Zeros of Jacobi Polynomials and associated Inequalities

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Abstract

This Dissertation focuses on the Jacobi polynomial. Specifically, it discusses certain aspects of the zeros of the Jacobi polynomial such as the interlacing property and quasi-orthogonality. Also found in the Dissertation is a chapter on the inequalities of the zeros of the Jacobi polynomial, mainly those developed by Walter Gautschi.

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Contents

Chapter 1

Introduction

The area of so called "special functions" is a vast field of mathematics that includes the well known logarithmic, exponential and trigonometric functions. Over many years, this field has been extended to include the beta, gamma and zeta functions along with classes of orthogonal polynomials. In essence, we think of special functions as organising points for mathematical calculations.

These special functions have many and varied applications in pure mathematics and numerous applied sciences such as astronomy, heat conduction, electric circuits, quantum mechanics, electrostatic interpretation and mathematical statistics, for examples see, [3] ,[36], [47], [52].

Many famous mathematicians are credited with creating new special functions and discovering new properties of existing functions. For example, in the 1700s, famed Swiss mathematician Leonhard Euler developed the gamma and the zeta functions as well as defining Bessel functions for circular drums [52]. In the 1800s, Carl Friedrich Gauss investigated the hypergeometric series and paid special attention to the ${}_2F_1$ series. By this time, most special functions that we use and recognise today had been established and known throughout the mathematical world.

Then, there is the class of orthogonal polynomials whose origins can be traced to Legendre's work on planetary motion [12]. Orthogonal polynomials are connected with trigonometric, hypergeometric, Bessel and elliptical functions [47]. The systems of orthogonal polynomials associated with the names of Hermite, Laguerre and Jacobi are the most extensively studied and widely applied systems. These functions are also referred to as *classical* orthogonal polynomials.

This dissertation aims to study aspects of Jacobi polynomials and their zeros. These topics will include the interlacing of zeros and the inequality conjectures associated with the zeros.

The dissertation is structured as follows. Chapter 2 provides an introduction to all the preliminary mathematics pertaining to special functions that is required for the work. Chapter 3 introduces the concept of orthogonal polynomials and gives properties of a few particular orthogonal polynomials that will be used in subsequent chapters. Chapter 4 concentrates on the zeros of Jacobi polynomials. The interlacing property is discussed along with the ideas of quadrature and quasi-orthogonality. Chapter 5 aims to discuss the inequalities associated with the zeros of Jacobi polynomials. Various papers are used to support the investigation.

Chapter 2

Mathematical Preliminaries

This chapter aims to introduce basic definitions and theorems that will be used in subsequent chapters. Readers may consult the relevant texts for further details relating to the proofs even though the proofs of the theorems are provided.

2.1 The Pochhammer Symbol

The Pochhammer symbol was introduced by Leo August Pochhammer (1841-1920) [31] and is often used in special functions. Typically, the gamma function uses the Pochhammer symbol extensively in certain definitions and theorems as will be shown later in this chapter.

There are various representations of the Pochhammer symbol used throughout the many fields of mathematics. Here are a few common notations:

$$(a, m) \equiv (a)_m \equiv (a|m),$$

where m is a non-negative integer and a is a real or complex number.

Note: In this dissertation, the $(a)_m$ notation will be used to denote the Pochhammer symbol.

Definition 2.1.1 *From, [2], [32] and [42], the* ***Pochhammer symbol*** *is defined as follows:*

$$(a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a+1)...(a+m-1) & \text{if } m = 1, 2, 3, \dots. \end{cases}$$

Note: It is interesting that the Pochhammer symbol is occasionally called the shifted or rising factorial function, as

$$(1)_n = n!$$

where n is a non-negative integer.

The theorem below from [31] expresses the binomial coefficients in terms of the Pochhammer symbol.

Theorem 2.1.1

$$\binom{a}{m} = \frac{a(a-1)(a-2)...(a-m+1)}{m!} = \frac{(-1)^m(-a)_m}{m!}$$

where m is a non-negative integer and $a \in \mathbb{R}$ where $m \leq a$.

Proof

$$\begin{aligned}\binom{a}{m} &= \frac{a(a-1)(a-2)...(a-m+1)(a-m)!}{m!(a-m)!} \\ &= \frac{a(a-1)(a-2)...(a-m+1)}{m!} \\ &= \frac{(-1)^m(-a)(-a+1)(-a+2)...(-a+m-1))}{m!} \qquad = \frac{(-1)^m(-a)_m}{m1}\end{aligned}$$

Listed below are various identities of the Pochhammer symbol that might be useful and can be found in [17].

Definition 2.1.2 *The following identities hold true for m, $n \in \mathbb{N}$:*

(1) $(x)_{m+n} = (x)_m(x+m)_n$ *for* $m, n \in \mathbb{N}$

(2) $(x)_m = (-1)^m(1-m-x)_m$

(3) $(x)_{m-k} = \frac{(x)_m(-1)^k}{(1-m-x)_n}, \quad 0 \leq k \leq m.$

2.2 The Gamma Function

The gamma function, denoted $\Gamma(x)$, was developed by Euler in the 1720s when he introduced it as an interpolating function for the factorials defined by $n! = \prod_{k=1}^{n} k$ [29]. However, through a series of letters sent to Christian Goldbach (1690-1764) from Daniel Bernoulli (1700-1782) in 1729, it was discovered that Bernoulli, in fact, gave the initial representation of a function that interpolated factorials. He did so in the form of the following infinite product, [7],

$$x! = \lim_{n\to\infty} \left(n + 1 + \frac{x}{2}\right)^{x-1} \prod_{i=1}^{n} \frac{i+1}{i+x}.$$

During this time, Leonard Euler (1707-1785) stayed with Bernoulli and his eagerness to help Goldbach and Bernoulli in their research on interpolating various sequences, paved the way for the creation of the gamma function [43]. He went above and beyond Bernoulli by obtaining new representations and theorems for the gamma function [29].

The gamma function is quite simple and has become important in the field of special functions. It is often a prerequisite for the study of other special functions such as the beta function.

The gamma function has several representations, which can be found in [2], [20], [6], [36].

From ([20], p57), we have Euler's Integral representation for the gamma function given by the definition below.

Definition 2.2.1 ***Euler's Integral*** *representation of the gamma function is an improper integral of the form*

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, \mathrm{d}t \ ,$$

where x may be a real or complex variable with $Re(x) > 0$.

From ([2], p3), we have Euler's limit formula defined as follows.

Definition 2.2.2 ***Euler's Limit Formula*** *for the gamma function has the following form for all complex numbers $x \neq 0, -1, -2, ...,$*

$$\Gamma(x) = \lim_{k\to\infty} \frac{k!k^{x-1}}{(x)_k} \ ,$$

where $(x)_k$ is the Pochhammer symbol and k is any real constant.

We have Weierstrass' Infinite Product, from ([20], p64), which is defined below.

Definition 2.2.3 ***Weierstrass' Infinite Product***

For all real numbers $n \neq 0, -1, -2, ...$ and finite values of $x \in \mathbb{C}$,

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \right),$$

where γ is Euler's constant given by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0,577215665.$$

Definition 2.2.1 is used as the primary definition for the gamma function.

The gamma function can also be represented as

$$\Gamma(x+1) = x!, \quad \text{for } \ x = 0, 1, 2, \ldots .$$

It follows from this identity that when $x = 0$ then $\Gamma(1) = 1$.

Apart from Euler, mathematicians such as Hermann Hankel (1839-1873) and Karl Theodor Weierstrass (1815-1897) developed other representations of the gamma function using various mathematical instruments. Hankel's representation can be found in ([20], p64), Weierstrass' formula is given above.

Two useful theorems for the gamma function are given below. The proofs of these standard results may be found in the following text, ([36], p3) even though the proofs are provided below.

Theorem 2.2.1 *The Reduction formula for the gamma function is given by*

$$\Gamma(x+1) = x\Gamma(x)$$

where $Re(x) \geq 0$, $x \neq 0, -1, -2, \dots$.

Proof

Using Euler's integral representation for the gamma function and integration by parts, we have

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\
&= -e^{-t}t^x|_0^\infty - \int_0^\infty -xt^{x-1}e^{-t}\mathrm{d}t \\
&= 0 + x\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \\
&= x\Gamma(x).
\end{aligned}$$

The following theorem demonstrates how the Pochhammer symbol can be written in terms of the gamma function.

Theorem 2.2.2 *For any positive integers* n *and* $x \neq 0, -1, -2, \dots$,

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

Proof

Using the reduction formula given in Theorem 2.2.1,

$$\begin{aligned}
&\Gamma(x+1) = x\Gamma(x) \\
&\Gamma(x+2) = (x+1)\Gamma(x+1) = x(x+1)\Gamma(x).
\end{aligned}$$

Repeated use of the reduction formula leads to the following

$$\begin{aligned}
\Gamma(x+n) &= x(x+1)(x+2)...(x+n-2)(x+n-1)\Gamma(x) \\
&= (x)_n\Gamma(x),
\end{aligned}$$

which leads us to the final identity

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

An interesting result of the gamma function is given below, the proof of which can be found in ([6], p25).

Theorem 2.2.3

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2}\mathrm{d}t = \sqrt{\pi}.$$

Proof

Using Euler's integral representation of the gamma function, we put $x = \frac{1}{2}$ and $t = r^2$, so $\mathrm{d}t = 2r\mathrm{d}r$. Then we have the following expression

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty (r^2)^{-\frac{1}{2}} e^{-r^2} 2r\mathrm{d}r \\ &= \int_0^\infty 2e^{-r^2}\mathrm{d}r.\end{aligned}$$

To evaluate this integral, proceed as follows: Let

$$I = \int_0^\infty e^{-x^2}\mathrm{d}x = \int_0^\infty e^{-y^2}\mathrm{d}y.$$

Then,

$$\begin{aligned}I^2 &= \int_0^\infty e^{-x^2}\mathrm{d}x \int_0^\infty e^{-y^2}\mathrm{d}y \\ &= \int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

which is clearly a double integral. In a geometric sense, the area of integration represents the whole of the first quadrant. We can change to polar coordinates. Therefore, we put $x = r\cos\theta$ and $y = r\sin\theta$ with unit area $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$ where $0 \leq \theta \leq \frac{\pi}{2}$.

Thus, we have the following,

$$\begin{aligned}
I^2 &= \int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r\mathrm{d}r\mathrm{d}\theta \\
&= \frac{\pi}{2}\int_0^{\infty} re^{-r^2}\mathrm{d}r \\
&= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= 2I \\
&= 2\sqrt{\frac{\pi}{4}} \\
&= \frac{2}{2}\sqrt{\pi} \\
&= \sqrt{\pi}\ ,
\end{aligned}$$

as required.

It is interesting to note that alternative integral representations of the gamma function exist and some can be found in ([2], p60). Two such representations are given below:

Theorem 2.2.4 *The gamma function has the following representations,*

(1) $\Gamma(x) = 2\int_0^{\infty} e^{-t^2} t^{2x-1}\mathrm{d}t$

(2) $\Gamma(x) = \alpha^x \int_0^{\infty} e^{-\alpha r} r^{x-1}\mathrm{d}r.$

Proof:

(1) Let $t = y^2$ then $\mathrm{d}t = 2y\mathrm{d}y$. We then use Euler's integral for the gamma function i.e. $\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}\mathrm{d}x$, and substitute the new value for t to obtain,

$$\begin{aligned}
\Gamma(x) &= \int_0^{\infty} e^{-y^2}(y^2)^{x-1}2y\mathrm{d}y \\
&= \int_0^{\infty} e^{-y^2}y^{2x-2}2y\mathrm{d}y
\end{aligned}$$

$$= 2\int_0^\infty e^{-y^2} y^{2x-1} \mathrm{d}y.$$

(2) Here, we set $t = \alpha r$ so that $\mathrm{d}t = \alpha \mathrm{d}r$. We, once again, subsitute the new value for t into the Euler integral definition of the gamma function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}x$, and get the following,

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty e^{-\alpha r} (\alpha r)^{x-1} \alpha \mathrm{d}r \\
&= \int_0^\infty e^{-\alpha r} (\alpha)^{x-1} r^x r^{-1} \alpha \mathrm{d}r \\
&= \int_0^\infty e^{-\alpha r} (\alpha)^x r^{x-1} \mathrm{d}r.
\end{aligned}$$

We then have the final result

$$\Gamma(x) = \alpha^x \int_0^\infty e^{-\alpha r} r^{x-1} \mathrm{d}r. \tag{2.1}$$

2.3 The Beta Function

The beta function was developed in connection with the gamma function. Certain useful results, that might be needed in later chapters are given below. It is interesting to note that the beta function is sometimes called the Euler integral of the first kind ([50], p103).

The following is a definition of the beta function that can be found in [2], [20], [6] and [36].

Definition 2.3.1 *The* ***beta function*** *has the form*

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \mathrm{d}t\ ,$$

for $Re(x) > 0$ *and* $Re(y) > 0$.

The theorem below represents the beta function in terms of the gamma function. It is extremely useful and its proof can be found in ([2], p5).

Theorem 2.3.1 *For $Re(x) > 0$ and $Re(y) > 0$, the beta function is given by*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof:

Use the definition of the beta function, $B(x,y) = \int\limits_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$, with the substitution $u = \frac{t}{1-t}$. In other words, put $t = \frac{u}{u+1}$ so that $\mathrm{d}t = \frac{du}{(1+u)^2}$ and change the limits since $t = 0 \Rightarrow u = 0$ and when $t \to 1$, $u \to \infty$.

Hence, we obtain

$$\begin{aligned}
B(x,y) &= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1} \left(1 - \frac{u}{u+1}\right)^{y-1} \left(\frac{1}{(1+u)^2}\right) \mathrm{d}u \\
&= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1} \left(\frac{1}{u+1}\right)^{y-1} \left(\frac{1}{(1+u)^2}\right) \mathrm{d}u \\
&= \int_0^\infty u^{x-1}(1+u)^{-(x+y)}\mathrm{d}u
\end{aligned}$$

We then use Equation (2.1), $\Gamma(x) = \alpha^x \int_0^\infty e^{-\alpha t}t^{x-1}\ \mathrm{d}t$, but replace α with $u+1$ and replace x with $x+y$. Using these substitutions we get

$$\int_0^\infty e^{-(1+u)t}t^{x+y-1}\ \mathrm{d}t = \frac{\Gamma(x+y)}{(1+u)^{(x+y)}}.$$

We now make $(1+u)^{-(x+y)}$ the subject of the equation which gives,

$$\frac{1}{\Gamma(x+y)} \int_0^\infty e^{-(1+u)t}t^{x+y-1}\ \mathrm{d}t = (1+u)^{-(x+y)}. \tag{2.2}$$

Next, substitute the left hand side of Equation (2.2) into the earlier result to obtain

$$B(x,y) = \int_0^\infty u^{x-1} \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-(1+u)t}t^{x+y-1}\ \mathrm{d}t\mathrm{d}u.$$

Swopping the order of integration yields

$$B(x,y) = \frac{1}{\Gamma(x+y)} \int_0^\infty t^{x+y-1}e^{-t} \int_0^\infty u^{x-1}e^{-ut}\ \mathrm{d}u\mathrm{d}t$$

$$= \frac{1}{\Gamma(x+y)} \int_0^\infty t^{x+y-1} e^{-t} \frac{\Gamma(x)}{t^x} \, \mathrm{d}t,$$

by Equation (2.1), with $r = u$ and $\alpha = t$. Hence,

$$\begin{aligned} B(x,y) &= \frac{\Gamma(x)}{\Gamma(x+y)} \int_0^\infty t^{y-1} e^{-t} \mathrm{d}t \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \end{aligned}$$

as required.

The next theorem illustrates the symmetry of the beta function and can be proved easily by making the substituting $u = 1 - t$ in the definition of the beta function. An explicit proof can be found in ([20], p77).

Theorem 2.3.2 *The symmetry property of the beta function is shown below for $Re(x) > 0$ and $Re(y) > 0$:*

$$B(x,y) = B(y,x).$$

Proof:

Let $u = 1 - t$, then $du = -1dt$ therefore $t = 1 - u$. We then have the following:

$$\begin{aligned} B(x,y) &= \int_1^0 -(u-1)^{x-1} u^{y-1} \mathrm{d}u \\ &= \int_0^1 (u-1)^{x-1} u^{y-1} \mathrm{d}u \\ &= B(y,x). \end{aligned}$$

The next two theorems contain interesting results pertaining to the beta function.
The proof of Theorem 2.3.3 below, makes use of the gamma function result, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and will not be proved in this text. However, a proof of this theorem is given in ([6], p24).

Theorem 2.3.3

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$$

The theorem below establishes a type of iterative formula for the beta function. This theorem, together with its proof, is given in ([2], p5).

Theorem 2.3.4 *For $Re(x) > 0$ and $Re(y) > 0$,*

$$B(x, y) = \frac{x+y}{y}\, B(x, y+1).$$

Proof: For $Re(x) > 0$ and $Re(y) > 0$,

$$\begin{aligned} B(x, y+1) &= \int_0^1 t^{x-1}(1-t)(1-t)^{y-1}\mathrm{d}t \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}dt - \int_0^1 t^{x}(1-t)^{y-1}\mathrm{d}t \\ &= B(x, y) - B(x+1, y), \end{aligned} \tag{2.3}$$

However, integration by parts gives

$$\begin{aligned} B(x, y+1) &= \int_0^1 t^{x-1}(1-t)^{y}\ \mathrm{d}t \\ &= \left[\frac{1}{x}t^{x}(1-t)^{y}\right]_0^1 + \frac{y}{x}\int_0^1 t^{x}(1-t)^{y-1}\ \mathrm{d}t \\ &= 0 + \frac{y}{x}\, B(x+1, y). \end{aligned}$$

In other words,

$$B(x+1, y) = \frac{x}{y}\, B(x, y+1). \tag{2.4}$$

Substitute Equation (2.4) into Equation (2.3) to get

$$B(x, y+1) = B(x, y) - \frac{x}{y}B(x, y+1)$$

and the result follows.

