The Simplest Gauge-String Duality

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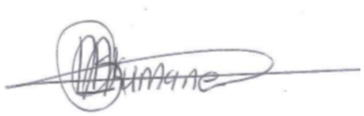
Declaration

I declare that the work contained in this thesis is my own work, any work done previously by others or by myself has been acknowledged and put to references. This thesis is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg and it has not been submitted before in any other tertiary institution.



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ABSTRACT

The gauge/gravity correspondence is a conjectured exact duality between quantum field theories and theories of quantum gravity. A very simple gauge/string duality, claims an equivalence between the Gaussian matrix model and the topological A-model string theory on $\mathbf{P^1}$. In this dissertation we study this duality, proposing concrete operators in the matrix model that are dual to gravitational descendants of the puncture operator of the topological string theory. We test our proposal by showing that a large number of matrix model correlators are in complete agreement with correlators in the dual topological string theory. Contact term interactions, as proposed by Gopakumar and Pius, play an interesting and non-trivial role in the duality.

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1 Introduction

Dualities between theories play an important role in theoretical physics. In many known examples, dualities relate the strong coupling limit of one theory to the weak coupling limit of another. Thus, they offer the hope of genuine progress in understanding the dynamics of strongly coupled field theories. Gauge-string duality [1, 2, 3] relates large N gauge theories to quantum gravity theories. We can hope to gain insights both into quantum gravity and into strongly coupled gauge theory from gauge-string duality. Towards this end, it makes sense to explore the simplest examples of these dualities. This is the motivation for this MSc dissertation.
Concretely, we study the Gaussian matrix model in planar limit as our gauge theory. The Gaussian matrix model is conjectured to be dual to the A-model topological string theory on $\mathbf{P^1}$, which is a theory of quantum gravity. We examine this duality by comparing correlators computed in the matrix model to correlators computed in the topological string. The relevant correlators of the matrix model that participate in this duality are all connected and include, for example

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle_{conn} \tag{1.1}$$

and

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_q})Tr(M^{2n_{q+1}}\ln M)Tr(M^{2n_{q+2}}\ln M)\cdots Tr(M^{2n_{q+m}}\ln M)\rangle_{conn}. \tag{1.2}$$

In an attempt to give a purely combinatorial foundation for the gauge/string duality, connections between Wick's theorem and topological string theories have been studied and promising results have been found [4, 5, 6, 7, 8, 9, 10]. This approach tells us that the Wick contractions of the fields are related to the sum over contributions from the clean Belyi maps. The Belyi maps take a point from a world sheet with genus g to $\mathbf{P}^1$. This map has three branching points at 0,1 and ∞. The clean Belyi maps are defined by $w = z^2$ where w is the local coordinate of $\mathbf{P}^1$ and z is the local coordinate of the world sheet. The only Riemann surfaces that admit Belyi maps are the arithmetic Riemann surfaces. It has been shown that the combinatorics of the Wick's contractions are reproduced by a string theory whose sum over world sheets only receives contributions from world sheet which are arithmetic [4].

Another matrix model dual to the topological A-model on $\mathbf{P}^1$ was put forward by [23, 24, 25]. Another set of interesting questions would revolve around the duality between these two matrix models. We will not pursue this further in this dissertation.

We start from the conjecture in [15] which states that the Gaussian matrix model is dual to the topological A-model on $\mathbf{P}^1$ and the proposal that the operator $Tr(M^{2n})$ in the Gaussian matrix model can be roughly identified with the operator $\sigma_{2n}(Q)$ in the topological string. The correlators of these theories are meant to be related by

$$\langle Tr(M^{2n_1})Tr(M^{2n_1})\cdots Tr(M^{2n_k})\rangle_{conn} \leftrightarrow \langle \sigma_{2n_1-1}(Q)\sigma_{2n_2-1}(Q)\prod_{i=3}^{k}\sigma_{2n_i}(Q)\rangle \tag{1.3}$$

where $\sigma_{2n}(Q)$ is the gravitational descendant of the Kähler class. This idea was extended in [26] where a number of correlators were computed and support for the conjecture in [15] was found. Another proposal was made in [26] stating that the operator $\sigma_{2n}(P)$ in topological A-model on $\mathbf{P}^1$ is related to the operator $Tr(M^{2n}\ln M)$ in the Gaussian matrix model i.e. $\sigma_{2n}(P) \leftrightarrow Tr(M^{2n}\ln M)$, where $\sigma_{2n}(P)$ is known as the gravitational descendant of the punture.

To explore this conjecture we study correlators with $\ln M$ insertions in the matrix model. In section 2, we discuss why we expect the duality between matrix model and topological string theories to exist. In section 3, we start by studying connected correlators of the form $\langle\prod_{i=1}^{k} Tr(M^{2n_i})\rangle_{conn,0}$. These correlators are determined by a set of recursion relations obtained by using a systematic $\frac{1}{N}$ expansion of well chosen Schwinger-Dyson equations. In section 4, we study correlators of the form $\langle\prod_{i=1}^{q} Tr(M^{2n_i})\prod_{j=q+1}^{k} Tr(M^{2n_j}\ln M)\rangle_{conn,0}$. In section 5, we compute these correlators using analytical continuation and verify that they solve the recursion relations obtained from the Schwinger-Dyson equations. In section 6 we use the Eguchi-Yang matrix model to compute connected correlators using recursion relations and compare to the correlators computed from topological string on $\mathbf{P}^1$. We find that the correlators computed from this matrix model are in perfect agreement with the correlators computed using the topological string. The topological string correlators with gravitational descendants of the puncture are related to matrix model correlators with $\ln M$ insertions. In section 7, we propose an identification of gravitational descendants of the puncture with Gaussian matrix model operators that reproduces a complete set of three point correlators. With this new identification there is a mismatch on the general set of correlators in the Gaussian matrix model and the topological string. The presence of these mismatches are due to contact terms in topological string [26]. However, in the large degree d limit, these mismatches vanish and this identification produces the correct three point correlators of matrix model and the topological string. In section 8, we discuss the results we have found in previous sections including the mismatches. Our final results, with the correct contact term contributions included, demonstrate complete agreement between the Gaussian matrix model and the topological A-model string on $\mathbf{P^1}$. This provides very strong support for the simplest gauge string duality. In the appendices we include a number of examples of how to compute connected correlators using the $\frac{1}{N}$ expansion of the Schwinger-Dyson equations and verify that the recursion relations are satisfied by the solutions obtained from analytical continuation.

Most of the work in this dissertation has been reported in [30]

2 Matrix model and string theories

Our aim is to study correlation functions for a matrix model, which are expressed as

$$\langle M_{ij} M_{kl} ... M_{rs} \rangle = \mathcal{N} \int dM e^{-\omega tr M^2} M_{ij} M_{kl} ... M_{rs} \tag{2.1}$$

The integral runs over the complete set of Hermitian matrices. Concretely, it is an integral over N real diagonal elements and $N(N-1)/2$ complex off diagonal elements. This gives N^2 real integrals in total. We define a partition function

$$Z = \mathcal{N} \int dM e^{-\omega tr M^2}, \tag{2.2}$$

and the generating function

$$Z[J] = \mathcal{N} \int dM e^{-\omega tr M^2 + tr JM}. \tag{2.3}$$

The generating function enables us to completely generate the n-point correlation functions by differentiating with respect to J_{ij} and setting $J = 0$, that is

$$\langle M_{ij} M_{kl} ... M_{rs} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} ... \frac{d}{dJ_{sr}} Z[J] \bigg|_{J=0}. \tag{2.4}$$

The normalization $\mathcal{N}$ is fixed by the condition $Z[J = 0] = 1$.

2.1 Normalisation of the partition function

As stated above, we normalise the partition function to 1, that is

$$1 = \mathcal{N} \int dM e^{-\omega tr M^2}. \tag{2.5}$$

The normalisation constant ($\mathcal{N}$) of this partition function depends on what type of matrix M (Hermitian, real symmetric or anti-symmetric) we use. For completeness, we will fix the normalization in all three cases. This is done to get a better understanding of what the matrix integral means.

2.1.1 Normalisation constant $\mathcal{N}$ for a Hermitian matrix M

Lets M be an $N \times N$ Hermitian matrix. All Hermitian matrices are self-adjoint, that is

$$M = M^\dagger. \tag{2.6}$$

The measure for this matrix integral is given by

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i>j} Re(dM_{ij}) Im(dM_{ij}) \tag{2.7}$$

and we integrate over all space, i.e. in the integral below integrate from $-\infty$ to $+\infty$

$$\begin{aligned}
1 =& \mathcal{N} \int dM e^{-\omega tr M^2} \\
1 =& \mathcal{N} \int \prod_{i=1}^{N} dM_{ii} \prod_{i>j} Re(dM_{ij}) Im(dM_{ij}) e^{-\omega \sum_{i=1}^{N} M_{ii}M_{ii} - 2\omega \sum_{i>j}^{N} M_{ij}M_{ij}} \\
=& \mathcal{N} \int \prod_{i=1}^{N} dM_{ii} e^{-\omega \sum_{i=1}^{N} M_{ii}M_{ii}} \prod_{i>j} Re(dM_{ij}) Im(dM_{ij}) e^{-2\omega \sum_{i>j}^{N} M_{ij}M_{ij}} \\
=& \mathcal{N} \left(\sqrt{\frac{\pi}{\omega}}\right)^{N} \left(\sqrt{\frac{\pi}{2\omega}}\right)^{N(N-1)} \\
=& \mathcal{N} \left(\sqrt{\frac{\pi}{\omega}}\right)^{N^2} \left(\frac{1}{2}\right)^{\dfrac{N(N-1)}{2}} \\
\mathcal{N} =& \left(\sqrt{\frac{\pi}{\omega}}\right)^{-N^2} 2^{\frac{N(N-1)}{2}}.
\end{aligned}$$

where $TrM^2 = \sum_{i=1}^{N} M_{ii}M_{ii} + 2\sum_{i>j}^{N} M_{ij}M_{ij}$. Absorbing this normalisation constant into the measure, our normalised partition function becomes

$$Z = \int dM e^{-\omega tr M^2} \tag{2.8}$$

2.1.2 Normalisation constant $\mathcal{N}$ for a real symmetric matrix M

Let M be a real symmetric $N \times N$ matrix. Real symmetric matrices satisfy the condition

$$M = M^T. \tag{2.9}$$

The measure for a real symmetric matrix is given by

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i>j} Re(dM_{ij}). \tag{2.10}$$

Now, $\mathcal{N}$ can be calculated as follows

$$
\begin{aligned}
1 =& \mathcal{N} \int dM e^{-\omega tr M^2} \\
1 =& \mathcal{N} \int \prod_{i=1}^{N} dM_{ii} \prod_{i>j} Re(dM_{ij}) e^{-\omega \sum_{i=1}^{N} M_{ii} M_{ii} - 2\omega \sum_{i>j}^{N} M_{ij} M_{ij} -} \\
=& \mathcal{N} \int \prod_{i=1}^{N} dM_{ii} e^{-\omega \sum_{i=1}^{N} M_{ii} M_{ii}} \prod_{i>j} Re(dM_{ij}) e^{-2\omega \sum_{i>j}^{N} M_{ij} M_{ij}} \\
=& \mathcal{N} \left(\sqrt{\frac{\pi}{\omega}} \right)^{N} \left(\sqrt{\frac{\pi}{2\omega}} \right)^{\frac{N(N-1)}{2}} \\
=& \mathcal{N} \left(\sqrt{\frac{\pi}{\omega}} \right)^{\frac{N(N+1)}{2}} \left(\sqrt{\frac{1}{2}} \right)^{\frac{N(N-1)}{2}} \\
\mathcal{N} =& \left(\sqrt{\frac{\pi}{\omega}} \right)^{-\frac{N(N+1)}{2}} 2^{\frac{N(N-1)}{4}}.
\end{aligned}
$$

2.1.3 Normalisation constant $\mathcal{N}$ for an anti-symmetric matrix M

Let M be an anti-symmetric $N \times N$ matrix. Anti-symmetric matrices satisfy the condition

$$M = -M^T. \tag{2.11}$$

The measure for an anti-symmetric matrix is given by

$$dM = \prod_{i>j} (dM_{ij}). \tag{2.12}$$

$\mathcal{N}$ can be evaluated as follows

$$
\begin{aligned}
1 =&\mathcal{N}\int dMe^{-\omega trM^2}\\
=&\mathcal{N}\int\prod_{i>j}(dM_{ij})e^{-2\omega\sum_{i>j}^{N}M_{ij}M_{ij}}\\
=&\mathcal{N}\left(\sqrt{\frac{\pi}{2\omega}}\right)^{\frac{N(N-1)}{2}}\\
=&\mathcal{N}\left(\sqrt{\frac{\pi}{\omega}}\right)^{\frac{N(N-1)}{2}}\left(\sqrt{\frac{1}{2}}\right)^{\frac{N(N-1)}{2}}\\
\mathcal{N} =&\left(\sqrt{\frac{\pi}{\omega}}\right)^{-\frac{N(N1)}{2}}2^{\frac{N(N-1)}{4}}.
\end{aligned}
$$

2.2 Correlation functions.

From now on we will treat M as an Hermitian matrix. In this section, we will develop two methods for computing correlators.

2.2.1 Correlation function from generating function: $Z[J]=\int dMe^{-\omega trM^2+trJM}$

As it was stated before, we can generate the n-point correlation functions using the generating function given by equation (2.3). This can be evaluated by completing the square in (2.3)

$$
\begin{aligned}
Z[J] =& \int dMe^{-\omega trM^2+trJM}\\
=& \quad e^{Tr\left(\frac{J^2}{4\omega}\right)}\int dMe^{-NTr\left(M-\frac{J}{2\omega}\right)^2}\\
=& \quad e^{Tr\left(\frac{J^2}{4\omega}\right)} \quad \text{(by the using the defination of equation (2.8))}\\
=& \quad e^{Tr\left(\frac{J^2}{4\omega}\right)}
\end{aligned}
$$

We will use this results to evaluate the following correlators: $\langle M_{ij}M_{kl}\rangle$; $\langle tr(M^2)\rangle$, $\langle tr(M^2)tr(M^2)\rangle$ and $\langle tr(M^2)tr(M^2)tr(M^2)\rangle$. Thanks to the summation convention $tr(J^2)=J_{ab}J_{ba}$

1. $\langle M_{ij} M_{kl} \rangle$

$$
\begin{aligned}
\langle M_{ij} M_{kl} \rangle =& \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} e^{Tr\left(\frac{J^2}{4\omega}\right)} \Big|_{J=0} \\
=& \frac{d}{dJ_{ji}} \left(\frac{d}{dJ_{lk}} \frac{J_{ab} J_{ba}}{4\omega} \right) e^{Tr\left(\frac{J^2}{4\omega}\right)} \Big|_{J=0} \\
=& \frac{1}{4\omega} \frac{d}{dJ_{ji}} (\delta_{la}\delta_{bk} J_{ba} + J_{ab}\delta_{bl}\delta_{ak}) e^{Tr\left(\frac{J^2}{4\omega}\right)} \Big|_{J=0} \\
=& \frac{1}{4\omega} (\delta_{la}\delta_{bk}\delta_{jb}\delta_{ia} + \delta_{aj}\delta_{ib}\delta_{bl}\delta_{ak}) e^{Tr\left(\frac{J^2}{4\omega}\right)} \\
& + \frac{1}{16\omega^2} (\delta_{la}\delta_{bk} J_{ba} + J_{ab}\delta_{bl}\delta_{ak})(\delta_{ja}\delta_{bi} J_{ba} + J_{ab}\delta_{bj}\delta_{ai}) e^{Tr\left(\frac{J^2}{4\omega}\right)} \Big|_{J=0} \\
=& \frac{1}{4\omega} (\delta_{la}\delta_{bk}\delta_{jb}\delta_{ia} + \delta_{aj}\delta_{ib}\delta_{bl}\delta_{ak}) \\
=& \frac{1}{2\omega} \delta_{la}\delta_{bk}\delta_{jb}\delta_{ia}
\end{aligned}
$$

finally,

$$
\langle M_{ij} M_{kl} \rangle = \frac{1}{2\omega} \delta_{jk} \delta_{il} \tag{2.13}
$$

$\langle M_{ij} M_{kl} \rangle$ is known as the propagator

2. $\langle tr(M^2)\rangle = \langle M_{ij}M_{ji}\rangle$

$$
\begin{aligned}
\langle tr(M^2)\rangle =& \frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}} e^{Tr\left(\frac{J^2}{4\omega}\right)}\Bigg|_{J=0} \\
=& \frac{1}{4\omega}\frac{d}{dJ_{ji}}(\delta_{ia}\delta_{bj}J_{ba} + J_{ab}\delta_{bi}\delta_{aj}) e^{Tr\left(\frac{J^2}{4\omega}\right)}\Bigg|_{J=0} \\
=& \frac{1}{4\omega}(\delta_{ia}\delta_{jk}\delta_{jb}\delta_{ia} + \delta_{aj}\delta_{ib}\delta_{bi}\delta_{aj}) e^{Tr\left(\frac{J^2}{4\omega}\right)}\Bigg|_{J=0} \\
&+ \frac{1}{16\omega^2}(\delta_{ia}\delta_{bj}J_{ba} + J_{ab}\delta_{bi}\delta_{aj})(\delta_{ja}\delta_{bi}J_{ba} + J_{ab}\delta_{bj}\delta_{ai}) e^{Tr\left(\frac{J^2}{4\omega}\right)}\Bigg|_{J=0} \\
=& \frac{1}{4\omega}(\delta_{ia}\delta_{jk}\delta_{jb}\delta_{ia} + \delta_{aj}\delta_{ib}\delta_{bi}\delta_{aj}) \\
=& \frac{1}{2\omega}(\delta_{ii}\delta_{jj})
\end{aligned}
$$

$$
\langle tr(M^2)\rangle = \frac{N^2}{2\omega} \tag{2.14}
$$

3. $\langle tr(M^2)tr(M^2)\rangle = \langle M_{kl}M_{lk}M_{ij}M_{ji}\rangle$

$$
\begin{aligned}
\langle tr(M^2)tr(M^2)\rangle =& \frac{d}{dJ_{lk}}\frac{d}{dJ_{kl}}\frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}} e^{Tr\left(\frac{J^2}{4\omega}\right)}\Bigg|_{J=0} \\
=& \frac{1}{4\omega^2}\left[(\delta_{ii}\delta_{jj})(\delta_{ll}\delta_{kk}) + \delta_{kj}\delta_{li}\delta_{il}\delta_{kj} + \delta_{ki}\delta_{lj}\delta_{lj}\delta_{ki}\right]
\end{aligned}
$$

$$
\langle tr(M^2)tr(M^2)\rangle = \frac{1}{4\omega^2}(2N^2 + N^4) \tag{2.15}
$$

4. $\langle tr(M^2)tr(M^2)tr(M^2)\rangle = \langle M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}\rangle$

$$
\begin{aligned}
\langle tr(M^2)tr(M^2)tr(M^2)\rangle =& \frac{d}{dJ_{sr}}\frac{d}{dJ_{rs}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{kl}}\frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}} e^{Tr\left(\frac{J^2}{4\omega}\right)}\Bigg|_{J=0} \\
=& \frac{1}{8\omega^3}[N^6 + 6N^4 + 8N^2]
\end{aligned} \tag{2.16}
$$

2.2.2 Correlation functions using Wick's Theorem

Wick's Theorem is a method that is used to calculate n-point correlation functions, by contracting over all possible fields. The total number of fields contracted must be even. If they are odd, then the correlation function vanishes. The computation of a n-point correlation function using Wick contractions is similar to the computation of n-point correlation functions of a zero dimension scaler field. Consider the n-point correlation function of a zero dimensional scaler field ϕ

$$\langle \phi^{2n} \rangle = \int_{\infty}^{\infty} d\phi \quad \phi^{2n} e^{-\alpha\phi^2} = \frac{(2n-1)!!}{(2\alpha)^n} \tag{2.17}$$

The term $(2n-1)!!$ counts the number of ways in which the fields can combine with each other. The Wick contraction also tells us how the fields are going to combine. Using this idea of a zero dimension scalar field, we write the correlation function as follows

$$\langle M^{2n} \rangle = \langle M_{ij} M_{kl} .....M_{rs} \rangle \tag{2.18}$$

$$= \frac{\text{(sum of all Wick contrations)}}{(2\omega)^n}. \tag{2.19}$$

For the matrix model, a Wick contraction of two fields is given by

$$\langle M_{ij} M_{kl} \rangle = \overbrace{M_{ij} M_{kl}} \tag{2.20}$$

Using this, we can now again compute the examples we considered above, but this time we will use Wick's Theorem.

