a vector $V^a$ that caries a Lorentz index $a$ is given by

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b.$$

The index $\alpha$ carried by the spinor field also transforms so that the covariant derivative of the spinor field is given by

$$D_\mu \Psi_\alpha = \partial_\mu \Psi_\alpha + (\omega_\mu)_{ab}\frac{[\Gamma^a, \Gamma^b]}{2}\Psi_\alpha.$$

The term that multiplies the spin connection (i.e. $\frac{[\Gamma^a, \Gamma^b]}{2}$) is the generator of the Lorentz group for the spinor representaion. The generators obey the Lorentz algebra which is given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = (\eta^{\sigma\mu}J^{\rho\nu} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\rho\mu}J^{\sigma\nu} - \eta^{\nu\rho}J^{\mu\sigma}).$$

To prove that $[\Gamma^a, \Gamma^b]$ satisfies the Lorentz algebra, lets first define $\Gamma^a\Gamma^b = \Gamma^{ab}$. We will work in 4-dimensional flat space where the gamma matrices are denoted by $\gamma$. Through out this proof we will use the identities

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\nu\mu}, \tag{A.0.61}$$

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \qquad \text{for } \mu \neq \nu. \tag{A.0.62}$$

Let $L^{\mu\nu}$ be given by

$$L^{\mu\nu} = \frac{1}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] \tag{A.0.63}$$

$$= \gamma^\mu\gamma^\nu \qquad \text{for } \mu \neq \nu \tag{A.0.64}$$

then

$$L^{\mu\nu}L^{\sigma\rho} = \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho \qquad \text{for } \mu \neq \nu \text{ and } \sigma \neq \rho. \tag{A.0.65}$$

Using the above identities we get

$$\begin{aligned}
L^{\mu\nu}L^{\sigma\rho} &= \gamma^\mu(-\gamma^\sigma\gamma^\nu + 2\eta^{\sigma\nu})\gamma^\rho \\
&= -(-\gamma^\sigma\gamma^\mu + 2\eta^{\sigma\mu})\gamma^\nu\gamma^\rho + 2\eta^{\sigma\nu}\gamma^\mu\gamma^\rho \\
&= \gamma^\sigma\gamma^\mu(-\gamma^\rho\gamma^\nu + 2\eta^{\rho\nu}) - 2\eta^{\sigma\mu}\gamma^\nu\gamma^\rho + 2\eta^{\sigma\nu}\gamma^\mu\gamma^\rho \\
&= -\gamma^\sigma(-\gamma^\rho\gamma^\mu + 2\eta^{\rho\mu})\gamma^\nu + 2\eta^{\rho\nu}\gamma^\sigma\gamma^\mu - 2\eta^{\sigma\mu}\gamma^\nu\gamma^\rho + 2\eta^{\sigma\nu}\gamma^\mu\gamma^\rho \\
&= L^{\sigma\rho}L^{\mu\nu} - 2\eta^{\rho\mu}\gamma^\sigma\gamma^\nu + 2\eta^{\rho\nu}\gamma^\sigma\gamma^\mu - 2\eta^{\sigma\mu}\gamma^\nu\gamma^\rho + 2\eta^{\sigma\nu}\gamma^\mu\gamma^\rho.
\end{aligned}$$

Notice that by using (A.0.64) we can write

$$2\eta^{\nu\rho}\gamma^\sigma\gamma^\mu = 2\eta^{\nu\rho}L^{\sigma\mu}$$

and this is only applicable when $\sigma \neq \mu$. This implies that we make a mistake when $\nu = \rho$ and $\sigma = \mu$, but this mistake is cancelled by the term

$$-2\eta^{\sigma\mu}\gamma^\nu\gamma^\mu \to -2\eta^{\sigma\mu}L^{\nu\mu}.$$

Finally we get

$$[L^{\mu\nu}, L^{\sigma\rho}] = 2\eta^{\nu\rho}L^{\sigma\mu} - 2\eta^{\sigma\mu}L^{\nu\rho} + 2\eta^{\sigma\nu}L^{\mu\rho} - 2\eta^{\mu\rho}L^{\sigma\nu}. \tag{A.0.66}$$

Therefore we see that the $L$'s satisfy the Lorentz algebra which implies that they are Lorentz generators. The covariant derivative is

$$D_\mu\Psi_\alpha = \partial_\mu\Psi_\alpha + (\omega_\mu)_{ab}\Gamma^{ab}\Psi_\alpha. \tag{A.0.67}$$

We have managed to defined a spinor and its covariant derivative. Now let investigate how the spinor covariant derivative transforms.

**Spinor transformation**

In this subsection, will focus on how spinors and their covariant derivative transforms. To be sure that we are arriving at sensible results, we will confirm our results by analogy with the $U(1)$ gauge theory. Although we would not expect our results to be the same as the $U(1)$ theory, we expect at least the structure to be same. Let's first investigate how the spinor transforms. Recall that in the $U(1)$ gauge theory the field $\phi$ transforms as

$$\phi(x) \to \phi'(x) = U(x)\phi(x)$$

where $U(x)$ is a unitary operator which is usually represented by $U(x) = e^{i\beta(x)}$ where $\beta(x)$ is real. In this case the $U(1)$ theory tells us that we should expect that when we transform the spinor field, the new spinor field must contain the old spinor field multiplied by the group transformation element

$$\Psi(x) \to \Psi(x)' = U(\Lambda)\Psi(x).$$

We would like to write the spinor transformation in terms of the exponential. We can achieve this by expanding $\Lambda$ as follows

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \lambda^\mu_\nu + O(\lambda^2).$$

To trust this expansion, we must make sure that it satisfies the property

$$\eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = \eta_{\rho\sigma}. \tag{A.0.68}$$

Let check if this is satisfied

$$\begin{aligned}
\eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma &= \eta_{\mu\nu}(\delta^\mu_\rho + \lambda^\mu_\rho)(\delta^\nu_\sigma + \lambda^\nu_\sigma) \\
&= \eta_{\mu\nu}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\rho\lambda^\nu_\sigma + \lambda^\mu_\rho\delta^\nu_\sigma + O(\lambda^2)) \\
&= \eta_{\rho\sigma} + \lambda_{\sigma\rho} + \lambda_{\rho\sigma}.
\end{aligned}$$

This last equation is equivalent to (A.0.68) if and only if $\lambda_{\sigma\rho}$ is anti-symmetric (i.e. $\lambda_{\sigma\rho} = -\lambda_{\rho\sigma}$). Now we can write

$$U(\Lambda) = e^{-\frac{1}{2}\lambda_{ab}(x)\gamma^{ab}}.$$

Therefore our spinor transformation is given by

$$\Psi(x) \to \Psi(x)' = e^{-\frac{1}{2}\lambda_{ab}(x)\gamma^{ab}}\Psi(x). \tag{A.0.69}$$

Now look at how the covariant derivative of a spinor transforms. Again, we will first look at the covariant derivative of the $U(1)$ field and see how it transforms. This covariant derivative is given by

$$D_\mu\phi(x) = \partial_\mu\phi + A_\mu\phi(x)$$

and (A.0.59) tells us how $A_\mu$ transforms. Now, considering the covariant derivative of the $U(1)$ field, we get

$$\begin{aligned}
D_\mu\phi \to [D_\mu\phi]' &= \partial_\mu(U\phi) + (UA_\mu U^\dagger - (\partial_\mu U)U^\dagger)U\phi \\
&= (\partial_\mu U)\phi + U(\partial_\mu\phi) + (UA_\mu U^\dagger U - (\partial_\mu U)U^\dagger U)\phi \\
&= (\partial_\mu U)\phi + U(\partial_\mu\phi) + (UA_\mu - \partial_\mu U)\phi \\
&= U\partial_\mu\phi + UA_\mu\phi \\
&= U(\partial_\mu\phi + A_\mu\phi) \\
&= UD_\mu\phi.
\end{aligned}$$

When we transform the covariant derivative of the $U(1)$ field we get the same covariant derivative multiplied by $U$. Therefore the covariant derivative transforms like the field $\phi(x)$. So we expect the same structure when we transform the covariant derivative of the spinor. We expect that the covariant derivative of the spinor transforms like a spinor and it turns out that this is correct

$$D_\mu\Psi_\alpha \to [D_\mu\Psi_\alpha]' = e^{-\frac{1}{2}\lambda_{ab}(x)\gamma^{ab}}D_\mu\Psi_\alpha.$$

### A.0.9 Type $II$B Supergravity

In type $II$B supergravity we have different types of fields which are classified as bosons or fermions. These bosonic and fermionic fields can further be classified as gauge fields and non-gauge fields. A summary of these fields is given in the table below

| Fields | $U(1)$ | $SU(1,1)$ | Dim | fermion/boson | gauge field/non gauge field |
|---|---|---|---|---|---|
| $e_\mu^a$ | 0 | 0 | $L^0$ | boson | gauge |
| $A_{\mu\nu}^\alpha$ | 0 | 2 | $L^{-4}$ | boson | gauge |
| $A_{\mu\nu\rho\sigma}$ | 0 | 0 | $L^{-4}$ | boson | gauge |
| $V_\pm^\alpha$ | $\pm 1$ | 2 | $L^0$ | boson | non gauge |
| $\psi_\mu$ | $\frac{1}{2}$ | 0 | $L^{-\frac{9}{2}}$ | fermion | gauge |
| $\lambda$ | $\frac{3}{2}$ | 0 | $L^{-\frac{9}{2}}$ | fermion | non gauge |
| $\varepsilon$ | $\frac{1}{2}$ | 0 | $L^{\frac{1}{2}}$ | fermion | non gauge |
| $\kappa$ | 0 | 0 | $L^4$ | - | - |