2.4 The Bessel Function

Bessel functions are important in pure mathematics as they are used for many problems in number theory, integral transforms and evaluations of integrals to name a few [21].

The astronomer Friedrich Wilhelm Bessel (1784-1846) was the first mathematician to systematise these functions [21]. However, the Bessel functions were first discovered by Daniel Bernoulli when he studied the oscillations of a heavy yet flexible chain that was suspended while its lower end was free [51]. He came up with an equation which was a particular case of an equation that Bessel formulated 100 years later, [39].

Bessel functions are solutions to the Bessel differential equation of order p which is given by

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0.$$

This definition can be found in [20]. As this is a second order differential equation, two linearly independent sets of functions, J_p and Y_p, form the general solution

$$y(x) = c_1J_p(x) + c_2Y_p(x),$$

where c_1 and c_2 are arbitrary constants.

The method used to solve the Bessel equation is the Fröbenius method and this is illustrated in ([20], p225-227). In the general solution, $J_p(x)$, is called the Bessel function of the first kind of order p. The following definition can be found in [2] and [51].

Definition 2.4.1 ***Bessel functions of the first kind*** *are given by*

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{p+2k},$$

where $k = 0, 1, 2, \dots$, *and* $|x| < \infty$ *and* $p \in \mathbb{Z}$.

The following definition, which illustrates the relationship between the Bessel functions when the parameter p is negative, can be found in [18].

Definition 2.4.2

$$J_{-p}(x) = (-1)^p J_p(x).$$

The next definition is the second solution to the Bessel equation, also known as the Bessel equation of the second kind. A method to obtain this function is found in ([6], p225).

Definition 2.4.3 ***Bessel functions of the second kind*** *have the form*

$$Y_p(x) = \frac{J_p(x)\cos\pi p - J_{-p}(x)}{\sin\pi p},$$

where $p \in \mathbb{Z}$.

Note: Higher order Bessel functions can be represented by Bessel functions of lower order, forming a recurrence relation. For example,

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x),$$

where $n \in \mathrm{N}$.

2.5 The Hypergeometric Series

In 1812, Gauss presented a paper to the Royal Society of Sciences at Göttingen. In this paper, he looked at the following infinite series:

$$1 + \frac{ab}{1.c}z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}z^3 + ...$$

as a function of a, b, c and z where c is not zero or a negative integer.
This series was given the name 'hypergeometric series' by Ernst Eduard Kummer (1810-1883) [22]. Along with Gauss, other mathematicians such as Kummer and Euler studied this series quite extensively. However, it was Bernhard Riemann who first characterised the series as satisfyng a second order differential equation with regular singularities [2]. The series can be used in applications of various real-world problems such as the simple pendulum and a one-dimensional oscillator [45].

The ${}_2F_1$ Gauss hypergeometric function is a particular solution to the differential equation

$$z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0.$$

A definition of the ${}_2F_1$ Gauss hypergeometric function, which can be found in ([2], p64), is given below.

Definition 2.5.1 ***The*** $_2F_1$ ***Gauss hypergeometric series*** *for* $a,\ b,\ c,\ z \in \mathbb{C},\ c \neq 0, -1, -2, ...$ *and* $|z| < 1$ *is given by*

$$_2F_1(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}.$$

Note: There are many notations for the $_2F_1$ Gauss hypergeometric function such as $_2F_1(z)$; others can be found in [12] and [42]. However, the notation on the left hand side of Definition 2.5.1 will be used.

In the Gauss hypergeometric function given in Definition 2.5.1, it is obvious that no parameter in the denominator can be zero or a negative integer.

The definition for the general $_pF_q$ hypergeometric function is given below and can be found in ([45], p34).

Definition 2.5.2 *The* ***generalised hypergeometric series,*** $_pF_q(z)$***,*** *is defined to be*

$$_pF_q(a_1, a_2, a_3, ..., a_p; b_1, b_2, ..., b_q; z) = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_k(a_2)_k...(a_p)_k}{(b_1)_k(b_2)_k...(b_q)_k}\frac{z^k}{k!}$$

where $a_1, a_2, a_3, ..., a_p, b_1, b_2, ..., b_q, z$ *can be part of the real or complex number set. However,* $b_1, b_2, ..., b_q$ *are not zero or negative integers.*

From the above definition, it is clear that the Gauss hypergeometric function is the hypergeometric function when $p = 2$ and when $q = 1$.

The hypergeometric function is quite useful in that it can be used to represent many well-known functions. For example: $\arcsin z = z\ {_2F_1}(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2)$ and $\ln(1+z) = z\ {_2F_1}(1, 1; 2; -z)$.

Below is a theorem that contains Euler's Integral representation of the $_2F_1$ series, a proof of which can be found in ([2], p65).

Theorem 2.5.1 *If $|z| < 1$ holds for the z-plane cut along the real axis from 1 to ∞ and $Re(c) > Re(b) > 0$ then Euler's integral representation has the form*

$$ {}_2F_1\left(a, b; c; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t. $$

Proof:

Begin by using the definition of the Gauss hypergeometric function and arranging it in terms of the gamma function by using the identity $\Gamma(z+n) = (z)_n\Gamma(z)$ to obtain

$$ \begin{aligned} {}_2F_1(a, b; c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}\frac{(a)_n z^n}{n!}. \end{aligned} $$

Furthermore, for $Re(c) > Re(b) > 0$,

$$ \begin{aligned} \frac{\Gamma(b+n)\Gamma(c)}{\Gamma(b)\Gamma(c+n)} &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(c-b)\Gamma(b+n)}{\Gamma(c+n)} \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(b+n, c-b) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\mathrm{d}t. \end{aligned} $$

We then have

$$ \begin{aligned} {}_2F_1\left(a, b; c; z\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}z^n\mathrm{d}t \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(tz)^n\mathrm{d}t. \end{aligned} $$

We recall the binomial theorem i.e $(1-y)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}y^n$.

Continue by letting $\alpha = a$ and $y = zt$ to get $(1-zt)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}(zt)^n$.

The following result is then obtained

$$_2F_1\,(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\mathrm{d}t.$$

When one or more of the upper parameters is a negative integer the $_2F_1$ series is finite, which is illustrated in the theorem below. The proof of Theorem 2.5.2 can be found in ([2], p67). It is also referred to as the the Chu-Vandermonde transformation.

Theorem 2.5.2 *The **Chu-Vandermonde transformation** is defined as*

$$_2F_1(-n,a;c;1) = \frac{(c-a)_n}{(c)_n},$$

where $n \in \mathbb{Z}+$.

The following transformation is called Pfaff's transformation and can be found in ([36], p247) along with its proof.

Theorem 2.5.3 ***Pfaff's Transformation*** *is given by*

$$F(a,b;c;x) = (1-x)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{-x}{1-x}\right),$$

where $|arg(1-x)| < \pi$.

Pfaff's transformation can be used to prove certain relationships as shown below.

$$\begin{aligned} \arctan x &= x\ {}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x^2\right) \\ &= \tfrac{x}{\sqrt{1+x^2}}\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; \tfrac{x^2}{1+x^2}\right) \\ &= \arcsin \tfrac{x}{1+x^2}\ . \end{aligned}$$

Chapter 3

Orthogonal Polynomials

Orthogonal polynomials appear to have been developed by Legendre and Laplace whilst working on celestial mechanics. This work dealt with specific sets of orthogonal polynomials. Some of the first polynomials to be proven as orthogonal were the symmetric beta functions on the interval $(-1, 1)$, and then the normal distribution on the whole real line. The general theory began with Pafnuty Lvovich Chebyshev (1821-1894) in the 1850s [7], as his work was motivated by analogies with Fourier series and the theory of continued fractions. Orthogonal polynomials are used in solving many problems such as the birth-death process [49].

Orthogonal polynomials can be classified further into various types of orthogonal polynomials, such as:

- Classical Orthogonal Polynomials (Jacobi, Laguerre, Hermite and their special cases i.e Gegenbauer, Chebyshev and Legendre)
- Wilson Polynomials - these generalise the Jacobi Polynomials
- Askey-Wilson Polynomials - these introduce an extra parameter q into the Wilson Polynomials.

The following definition of the orthogonality of polynomials is found in [2] and [32].

Definition 3.0.1 *A sequence of polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *where* $p_n(x)$ *is of exact degree* n, *is* ***orthogonal*** *on the interval* $[a, b]$ *with respect to a weight function* $w(x)$, *which is* *greater than zero if*

$$\int_a^b p_n(x)p_m(x)w(x)dx = h_n\delta_{nm}$$

where

$$\delta_{nm} = \begin{cases} 0 & \textit{if } n \neq m \\ 1 & \textit{if } n = m. \end{cases}$$

Note:

1. A "weight function" is a mathematical instrument used when performing a sum, integral, or average to give some elements more weight or influence on the final result than other elements found in the set.
   Instead of the term "weight function" the term "norm function" is sometimes seen in literature. The expression "distribution" also occurs in the classical interpretaiton of $d\alpha(x)$ as a mass distribution (continuous or discontinuous) of a continuous function $\alpha(x)$ in the interval $[a, b]$. We will consider a non- negative weight function $w(x)$ for which $\int_a^b w(x)\mathrm{d}x > 0$, [47].

2. With the above definition, if $m = 0$ and $n \geq 1$, we have the integral

   $$\int_a^b p_n(x)w(x)dx = 0,$$

   since $p_0(x)$ is constant.

The definition below is worth noting and it can be found in reference text [42].

Definition 3.0.2 *A sequence of orthogonal polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *is orthogonal on the interval* $[a, b]$ *with respect to the weight function* $w(x) > 0$ *if and only if*

$$\int_a^b p_n(x)w(x)x^k dx = 0$$

for all $k = 0, 1, ..., (n-1)$ *and* n *is non-negative.*

The following definition can be found in [[2], p283].

Definition 3.0.3 *Given a sequence $a_0, a_1, a_2, ...$ the function*

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + = \sum_{n=0}^{\infty} a_n x^n,$$

is called the ***generating function*** *for the given sequence.*

Orthogonal polynomials satisy three-term recurrence relations which illustrate their connection to continued fractions.

The following theorem, the proof of which can be found in ([2], p245), says that a sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation.

Theorem 3.0.1 *A sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation*

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x)$$

for $n \geq 0$, where we set $p_{-1}(x) = 0$. Here A_n, B_n and C_n are real constants, $n = 0, 1, 2, ...$ and $A_{n-1} A_n C_n > 0$ for $n = 1, 2, 3, ...$
If the highest coefficient of $p_n(x)$ is $K_n > 0$, then:

$$A_n = \frac{K_{n+1}}{K_n},$$

$$C_{n+1} = \frac{A_{n+1}}{A_n} \frac{h_{n+1}}{h_n}$$

where h_n is given by Definition 3.0.1.

Proof:

Since $p_{n+1}(x)$ has degree exactly $(n+1)$ as does $x \, p_n(x)$, we can determine A_n such that

$$p_{n+1} \, (x) - A_n \, x \, p_n(x)$$

is a polynomial of degree at most n. Thus for some constants b_k,

$$p_{n+1}(x) - A_n x \, p_n(x) = \sum_{k=0}^{n} \, b_k \, p_k(x). \tag{3.1}$$

If $Q(x)$ is any polynomial of degree $m < n$, we know from Definition 3.0.1 that $\int_a^b P_n(x)Q(x)w(x)\,\mathrm{d}x = 0$.

Multiply both sides of Equation (3.1) by $w(x)p_m(x)$ where $m \in \{0, 1... , n-2\}$ to get (on integrating)

$$\int_a^b p_{n+1}(x)p_m(x)w(x)\mathrm{d}x - \int_a^b A_n x p_m(x)p_n(x)w(x)\mathrm{d}x = \sum_{k=0}^{m} \int_a^b b_k p_k(x)p_m(x)w(x)\,\mathrm{d}x.$$

But, the left hand side of the above equation is zero for each $m \in \{0, 1, ... \, n-2\}$ noting that $xp_m(x)$ is then a polynomial of degree $(m+1)$ which is less than or equal to $(n-1)$. On the right hand side of the equation, as K runs from 0 to n, the only integral in the sum that is not equal to zero, is the one involving $K = m$.

Thus, $b_m h_m = 0$ for each $m \in \{0, 1, ... , n-2\}$ and since $h_m \neq 0$, we have $b_m = 0$ for $m = 0, 1, ... , n-2$.

Therefore,

$p_{n+1}(x) - A_n \; xp_n(x) = b_{n-1}p_{n-1}(x) + b_n p_n(x)$ or $p_{n+1}(x) = (A_n \; x + b_n)p_n(x) + b_{n-1}p_{n-1}(x)$.

Now let $b_n = B_n$ and $b_{n-1} = -C_n$ then we obtain

$$p_{n+1}(x) = (A_n \; x + B_n)p_n(x) - C_n p_{n-1}(x), \tag{3.2}$$

which is the required three-term recurrence relation.

It should be clear from Equation (3.2) that $A_n = \frac{K_{n+1}}{K_n}$.

To prove the final part multiply Equation (3.2) by $p_{n-1}(x)w(x)$ and integrate. Then

$$0 = A_n \int_a^b x \; p_n(x)p_{n-1}(x)w(x)\mathrm{d}x - C_n \int_a^b p_{n-1}^2(x)w(x)\,\mathrm{d}x. \tag{3.3}$$

Now, $p_{n-1} = K_{n-1}x^{n-1}+$ a polynomial of degree $\leq n-2$.

Thus, $p_n(x) = K_n x^n +$ a polyomial of degree $\leq n-1$.

Then $x \; p_{n-1}(x) = \frac{K_{n-1}}{K_n}p_n(x) + \sum_{k=0}^{n-1} d_k p_k(x)$.

We see that from Equation (3.3)

$$0 = A_n \frac{K_{n-1}}{K_n} \; h_n - C_n \; h_{n-1}.$$

Since

$$A_n = \frac{K_{n+1}}{K_n},$$

we have

$$C_{n+1} = \frac{A_{n+1}}{A_n} \frac{h_{n+1}}{h_n}$$

which proves the result.

This form of the recurrence relation becomes useful when calculating p_{n+1} as you already have p_n and p_{n-1}.

An important result of the three-term recurrence relation is the Christoffel-Darboux formula. It is found in, ([2], p246) ([12], p23) and ([47], p43) along with its proof.

Theorem 3.0.2 *Suppose $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to the weight function $w(x)$ on interval $[a, b]$ where $a, b \in \mathbb{R}$. Suppose also, that the leading coefficient of $p_n(x)$ is k_n and that*

$$\int_a^b p_n^2(x) w(x) dx = h_n \neq 0.$$

Then from Definition 3.0.1

$$\sum_{m=0}^{n} \frac{p_m(x) p_m(y)}{h_m} = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(y) - p_{n+1}(y) p_n(x)}{(x-y) h_n}.$$

Proof:

The recurrence relation given in Theorem 3.0.1 gives

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x),$$

where $A_n = \frac{K_{n+1}}{K_n}$ and $C_{n+1} = \frac{A_{n+1}}{A_n} \frac{h_{n+1}}{h_n}$.

Multiply through by $p_n(y)$ to get

$$p_{n+1}(x)p_n(y) = (A_n x + B_n)p_n(x)p_n(y) - C_n p_{n-1}(x)p_n(y) \tag{3.4}$$

and similiarly

$$p_{n+1}(y)p_n(x) = (A_n y + B_n)p_n(y)p_n(x) - C_n p_{n-1}(y)p_n(x). \tag{3.5}$$

Subtract Equation (3.5) from Equation (3.4) to obtain

$$p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x) = A_n(x-y)p_n(x)p_n(y) - C_n[p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)].$$

Divide through by $A_n h_n(x-y)$ to get

$$\frac{1}{A_n}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{h_n(x-y)} = \frac{p_n(x)p_n(y)}{h_n} - \frac{1}{A_{n-1}}\left[\frac{p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)}{h_{n-1}(x-y)}\right].$$

Repeated application of this gives the required result, when we observe that $A_n = \frac{K_{n+1}}{K_n}$.

In this work, we are only interested in the group of classical orthogonal polynomials such as the Hermite, Laguerre and Jacobi polynomials.

Classical orthogonal polynomials have the following characteristics:

1. The family of derivatives, $p_n'(x)$, is also an orthogonal system.

2. The polynomial $p_n(x)$ satisfies a second order linear differential equation of the type:

   $$A(x)y'' + B(x)y' + \lambda_n y = 0$$

   where A and B are variables that do not depend on n and λ_n is also independent of x.

3. There is a Rodrigues formula having the general form:

$$p_n(x) = \frac{1}{a_n w_n(x)} \frac{d^n}{dx^n}\{w(x)[Q(x)]^n\},$$

where $Q(x)$ is a polynomial in x with coefficients that do not depend on n. Furthermore a_n does not depend on x as it is a constant and $w(x)$ is a positive weight function.

The Rodrigues formula is useful in providing immediate information on the following:

1. the interval of orthogonality
2. the weight function
3. the range of parameters for which the orthogonality holds.

3.1 Hermite Polynomials

The Hermite polynomials, denoted $H_n(x)$, were first studied by Laplace and Chebyshev but were named after Charles Hermite (1822-1901). They appear in probability theory, combinatorics [2] and even in physics, such as in the quantum harmonic oscillator.

The generating function of the Hermite polynomial is provided below. This definition can be found in ([20], p178).

Definition 3.1.1 *The Hermite polynomials, denoted $H_n(x)$, can be defined by their generating function*

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x),$$

where $|t|, |x| < \infty$.

The orthogonality property of $H_n(x)$ is shown in the following theorem, found in ([47], p105) along with its proof.

Theorem 3.1.1 *The orthogonality property of the Hermite polynomials is given below:*

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

where n, m are real, e^{-x^2} is the positive weight function and δ_{nm} is defined as in Definition 3.0.1.