1. $\langle M_{ij} M_{kl} \rangle$

$$\begin{aligned} \langle M_{ij} M_{kl} \rangle =& \overbrace{M_{ij} M_{kl}} \\ =& \frac{\delta_{il}\delta_{jk}}{(2\omega)} \end{aligned}$$

2. $\langle tr(M^2) \rangle = \langle M_{ij} M_{ji} \rangle$

$$\begin{aligned} \int [dM] e^{-\omega Tr(M^2)} tr(M^2) &= \int [dM] e^{-\omega Tr(M^2)} M_{ij} M_{ji} \\ &= \langle M_{ij} M_{ji} \rangle \\ &= \frac{\text{(sum of all Wick contrations)}}{2\omega} \\ &= \overbrace{M_{ij} M_{ji}} \\ &= \frac{\delta_{ii}\delta_{jj}}{(2\omega)} \end{aligned}$$

$$\langle M_{ij}M_{ji}\rangle = \frac{N^2}{(2\omega)} \tag{2.21}$$

3. $\langle tr(M^2)tr(M^2)\rangle = \langle M_{kl}M_{lk}M_{ij}M_{ji}\rangle$

$$\begin{aligned}\int [dM]e^{-\omega Tr(M^2)}tr(M^2)tr(M^2) &= \int [dM]e^{-\omega Tr(M^2)}M_{kl}M_{lk}M_{ij}M_{ji}\\ &= \langle M_{kl}M_{lk}M_{ij}M_{ji}\rangle\\ &= \frac{\text{(sum of all Wick contrations)}}{(2\omega)^2}\end{aligned}$$

Now, summing all the possible Wick contractions, we find

$$\langle M_{kl}M_{lk}M_{ij}M_{ji}\rangle = \frac{1}{(2\omega)^2}\left(M_{kl}M_{lk}M_{ij}M_{ji} + M_{kl}M_{lk}M_{ij}M_{ji} + M_{kl}M_{lk}M_{ij}M_{ji}\right)$$

We find that this is equal to

$$\begin{aligned}\langle M_{kl}M_{lk}M_{ij}M_{ji}\rangle &= \frac{1}{(2\omega)^2}\left(\delta_{kk}\delta_{ll}\delta_{jj}\delta_{ii} + \delta_{kj}\delta_{li}\delta_{li}\delta_{kj} + \delta_{ki}\delta_{lj}\delta_{lj}\delta_{ki}\right)\\ &= \frac{1}{(2\omega)^2}\left(N^4 + N^2 + N^2\right)\\ &= \frac{1}{(2\omega)^2}\left(N^4 + 2N^2\right)\end{aligned} \tag{2.22}$$

4. $\langle tr(M^2)tr(M^2)tr(M^2)\rangle = \langle M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}\rangle$

$$\langle M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}\rangle = \frac{\text{(sum of all Wick contrations)}}{(2\omega)^3}$$

The possible Wick contractions are

(i)

$$M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}$$

$$\begin{aligned}&=\delta_{rr}\delta_{ss}\delta_{kk}\delta_{ll}\delta_{ii}\delta_{jj} + \delta_{rr}\delta_{ss}\delta_{kj}\delta_{li}\delta_{li}\delta_{kj} + \delta_{rr}\delta_{ss}\delta_{ki}\delta_{lj}\delta_{lj}\delta_{ki}\\ &=N^6 + N^4 + N^4\\ &=2N^4 + N^6\end{aligned}$$

(ii)

$$M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}$$

$$\begin{aligned}
=&\delta_{rl}\delta_{sk}\delta_{sk}\delta_{rl}\delta_{ii}\delta_{jj} + \delta_{rl}\delta_{sk}\delta_{sj}\delta_{ri}\delta_{li}\delta_{kj} + \delta_{rl}\delta_{sk}\delta_{si}\delta_{rj}\delta_{lj}\delta_{ki}\\
=&N^4 + N^2 + N^2\\
=&2N^2 + N^4
\end{aligned}$$

(iii)

$$M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}$$

$$\begin{aligned}
=&\delta_{rk}\delta_{sl}\delta_{sl}\delta_{rk}\delta_{ii}\delta_{jj} + \delta_{rk}\delta_{sl}\delta_{sj}\delta_{ri}\delta_{ki}\delta_{lj} + \delta_{rk}\delta_{sl}\delta_{si}\delta_{rj}\delta_{kj}\delta_{li}\\
=&N^4 + N^2 + N^2\\
=&2N^2 + N^4
\end{aligned}$$

(iv)

$$M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}$$

$$\begin{aligned}
=&\delta_{rj}\delta_{si}\delta_{sl}\delta_{rk}\delta_{li}\delta_{kj} + \delta_{rj}\delta_{si}\delta_{sk}\delta_{rl}\delta_{ki}\delta_{lj} + \delta_{rj}\delta_{si}\delta_{si}\delta_{rj}\delta_{kk}\delta_{ll}\\
=&N^2 + N^2 + N^4\\
=&2N^2 + N^4
\end{aligned}$$

(v)

$$M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji} + {}_{ij}M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}$$

$$\begin{aligned}
=&\delta_{ri}\delta_{sj}\delta_{sl}\delta_{rk}\delta_{lj}\delta_{ki} + \delta_{ri}\delta_{sj}\delta_{sk}\delta_{rl}\delta_{kj}\delta_{li} + \delta_{ri}\delta_{sj}\delta_{sj}\delta_{ri}\delta_{kk}\delta_{ll}\\
=&N^2 + N^2 + N^4\\
=&2N^2 + N^4
\end{aligned}$$

$$\therefore \langle M_{rs}M_{sr}M_{kl}M_{lk}M_{ij}M_{ji}\rangle = \frac{(\text{ sum of all Wick contration})}{(2N)^3}$$
$$= \frac{1}{(2\omega)^3}(6N^4 + 8N^2 + N) \qquad (2.23)$$

5. $\langle tr(M^2)tr(M^4)\rangle = \langle M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}\rangle$

$$\langle M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}\rangle = \frac{(\text{sum of all Wick contration})}{(2\omega)^3}$$

(i)

$$M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}$$

$$=\delta_{ii}\delta_{jj}\delta_{lr}\delta_{kk}\delta_{rl}\delta_{ss} + \delta_{ii}\delta_{jj}\delta_{ls}\delta_{kr}\delta_{kl}\delta_{rs} + \delta_{ii}\delta_{jj}\delta_{ll}\delta_{ks}\delta_{ks}\delta_{rr}$$
$$=N^5 + N^3 + N^5$$

(ii)

$$M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}$$

$$=\delta_{ik}\delta_{jl}\delta_{jr}\delta_{ik}\delta_{rl}\delta_{ss} + \delta_{ik}\delta_{jl}\delta_{js}\delta_{ir}\delta_{kl}\delta_{rs} + \delta_{ik}\delta_{jl}\delta_{jl}\delta_{is}\delta_{ks}\delta_{rr}$$
$$=N^3 + N + N^3$$
$$=2N^3 + N$$

(iii)

$$M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}$$

$$=\delta_{ir}\delta_{jk}\delta_{jk}\delta_{il}\delta_{rl}\delta_{ss} + \delta_{ir}\delta_{jk}\delta_{sj}\delta_{ir}\delta_{ll}\delta_{ks} + \delta_{ir}\delta_{jk}\delta_{jl}\delta_{is}\delta_{ls}\delta_{rk}$$
$$=N^3 + N^3 + N$$
$$=2N^3 + N$$

(iv)

$$M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}$$

$$\begin{aligned}
&=\delta_{is}\delta_{jr}\delta_{jk}\delta_{il}\delta_{kl}\delta_{rs} + \delta_{is}\delta_{jr}\delta_{jr}\delta_{ik}\delta_{ll}\delta_{ks} + \delta_{is}\delta_{jr}\delta_{jl}\delta_{is}\delta_{lr}\delta_{kk} \\
&=N + N^3 + N^3 \\
&=2N^3 + N
\end{aligned}$$

(v)

$$M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl} + {}_{rs}M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}$$

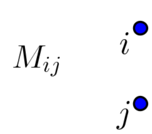
$$\begin{aligned}
&=\delta_{il}\delta_{js}\delta_{jk}\delta_{il}\delta_{ks}\delta_{rr} + \delta_{il}\delta_{js}\delta_{jr}\delta_{ik}\delta_{ls}\delta_{kr} + \delta_{il}\delta_{js}\delta_{js}\delta_{ir}\delta_{lr}\delta_{kk} \\
&=N^3 + N + N^3 \\
&=2N^3 + N
\end{aligned}$$

Summing all the terms together, we get

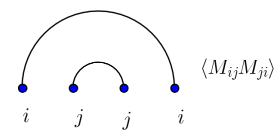
$$\begin{aligned}
\langle M_{ij}M_{ji}M_{lk}M_{kr}M_{rs}M_{sl}\rangle &= \frac{(\text{sum of all Wick contrations})}{(2\omega)^3} \\
&= \frac{1}{(2\omega)^3}(2N^5 + 9N^3 + 4N) \qquad (2.24)
\end{aligned}$$

We are now in a position to state Feynman rules for this matrix model.

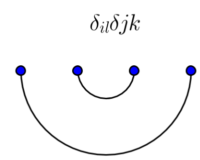
1. For each matrix element M_{ij} draw a pair of dots, one dot for each of i and j



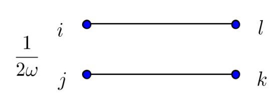
2. Join dots of indices that are being traced



3. Connect pairs of dots using a ribbon. Do not twist the ribbon.



4. For each ribbon we include a factor of $\dfrac{1}{2\omega}$



2.3 Large N factorisation

If one computes the correlator $\langle \prod_i Tr(M^{2n_i}) \rangle$, the Feynman diagrams that are produced by this correlator include connected (planar and non-planar) and disconnected diagrams. Planar diagrams are diagrams that do not cross when drawn on a plane, while non-planar diagrams are diagrams that do cross when drawn on a plane. We are interested in computing planar diagrams without having to deal with the non-planar and disconnected diagrams. This is done using a method called large N factorization.

Large N factorization is a way of organising our theory so that after taking the limit $N \to \infty$, we are left with the planar diagrams.
To begin, normalise our operator TrM^{2n} such that its expectation value in the large N limit is of order 1.

$$\mathcal{N} TrM^{2n} = \frac{TrM^{2n}}{N^{\frac{2n+2}{2}}}. \tag{2.25}$$

Using this normalisation and dropping terms that vanishes as $N \to \infty$,

$$\langle Tr(M^2)\rangle = \frac{1}{2\omega} \tag{2.26}$$

$$\begin{aligned}\langle Tr(M^2)Tr(M^2)\rangle &= \lim_{N\to\infty}\frac{1}{(2\omega)^2}\frac{(2N^2+N^4)}{N^4} \\ &= \frac{1}{(2\omega)^2} \\ &= \langle Tr(M^2)\rangle^2 \end{aligned}\tag{2.27}$$

$$\begin{aligned}\langle Tr(M^2)Tr(M^2)Tr(M^2)\rangle &= \lim_{N\to\infty}\frac{1}{(2\omega)^3}\frac{N^6+6N^4+8N^2}{N^6} \\ &= \frac{1}{(2\omega)^3} \\ &= \langle Tr(M^2)\rangle^3. \end{aligned}\tag{2.28}$$

From the above calculation we conclude that the expectation value of the product is the product of the expectation values,

$$\langle\prod_i Tr(M^{2n_i})\rangle = \prod_i\langle Tr(M^{2n_i})\rangle. \tag{2.29}$$

Although we have not demonstrated it, this statement is rather general

2.4 Including interactions

So far, we have been studying a free theory where the action is written as

$$S = \omega Tr(M^2)$$

Now, we want to include interactions in our theory. We will study a matrix model, which is weakly coupled ($g \ll 1$), with an action that takes the form

$$S = \omega Tr(M^2) + gTr(M^4).$$

We normalise the measure using the same procedure as before and note that the generating function is given by

$$Z[J,g] = \int [dM]e^{-\omega Tr(M^2)-gTr(M^4)+Tr(JM)} \tag{2.30}$$

which means

$$Z[J,g]\Big|_{J=0,g=0} = 1 \tag{2.31}$$

To compute correlators in the interacting theory, we need to first expand the term in the action that contains g. Consider correlators accurate to order g. We will illustrate the idea with two examples

0-point function $\langle 1 \rangle$

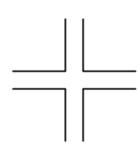
$$\begin{aligned}\langle 1 \rangle &= \int [dM] e^{-\omega Tr(M^{2n})}(1 - gTr(M^4) + \mathcal{O}(g^2)) \\ &= 1 - g\langle Tr(M^4)\rangle \\ &= 1 - \frac{g}{(2\omega)^2}(2N^3 + N) \end{aligned} \tag{2.32}$$

2-point function $\langle Tr(M^2) \rangle$

$$\begin{aligned}\langle Tr(M^2) \rangle &= \int [dM] e^{-\omega Tr(M^{2n})} Tr(M^2)(1 - gTr(M^4) + \mathcal{O}(g^2)) \\ &= \langle Tr(M^2)\rangle - g\langle Tr(M^2)Tr(M^4)\rangle \\ &= \frac{N^2}{2\omega} - \frac{g}{(2\omega)^3}(2N^5 + 9N^3 + 4N) \end{aligned} \tag{2.33}$$

We state the new Feynman rule for the interacting theory as

5. Each vertex is given by $-g$



$= -g$

We see that if we include perturbative interactions in our theory, we have to introduce a g which is small so that perturbation theory is valid. If we look at (2.32) in the large N limit, the correction term $(\frac{g}{(2\omega)^2}(2N^3 + N))$ is no longer small due to the presence of N^3 in front of g. To overcome this problem, we need a new parameter in the theory which will ensure that the correction term is still small.

2.5 t'Hooft limit

To make sure our perturbation theory produces sensible results, we introduce the t'Hooft parameter λ

$$\lambda = gN \tag{2.34}$$

We use this parameter when we are computing correlators. This parameter will be kept fixed in the large N limit. Thus, we are sending $g \to 0$ as $N \to \infty$. Now, let compute the 0-point function using the t'Hooft parameter

$$\begin{aligned}\langle 1 \rangle_{conn} &= 1 - \frac{g}{(2\omega)^2}(2N^3 + N) \\ &= 1 + N^2\left(\frac{-2\lambda}{2\omega}\right) - \frac{\lambda}{2\omega}\end{aligned} \tag{2.35}$$

Now, treating this last expression as an expansion in $\frac{1}{N}$ and λ, we have

$$\langle 1 \rangle_{conn} = 1 + N^2\left(\frac{-2\lambda}{2\omega} + \mathcal{O}(\lambda^2)\right) + N^0\left(-\frac{\lambda}{2\omega} + \mathcal{O}(\lambda^2)\right) + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{2.36}$$

The 2-point function in terms of the λ parameter is given by

$$\begin{aligned}\langle Tr(M^2) \rangle_{conn} &= \frac{N^2}{2\omega} - \frac{g}{(2\omega)^3}(8N^3 + 4N) \\ &= \frac{N^2}{2\omega} - \frac{1}{(2\omega)^3}(8\lambda N^2 + 4\lambda) \\ &= N^2\left(\frac{1}{2\omega} - \frac{8\lambda}{(2\omega)^3} + \mathcal{O}(\lambda^2)\right) + N^0\left(-\frac{4\lambda}{(2\omega)^3} + \mathcal{O}(\lambda^2)\right) + \mathcal{O}\left(\frac{1}{N^2}\right)\end{aligned} \tag{2.37}$$

Taking the planar limit, the normalised correlator becomes

$$\mathcal{N}\langle Tr(M^2) \rangle_{conn} = \frac{1}{2\omega} - \frac{8\lambda}{(2\omega)^3} + \mathcal{O}(\lambda^2) \tag{2.38}$$

Therefore, we see that this approach of defining a new parameter gives us sensible results, by eliminating the correction that seemed to be bigger than the leading term.