The $\mathcal{N}=2$, $D=10$ theory [78] has a linearly realized $U(1)$ symmetry and a complex scalar. This complex scalar is made out of two real fields $\phi$ and $\chi$ and is identified with a coset space $SU(1,1)/U(1)$. It is denoted $V_\pm^\alpha$. The $\pm$ on $V_\pm^\alpha$ tells us about the $U(1)$ charge and $\alpha = 1,2$ represents the $SU(1,1)$ index. We would like to derive the variation of the fields in the above table. We will mainly use the $U(1)$ and $SU(1,1)$ to guide us in guessing the expressions and later we will check if the dimensions of the left hand side matches the dimensions of the right hand side. When doing the variation for terms that are not gauge fields, we will use a selection rule based on the fact that gauge fields have derivative terms in their transformation rule while non gauge fields do not have these terms. This can be seen in the following example: consider the transformation of the gauge field

$$A_{\mu\nu}^\alpha \to A_{\mu\nu}^\alpha + \partial_\mu \Lambda_\nu^\alpha - \partial_\nu \Lambda_\mu^\alpha \tag{A.0.70}$$

$$A_{\mu\nu\rho\lambda} \to A_{\mu\nu\rho\lambda} - \frac{i\kappa}{4}\epsilon_{\alpha\beta}\Lambda_{[\mu}^\alpha F_{\nu\rho\lambda]}^\beta \tag{A.0.71}$$

where $F_{\nu\rho\lambda}^\beta = 3\partial_{[\nu}A_{\rho\lambda]}^\beta$. We see that gauge transformation of $A_{\mu\nu\rho\lambda}$ is not as trivial as one might have thought. One would have naively guessed that the transformation should be

$$A_{\mu\nu\rho\lambda} \to A_{\mu\nu\rho\lambda} + 4\partial_{[\mu}\Lambda_{\nu\rho\lambda]}$$

where

$$4\partial_{[\mu}\Lambda_{\nu\rho\lambda]} = \partial_\mu\Lambda_{\nu\rho\lambda} - \partial_\nu\Lambda_{\rho\lambda\mu} + \partial_\rho\Lambda_{\lambda\mu\nu} - \partial_\lambda\Lambda_{\mu\nu\rho}$$

but due to constrains that the type $IIB$ supegravity obeys, the correct gauge transformation is (A.0.71). From these examples we see that the gauge transformation of the gauge fields contains derivative terms. So any term that contains gauge fields in a non gauge field variation will be set to zero. We will compute the variation of $\delta V_+^\alpha$, $\delta V_-^\alpha$ and $\delta A_{\mu\nu}^\alpha$.

**Variation of the fields**

We will start with $\delta V_+^\alpha$

**Variation of $V_+^\alpha$**

The possible terms are

$$\delta_\epsilon V_+^\alpha = V_-^\alpha \bar{\varepsilon}^* \lambda + V_+^\alpha \bar{\varepsilon}\gamma^\mu\psi_\mu + A_{\beta\rho}^\alpha \bar{\varepsilon}^*\gamma^{\beta\rho}\gamma^\mu\psi_\mu + V_+^\alpha(\bar{\varepsilon}\gamma^\mu\psi_\mu)^* + \bar{\varepsilon}\lambda\gamma^{\mu\nu}A_{\mu\nu}^\alpha.$$

Since $V_-^\alpha$ is not a gauge field, we have to eliminate all the terms that contains the gauge fields. This expression then becomes

$$\delta_\epsilon V_+^\alpha = V_-^\alpha \bar{\varepsilon}^* \lambda.$$

Now to fix the dimensions of this equation we need to include the Newton's constant $\kappa$. This equation is then expressed as

$$\delta_\epsilon V_+^\alpha = \kappa V_-^\alpha \bar{\varepsilon}^* \lambda. \tag{A.0.72}$$

**Variation of $V_-^\alpha$**

$$\delta_\varepsilon V_-^\alpha = V_+^\alpha \bar{\varepsilon}\lambda^* + V_-^\alpha \bar{\varepsilon}\gamma^\mu \psi_\mu + V_-^\alpha (\bar{\varepsilon}\gamma^\mu \psi_\mu)^* + \bar{\varepsilon}\gamma^{\beta\rho} A_{\beta\rho}^\alpha \gamma^\mu \bar{\psi}_\mu + \bar{\epsilon}^* \lambda^* \gamma^{\mu\nu} A_{\mu\nu}^\alpha.$$

Applying the same reasoning as above, we find

$$\delta_\varepsilon V_-^\alpha = \kappa V_+^\alpha \bar{\varepsilon}\lambda^*. \tag{A.0.73}$$

**Variation of $A_{\mu\nu}^\alpha$**

$$\delta_\varepsilon A_{\mu\nu}^\alpha = \kappa A_{\mu\nu}^\alpha \bar{\varepsilon}\gamma^\beta \psi_\beta + \kappa A_{\mu\nu}^\alpha (\bar{\varepsilon}\gamma^\beta \psi_\beta)^* + V_-^\alpha \bar{\varepsilon}\lambda\gamma_{\mu\nu} + V_+^\alpha \bar{\varepsilon}^* \lambda^* \gamma_{\mu\nu} + V_-^\alpha \bar{\varepsilon}^* \gamma_{[\mu}\psi_{\nu]} + V_+^\alpha \bar{\varepsilon}\gamma_\mu \psi_\nu^*.$$

Notice that $A_{\mu\nu}^\alpha$ is a gauge field, so in this case the terms that contains gauge fields will be allowed, unlike previous cases. We expect that when we take the second variation the left hand side must depend on terms that have at most first order derivatives in $\varepsilon$ (i.e. $\partial\varepsilon$). However the first and the second terms will give us second order derivatives, so we need to exclude them. This leads us to the expression

$$\delta_\varepsilon A_{\mu\nu}^\alpha = V_-^\alpha \bar{\varepsilon}\lambda\gamma_{\mu\nu} + V_+^\alpha \bar{\varepsilon}^* \lambda^* \gamma_{\mu\nu} + 4i V_-^\alpha \bar{\varepsilon}^* \gamma_{[\mu}\psi_{\nu]} + 4i V_+^\alpha \bar{\varepsilon}\gamma_\mu \psi_\nu^*. \tag{A.0.74}$$

One needs to be careful when writing the variation of the gauge field. Checking if the $U(1)$ and $SU(1,1)$ charges match is not enough. One also needs to check if the commutator of two supersymmetry transformation is consistent with the set of gauge transformations and this is why we have added the factor "$4i$" on the last two terms. It is clear that for the fields to have a consistent gauge transformation, they must obey the following algebra

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \delta(\xi) + \delta(l) + \delta(\varepsilon) + \delta(\Lambda_{.}) + \delta(\Lambda_{...}) + \delta(\Sigma), \tag{A.0.75}$$

and all of these parameters represent local symmetries. $\delta(\varepsilon)$ is the local SUSY, $\delta(\xi)$ is the local change of coordinate, $\delta(l)$ is the local Lorentz, $\delta(\Lambda_{.})$ is the string gauge transformation, $\delta(\Lambda_{...})$ is the 3 brane gauge transformation and $\delta(\Sigma)$ is the local $U(1)$. This commutator plays an important role in this theory. The fields of this theory satisfy [78]

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] \begin{pmatrix} e_\mu^a \\ A_{\mu\nu}^\alpha \\ V_\pm^\alpha \end{pmatrix} = \Big(\delta(\xi) + \delta(l) + \delta(\varepsilon) + \delta(\Lambda_{.}) + \delta(\Lambda_{...}) + \delta(\Sigma)\Big) \begin{pmatrix} e_\mu^a \\ A_{\mu\nu}^\alpha \\ V_\pm^\alpha \end{pmatrix}. \tag{A.0.76}$$

Notice that we didn't include the fields $A_{\mu\nu\rho\sigma}, \psi_\mu$ and $\lambda$. This is because for these fields to satisfy the above equation, there must be an extra condition that must be imposed. For $A_{\mu\nu\rho\sigma}$ to satisfy (A.0.76), the five form of this theory must be self dual $F_{(5)} =^* F_{(5)}$. $\psi_\mu$ and $\lambda$ only satisfy equation (A.0.76) if some field equations are true and these field equations give the equations of motion of the $IIB$ supergravity.

# Appendix B

# Projectors

In order to evaluate the Schur polynomials and restricted Schur polynomials, we need to know the characters and the restricted characters of the symmetric group. Projectors become handy when we want to compute these characters. In this appendix, we will show how to construct projectors of arbitrary Young diagram with two removable boxes. This method can be used to construct projectors with any number of removable boxes. The constructed projectors will then be tested to see if they satisfy the projector conditions.