Proof:

Begin by using the generating function of the Hermite polynomial, that is

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{3.6}$$

This can be written as

$$e^{2sx-s^2} = \sum_{m=0}^{\infty} \frac{s^m}{m!} H_m(x), \tag{3.7}$$

Next multiply Equations (3.6) and (3.7) to obtain

$$e^{2xt-t^2+2sx-s^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n s^m}{n!m!} H_n(x) H_m(x). \tag{3.8}$$

Then multiply both sides of Equation (3.8) by e^{-x^2} and integrate it with respect to x from $-\infty$ to ∞ to obtain

$$\int_{-\infty}^{\infty} e^{[-(-x+s+t)^2+2st]} \mathrm{d}x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n s^m}{n!m!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, \mathrm{d}x. \tag{3.9}$$

In the above Equation (3.9), the left hand side is equal to

$$e^{2st} \int_{-\infty}^{\infty} e^{-(-x+s+t)^2} \, \mathrm{d}x.$$

But from Theorem 2.2.3, we have that $\int_{-\infty}^{\infty} e^{-(x+s+t)^2} \mathrm{d}x = \sqrt{\pi}$. Therefore, the left hand side becomes

$$e^{2st} \sqrt{\pi} = \sqrt{\pi} \sum_{m=0}^{\infty} \frac{2^m s^m t^m}{m!} \tag{3.10}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n s^m}{n!m!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, \mathrm{d}x. \tag{3.11}$$

Now, let $m = n$ and equate the coefficients of $s^n t^n$ as follows,

$$\frac{2^n \sqrt{\pi}}{n!} = \frac{1}{(n!)^2} \int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) \, \mathrm{d}x.$$

However, when $m \neq n$, the powers of t and s are always equal for each term on the right hand side of Equation (3.8). So, if one equates the coefficients of $t^n t^m$ when $m \neq n$, we obtain

$$\frac{1}{n!m!}\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,\mathrm{d}x = 0,$$

which proves the result.

Now, we give the Rodrigues formula which is found in [20].

Theorem 3.1.2 *The Rodrigues formula for Hermite polynomials has the form:*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Proof:

Start by using the generating function for a Hermite polynomial.

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

Next, use Taylor's expansion around $t = 0$ to obtain

$$\sum_{n=0}^{\infty} \left[\frac{\partial^n}{\partial t^n} e^{2tx-t^2}\right]_{t=0} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

Continue by equating coefficients of t^n to obtain

$$\begin{aligned} H_n(x) &= \left[\frac{\partial^n}{\partial t^n} e^{2xt-t^2}\right]_{t=0} \\ &= \left[\frac{\partial^n}{\partial t^n} e^{x^2-(x-t)^2}\right]_{t=0} \\ &= e^{x^2} \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} \end{aligned}$$

We note that $\frac{\partial^n}{\partial t^n} f(x-t) = (-1)^n \frac{\partial^n}{\partial x^n} f(x-t)$ for all functions f.

This leads to

$$H_n(x) = e^{x^2} \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0}$$

$$= (-1)^n e^{x^2} \left[\frac{\partial^n}{\partial x^n} e^{-(x-t)^2} \right]_{t=0}$$

Thus we have the required result.

The next theorem gives us the three-term recurrence relation for the Hermite polynomials, given in([23], p31).

Theorem 3.1.3 *Hermite polynomials satisfy the recurrence relation*

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),$$

where $n = 1, 2, ...$ *and* $x \in \mathbb{R}$.

Proof:

Differentiating the generating function, given in Definition 3.1.1, with respect to t gives

$$2(x-t)e^{2xt-t^2} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x),$$

and then changing the running index on the right hand side yields

$$2(x-t) \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+1}(x).$$

Equate coefficients of t^n on both sides so that

$$2x \frac{H_n(x)}{n!} - \frac{2H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{n!},$$

which implies that $2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)$. This proves the result.

Another recurrence relation which is interesting to note, is found in ([2], p280).

Theorem 3.1.4

$$H_n'(x) = 2nH_{n-1}(x),$$

where $n \in \mathbb{R}$ *and* $n \geq 1$.

Proof:

Differentiating the generating function, given in Definition 3.1.1, with respect to x gives

$$2t\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}=\sum_{n=0}^{\infty}\frac{H_n'(x)t^n}{n!}.$$

Equating coefficients of t^n on both sides gives us

$$\frac{2H_{n-1}(x)}{(n-1)!}=\frac{H_n'(x)}{n!}$$

from which it follows that

$$2nH_{n-1}(x)=H_n'(x),$$

thus proving the result.

The Hermite differential equation (see [42], p188) is as follows:

$$y''-2xy'+2ny=0,$$

where $y=H_n$.

3.2 Laguerre Polynomials

Laguerre polynomials, denoted $L_n^{\alpha}(x)$, are orthogonal with respect to the gamma distribution, $e^x x^{-\alpha}$. Already, we notice that the Laguerre polynomial has a parameter α, unlike the Hermite polynomial.

The following definition may be found in [2] and [42].

Definition 3.2.1 *The* ***Laguerre polynomial*** *is defined as*

$$L_n^{\alpha}(x)=\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{(\alpha+1)_k k!},$$

where $\alpha>-1$ *and* $n=0,1,2,\ldots$.

One can see that the right hand side of the equation is a ${}_1F_1$ hypergeometric series, that is, $L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n, \alpha+1; x)$.

The Rodrigues formula for the Laguerre polynomial can be found in ([42], p204).

Theorem 3.2.1 *The Rodrigues formula for Laguerre polynomials is*

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}],$$

where $\alpha > -1$ *and* $n = 1, 2, 3, \ldots$.

Proof:

We use Leibnitz's rule for the n^{th} derivative of a product, which is given by

$$\frac{d^n}{dx^n}\{f(x)g(x)\} = \sum_{k=0}^{n}\binom{n}{k}\left[\frac{d^k}{dx^k}f(x)\right]\left[\frac{d^{n-k}}{dx^{n-k}}g(x)\right].$$

Applying this rule to the right hand side of the Rodrigues formula and letting $D = \frac{d}{dx}$, gives

$$\begin{aligned}
\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}] &= \frac{x^{-\alpha}e^x}{n!}\sum_{k=0}^{n}\binom{n}{k} D^k(e^{-x})D^{n-k}(x^{n+\alpha}) \\
&= \frac{x^{-\alpha}e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}\frac{(1+\alpha)_n}{(1+\alpha)_k}x^{k+\alpha} \\
&= \sum_{k=0}^{n}\frac{(-x)^k(1+\alpha)_n}{(n-k)!k!(1+\alpha)_k} \\
&= \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n}\frac{n!(-1)^k}{(n-k)!}\frac{1}{(1+\alpha)_k}\frac{x^k}{k!} \\
&= \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(1+\alpha)_k}\frac{x^k}{k!} \\
&= L_n^\alpha(x),
\end{aligned}$$

as required.

Below is the orthogonality property of the Laguerre polynomial ([44], p301), along with a brief proof. It can be seen that the Laguerre polynomial is orthogonal on the real line with respect to the gamma distribution.

Theorem 3.2.2

$$\int_{n}^{\alpha} L_n^\alpha(x) L_m^\alpha(x) e^{-x} x^\alpha = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{nm}$$

for $\alpha > -1$ *and* $n, m = 0, 1, 2, \ldots$.

Proof:

We consider three cases:

(1) The first case is when $m < n$. Begin by using the Rodrigues formula for the Laguerre polynomial, given in Theorem 3.2.1. Hence we have

$$\begin{aligned}
&\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) x^\alpha e^{-x}\,\mathrm{d}x \\
&\quad = \frac{1}{n!}\int_0^\infty L_m^\alpha(x)\left[\frac{d^n}{dx^n} e^{-x} x^{n+\alpha}\right]\,\mathrm{d}x \\
&\quad = \frac{1}{n!}\left[L_m^\alpha(x)\frac{d^{n-1}}{dx^{n-1}} e^{-x} x^{n+\alpha}\right]_0^\infty - \frac{1}{n!}\int_0^\infty \left[\frac{d^{n-1}}{dx^n} e^{-x} x^{n+\alpha}\right]\left[\frac{d}{dx} L_m^\alpha(x)\right]\,\mathrm{d}x,
\end{aligned}$$

The first term is zero. We integrate by parts n times. This gives the value zero due to the fact that $L_m^\alpha(x)$ is a polynomial of degree m and so the derivative vanishes because $m < n$.

(2) The second case is when $m > n$. To do this, we reverse the order of the argument in case 1.

(3) The third case takes $m = n$. When $m = n$, we observe that $\frac{d^n}{dx^n} L_n^\alpha(x) = (-1)^n$. Then

$$\int_0^\infty \left(L_n^\alpha(x)\right)^2 x^\alpha e^{-x}\mathrm{d}x = \frac{1}{n!}\int_0^\infty L_n^\alpha(x)\left[\frac{d^n}{dx^n} e^{-x} x^{n+\alpha}\right]\mathrm{d}x$$

and integration by parts n times gives

$$\begin{aligned}
\int_0^\infty \left(L_n^\alpha(x)\right)^2 x^\alpha e^{-x}\mathrm{d}x &= \frac{(-1)^n}{n!}\int_0^\infty e^{-x} x^{n+\alpha}\left[\frac{d^n}{dx^n} L_n^\alpha(x)\right]\mathrm{d}x \\
&= \frac{(-1)^n}{n!}(-1)^n\int_0^\infty e^{-x} x^{n+\alpha}\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{n!}\Gamma(n+\alpha+1),$$

which proves the result.

The definition given below is that of the generating function of the Laguerre polynomial. This definition can be found in [[32], p99] and is used in the proof of the following theorem.

Definition 3.2.2 *The* ***generating function of the Laguerre polynomial*** *is given as*

$$\frac{1}{(1-t)^{\alpha+1}}e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n,$$

where $|t| < 1$.

The next theorem gives the three-term recurrence relation. The proof can be found in ([2], p283).

Theorem 3.2.3 *The Laguerre polynomial satisfies the following three- term recurrence relation*

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+\alpha+1-x)L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x),$$

when $\alpha > -1$ *and* $n = 1, 2, 3, ...$

Proof:

We use the generating function for a Laguerre polynomial given by

$$\frac{1}{(1-t)^{\alpha+1}}e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n,$$

where $|t| < 1$.

Next, differentiate both sides of the generating function with respect to t to get

$$-\frac{1}{1-t}\frac{x}{(1-t)^2}e^{\frac{-xt}{1-t}} + \frac{1}{(1-t)^2}e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} nL_n^{\alpha}(x)t^{n-1}.$$

Once again, using the generating function gives

$$-\frac{x}{(1-t)^2}\sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n + \frac{1}{(1-t)}\sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n = \sum_{n=0}^{\infty} nL_n^{\alpha}(x)t^{n-1}.$$

Now, multiply both sides by $(1-t)^2$,

$$-x\sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n + (1-t)\sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n = (1-2t+t^2)\sum_{n=0}^{\infty} nL_n^{\alpha}(x)t^{n-1}.$$

Lastly, equate coefficients of t^n so that one has

$$(n+1)L_n^{\alpha}(x) = (2n+1-x)L_n^{\alpha} - nL_n(x)t^{n-1}$$

which proves the result.

There are a few useful relationships between the Laguerre and Hermite polynomials, which can also be found in ([12], p284).

- $H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{-\frac{1}{2}}(x^2), x \in \mathbb{R}, m = 0, 1, 2, ...$
- $H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{\frac{1}{2}}(x^2), x \in \mathbb{R}, m = 0, 1, 2, ...$

The Laguerre differential equation from ([12], p149) is given below:

$$xy'' + (\alpha + 1 - x)y' + ny = 0,$$

where $y = L_n^{\alpha}(x)$. It should be noted that the Laguerre polynomial is the solution to the Laguerre second-order differential equation.

3.3 Jacobi Polynomials

As the Jacobi polynomial is the central topic of study in this paper, this section shall be slightly more detailed than previous sections.

The Jacobi polynomial is an orthogonal polynomial that has two parameters, usually denoted by α and β. The formal definition can be found below and is given in reference texts ([4], p7), ([42], p143) and ([34], p218).

Definition 3.3.1 *The **Jacobi polynomials** of degree n are defined by*

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

where $\alpha, \beta > -1$ and $-1 < x < 1$.

Note:

1. When $\alpha = \beta = 0$, we have the Legendre polynomial.
2. When $\alpha = \beta = \frac{1}{2}$, we have the Chebyshev polynomials of the second kind.
3. When $\alpha = \beta = -\frac{1}{2}$, we have the Chebyshev polynomials of the first kind.
4. When $\alpha = \beta$, we have the Gegenbauer polynomial.

Jacobi polynomials can also be defined via a hypergeometric series:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right). \tag{3.12}$$

Orthogonality is given by the following theorem and is found, along with the proof in ([42], p259).

Theorem 3.3.1 *The orthogonality property of the Jacobi polynomials is given as follows*

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = 2^{\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)(\alpha+\beta+2n+1)n!}\delta_{nm},$$

where $\alpha, \beta > -1$ and $n, m \geq 0$.

Proof:

Consider the case when $m = n$. We also use the identity given in Equation (3.12) along with the definition of the beta function. The following is then calculated.

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\{P_n^{\alpha,\beta}(x)\}^2\mathrm{d}x = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} P_n^{\alpha,\beta}(x)(1-x)^{n+\alpha}(1+x)^{n+\beta}\mathrm{d}x \\
&= \frac{(n+\alpha+\beta+1)_n}{2^{2n}n!}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\mathrm{d}x \\
&= \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)2^{2n}n!}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\mathrm{d}x.
\end{aligned}
$$

We then use the substitution $1-x=2t$ then $\mathrm{d}x=-2\mathrm{d}t$. This gives

$$
\begin{aligned}
\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\mathrm{d}x &= \int_{0}^{1}(2t)^{n+\alpha}(2-2t)^{n+\beta}2\mathrm{d}t \\
&= 2^{2n+\alpha+\beta+1}\int_{0}^{1} t^{n+\alpha}(1-t)^{n+\beta}\mathrm{d}t \\
&= B(n+\alpha+1, n+\beta+1)2^{2n+\alpha+\beta+1} \\
&= 2^{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}
\end{aligned}
$$

.

Now, if we had the case where $m \neq n$, we may choose m to be the larger (or interchange n and m) and conclude that

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x) = 0.$$

This occurs as $P_n^{(\alpha,\beta)}(x)$, on the right hand side of the orthogonality condition, is a polynomial of degree n.
Thus proving the result.

Note that when $\alpha=\beta=0$, we have the Legendre case for orthogonality.

An important property of the Jacobi polynomial is that the polynomial is only orthogonal with respect to the weight function $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$ when $\alpha>-1$ and $\beta>-1$.

Below, we have the three-term recurrence relation for the Jacobi polynomial given in ([12], p153) as well as ([32], p84), in which the proof can be found.

Theorem 3.3.2 *The Jacobi polynomials have the following three-term recurrence rela-*

tion for $n = 2, 3, 4 ...$*:*

$$2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x) = (2n+\alpha+\beta-1)[(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2]$$

$$\times P_{n-1}^{(\alpha,\beta)}(x) - 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x).$$

The recurrence relation can be useful in estimating the Jacobi polynomial at the x-coordinate (abscissa) of a point in the interval $[-1, 1]$.

The Rodrigues formula for the Jacobi polynomial is found in ([42], p257) and is provided below.

Theorem 3.3.3 *The Rodrigues formula for the Jacobi polynomials is of the form*

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\},$$

for $n \geq 0$ *and* $\alpha, \beta \in \mathbb{R}$*.*

Proof:

The Jacobi polynomial can be written in the form

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\beta)_n(x-1)^k(x+1)^{n-k}}{2^n k!(n-k)!(1+\alpha)_k(1+\beta)_{n-k}}. \tag{3.13}$$

Now, if $D = \frac{d}{dx}$ then for non-negative integrals f and g,

$$\begin{aligned} D^f x^{g+\alpha} &= (g+\alpha)(g+\alpha-1)...\ (g+\alpha-f+1)x^{g-f+\alpha} \\ &= \frac{(1+\alpha)_g x^{g-f+\alpha}}{(1+\alpha)_{g-f}}. \end{aligned}$$

From the above equation, we obtain the following identities,

$$D^k(x+1)^{n+\beta} = \frac{(1+\beta)_n(x+1)^{n-k+\beta}}{(1+\beta)_{n-k}}$$

and

$$D^{n-k}(x-1)^{n+\alpha} = \frac{(1+\alpha)_n (x-1)^{k+\alpha}}{(1+\alpha)_k}.$$

Hence, Equation (3.13) can be put into the form

$$P_n^{(\alpha,\beta)}(x) = \frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{2^n n!} \times \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}[D^{n-k}(x-1)^{n+\alpha}][D^k(x+1)^{n+\beta}].$$

Now use Leibnitz' rule for the derivative of a product, given below as

$$\frac{d^n}{dx^n}\{f(x)g(x)\} = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{d^k}{dx^k}f(x)\right] \left[\frac{d^{n-k}}{dx^{n-k}}g(x)\right],$$

which yields the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\}.$$

A consequence of the Rodrigues' formula is the symmetry property of the Jacobi polynomial.

To show this, we must take the n^{th} order derivative of the Rodrigues' formula and expand it to get the identity

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k.$$

We then replace x with $-x$ in the above formula to obtain

$$P_n^{(\alpha,\beta)}(-x) = -P_n^{(\beta,\alpha)}(x). \tag{3.14}$$

From calculus, we know that

- If $y = f(x)$ is an even function then $f(-x) = f(x)$
- If $y = f(x)$ is an odd function then $f(-x) = -f(x)$.

Further reading on odd and even functions can be found in ([19], p26).

Hence, we can conclude that the Jacobi polynomial is an odd or even function depending on the degree, n, of the polynomial.