2.6 How the powers of N in matrix model correspond to different topologies

Recall that we have a method to compute correlators by drawing planar diagrams. Now, we want to investigate the powers of N that appear when we compute correlators using the $\frac{1}{N}$ expansion. Recall the 2-point function

$$\langle Tr(M^2)\rangle_{conn} = N^2\frac{1}{2\omega} - N^2\left(\frac{8\lambda}{(2\omega)^3}+\frac{4\lambda}{(2\omega)^3N^2}\right) + \mathcal{O}(\lambda^2)$$

In this expansion we have terms of order N^2 and N^0. If we have to include all terms in this expansion, there will be terms of order N^{-2n}. In general, given an observable O, the expansion of this observable is given by

$$\langle O\rangle = \sum_{n=0}^{\infty} f_n(\lambda)N^{2-2n} \tag{2.39}$$

Since planar diagrams can be drawn on a sphere and non-planar diagrams can be drawn on surfaces of higher genus, we can use this knowledge of topology to explain the powers of N in our correlator expansion. The terms with coefficient N^2 are the ones that come from planar diagrams and terms with power N^0 come from diagrams on the torus.

In terms of the surfaces, the 0-point function (2.36) expansion can be represented as follows

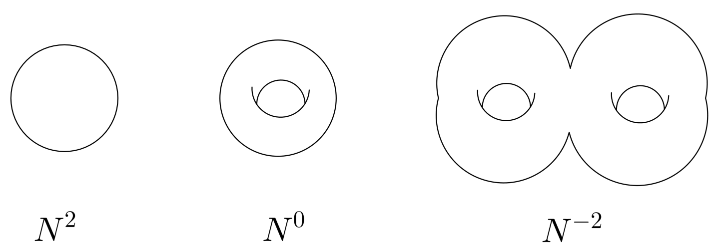


Figure 1: The large N expansion is an expansion in surfaces.

We can relate the powers of N to the Euler characteristic χ as N^χ. The Euler characteristic is topological invariant. χ for the sphere is two ($\chi_{sphere} = 2$) and for a torus is zero ($\chi_{torus} = 0$). In general, the **Euler characteristic of closed orientable surfaces** is given by

$$\chi = 2 - 2g$$

where g is the genus which counts the number of handles on a surface. The general equation for χ is given by

$$\chi = 2 - 2g - b, \tag{2.40}$$

where b is the number of boundaries each surface has. For closed orientable surfaces $b = 0$. Other possible diagrams that can be drawn that obey this last equation are shown in the figure below

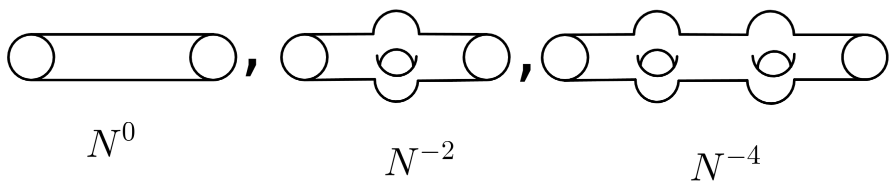


Figure 2: Example of surfaces and N^χ.

This relation tells us that instead of summing all the Feynman diagrams that comes from the connected correlator we can sum topological surfaces with χ determining the powers of N. These results tell us that matrix model correlators can be interpreted in terms of a sum over surfaces.

Consider a point particle. The propagator is drawn as a line because the amplitude can be obtained by summing over all possible worldlines. The loop expansion for the 2-point function is drawn as follows

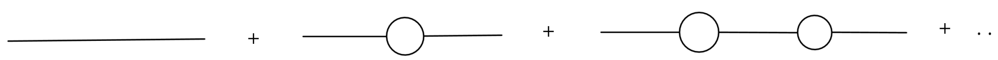


Figure 3: Sum of particle worldlines.

If we change the particle to strings, the string propagator will no longer be represented by a line, instead it will be represented by a sheet. The loop expansion for this string propagator will be a sum of diagrams for the worldsheet as shown below

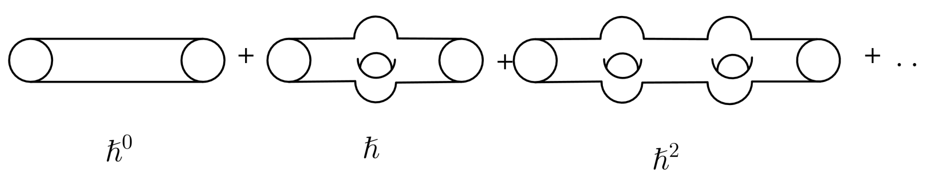


Figure 4: Sum of the string worldsheets.

This represents a loop expansion in $\hbar$. The one loop diagram is of order $\hbar$, two loop is of order $\hbar^2$ and so on. Comparing fig 2 and fig 4, we conclude that $\hbar$ in string theory is equal to $\frac{1}{N^2}$ in the matrix model. These results gives us strong evidence that matrix models can be related to string theories.

3 Matrix Model Correlators

We study a Gaussian matrix model. Correlators of this model are defined to be

$$\langle O\rangle \equiv \int [dM] e^{-\frac{1}{2}TrM^2} O \tag{3.1}$$

where M is an $N \times N$ Hermitian matrix.

Recursions relations for correlators are obtained by using Schwinger-Dyson equations

$$0 = \int [dM] \frac{d}{dM_{ij}} \left(e^{-\frac{1}{2}TrM^2} O \right). \tag{3.2}$$

As an example, the Schwinger-Dyson equation for correlators of the form (1.1) follow from

$$0 = \int [dM] \frac{d}{dM_{ij}} \left((M^{2n_1-1})_{ij} Tr(M^{2n_2}) \cdots Tr(M^{2n_k}) e^{-\frac{1}{2}Tr(M^2)} \right). \tag{3.3}$$

The presence of the term $(M^{2n_1-1})_{ij}$ is needed to make sure that after taking the derivative with respect to M_{ij} all indices contract with each other. We then obtain the recursion formula. After obtaining the recursion form of (3.3), we restrict these correlators to only connected correlators. To find these connected correlators we first need to look at the expansion of $\langle Tr(M^{2n})\rangle$. This expansion is given by

$$\langle Tr(M^{2n})\rangle = c_o N^{n+1} + c_1 N^{n-1} + \cdots + c_p N^{n+1-2p} + \cdots \tag{3.4}$$

where we have used the notation

$$c_p N^{n+1-2p} \leftrightarrow \langle Tr(M^{2n})\rangle_p. \tag{3.5}$$

Knowing the expansion of $\langle Tr(M^{2n})\rangle$, we can compute connected correlators from different orders of the large N expansion. The coefficients $(c_0, c_1, \cdots, c_p)$ play an important role in counting the number of connected Feynman diagrams.
Using this notation, we find the leading order of the correlators are

$$\langle Tr(M^{2n_1}) Tr(M^{2n_2}) \cdots Tr(M^{2n_k})\rangle_0 = O(N^{n_1+n_2+\cdots+n_k+k}), \tag{3.6}$$

$$\langle Tr(M^{2n_1}) Tr(M^{2n_2}) \cdots Tr(M^{2n_k})\rangle_{conn,0} = O(N^{n_1+n_2+\cdots+n_k-k+2}). \tag{3.7}$$

Deriving a recursion relation for the correlator (1.1) using the Schwinger-Dyson equations allows us to extract the connected correlators by using the above expansion . Later, we will employ the same procedure to find correlators of form $\langle Tr(M^n \ln M)\rangle_{conn}$. Let's compute a few example of connected correlators of the form (1.1).

3.1 The Schwinger-Dyson equation; k=2 case

The relevant Schwinger-Dyson equation follows from

$$0 = \int [dM] \frac{d}{dM_{ij}} \left((M^{2n_1-1})_{ij} Tr(M^{2n_2}) e^{-\frac{1}{2}Tr(M^{2n})} \right). \tag{3.8}$$

This equation reduces to

$$0 = \langle \sum_{r=0}^{2n_1-2} Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2})\rangle + 2n_2 \langle Tr(M^{2n_1+2n_2-2})\rangle - \langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle. \tag{3.9}$$

From (3.7), it is clear that the order of the expansion that will give us the connected correlator is order $O(N^{n_1+n_2})$. Let us now consider the systematic $\frac{1}{N}$ expansion of (3.8)

Leading order $O(N^{n_1+n_2+2})$:

$$0 = \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_0 - \langle Tr(M^{2n_1})\rangle_0 \langle Tr(M^{2n_2})\rangle_0. \tag{3.10}$$

To simplify this expression, consider the following Schwinger-Dyson equation

$$0 = \int dM \frac{d}{dM_{ij}} \left((M^{2n_1-1})_{ij} e^{-\frac{1}{2}Tr(M^2)} \right). \tag{3.11}$$

This implies that

$$0 = \langle \sum_{r=0}^{2n_1-2} Tr(M^r)Tr(M^{2n_1-r-2})\rangle - \langle Tr(M^{2n_1})\rangle. \tag{3.12}$$

The leading order of this equation is

$$0 = \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0 - \langle Tr(M^{2n_1})\rangle_0, \tag{3.13}$$

so that (3.10) is obeyed. We will also need the first subleading order of (3.12), which is given by

$$\begin{aligned} 0 = &\sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_1-2r-2})\rangle_0 + \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_1 \\ &+ \sum_{r=0}^{2n_1-2} \langle Tr(M^r)Tr(M^{2n_1-r-2})\rangle_{conn,0} - \langle Tr(M^{2n_1})\rangle_1. \end{aligned} \tag{3.14}$$

Note, if the traces $Tr(M^r)$ and $Tr(M^{2n_1-r-2})$ are not in the same expectation i.e. they appear as $\langle Tr(M^r)\rangle$ and $\langle Tr(M^{2n_1-r-2})\rangle$, we will take the upper limit of the sum to be n_1-1 instead $2n_1-2$, keeping only even powers

$$\sum_{r=0}^{2n_1-2}\langle Tr(M^r)\rangle\langle Tr(M^{2n_1-r-2})\rangle = \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle\langle Tr(M^{2n_1-2r-2})\rangle + 0.$$

The odd powers vanishes because they are odd Gaussian integrals.

First subleading order: $O(N^{n_1+n_2})$.

Computing terms of orders $O(N^{n_1+n_2})$ contributing to (3.9), we get

$$\begin{aligned}
0 =& \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1 \\
&+ \sum_{r=0}^{2n_1-2}\langle Tr(M^{r})Tr(M^{2n_1-r-2})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})Tr(M^{2r})\rangle_{conn,0} \\
&+ 2n_2\langle Tr(M^{2n_1+2n_2-2})\rangle_0 - \langle Tr(M^{2n_2})Tr(M^{2n_1})\rangle_{conn,0} - \langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_0 \\
&- \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_1
\end{aligned}$$

The bold terms below vanishes after applying (3.14)

$$\begin{aligned}
0 &= \sum_{\mathbf{r=0}}^{\mathbf{n_1-1}} \langle \mathbf{Tr(M^{2r})} \rangle_{\mathbf{1}} \langle \mathbf{Tr(M^{2n_1-2r-2})} \rangle_{\mathbf{0}} \langle \mathbf{Tr(M^{2n_2})} \rangle_{\mathbf{0}} \\
&+ \sum_{\mathbf{r=0}}^{\mathbf{n_1-1}} \langle \mathbf{Tr(M^{2r})} \rangle_{\mathbf{0}} \langle \mathbf{Tr(M^{2n_1-2r-2})} \rangle_{\mathbf{1}} \langle \mathbf{Tr(M^{2n_2})} \rangle_{\mathbf{0}} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_1 \\
&+ \sum_{\mathbf{r=0}}^{\mathbf{2n_1-2}} \langle \mathbf{Tr(M^{r})Tr(M^{2n_1-r-2})} \rangle_{\mathbf{conn,0}} \langle \mathbf{Tr(M^{2n_2})} \rangle_{\mathbf{0}} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_2}) \rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) Tr(M^{2r}) \rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2}) \rangle_0 - \langle Tr(M^{2n_2}) Tr(M^{2n_1}) \rangle_{conn,0} - \langle \mathbf{Tr(M^{2n_1})} \rangle_{\mathbf{1}} \langle \mathbf{Tr(M^{2n_2})} \rangle_{\mathbf{0}} \\
&- \langle Tr(M^{2n_1}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_1
\end{aligned}$$

The terms in bold sum to zero after using (3.13)

$$\begin{aligned}
0 &= \sum_{\mathbf{r=0}}^{\mathbf{n_1-1}} \langle \mathbf{Tr(M^{2r})} \rangle_{\mathbf{0}} \langle \mathbf{Tr(M^{2n_1-2r-2})} \rangle_{\mathbf{0}} \langle \mathbf{Tr(M^{2n_2})} \rangle_{\mathbf{1}} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_2}) \rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) Tr(M^{2r}) \rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2}) \rangle_0 - \langle Tr(M^{2n_2}) Tr(M^{2n_1}) \rangle_{conn,0} - \langle \mathbf{Tr(M^{2n_1})} \rangle_{\mathbf{0}} \langle \mathbf{Tr(M^{2n_2})} \rangle_{\mathbf{1}}.
\end{aligned}$$

Finally, this gives

$$\begin{aligned}
0 = &\sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_2}) \rangle_{conn,0} + \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) Tr(M^{2r}) \rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2}) \rangle_0 - \langle Tr(M^{2n_2}) Tr(M^{2n_1}) \rangle_{conn,0}. \qquad (3.15)
\end{aligned}$$

The basic quantity we are interested in is $\gamma(n_1, n_2, \cdots, n_k)$, where

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle_0 = N^{n_1+n_2+\cdots+n_k+k}\gamma(n_1, n_2, \cdots, n_k), \qquad (3.16)$$

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle_{conn,0} = N^{n_1+n_2+\cdots+n_k-k+2}\gamma(n_1, n_2, \cdots, n_k). \qquad (3.17)$$

This quantity counts the number of connected genus zero Feynman diagrams since $\langle Tr(M^{2n_2})Tr(M^{2n_1})\rangle_{conn,0} = \gamma(n_1, n_2)$. Re-writing (3.15) in terms of this quantity

$$\begin{aligned}\gamma(n_1, n_2) = &\sum_{r=0}^{n_1-1}\gamma(r)\gamma(n_1-r-1, n_2) + \sum_{r=0}^{n_1-1}\gamma(n_1-r-1)\gamma(r, n_2)\\ &+ 2n_2\gamma(n_1+n_2-1). \end{aligned} \qquad (3.18)$$

This is the recursion formula for $k = 2$. It is in perfect agreement with the recursion formula derived in [27].

3.2 Schwinger-Dyson equation; k=3 case.

The Schwinger-Dyson equation for the correlator $\langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle$ follows from

$$0 = \int dM \frac{d}{dM_{ij}}\left((M^{2n_1-1})_{ij}Tr(M^{2n_2})Tr(M^{2n_3})e^{-\frac{1}{2}Tr(M^2)}\right). \qquad (3.19)$$

This implies

$$\begin{aligned}0 = &\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2})Tr(M^{2n_3})\rangle + 2n_2\langle Tr(M^{2n_1+2n_2-2})Tr(M^{2n_3})\rangle\\ &+ 2n_3\langle Tr(M^{2n_1+2n_3-2})Tr(M^{2n_2})\rangle - \langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle. \end{aligned} \qquad (3.20)$$

We again develop this equation using a $\frac{1}{N}$ expansion . Our task is to compute the second subleading order of equation (3.20) which is of order $O(N^{n_1+n_2+n_3-1})$. This follows because we are interested in the quantity $\gamma(n_1, n_2, n_3)$, obtained from the relation

$$\begin{aligned}\langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0} &= N^{n_1+n_2+n_3-3+2}\gamma(n_1, n_2, n_3)\\ &= N^{n_1+n_2+n_3-1}\gamma(n_1, n_2, n_3).\end{aligned}$$

A recursion relation for $\gamma(n_1, n_2, n_3)$ is obtained by evaluating and grouping the same orders in the expansion of (3.20). The $\frac{1}{N}$ expansion of (3.20) is carried out below. We first expand each of the terms appearing in (3.20). Once this is achieved, we recombine the terms to give the systematic expansion of the full Schwinger-Dyson equation, (3.20). The expansion of the terms in (3.20) now follows.

First Term : Leading Order $O(N^{n_1+n_2+n_3+3})$

$$\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_0 = \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \tag{3.21}$$

First Term : First Subleading Order $O(N^{n_1+n_2+n_3+1})$.

$$\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_1$$

$$\begin{aligned}
&= \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1 \\
&+ \sum_{r=0}^{2n_1-2}\langle Tr(M^{r})Tr(M^{2n_1-r-2})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0 .
\end{aligned}$$

First Term : Second Subleading Order $O(N^{n_1+n_2+n_3-1})$.

$$
\begin{aligned}
&\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_2 \\
=&\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_1 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_2\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_2\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_2\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_2 \\
&+\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_0 \\
&+\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0 \\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0
\end{aligned}
$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2r})\rangle_0$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}$$

$$+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{2n_1-r-2})Tr(M^{r})\rangle_{conn,0}\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}$$

$$+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{r})Tr(M^{2n_1-r-2})\rangle_{conn,0}[\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0 + \langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1]$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}[\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_3})\rangle_0 + \langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_3})\rangle_1]$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}[\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0 + \langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1]$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})\rangle_{conn,0}[\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_3})\rangle_0 + \langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_3})\rangle_1]$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_3})Tr(M^{2n_1-2r-2})\rangle_{conn,0}[\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_2})\rangle_0 + \langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_2})\rangle_1]$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}[\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2r})\rangle_0 + \langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2r})\rangle_1]$$

$$+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{r})Tr(M^{2n_1-r-2})\rangle_{conn,1}\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0$$

$$+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})\rangle_{conn,1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$

$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_3})Tr(M^{2n_1-2r-2})\rangle_{conn,1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_2})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2r})\rangle_0.$$

Second Term : Leading Order $O(N^{n_1+n_2+n_3+1})$.

$$2n_2\langle Tr(M^{2n_1-2n_2-2})Tr(M^{2n_3})\rangle_0 = 2n_2\langle Tr(M^{2n_1-2n_2-2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \tag{3.22}$$

Second Term : first Subleading Order $O(N^{n_1+n_2+n_3-1})$.

$$\begin{aligned}2n_2&\langle Tr(M^{2n_1-2n_2-2})Tr(M^{2n_3})\rangle_1\\ &=2n_2\Bigg[\langle Tr(M^{2n_1-2n_2-2})\rangle_0\langle Tr(M^{2n_3})\rangle_1+\langle Tr(M^{2n_1-2n_2-2})\rangle_1\langle Tr(M^{2n_3})\rangle_0\\ &\qquad\langle Tr(M^{2n_1-2n_2-2})Tr(M^{2n_3})\rangle_{conn,0}\Bigg]\end{aligned} \tag{3.23}$$

Third Term : Leading Order $O(N^{n_1+n_2+n_3+1})$.

$$2n_3\langle Tr(M^{2n_1-2n_3-2})Tr(M^{2n_2})\rangle_0 = 2n_3\langle Tr(M^{2n_1-2n_3-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0 \tag{3.24}$$

Third Term : first Subleading Order $O(N^{n_1+n_2+n_3-1})$.

$$\begin{aligned}2n_3&\langle Tr(M^{2n_1-2n_3-2})Tr(M^{2n_2})\rangle_1\\ &=2n_3\Bigg[\langle Tr(M^{2n_1-2n_3-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1+\langle Tr(M^{2n_1-2n_3-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0\\ &\qquad\langle Tr(M^{2n_1-2n_3-2})Tr(M^{2n_2})\rangle_{conn,0}\Bigg]\end{aligned} \tag{3.25}$$

Fourth Term : Leading Order $O(N^{n_1+n_2+n_3+3})$.