The aim of this appendix is to explicitly demonstrate the following projector properties that were quoted in chapter 3.4

$$P_{\square\square} \cdot P_{\square\square} = P_{\square\square}, \qquad\qquad P_{\square\square} \cdot P_{\substack{\square\\\square}} = 0. \tag{B.0.1}$$

Given the projectors

$$\begin{aligned}P_{\square\square} = \frac{d_s}{2}\Bigg[&|1\rangle\langle 1| + |2\rangle\langle 2| + \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle\langle 1| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle\langle 1| \\ &- \frac{1}{c_{(ab)} - c_{(cd)}}|2\rangle\langle 2| + \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|1\rangle\langle 2|\Bigg].\end{aligned} \tag{B.0.2}$$

$$\begin{aligned}P_{\substack{\square\\\square}} = \frac{d_s}{2}\Bigg[&|1\rangle\langle 1| + |2\rangle\langle 2| - \frac{1}{c_{(ab)} - c_{(cd)}}|1\rangle\langle 1| - \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|2\rangle\langle 1| \\ &+ \frac{1}{c_{(ab)} - c_{(cd)}}|2\rangle\langle 2| - \sqrt{1 - \frac{1}{(c_{(ab)} - c_{(cd)})^2}}|1\rangle\langle 2|\Bigg],\end{aligned} \tag{B.0.3}$$

we want to check that the defining properties true for any projector, are obeyed.

$$
\begin{aligned}
P_{\square\square}\cdot P_{\square\square} = \frac{1}{4}\Bigg[ & |1\rangle\langle 1| + |2\rangle\langle 2| + \frac{1}{c_{(ab)}-c_{(cd)}}|1\rangle\langle 1| + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|2\rangle\langle 1| \\
& - \frac{1}{c_{(ab)}-c_{(cd)}}|2\rangle\langle 2| + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|1\rangle\langle 2|\Bigg]\cdot\Bigg[|1\rangle\langle 1| + |2\rangle\langle 2| \\
& + \frac{1}{c_{(ab)}-c_{(cd)}}|1\rangle\langle 1| + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|2\rangle\langle 1| - \frac{1}{c_{(ab)}-c_{(cd)}}|2\rangle\langle 2| \\
& + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|1\rangle\langle 2|\Bigg] \\
= {} & \frac{1}{4}\Bigg[|1\rangle\langle 1|\left(2+\frac{2}{c_{(ab)}-c_{(cd)}}\right) + |2\rangle\langle 2|\left(2-\frac{2}{c_{(ab)}-c_{(cd)}}\right) \\
& + |1\rangle\langle 2|\left(2\sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\right) + |2\rangle\langle 1|\left(2\sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\right)\Bigg] \\
= {} & P_{\square\square}.
\end{aligned}
$$

$$
\begin{aligned}
P_{\square\square}\cdot P_{\substack{\square\\\square}} = \frac{1}{4}\Bigg[ & |1\rangle\langle 1| + |2\rangle\langle 2| + \frac{1}{c_{(ab)}-c_{(cd)}}|1\rangle\langle 1| + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|2\rangle\langle 1| \\
& - \frac{1}{c_{(ab)}-c_{(cd)}}|2\rangle\langle 2| + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|1\rangle\langle 2|\Bigg]\cdot\Bigg[|1\rangle\langle 1| + |2\rangle\langle 2| \\
& - \frac{1}{c_{(ab)}-c_{(cd)}}|1\rangle\langle 1| - \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|2\rangle\langle 1| + \frac{1}{c_{(ab)}-c_{(cd)}}|2\rangle\langle 2| \\
& - \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}|1\rangle\langle 2|\Bigg] \\
= {} & \frac{1}{4}\Bigg[|1\rangle\langle 1|\left(1-1+\frac{1}{c_{(ab)}-c_{(cd)}}-\frac{1}{c_{(ab)}-c_{(cd)}}\right) + |2\rangle\langle 2|\left(1-1-\frac{1}{c_{(ab)}-c_{(cd)}}+\frac{1}{c_{(ab)}-c_{(cd)}}\right) \\
& + |1\rangle\langle 2|\Bigg(\sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\cdot\frac{1}{c_{(ab)}-c_{(cd)}} - \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\cdot\frac{1}{c_{(ab)}-c_{(cd)}} \\
& + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}} - \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\Bigg) \\
& + |2\rangle\langle 1|\Bigg(\sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\cdot\frac{1}{c_{(ab)}-c_{(cd)}} - \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\cdot\frac{1}{c_{(ab)}-c_{(cd)}} \\
& + \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}} - \sqrt{1-\frac{1}{(c_{(ab)}-c_{(cd)})^2}}\Bigg)\Bigg] \\
= {} & 0.
\end{aligned}
$$

So we have shown that the projectors that we constructed using two removable boxes do satisfy the projector conditions. We conclude that $P_{\square\square}$ and $P_{\substack{\square\\\square}}$ are indeed projectors.

# Appendix C

# Real Hermittian matrix

Our goal here is to study the Jacobian that arises when one changes variables from the real Hermittian matrix to eigenvalues. We do this as follows

$$\begin{aligned}0 &= \int [dM] \frac{d}{dM_{ij}} \Big( (M^n)_{ij} e^{-\frac{1}{2}tr(M^2)} \Big) \\ &= \int [dM] \Big( \sum_{r=0}^{n-1} tr(M^r) tr(M^{n-r-1}) - tr(M^{n+1}) \Big) e^{-\frac{1}{2}tr(M^2)}. \end{aligned} \tag{C.0.1}$$

The first term of the above equation is obtain by using the following relation

$$\begin{aligned}\frac{d}{dM_{ij}}(M^n)_{ij} &= \frac{d}{dM_{ij}} \Big( M_{ia}M_{ab}M_{bc}M_{cd}\cdots M_{yz}M_{zj} \Big) \\ &= \delta_{ii}\delta_{ja}M_{ab}M_{bc}M_{cd}\cdots M_{yz}M_{zj} + \delta_{ia}\delta_{bj}M_{ia}M_{bc}M_{cd}\cdots M_{yz}M_{zj} + \delta_{ib}\delta_{jc}M_{ia}M_{ab}M_{cd}\cdots M_{yz}M_{zj} \\ &+ \delta_{ic}\delta_{jd}M_{ia}M_{ab}M_{bc}\cdots M_{yz}M_{zj} + \cdots + \delta_{iy}\delta_{jz}M_{ia}M_{ab}M_{bc}M_{cd}\cdots M_{zj} + M_{ia}M_{ab}M_{bc}M_{cd}\cdots M_{yz}\delta_{iz}\delta_{jj} \\ &= tr(M^0)M_{jb}M_{bc}M_{cd}\cdots M_{yz}M_{zj} + M_{ii}M_{jc}M_{cd}\cdots M_{yz}M_{zj} + M_{ia}M_{ai}M_{cd}\cdots M_{yz}M_{zj} \\ &+ M_{ia}M_{ab}M_{bi}\cdots M_{yz}M_{zj} + \cdots + M_{ia}M_{ab}M_{bc}M_{cd}\cdots M_{jj} + M_{ia}M_{ab}M_{bc}\cdots M_{yz}M_{yi}tr(M^0) \\ &= tr(M^0)tr(M^{n-1}) + tr(M)tr(^{n-2}) + tr(M^2)tr(M^{n-3}) + tr(M^3)tr(M^{n-4}) + \cdots + tr(M^{n-2})tr(M) \\ &+ tr(M^{n-1}tr(M^0)) \\ &= \sum_{r=0}^{n-1} tr(M^r)tr(M^{n-r-1}). \end{aligned}$$

Equation (C.0.1) then becomes

$$\Big\langle \sum_{r=0}^{n-1} tr(M^r)tr(M^{n-r-1}) \Big\rangle = \Big\langle tr(M^{n+1}) \Big\rangle. \tag{C.0.2}$$

In terms of eigenvalues, we can write this expression as

$$\Big\langle \sum_{r=0}^{n-1} \sum_{i=1}^{N} \lambda_i^r \sum_{j=1}^{N} \lambda_j^{n-1-r} \Big\rangle = \Big\langle \sum_{k=1}^{N} \lambda_k^{n+1} \Big\rangle. \tag{C.0.3}$$

Now let us repeat the same procedure, but instead of starting from the Hermittian matrix, we will directly start by representing our real Hermittian matrix in terms of its eigenvalues and this means that we need to also include the Jacobian, that is

$$
\begin{aligned}
0 &= \int \prod_{i=1}^{N} d\lambda_i \sum_{j=1}^{N} \frac{\partial}{\partial \lambda_j}\left(\lambda_j^n J e^{-\frac{1}{2}\sum_{k=1}^{N}\lambda_k^2}\right) && \text{(C.0.4)}\\
&= \int \prod_{i=1}^{N} d\lambda_i J e^{-\frac{1}{2}\sum_{k=1}^{N}\lambda_k^2}\left(n\sum_{j=1}^{N}\lambda_j^{n-1} + \sum_{j=1}^{N}\frac{1}{J}\frac{\partial J}{\partial \lambda_j}\lambda_j^n - \sum_{j=1}^{N}\lambda_j^{n+1}\right) && \text{(C.0.5)}\\
&= \int \prod_{i=1}^{N} d\lambda_i J e^{-\frac{1}{2}\sum_{k=1}^{N}\lambda_k^2}\left(n\sum_{j=1}^{N}\lambda_j^{n-1} + \sum_{j=1}^{N}\frac{\partial \ln J}{\partial \lambda_j}\lambda_j^n - \sum_{j=1}^{N}\lambda_j^{n+1}\right). && \text{(C.0.6)}
\end{aligned}
$$

Comparing (C.0.3) and (C.0.6) we learn that

$$
0 = \int \prod_{i=1}^{N} J e^{-\frac{1}{2}\sum_{k=1}^{N}\lambda_k^2}\left(n\sum_{j=1}^{N}\lambda_j^{n-1} + \sum_{j=1}^{N}\frac{\partial \ln J}{\partial \lambda_j}\lambda_j^n - \sum_{r=0}^{n-1}\sum_{i=1}^{N}\lambda_i^r\sum_{j=1}^{N}\lambda_j^{n-1-r}\right). \tag{C.0.7}
$$