The Jacobi polynomial is the solution the following second-order, homogeneous equation;

$$(1-x^2)y'' + [\beta - \alpha - x(\alpha+\beta+2)]y' + n(n+\alpha+\beta+1)y = 0,$$

where $y = P_n^{(\alpha,\beta)}(x)$. The above equation is also known as the Jacobi differential equation, ([42], p258).

One can use any mathematical programming software (such as Maple, Matlab or Mathematica) to obtain the Jacobi polynomial of n^{th} degree. Calculating the polynomial by hand is harder and prone to error, therefore using a program is preferred.

For example, if one wanted to work out the 3rd Jacobi polynomial (i.e when n=3), in Wolfram Mathematica 9, one would write the following code:

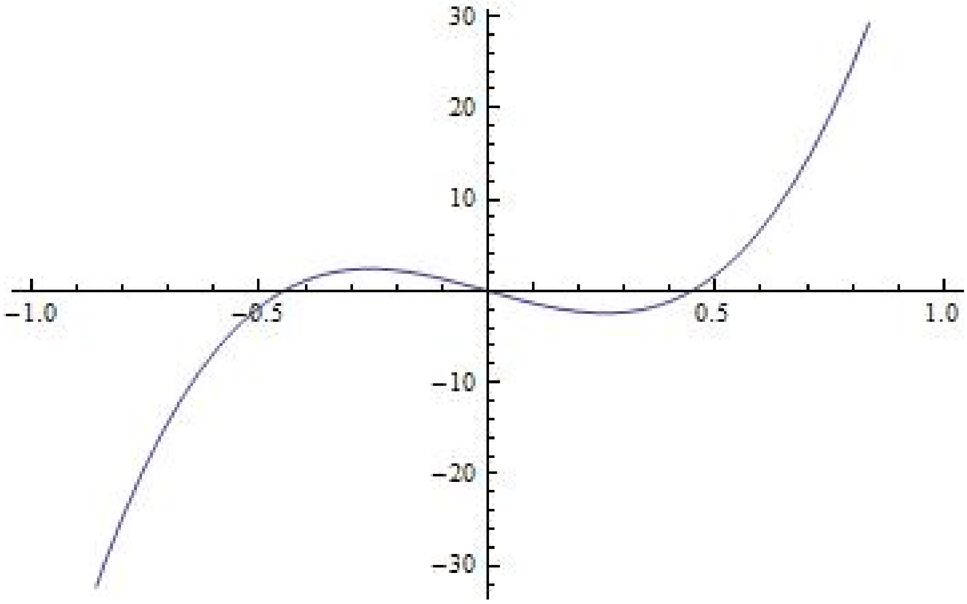
JacobiP[3, a, b, z],

and receive the following output:

$$\frac{1}{6}(1+a)(2+a)(3+a)+\frac{1}{4}(2+a)(3+a)(4+a+b)(-1+z)+\frac{1}{8}(3+a)(4+a+b)(5+a+b)(-1+z)^2$$
$$+\frac{1}{48}(4+a+b)(5+a+b)(6+a+b)(-1+z)^3.$$

The above polynomial is the resulting Jacobi polynomial of degree 3. Mathematica has a built-in function, JacobiP to calculate the polynomial which is most useful. We can plot the Jacobi polynomial of degee 3 where $\alpha = 5$ and $\beta = 5$ as follows

Plot[JacobiP[3, 5, 5,x], x, -1,1]



We see that in the interval $[-1, 1]$ the graph cuts the x-axis at three points, hence implying the polynomial has 3 roots, that is 3 zeros.

Chapter 4

Zeros of the Jacobi Polynomial

The problem of determining the zeros (roots) of a polynomial dates back to approximately 1700 B.C. Evidence of this is provided in the Yale Babylon collection in which a cuneiform (writing system that was used in ancient Middle East) table dating to roughly the same time period is stored. This table gives a 60-base number equivalent to the approximation of the number $\sqrt{2}$, ([10], p46).

In general, a polynomial of degree n has at most n distinct complex zeros ([53], p39). This concept can be extended to the set of orthogonal polynomials. The theorem given below is that of the zeros of orthogonal polynomials (and therefore of the Jacobi polynomial). The theorem and its proof is provided by ([42], p149) and ([2], p253).

Theorem 4.0.1 *If the sequence of real polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to a weight function $w(x) > 0$ over the interval $[a,\ b\ ]$, the zeros of $\{p_n(x)\}_{n=0}^{\infty}$ are distinct and all lie in the interval $[a,b]$.*

Proof:

Suppose that $\int_a^b p_n(x)x^k w(x)dx = 0$ for all $k = 0, 1, ..., (n-1)$. Since x^k forms a simple set, there exists constant $b(k,m)$ such that $p_m(x) = \sum_{k=0}^{m} b(k,m)$. This result is derived from the well known result which states that any polynomial of degree $\leq n$ can be expressed as a linear combination of the elements of that simple set.

We continute by letting $m < n$ then

$$\int_a^b p_n(x)p_m(x)w(x)\mathrm{d}x = \sum_{k=0}^{m} \int_a^b p_n(x)x^k w(x)dx$$
$$= 0.$$

The result follows as m (and hence) k are $< n$.

If $m > n$ then we interchange m and n in the above argument.

Now,we suppose that $\int_a^b p_n(x)x^k w(x)dx = 0$ where $n \neq m$. Then $p_n(x)$ forms a simple set, so that there exist constants $a(m,k)$ such that $x^k = \sum_{m=0}^{k} a(m,n)p_m(x)$. For any k in the range $0 \leq k < n$ we have

$$\int_a^b p_n(x)x^k w(x)dx = \sum_{m=0}^{\infty} a(m,k) \int_a^b p_n(x)p_m(x)w(x)dx$$
$$= 0 \; since \; m \leq k < n \; so \; m \neq n,$$

thus proving the result.

There are many varied methods used to calculate the zeros of orthogonal polynomials. Some of these methods include numerical analysis techniques. This includes the implementation of the Newton-Raphson method as shown in ([33], p230). Further reading on numerical analysis of polynomials can be found in ([10], p45-p100). The asymptotic approximation of zeros is another method that deserves mention. This is implemented in great detail for orthogonal polynomials in [5] and for Jacobi polynomials in ([35], p195-p234) and ([38], p98-p113).

A concept related to the calculation of the zeros of Jacobi polynomials is that of Gaussian quadrature. Quadrature is the method of estimating the definite integral of a function. This is usually stated as a sum of the weighted values of a function at specified points, evaluated within the domain of integration. The definition of quadrature from [30] is given below.

Definition 4.0.1 *An n-point quadrature rule is given below:*

$$\int_a^b f(x)dx \cong \sum_{k=1}^{n} w_k f(x_k),$$

for a set of nodes x_k and weights, denoted w_k.

However, a quadrature rule is only termed as 'Gaussian' if the following equation holds true for polynomials of degree $\leq 2n-1$:

$$\int_a^b w(x)f(x)dx \cong \sum_{k=1}^{n} w_k f(x_k).$$

That is for all polynomials of degree $\leq 2n-1$, the integral on the left hand side of the equation exactly equals the expression given on the right hand side of the equation [48]. Therefore, it is sensible to note that the Gauss-Jacobi quadrature rule is associated with the Jacobi weight function, $w(x) = (1+x)^{\alpha}(1-x)^{\beta}$ and the interval $[-1, 1]$.

When considering the Gaussian quadratures, one can see that the concept lets one have the freedom to pick the weight functions and also the nodes at which the function will be approximated. This allows higher order polynomials to be integrated. By choosing the correct weight, one can make the integral exact for a class of integrands associated with that weight. It must be noted that the weights do not have to be simple numbers. The Fundamental Theorem of Gaussian quadratures essentially says that the nodes of the Gaussian quadrature along with the weight function $w(x)$ in the interval (a, b) are exactly the zeros of the orthogonal polynomial $p_n(x)$ for the same weight function and interval [40].

Gaussian quadrature will only produce exact and accurate results if the function is well approximated by a polynomial function within the range $[-1, 1]$. It is logical, therefore, to assume that the Gaussian quadrature will not work for singularities. Also, Gaussian quadrature can be derived for intervals $[a, b]$ but these intervals will have to be mapped to the interval $[-1, 1]$.

However, the fastest and most elegant way to calculate the zeros of the Jacobi polynomial is by using mathematical programs such as Matlab, Mathematica and Maple. These programs are only a few of the type available to help solve mathematical problems. Programs such as Mathematica have built-in functions to compute the zeros of polynomials. In the example below, the input is coded to calculate the zeros of the Jacobi function of order 10 with $\alpha = 5$ and $\beta = 5$. This code should return 10 roots.

```
N[x/.NSolve[JacobiP[10,5,5,x]=0,x],],
```

gives the output

$$\{-0.838508, -0.688808, -0.512678, -0.316109, -0.106808, 0.106808,$$

$$0.316109, 0.512678, 0.688808, 0.838508\},$$

i.e. 10 roots as required.

4.1 Properties of Zeros of Jacobi Polynomials

This section concentrates on the interlacing property of zeros. Much of the material considered for this section may be found in [16] which investigated the constraints of the interlacing property on the zeros of the Jacobi polynomials where α and β differ.

Before proceeding with the analysis of the results of this paper, a few theorems and identities need to be introduced. These are essential in the further understanding of the interlacing property and its effects on the zeros of the Jacobi polynomial.

The theorem given below can be found in [[2], p253]

Theorem 4.1.1 *Given that $\{P_n(x)\}$ is a sequence of orthogonal polynomials with respect to the weight function, $d\alpha(x)$ on the interval $[a, b]$. Then the Jacobi polynomial, $P_n(x)$ has n simple zeros in the closed interval $[a, b]$.*

Proof:

We are given that Jacobi polynomial $P_n(x)$ has m distinct zeros $x_1, x_2, ..., x_m$ in the closed interval $[a, b]$ that have odd order. Then we have

$$Q(x) = P_n(x)(x - x_1)(x - x_2)...(x - x_m) \geq 0,$$

for x in $[a, b]$. If $m < n$, then by the property of orthogonality

$$\int_a^b Q(x)dx = 0.$$

The inequality $P_n(x)(x-x_1)(x-x_2)...(x-x_m) \geq 0$ implies that the integral $\int_a^b Q(x)dx = 0$ should only be positive. This results in a contradiction which implies that $m = n$ and that the zeros of the Jacobi polynomial $P_n(x)$ are simple.

The following theorem and its proof is provided by([2], p253).

Theorem 4.1.2 *The zeros of the polynomials $p_n(x)$ and $p_{n+1}(x)$ separate each other where $n \in \mathbb{R}$.*

Proof:

The proof requires the following result which can be found as a corollary in [[4], p247].

$$p_n'(x)p_n(x) - p_n(x)p_n'(x) > 0,$$

for all x.

Using the result,

$$p_n'(x)p_n(x) - p_n(x)p_n'(x) < 0.$$

Let $x_{k,n+1}$ be a zero of the Jacobi polynomial $p_n(x)$, we observe that

$$p_n'(x_{k,n+1})p_{n+1}'(x_{k,n+1}) > 0.$$

Owing to the simplicity of the zeros, $p_n'(x_{k,n+1})$ and $p_{n+1}'(x_{k,n+1})$ have different signs. As the Jacobi polynomial, $p_n(x)$ is continuous, we know it has a zero between $x_{k,n+1}$ and $x_{k+1,n+1}$ for $k = 1, 2, ..., n$. This proves the result.

Furthermore, from ([2], p254) and ([12], p253), we have another property of the zeros of orthogonal polynomials. The proof can be found in the relevant reference texts. Interestingly, the proof of the theorem below involves the use of Gaussian quadrature and is an extension of Theorem 4.1.1.

Theorem 4.1.3 *If $m < n$, then between any two zeros of $p_m(x)$ there is a zero of $p_n(x)$.*

Proof:

We now suppose that there is no zero between $x_{k,m}$ and $x_{k+1,m}$ of $p_n(x)$. Consider the polynomial given below

$$r(x) = \frac{p_m(x)}{(x - x_{k,m})(x - x_{k+1,m})}.$$

It is observed that $r(x)p_m(x) \geq 0$ where $x \notin (x_{k,m}, x_{k+1,m})$. Using the Gaussian quadrature formula given in Theorem 4.0.1, we obtain

$$\int_a^b p_m(x) d\alpha(x) = \sum_{j=1}^{n} g(x_{j,n}) p(x_{j,n}).$$

Since $g(x_{j,n})p(x_{j,n}) \geq 0$ and exists for all $j = 1, ..., n$, the sum on the left hand side is positive. However, due to orthogonality properties, the integral is equal to 0. This is a contradiction and hence the result is proved.

Consider a sequence of orthogonal polynomials $p_n(x)_{n=0}^{\infty}$ where the zeros are real and simple [16]. Let the zeros of the polynomial $p_n(x)$ be denoted by

$$x_{i,n}, \quad \text{where} \quad i = 1, ..., n.$$

Suppose that the zeros of $p_k(x)$ where $k < n$ are

$$x_{1,k} < x_{2,k} < ... < x_{k,k},$$

and that

$$x_{1,k-1} < x_{2,k-1} < ... < x_{k-1,k-1},$$

are the zeros of polynomial $p_{k-1}(x)$.

The zeros are clearly decreasing over the sequence of polynomials and by Theorem 4.1.2 for $k < n$, it follows that between any two consecutive zeros of $p_k(x)$ there is a zero of $P_{k-1}(x)$, that is

$$x_{1,k} < x_{1,k-1} < x_{2,k} < x_{2,k-1} < ... < x_{k-1,k-1} < x_{k,k}.$$

This is generally known as the interlacing of zeros property. This property can be extended to the zeros of the Jacobi polynomial, see [14].

A lemma provided by the paper ([16], p321) is used repeatedly throughout the proofs of the results.

It should be noted that in many instances, the notation used in [16] is $p_n^{(\alpha,\beta)}$ where it is understood that the variable is x. This notation is retained in the statment of the theorems. However, in the proofs of theorems $p_n^{(\alpha,\beta)}(x)$ will be used.

Lemma 4.1.1 *We let $\{p_n(x)\}_n^\infty$ be a sequence of orthogonal polynomials on the open interval (c, d). The interval can be finite or infinite in nature. Let f_{n-1} be a polynomial of degree $n-1$ that satisfies the equation*

$$f_{n-1}(x) = a_n(x)p_{n+1}(x) - (x - A_n)b_n(x)p_n(x) \tag{4.1}$$

for each $n \in \mathbb{N}$, some constant A_n, and some functions of x, $a_n(x)$ and $b_n(x)$ with $b_n(x) \neq 0$ for $x \in (c, d)$. Then, for each $n \in \mathbb{N}$ the following hold true:

(i) The zeros of f_{n-1} are all real and simple. The zeros, together with the point A_n interlace with the zeros of the polynomial p_{n+1} if f_{n-1} and p_{n+1} are co-prime.

(ii) If f_{n-1} and p_{n+1} are not co-prime, then at $x = A_n$, a common zero is located and the $n-1$ zeros of f_{n-1} interlace with the remaining n zeros of p_{n+1}.

Proof:

Let $w_1 < w_2 < ... < w_{n+1}$ denote the zeros of p_{n+1}.

(i) Since p_n and p_{n+1} are always co-prime and hence, by assumption, $b_n \neq 0$ for all $x \in (c, d)$, and p_{n+1} and f_{n-1} are co-prime, it is implied that from (3.12) that $A_n \neq w_k$ for any $k \in 1, 2, ..., n+1$. We now evaluate (3.12) at the zeros w_k and w_{k+1} to obtain

$$\frac{f_{n-1}(w_k)g_{n-1}(w)_{k+1}}{p_n(w_k)p_n(w_{k+1})} = (w_k - A_n)(w_{k+1} - A_n)b_n(w_k)b_n(w_{k+1}), \tag{4.2}$$

for each $k \in 1, 2, ..., n$. Since w_k and $w_{k+1} \in (c, d)$ while b_n keeps constant in terms of signage in the interval (c, d). Therefore, the right hand side of (4.2) is positive iff $A_n \notin (w_k, w_{k+1})$. As the zeros of p_{n+1} and p_n are interlacing, f_{n-1} has a different sign at consecutive zeros of p_{n+1} and hence, has an odd number of zeros in each interval (w_k, w_{k+1}) where $k \in 1, 2, ..., n$, apart from one interval that may contain A_n. We then use the Intermediate Value Theorem to deduce that for each $n \in \mathrm{N}$ the $n-1$ simple zeros of f_{n-1}, together with A_n, interlace with the $n+1$ zeros of p_{n+1}.

(ii) If p_{n+1} and f_{n-1} have common zeros, it follows from (3.12)that there can only be a single common zero at $x = A_n$ as p_n and p_{n+1} are co-prime. For $x \neq A_n$, one can rewrite (3.12) as

$$F_{n-2}(x) = a_n(x)p_n(x) - b_n(x)p_n(x), \tag{4.3}$$

where $(x - A_n)F_{n-2}(x) = f_{n-1}(x)$ and $(x - A_n)p_n(x) = p_{n+1}(x)$. Let v_j where $j = 1, 2, ..., n$ and $v_1 < v_2 < ... < v_n$ be the n non-common zeros of p_{n+1}. At most one interval of the form $(v_k, v_{k+1)}$ can contain the point A_n where $k \in \{1, ..., n-1\}$. We evaluate (4.3) at v_k and v_{k+1} for each $k \in \{1, ..., n-1\}$ such that the point $A_n \notin (v_k, v_{k+1})$ to obtain

$$F_{n-2}(v_k)F_{n-2}(v_{k+1}) = b_n(v_k b_n(v_{k+1)}p_n(v_k)p_n(v_{k+1}) < 0.$$

It follows that F_{n-2} has an odd number of zeros in each interval (v_k, v_{k+1}) that does not contain A_n and for $k \in \{1, ..., n-1\}$. Since there are $n-2$ of these intervals and the degree of F_{n-2} is $n-2$, there are at most $n-2$ such intervals. We can then deduce that $A_n = w_j$ and that the zeros of F_{n-2} together with the point A_n

interlace with the n zeros of p_n. Hence we have our result from the result of F_{n-2} and P_n.