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_0 = \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$

Fourth Term : First Subleading Order $O(N^{n_1+n_2+n_3+1})$.

$$\begin{aligned}&\langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_1\\ &=\langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1\\ &+\langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0\\ &+\langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0\\ &+\langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_0\\ &+\langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0\\ &+\langle Tr(M^{2n_3})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_1})\rangle_0\end{aligned}$$

Fourth Term : Second Subleading Order

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_2$$

$$\begin{aligned}
&= \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_1 \\
&+ \langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0 \\
&+ \langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1 \\
&+ \langle Tr(M^{2n_1})\rangle_2\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&+ \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_2\langle Tr(M^{2n_3})\rangle_0 \\
&+ \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_2 \\
&+ \langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_1 \\
&+ \langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_1 \\
&+ \langle Tr(M^{2n_3})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_1})\rangle_1 \\
&+ \langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0} \\
&+ \langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_1})\rangle_0 \\
&+ \langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,1}\langle Tr(M^{2n_3})\rangle_0 \\
&+ \langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_2})\rangle_0.
\end{aligned}$$

This completes the expansion of the individual terms in the Schwinger-Dyson equation. We will now sum the different orders to obtain the $\frac{1}{N}$ expansion of the Schwinger-Dyson equation (3.20) itself.

Leading order of the Schwinger-Dyson equation $O(N^{2n_1+2n_2+2n_3+3})$.

$$\begin{aligned}
0 &= \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&- \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0 \\
&= \left[\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0 - \langle Tr(M^{2n_1})\rangle_0\right]\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0.
\end{aligned}$$

Using (3.13) the above equation is satisfied.

Recall that we expanded (3.12) up to first subleading order. We now need to expand

(3.12) to second subleading order $O(N^{n_1-3})$, which gives

$$0 = \langle \sum_{r=0}^{n_1-1} Tr(M^{2r}) \rangle_1 \langle Tr(M^{2n_1-2r-2}) \rangle_1 + \langle \sum_{r=0}^{n_1-1} Tr(M^{2r}) \rangle_2 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \\ + \langle \sum_{r=0}^{n_1-1} Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_2 + \langle \sum_{r=0}^{2n_1-2} Tr(M^{r}) Tr(M^{2n_1-r-2}) \rangle_{conn,1} - \langle Tr(M^{2n_1}) \rangle_2. \tag{3.26}$$

We also need the second subleading order of (3.9) which is of order $O(N^{n_1+n_2-2})$. The second subleading order of (3.9) is given by

$$
\begin{aligned}
0 &= \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_1 \langle Tr(M^{2n_2})\rangle_1 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_1 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_1-2r-2})\rangle_1 \langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_2 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_2 \langle Tr(M^{2n_2})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_2 \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{2n_2})\rangle_1 \langle Tr(M^{r}) Tr(M^{2n_1-r-2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_1 \langle Tr(M^{2r}) Tr(M^{2n_2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_2}) Tr(M^{2n_1-2r-2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{2n_2})\rangle_0 \langle Tr(M^{r}) Tr(M^{2n_1-r-2})\rangle_{conn,1} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2r}) Tr(M^{2n_2})\rangle_{conn,1} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_2}) Tr(M^{2n_1-2r-2})\rangle_{conn,1} \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{2n_2}) Tr(M^{r}) Tr(M^{2n_1-r-2})\rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2})\rangle_1 - \Bigg[\langle Tr(M^{2n_1})\rangle_1 \langle Tr(M^{2n_2})\rangle_1 \\
&+ \langle Tr(M^{2n_1})\rangle_2 \langle Tr(M^{2n_2})\rangle_0 + \langle Tr(M^{2n_1})\rangle_0 \langle Tr(M^{2n_2})\rangle_2 + \langle Tr(M^{2n_1}) Tr(M^{2n_2})\rangle_{conn,1} \Bigg].
\end{aligned}
\tag{3.27}
$$

Simplifying this, we get

$$
\begin{aligned}
0 &= \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_1 \langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,1} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})\rangle_{conn,1} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2})Tr(M^{2r})Tr(M^{2n_1-2r-2})\rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2})\rangle_1 - \langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,1}.
\end{aligned}
$$

Finally, we also need to consider the Schwinger-Dyson equations following from

$$
0 = \int [dM] \frac{d}{dM_{ij}} \left((M^{2n_1})_{ij} Tr(M^{2n_3}) e^{-\frac{1}{2}Tr(M^{2n})} \right). \tag{3.28}
$$

The expressions for the different orders of the expansion of this equation looks the same as the results we found in (3.8), except that we need to replace $\langle Tr(M^{2n_2})\rangle \rightarrow \langle Tr(M^{2n_3})\rangle$. We need three equations following from (3.28). They are given below.

Leading order : $O(N^{n_1+n_3+2})$

$$
0 = \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_3})\rangle_0 - \langle Tr(M^{2n_1})\rangle_0 \langle Tr(M^{2n_3})\rangle_0. \tag{3.29}
$$

First subleading order: $O(N^{n_1+n_3})$

$$
\begin{aligned}
0 &= \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0} \\
&\sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_3})Tr(M^{2r})\rangle_{conn,0} \\
&+ 2n_3 \langle Tr(M^{2n_1+2n_3-2})\rangle_0 - \langle Tr(M^{2n_3})Tr(M^{2n_1})\rangle_{conn,0}.
\end{aligned} \tag{3.30}
$$

Second subleading order : $O(N^{n_1+n_3-2})$

$$
\begin{aligned}
0 =& \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_1 \langle Tr(M^{2r}) Tr(M^{2n_3})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_3}) Tr(M^{2n_1-2r-2})\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2r}) Tr(M^{2n_3})\rangle_{conn,1} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_3}) Tr(M^{2n_1-2r-2})\rangle_{conn,1} \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{2n_3}) Tr(M^{r}) Tr(M^{2n_1-r-2})\rangle_{conn,0} \\
&+ 2n_3 \langle Tr(M^{2n_1+2n_3-2})\rangle_1 - \langle Tr(M^{2n_1}) Tr(M^{2n_3})\rangle_{conn,1}.
\end{aligned} \tag{3.31}
$$

Now, we are in a position to find the $\frac{1}{N}$ expansion of (3.20), which will give us the recursion relation obeyed by the connected correlators for $k = 3$.

First subleading order: $O(N^{n_1+n_2+n_3+1})$

Writing down all the terms of order $O(N^{n_1+n_2+n_3+1})$ of (3.20), we find

$$
\begin{aligned}
0 =& \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_0 \langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_1 \langle Tr(M^{2n_2})\rangle_0 \langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_1 \langle Tr(M^{2n_3})\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2})\rangle_0 \langle Tr(M^{2n_2})\rangle_0 \langle Tr(M^{2n_3})\rangle_1 \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{r}) Tr(M^{2n_1-r-2})\rangle_{conn,0} \langle Tr(M^{2n_2})\rangle_0 \langle Tr(M^{2n_3})\rangle_0
\end{aligned}
$$

$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$
$$\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0$$
$$+2n_2\langle Tr(M^{2n_1+2n_2-2})\rangle_0\langle Tr(M^{2n_3})\rangle_0+2n_3\langle Tr(M^{2n_1+2n_3-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0$$
$$-\Big[\langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1$$
$$+\langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0$$
$$+\langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$
$$+\langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_0$$
$$+\langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_0$$
$$+\langle Tr(M^{2n_3})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_1})\rangle_0\Big].$$

Simplifying this we get

$$=\sum_{r=0}^{n_1-1}\Big(\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}$$
$$+2n_3\langle Tr(M^{2n_1+2n_3-2})\rangle_0$$
$$-\langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,0}\Big)\langle Tr(M^{2n_2})\rangle_0. \tag{3.32}$$

Second subleading order of (3.20) $O(N^{n_1+n_2+n_3-1})$

$$
\begin{aligned}
0 = &\sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_1 \langle Tr(M^{2n_1-2r-2}) \rangle_1 \langle Tr(M^{2n_2}) \rangle_0 \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_1 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_1 \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_1 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_0 \langle Tr(M^{2n_3}) \rangle_1 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_1 \langle Tr(M^{2n_2}) \rangle_1 \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_1 \langle Tr(M^{2n_2}) \rangle_0 \langle Tr(M^{2n_3}) \rangle_1 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_1 \langle Tr(M^{2n_3}) \rangle_1 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_2 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_0 \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_2 \langle Tr(M^{2n_2}) \rangle_0 \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_2 \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) \rangle_0 \langle Tr(M^{2n_1-2r-2}) \rangle_0 \langle Tr(M^{2n_2}) \rangle_0 \langle Tr(M^{2n_3}) \rangle_2 \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{r}) Tr(M^{2n_1-r-2}) Tr(M^{2n_2}) \rangle_{conn,0} \langle Tr(M^{2n_3}) \rangle_0 \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{r}) Tr(M^{2n_1-r-2}) Tr(M^{2n_3}) \rangle_{conn,0} \langle Tr(M^{2n_2}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) Tr(M^{2n_2}) Tr(M^{2n_3}) \rangle_{conn,0} \langle Tr(M^{2n_1-2r-2}) \rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2}) Tr(M^{2n_1-2r-2}) Tr(M^{2n_3}) \rangle_{conn,0} \langle Tr(M^{2r}) \rangle_0
\end{aligned}
$$

$$+\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})\rangle_{conn,0}[\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0+\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1]$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}[\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_3})\rangle_0+\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_3})\rangle_1]$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}[\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0+\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1]$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})\rangle_{conn,0}[\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_3})\rangle_0+\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_3})\rangle_1]$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_3})Tr(M^{2n_1-2r-2})\rangle_{conn,0}[\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_2})\rangle_0+\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_2})\rangle_1]$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}[\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2r})\rangle_0+\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2r})\rangle_1]$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}$$
$$+\sum_{r=0}^{2n_1-2}\langle Tr(M^{2n_1-r-2})Tr(M^{r})\rangle_{conn,0}\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}$$
$$+\sum_{r=0}^{2n_1-2}\langle Tr(M^r)Tr(M^{2n_1-r-2})\rangle_{conn,1}\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})\rangle_{conn,1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_3})Tr(M^{2n_1-2r-2})\rangle_{conn,1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_2})\rangle_0$$
$$+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2r})\rangle_0$$

$$+2n_2\Bigg[\langle Tr(M^{2n_1+2n_2-2})\rangle_0\langle Tr(M^{2n_3})\rangle_1 + \langle Tr(M^{2n_1+2n_2-2})\rangle_1\langle Tr(M^{2n_3})\rangle_0$$
$$+ \langle Tr(M^{2n_1+2n_2-2})Tr(M^{2n_3})\rangle_{conn,0}\Bigg]$$
$$+ 2n_3\Bigg[\langle Tr(M^{2n_1+2n_3-2})\rangle_0\langle Tr(M^{2n_2})\rangle_1 + \langle Tr(M^{2n_1+2n_3-2})\rangle_1\langle Tr(M^{2n_2})\rangle_0$$
$$+ \langle Tr(M^{2n_1-2n_3-2})Tr(M^{2n_2})\rangle_{conn,0}\Bigg]$$
$$- \Bigg[\langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_1$$
$$+ \langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_1\langle Tr(M^{2n_3})\rangle_0$$
$$+ \langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_1$$
$$+ \langle Tr(M^{2n_1})\rangle_2\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_0$$
$$+ \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_2\langle Tr(M^{2n_3})\rangle_0$$
$$+ \langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2})\rangle_0\langle Tr(M^{2n_3})\rangle_2$$
$$+ \langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_3})\rangle_1$$
$$+ \langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_2})\rangle_1$$
$$+ \langle Tr(M^{2n_3})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2n_1})\rangle_1$$
$$+ \langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}$$
$$+ \langle Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_1})\rangle_0$$
$$+ \langle Tr(M^{2n_1})Tr(M^{2n_2})\rangle_{conn,1}\langle Tr(M^{2n_3})\rangle_0$$
$$+ \langle Tr(M^{2n_1})Tr(M^{2n_3})\rangle_{conn,1}\langle Tr(M^{2n_2})\rangle_0\Bigg].$$

Simplifying this we get

$$0 = \sum_{r=0}^{n_1-1}\langle Tr(M^{2r})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2n_1-2r-2})\rangle_0$$
$$+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2n_2})Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2r})\rangle_0$$
$$+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2})\rangle_{conn,0}\langle Tr(M^{2r})Tr(M^{2n_3})\rangle_{conn,0}$$
$$+ \sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_3})\rangle_{conn,0}\langle Tr(M^{2r})Tr(M^{2n_2})\rangle_{conn,0}$$

$$+2n_2\left[\langle Tr(M^{2n_1+2n_2-2})Tr(M^{2n_3})\rangle_{conn,0}\right]$$
$$+2n_3\left[\langle Tr(M^{2n_1+2n_3-2})Tr(M^{2n_2})\rangle_{conn,0}\right]$$
$$-\left[\langle Tr(M^{2n_1})Tr(M^{2n_2})Tr(M^{2n_3})\rangle_{conn,0}\right]. \tag{3.33}$$

Now, using (3.16) and (3.17) we reduce (3.33) to

$$0=\sum_{r=0}^{n_1-1}\gamma(r,n_2,n_3)\gamma(n_1-r-2)+\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_2,n_3)\gamma(r)+\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_2)\gamma(r,n_3)$$
$$+\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_3)\gamma(r,n_2)+2n_2\gamma(n_1+n_2-1,n_3)+2n_3\gamma(n_1+n_3-1,n_2)-\gamma(n_1,n_2,n_3). \tag{3.34}$$

Which is

$$\gamma(n_1,n_2,n_3)=\sum_{r=0}^{n_1-1}\gamma(r,n_2,n_3)\gamma(n_1-r-2)+\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_2,n_3)\gamma(r)+\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_2)\gamma(r,n_3)$$
$$+\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_3)\gamma(r,n_2)+2n_2\gamma(n_1+n_2-1,n_3)+2n_3\gamma(n_1+n_3-1,n_2). \tag{3.35}$$

This recursion relation is again in complete agreement with the results of the combinatorial analysis given in [27].

3.3 Schwinger-Dyson equation; General case.

The Schwinger-Dyson equation for $\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle$ is given by

$$0=\int dM\frac{d}{dM_{ij}}\left((M^{2n_1-1})_{ij}Tr(M^{2n_2})\cdots Tr(M^{2n_k})e^{-\frac{1}{2}Tr(M^2)}\right). \tag{3.36}$$

This equation implies

$$0 = \sum_{r=0}^{2n_1-2} \langle Tr(M^r) Tr(M^{2n_1-r-2}) \prod_{i=2}^{k} Tr(M^{2n_i}) \rangle$$
$$+ \sum_{j=2, j\neq i}^{k} 2n_j \langle Tr(M^{2n_1+2n_j-2}) \prod_{i=2}^{k} Tr(M^{2n_i}) \rangle \quad (3.37)$$
$$- \langle \prod_{i=1}^{k} Tr(M^{2n_i}) \rangle. \quad (3.38)$$

To find the recursion relation obeyed by the connected correlator of this general case, we need to expand the Schwinger-Dyson equation to order $O(N^{n_1+n_2+\cdots+n_k-k+2})$. To simplify the **leading order** of this case we need to use (3.13). To simplify the **first subleading order**, we have to use the results found from the leading and subleading orders of all the previous Schwinger-Dyson equations (the Schwinger-Dyson equations used to study correlators that comes before the k^{th} correlator, that is $\gamma(n_1, n_2), \gamma(n_1, n_3), \cdots, \gamma(n_1, n_2, n_3, n_4)$ $, \cdots, \gamma(n_1, n_2, n_3, n_4, \cdots, n_{k-1})$). To find the **second subleading order**; we use the leading order, first subleading and second subleading order of all the previous Schwinger-Dyson equations. To find the **third subleading order**; we use the leading order, first subleading, second subleading order and the third subleading order of all the previous Schwinger-Dyson equations. To find the k^{th} **subleading order**; we have to use the leading order, first subleading, second subleading order, the third subleading order up to k^{th} subleading order of all the previous Schwinger-Dyson equations.