For this above equation to be true, the term inside the bracket should be set to zero

$$
\begin{aligned}
0 &= n\sum_{j=1}^{N}\lambda_j^{n-1} + \sum_{j=1}^{N}\frac{\partial \ln J}{\partial \lambda_j}\lambda_j^n - \sum_{r=0}^{n-1}\sum_{i=1}^{N}\lambda_i^r\sum_{j=1}^{N}\lambda_j^{n-1-r} && \text{(C.0.8)}\\
\sum_{j=1}^{N}\frac{\partial \ln J}{\partial \lambda_j}\lambda_j^n &= \sum_{r=0}^{n-1}\sum_{i=1}^{N}\lambda_i^r\sum_{j=1}^{N}\lambda_j^{n-1-r} - n\sum_{j=1}^{N}\lambda_j^{n-1} && \text{(C.0.9)}\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j^{n-1}\sum_{r=0}^{n-1}\left(\frac{\lambda_i}{\lambda_j}\right)^r - n\sum_{j=1}^{N}\lambda_j^{n-1} && \text{(C.0.10)}\\
&= \sum_{j=1}^{N}\lambda_j^{n-1}\sum_{r=0}^{n-1}\left(\frac{\lambda_j}{\lambda_j}\right)^r + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\lambda_j^{n-1}\sum_{r=0}^{n-1}\left(\frac{\lambda_i}{\lambda_j}\right)^r - n\sum_{j=1}^{N}\lambda_j^{n-1} && \text{(C.0.11)}\\
&= \sum_{j=1}^{N}\lambda_j^{n-1} n + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\lambda_j^{n-1}\sum_{r=0}^{n-1}\left(\frac{\lambda_i}{\lambda_j}\right)^r - n\sum_{j=1}^{N}\lambda_j^{n-1} && \text{(C.0.12)}\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N}\lambda_j^{n-1}\sum_{r=0}^{n-1}\left(\frac{\lambda_i}{\lambda_j}\right)^r. && \text{(C.0.13)}
\end{aligned}
$$

Now using the results that one gets when summing the geometric series, that is

$$
\sum_{r=0}^{n-1}\left(\frac{\lambda_i}{\lambda_j}\right)^r = \frac{1-\frac{\lambda_i^n}{\lambda_j^n}}{1-\frac{\lambda_i}{\lambda_j}} = \frac{\lambda_j^n - \lambda_i^n}{\lambda_j^{n-1}(\lambda_j - \lambda_i)}, \tag{C.0.14}
$$

we find

$$\sum_{j=1}^{N} \frac{\partial \ln J}{\partial \lambda_j} \lambda_j^n = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \lambda_j^{n-1} \frac{\lambda_j^n - \lambda_i^n}{\lambda_j^{n-1}(\lambda_j - \lambda_i)} \tag{C.0.15}$$

$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n - \lambda_i^n}{\lambda_j - \lambda_i} \tag{C.0.16}$$

$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_j - \lambda_i} - \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_i^n}{\lambda_j - \lambda_i} \tag{C.0.17}$$

$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_j - \lambda_i} - \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_i - \lambda_j} \tag{C.0.18}$$

$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_j - \lambda_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_j - \lambda_i} \tag{C.0.19}$$

$$= 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_j - \lambda_i}, \tag{C.0.20}$$

where $n = 0, 1, \cdots, N-1$. Now it is straight forward to solve for $J$,

$$\sum_{j=1}^{N} \frac{\partial \ln J}{\partial \lambda_j} \lambda_j^n = 2 \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{\lambda_j^n}{\lambda_j - \lambda_i}. \tag{C.0.21}$$

This last equation holds for any $n$. This tells us that

$$\frac{\partial \ln J}{\partial \lambda_j} = 2 \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{1}{\lambda_j - \lambda_i}. \tag{C.0.22}$$

The solution to this differential equation is

$$\ln J = \sum_{\substack{i,j \\ i \neq j}} \ln(\lambda_i - \lambda_j)^2 \tag{C.0.23}$$

and this implies that

$$J = \prod_{\substack{i,j \\ i \neq j \\ i > j}} (\lambda_i - \lambda_j)^2. \tag{C.0.24}$$

So we see that the Jacobian that is used to change variable from the real Hermittian matrices to eigenvalues is given by (C.0.24).

# Appendix D

# BPS Gauss graphs

In this appendix, we want to investigate the relation between the operators $O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$ and $O^{\vec{m},\vec{p}}_{R,r}(\sigma)$. The operator $O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$ (which it is a BPS operator) is given by

$$O^{\vec{m},\vec{p}}_{R,r}(X,Y.Z) = \frac{1}{m!n!p!}\sqrt{\frac{|H_X \times H_Y| hooks_r}{hooks_R f_R}} \sum_{\sigma \in S_{n+m+p}} Tr(P_{R,r}\Gamma_R(\sigma)) Tr(\sigma X^{\otimes p} Y^{\otimes m} Z^{\otimes n}) \tag{D.0.1}$$

and the operator $O^{\vec{m},\vec{p}}_{R,r}(\sigma)$ is given by

$$O^{\vec{m},\vec{p}}_{R,r}(\sigma) = \frac{|H_X \times H_Y|}{\sqrt{p!m!}} \sum_{j,k}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}_1,\vec{\mu}_2} \sqrt{d_s d_t}\,\Gamma^{(s,t)}_{jk}(\sigma) B^{(s,t)\to 1_{|H_X\times H_Y|}}_{j\vec{\mu}_1} B^{(s,t)\to 1_{|H_X\times H_Y|}}_{k\vec{\mu}_2} O_{R,(r,s,t)_{\vec{\mu}_1\vec{\mu}_2}}. \tag{D.0.2}$$

The easy way to find out how these two operators are related is to compute the two point function of both operators and compare the results. The operators $O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$ are normalized to have a unit two point function, that is

$$\langle O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)\rangle = 1. \tag{D.0.3}$$

The two point function of the other operator is given by

$$\begin{aligned}\langle O^{\vec{m},\vec{p}}_{R,r}(\sigma_1) O^{\vec{m}',\vec{p}'}_{R',r'}(\sigma_2)\rangle &= \frac{|H_X\times H_Y|^2}{\sqrt{p!m!p'!m'!}} \sum_{j,k}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\vec{\mu}_1,\vec{\mu}_2} \sqrt{d_s d_t}\,\Gamma^{(s,t)}_{jk}(\sigma_1) B^{(s,t)\to 1_{|H_X\times H_Y|}}_{j\vec{\mu}_1} B^{(s,t)\to 1_{|H_X\times H_Y|}}_{k\vec{\mu}_2} \\ &\times \sum_{j',k'}\sum_{s'\vdash m'}\sum_{t'\vdash p'}\sum_{\vec{\mu}'_1,\vec{\mu}'_2} \sqrt{d_{s'} d_{t'}}\,\Gamma^{(s',t')}_{j'k'}(\sigma_2) B^{(s',t')\to 1_{|H_X\times H_Y|}}_{j'\vec{\mu}'_1} B^{(s',t')\to 1_{|H_X\times H_Y|}}_{k'\vec{\mu}'_2} \langle O_{R,(r,s,t)_{\vec{\mu}_1\vec{\mu}_2}} O_{R',(r',s',t')_{\vec{\mu}'_1\vec{\mu}'_2}}\rangle.\end{aligned} \tag{D.0.4}$$

Now using the identity

$$\frac{1}{|H_X\times H_Y|}\sum_{\gamma\in H_X\times H_Y} \Gamma^{(s,t)}_{ik}(\gamma) = \sum_{\vec{\mu}} B^{(s,t)\to 1_{|H_X\times H_Y|}}_{i\vec{\mu}} B^{(s,t)\to 1_{|H_X\times H_Y|}}_{k\vec{\mu}} \tag{D.0.5}$$

we find

$$\langle O^{\vec{m},\vec{p}}_{R,r}(\sigma_1) O^{\vec{m}',\vec{p}'}_{R',r'}(\sigma_2)\rangle = \frac{1}{p!m!} \sum_{j,k}\sum_{j',k'}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\gamma_1,\gamma_2\in H_X\times H_Y} d_s d_t \Gamma^{(s,t)}_{jk}(\sigma_1)\Gamma^{(s,t)}_{jj'}(\gamma_1)\Gamma^{(s,t)}_{j'k'}(\sigma_2)\Gamma^{(s,t)}_{kk'}(\gamma_2). \tag{D.0.6}$$

Above we have used the following results

$$\langle O_{R,(r,s,t)_{\vec{\mu}_1\vec{\mu}_2}} O_{R',(r',s',t')_{\vec{\mu}'_1\vec{\mu}'_2}}\rangle = \delta_{RR'}\delta_{rr'}\delta_{ss'}\delta_{\mu_1\mu'_1}\delta_{\mu_2\mu'_2}. \tag{D.0.7}$$

We therefore have

$$\langle O^{\dagger\vec{m},\vec{p}}_{R,r}(\sigma_1)O^{\vec{m}',\vec{p}'}_{R',r'}(\sigma_2)\rangle = \frac{1}{p!m!}\sum_s\sum_t\sum_{\gamma_1,\gamma_2\in H_X\times H_Y} d_s d_t \chi_{(s,t)}(\sigma_1\gamma_1\sigma_2\gamma_2) \tag{D.0.8}$$

$$= \sum_{\gamma_1,\gamma_2\in H_X\times H_Y}\delta(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2). \tag{D.0.9}$$

Now that we have managed to compute this two point function, we will re-write the operator $O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$ in terms of $O^{\vec{m},\vec{p}}_{R,r}(\sigma)$ keeping in mind that we are interested only in BPS operators.