We again consider the Equation (3.12) below which expresses the Jacobi polynomial in terms of the ${}_2F_1$ hypergeometric polynomial, that is

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right).$$

The identities given below are extensions of the contiguous relations of the ${}_2F_1$ hypergeometric formula. They are also referred to as mixed recurrence relations. They can be found in ([42], p50-p53). A summary of the equations can be found in ([42], p71).

Note that for the following expressions, we represent ${}_2F_1(-n, b; c; z)$ by F_n, ${}_2F_1(-n-1, b+1; c; z)$ by $F_{n+1}(b+)$, ${}_2F_1(-n+1, b+1; c-3; z)$ by $F_{n-1}(b+, c-3)$ and so on. We then have the following identities which will aid in the proof of subsequent theorems in this section.

(A) $\left(\frac{b(c+n)}{(b+n)(b+n+1)} - z\right) F_n = \frac{b(c+n)}{(b+n)(b+n+1)} F_{n+1}(b+) + \frac{n(b-c)z}{c(b+n)} F_{n-1}(c+)$

(B) $\left(\frac{c)}{b+n+1} - z\right) F_n = \frac{c}{(b+n+1)} F_{n+1}(b+) + \frac{(b-c)nz^2}{c(c+1)} F_{n-1}(b+, c+2)$

(C) $\left(\frac{c(c+1)}{(b+1)(c+n+1)} - z\right) F_n = \frac{c+c^2-bnz+cnz}{(b+1)(c+n+1)} F_{n+1}(b+) + \frac{n(b-c)(b+n+1)z^3}{c(c+1)(c+2)} F_{n-1}(b+2, c+3)$

(D) $\left(1 - \frac{(b+1)(2+c+2n)-cn}{c(c+2)} z\right) F_n = \left(1 - \frac{2(b-c)n}{c(c+2)} z - \frac{n(b-c)(1+b+n)}{c(c+1)(c+2)} z^2\right) F_{n+1}(b+)$
$+ \frac{a}{c^2(c+1)^2(c+2)^2(c+3)} F_{n-1}(b+3)(c+4)$, where $a = (b+1)(b+2)(b-c)(c+n+1)(c+n+2)(1+b+n)z^4 n$.

The first theorem of [16] considers the results obtained when interlacing occurs between the zeros of Jacobi polynomials of different sequences. These polynomials have degrees that differ by two.

Theorem 4.1.4 *(i) If the Jacobi polynomials $P_{n-1}^{(\alpha+t,\beta)}$ and $P_{n+1}^{(\alpha,\beta)}$ are co-prime then*

(a) the zeros of the polynomial $P_{n-1}^{(\alpha+t,\beta)}$ and $\frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}$ when $t = 0, 1, 2$;

(b) *the zeros of the polynomial* $P_{n-1}^{(\alpha+3,\beta)}$ *and* $\frac{n(n+\alpha+\beta+2)+(\alpha+2)(n-\alpha+\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}$ *interlace with the zeros of* $P_{n+1}^{(\alpha,\beta)}$*;*

(c) *the zeros of the polynomial* $P_{n-1}^{(\alpha+4,\beta)}$ *and* $\frac{2n(n+\alpha+\beta+3)+(\alpha+3)(\beta-\alpha)}{2n(n+\alpha+\beta+3)+(\alpha+3)(\alpha+\beta+2)}$ *interlace with the zeros of* $P_{n+1}^{(\alpha,\beta)}$*.*

(ii) *If the Jacobi polynomials* $P_{n-1}^{(\alpha+t,\beta)}$ *and* $P_{n+1}^{(\alpha,\beta)}$ *are not co-prime, then these polynomials have only one common zero. This zero has been located at the respective points established in (i)(a) through to (i)(c). Hence, the (n-1) zeros of* $P_{n-1}^{(\alpha+t,\beta)}$ *interlace with the remaining n zeros of* $P_{n+1}^{(\alpha,\beta)}$*.*

As the proof is extremely long, we prove each section separately.

Proof (i)(a):

To prove that the zeros of $P_{n-1}^{(\alpha+t,\beta)}(x)$ and $\frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)(x)}$, we must use the three-term recurrence relation of the Jacobi polynomial, given in Theorem 3.3.2, to get an equation of the form shown in (4.1) so that the results of Lemma 4.1.1 can be implemented.

When $t = 0$, it is given that the polynomials $P_{n-1}^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta)}(x)$ are co-prime (i.e. have no common zeros). We need to prove that the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$ and $\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)(x)}$. When $t = 0$, $\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$ is constant and corresponds to $A_n$in (4.1).

We then write the alternate form of the three-term recurrence relation of the Jacobi polynomial found in [47]

$$\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x) = \left(x - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)P_n^{(\alpha,\beta)}(x)$$

$$-\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x),$$

in an equation of the form given in (4.1) where $A_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$. Now rearranging the terms in the above difference equation, we obtain

$$\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x) = \left(\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta(x))}\right)$$

$$-\left(x - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)P_n^{(\alpha,\beta(x))}.$$

Then comparing this expression with Lemma 4.1.1, we see that the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$ and $\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$. As $P_{n-1}^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta)}(x)$ are co-prime, the conditions for Lemma 4.1.1 are satisfied.

For $t=1$, we must use identity (A) from the listed identities following Lemma 4.1.1. We want (A) to resemble the three term recurrence relation, to be able to apply the results of Lemma 4.1.1.

It must be proved that the zeros of $P_{n-1}^{(\alpha+1,\beta)(x)}$ and $\frac{\beta^2-\alpha^2+(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$.

We begin by looking at the first term in identity (A), $\left(\frac{b(c+n)}{(b+n)(b+n+1)} - z\right)F_n$. We then let $b=\alpha+\beta+n+1$, $c=\alpha+1$ and $z=\frac{1-x}{2}$. We obtain the expression

$$\left(\frac{2(\alpha+\beta+n+1)(\alpha+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)} - \frac{1}{2} + \frac{x}{2}\right)F_n,$$

which, when reduced, yields

$$\frac{1}{2}\left(x - \frac{\beta^2-\alpha^2+(\beta-\alpha+2n(\alpha+\beta+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\right)F_n.$$

We can replace the F_n with ${}_2F_1(-n,b;c;z)$. Then use the identity given in Equation (3.12) to get a final expression for the first term, namely

$$\frac{1}{2}\left(x - \frac{\beta^2-\alpha^2+(\beta-\alpha+2n(\alpha+\beta+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\right)\frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(x). \tag{4.4}$$

The second term of identity (A), $\left(\frac{b(c+n)}{(b+n)(b+n+1)}\right) F_{n+1}(b+)$ is now considered. Keeping the following values, $b = \alpha+\beta+n+1$, $c = \alpha+1$ and $z = \frac{1-x}{2}$, the second term becomes

$$\left(\frac{(\alpha+\beta+n+1)(\alpha+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\right) F_{n+1}(b+),$$

where $F_{n+1}(b+) = {}_2F_1\left(-n-1, \alpha+\beta+n+2; \alpha+1; \frac{1-x}{2}\right)$. Using the identity from (3.12), the second term can be simplified to

$$\left(\frac{\alpha^2+2\alpha+2\alpha n+\alpha\beta+\beta+\beta n+2n+n^2+1}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\right)\frac{(n+1)!}{(\alpha+1)_{n+1}} P_{n+1}^{(\alpha,\beta)}(x). \tag{4.5}$$

Finally, the last term of identity (A), $\left(\frac{n(b-c)z}{c(b+n)}\right) F_{n-1}(c+)$, is evaluated using the same values previously assigned to b, c, and z. Therefore, the third term of (A) will become

$$\left(\frac{n(\beta+n)(\frac{1-x}{2})}{(\alpha+1)(\alpha+\beta+2n+1)}\right) F_{n-1}(c+).$$

Recall that $F_{n-1}(c+) = {}_2F_1\left(-n+1, \alpha+\beta+n+1; \alpha+2; \frac{1-x}{2}\right)$. Applying the identity in (3.12) to the expression, the final term will be

$$\left(\frac{n(\beta+n)\frac{1-x}{2}}{(\alpha+1)(\alpha+\beta+2n+1)}\right)\frac{(n-1)!}{(\alpha+2)_{n-1}} P_{n-1}^{(\alpha+1,\beta)}(x). \tag{4.6}$$

When we put the reduced terms from (4.3), (4.4) and (4.5) together again and simplify, we get the equation given below.

$$\begin{aligned} P_{n-1}^{(\alpha+1,\beta)}(x) &= \frac{n+1}{(\alpha+1)(\alpha+n)(\alpha+n+1)(1-x)(\alpha+\beta+n+1)} \\ &\quad \times\left(\frac{\alpha^2+2\alpha+2\alpha n+\alpha\beta+\beta+\beta n+2n+n^2+1}{\alpha+\beta+2n+2}\right) P_{n+1}^{(\alpha,\beta)}(x) \\ &\quad -\frac{P_n^{(\alpha,\beta)}(x)}{(\alpha+n)(\alpha+\beta+n+1)(1-x)}\left(x-\frac{\beta^2-\alpha^2+(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+2)}\right). \end{aligned}$$

The above equation is of the form required by Lemma 4.1.1. That is, we have

$f_{n-1} = P_{n-1}^{(\alpha+1,\beta)}(x)$,

$a_n(x) = \frac{n+1}{(\alpha+n)^2(1-x)(\alpha+\beta+2n+2)}$,

$A_n = \left(\frac{\beta^2-\alpha^2+(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+2)}\right)$

and $b_n(x) = \frac{\alpha+\beta+2n+1}{(\alpha+n)(\alpha+\beta+n+1)(1-x)}$.

Since we have reduced the original mixed recurrence relation, (A), to the form necessary for Lemma 4.1.1 and all conditions stated by said lemma are satisfied, we can assume that the zeros of the polynomial $P_{n-1}^{(\alpha+t,\beta)}(x)$ and $\frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$ when $t=1$.

For $t=2$, mixed recurrence (B) is utilised. It must be proved that the zeros of $P_{n-1}^{(\alpha+t,\beta)}(x)$ and $\frac{\beta^2-\alpha^2+t(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$ when $t=2$.
Start with the first term of the recurrence relation,

$$\left(\frac{c}{b+n+1}-z\right)F_n.$$

Let $b=\alpha+\beta+n+1$, $c=\alpha+1$ and $z=\frac{1-x}{2}$.
Proceed to substitute these values into the first term, which yields

$$\left(\frac{\alpha+1}{\alpha+\beta+2n+2}-\frac{1-x}{2}\right)F_n,$$

expanding and simplifying the expression, as well as letting $F_n = {}_2F_1(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2})$ gives the equation

$$\frac{1}{2}\left(x-\frac{\beta+2n-\alpha}{2(\alpha+\beta+2n+2)}\right)F_n.$$

By applying identity (3.12) to $F_n = {}_2F_1(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2})$, the first term of mixed recurrence (B) is reduced to

$$\frac{1}{2}\left(x-\frac{\beta+2n-\alpha}{2(\alpha+\beta+2n+2)}\right)\frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(x). \tag{4.7}$$

The second term of (B) is

$$\left(\frac{c}{b+n+1}\right)F_{n+1}(b+).$$

Using the same values for b, c, and z as for the first term, the second term becomes

$$\left(\frac{\alpha+1}{\alpha+\beta+2n+2}\right)F_{n+1}(b+).$$

We know that $F_{n+1}(b+) = {}_2F_1\left(-n-1, \alpha+\beta+n+2; \alpha+1; \frac{1-x}{2}\right)$. Using Equation (3.12), the second term can finally be written in the form

$$\left(\frac{\alpha+1}{\alpha+\beta+2n+2}\right)\frac{(n+1)!}{(\alpha+1)_n}P_{n+1}^{(\alpha,\beta)}(x). \tag{4.8}$$

Now, evaluate the last term of mixed recurrence relation (B):

$$\frac{(b-c)z^2 n}{c(c+1)}F_{n-1}(b+, c+2).$$

Once again, let $b = \alpha+\beta+n+1$, $c = \alpha+1$ and $z = \frac{1-x}{2}$. With the new values of b, c, and z, the third term becomes

$$\left(\frac{n(\beta+n)(1-x)^2}{4(x+2)(x+1)}\right)F_{n-1}(b+, c+2).$$

Replace $F_{n-1}(b+, c+2)$ with ${}_2F_1\left(-n+1, b+1; \alpha+2; \frac{1-x}{2}\right)$ and simplify the equation. This yields

$$\left(\frac{n(\beta+n)(1-x)^2}{4(\alpha+1)(\alpha+2)}\right)\frac{(n-1)!}{(\alpha+1)_{n-1}}P_{n-1}^{(\alpha+2,\beta)}(x). \tag{4.9}$$

Take (4.7), (4.8) and (4.9) and manipulate them so we have an expression of the form required by Lemma 4.1.1. We put the equations in an expression that is analagous to that of Equation (3.12) to get the following

$$\begin{aligned}&\left(\frac{n(\beta+n)(1-x)^2}{4(\alpha+1)(\alpha+2)}\right)\frac{(n-1)!}{(\alpha+1)_{n-1}}P_{n-1}^{(\alpha+2,\beta)}(x)\\ &\quad= \frac{1}{2}\left(x - \frac{\beta+2n-\alpha}{2(\alpha+\beta+2n+2)}\right)\frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(x) - \left(\frac{\alpha+1}{\alpha+\beta+2n+2}\right)\frac{(n+1)!}{(\alpha+1)_n}P_{n+1}^{(\alpha,\beta)}(x).\end{aligned}$$

We then multiply each term of the expression by $\frac{n(n-1)!}{(\alpha+1)_n}$ to get

$$\begin{aligned}\left(\frac{(\beta+n)(1-x)^2(\alpha+n+1)}{4(\alpha+1)(\alpha+2)}\right)P_{n-1}^{(\alpha+2,\beta)}(x) &= \frac{1}{2}\left(x - \frac{\beta+2n-\alpha}{4(\alpha+\beta+2n+2)}\right)P_n^{(\alpha,\beta)}(x)\\ &\quad - \left(\frac{(\alpha+1)(n+1)}{\alpha+\beta+2n+2}\right)P_{n+1}^{(\alpha,\beta)}(x).\end{aligned}$$

Simplifying the equation in terms of $P_{n-1}^{(\alpha+2,\beta)}(x)$ gives us

$$\begin{aligned} P_{n-1}^{(\alpha+2,\beta)}(x) &= \left(2x - \frac{\beta+2n-\alpha}{4(\alpha+\beta+2n+2)}\right) \frac{(\alpha+1)(\alpha+2)}{(\beta+n)(1-x)^2(\alpha+n+1)} P_n^{(\alpha,\beta)}(x) \\ &\quad - \left(\frac{4(\alpha+1)^2(\alpha+2)(n+1)}{(\alpha+\beta+2n+2)(\beta+n)(1-x)^2}\right) P_{n+1}^{(\alpha,\beta)}(x). \end{aligned}$$

This equation satisfies the conditions of Lemma 4.1.1 where

$f_{n-1}(x) = P_{n-1}^{\alpha+2}(x)$,

$a_n(x) = -\frac{4(\alpha+2)(n+1)(\alpha+1)^2}{(\alpha+\beta+2n+2(\alpha+n-1)1-x^2)}$,

$A_n(x) = \frac{\beta+2n-\alpha}{4(\alpha+\beta+2n+2)}$,

and $b_n(x) = \frac{(\alpha+n+1)(\alpha+2)}{(n+\beta)(1-x)^2(\alpha+n+1)}$.

Therefore, it can be assumed that the zeros of $P_{n-1}^{(\alpha+2,\beta)}(x)$ and $\frac{\beta^2-\alpha^2+2(\beta-\alpha+2n(n+\beta+1))}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$ when $P_{n-1}^{(\alpha+2)}(x)$ and $P_{n+1}^{(\alpha,\beta)}(x)$ are co-prime.

Proof (i)(b): To prove that the zeros of $P_{n-1}^{(\alpha+3,\beta)}(x)$ and $\frac{n(n+\alpha+\beta+2)+(\alpha+2)(n-\alpha+\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}$ interlace with the zeros of the Jacobi polynomial $P_{n+1}^{(\alpha,\beta)}(x)$, use mixed recurrence relation (C). In (C) replace b with $\alpha+\beta+n+1$, c with $\alpha+1$ and z with $\frac{1-x}{2}$ and use identity (3.12). Simplify the equation using appropriate identities and methods to obtain the following equation

$$\begin{aligned} &\left(x - \frac{n^2+(2\alpha+\beta+4)-(\alpha+2)(\alpha-\beta)}{(n+\alpha+2)(n+\alpha+\beta+2)}\right) P_n^{(\alpha,\beta)}(x) = \\ &\frac{(n+1)Q(x)P_{n+1}^{(\alpha,\beta)}(x)}{(n+\alpha+1)(n+\alpha+2)(n+\alpha+\beta+2)} + \frac{(1-x)^3(2n+\alpha+\beta+2)(n+\beta)}{4(n+\alpha+1)(n+\alpha+2)} P_{n-1}^{(\alpha+3,\beta)}(x). \end{aligned}$$

Note that in the above expression, $Q(x) = n(n+\beta)(x-1) + 2(\alpha+1)(\alpha+2)$.

Lemma 4.1.1 supplies the required result.