The formula for the connected correlator is then be given by

$$\gamma(n_1, n_2, \cdots, n_k) = \sum_{P} \sum_{j=0}^{n_1-1} \gamma(j, P) \gamma(n_1 - j - 1, \tilde{P})$$
$$+ \sum_{r=2}^{k} 2n_r \gamma(n_1 + n_r - 1, S'_r). \quad (3.39)$$

In the above equation, S represents a set $\{n_2, n_3, \cdots, n_k\}$ and $S'_r = S - \{n_r\}$. This recursion relation is equivalent to the one derived in [27]. It is worth stressing that our derivation of this recursion relation has employed matrix model dynamics. We have simply performed a systematic $1/N$ expansion of the Schwinger-Dyson equation. The arguments of [27] use sophisticated combinatoric methods. Trying to generalize these methods to non-polynomial matrix model operators seems hopeless. As we will see in the next chapter, the Schwinger-Dyson approach has an easy generalization.

4 Matrix model correlators with $\ln M$ insertions.

Correlators of form

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle, \tag{4.1}$$

have been evaluated using different methods. [27] uses a recursion relation. This recursion relation was obtained by thinking about how many ways one can connect vertices inequivalently on bounding curves of a band. In a separate study, [26] uses orthogonal polynomials to evaluate these correlators. Our goal is to study correlators of form

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_q})Tr(M^{2n_{q+1}}\ln M)Tr(M^{2n_{q+2}}\ln M)\cdots Tr(M^{2n_{q+m}}\ln M)\rangle_{conn,0}. \tag{4.2}$$

[26] has computed a special case of (4.2), given by

$$\langle (Tr\ln M)^2 Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle_{conn,0} \tag{4.3}$$

by using orthogonal polynomials. In this chapter, using the Schwinger-Dyson equation approach, we will derive a set of recursion relations that can be used to solve for the complete set of correlators of the form (4.2). In the appendix, we will show how to recover the results of [26] using the recursion relations obtained from the Schwinger-Dyson equations.

4.1 The Schwinger-Dyson equation for correlators with one $\ln M$ insertion, for $k=2$

Consider the Schwinger-Dyson equation

$$0=\int dM\frac{d}{dM_{ij}}\left((M^{2n_1-1})_{ij}Tr(M^{2n_2}\ln M)e^{-\frac{1}{2}Tr(M^2)}\right). \tag{4.4}$$

This implies

$$\begin{aligned}0=&\langle\sum_{r=0}^{2n_1-2}Tr(M^r)Tr(M^{2n_1-r-2})Tr(M^{2n_2}\ln M)\rangle+2n_2\langle Tr(M^{2n_1+2n_2-2})\ln M\rangle+\langle Tr(M^{2n_1+2n_2-2})\rangle\\&-\langle Tr(M^{2n_1})Tr(M^{2n_2}\ln M)\rangle.\end{aligned} \tag{4.5}$$

The recursion relation for the connected correlator is obtained by performing a systematic $\frac{1}{N}$ expansion of (4.5). For the $k=2$ case, the connected correlators occur at order $O(N^{n_1+n_2})$.

Leading Order: $O(N^{n_1+n_2+2})$

$$0=\left[\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0-\langle Tr(M^{2n_1})\rangle_0\right]\langle Tr(M^{2n_2}\ln M)\rangle_0. \qquad (4.6)$$

First Subleading Order: $O(N^{n_1+n_2})$

Writing all the terms of order $O(N^{n_1+n_2})$ of (4.5), we get

$$\begin{aligned}
0=&\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_1\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2}\ln M)\rangle_0\\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_1\langle Tr(M^{2n_2}\ln M)\rangle_0\\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2}\ln M)\rangle_1\\
&+\sum_{r=0}^{2n_1-2}\langle Tr(M^{2r})Tr(M^{2n_1-2r-2})\rangle_{conn,0}\langle Tr(M^{2n_2}\ln M)\rangle_0\\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2}\ln M)\rangle_{conn,0}\\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2}\ln M)Tr(M^{2r})\rangle_{conn,0}\\
&+2n_2\langle Tr(M^{2n_1+2n_2-2}\ln M)\rangle_0+\langle Tr(M^{2n_1+2n_2-2})\rangle_0-\langle Tr(M^{2n_2}\ln M)Tr(M^{2n_1})\rangle_{conn,0}\\
&-\langle Tr(M^{2n_1})\rangle_1\langle Tr(M^{2n_2}\ln M)\rangle_0-\langle Tr(M^{2n_1})\rangle_0\langle Tr(M^{2n_2}\ln M)\rangle_1.
\end{aligned}$$

Simplifying this expression by using the equations we have derived for the other Schwinger-Dyson equations, we get

$$\begin{aligned}
0=&\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0\langle Tr(M^{2n_1-2r-2})Tr(M^{2n_2}\ln M)\rangle_{conn,0}\\
&+\sum_{r=0}^{n_1-1}\langle Tr(M^{2n_1-2r-2})\rangle_0\langle Tr(M^{2n_2}\ln M)Tr(M^{2r})\rangle_{conn,0}\\
&+2n_2\langle Tr(M^{2n_1+2n_2-2}\ln M)\rangle_0+\langle Tr(M^{2n_1+2n_2-2})\rangle_0-\langle Tr(M^{2n_2}\ln M)Tr(M^{2n_1})\rangle_{conn,0}.
\end{aligned} \qquad (4.7)$$

To write this last expression in terms of our γ notation, we need to further extend our notation as follows

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k}\ln M)\rangle_{conn}=\gamma(n_1,n_2,\cdots,\overline{n_k}), \qquad (4.8)$$

Using this notation, the recursion relation for the connected correlator for $k = 2$ becomes

$$\gamma(n_1, \overline{n_2}) = \sum_{r=0}^{n_1-1} \gamma(n_1 - r - 1, \overline{n_2})\gamma(r) + \sum_{r=0}^{n_1-1} \gamma(r, \overline{n_2})\gamma(n_1 - r - 1) + 2n_2\gamma(\overline{n_1 + n_2 - 1}) + \gamma(n_1 + n_2 - 1). \tag{4.9}$$

4.2 The Schwinger-Dyson equation for correlators with one $\ln M$ insertion, for $k = 3$.

The Schwinger-Dyson equation we will need is

$$0 = \int dM \frac{d}{dM_{ij}} \left((M^{2n_1-1})_{ij} Tr(M^{2n_2}) Tr(M^{2n_3} \ln M) e^{-\frac{1}{2}Tr(M^2)} \right). \tag{4.10}$$

This implies

$$0 = \sum_{r=0}^{2n_1-2} \langle Tr(M^r) Tr(M^{2n_1-r-2}) Tr(M^{2n_2}) Tr(M^{2n_3} \ln M)\rangle + 2n_2 \langle Tr(M^{2n_1+2n_2-2}) Tr(M^{2n_3} \ln M)\rangle + 2n_3 \langle Tr(M^{2n_1+2n_3-2} \ln M) Tr(M^{2n_2})\rangle + \langle Tr(M^{2n_1+2n_3-2}) Tr(M^{2n_2})\rangle - \langle Tr(M^{2n_1}) Tr(M^{2n_2}) Tr(M^{2n_3} \ln M)\rangle. \tag{4.11}$$

Performing the $\frac{1}{N}$ expansion, we find the connected correlators at order $O(N^{n_1+n_2+n_3-1})$. The final results are

$$\begin{aligned} 0 = &\sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) Tr(M^{2n_2}) Tr(M^{2n_3} \ln M)\rangle_{conn,0} \langle Tr(M^{2n_1-2r-2})\rangle_0 \\ &+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2}) Tr(M^{2n_1-2r-2}) Tr(M^{2n_3} \ln M)\rangle_{conn,0} \langle Tr(M^{2r})\rangle_0 \\ &+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_2})\rangle_{conn,0} \langle Tr(M^{2r}) Tr(M^{2n_3} \ln M)\rangle_{conn,0} \\ &+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_3} \ln M)\rangle_{conn,0} \langle Tr(M^{2r}) Tr(M^{2n_2})\rangle_{conn,0} \\ &+ 2n_2 \langle Tr(M^{2n_1+2n_2-2}) Tr(M^{2n_3} \ln M)\rangle_{conn,0} \\ &+ 2n_3 \langle Tr(M^{2n_1-2n_3-2} \ln M) Tr(M^{2n_2})\rangle_{conn,0} \\ &+ \langle Tr(M^{2n_1-2n_3-2}) Tr(M^{2n_2})\rangle_{conn,0} \\ &- \langle Tr(M^{2n_1}) Tr(M^{2n_2}) Tr(M^{2n_3} \ln M)\rangle_{conn,0}. \end{aligned} \tag{4.12}$$

In terms of the γ notation this becomes

$$\begin{aligned}\gamma(n_1,n_2,\overline{n_3}) = &\sum_{r=0}^{n_1-1}\gamma(r,n_2,\overline{n_3})\gamma(n_1-r-1) + \sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_2,\overline{n_3})\gamma(r)\\ &\sum_{r=0}^{n_1-1}\gamma(n_1-r-1,n_2)\gamma(r,\overline{n_3}) + \sum_{r=0}^{n_1-1}\gamma(n_1-r-1,\overline{n_3})\gamma(r,n_2)\\ &2n_2\gamma(n_1+n_2-1,\overline{n_3}) + 2n_3\gamma(\overline{n_1+n_3-1},n_2)\\ &+\gamma(n_1+n_3-1,n_2). \end{aligned} \tag{4.13}$$

4.3 The Schwinger-Dyson equation for correlators with two $\ln M$ insertions, for $k = 2$.

The Schwinger-Dyson equation we will need is given by

$$0 = \int dM \frac{d}{dM_{ij}}\left((M^{2n_1-1}\ln M)_{ij} Tr(M^{2n_2}\ln M) e^{-\frac{1}{2}Tr(M^2)}\right). \tag{4.14}$$

This implies

$$\begin{aligned}0 = &\langle \sum_{r=0}^{2n_1-2} Tr(M^r)Tr(M^{2n_1-r-2}\ln M)Tr(M^{2n_2}\ln M)\rangle + \langle Tr(M^{2n_1-2})Tr(M^{2n_2}\ln M)\rangle\\ &+2n_2\langle Tr(M^{2n_1+2n_2-2}(\ln M)^2)\rangle + \langle Tr(M^{2n_1+2n_2-2}\ln M)\rangle - \langle Tr(M^{2n_1}\ln M)Tr(M^{2n_2}\ln M)\rangle. \end{aligned} \tag{4.15}$$

To find the recursion relation for the connected correlator from (4.15), we again perform the $\frac{1}{N}$ expansion. The connected piece occurs at order $O(N^{n_1+n_2})$.

Leading Order: $O(N^{n_1+n_2+2})$

$$0 = \left[\sum_{r=0}^{n_1-1}\langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2}\ln M)\rangle_0 - \langle Tr(M^{2n_1}\ln M)\rangle_0\right]\langle Tr(M^{2n_2}\ln M)\rangle_0. \tag{4.16}$$

Subleading Order: $O(N^{n_1+n_2})$

$$
\begin{aligned}
0 =& \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_1 \langle Tr(M^{2n_1-2r-2}\ln M)\rangle_0 \langle Tr(M^{2n_2}\ln M)\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2}\ln M)\rangle_1 \langle Tr(M^{2n_2}\ln M)\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2}\ln M)\rangle_0 \langle Tr(M^{2n_2}\ln M)\rangle_1 \\
&+ \sum_{r=0}^{2n_1-2} \langle Tr(M^{2r}) Tr(M^{2n_1-2r-2}\ln M)\rangle_{conn,0} \langle Tr(M^{2n_2}\ln M)\rangle_0 \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2}\ln M) Tr(M^{2n_2}\ln M)\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}\ln M)\rangle_0 \langle Tr(M^{2n_2}\ln M) Tr(M^{2r})\rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2}(\ln M)^2)\rangle_0 + \langle Tr(M^{2n_1+2n_2-2}\ln M)\rangle_0 - \langle Tr(M^{2n_2}\ln M) Tr(M^{2n_1}\ln M)\rangle_{conn,0} \\
&- \langle Tr(M^{2n_1}\ln M)\rangle_1 \langle Tr(M^{2n_2}\ln M)\rangle_0 - \langle Tr(M^{2n_1}\ln M)\rangle_0 \langle Tr(M^{2n_2}\ln M)\rangle_1 .
\end{aligned}
$$

By employing the previous Schwinger-Dyson equations, this simplifies to

$$
\begin{aligned}
0 =& \sum_{r=0}^{n_1-1} \langle Tr(M^{2r})\rangle_0 \langle Tr(M^{2n_1-2r-2}\ln M) Tr(M^{2n_2}\ln M)\rangle_{conn,0} \\
&+ \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}\ln M)\rangle_0 \langle Tr(M^{2n_2}\ln M) Tr(M^{2r})\rangle_{conn,0} \\
&+ 2n_2 \langle Tr(M^{2n_1+2n_2-2}(\ln M)^2)\rangle_0 + \langle Tr(M^{2n_1+2n_2-2}\ln M)\rangle_0 - \langle Tr(M^{2n_2}\ln M) Tr(M^{2n_1}\ln M)\rangle_{conn,0} .
\end{aligned}
\tag{4.17}
$$

Using the γ notation, this can be written as

$$
\begin{aligned}
\gamma(\overline{n_1}, \overline{n_2}) =& \sum_{r=0}^{n_1-1} \gamma(r)\gamma(\overline{n_1-r-1}, \overline{n_2}) + \sum_{r=0}^{n_1-1} \gamma(r, \overline{n_2})\gamma(\overline{n_1-r-1}) \\
&+ 2n_2\gamma(\overline{\overline{n_1+n_2-1}}) + \gamma(\overline{n_1+n_2-1}).
\end{aligned}
\tag{4.18}
$$

where $\gamma(\overline{\overline{n_1+n_2-1}}) = \langle Tr(M^{2n_1+2n_2-2}\ln M \ln M)\rangle_{conn,0}$

4.4 The Schwinger-Dyson equation for correlators with two $\ln M$ insertion, for $k = 3$.

The Schwinger-Dyson equation we will need is given by

$$0 = \int dM \frac{d}{dM_{ij}} \left((M^{2n_1-1})_{ij} Tr(M^{2n_2} \ln M) Tr(M^{2n_3} \ln M) e^{-\frac{1}{2}Tr(M^2)} \right). \tag{4.19}$$

This implies

$$\begin{aligned}
0 = & \sum_{r=0}^{2n_1-2} \langle Tr(M^r) Tr(M^{2n_1-r-2}) Tr(M^{2n_2} \ln M) Tr(M^{2n_3} \ln M) \rangle \\
& + 2n_2 \langle Tr(M^{2n_1+2n_2-2} \ln M) Tr(M^{2n_3} \ln M) \rangle \qquad (4.20) \\
& + \langle Tr(M^{2n_1+2n_2-2}) Tr(M^{2n_3} \ln M) + 2n_3 \langle Tr(M^{2n_1+2n_3-2} \ln M) Tr(M^{2n_2} \ln M) \rangle \\
& + \langle Tr(M^{2n_1+2n_3-2}) Tr(M^{2n_2} \ln M) \rangle - \langle Tr(M^{2n_1}) Tr(M^{2n_2} \ln M) Tr(M^{2n_3} \ln M) \rangle.
\end{aligned} \tag{4.21}$$

Extracting the recursion relation for the connected correlator, which occurs at order $O(N^{n_1+n_2+n_3-1})$ of the $\frac{1}{N}$ expansion, we find

$$\begin{aligned}
0 = & \sum_{r=0}^{n_1-1} \langle Tr(M^{2r}) Tr(M^{2n_2} \ln M) Tr(M^{2n_3} \ln M) \rangle_{conn,0} \langle Tr(M^{2n_1-2r-2}) \rangle_0 \\
& + \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_2} \ln M) Tr(M^{2n_1-2r-2}) Tr(M^{2n_3} \ln M) \rangle_{conn,0} \langle Tr(M^{2r}) \rangle_0 \\
& + \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_2} \ln M) \rangle_{conn,0} \langle Tr(M^{2r}) Tr(M^{2n_3} \ln M) \rangle_{conn,0} \\
& + \sum_{r=0}^{n_1-1} \langle Tr(M^{2n_1-2r-2}) Tr(M^{2n_3} \ln M) \rangle_{conn,0} \langle Tr(M^{2r}) Tr(M^{2n_2} \ln M) \rangle_{conn,0} \\
& + 2n_2 \langle Tr(M^{2n_1+2n_2-2} \ln M) Tr(M^{2n_3} \ln M) \rangle_{conn,0} + \langle Tr(M^{2n_1+2n_2-2}) Tr(M^{2n_3} \ln M) \rangle_{conn,0} \\
& + 2n_3 \langle Tr(M^{2n_1-2n_3-2} \ln M) Tr(M^{2n_2} \ln M) \rangle_{conn,0} + \langle Tr(M^{2n_1-2n_3-2}) Tr(M^{2n_2} \ln M) \rangle_{conn,0} \\
& - \langle Tr(M^{2n_1}) Tr(M^{2n_2} \ln M) Tr(M^{2n_3} \ln M) \rangle_{conn,0}.
\end{aligned} \tag{4.22}$$

Re-writing this equation using the γ notation, we get

$$
\begin{aligned}
\gamma(n_1,\overline{n_2},\overline{n_3}) = &\sum_{r=0}^{n_1-1}\gamma(r,\overline{n_2},\overline{n_3})\gamma(n_1-r-1) + \sum_{r=0}^{n_1-1}\gamma(r)\gamma(n_1-r-1,\overline{n_2},\overline{n_3}) \\
&+ \sum_{r=0}^{n_1-1}\gamma(r,\overline{n_3})\gamma(n_1-r-1,\overline{n_2}) + \sum_{r=0}^{n_1-1}\gamma(r,\overline{n_2})\gamma(n_1-r-1,\overline{n_3}) \\
&+ 2n_2\gamma(\overline{n_1+n_2-1},\overline{n_3}) + 2n_3\gamma(\overline{n_1+n_3-1},\overline{n_2}) + \gamma(n_1+n_2-1,\overline{n_3}) \\
&+ \gamma(n_1+n_3-1,\overline{n_2}). \qquad (4.23)
\end{aligned}
$$