$$O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) = \frac{1}{m!n!p!}\sqrt{\frac{|H_X\times H_Y|hooks_r}{hooks_R f_R}}\sum_{\sigma\in S_{n+m+p}} Tr(P_{R,r}\Gamma_R(\sigma))Tr(\sigma X^{\otimes p}Y^{\otimes m}Z^{\otimes n}) \tag{D.0.10}$$

$$= \frac{1}{m!n!p!}\sqrt{\frac{|H_X\times H_Y|hooks_r}{hooks_R f_R}}\sum_{\sigma\in S_{n+m+p}} Tr\Big(\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\mu} P_{R,(r,s,t)_{\mu\mu}}\Gamma_R(\sigma)\Big)Tr(\sigma X^{\otimes p}Y^{\otimes m}Z^{\otimes n}) \tag{D.0.11}$$

$$= \sqrt{\frac{|H_X\times H_Y|hooks_r}{hooks_R f_R}}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\mu}\chi_{R,(r,s,t)_{\mu\mu}}(X,Y,Z) \tag{D.0.12}$$

$$= \sqrt{\frac{|H_X\times H_Y|hooks_r}{hooks_R f_R}}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\mu}\sqrt{\frac{hooks_R f_R}{hooks_r hooks_s hooks_t}}O_{R,(r,s,t)_{\mu\mu}}(X,Y,Z) \tag{D.0.13}$$

$$= \sqrt{|H_X\times H_Y|}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\mu}\sqrt{\frac{1}{hooks_s hooks_t}}O_{R,(r,s,t)_{\mu\mu}}(X,Y,Z) \tag{D.0.14}$$

$$= \sqrt{\frac{|H_X\times H_Y|}{m!p!}}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\mu}\sqrt{d_s d_t}O_{R,(r,s,t)_{\mu\mu}}(X,Y,Z). \tag{D.0.15}$$

Now looking at the BPS operator quoted in (D.0.2), which is obtained by setting $\sigma = 1$, we have

$$O^{\vec{m},\vec{p}}_{R,r}(1) = \frac{|H_X\times H_Y|}{\sqrt{m!p!}}\sum_{s\vdash m}\sum_{t\vdash p}\sum_{\mu}\sqrt{d_s d_t}O_{R,(r,s)_{\mu\mu}}. \tag{D.0.16}$$

Compare this with (D.0.15), to learn [1]

$$O^{\vec{m},\vec{p}}_{R,r}(X,Y.Z) = \frac{1}{\sqrt{|H_X\times H_Y|}}O^{\vec{m},\vec{p}}_{R,r}(1). \tag{D.0.17}$$

This is the relation we wanted. Taking the two point function we find

$$\langle O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)\rangle = \frac{1}{|H_X\times H_Y|}\langle O^{\dagger\vec{m},\vec{p}}_{R,r}(1)O^{\vec{m},\vec{p}}_{R,r}(1)\rangle \tag{D.0.18}$$

and this tells us that we have the following relationship

$$\sum_{\gamma_1,\gamma_2\in H_X\times H_Y}\delta(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2)\Big|_{\sigma_1=\sigma_2=1} = |H_X\times H_Y|. \tag{D.0.19}$$

[1]One should bare in mind that this operator $O^{\vec{m},\vec{p}}_{R,r}(1)$ is a function of $X, Y, Z$.

# Appendix E

# Derivatives of the Gauss graphs

Here, we want to derive the expression obtained when one takes derivative of $O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$. The derivatives we are interested in are $Tr\left(\frac{d}{dX}\right)$, $Tr\left(\frac{d}{dY}\right)$ and $Tr\left(\frac{d}{dZ}\right)$. Consider the operator $\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$ which is given by

$$O^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) = \sqrt{\frac{|H_X \times H_Y| hooks_r}{hooks_R f_R}}\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z). \tag{E.0.1}$$

First we use the following form of this operator

$$\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) = Tr\left(Y\frac{d}{dZ}\right)^m Tr\left(X\frac{d}{dZ}\right)^p \chi_{(n+m+p)}(Z). \tag{E.0.2}$$

Taking the derivative of $\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$ with respect to $Tr\left(\frac{d}{dX}\right)$, $Tr\left(\frac{d}{dY}\right)$ and $Tr\left(\frac{d}{dZ}\right)$ will entail evaluating $Tr\left(\frac{d}{dZ}\right)\chi_{(n+m+p)}(Z)$. We will now develop a way of evaluating this expression. We follow [60].

## E.1 Evaluating: $Tr\left(\frac{d}{dZ}\right)\chi_R(Z)$

Recall the definition of the Schur polynomial

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}}. \tag{E.1.1}$$

The derivative of the Schur polynomial is

$$Tr\left(\frac{d}{dZ}\right)\chi_R(Z) = Tr\left(\frac{d}{dZ}\right)\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \tag{E.1.2}$$

$$= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\Big(\delta^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}} + Z^{i_1}_{i_{\sigma(1)}} \delta^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}} + Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \delta^{i_3}_{i_{\sigma(3)}} \cdots Z^{i_n}_{i_{\sigma(n)}} + \cdots + Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \cdots \delta^{i_n}_{i_{\sigma(n)}}\Big). \tag{E.1.3}$$

To simplify this, make use of a change of variables by replacing $\sigma \to \rho\tau\rho^{-1}$. We replace the $\sigma$ of the second term, third term,$\cdots$, $n$-term by $(12)\tau(12)$, $(13)\tau(13)$,$\cdots$, $(1n)\tau(1n)$ respectively. Doing that we find

$$Tr\left(\frac{d}{dZ}\right)\chi_R(Z) = \frac{1}{n!}\Bigg(\sum_{\tau\in S_n}\chi_R(\tau)\delta^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \sum_{\tau\in S_n}\chi_R((12)\tau(12))Z^{i_1}_{i_{(12)\tau(12)(1)}}\delta^{i_2}_{i_{(12)\tau(12)(2)}}Z^{i_3}_{i_{(12)\tau(12)(3)}}$$
$$\cdots Z^{i_n}_{i_{(12)\tau(12)(n)}} + \sum_{\tau\in S_n}\chi_R((13)\tau(13))Z^{i_1}_{i_{(13)\tau(13)(1)}}Z^{i_2}_{i_{(13)\tau(13)(2)}}\delta^{i_3}_{i_{(13)\tau(13)(3)}}\cdots Z^{i_n}_{i_{(13)\tau(13)(n)}} +$$
$$\cdots + \sum_{\tau\in S_n}\chi_R((1n)\tau(1n))Z^{i_1}_{i_{(1n)\tau(1n)(1)}}Z^{i_2}_{i_{(1n)\tau(1n)(2)}}Z^{i_3}_{i_{(1n)\tau(1n)(3)}}\cdots\delta^{i_n}_{i_{(1n)\tau(1n)(n)}}\Bigg). \tag{E.1.4}$$

Making use of the identities

$$\prod_{a=1}^{n} v(\sigma(a))^{i_a} = \prod_{a=1}^{n} v(a)^{i}_{\sigma^{-1}(a)} \tag{E.1.5}$$
$$\prod_{a=1}^{n}\delta^{i_a}_{j_{\sigma(a)}} = \prod_{a=1}^{n}\delta^{i_{\sigma^{-1}(a)}}_{j_a} \tag{E.1.6}$$

gives

$$Tr\left(\frac{d}{dZ}\right)\chi_R(Z) = \frac{1}{n!}\Bigg(\sum_{\tau\in S_n}\chi_R(\tau)\delta^{i_1}_{i_{\tau(1)}}Z^{i_2}_{i_{\tau(2)}}Z^{i_3}_{i_{\tau(3)}}\cdots Z^{i_n}_{i_{\tau(n)}} + \sum_{\tau\in S_n}\chi_R((12)\tau(12))Z^{i_{(12)(1)}}_{i_{\tau(2)}}\delta^{i_{(12)(2)}}_{i_{\tau(1)}}Z^{i_{(12)(3)}}_{i_{\tau(3)}}$$
$$\cdots Z^{i_{(12)(n)}}_{i_{\tau(n)}} + \sum_{\tau\in S_n}\chi_R((13)\tau(13))Z^{i_{(13)(1)}}_{i_{\tau(3)}}Z^{i_{(13)(2)}}_{i_{\tau(2)}}\delta^{i_{(13)(3)}}_{i_{\tau(1)}}\cdots Z^{i_{(13)(n)}}_{i_{\tau(n)}} +$$
$$\cdots + \sum_{\tau\in S_n}\chi_R((1n)\tau(1n))Z^{i_{(1n)(1)}}_{i_{\tau(n)}}Z^{i_{(1n)(2)}}_{i_{\tau(2)}}Z^{i_{(1n)(3)}}_{i_{\tau(3)}}\cdots\delta^{i_{(1n)(n)}}_{i_{\tau(1)}}\Bigg)$$
$$= \frac{1}{n!}\Bigg(\sum_{\sigma\in S_n}\chi_R(\sigma)\delta^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \sum_{\tau\in S_n}\chi_R((12)\tau(12))Z^{i_{(2)}}_{i_{\tau(2)}}\delta^{i_{(1)}}_{i_{\tau(1)}}Z^{i_{(3)}}_{i_{\tau(3)}} \tag{E.1.7}$$
$$\cdots Z^{i_{(n)}}_{i_{\tau(n)}} + \sum_{\tau\in S_n}\chi_R((13)\tau(13))Z^{i_{(3)}}_{i_{\tau(3)}}Z^{i_{(2)}}_{i_{\tau(2)}}\delta^{i_{(1)}}_{i_{\tau(1)}}\cdots Z^{i_{(n)}}_{i_{\tau(n)}} + \cdots + \sum_{\tau\in S_n}\chi_R((1n)\tau(1n))Z^{i_{(n)}}_{i_{\tau(n)}}Z^{i_{(2)}}_{i_{\tau(2)}}Z^{i_{(3)}}_{i_{\tau(3)}}\cdots\delta^{i_{(1)}}_{i_{\tau(1)}}\Bigg).$$