Proof (i)(c): This requires the use of mixed recurrence relation (D) with the usual substitutions of b, c and z. This, along with the correct use of the identity (3.12), yields

$$\left(x - \frac{2n^2 - (\alpha+3)(\alpha-\beta) + 2n(\alpha+\beta+3)}{R_n}\right) P_n^{(\alpha,\beta)}(x) = -\frac{(S(x))(n+1)}{2(n+\alpha+1)(\alpha+2)R_n} P_{n+1}^{(\alpha,\beta)}(x) + \frac{(1-x)^4 T_n}{8(n+\alpha+1)(\alpha+2)R_n} P_{n-1}^{(\alpha+4,\beta)}(x),$$

where $R_n = 2n(n+\alpha+\beta+3) + (\alpha+3)(\alpha+\beta+2)$,
$T_n = (2n+\alpha+\beta+2)(n+\beta)(n+\alpha+\beta+2)(n+\alpha+\beta+3)$ and $S(x)$ represents a polynomial of degree 2.
The required result is supplied from Lemma 4.1.1.

Proof (ii): This statement easily follows from the second part of Lemma 4.1.1 and the proofs of the (i)(a), (i)(b) and (i)(c) given above.
This ends the proof of Theorem 4.1.4.

The next corollary follows directly from Theorem 4.1.1. In the previous chapter, the symmetry property of Jacobi polynomials is described by identity (3.14). Using this property

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

the following corollary is generated.

Corollary 4.1.1 *(i) If the Jacobi polynomials $P_{n-1}^{(\alpha,\beta+t)}$ and $P_{n+1}^{(\alpha,\beta)}$ are co-prime then*

(a) the zeros of the polynomial $P_{n-1}^{(\alpha,\beta+t)}$ and $\frac{\beta^2-\alpha^2-t(\alpha-\beta+2n(n+\alpha+1))}{(2n+\alpha+\beta+t)(2n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}$ when $t = 1, 2$,

(b) the zeros of the polynomial $P_{n-1}^{(\alpha,\beta+3)}$ and $-\frac{n(n+\alpha+\beta+2)+(\beta+2)(n+\alpha-\beta)}{(n+\beta+2)(n+\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}$,

(c) the zeros of the polynomial $P_{n-1}^{(\alpha,\beta+4)}$ and $-\frac{2n(n+\alpha+\beta+3)+(\beta+3)(\alpha-\beta)}{2n(n+\alpha+\beta+3)+(\beta+3)(\alpha+\beta+2)}$ interlace with the zeros of $P_{n+1}^{(\alpha,\beta)}$.

(ii) If the Jacobi polynomials $P_{n-1}^{(\alpha,\beta+t)}$ and $P_{n+1}^{(\alpha,\beta)}$ are not co-prime, then these polynomials have only one common zero. This zero has been located at the respective points

established in (i)(a) through to (i)(c). Hence, the (n-1) zeros of $P_{n-1}^{(\alpha,\beta+t)}$ *interlace with the n zeros of* $P_{n+1}^{(\alpha,\beta)}$ *that are left behind.*

Note that in Theorem 4.1.4 only α was varied but in Corollary 4.1.1, only β was varied.

The next theorem looks at the effect of interlacing on the zeros of Jacobi polynomials in which α and β are varied. The proof of the following theorem is not provided as it follows much the same reasoning as the proofs given above in Theorem 4.1.4. This theorem can be found in [16].

Theorem 4.1.5 *(i) For each fixed* $i, j \in \{1, 2\}$, *if polynomials* $P_{n+1}^{(\alpha,\beta)}$ *and* $P_{n-1}^{(\alpha+i,\beta+j)}$

(a) are co-prime, then the zeros of the polynomial $P_{n-1}^{(\alpha+x,\beta+y)}$ *and* $\frac{\beta-\alpha-n(y-x)}{\alpha+\beta+2+n(4-x-y)}$ *interlace with the zeros of* $P_{n+1}^{(\alpha,\beta)}$*;*

(b) are not co-prime, they have one common zero that is located at a point in (i)(a). The (n-1) non-common zeros of $P_{n-1}^{(\alpha+x,\beta+y)}$ *interlace with the n remaining zeros of* $P_{n+1}^{(\alpha,\beta)}$.

(ii) If polynomials $P_{n+1}^{(\alpha,\beta)}$ *and* $P_{n-1}^{(\alpha+3,\beta+1)}$

(a) are co-prime, then the zeros of the polynomial $P_{n-1}^{(\alpha+3,\beta+1)}$ *and* $\frac{n^2+n(\alpha+\beta+3)-(\alpha+2)(\alpha-\beta)}{n^2+n(\alpha+\beta+3)+(\alpha+2)(\alpha+\beta+2)}$ *interlace with the zeros of* $P_{n+1}^{(\alpha,\beta)}$*;*

(b) are not co-prime, they have one common zero that is located at a point in (ii)(a). The (n-1) zeros of $P_{n-1}^{(\alpha+3,\beta+1)}$ *interlace with n remaining non-common zeros of* $P_{n+1}^{(\alpha,\beta)}$.

(iii) If polynomials $P_{n+1}^{(\alpha,\beta)}$ *and* $P_{n-1}^{(\alpha+1,\beta+3)}$

(a) are co-prime, then the zeros of the polynomial $P_{n-1}^{(\alpha+1,\beta+3)}$ *and* $\frac{-n^2-n(\alpha+\beta+3)-(\beta+2)(\alpha-\beta)}{n^2+n(\alpha+\beta+3)+(\beta+2)(\alpha+\beta+2)}$ *interlace with the zeros of* $P_{n+1}^{(\alpha,\beta)}$*;*

(b) are not co-prime, then one common zero can be found at the point located in (iii)(a). The (n-1) non-common zeros of $P_{n-1}^{(\alpha+1,\beta+3)}$ *interlace with n remaning zeros of* $P_{n+1}^{(\alpha,\beta)}$.

It should be noted that part (iii) is just part(ii) with the symmetrical property put into play.

It must be remarked upon that the values of t, considered in Corollary 4.1.1, and the values of y, considered in Theorem 4.1.4, must be restricted with respect to their ranges. Although the degrees of the Jacobi polynomials may differ by two, the interlacing property does not hold in general and hence restrictions on the values of t and y are necessary. This will be illustrated by means of the example given below.

For example, if Mathematica is used to calculate the zeros of $P_4^{(19.7,-0.5)}$ and $P_6^{(20.7,0.5)}$, it will be shown that the interlacing property does not hold when $t = y = -1$. The degrees of the polynomials differ by two and the values of α and β are different in each Jacobi polynomial. However, the aim is to evaluate whether interlacing holds for these two Jacobi polynomials with degrees differing by two.

The code below provides the zeros for the Jacobi polynomial, $P_4^{(19.7,-0.5)}$:
solns2=N[x/.NSolve[JacobiP[4,19.7,-0.5,x]=0,x]]
The zeros are {-0.987902,-0.891163,-0.696881,-0.394643}.

The following code provides the zeros for the Jacobi polynomial $P_6^{(20.7,0.5)}$:
solns3=N[x/.NSolve[JacobiP[6,20.7,0.5,x]=0,x]]
The zeros are {-0.973436,-0.894324,-0.764257,-0.585269,-0.358125,-0.0751924}.

As can be seen from the computed zeros,, the interlacing property breaks down between the second zeros of the polynomials, therefore providing numerical evidence to the claim that interlacing doesn't hold when $t = y = -1$.

If we write out the zeros in increasing numerical order, using bold font for the zeros for $P_4^{(19.7,-0.5)}$ and rest being the zeros for $P_6^{20.7,0.5}$ then if the property holds, we should see that a bold font number is followed by a number that is not bold and so on. In increasing order, the zeros are

-0.987902, -0.973436, -0.894324, **-0.891163**,....

It is not necessary to continue as we see that after the value -0.973436, the property does not hold as two zeros of $P_6^{(20.7,0.5)}$ follow each other.

4.2 Quasi-Orthogonality and its Applications to the Jacobi Polynomial

It has been discussed that Jacobi polynomials are orthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ for $\alpha, \beta > -1$. This is known as 'formal orthogonality' which is expanded upon in [8]. The paper considered in this section, [9], looks at what happens when certain orthogonal polynomials are not orthogonal when $\alpha, \beta > -1$.

A definition for quasi-orthogonality of polynomials is given below and can be found in [11] and [13].

Definition 4.2.1 *Let the polynomial, Q_n, be of degree $n \geq k$. If Q_n satisfies the following conditions*

$$\int_a^b x^k Q_n(x) w(x) \begin{cases} = 0 & \text{for } k = 0, ..., n-1-r, \\ \neq 0 & \text{for } k = n-r, \end{cases}$$

where $w(x) > 0$ on the closed interval $[a, b]$, it is then said that Q_n is quasi-orthogonal of order r on the closed interval $[a, b]$ with respect to the weight function, $w(x)$.

Paper [11] contains a property of the quasi-orthogonal polynomials. T.S Chihara says that a family of orthogonal polynomials on a closed interval $[c, d]$ with respect to a positive weight function has to have coefficients which are numbers that can depend on the degree n of the polynomial. In addition, the product of the first and last coefficients is non-zero. This is referred to as the necessary and sufficient condition for a polynomial to be quasi-orthogonal and is given as Theorem 1 in [9].

The theorem below, found in [[9], p164], states which values for the parameters α and β allow quasi-orthogonality to occur with respect to the Jacobi polynomial.

Theorem 4.2.1 *The Jacobi polynomials $P_n^{(\alpha-w,\beta-v)}$ on the interval $[-1,1]$ are quasi-orthogonal of order $(w+v)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ where $0>\alpha>-1$ and $0>\beta>-1$, $w,v\in\mathbb{N}$ and $w+v<n$,.*

Proof:

The recurrence relations used in this proof can be found in ([1], Formulae 22.7.18 and 22.7.19) and in ([42], p265, eqns (14) and (15)) and are given below. Firstly,

$$(\alpha+\beta+2n)P_n^{(\alpha,\beta-1)}(x)=(\alpha+\beta+n)P_n^{(\alpha,\beta)}(x)+(\alpha+n)P_{n-1}^{(\alpha,\beta)}(x) \tag{4.10}$$

and secondly,

$$(\alpha+\beta+2n)P_n^{(\alpha-1,\beta)}(x)=(\alpha+\beta+n)P_n^{(\alpha,\beta)}(x)-(\beta+n)P_{n-1}^{(\alpha,\beta)}(x). \tag{4.11}$$

We now use these recurrence relations to show that the Jacobi polynomial, $P_n^{(\alpha-w,\beta-v)}(x)$., can be expressed as a linear combination of the polynomials $p_n^{(\alpha,\beta)}(x)$,...,$P_{n-(w+v)}^{(\alpha,\beta)}(x)$. Then using the result below and the fact that $\alpha>-1$ and $\beta>-1$,

$$\int_{-1}^{1} x^j p_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx=0, \quad j=0,1,...n-1$$

it follows that

$$\int_{-1}^{1} x^j p_n^{(\alpha-w,\beta-v)}(x)(1-x)^\alpha(1+x)^\beta dx=0$$

when $j=0,1,...n-(v+w)-1$.

We now use this result to conclude how many zeros still exist in the interval $(-1,1)$ when the restraints on the parameter are changed. The theorem provided below can be found in [[9], p160].

Theorem 4.2.2 *If Q_n is a quasi-orthogonal polynomial of order r, with respect to a positive weight function, on the closed interval $[c,d]$, then Q_n has at least $(n-r)$ distinct zeros in the open interval (c,d).*

The following theorem can be found in [9].

Theorem 4.2.3 *When $0 > \alpha > -1$, $0 > \beta > -1$ where $w, v \in \mathbb{N}$ and $w + v < n$, the Jacobi polynomial $P_n^{(\alpha-w,\beta-v)}(x)$ has at least $n - (w + v)$ zeros in the interval $(-1, 1)$.*

Proof:

The result follows from the Theorems 4.2.2 and 4.2.1. From Theorem 4.2.1, we know that $P_n^{(\alpha-w,\beta-v)}(x)$ is a quasi-orthogonal polynomial of order $(w+v)$ in the closed interval $[-1, 1]$. From Theorem 4.2.2, we see that if a polynomial is quasi-orthogonal of order r, it has at least $(n - r)$ distinct zeros in (c, d). Therefore if $P_n^{(\alpha-w,\beta-v)}(x)$ is quasi-orthogonal on [-1,1], with order $(w + v)$,, the polynomial has at least $n - (w + v)$ distinct zeros in the open interval (-1,1). Thus, the result is proved.

Theorem 4.2.7 has uses that extend to the interlacing property of Jacobi polynomials and is thus central to the ideas and concepts in this dissertation.

Before stating Theorem 4.2.7, however, certain other results are required. The proofs of the next three results will not be provided but are given in [9].

Theorem 4.2.4 *Consider the polynomial $Q_n(x) = P_n(x) + a_n P_{n-1}(x)$. $\{P_n\}$ is a family of orthogonal polynomials with respect to a positive weight function on a closed interval $[a, b]$. Let P_n have zeros $a < x_{1,n} < ... < x_{n,n} < b$ and P_{n-1} have zeros $a < x_{1,n-1} < ... < x_{n-1,n-1} < b$. If Q_n is a quasi-orthogonal polynomial of order 1 and $g_n = \frac{P_n(x)}{P_{n-1}(x)}$, the following properties are held by Q_n.*

(i) The zeros $y_1 < ... < y_n$ of Q_n are simple, real and distinct. At most one of the polynomial's zeros lies outside the interval (a, b).

(ii) (a) If $-a_n > 0$, where a_n is non-zero number, then $x_{i,n} < y_i < x_{i,n-1}$ where $x_{n,n} < y_n$ and $i = 1, ..., n$,

(b) If $-a_n < 0$, where a_n is non-zero number, then $y_1 < x_{i,n}$ and $x_{i-1,n-1} < y_i < x_{i,n}$ where $i = 2, ..., n$.

(iii) If $-a_n < g_n(a) < 0$ then $y_1 < a$.

(iv) If $-a_n > g_n(b)$ then $y_n > b$.

The next two results will consider the polynomial,

$$Q_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x)$$

, where $a_n \neq 0$ and $b_n \neq 0$. Q_n is quasi-orthogonal of order 2. The first result, that is Theorem 4.2.5 may be found in [46], while Theorem 4.2.6 is given in [9].

Theorem 4.2.5 *If the constant $b_n < 0$, Q_n has at most two of its real and distinct zeros outside the interval (a, b).*

Theorem 4.2.6 *Let $y_1 < ... < y_n$ be the zeros of Q_n and $g_n = \frac{P_n}{P_{n-1}}$. If $b_n < 0$ then the zeros of quasi-orthogonal polynomial, Q_n, are such that $y_1 < x_{1,n-1}$ and $x_{i-1,n-1} < y_i < x_{i,n-1}$ where $i = 2, ..., n-1$. Also, $x_{n-1,n-1} < y_n$. The following also holds true:*

(i) $y_n < x_{n,n}$ *if* $-a_n - \frac{b_n}{g_{n-1}(x_{n,n})} < 0$ *and* $y_n > x_{n,n}$ *if* $-a_n - \frac{b_n}{g_{n-1}(x_{n,n})} > 0$,

(ii) $y_n < b$ *if* $-a_n - \frac{b_n}{g_{n-1}(b)} < g_n(b)$ *and* $y_n > b$ *if* $-a_n - \frac{b_n}{g_{n-1}(b)} > g_n(b)$,

(iii) $y_n < x_{1,n}$ *if* $-a_n - \frac{b_n}{g_{n-1}(x_{1,n})} < 0$ *and* $y_n > x_{1,n}$ *if* $-a_n - \frac{b_n}{g_{n-1}(x_{1,n})} > 0$,

(iv) $y_n < a$ *if* $-a_n - \frac{b_n}{g_{n-1}(a)} < g_n(a)$ *and* $y_n > b$ *if* $-a_n - \frac{b_n}{g_{n-1}(a)} > g_n(a)$.

Proof:

As a consequence of the Christoffel-Darboux Theorem (Theorem 3.0.2), we have the below equality, which can also be found in [[47], Formula 3.2.4],

$$P_n'(x)P_{n-1}(x) - P_n(x)P_{n-1}'(x) = \frac{t_n}{t_{n-1}} h_{n-1} \sum_{i=0}^{n-1} h_i^{-1} P_i^2(x),$$

where t_n is the coefficient of x^n in $P_n(x)$ and $h_n = \int_a^b P_n^2(x)w(x)dx > 0$. Now let y and z be two consecutive zeros of P_{n-1}. The equality then gives the below two expressions,

$$P_n'(y)P_{n-1}(y) - P_n(y)P_{n-1}'(y) = -P_n(y)P_{n-1}'(y) > 0$$

and

$$P_{n-1}'(y)P_{n-2}(y) - P_{n-1}(y)P_{n-2}'(y) = P_{n-1}(y)P_{n-2}'(y) > 0.$$

Hence $P_n(y)$ and $P_{n-2}(y)$ have opposite signs. This is also true for $P_n(z)$ and $P_{n-2}(z)$. As $b_n < 0$ if is clear that $P_n(y)$ and $b_nP_{n-2}(y)$ have the same sign. This is, once again, true for $P_n(z)$ and $b_nP_{n-2}(z)$ Furthermore, $Q_n(y)$ and $Q_n(z)$ have opposite signs. Then Q_n has a zero between y and z hence proving the interlacing property.
We have that Q_n=0 if and only if $g_n(x) = -a_n - \frac{b_n}{g_{n-1}(x)}$. In the open interval, $(x_{n-1,n-2}, \infty), g_{n-1}(x)$ increases from $-\infty$ to∞ and $g_{n-1}(x_{n-1,n-1}) = 0$. So if $b_n < 0, -a_n - \frac{b_n}{f_{n-1}(x)}$ decreases from $-a_n$ to $-\infty$ in $(x_{n-2,n-2}, x_{n-1,n-1})$ and from ∞ to $-a_n$ in $(x_{n-1,n-1}, \infty)$. In the open interval $(x_{n-1,n-1}, \infty)$, we see that $f_n(x)$ increases from negative infinity to positive infinity. Therefore, y_n is greater than the zero $x_{n-1,n-1}$ and the remaining results are proved.

We are now in a position to prove the final theorem of this chapter, also found in [9].