4.5 The Schwinger-Dyson equation for correlators with two $\ln M$ insertions, for k^{th} case.

The recursion relation of the correlator $\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_{k-2}})Tr(M^{2n_{k-1}}\ln M)Tr(M^{2n_k}\ln M)\rangle_{conn,0}$ is given by

$$
\begin{aligned}
\gamma(n_1,n_2,\cdots,n_{k-2},\overline{n_{k-1}},\overline{n_k}) = &\sum_P\sum_{j=0}^{n_1-1}\gamma(j,P)\gamma(n_1-j-1,\tilde{P}) \\
&+\sum_{r=2}^{k-2}2n_r\gamma(n_1+n_r-1,S'_r) + \sum_{r=k-1}^{k}2n_r\gamma(\overline{n_1+n_r-1},S'_r) \\
&+\sum_{r=k-1}^{k}\gamma(n_1+n_r-1,S'_r) \qquad (4.24)
\end{aligned}
$$

4.6 The Schwinger-Dyson equation for correlators with k $\ln M$ insertions, for a general case

We have derived the general formula for correlators with a single $\ln M$ insertion $\gamma(n_1,n_2,\cdots,\overline{n}_k)$ and correlators with double $\ln M$ insertions $\gamma(n_1,n_2,\cdots,\overline{n}_{k-1},\overline{n}_k)$. Now, we want to derive a general formula for correlators with k $\ln M$ insertions $\gamma(\overline{n_1},\overline{n_2},\cdots,\overline{n}_{k-1},\overline{n}_k)$ using the Schwinger-Dyson equations. Let's introduce a new notation that will simplify the presentation of the derivation. Let $k=q+m$ and

$$\gamma^1(n_1,n_2,\cdots,n_q,n_{q+1}) = \gamma(n_1,n_2,\cdots,n_q,\overline{n}_{q+1}). \qquad (4.25)$$

$$\gamma^2(n_1,n_2,\cdots,n_q,n_{q+1},n_{q+2}) = \gamma(n_1,n_2,\cdots,n_q,\overline{n}_{q+1},\overline{n}_{q+2}). \qquad (4.26)$$

$$\gamma^m(n_1,n_2,\cdots,n_q,n_{q+1},n_{q+2},\cdots,n_{q+m}) = \gamma(n_1,n_2,\cdots,n_q,\overline{n}_{q+1},\overline{n}_{+2},\cdots,\overline{n}_{q+m-1},\overline{n}_{q+m}). \qquad (4.27)$$

Using exactly the same ideas and methods we develop above, it is rather simple to obtain the following recursion relations

For $\mathbf{k} = \mathbf{m}$:

$$\gamma^m(n_1, n_2, \cdots, n_q, n_{q+1}, n_{q+2}, \cdots, n_{q+m}) = \sum_P \sum_{r=0}^{n_1-1} \gamma(r, P) \quad \gamma(n_1 - r - 1, \tilde{P})$$
$$+ \sum_{r=2}^{q} 2n_r \gamma(n_1 + n_r - 1, S'_r) + \sum_{r=q+1}^{q+m} 2n_r \gamma(\overline{\overline{n_1 + n_r - 1}}, S'_r) + \sum_{r=q+1}^{q+m} \gamma(\overline{n_1 + n_r - 1}, S'_r).$$

For $\mathbf{k} > \mathbf{m}$:

$$\gamma^m(n_1, n_2, \cdots, n_q, n_{q+1}, n_{q+2}, \cdots, n_{q+m})) = \sum_P \sum_{r=0}^{n_1-1} \gamma(r, P) \quad \gamma(n_1 - r - 1, \tilde{P})$$
$$+ \sum_{r=2}^{q} 2n_r \gamma(n_1 + n_r - 1, S'_r) + \sum_{r=q+1}^{q+m} 2n_r \gamma(\overline{n_1 + n_r - 1}, S'_r) + \sum_{r=q+1}^{q+m} \gamma^{m-1}(n_1 + n_r - 1, S'_r). \tag{4.28}$$

m counts the number of $\ln M$ insertions in the correlators. The recursion relations that we have obtained in this section are new. As we will see in the next section, they completely determine the correlators of the form (4.2). These correlators have not been reported before in the literature. Since they are needed to explore the simplest gauge/string duality, determining them is useful and one of the main goals motivating the study presented in this dissertation.

5 Solution to the recursion relation

Previous chapters have written down the recursion relations for various connected correlators using the Schwinger-Dyson equation. In this chapter we solve these recursion equations and find the general solution for the correlators given in (4.2) that we are interested in.

We generalise the types of correlators that have been studied so far as follows

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_q})Tr(M^{2n_{q+1}}\ln M),\cdots,Tr(M^{2n_{q+m}}\ln M)\rangle_{conn,0} \quad (5.1)$$

where $q+m=k$. When $m=0$, one obtains the correlator

$$\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_k})\rangle_{conn,0}. \quad (5.2)$$

The correlators (5.2) were computed in [27] using recursion relations and [26] computed them using orthogonal polynomials. The solution is

$$\gamma(n_1,n_2,\cdots,n_k)=\frac{(n-1)!}{(n-k+2)!}\prod_{i=1}^{k}\frac{(2n_i)!}{n_i!(n_i-1)!} \quad (5.3)$$

where $n=\sum_i n_i$. In addition $\gamma(n_1,n_2,\cdots,n_k)$ satisfies the following conditions

- If $k=1$ then $\gamma(0)=1$.
- For $k>1$, $\gamma(n_1,n_2,n_3,\cdots,n_k)=0$ if $n_i=0$.

The general correlators (5.1) have not been computed before this study. In this section, we are going to compute the correlators in (5.1) by solving the recursion relations we found in chapter 3.

5.1 Solution to the recursion relation for $m=1$

The connected correlator for $m=1$ is given by

$$\gamma(n_1,n_2,\cdots,n_q,\overline{n_{q+1}})=\langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_q})Tr(M^{2n_{q+1}}\ln M)\rangle_{conn,0} \quad (5.4)$$

We could try to compute this correlator by analytic continuation. Indeed, note that

$$\begin{aligned}M^{2n+\epsilon}&=M^{2n}M^{\epsilon}\\&=M^{2n}e^{\epsilon\ln M}\\&=M^{2n}(1+\epsilon\ln M+\mathcal{O}(\epsilon^2))\end{aligned}$$

Thus, by replacing the power of a matrix by the power plus ϵ and then expanding in ϵ and keeping the first order in this expansion, we can generate $\ln M$ insertions in the trace.

To do the expansion, we change the factorial representation of the solution to the gamma function representation. For example

$$\frac{(n-1)!}{(n-k+2)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!} = \frac{\Gamma(n)}{\Gamma(n-k+3)}\frac{\Gamma(2n_{q+1}+1)}{\Gamma(n_{q+1}+1)\Gamma(n_{q+1})}$$

To get the solution of the correlators that contains the logarithm, we analytically continue the above expression as follow

$$\frac{\Gamma(n)}{\Gamma(n-k+3)}\frac{\Gamma(2n_{q+1}+1)}{\Gamma(n_{q+1}+1)\Gamma(n_{q+1})} \to \frac{\Gamma(n+\epsilon)}{\Gamma(n+\epsilon-k+3)}\frac{\Gamma(2n_{q+1}+2\epsilon+1)}{\Gamma(n_{q+1}+\epsilon+1)\Gamma(n_{q+1}+\epsilon)}$$

The expansion of the gamma function to first order in ϵ is given as

$$\Gamma(n+\epsilon) = \Gamma(n) + \epsilon\Gamma'(n),$$

where the derivative of the gamma function is given by

$$\Gamma'(n+1) = n!(-\gamma + \sum_{j=1}^{n}\frac{1}{j}) \tag{5.5}$$

$$= \Gamma(n+1)\psi^{(0)}(n+1). \tag{5.6}$$

γ is a constant number known as the **Euler-Mascheroni constant** and $\psi^{(0)}(n+1)$ is the **digamma function**. The m^{th} derivative of the digamma function is given by a polygamma function of order m

$$\psi^{(m)}(n) = \frac{d^m}{dn^m}\psi^{(0)}(n). \tag{5.7}$$

To perform the analytic continuation it is useful to consider the expansion

$$\frac{\Gamma(n+\epsilon)}{\Gamma(n+\epsilon-k+3)}\frac{\Gamma(2n_{q+1}+2\epsilon+1)}{\Gamma(n_{q+1}+\epsilon+1)\Gamma(n_{q+1}+\epsilon)}$$

$$= \frac{\Gamma(n)+\epsilon(n-1)!(-\gamma+\sum_{j=1}^{n-1}\frac{1}{j})}{\Gamma(n-k+3)+\epsilon(n-k+2)!(-\gamma+\sum_{j=1}^{n-k+2}\frac{1}{j})}$$

$$\times \frac{\Gamma(2n_{q+1}+1)+2\epsilon(2n_{q+1})!(-\gamma+\sum_{j=1}^{2n_{q+1}-1}\frac{1}{j})}{\left(\Gamma(n_{q+1})+\epsilon(n_{q+1}-1)!)(-\gamma+\sum_{j=1}^{n_{q+1}-1}\frac{1}{j})\right)\left(\Gamma(n_{q+1}+1)+\epsilon(n_{q+1})!(-\gamma+\sum_{j=1}^{n_{q+1}}\frac{1}{j})\right)}$$

$$= \frac{\Gamma(n)+\epsilon\Gamma(n)(-\gamma+\sum_{j=1}^{n-1}\frac{1}{j})}{\Gamma(n-k+3)+\epsilon\Gamma(n-k+3)(-\gamma+\sum_{j=1}^{n-k+2}\frac{1}{j})}$$

$$\times \frac{\Gamma(2n_{q+1}+1)+2\epsilon\Gamma(2n_{q+1}+1)(-\gamma+\sum_{j=1}^{2n_{q+1}-1}\frac{1}{j})}{(\Gamma(n_{q+1})+\epsilon\Gamma(n_{q+1}))(-\gamma+\sum_{j=1}^{n_{q+1}-1}\frac{1}{j})(\Gamma(n_{q+1}+1)+\epsilon\Gamma(n_{q+1}+1)(-\gamma+\sum_{j=1}^{n_{q+1}}\frac{1}{j}))}$$

$$= \frac{\Gamma(n)\left(1+\epsilon(-\gamma+\sum_{j=1}^{n-1}\frac{1}{j})\right)}{\Gamma(n-k+3)\left(1+\epsilon(-\gamma+\sum_{j=1}^{n-k+2}\frac{1}{j})\right)}$$

$$\times \frac{\Gamma(2n_{q+1}+1)\left(1+2\epsilon(-\gamma+\sum_{j=1}^{2n_{q+1}-1}\frac{1}{j})\right)}{\Gamma(n_{q+1})\left(1+\epsilon(-\gamma+\sum_{j=1}^{n_{q+1}-1}\frac{1}{j})\right)\Gamma(n_{q+1}+1)\left(1+\epsilon(-\gamma+\sum_{j=1}^{n_{q+1}}\frac{1}{j}\right)}$$

$$= \frac{\Gamma(n)\left(1+\epsilon(-\gamma+\sum_{j=1}^{n-1}\frac{1}{j})\right)\left(1-\epsilon(-\gamma+\sum_{j=1}^{n-k+2}\frac{1}{j})\right)}{\Gamma(n-k+3)}$$

$$\times \frac{\Gamma(2n_{q+1}+1)\left(1+2\epsilon(-\gamma+\sum_{j=1}^{2n_{q+1}-1}\frac{1}{j})\right)\left(1-\epsilon(-\gamma+\sum_{j=1}^{n_{q+1}-1}\frac{1}{j})\right)\left(1-\epsilon(-\gamma+\sum_{j=1}^{n_{q+1}}\frac{1}{j}\right)}{\Gamma(n_{q+1})\Gamma(n_{q+1}+1)}$$

Collecting the terms of order ϵ, we get

$$\frac{\Gamma(n+\epsilon)}{\Gamma(n+\epsilon-k+3)}\frac{\Gamma(2n_{q+1}+2\epsilon+1)}{\Gamma(n_{q+1}+\epsilon+1)\Gamma(n_{q+1}+\epsilon)}\bigg|_{\epsilon \text{ term}}$$
$$= \frac{\Gamma(n)\Gamma(2n_{q+1}+1)}{\Gamma(n-k+3)\Gamma(n_{q+1})\Gamma(n_{q+1}+1)}\left(\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n-k+2}\frac{1}{j}+2\sum_{j=1}^{2n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}-1}\frac{1}{j}\right)$$
$$= \frac{(n-1)!(2n_{q+1})!}{(n-k+2)!n_{q+1}!(n_{q+1}-1)!}\left(\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n-k+2}\frac{1}{j}+2\sum_{j=1}^{2n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}-1}\frac{1}{j}\right)$$

It is now simple to obtain the following expression of the correlator $\gamma(n_1, n_2, \cdots, n_q, \overline{n_{q+1}})$.

$$\gamma(n_1, n_2, \cdots, n_q, \overline{n_{q+1}}) = \frac{1}{2}\frac{(n-1)!}{(n-k+2)!}\prod_{i=1}^{q}\frac{(2n_i)!}{n_i!(n_i-1)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!}$$
$$\left(\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n-k+2}\frac{1}{j}+2\sum_{j=1}^{2n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}-1}\frac{1}{j}\right) \tag{5.8}$$

This computation is not a derivation of the correlator. It does however give us a very plausible guess for this correlator. We can now check this guess by verifying that it does indeed solve the recursion relation we have derived. It is an easy exercise to check that this is indeed the case, so that (5.8) is indeed correct.

5.2 Solution to the recursion relation for any $q+m$ and $m=2$

For $m=2$, we study the correlator

$$\gamma(n_1, n_2, \cdots, n_q, \overline{n_{q+1}}, \overline{n_{q+2}}) = \langle Tr(M^{2n_1})Tr(M^{2n_2})\cdots Tr(M^{2n_q})Tr(M^{2n_{q+1}}\ln M)Tr(M^{2n_{q+2}}\ln M)\rangle_{conn,0} \tag{5.9}$$

To get the part of the correlator that contains the logarithms, note that

$$\begin{aligned} Tr(M^{2n_1+\epsilon_1})Tr(M^{2n_2+\epsilon_2}) &= Tr(M^{2n_1}M^{\epsilon_1})Tr(M^{2n_2}M^{\epsilon_2}) \\ &= Tr(M^{2n_1}e^{\epsilon_1\ln M})Tr(M^{2n_2}e^{\epsilon_2\ln M}) \\ &= Tr(M^{2n_1}(1+\epsilon_1\ln M+\mathcal{O}(\epsilon_1^2)))Tr(M^{2n_2}(1+\epsilon_2\ln M+\mathcal{O}(\epsilon_2^2)). \end{aligned}$$

We clearly need the terms of order $\epsilon_1\epsilon_2$, that is

$$Tr(M^{2n_1+\epsilon_1})Tr(M^{2n_2+\epsilon_2})\bigg|_{\epsilon_1\epsilon_2 \text{ term}} = Tr(\epsilon_1 M^{2n_1}\ln M)Tr(\epsilon_2 M^{2n_2}\ln M).$$

Consider the expression

$$\frac{(n-1)!}{(n-k+2)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!}\frac{(2n_{q+2})!}{n_{q+2}!(n_{q+2}-1)!}.$$

Analytic continuation of this expression gives

$$\frac{(n-1)!}{(n-k+2)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!}\frac{(2n_{q+2})!}{n_{q+2}!(n_{q+2}-1)!} \to \frac{(n+\epsilon_1+\epsilon_2-1)!}{(n+\epsilon_1+\epsilon_2-k+2)!}$$
$$\frac{(2n_{q+1}+2\epsilon_1)!}{(n_{q+1}+\epsilon_1)!(n_{q+1}+\epsilon_1-1)!}\frac{(2n_{q+2}+2\epsilon_2)!}{(n_{q+2}+\epsilon_2)!(n_{q+2}+\epsilon_2-1)!}$$

Changing from the factorial representation to the gamma function representation we have

$$\frac{(n+\epsilon_1+\epsilon_2-1)!}{(n+\epsilon_1+\epsilon_2-k+2)!}\frac{(2n_{q+1}+2\epsilon_1)!}{(n_{q+1}+\epsilon_1)!(n_{q+1}+\epsilon_1-1)!}\frac{(2n_{q+2}+2\epsilon_2)!}{(n_{q+2}+\epsilon_2)!(n_{q+2}+\epsilon_2-1)!}$$
$$=\frac{\Gamma(n+\epsilon_1+\epsilon_2)}{\Gamma(n+\epsilon_1+\epsilon_2-k+3)}\frac{\Gamma(2n_{q+1}+2\epsilon_1+1)}{\Gamma(n_{q+1}+\epsilon_1+1)\Gamma(n_{q+1}+\epsilon_1)}\frac{\Gamma(2n_{q+2}+2\epsilon_2+1)!}{\Gamma(n_{q+2}+\epsilon_2+1)\Gamma(n_{q+2}+\epsilon_2)!}$$

Now, we expand the gamma function to second order, to extract the term that contains $\epsilon_1\epsilon_2$

$$\frac{\Gamma(n+\epsilon_1+\epsilon_2)}{\Gamma(n+\epsilon_1+\epsilon_2-k+3)}\frac{\Gamma(2n_{q+1}+2\epsilon_1+1)}{\Gamma(n_{q+1}+\epsilon_1+1)\Gamma(n_{q+1}+\epsilon_1)}\frac{\Gamma(2n_{q+2}+2\epsilon_2+1)!}{\Gamma(n_{q+2}+\epsilon_2+1)\Gamma(n_{q+2}+\epsilon_2)!}$$

$$= \frac{\Gamma(n)+\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}+\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)\Gamma(n)+\frac{1}{2}\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}+\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)^2\Gamma(n)}{\Gamma(n-k+3)+\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}+\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)\Gamma(n-k+3))+\frac{1}{2}\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}+\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)^2\Gamma(n-k+3)}$$

$$\times \frac{\Gamma(2n_{q+1})+\left(2\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)\Gamma(2n_{q+1})+\frac{1}{2}\left(2\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)^2\Gamma(2n_{q+1})}{\left(\Gamma(n_{q+1}+1)+\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)\Gamma(n_{q+1}+1)+\frac{1}{2}\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)^2\Gamma(n_{q+1}+1)\right)}$$

$$\times \frac{1}{\left(\Gamma(n_{q+1})+\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)\Gamma(n_{q+1})+\frac{1}{2}\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)^2\Gamma(n_{q+1})\right)}$$

$$\times \frac{\Gamma(2n_{q+2})+\left(2\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)\Gamma(2n_{q+2})+\frac{1}{2}\left(2\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)^2\Gamma(2n_{q+2})}{\left(\Gamma(n_{q+2}+1)+\left(\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)\Gamma(n_{q+2}+1)+\frac{1}{2}\left(\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)^2\Gamma(n_{q+2}+1)\right)}$$

$$\times \frac{1}{\left(\Gamma(n_{q+2})+\left(\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)\Gamma(n_{q+2})+\frac{1}{2}\left(\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)^2\Gamma(n_{q+2})\right)}.$$