Using the cyclicity of the trace, we have

$$Tr\left(\frac{d}{dZ}\right)\chi_R(Z) = \frac{1}{n!}\Bigg(\sum_{\psi\in S_n}\chi_R(\psi)\delta^{i_1}_{i_{\psi(1)}}Z^{i_2}_{i_{\psi(2)}}Z^{i_3}_{i_{\psi(3)}}\cdots Z^{i_n}_{i_{\psi(n)}} + \sum_{\psi\in S_n}\chi_R(\psi)Z^{i_{(2)}}_{i_{\psi(2)}}\delta^{i_{(1)}}_{i_{\psi(1)}}Z^{i_{(3)}}_{i_{\psi(3)}}$$
$$\cdots Z^{i_{(n)}}_{i_{\psi(n)}} + \sum_{\psi\in S_n}\chi_R(\psi)Z^{i_{(3)}}_{i_{\psi(3)}}Z^{i_{(2)}}_{i_{\psi(2)}}\delta^{i_{(1)}}_{i_{\psi(1)}}\cdots Z^{i_{(n)}}_{i_{\psi(n)}} + \cdots + \sum_{\psi\in S_n}\chi_R(\psi)Z^{i_{(n)}}_{i_{\psi(n)}}Z^{i_{(2)}}_{i_{\psi(2)}}Z^{i_{(3)}}_{i_{\psi(3)}}\cdots\delta^{i_{(1)}}_{i_{\psi(1)}}\Bigg) \tag{E.1.8}$$
$$= \frac{n}{n!}\sum_{\psi\in S_n}\chi_R(\psi)\delta^{i_1}_{i_{\psi(1)}}Z^{i_2}_{i_{\psi(2)}}Z^{i_3}_{i_{\psi(3)}}\cdots Z^{i_n}_{i_{\psi(n)}}. \tag{E.1.9}$$

Now, our next step is to fix the delta-function. This is done by moving from $S_n$ to $S_{n-1}$ where $S_{n-1}\subset S_n$. To move from the space of $S_n$ to the subspace of $S_{n-1}$ we restrict $i_1 = i_{\sigma(1)}$ which implies that $\sigma(1) = 1$. Now, using the coset decomposition

$$S_n = S_{n-1}\oplus(12)S_{n-1}\oplus(13)S_{n-1}\oplus(14)S_{n-1}\oplus\cdots\oplus(1n)S_{n-1} \tag{E.1.10}$$

for $\sigma \in S_{n-1}$, we find

$$Tr\bigg(\frac{d}{dZ}\bigg)\chi_R(Z) = \frac{n}{n!}\sum_{\psi\in S_n}\chi_R(\psi)\delta^{i_1}_{i_{\psi(1)}}Z^{i_2}_{i_{\psi(2)}}Z^{i_3}_{i_{\psi(3)}}\cdots Z^{i_n}_{i_{\psi(n)}} \tag{E.1.11}$$

$$= \frac{n}{n!}\sum_{\sigma\in S_{n-1}}\bigg(\chi_R(\sigma)\delta^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \chi_R((12)\sigma)\delta^{i_1}_{i_{(12)\sigma(1)}}Z^{i_2}_{i_{(12)\sigma(2)}}Z^{i_3}_{i_{(12)\sigma(3)}}\cdots Z^{i_n}_{i_{(12)\sigma(n)}}$$

$$+ \chi_R((13)\sigma)\delta^{i_1}_{i_{(13)\sigma(1)}}Z^{i_2}_{i_{(13)\sigma(2)}}Z^{i_3}_{i_{(13)\sigma(3)}}\cdots Z^{i_n}_{i_{(13)\sigma(n)}} + \cdots$$

$$+ \chi_R((1n)\sigma)\delta^{i_1}_{i_{(1n)\sigma(1)}}Z^{i_2}_{i_{(1n)\sigma(2)}}Z^{i_3}_{i_{(1n)\sigma(3)}}\cdots Z^{i_n}_{i_{(1n)\sigma(n)}}\bigg). \tag{E.1.12}$$

The delta function $\delta^{i_1}_{i_{\sigma(1)}}$ becomes $\delta^{i_1}_{i_{\sigma(1)}} = N$ and this expression is reduced to

$$Tr\bigg(\frac{d}{dZ}\bigg)\chi_R(Z) = \frac{n}{n!}\sum_{\sigma\in S_{n-1}}\bigg(\chi_{R'}(\sigma)NZ^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \chi_{R'}((12)\sigma)\delta^{i_2}_{i_{\sigma(1)}}Z^{i_1}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}}$$

$$+ \chi_{R'}((13)\sigma)\delta^{i_3}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Z^{i_1}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \cdots + \chi_{R'}((1n)\sigma)\delta^{i_n}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_1}_{i_{\sigma(n)}}\bigg) \tag{E.1.13}$$

$$= \frac{n}{n!}\sum_{\sigma\in S_{n-1}}\bigg(\chi_{R'}(\sigma)NZ^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \chi_{R'}((12)\sigma)Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}}$$

$$+ \chi_{R'}((13)\sigma)Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} + \cdots + \chi_{R'}((1n)\sigma)Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}}\bigg) \tag{E.1.14}$$

where $R'$ denotes the Young diagram obtained when we remove a single box from the Young diagram $R$. We can write this last expression as follows

$$Tr\bigg(\frac{d}{dZ}\bigg)\chi_R(Z) = \frac{n}{n!}\sum_{\sigma\in S_{n-1}}Tr\bigg(\Gamma_{R'}(N+(12)+(13)+(14)+\cdots+(1n))\Gamma_{R'}(\sigma)\bigg)Z^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}}. \tag{E.1.15}$$

This can nicely be written in terms of symmetric group Casimirs. The Casimirs $\mathcal{C}_{(\cdot,\cdot)}$ we need are defined as follows

$$\mathcal{C}_{(\cdot,\cdot)} = \sum_{i=2}^{n}\sum_{j=1}^{i-1}\Gamma_R((ij)) = \lambda_R\mathbf{1}_R \tag{E.1.16}$$

where $\lambda_R$ is given by

$$\lambda_R = \sum\frac{r_i(r_i-1)}{2} - \sum_j\frac{c_i(c_i-1)}{2}. \tag{E.1.17}$$

Using the definition of the Casimir, we can write the following relation

$$(12)+(13)+(14)+\cdots+(1n) = \sum_{i<j,i=1}^{j=n}(ij) - \sum_{i<j,i=2}^{j=n}(ij) \tag{E.1.18}$$

$$= \lambda_R - \lambda_{R'} \tag{E.1.19}$$

We then have

$$Tr\left(\frac{d}{dZ}\right)\chi_R(Z) = \frac{n}{n!}\sum_{\sigma\in S_{n-1}} Tr\bigg((N+\lambda_R-\lambda_{R'})\Gamma_{R'}(\sigma)\bigg) Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} \tag{E.1.20}$$

$$= \frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} \chi_{R'}(\sigma)(N+\lambda_R-\lambda_{R'}) Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}}\cdots Z^{i_n}_{i_{\sigma(n)}} \tag{E.1.21}$$

$$= \sum_{R'} c_{RR'}\chi_{R'}(Z) \tag{E.1.22}$$

$$= \sum_{i=1}^{q} c_{RR_i^{(1)}}\chi_{R_i^{(1)}}(Z) \tag{E.1.23}$$

where

$$c_{RR'} = N+\lambda_R-\lambda_{R'} \quad \text{and} \quad R\vdash n, \qquad R'\vdash n-1$$

Now that we have managed to compute $Tr\left(\frac{d}{dZ}\right)\chi_R(Z)$, it is easy to generalized this to $Tr\left(\frac{d}{dZ}\right)^m \chi_R(Z)$. This general case is

$$Tr\left(\frac{d}{dZ}\right)^m \chi_R(Z) = Tr\left(\frac{d}{dZ}\right)^{m-1}\left[Tr\left(\frac{d}{dZ}\right)\chi_R(Z)\right] \tag{E.1.24}$$

$$= Tr\left(\frac{d}{dZ}\right)^{m-1}\left[\sum_{R'} c_{RR'}\chi_{R'}(Z)\right] \tag{E.1.25}$$

$$= Tr\left(\frac{d}{dZ}\right)^{m-2}\left[\sum_{R''}\sum_{R'} c_{R'R''}c_{RR'}\chi_{R''}(Z)\right] \tag{E.1.26}$$

$$= Tr\left(\frac{d}{dZ}\right)^{m-3}\left[\sum_{R'''}\sum_{R''}\sum_{R'} c_{R''R'''}c_{R'R''}c_{RR'}\chi_{R'''}(Z)\right] \tag{E.1.27}$$

$$\vdots$$

$$= \sum_{R^m}\cdots\sum_{R'''}\sum_{R''}\sum_{R'} c_{R^{m-1}R^m}\cdots c_{R''R'''}c_{R'R''}c_{RR'}\chi_{R^m}(Z). \tag{E.1.28}$$