Theorem 4.2.7 *(i) Let $0 > \alpha > -1$ and $0 > \beta > -1$. The polynomial $P_n^{(\alpha-1,\beta-1)}$ has real and distinct zeros and $(n-2)$ of these zeros lie in the open interval $(-1,1)$. The smallest zero of this polynomial is smaller than -1 and the largest zero of the polynomial is larger than 1.*

(ii) Let $P_n^{(\alpha,\beta)}$ have zeros $x_{1,n} < ... < x_{n,n}$ and $P_{n-1}^{(\alpha,\beta)}$ have zeros $x_{1,n-1} < ... < x_{n-1,n-1}$:

(a) If $\alpha > -1$ and $-1 < \beta < 0$, the zeros of $P_n^{(\alpha,\beta-1)}$, $l_1 < ... < l_n$ are distinct and real with $(n-1)$ of these zeros located in the open interval $(-1,1)$. Also $l_1 < -1$ and the zeros of $P_n^{(\alpha,\beta-1)}$ interlace with the zeros of polynomials $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ as stated in Theorem 4.2.2(ii)(b).

(b) If $\beta > -1$ and $-1 < \alpha < 0$, the zeros of $P_n^{(\alpha-1,\beta)}$, $l_1 < ... < l_n$ are distinct and real with $(n-1)$ of these zeros located in the open interval $(-1,1)$. Also $l_1 > 1$ and the zeros of $P_n^{(\alpha-1,\beta)}$ interlace with the zeros of polynomials $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ as stated in Theorem 4.2.2(ii)(a).

Proof:

(i)From Equations (4.9) and (4.10), the following expression is obtained for $a_n > 0$ and $c_n < 0$;

$$P_n^{(\alpha-1,\beta-1)}(x) = a_n[P_n^{(\alpha,\beta)}(x) + b_nP_{n-1}^{(\alpha,\beta)}(x) + c_nP_{n-2}^{(\alpha,\beta)}(x)].$$

This yields

$$b_n = \frac{(\alpha+n-1)(\beta+\alpha+2n)-(n+\beta)(\beta+\alpha+2n-2)}{(n+\beta+\alpha)(\alpha+\beta+2n-2)}$$

$$c_n = \frac{-(\alpha+\beta+2n)(n+\alpha-1)(n+\beta-1)}{(\alpha+\beta+n-1)(\alpha+\beta+n)(\alpha+\beta+2n-2)}.$$

Therefore we can get $\frac{P_n^{(\alpha,\beta)}(1)}{P_{n-1}^{(\alpha,\beta)}(1)} = \frac{n+\alpha}{n}$. After manipulating the expression, we observe that the second inequality in Theorem 4.2.6(i) is satisfied. Hence, we have that the largest zero of $P_n^{(\alpha-1,\beta-1)} > 1$.

A similar proof can be done for proving that the smallest zero of $P_n^{(\alpha-1,\beta-1)} < -1$, using the first inequality of Theorem 4.2.6(ii).

(ii)(a) We use the identity provided in (4.9) as it is associated with the second case, part (b) of Theorem 4.2.5. Here, we have that $a_n = \frac{n+\alpha}{\alpha+\beta+n} > 0$. Therefore, the results of Theorem 4.2.5 hold. Calculating $\frac{P_n^{(\alpha,\beta)}(-11)}{P_n^{(\alpha,\beta)}(-1)} = \frac{-(\beta+n)}{n}$ and using it in Theorem 4.2.5(iii), the result follows as $\beta < 0$.

(ii)(b)Here use the same principles as for (ii)(a). Instead of using relation (4.9), however, we use relation (4.10) in conjunction with Theorem 4.2.5(iv) with $\alpha < 0$.

In conclusion, we observe that the Jacobi polynomial still retains some form of orthogonality when the parameter restraints are changed. This is evidenced in the fact that some zeros still remain in the interval $[c, d]$ after the new parameter restraints have been imposed.

Chapter 5

Inequalities of Jacobi Polynomials

This chapter will be a discussion on the inequalities of the Jacobi polynomial.

Walter Gautschi (1927-) has done extensive research on the inequalities of Jacobi polynomials and how they affect the zeros of these polynomials. Gautschi wrote five papers, [24], [25], [26], [27] and [28], of interest, the first being published in 2007, [24], which he co-authored with Paul Leopardi. Each paper considers an inequality related to the Jacobi polynomials and their zeros. The papers then further explore the properties of this inequality through various conjectures.

Please note that in [24], the inequalities are conjectured to hold and numerical and analytical evidence is provided to support the validity and truth of these statements.

Conjectures 5.0.1 and 5.0.2 consider the special Jacobi polynomial where $\beta = \alpha - 1$.

The main aim of [24] is to propose inequalities for the largest zero for the Jacobi polynomial given as, $x_n = cos\theta_n^{(\alpha)}$ where $0 < \theta_n^{(\alpha)}\pi < \theta$. The various conjectures rely on the inequality

$$\tilde{P}_n^{(\alpha,\beta)}\left(\cos\left(\frac{\theta}{n}\right)\right) < \tilde{P}_{n+1}^{(\alpha,\alpha-1)}\left(\cos\left(\frac{\theta}{n+1}\right)\right),$$

where $\tilde{P}_n$ is the scaled polynomial given as

$$\tilde{P}_n^{(\alpha,\alpha-1)} = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}.$$

The most natural way to attempt proving the above inequality is by determining if the inequality given below,

$$n\theta_n^{(\alpha)} < (n+1)\theta_{n+1}^{(\alpha)}, \tag{5.1}$$

holds or or analagous forms of (5.1) hold.

Natural questions that arise from (5.1) are:

(1) Does this inequality hold for all zeros of the Jacobi polynomial?

(2) If so, then for which domain in the (α, β)-plane is this inequality true?

Both questions are addressed by Gautschi in [26] and [27] respectively.

The conjecture below, which can be found in [24] is formed to establish for which n the inequality (5.1) holds if $\alpha > 0$.

Conjecture 5.0.1 *Given that $\alpha > 0$, the inequality $n\theta^{(\alpha)} < (n+1)\theta_{n+1}^{(\alpha)}$ either holds for all $n = 1, 2, 3, \ldots$ or is false for $n = 1$. This means that the validity of inequality (5.1) when $n = 1$ implies that the inequality is valid when $n \geq 1$.*

Gautshi does not prove Conjecture 5.0.1, however, the result is rather supported by numerical and analytical evidence.

A Matlab code is generated to test the conjecture at various values of n and α. This code can be found in [24].

Since this is a short code, it will be included here (as well as Appendix A) for ease of reference.

```
ab=r_jacobi(n+1,a,a-1);


    for j=1:n
```

```
      fg     = gauss(j, ab);
      fg1    = gauss(j+1, ab);
      theta  = acos(fg(j,1));
      theta1 = acos(fg1(j+1,1));

       if
           j*theta >= (j+1)*theta1
           [j*theta,(j+1)*theta1], a, k, error('Conjecture 5.0.1 is false')
       end
   end
```

The first line, $ab = r\_jacobi(n+1, a, a-1)$, creates coefficients used for the Gauss command. The Gauss command calculates the weights and nodes of the Gaussian quadratures.

The code generates an error message 'Conjecture 5.0.1 is false' if for certain values of α and n the inequality $n\theta_n^{(\alpha)} \geq (n+1)\theta_{n+1}^{(\alpha)}$, holds For example when we input $n = 100$ for $\alpha = [0.5 : 0.01 : 1, 1.1 : 0.1 : 10; 5 : 0.5 : 20]$, the error statement is not generated, hence the inequailty (5.1) is not true for those values.

We evaluate the code by varying a. We let $n = 1$, as that is the point of interest in the conjecture. Then we vary a. After trial and error, it is observed that the error message occurs for $n = 1$ and $\alpha = 0.14$ as well as when $n = 1$ and $\alpha = 0.135$. Further manipulation of a reveals that the inequality (5.1) holds for all $n \geq 1$ when $\alpha > \alpha_0$. Due to the code, one can narrow the interval in which α_0 sits, that is $0.1351 > \alpha_0 > 0.1350$.

There is a more efficient way to examine the exact value of α_0 when $n = 1$ as shown below.

From a recurrence relation for Jacobi polynomials found in ([47], eqn (4.5.1)), we obtain

$$P_1^{(\alpha,\alpha-1)}(x) = \frac{1}{2}((2\alpha+1)x+1),$$

$$4P_2^{(\alpha,\alpha-1)}(x) = (\alpha+1)((2\alpha+3)x^2+2x-1).$$

Therefore, the roots of these polynomials are

$$x_1^{(\alpha)} = -\frac{1}{2\alpha+1},$$

and

$$x_2^{(\alpha)} = \frac{1}{1+\sqrt{2\alpha+4}}.$$

This means that for $n=1$, the inequality (5.1) is equivalent to

$$\arccos\left(-\frac{1}{2\alpha+1}\right) < 2\arccos\left(\frac{1}{1+\sqrt{2\alpha+4}}\right).$$

Using the fact that $\arccos(-t) = \pi - \arccos(t)$, the above expression can be written as

$$\arccos\left(\frac{1}{2\alpha+1}\right) + 2\arccos\left(\frac{1}{1+\sqrt{2\alpha+4}}\right) - \pi > 0. \qquad (5.2)$$

The left hand side of the inequality (5.2) is an increasing function of α, as the derivative is positive. It is negative when $\alpha = 0$, as replacing α with 0, one gets $\arccos(1) + \arccos\left(\frac{1}{3}\right) - \pi = -0.68$ radians. If we let α tend to infinity, the left hand side of the inequality tends to $\frac{1}{2}\pi$, since $\arccos(0) + 2\arccos(0) - \pi = \frac{\pi}{2} - 2(\frac{\pi}{2}) - \pi = \frac{\pi}{2}$.

Hence, if α_0 is the unique zero of

$$\arccos\left(\frac{1}{2\alpha+1}\right) + 2\arccos\left(\frac{1}{1+\sqrt{2\alpha+4}}\right) - \pi = 0,$$

we find, using Matlab, that $\alpha_0 = 0.13507978085964$. Hence if Conjecture 5.0.1 is true then inequality (5.1) holds for all $n \geq 1$ if $\alpha > \alpha_0$.

This concept can be extended to the scaled Jacobi polynomials, which are given by

$$\tilde{P}_n^{(\alpha,\beta)} = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \qquad (5.3)$$

where θ is considered to be in one of two intervals. These intervals are

$$0 < \theta_1^{(\alpha)} < \theta, \text{ and } 0 < \theta < \pi. \tag{5.4}$$

The conjecture used for Jacobi polynomials involves the inequality

$$\tilde{P}_n^{(\alpha,\alpha-1)}\left(\cos\left(\frac{\theta}{n}\right)\right) < \tilde{P}_{n+1}^{(\alpha,\alpha-1)}\left(\cos\left(\frac{\theta}{n+1}\right)\right), \tag{5.5}$$

where $\cos\theta_1^{(\alpha)} = x_1^{(\alpha)} = -\frac{1}{2\alpha+1}$.

A conjecture is given for the scaled polynomials given in (5.3). This can be found in [24].

Conjecture 5.0.2 *If $\alpha > 0$, two alternatives exist for each of the intervals described in (5.4). Either (5.5) holds for all θ in the respective interval and for all $n = 1, 2, 3...$ or (5.5) is false for some θ in the respective interval when $n = 1$. So, if (5.5) is valid for $n = 1$ it is implied that (5.5) is valid for all $n \geq 1$.*

To verify Conjecture 5.0.2 numerically, use the same reasoning as in the proof of Conjecture 5.0.1 (Appendix A.) The extended code is given in the Appendix B.

When the code is run with values $n = 100$, $N = 100$ and $\alpha = 10.5 : 0.5 : 20$ for the interval $0 < \theta < \theta_1^{(\alpha)}$, we find that the error message is generated. Hence, we see that Conjecture 5.0.2 appears to be true for inequality (5.5) in interval $0 < \theta < \theta_1^{(\alpha)}$.

In the case of interval $0 < \theta < \pi$, when $N = 1000$ and randomly chosen $\alpha = 1.1 : 0.1 : 10$, an error message appears when $\alpha = 0.28$ and $\alpha = 0.29$. Hence Conjecure 5.0.2 is valid.

Extending the Matlab routine used in the proofs of Conjecture 5.0.1 and 5.0.2 leads to the conjecture below found in [24]. The conjecture is similiar to that of Conjecture 5.0.1, but places restrictions on α and introduces a new parameter, β.

We now denote the largest zero of Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$ as $x_n^{(\alpha,\beta)} = cos\theta_n^{(\alpha,\beta)}$ where $\alpha > -1, \beta > -1$.

The inequality considered in Conjecture 5.0.3 is given below

$$n\theta_n^{(\alpha,\beta)} < (n+1)\theta_{n+1}^{(\alpha,\beta)}. \tag{5.6}$$

Conjecture 5.0.3 *If $\alpha > -1$ and $\beta > -1$, two alternatives exist. Either inequality (5.6) holds for all $n = 1, 2, 3, ...$ or inequality (5.6) is false for $n = 1$. Therefore, if (5.6) is valid for $n = 1$, it is valid for all $n \geq 1$.*

Once again, the numerical verification of Conjecture 5.0.3 can be found in [24]. The reasoning is similar to that of the proof of Conjecture 5.0.1 and Conjectue 5.0.2.

It must be noted that when $\alpha = \beta = \frac{-1}{2}$, we consider it an exception as then left and right sides of (5.6) will be equal to $\frac{\pi}{2}$.
Gautschi went on to write three more papers that considered inequality (5.6) alone, [25], [26], [27].

In [25], he investigated the validity of the inequality (5.6) for a change in the domain of the parameter space (α, β).

Paper [25] uses Conjecture 5.0.3 (from [24]) to describe the conjectured domain in which (5.6) is valid showing that the conjecture is, in fact, false in a small subregion of the domain of parameter space (α, β). Firstly, a curve is defined as

$$\mathcal{B} : \beta = \beta(\alpha), \quad -1 < \alpha < 1. \tag{5.7}$$

The curve is monotonically decreasing from the point (-1,0) to the point (1,-1). Between these two points, (5.6) is true for all $n = 1$. However, on and below these points, the inequality (5.6) does not hold when $n = 1$. Now, $\beta = \beta(\alpha)$ is the solution of the equation

$$2 \arccos\left(\frac{1}{\alpha+\beta+4}\left(-(\alpha-\beta) + 2\sqrt{2 + \frac{\alpha\beta - 2}{\alpha+\beta+3}}\right)\right) + \arccos\frac{\alpha-\beta}{\alpha+\beta+2} - \pi = 0. \tag{5.8}$$

In [28], it is shown that Gatteschi developed results for the asymptotic behavior of the decreasing zeros of the Jacobi polynomial that have large degree n.

A particular result (given in [25]) is true for the k^{th} zero where k is fixed and also for the special case where $k = 1$. If $\alpha > -1$ and $\beta > -1$ then we have

$$\theta_n^{(\alpha,\beta)} = \frac{J_{\alpha,1}}{v} + O(n^{-5}), \tag{5.9}$$

as n tends to ∞ where $J_{\alpha,1}$ is the first positive zero of the Bessel function J_α. Also

$$v = \left[\left(n + \frac{\alpha+\beta+1}{2} \right)^2 + \frac{1-\alpha^2-3\beta^2}{12} \right]^{\frac{1}{2}} = v(n).$$

From Equation (5.9), the following is obtained

$$\frac{\theta_n^{(\alpha,\beta)}}{\theta_{n+1}^{(\alpha,\beta)}} = \frac{\frac{J_{\alpha,1}}{v(n)} + O(n^{-5})}{\frac{J_{\alpha,1}}{v(n+1)} + O(n^{-5})} = \frac{v(n+1)}{v(n)} + O(n^{-4}).$$

Now, expand the expression above using Mathematica or Matlab to get the expression below in descending powers of n,

$$\frac{\theta_n^{(\alpha,\beta)}}{\theta_{n+1}^{(\alpha,\beta)}} = 1 + n^{-1} - \frac{1}{2}(\alpha+\beta+1)n^{-2} + \frac{1}{6}(2\alpha^2+3\alpha\beta+3\beta^2+3\alpha+3\beta+1)n^{-3} + O(n^{-4}). \tag{5.10}$$

Using these results the following theorem in [28] states in which domain the parameters must lie to make sure that the inequality (5.6) is valid.

Theorem 5.0.1 *The inequality (5.6) is valid for sufficiently large n when* $\alpha+\beta+1 > 0$. *The same is true when* (α, β) *is located on an open line segment from (-1,0) to* $(-\frac{1}{2}, -\frac{1}{2})$ *if* $\alpha+\beta+1 = 0$. *However, the inequality does not hold for n large enough on the half-open line segment from* $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ *inclusive to (0,-1). The inequality (5.6) is false when* $\alpha+\beta+1 < 0$ *and for n sufficiently large.*

Proof:

Inequality (5.6) can be written as follows,

$$\frac{\theta_n^{(\alpha,\beta)}}{\theta_{n+1}^{(\alpha,\beta)}} < 1 + n^{-1}. \tag{5.11}$$

To prove the first part of the theorem, consider Equation (5.10). In the equation, the left hand side's ratio is less than $1 + n^{-1}$ when n is sufficiently large and $\alpha + \beta + 1 > 0$. Hence, we know that inequalities (5.6) and (5.11) hold true for sufficiently large n. Now, to prove the second part of the theorem, evaluate the coeffiicent of n^{-3}, i.e. $\frac{1}{6}(2\alpha^2 + 3\alpha\beta + 3\beta^2 + 3\alpha + 3\beta + 1)$. By the substitution $\beta = -1 - \alpha$, the expression becomes $\frac{1}{6}(2\alpha + 1)(\alpha + 1)$. For $\alpha > -1$, expression is only negative, which makes the inequality (5.6) false, when $\alpha < -\frac{1}{2}$. The last part is proved using the same reasoning as in the first part of the proof.

Therefore, Theorem 5.0.1 is proved and the domain in which the inequality (5.6) is valid is determined.