Using the following derivatives of the gamma function

$$\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}+\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)\Gamma(n) = \left(\epsilon_1\psi^{(0)}(n)+\epsilon_2\psi^{(0)}(n)\right)\Gamma(n)$$

$$\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}+\epsilon_2\frac{\partial}{\partial n_{q+2}}\right)^2\Gamma(n) = \left(\epsilon_1^2\psi^{(1)}(n)+\epsilon_2^2\psi^{(1)}(n)+2\epsilon_1\epsilon_2\psi^{(0)}(n)\psi^{(0)}(n)+2\epsilon_1\epsilon_2\psi^{(1)}(n)\right)\Gamma(n)$$

$$\left(\epsilon_1\frac{\partial}{\partial n_{q+1}}\right)\Gamma(n_{q+1}) = \left(\epsilon_1\psi^{(0)}(n_{q+1})\right)\Gamma(n_{q+1})$$

and collecting all the terms of order $\epsilon_1\epsilon_2$ we get

$$\frac{(n+\epsilon_1+\epsilon_2-1)!}{(n+\epsilon_1+\epsilon_2-k+2)!}\frac{(2n_{q+1}+2\epsilon_1)!}{(n_{q+1}+\epsilon_1)!(n_{q+1}+\epsilon_1-1)!}\frac{(2n_{q+2}+2\epsilon_2)!}{(n_{q+2}+\epsilon_2)!(n_{q+2}+\epsilon_2-1)!}\Bigg|_{\epsilon_1\epsilon_2 \text{ term}}$$
$$= \frac{(n-1)!}{(n-k+2)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!}\frac{(2n_{q+2})!}{n_{q+2}!(n_{q+2}-1)!}$$
$$\psi^{(0)}(n)\Big(\psi^{(0)}(2n_2+1)-2\psi^{(0)}(n_2+1)+\psi^{(0)}(n)-\psi^{(0)}(n-k+3)-\psi^{(0)}(n_2)\Big)+\psi^{(1)}(n)$$
$$+\psi^{(0)}(n-k+3)\Big(-\psi^{(0)}(2n_2+1)+2\psi^{(0)}(n_2+1)-\psi^{(0)}(n)+\psi^{(0)}(n-k+3)+\psi^{(0)}(n_2)\Big)$$
$$-\psi^{(1)}(n-k+3)+\psi^{(0)}(2n_1+1)\Big(\psi^{(0)}(2n_2+1)-2\psi^{(0)}(n_2+1)+\psi^{(0)}(n)-\psi^{(0)}(n-k+3)-\psi^{(0)}(n_2)\Big)$$
$$+\psi^{(0)}(n_1+1)\Big(-\psi^{(0)}(2n_2+1)+2\psi^{(0)}(n_2+1)-\psi^{(0)}(n)+\psi^{(0)}(n-k+3)+\psi^{(0)}(n_2)\Big)$$
$$+\psi^{(0)}(n_1)\Big(-\psi^{(0)}(2n_2+1)+2\psi^{(0)}(n_2+1)-\psi^{(0)}(n)+\psi^{(0)}(n-k+3)+\psi^{(0)}(n_2)\Big).$$

Simplifying this expression and use the definition of the digamma function we get

$$\frac{(n+\epsilon_1+\epsilon_2-1)!}{(n+\epsilon_1+\epsilon_2-k+2)!}\frac{(2n_{q+1}+2\epsilon_1)!}{(n_{q+1}+\epsilon_1)!(n_{q+1}+\epsilon_1-1)!}\frac{(2n_{q+2}+2\epsilon_2)!}{(n_{q+2}+\epsilon_2)!(n_{q+2}+\epsilon_2-1)!}\Bigg|_{\epsilon_1\epsilon_2 \text{ term}}$$
$$= \frac{(n-1)!}{(n-k+2)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!}\frac{(2n_{q+2})!}{n_{q+2}!(n_{q+2}-1)!}$$
$$\Bigg[\Bigg(2\sum_{j=1}^{2n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}-1}\frac{1}{j}+\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n}\frac{1}{j}\Bigg)$$
$$\times\Bigg(2\sum_{j=1}^{2n_{q+2}}\frac{1}{j}-\sum_{j=1}^{n_{q+2}}\frac{1}{j}-\sum_{j=1}^{n_{q+2}-1}\frac{1}{j}+\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n}\frac{1}{j}\Bigg)+(\psi^1(n)-\psi^1(n-k+3)).\Bigg]$$

This then suggests

$$\gamma(n_1,n_2,\cdots,n_q,\overline{n}_{q+1},\overline{n}_{q+2}) = \frac{(n-1)!}{2^2(n-k+2)!}\prod_{i=1}^{q}\frac{(2n_i)!}{n_i!(n_i-1)!}\frac{(2n_{q+1})!}{n_{q+1}!(n_{q+1}-1)!}\frac{(2n_{q+2})!}{n_{q+2}!(n_{q+2}-1)!}$$
$$\Bigg[\Bigg(2\sum_{j=1}^{2n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}}\frac{1}{j}-\sum_{j=1}^{n_{q+1}-1}\frac{1}{j}+\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n}\frac{1}{j}\Bigg)$$
$$\Bigg(2\sum_{j=1}^{2n_{q+2}}\frac{1}{j}-\sum_{j=1}^{n_{q+2}}\frac{1}{j}-\sum_{j=1}^{n_{q+2}-1}\frac{1}{j}+\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n}\frac{1}{j}\Bigg)+(\psi^1(n)-\psi^1(n-k+3))\Bigg]. \quad (5.10)$$

One can again prove that this is the correct result by showing that (5.10) does indeed solve the recursion relation we have derived.

5.3 Solution to the recursion relation for $q + m = 3$

The solution for this case (k=3) is much simpler since the term $\frac{(n-1)!}{(n-k+2)!} = 1$. The solution is given by

$$\gamma^{3-q}(n_{q+m-2}, n_{q+m-1}, n_{q+m}) = \frac{1}{2^q} \prod_{i=1}^{q} \frac{(2n_i)!}{n_i!(n_i-1)!} \prod_{r=1}^{m} \frac{(2n_{3-m+r})!}{n_{3-m+r}!(n_{3-m+r}-1)!} \left(2 \sum_{j=1}^{2n_{3-m+r}} \frac{1}{j} - \sum_{j=1}^{n_{3-m+r}} \frac{1}{j} - \sum_{j=1}^{n_{3-m+r}-1} \frac{1}{j} \right). \tag{5.11}$$

5.4 Solution to the recursion relation for $q + m \neq 3$

When writing the solution for the correlator (5.1) for $m + q \neq 3$ the polygamma functions $\psi^{(m-1)}(n)$ and $\psi^{(m-1)}(n-k+3)$ will appear in the solution. These polygamma's are generated by the term $\mathcal{B}$ from (5.12). These polygamma functions appears because we analytically continue (5.12) from n to $n + \epsilon$.

The solution to the recursion equation of the correlator (5.1) i.e. $\gamma(n_1, n_2, \cdots, n_q, \overline{n}_{q+1}, \cdots, \overline{n}_{q+m})$ is given by

$$\gamma(n_1, n_2, \cdots, n_q, \overline{n}_{q+1}, \cdots, \overline{n}_{q+m}) = \frac{(n-1)!}{(n-k+2)!} \frac{\mathcal{A}\mathcal{B}}{2^m}. \tag{5.12}$$

n and $\mathcal{A}$ are always expressed as

$$\begin{aligned} n &= \sum_{i=1}^{q} n_i + \sum_{j=q+1}^{m} n_j. \\ \mathcal{A} &= \prod_{i=1}^{q} \frac{(2n_i)!}{n_i!(n_i-1)!}, \quad \text{for} \quad q > 0 \quad \text{and} \\ &= 1 \quad \text{for} \qquad\qquad q = 0. \end{aligned}$$

The structure of expression $\mathcal{B}$ changes with m. When $m = 0$, $\mathcal{B} = 1$ and when $m > 0$, $\mathcal{B}$ is expressed in terms of the polygamma functions. In general the expression for $\mathcal{B}$ is not simple and we will not discuss it further

6 Topological String Correlators for the A Model on $\mathbf{P}^1$

Eguchi and Yang proposed a matrix model which reproduces the A-model on $\mathbf{P}^1$[23, 24, 25]. The partition function of this matrix model is

$$Z = \int [dM] e^{N \mathrm{Tr} V(M)} \tag{6.1}$$

where the action $\mathrm{Tr}V(M)$ is given by

$$\mathrm{Tr}V(M) = -2\mathrm{Tr}(M \ln M - M) + \sum_{n=1} 2t_{n,P} \mathrm{Tr}(M^n \ln M - c_n M^n) + \sum_{n=1} t_{n-1,Q} \frac{\mathrm{Tr}M^n}{n} \tag{6.2}$$

and

$$c_n = \sum_{j=1}^{n} \frac{1}{j}. \tag{6.3}$$

There is a primary field for each cohomology class of the target manifold in the topological A-model string theory. For the A-model on $\mathbf{P}^1$ there are thus two primaries, the puncture P and the Kähler class Q. The observables of the theory are these primaries and their gravitational descendents $\sigma_n(P)$, $\sigma_n(Q)$, $n \geq 1$. By setting $O_1 = P$, $O_2 = Q$ and $O_3 = 0$, we describe the complete set of observables using the notation O_α and $\sigma_n(O_\alpha)$. The $U(1)$ charges of the primary fields are $q_1 = 0$ and $q_2 = 1$. We raise and lower the indices of the primary fields using the metric $\eta_{11} = \eta_{22} = 0$ and $\eta_{12} = \eta_{21} = 1$.

The correlators of the theory are determined by solving recursion relations, which express n-point correlators in terms of lower point correlators. The partition function of the string theory contains a coupling constant for each observable in the theory. The coupling constants for P and Q are $t_{n,P}$ and $t_{n-1,Q}$, respectively. One distinguishes between recursion relations that hold in the large phase space (when all couplings are turned on) and recursion relations that hold only after certain couplings are set to zero. When all the couplings are set to zero the genus g correlation functions

$$\langle \sigma_{n_1}(O_{\alpha_1}) \cdots \sigma_{n_s}(O_{\alpha_s}) \rangle_g \tag{6.4}$$

only recieve contributions from holomorphic maps of degree d.

$$2d + 2(g-1) = \sum_{i=1}^{s} (n_i + q_{\alpha_i} - 1) \tag{6.5}$$

This is a consequence of the ghost number conservation law. (6.5) holds if one works on $\mathbf{P}^1$. In general if one works on $\mathbf{P}^N$, the degree is obtained from

$$(N+1)d + (N-3)(1-g) = \sum_{i=1}^{s}(n_i + q_{\alpha_i} - 1). \tag{6.6}$$

When we want to indicate the contribution to a correlator from maps of a specific degree d at genus g, we will write $\langle \cdots \rangle_{g,d}$. We will use $\langle \cdots \rangle_g$ to denote the genus g contribution to the topological string correlator.

6.1 String correlators

The one point Eguchi-Hori-Yang relation

$$d^2\langle \sigma_n(O_\alpha)\rangle_{0,d} = -2nd\langle \sigma_{n-1}(O_{\alpha+1})\rangle_{0,d} + \sum_{\bar{d}} n\,\bar{d}^2\langle \sigma_{n-1}(O_\alpha)O_\beta\rangle_{0,d-\bar{d}}\langle O^\beta\rangle_{0,\bar{d}} \tag{6.7}$$

holds in the large phase space. Setting all couplings to zero and using $\langle P\rangle_{0,d} = \langle Q\rangle_{0,d} = 0$ except $\langle Q\rangle_{0,1} = 1$, the relation (6.7) implies

$$\langle \sigma_{2m}(Q)\rangle_{0,d} = \delta_{d,m+1}\frac{(2m)!}{(m+1)!(m+1)!} \qquad \langle \sigma_{2m+1}(Q)\rangle_0 = 0, \tag{6.8}$$

$$\langle \sigma_{2m+1}(P)\rangle_{0,d} = -2c_{m+1}\delta_{d,m+1}\frac{(2m+1)!}{(m+1)!(m+1)!} \qquad \langle \sigma_{2m}(P)\rangle_0 = 0. \tag{6.9}$$

As a consequence of ghost number conservation, the above correlators only get a contribution from maps of degree $m+1$.

The two point correlator is computed using the recursion relation

$$\begin{aligned}\langle \sigma_n(O_\alpha)\sigma_m(O_\beta)\rangle &= \frac{2mn\delta_{\phi\sigma}\eta^{\delta\phi}\eta^{\gamma\sigma}}{n+m+q_\alpha+q_\beta}\langle \sigma_{n-1}(O_\alpha)O_\gamma\rangle\langle \sigma_{m-1}(O_\beta)O_\delta\rangle \\ &- \frac{2n}{n+m+q_\alpha+q_\beta}\langle \sigma_{n-1}(O_{\alpha+1})\sigma_m(O_\beta)\rangle \\ &- \frac{2m}{n+m+q_\alpha+q_\beta}\langle \sigma_n(O_\alpha)\sigma_{m-1}(O_{\beta+1})\rangle. \end{aligned}\tag{6.10}$$

Using this two point correlator one finds the following results

$$\langle \sigma_{2m_1}(Q)\sigma_{2m_2}(Q)\rangle_0 = \frac{1}{m_1+m_2+1}\frac{(2m_1)!}{m_1!m_1!}\frac{(2m_2)!}{m_2!m_2!} \tag{6.11}$$

$$\langle \sigma_{2m_1-1}(Q)\sigma_{2m_2-1}(Q)\rangle_0 = \frac{1}{4(m_1+m_2)}\frac{(2m_1)!}{m_1!m_1!}\frac{(2m_2)!}{m_2!m_2!} \tag{6.12}$$

$$\langle\sigma_{2m_1-1}(Q)\sigma_{2m_2}(Q)\rangle_0 = 0 \tag{6.13}$$

$$\langle\sigma_{2m_1}(Q)\sigma_{2m_2+1}(P)\rangle_0 = \frac{1}{m_1+m_2+1}\frac{(2m_1)!}{m_1!m_1!}\frac{(2m_2+1)!}{m_2!m_2!}\left[-2c_{m_2} - \frac{1}{m_1+m_2+1}\right] \tag{6.14}$$

$$\begin{aligned}\langle\sigma_{2m_1+1}(Q)\sigma_{2m_2}(P)\rangle_0 = \frac{1}{2(m_1+m_2+1)}\frac{(2m_1+2)!}{(m_1+1)!(m_1+1)!}\frac{(2m_2)!}{m_2!(m_2-1)!}\times \\ \times\left[-2c_{m_2} + \frac{1}{m_2} - \frac{1}{m_1+m_2+1}\right]\end{aligned} \tag{6.15}$$

The higher point correlators are calculated using the topological recursion relation,

$$\langle\sigma_n(O_\gamma)XY\rangle_0 = n\langle\sigma_{n-1}(O_\gamma)O_\alpha\rangle_0\eta^{\alpha\beta}\langle O_\beta XY\rangle_0. \tag{6.16}$$

This recursion relation holds in large phase space.

Three point correlators

$$\langle\sigma_{2k_1}(Q)\sigma_{2k_2}(Q)\sigma_{2k_3}(Q)\rangle = \frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3)!}{(k_3!)^2} \tag{6.17}$$

$$\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3}(Q)\rangle = \frac{1}{4}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3)!}{(k_3!)^2} \tag{6.18}$$

$$\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3+1}(P)\rangle = -\frac{2c_{k_3}}{4}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3+1)!}{(k_3!)^2} \tag{6.19}$$

$$\langle\sigma_{2n}(P)\sigma_{2k_1-1}(Q)\sigma_{2k_2}(Q)\rangle = \frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2n-1)!}{((n-1)!)^2}\left(-2c_n + \frac{1}{n}\right) \tag{6.20}$$

Higher than three point correlators

$$\langle\prod_{i=1}^{n}\sigma_{2k_i}(Q)\rangle = (d+1)^{n-3}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!} \tag{6.21}$$

$$\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle = \frac{d^{n-3}}{4}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}$$

$$(6.22)$$

$$\langle\prod_{i=1}^{n}\sigma_{2k_i}(Q)\sigma_{2k+1}(P)\rangle = (d+1)^{n-2}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}\frac{(2k+1)!}{k!k!}\left(-2c_k+\frac{n-2}{d+1}\right) \tag{6.23}$$

$$\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\sigma_{2k+1}(P)\rangle = \frac{d^{n-2}}{4}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}\frac{(2k+1)!}{k!k!}\left(-2c_k+\frac{n-2}{d}\right) \tag{6.24}$$

where $d = \sum_i k_i + k$.