## E.2 The derivatives of $\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$

We will consider $Tr\left(\frac{d}{dX}\right)\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z)$, which is given by

$$Tr\left(\frac{d}{dX}\right)\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) = Tr\left(\frac{d}{dX}\right)Tr\left(Y\frac{d}{dZ}\right)^m Tr\left(X\frac{d}{dZ}\right)^p \chi_{(n+m+p)}(Z) \tag{E.2.1}$$

$$= pTr\left(Y\frac{d}{dZ}\right)^m Tr\left(X\frac{d}{dZ}\right)^{p-1} Tr\left(\frac{d}{dZ}\right)\chi_{(n+m+p)}(Z) \tag{E.2.2}$$

$$= p\sum_{i=1}^{q} c_{RR_i^{(1)}} Tr\left(Y\frac{d}{dZ}\right)^m Tr\left(X\frac{d}{dZ}\right)^{p-1}\chi_{(n+m+p-1)}(Z) \tag{E.2.3}$$

$$= p\sum_{i=1}^{q} c_{RR_i^{(1)}} \hat{O}^{\vec{m},\vec{p}_i^{(1)}}_{R_i^{(1)},r}(X,Y,Z) \tag{E.2.4}$$

where $q$ is the number of rows in the Young diagram $R$

$$Tr\left(\frac{d}{dY}\right)\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) = Tr\left(\frac{d}{dY}\right)Tr\left(Y\frac{d}{dZ}\right)^{m}Tr\left(X\frac{d}{dZ}\right)^{p}\chi_{(n+m+p)}(Z) \tag{E.2.5}$$

$$= mTr\left(Y\frac{d}{dZ}\right)^{m-1}Tr\left(X\frac{d}{dZ}\right)^{p}Tr\left(\frac{d}{dZ}\right)\chi_{(n+m+p)}(Z) \tag{E.2.6}$$

$$= m\sum_{i=1}^{q} c_{RR_i^{(1)}}Tr\left(Y\frac{d}{dZ}\right)^{m-1}Tr\left(X\frac{d}{dZ}\right)^{p}\chi_{(n+m+p-1)}(Z) \tag{E.2.7}$$

$$= m\sum_{i=1}^{q} c_{RR_i^{(1)}}\hat{O}^{\vec{m}_i^{(1)},\vec{p}}_{R_i^{(1)},r}(X,Y,Z) \tag{E.2.8}$$

and

$$Tr\left(\frac{d}{dZ}\right)\hat{O}^{\vec{m},\vec{p}}_{R,r}(X,Y,Z) = Tr\left(\frac{d}{dZ}\right)Tr\left(Y\frac{d}{dZ}\right)^{m}Tr\left(X\frac{d}{dZ}\right)^{p}\chi_{(n+m+p)}(Z) \tag{E.2.9}$$

$$= \sum_{i=1}^{q} c_{RR_i^{(1)}}\hat{O}^{\vec{m},\vec{p}}_{R_i^{(1)},r_i^{(1)}}(X,Y,Z). \tag{E.2.10}$$

# Bibliography


[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].

[4] G. ’t Hooft, “A Planar Diagram Theory for Strong Interactions,” Nucl. Phys. B **72**, 461 (1974). doi:10.1016/0550-3213(74)90154-0

[5] D. Garner, S. Ramgoolam and C. Wen, “Thresholds of large N factorization in $\mathrm{CFT}_4$: exploring bulk spacetime in $\mathrm{AdS}_5$,” JHEP **1411**, 076 (2014) doi:10.1007/JHEP11(2014)076 [arXiv:1403.5281 [hep-th]].

[6] J. A. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” JHEP **0303**, 013 (2003) doi:10.1088/1126-6708/2003/03/013 [hep-th/0212208].

[7] I. Bena, J. Polchinski and R. Roiban “Hidden symmetries of the $AdS_5 \times S^5$ superstring,” Phys. Rev. D 69, 046002 (2004) doi:10.1103/PhysRevD.69.046002 [arXiv:hep-th/0305116[hep-th]]

[8] S. Corley, A. Jevicki and S. Ramgoolam, “Exact correlators of giant gravitons from dual N=4 SYM theory,” Adv. Theor. Math. Phys. **5**, 809 (2002) [hep-th/0111222].

[9] D. Berenstein, “A Toy model for the AdS / CFT correspondence,” JHEP **0407**, 018 (2004) doi:10.1088/1126-6708/2004/07/018 [hep-th/0403110].

[10] H. Lin, O. Lunin and J. M. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP **0410**, 025 (2004) [hep-th/0409174].

[11] V. Balasubramanian, M. Berkooz, A. Naqvi and M. J. Strassler, “Giant gravitons in conformal field theory,” JHEP **0204**, 034 (2002) [hep-th/0107119].

[12] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from Anti-de Sitter space,” JHEP **0006**, 008 (2000) doi:10.1088/1126-6708/2000/06/008 [hep-th/0003075].

[13] A. Hashimoto, S. Hirano and N. Itzhaki, “Large branes in AdS and their field theory dual,” JHEP **0008**, 051 (2000) doi:10.1088/1126-6708/2000/08/051 [hep-th/0008016].

[14] M. T. Grisaru, R. C. Myers and O. Tafjord, “SUSY and goliath,” JHEP **0008**, 040 (2000) doi:10.1088/1126-6708/2000/08/040 [hep-th/0008015].

[15] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. Math. Phys. **59**, 35 (1978). doi:10.1007/BF01614153

[16] J. Ginibre, “Statistical Ensembles of Complex, Quaternion and Real Matrices,” J. Math. Phys. **6**, 440 (1965). doi:10.1063/1.1704292

[17] A. Jevicki and B. Sakita, "The Quantum Collective Field Method and Its Application to the Planar Limit," Nucl. Phys. B **165**, 511 (1980). doi:10.1016/0550-3213(80)90046-2

[18] A. Jevicki and B. Sakita, "Collective Field Approach to the Large $N$ Limit: Euclidean Field Theories," Nucl. Phys. B **185**, 89 (1981). doi:10.1016/0550-3213(81)90365-5

[19] K. Demeterfi, A. Jevicki and J. P. Rodrigues, "Scattering amplitudes and loop corrections in collective string field theory," Nucl. Phys. B **362**, 173 (1991). doi:10.1016/0550-3213(91)90561-B

[20] K. Demeterfi, A. Jevicki and J. P. Rodrigues, "Scattering amplitudes and loop corrections in collective string field theory. 2.," Nucl. Phys. B **365**, 499 (1991). doi:10.1016/S0550-3213(05)80030-6

[21] K. Demeterfi, A. Jevicki and J. P. Rodrigues, "Perturbative results of collective string field theory," Mod. Phys. Lett. A **6**, 3199 (1991). doi:10.1142/S0217732391003699

[22] A. Donos, A. Jevicki and J. P. Rodrigues, "Matrix model maps in AdS/CFT," Phys. Rev. D **72**, 125009 (2005) doi:10.1103/PhysRevD.72.125009 [hep-th/0507124].

[23] M. Masuku and J. P. Rodrigues, "Laplacians in polar matrix coordinates and radial fermionization in higher dimensions," J. Math. Phys. **52**, 032302 (2011) doi:10.1063/1.3553456 [arXiv:0911.2846 [hep-th]].

[24] Y. Kimura, S. Ramgoolam and D. Turton, "Free particles from Brauer algebras in complex matrix models," JHEP **1005**, 052 (2010) doi:10.1007/JHEP05(2010)052 [arXiv:0911.4408 [hep-th]].

[25] M. Masuku and J. P. Rodrigues, "How universal is the Wigner distribution?," J. Phys. A **45**, 085201 (2012) doi:10.1088/1751-8113/45/8/085201 [arXiv:1107.3681 [hep-th]].

[26] M. Masuku, M. Mulokwe and J. P. Rodrigues, "Large N Matrix Hyperspheres and the Gauge-Gravity Correspondence," JHEP **1512**, 035 (2015) doi:10.1007/JHEP12(2015)035 [arXiv:1411.5786 [hep-th]].

[27] M. Masuku and J. P. Rodrigues, "De Alfaro, Fubini and Furlan from multi Matrix Systems," JHEP **1512**, 175 (2015) doi:10.1007/JHEP12(2015)175 [arXiv:1509.06719 [hep-th]].

[28] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, "A New double scaling limit of N=4 superYang-Mills theory and PP wave strings," Nucl. Phys. B **643**, 3 (2002) doi:10.1016/S0550-3213(02)00749-6 [hep-th/0205033].

[29] Y. Takayama and A. Tsuchiya, "Complex matrix model and fermion phase space for bubbling AdS geometries," JHEP **0510**, 004 (2005) doi:10.1088/1126-6708/2005/10/004 [hep-th/0507070].

[30] V. Balasubramanian, D. Berenstein, B. Feng and M. x. Huang, "D-branes in Yang-Mills theory and emergent gauge symmetry," JHEP **0503**, 006 (2005) [hep-th/0411205].

[31] R. Bhattacharyya, S. Collins and R. d. M. Koch, "Exact Multi-Matrix Correlators," JHEP **0803**, 044 (2008) [arXiv:0801.2061 [hep-th]].

[32] R. Bhattacharyya, R. de Mello Koch and M. Stephanou, "Exact Multi-Restricted Schur Polynomial Correlators," JHEP **0806**, 101 (2008) [arXiv:0805.3025 [hep-th]].

[33] V. De Comarmond, R. de Mello Koch and K. Jefferies, "Surprisingly Simple Spectra," JHEP **1102**, 006 (2011) [arXiv:1012.3884 [hep-th]].

[34] R. d. M. Koch, M. Dessein, D. Giataganas and C. Mathwin, "Giant Graviton Oscillators," JHEP **1110**, 009 (2011) [arXiv:1108.2761 [hep-th]].

[35] R. de Mello Koch and S. Ramgoolam, "A double coset ansatz for integrability in AdS/CFT," JHEP **1206**, 083 (2012) doi:10.1007/JHEP06(2012)083 [arXiv:1204.2153 [hep-th]].

[36] D. Berenstein, "Large N BPS states and emergent quantum gravity," JHEP **0601**, 125 (2006) doi:10.1088/1126-6708/2006/01/125 [hep-th/0507203].

[37] D. Berenstein and R. Cotta, "Aspects of emergent geometry in the AdS/CFT context," Phys. Rev. D **74**, 026006 (2006) doi:10.1103/PhysRevD.74.026006 [hep-th/0605220].

[38] D. Berenstein and R. Cotta, "A Monte-Carlo study of the AdS/CFT correspondence: An Exploration of quantum gravity effects," JHEP **0704**, 071 (2007) doi:10.1088/1126-6708/2007/04/071 [hep-th/0702090].