One can also disprove the conjecture. To disprove the conjecture, one needs to prove that the domains $\alpha + \beta + 1 < 0$ and $\beta(\alpha) > 0$ have a non-empty intersection in the square $-1 < \alpha < 0, -1 < \beta < 0$. The graph provided, Figure 5.1, is of the equation $\beta = -\alpha - 1 - \beta(\alpha)$. The open circle in Figure 5.1 corresponds to the point where $\alpha = -0.75$ and $\beta = \frac{1}{2}(\beta(\alpha) - \alpha - 1)$, which is in the intersection. We see that (5.6) is true for $n = 1, 2$ but false for $3 \leq n \leq 100$. The intersection is very slim.

A new revised conjecture in [28], is therefore formed.

Conjecture 5.0.4 *With the exception of the point $\alpha = \beta = -\frac{1}{2}$, the domain of validity in the (α, β)-plane of the inequality (5.6) for all $n \geq 1$ is the subdomain D of all admissible $\{(\alpha, \beta) : \alpha > -1, \beta > -1\}$ bounded below by the line segment C_1 from the point $(-1, 0)$ to $(-\frac{1}{2}, -\frac{1}{2})$, the part $C_2 = \{(\alpha, \beta) : \beta = \beta(\alpha), -\frac{1}{2} \leq \alpha < 1\}$ of the curve B and the line $C_3\{(\alpha, \beta) : 1 \leq \alpha < \infty, \beta = -1\}$.*

The only distinct difference between the Conjecture 5.0.3 and the revised Conjecture 5.0.4 above, is the replacement of the curved segment $\{(\alpha, \beta) : \beta = \beta(\alpha), -1 < \alpha < -\frac{1}{2}\}$

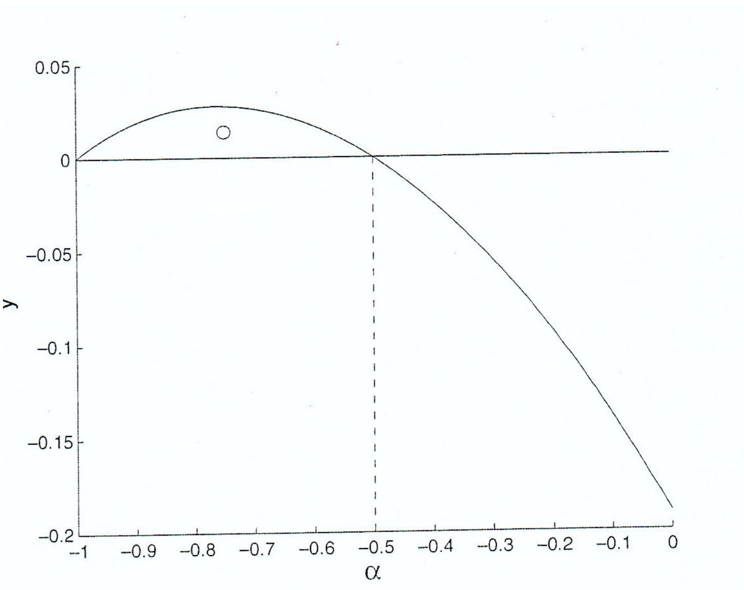


Figure 5.1:

in the original boundary of the domain of validity. Conjecture 5.0.1 remains unaffected by this change in domain however.

The Gautschi paper, [26] extends the inequality (5.6) to all the zeros of Jacobi polynomials, not just the largest zero. He determines which domain in the (α, β)-plane makes the following inequalities valid:

$$n\theta_{n,r}^{(\alpha,\beta)} < (n+1)\theta_{n+1,r}^{(\alpha,\beta)}, \quad r = 1, 2, ..., n. \tag{5.12}$$

The following strategy is used to evaluate the domain for the (α, β)-plane with respect to the Jacobi polynomial, $P_n^{(\alpha,\beta)}$ when $\alpha > -1$ and $\beta > -1$ if $1 \leq n \leq N$. We consider the cases where $N = 50$, $N = 100$ and $N = 200$. The code uses a bisection method with steps of length 0.2 to show that inequalitiy (5.12) is valid on horizontal segments defined as follows, $\mathcal{H} = \{(\alpha, \beta) : -0.5 < \alpha \leq -0.5, \beta = 0.5\}$. Use the same step length for α to determine that the inequality (5.12) is valid for the diagonal downward segment $\mathcal{D} = \{(\alpha, \beta) : -1 < \alpha < -0.5, \beta = -\alpha - 1\}$. The vertical lines are denoted $\mathcal{L}_\alpha$.

The domain can be better found using the Matlab code which is also given in [28] on website **www.cs.purdue.edu/archives/2002/wxg/codes/jacconj.m** and is given in Appendix C.

After implementing the Matlab code, one finds that the inequalities (5.12) are valid for all n inside horizontal strips,

$$H = \{(\alpha, \beta) : \alpha > -1, |\beta| \leq 0.5\}.$$

These horizontal strips are cut off on the left hand side by $\mathcal{D}$.

The paper, [27], extends inequality (5.6) to all zeros of the Jacobi polynomial, not just the largest zero. This can be tested using the codes provided, while altering values for the zero of the Jacobi polynomial.

Chapter 6

Conclusion

This dissertation has considered the basic concepts of Special Functions such as the Pochhammer symbol, gamma function, beta function, Bessel function and the hypergeometric function. Properties of the various classical orthogonal polynomials were also considered, results explored amongst others are the orthongality property, Rodrigues formula for the Jacobi, Hermite and Laguerre polynomials. Mention is made of the interlacing property of zeros of the Jacobi polynomial as well as a brief discussion on quasi-orthogonality and Guassian quadrature is also considered. The final chapter concentrates on the results of some inequalities of the Jacobi polynomial.

As can be seen throughout the paper, the Jacobi polynomial is a very versatile and well developed function. There is much known about the polynomial and various facets have been studied such as the polynomial's application to Gaussian quadrature and the solution to equations of motion of the symmetric top. Some topics have been studied in greater detail than others.

Some interesting questions that are not covered in this paper in detail have been asked and answered in papers such as [14] in which the authors consider the following. For a given pair of numbers (α, β) where $\alpha, \beta > -1$, Dimitrov and Rodrigues ask for what other pairs of numbers (a, b) such that $a, b > -1$ are the zeros of the Jacobi polynomial $P_n^{(a,b)}(x)$ are greater than (or smaller than) the zeros of the Jacobi polynomials of the form $P_n^{(\alpha,\beta)}(x)$. They answer this question by using certain results established by A.Markov in

[37] and the Routh-Hurwitz matrix. As this concept was beyond the scope of the paper, it was not explored further. It is eventually found that the zeros of $P_n^{(a,b)}(x)$ are less than the zeros of the polynomial $P_n^{(\alpha,\beta)}(x)$ when $a > \alpha$ and $b < \beta$ and vice versa.

Another interesting idea crossed the research path. While perusing [41], a table of the products of Jacobi polynomials stood out. In particular,

$$ {}_4F_3\left(a, b, b + \frac{1}{2}, 2b + n; 2b, c, b - c + 1; z\right) $$

can be expressed as

$$ \frac{(1)^n (n!)^2}{(c)_n (2b - c + 1)_n} P_n^{(c-1,2b-c)}(\sqrt{1-z}) P_n^{(c-1,2b-c)}(-\sqrt{1-z}). $$

The idea being the investigation as to what the zeros of the above equation were and to outline the behaviour of said zeros.

In summary, it is clear that there exists further research scope regarding the Jacobi polynomials and its various properties and uses, for example one can investigate the behaviour of zeros of α and β and then provide corresponding information on the zeros of the respective ${}_4F_3$ polynomials. To my knowledge, this has not been attempted before. A good reference for this would be [15]. This paper dealt with the products and the zeros of two ${}_4F_3$ polynomials. This work could be continued in my further studies.

Appendices

Appendix A

Code found below is used in proof of Conjecture 5.0.1 on pg 65 of the dissertation.

```
ab=r_jacobi(n+1,a,a-1);


    for j=1:n
       fg     = gauss(j, ab);
       fg1    = gauss(j+1, ab);
       theta  = acos(fg(j,1));
       theta1 = acos(fg1(j+1,1));


        if
            j*theta >= (j+1)*theta1
            [j*theta,(j+1)*theta1], a, k, error('conjecture false')
        end
    end
```

Appendix B

Code given in this appendix is used in proof of Conjecture 5.0.2 on pg 68.

```
ab=r_jacobi(n+1,a,a-1);

th1=acos(-1(2*a+1)); {note that th1=pi}

    for nu=1:N

     th=nu*th1/(N+1);

     for k=1:n

     x0=1; x=cos(th/k); y=cos(th/(k+1));

     p0=0; p01=1; px=0; px1=1; py=0; py1=1;

     for r=1:k+1

     p0m1=p0; p0=p01; pxm1=px; px=px1; pym1=py; py=py1;

     p01=(x0-ab(r,1))*p0-ab(r,2)*p0m1;

     px1=(x-ab(r,1))*px-ab(r,2)*pxm1;

     py1=(y-ab(r,1))*py-ab(r,2)*pym1;

     end

     if px/p0 >= py1/p01

     [px/p0, py1/p01], a, k, nu, error('Conjecture 5.0.2 is false')

     end

     end

    end
```

Appendix C

The following code is used in determining the domain of validity of Conjecture 5.0.4 on pg 71.

```
f0='%6.2f %12.8f %6.0f\n';
n=50; db=.02;
aa=zeros(25,1); bb=zeros(25,1);
foria=1:25
  a=.02+(ia-1)*.02;
  [b0,diff]=jacconj(n,a,db);
fprintf(f0,a,b0,diff)
aa(ia)=a; bb(ia)=b0;
end
```

However, this function uses the following supporting codes:

```
function [b0,diff]=jacconj(n,a,db)
eps0=1e10*eps;
if a=-.5, b=a; else b=-a-1-db; end
b=-.5-db;
b=-.5+db;
k0=0; diff=0;
while k0==0 & b>-1
   b=b+db;
   b=b-db;
  [k0,r0]=ineq(n,a,b);
```

```
if k0-r0>diff, diff=k0-r0; end
end
b0=0;
if k0>0
br=b; bl=b-db;
br=b+db; bl=b;

whilebr-bl>eps0
    b0=.5*(bl+br);
    [k0,r0]=ineq(n,a,b0);
if k0-r0>diff, diff=k0-r0; end
if k0>0
br=b0;
      bl=b0;
else
bl=b0;
       br=b0;
end
end
end
```

and

```
function [k0,r0]=ineq(n,a,b)
ab=r_jacobi(n+1,a,b);
k0=0; r0=0;
for k=1:n
xw=gauss(k,ab); xw1=gauss(k+1,ab);
  for r=k
for r=1:k
th=acos(xw(k+1-r,1)); th1=acos(xw1(k+2-r,1));
if k*th>=(k+1)*th1
```

```
      k0=k; r0=r;
break
end
end
if k0>0
break
end
end
```

Bibliography

[1] M. Abramowits, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Nat. Bur Standards Appl. Series, vol 55, US Government Printing Office, Washington DC, 1964.

[2] G. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge 1999.

[3] I. Area, D. Dimitrov, E. Godoy, F. R. Rafaeli, Inequalities for Zeros of Jacobi Polynomials via Obrechkoff's Theorem, *Mathematics of Computation*, Vol.81 (2012), 991-1004.

[4] R. Askey, *Orthogonal Polynomials and Special Functions*, Socitey of Industrial and Applied Mathematics, 1975.

[5] V. M. Badkov, Asymptotic Formulae for the Zeros of Orthogonal Polynomials,*Mat. Sb.*, 203:9, 2012, 13-14.

[6] R. Beals, R. Wong, *Special Functions: A Graduate Text*, Cambridge University Press, 2010.

[7] Letters from Daniel Bernoulli to Christian Goldbach, $www.luschny.de/math/factorial/history.hml$.

[8] C. Brezinski, *An Introduction to Formal Orthogonality and Some of its Applications*, The Pennsylvania State University, 2002.

[9] C. Brezinski, K. A. Driver, M. Redivo-Zaglia, Quasi-Orthogonality with applications to some families of classical orthogonal polynomials, *Applied Numerical Mathematics* 48 (0004),2002, 157-168.

[10] R. L. Burden, J. D. Faires, *Numerical Analysis*, Brooks /Cole, Cengage Learning, 2005.

[11] T. S. Chihara, On Quasi-Orthogonal Polynomials, *Proc, American Math. Society.* 8(1957), 765-767.

[12] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Dover Publications Inc., New York, 1978.

[13] D. Dickinson, On Quasi-Orthgonal Polynomials, Proc. *American Math. Soc.* 12(1961), 185-194.

[14] D. K. Dimitrov, On the Behaviour of Zeros of Jacobi Polynomials, *Journal of Approximation Theory*, 2002, 224-239.

[15] K. A. Driver, A. D. Love, Products of hypergeometric functions and the zeros of ${}_4F_3$ polynomials, *Numerical Algorithms*, 26(2001),1-9.

[16] K. Driver, A. Jooste, K. Jordaan Stieltjies Interlacing of Zeros of Jacobi Polynomials from Different Sequences, *Electronic Transactions on Numerical Analysis*, Volume 38, p317-326, Kent State University, 2011.

[17] C. F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables*, Cambridge University Press, 2001.

[18] J. Dutka, On the Early History of Bessel Functions, *History of Exact Sciences*, 49(1995), 105-134.

[19] H. Edwards, R. P Hostetler, R. Larson, *Calculus of a Single Variable*, Houghton Mifflin Company, 2004.

[20] R. El Attar, *Special Functions and Orthogonal Polynomials*, Lulu Press, USA, 2006.

[21] Efunda, Bessel Function, $www.efunda.com/math/bessel/bessel.cfm$, 2015.

[22] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, 2008.

[23] W. Gautschi, *Orthogonal Polynomials-Computation and Approximation*, Oxford University Press, 2004.

[24] W. Gautschi, P. Leopardi, *Conjectured Inequalities for Jacobi polynomials and their Largest Zeros*, Springer Science+Business Media, 2007.

[25] W. Gautschi, On a Conjectured Inequality for the largest zero of Jacobi polynomials, *Numer. Algor.* (2008) 49:195-198.

[26] W. Gautschi, On conjectured inequalities for zeros of Jacobi polynomials, *Numer. Algor.* (2009) 450:93-96.

[27] W. Gautschi, New conjectured inequalities for the zeros of Jacobi polynomials, *Numer. Algor.* (2009) 50:293-296.

[28] W. Gautschi, C. Giordano, Luigi Gatteschi's work on asymptotics of special functions and their zeros, *Numer. Algor.* doi:10.1007/s11075-008-9208-5.

[29] D. Gronau, Why is the Gamma Function so as it is?, *Teaching Mathematics and Computer Science*, 1/1, 43-45, 2003.

[30] N. Hale, A. Townsend, Fast and Accurate computation of Gauss-Legendre and Gauss-Jacobi Quadrature nodes and weights, 35 (2), A652-A674, 2013.

[31] M. Huber, *Infrared Behaviour of Vertex Functions in d-Dimensional Yang-Mills Theory*, Karl-Franzens University, 2007.

[32] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, 2005.

[33] G. E. Karniadakis, *Parallel; Scientific Computing in C++ and MPI: A Seamless Approach to Parallel Algorithms and their Implementation*, Cambridge University Press, 2003.

[34] R. Koekoek, P. A. Lesky, R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer, 2010.

[35] A. B. J. Kuijlaars, A. Martinez-Finkelshtein, Strong asymptotics with varying non-standard parameters, *J. Anal Math.* 94, 195-234, 2004.

[36] N. N. Lebedev, *Special Functions and their Applications* (translated and edited by Richard A. Silverman), Dover Publications Inc., 1972.

[37] A. Markov, Sur les rancines de certain equations (second note), *Math. Anne.* 27 (1886), 177-182.

[38] A. Martinez-Finkelshtein, P. Martinez-Gonzalez, R. Orive, Actas del International Workshop on Special Functions: Asymptotics, Harmonic Analysis and Mathematical Physics, City University of Hong Kong, 21-25 June 1999, World Scientific, 2000, 98-113.

[39] N. W. McLachlan, *Bessel Function for Engineers*, Oxford University Press, 1941

[40] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, *Numerical Recipes in C; The Art of Scientific Computing*, Cambridge University Press, 1988.

[41] A. P. Prudnikov, Y. A. Brychkov, O. I. Markov, *Integrals and Series Volume 2: Special Functions*, CRC Press, 1986.

[42] E. D. Rainville, *Special Functions*, Chelsea Publications Company, 1960.

[43] E. Sandifer, *How Euler Did It*, Mathematical Association of America, February 2005.

[44] G. Sansone, *Orthogonal Functions*, Robert E. Krieger Publishing Co., New York, 1977.

[45] J. B. Seaborn, *Hypergeometric Functions and their Applications*, Springer-Verlag New York Inc, 1991.

[46] J. A. Shohat, On Mechanical Quadratures, in particular, with Postive Coefficients, *Trans. American Math. Soc.* 42 (1973), 461-496.

[47] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, Colloquiam Publications, 2003.

[48] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Center for Mathematics and Computer Science, John Wiley and Sonds, Inc., New York, 1996.

[49] E. A. van Doorn, *Orthogonal Polynomials and Birth-Death Process*, Orthogonal Polynomials and their Applications,in Lecture Notes in Pure and Applied Mathematics Volume 117 (J. Vinuesa, ed.), Marcel Dekker Inc., New York, 23-24, 1989.

[50] Z. X. Wang, D. R. Guo, *Special Functions*, World Scientific Publishing, 2010.

[51] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1952.

[52] S. Wolfram, *The History and Future of Special Functions*, $www.stephenwolfram.com?publications/history\ future\ special\ functions/$, 2005.

[53] D. Xin, *Numerical Methods for Stochastic Computations: A Spectral Method Approach*, Princeton University Press, 2010.