[39] D. Berenstein, R. Cotta and R. Leonardi, "Numerical tests of AdS/CFT at strong coupling," Phys. Rev. D **78**, 025008 (2008) doi:10.1103/PhysRevD.78.025008 [arXiv:0801.2739 [hep-th]].

[40] D. Berenstein, "A Strong coupling expansion for N=4 SYM theory and other SCFT's," Int. J. Mod. Phys. A **23**, 2143 (2008) doi:10.1142/S0217751X08040688 [arXiv:0804.0383 [hep-th]].

[41] D. Berenstein and Y. Nakada, "The Shape of Emergent Quantum Geometry from an N=4 SYM Minisuperspace Approximation," arXiv:1001.4509 [hep-th].

[42] D. Berenstein, "Sketches of emergent geometry in the gauge/gravity duality," Fortsch. Phys. **62**, 776 (2014) [arXiv:1404.7052 [hep-th]].

[43] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, "The Library of Babel: On the origin of gravitational thermodynamics," JHEP **0512**, 006 (2005) doi:10.1088/1126-6708/2005/12/006 [hep-th/0508023].

[44] S. E. Vazquez, "Reconstructing 1/2 BPS space-time metrics from matrix models and spin chains," Phys. Rev. D **75**, 125012 (2007) [arXiv:hep-th/0612014].

[45] H. Y. Chen, D. H. Correa and G. A. Silva, "Geometry and topology of bubble solutions from gauge theory," Phys. Rev. D **76**, 026003 (2007) [arXiv:hep-th/0703068].

[46] R. de Mello Koch, "Geometries from Young Diagrams," JHEP **0811**, 061 (2008) doi:10.1088/1126-6708/2008/11/061 [arXiv:0806.0685 [hep-th]].

[47] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, "Three point functions of chiral operators in D = 4, N=4 SYM at large N," Adv. Theor. Math. Phys. **2**, 697 (1998) [hep-th/9806074].

[48] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory," Nucl. Phys. B **641**, 131 (2002) doi:10.1016/S0550-3213(02)00573-4 [hep-th/0205221].

[49] R. C. Myers and O. Tafjord, "Superstars and giant gravitons," JHEP **0111**, 009 (2001) doi:10.1088/1126-6708/2001/11/009 [hep-th/0109127].

[50] A. Donos, "A Description of 1/4 BPS configurations in minimal type IIB SUGRA," Phys. Rev. D **75**, 025010 (2007) doi:10.1103/PhysRevD.75.025010 [hep-th/0606199].

[51] A. Donos, "BPS states in type IIB SUGRA with SO(4) x SO(2)(gauged) symmetry," JHEP **0705**, 072 (2007) doi:10.1088/1126-6708/2007/05/072 [hep-th/0610259].

[52] B. Chen, S. Cremonini, A. Donos, F. L. Lin, H. Lin, J. T. Liu, D. Vaman and W. Y. Wen, "Bubbling AdS and droplet descriptions of BPS geometries in IIB supergravity," JHEP **0710**, 003 (2007) doi:10.1088/1126-6708/2007/10/003 [arXiv:0704.2233 [hep-th]].

[53] O. Lunin, "Brane webs and 1/4-BPS geometries," JHEP **0809**, 028 (2008) doi:10.1088/1126-6708/2008/09/028 [arXiv:0802.0735 [hep-th]].

[54] S. R. Das and A. Jevicki, "String Field Theory and Physical Interpretation of $D = 1$ Strings," Mod. Phys. Lett. A **5**, 1639 (1990). doi:10.1142/S0217732390001888

[55] D. Berenstein, "Strings on conifolds from strong coupling dynamics, part I," JHEP **0804**, 002 (2008) doi:10.1088/1126-6708/2008/04/002 [arXiv:0710.2086 [hep-th]].

[56] D. E. Berenstein and S. A. Hartnoll, "Strings on conifolds from strong coupling dynamics: Quantitative results," JHEP **0803**, 072 (2008) doi:10.1088/1126-6708/2008/03/072 [arXiv:0711.3026 [hep-th]].

[57] D. E. Berenstein, M. Hanada and S. A. Hartnoll, "Multi-matrix models and emergent geometry," JHEP **0902**, 010 (2009) doi:10.1088/1126-6708/2009/02/010 [arXiv:0805.4658 [hep-th]].

[58] R. de Mello Koch, P. Diaz and H. Soltanpanahi, "Non-planar Anomalous Dimensions in the sl(2) Sector," Phys. Lett. B **713**, 509 (2012) doi:10.1016/j.physletb.2012.06.057 [arXiv:1111.6385 [hep-th]].

[59] R. de Mello Koch, P. Diaz and N. Nokwara, "Restricted Schur Polynomials for Fermions and integrability in the su(2—3) sector," JHEP **1303**, 173 (2013) doi:10.1007/JHEP03(2013)173 [arXiv:1212.5935 [hep-th]].

[60] R. de Mello Koch, J. Smolic and M. Smolic, "Giant Gravitons - with Strings Attached (I)," JHEP **0706**, 074 (2007) [hep-th/0701066].

[61] R. de Mello Koch, J. Smolic and M. Smolic, "Giant Gravitons - with Strings Attached (II)," JHEP **0709**, 049 (2007) [hep-th/0701067].

[62] D. Bekker, R. de Mello Koch and M. Stephanou, "Giant Gravitons - with Strings Attached. III.," JHEP **0802**, 029 (2008) [arXiv:0710.5372 [hep-th]].

[63] A. Mikhailov, "Giant gravitons from holomorphic surfaces," JHEP **0011**, 027 (2000) doi:10.1088/1126-6708/2000/11/027 [hep-th/0010206].

[64] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, "Supersymmetric states of N=4 Yang-Mills from giant gravitons," JHEP **0712**, 006 (2007) doi:10.1088/1126-6708/2007/12/006 [hep-th/0606087].

[65] C. E. Beasley, "BPS branes from baryons," JHEP **0211**, 015 (2002) doi:10.1088/1126-6708/2002/11/015 [hep-th/0207125].

[66] G. Mandal and N. V. Suryanarayana, "Counting 1/8-BPS dual-giants," JHEP **0703**, 031 (2007) doi:10.1088/1126-6708/2007/03/031 [hep-th/0606088].

[67] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, "An Index for 4 dimensional super conformal theories," Commun. Math. Phys. **275**, 209 (2007) doi:10.1007/s00220-007-0258-7 [hep-th/0510251].

[68] C. Romelsberger, "Counting chiral primaries in N = 1, d=4 superconformal field theories," Nucl. Phys. B **747**, 329 (2006) doi:10.1016/j.nuclphysb.2006.03.037 [hep-th/0510060].

[69] T. W. Brown, P. J. Heslop and S. Ramgoolam, "Diagonal multi-matrix correlators and BPS operators in N=4 SYM," JHEP **0802**, 030 (2008) doi:10.1088/1126-6708/2008/02/030 [arXiv:0711.0176 [hep-th]].

[70] Y. Kimura and S. Ramgoolam, "Branes, anti-branes and brauer algebras in gauge-gravity duality," JHEP **0711**, 078 (2007) doi:10.1088/1126-6708/2007/11/078 [arXiv:0709.2158 [hep-th]].

[71] Y. Kimura and S. Ramgoolam, "Enhanced symmetries of gauge theory and resolving the spectrum of local operators," Phys. Rev. D **78**, 126003 (2008) doi:10.1103/PhysRevD.78.126003 [arXiv:0807.3696 [hep-th]].

[72] T. W. Brown, P. J. Heslop and S. Ramgoolam, "Diagonal free field matrix correlators, global symmetries and giant gravitons," JHEP **0904**, 089 (2009) doi:10.1088/1126-6708/2009/04/089 [arXiv:0806.1911 [hep-th]].

[73] W. Carlson, R. de Mello Koch and H. Lin, "Nonplanar Integrability," JHEP **1103**, 105 (2011) doi:10.1007/JHEP03(2011)105 [arXiv:1101.5404 [hep-th]].

[74] J. Pasukonis and S. Ramgoolam, "Quantum states to brane geometries via fuzzy moduli spaces of giant gravitons," JHEP **1204**, 077 (2012) doi:10.1007/JHEP04(2012)077 [arXiv:1201.5588 [hep-th]].

[75] M. C. Abbott, J. Murugan, A. Prinsloo and N. Rughoonauth, "Meromorphic Functions and the Topology of Giant Gravitons," Phys. Lett. B **730**, 215 (2014) doi:10.1016/j.physletb.2014.01.052 [arXiv:1312.4900 [hep-th]].

[76] R. de Mello Koch and R. Gwyn, "Giant graviton correlators from dual SU(N) super Yang-Mills theory," JHEP **0411**, 081 (2004) doi:10.1088/1126-6708/2004/11/081 [hep-th/0410236].

[77] D. Berenstein, "Giant gravitons: a collective coordinate approach," Phys. Rev. D **87**, no. 12, 126009 (2013) doi:10.1103/PhysRevD.87.126009 [arXiv:1301.3519 [hep-th]].

[78] J.H. Schwarz "Covariant field equations of chiral $N = 2$ $D = 10$ supergravity," Nucl. Phys. **B226** (1983),269.

[79] R. de Mello Koch, D. Gossman, L. Nkumane and L. Tribelhorn, "Eigenvalue Dynamics for Multimatrix Models," Phys. Rev. D **96**, no. 2, 026011 (2017) doi:10.1103/PhysRevD.96.026011 [arXiv:1608.00399 [hep-th]].

[80] R. de Mello Koch and L. Nkumane, "From Gauss Graphs to Giants," **JHEP 1802**, 005 (2018) doi:10.1007/JHEP02(2018)005 [arXiv:1710.09063 [hep-th]].