6.2 Matrix model and topological A-model correlators

We now know how to compute correlators in the matrix model and in the A-model topological string theory. We need to know how to relate correlators of the matrix model to correlators of the topological string. The identification is as follows

$$\sigma_m(Q) \leftrightarrow \frac{\mathrm{Tr}M^{m+1}}{m+1} \qquad \text{and} \qquad \sigma_m(P) \leftrightarrow 2\mathrm{Tr}\left(M^m \ln M - c_m M^m\right). \tag{6.25}$$

We can use the topological string correlators (6.8) - (6.14) to solve for the expected matrix model correlators. In this way, the topological string theory makes the following predictions (all γs predicted using the topological string correlators are hatted)

$$\hat{\gamma}(2m+1) = \frac{(2m+1)!}{(m+1)!(m+1)!} \qquad \hat{\gamma}(2m) = 0 \tag{6.26}$$

$$\hat{\gamma}(\overline{2m+1}) = (c_{2m+1} - c_{m+1})\frac{(2m+1)!}{(m+1)!(m+1)!} \qquad \hat{\gamma}(\overline{2m}) = 0 \tag{6.27}$$

$$\hat{\gamma}(2m+1, 2n) = 0 \tag{6.28}$$

$$\hat{\gamma}(2m+1, 2n+1) = \frac{1}{m+n+1}\frac{(2m+1)!}{m!m!}\frac{(2n+1)!}{n!n!} \tag{6.29}$$

$$\hat{\gamma}(2m, 2n) = \frac{mn}{m+n}\frac{(2m)!}{m!m!}\frac{(2n)!}{n!n!} \tag{6.30}$$

$$\hat{\gamma}(2n+1,\overline{2m+1}) = \left(\frac{c_{2m_2+1}-c_{m_2}}{m_1+m_2+1} - \frac{1}{2(m_1+m_2+1)^2}\right)\frac{(2m_1+1)!}{m_1!m_1!}\frac{(2m_2+1)!}{m_2!m_2!} \quad (6.31)$$

Using the Schwinger-Dyson equation, we will derive recursion relations in this section that will test these predictions. Starting from

$$0 = \int [dM]\frac{d}{dM_{ij}}\left((M^n)_{ij}e^{N\mathrm{Tr}V}\right) \quad (6.32)$$

we find (recall that we have set all couplings to zero)

$$2N\langle \mathrm{Tr}(M^n \ln M)\rangle = \sum_{r=0}^{n-1}\langle \mathrm{Tr}(M^{n-1-r})\mathrm{Tr}(M^r)\rangle \quad (6.33)$$

The leading order of this Schwinger-Dyson equation implies that

$$2N\langle \mathrm{Tr}(M^n \ln M)\rangle_0 = \sum_{r=0}^{n-1}\langle \mathrm{Tr}(M^{n-1-r})\rangle_0\langle \mathrm{Tr}(M^r)\rangle_0 \quad (6.34)$$

$$2\gamma(\bar{n}) = \sum_{r=0}^{n-1}\gamma(r)\gamma(n-1-r) \quad (6.35)$$

Inserting (6.26) into (6.35) we recover (6.27). Now consider the two point correlators. Starting from

$$0 = \int [dM]\frac{d}{dM_{ij}}\left((M^n)_{ij}\mathrm{Tr}(M^m)e^{N\mathrm{Tr}V}\right) \quad (6.36)$$

we find

$$2N\langle \mathrm{Tr}(M^m)\mathrm{Tr}(M^n \ln M)\rangle = \sum_{r=0}^{n-1}\langle \mathrm{Tr}(M^{n-1-r})\mathrm{Tr}(M^r)\mathrm{Tr}(M^m)\rangle + m\langle \mathrm{Tr}(M^{n+m-1})\rangle \quad (6.37)$$

Expanding this Schwinger-Dyson equation to subleading order, after using both the leading and subleading order of (6.33), we find

$$\begin{aligned} 2N\langle \mathrm{Tr}(M^m)\mathrm{Tr}(M^n \ln M)\rangle_{conn,0} = \sum_{r=0}^{n-1}\Big(&\langle \mathrm{Tr}(M^r)\mathrm{Tr}(M^m)\rangle_{conn,0}\langle \mathrm{Tr}(M^{n-1-r})\rangle_0 \\ &+ \langle \mathrm{Tr}(M^{n-1-r})\mathrm{Tr}(M^m)\rangle_{conn,0}\langle \mathrm{Tr}(M^r)\rangle_0\Big) + m\langle \mathrm{Tr}(M^{n+m-1})\rangle_0 \end{aligned} \quad (6.38)$$

which is

$$2\gamma(m,\bar{n}) = \sum_{r=0}^{n-1} (\gamma(r,m)\gamma(n-1-r) + \gamma(n-1-r,m)\gamma(r)) + m\gamma(n+m-1) \quad (6.39)$$

Using (6.28)-(6.30) in (6.39) we recover (6.31). For all of the examples we have considered, the matrix model correlators and topological string theory correlators are in perfect agreement. Notice that there is a natural way to understand the result (6.31) by analytic continuation. Indeed, it is simple to verify that

$$\begin{aligned}\gamma(n,\overline{m}) \quad &= \langle \mathrm{Tr}(M^n)\mathrm{Tr}(M^m \ln M)\rangle_0 \\ &= \frac{d}{d\epsilon}\langle \mathrm{Tr}(M^n)\mathrm{Tr}(M^{m+\epsilon})\rangle_0\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\gamma(n,m+\epsilon)\Big|_{\epsilon=0}\end{aligned} \quad (6.40)$$

In general we expect the $\gamma(n_1,n_2,\cdots,n_{k-1},n_k)$ and $\gamma(n_1,n_2,\cdots,n_{k-1},\bar{n}_k)$ that solve the recursion relations, to be related by analytic continuation. With the assumption (6.40), we have managed to solve the Schwinger-Dyson equations for the $\gamma(n_1,n_2,\cdots,n_k)$ and have verified that the correlators of the gravitational descendants computed in the matrix model agree with the same correlators computed using the topological string. The translation between the two is given by the first of (6.25). As an example, the topological string correlators[26]

$$\langle \prod_{i=1}^{k} \sigma_{2m_i}(Q)\rangle_0 = (d+1)^{k-3}\prod_{i=1}^{k}\frac{(2m_i)!}{m_i!m_i!} \qquad d=\sum_{i=1}^{k} m_i \quad (6.41)$$

predict

$$\gamma(2m_1+1,2m_2+1,\cdots,2m_k+1) = (d+1)^{k-3}\prod_{i=1}^{k}\frac{(2m_i+1)!}{m_i!m_i!} \quad (6.42)$$

Setting

$$\gamma(2m_1+1,\cdots,\overline{2m_k+1}) = \frac{d}{d\epsilon}\gamma(2m_1+1,\cdots,2m_k+1+\epsilon)\Big|_{\epsilon=0} \quad (6.43)$$

we find that $\gamma(2m_1+1,2m_2+1,\cdots,2m_k+1)$ and $\gamma(2m_1+1,\cdots,\overline{2m_k+1})$ do indeed satisfy the Schwinger-Dyson equations. Thus, the Eguchi-Yang matrix model does indeed correctly reproduce the correlators of the topological string theory. Apart from this agreement, we now have good evidence that the recursion relations we have obtained for the correlators that include $\mathrm{Tr}M^n\ln M$ insertions are indeed correct. This is a non-trivial test of our technology.

7 The Simplest Gauge String Duality

In this section we return to the Gaussian matrix model. The combinatorics of the Wick contractions in the Gaussian matrix model suggests that correlators are computed by a sum over branched covers from a genus g worldsheet to a target $\mathbf{P}^1$ "spacetime". These holomophic maps have exactly three branchpoints [4], and are known as Belyi maps. Using mathematical results from [29], the explicit form of these Belyi maps has been given in [15]. From the detailed form of the map, it is clear that $\langle \prod_{i=1}^n \mathrm{Tr}(M^{2k_i}) \rangle_{\mathrm{conn},0}$ only receives planar contributions only from degree $d = \sum_i k_i$ maps. As a consequence of the ghost number conservation law (see [25] for more details) the genus 0 contribution to the correlator of descendants of the Kähler class

$$\langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3}(Q)\cdots\sigma_{2k_n}(Q)\rangle \tag{7.1}$$

only receives contributions from maps of degree

$$d = \sum_{i=1}^{k} k_i \tag{7.2}$$

which strongly suggests [15, 26] the rough identification

$$\sigma_{2k}(Q) \sim \mathrm{Tr}\left(M^{2k}\right) \tag{7.3}$$

With this identification, the two point and three point functions in the Gaussian matrix model and in the topological string theory match[15] for any k_i

$$\begin{aligned}\langle \frac{1}{2k_1}\mathrm{Tr}M^{2k_1}\frac{1}{2k_2}\mathrm{Tr}M^{2k_2}\rangle_{\mathrm{conn},0} &= \frac{1}{4(k_1+k_2)}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\\ &= \langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\rangle_0\end{aligned} \tag{7.4}$$

$$\begin{aligned}\langle \frac{1}{2k_1+1}\mathrm{Tr}M^{2k_1+1}\frac{1}{2k_2+1}\mathrm{Tr}M^{2k_2+1}\rangle_{\mathrm{conn},0} &= \frac{1}{k_1+k_2+1}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\\ &= \langle \sigma_{2k_1}(Q)\sigma_{2k_2}(Q)\rangle_0\end{aligned} \tag{7.5}$$

$$\begin{aligned}\langle \frac{1}{2k_1}\mathrm{Tr}M^{2k_1}\frac{1}{2k_2}\mathrm{Tr}M^{2k_2}\frac{1}{k_3}\mathrm{Tr}M^{2k_3}\rangle_{\mathrm{conn},0} &= \frac{1}{4}\frac{(2k_1)!}{k_1!k_1!}\frac{(2k_2)!}{k_2!k_2!}\frac{(2k_3)!}{k_3!k_3!}\\ &= \langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3}(Q)\rangle_0\end{aligned} \tag{7.6}$$

$$\begin{aligned}\langle \frac{1}{2k_1+1}\mathrm{Tr}M^{2k_1+1}\frac{1}{2k_2+1}\mathrm{Tr}M^{2k_2+1}\frac{1}{k_3}\mathrm{Tr}M^{2k_3}\rangle_{\mathrm{conn},0} &= \frac{(2k_1)!}{k_1!k_1!}\frac{(2k_2)!}{k_2!k_2!}\frac{(2k_3)!}{k_3!k_3!}\\ &= \langle \sigma_{2k_1}(Q)\sigma_{2k_2}(Q)\sigma_{2k_3}(Q)\rangle_0\end{aligned} \tag{7.7}$$

The general map is as follows

$$\langle\frac{\mathrm{Tr}M^{2k_1}}{2k_1}\frac{\mathrm{Tr}M^{2k_2}}{2k_2}\frac{\mathrm{Tr}M^{2k_3}}{k_3}\cdots\frac{\mathrm{Tr}M^{2k_n}}{k_n}\rangle_{\mathrm{conn},0}\ \leftrightarrow\ \langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3}\cdots\sigma_{2k_n}(Q)\rangle_0 \tag{7.8}$$

These correlators appear to agree, up to contact terms[26]. Notice that when computing the topological string correlator, we treat two of the vertex operators differently to the remaining operators. The need for this is simply to ensure that the degree of the Belyi maps contributing to the matrix model correlator matches the degree of the holomorphic map contributing to the topological string theory correlator - so that the closed surface obtained after closing the holes in the ribbon graph is the topological closed string worldsheet. This is very much like usual worldsheet descriptions, where we fix the positions of three vertex operators. Here a difference of 2 in the degree of the two maps is being accounted for by putting two of the vertex operators at fixed points on the worldsheet. These results present a rather compelling case for the duality.

We would now like to propose the operators dual to the gravitational descendents of the puncture. In [26] it was suggested that $\sigma_n(P)\sim\mathrm{Tr}(M^n\ln M)$, which we will see, is essentially correct. We are again guided by matching correlators. It is the three point correlators that do not receive contact term contributions[26], so that we should require that the three point topological string correlators match with the corresponding matrix model correlators.

To reproduce the complete set of three point correlators, we propose that

$$\sigma_{2k_3+1}(P)\leftrightarrow\frac{2k_3+1}{k_3}\left(\mathrm{Tr}\left(2M^{2k_3}\ln M-\left[\frac{1}{k_3}+2c_{2k_3}\right]M^{2k_3}\right)\right) \tag{7.9}$$

Notice that there is a shift of 1 between the level of the gravitational descendant $(2k_3+1)$ of the puncture and the power of the matrix $(2k_3)$ in the matrix model operator. To see why this shift is necessary, recall that when all the couplings are set to zero the genus g correlation functions $\langle\sigma_{n_1}(O_{\alpha_1})\cdots\sigma_{n_s}(O_{\alpha_s})\rangle_g$ only receive contributions from holomorphic maps of degree d with

$$2d+2(g-1)=\sum_{i=1}^{s}(n_i+q_{\alpha_i}-1) \tag{7.10}$$

Since the $U(1)$ charge of the puncture $q_P=0$, including $\sigma_{2k_3+1}(P)$ in the correlator adds k_3 to the degree. This matches the matrix model computation since including $\mathrm{Tr}(M^{2k_3})$ (and hence also $\mathrm{Tr}(M^{2k_3}\ln M)$) adds degree k_3 to the Belyi maps summed to reproduce the matrix model correlator. It is now straight forward to verify that

$$\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3+1}(P)\rangle=$$
$$\left\langle\frac{\mathrm{Tr}M^{2k_1}}{2k_1}\frac{\mathrm{Tr}M^{2k_2}}{2k_2}\frac{2k_3+1}{k_3}\left(\mathrm{Tr}\left(2M^{2k_3}\ln M-\left[\frac{1}{k_3}+2c_{2k_3}\right]M^{2k_3}\right)\right)\right\rangle_{\mathrm{conn},0} \tag{7.11}$$

The fact that (7.9) reproduces an infinite number of correlators is strong evidence that it is indeed on the right track.

Lets now consider more general correlators. Comparing the topological string correlator

$$\begin{aligned}&\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\sigma_{2k+1}(P)\rangle\\&=\frac{d^{n-2}}{4}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}\frac{(2k+1)!}{k!k!}\left(-2c_k+\frac{n-2}{d}\right)\end{aligned}\tag{7.12}$$

to the matrix model correlator

$$\begin{aligned}&\left\langle\frac{1}{4}\prod_{i=1}^{n}\frac{\mathrm{Tr}M^{2k_i}}{k_i}\frac{2k+1}{k}\mathrm{Tr}\left(2M^{2k}\ln M-\left[\frac{1}{k}+2c_{2k}\right]M^{2k}\right)\right\rangle_{\mathrm{conn},0}\\
&=\frac{1}{4}\prod_{i=1}^{n}\frac{1}{k_i}\frac{(2k+1)}{k}\left(2\langle Tr(M^{2k_i})Tr(M^{2k}\ln M)\rangle_{conn,0}-\left[\frac{1}{k}+2c_{2k}\right]\langle Tr(M^{2k_i})Tr(M^{2k})\rangle_{conn,0}\right)\\
&=\frac{1}{4}\Bigg(2\times\frac{1}{2}\frac{(d-1)!}{(d-n+1)!}\prod_{i=1}^{n}\frac{1}{k_i}\frac{(2k+1)}{k}\frac{(2k_i)!(2k)!}{k_i!(k_i-1)!k!(k-1)!}(c_{d-1}-c_{d-k+2}+2c_{2k}-c_k-c_{k-1})\\
&-\left[\frac{1}{k}+2c_{2k}\right]\frac{(d-1)!}{(d-n+1)!}\prod_{i=1}^{n}\frac{1}{k_i}\frac{(2k+1)}{k}\frac{(2k_i)!(2k)!}{k_i!(k_i-1)!k!(k-1)!}\Bigg)\\
&=\frac{1}{4}\frac{(d-1)!}{(d-n+1)!}\prod_{i=1}^{n}\frac{(2k_i)!(2k+1)!}{k_i!k_i!k!k!}\left((c_{d-1}-c_{d-n+1}+2c_{2k}-c_k-c_{k-1})-\left[\frac{1}{k}+2c_{2k}\right]\right)\\
&=\frac{1}{4}\frac{(d-1)!}{(d-n+1)!}\prod_{i=1}^{n}\frac{(2k_i)!(2k+1)!}{k_i!k_i!k!k!}\left(c_{d-1}-c_{d-n+1}+2c_{2k}-2c_{2k}-c_k-(c_{k-1}+\frac{1}{k})\right)\\
&=\frac{(d-1)!}{(d-n+1)!}\frac{1}{4}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}\frac{(2k+1)!}{k!k!}\left(-2c_k+c_{d-1}-c_{d-n+1}\right)\end{aligned}\tag{7.13}$$

where $d=k_1+\cdots+k_n+k$, we see two sources of mismatch. First, there is an overall factor of d^{n-2} versus $\frac{(d-1)!}{(d-n+1)!}$. In the large d limit these two factors are identical. This factor has an elegant interpretation in terms of contact term corrections, as explained in [26]. Further, in the large d limit, we can replace

$$\frac{n-2}{d}\quad\rightarrow\quad 0\tag{7.14}$$

in the topological string correlator and

$$c_{d-1}-c_{d-n+1}\approx\int_1^{d-1}\frac{dx}{x}-\int_1^{d-n+1}\frac{dx}{x}=\ln\frac{d-1}{d-n+1}\rightarrow 0\tag{7.15}$$

in the matrix model correlator. Both of these numbers grow linearly with the number of operators in the correlator that are gravitational descendents of the puncture operator. Consequently, this term continues to go to zero in the large d limit for correlators with an arbitrary but finite number of the gravitational descendants of the puncture. This is a BMN like limit. In this limit there is perfect agreement between the topological string correlators and the matrix model correlators.

8 Discussion

In trying to further explore the string-guage duality, we consider the Gaussian matrix model and topological string on $\mathbf{P}^1$ as a toy model for the duality. The relevant correlators participating in this duality are the connected correlators. We have developed a method for computing connected correlators in the matrix model, from recursion relations which follow from a systematic $\frac{1}{N}$ expansion of the Schwinger-Dyson equations. This method provides us with an alternative way to calculate connected correlators which is complementary to methods given in [26] using orthogonal polynomial.

This method of computing connected correlators by applying Schwinger-Dyson equations allows us to also compute correlators with $\ln M$ insertion without difficulty. The result of our analysis is a set of recursion relations that determine the correlators with an arbitrary number of $\ln M$ insertions.

The solution for the connected correlators $\langle \prod_{i=1}^{k} Tr(M^{2n_i})\rangle_{conn,0}$ was already known [27] and later reproduced using orthogonal polynomials [26]. By analytically continuing the known correlators, we have given a guess for the connected correlators with $\ln M$ insertions. We have proved that these correlators are indeed correct by verifying that they solve the recursion relations we derived above.

The Eguchi-Yang matrix model together with the identification (6.25) reproduces the correct correlators of the topological string. In our attempt to reproduce the topological string correlators with gravitational descendant of the puncture using the Gaussian matrix model, we first wrote down a three point correlator in the Gaussian matrix model (7.9) which matches the three point correlator of the topological A-model on $\mathbf{P}^1$. Even though the three point correlator of the Gaussian matrix model agrees with the topological A-model, there are disagreements between the two complete sets of correlation functions. Let's review the sources of disagreement of these complete sets of correlators.

In this duality we expect the correlators $\langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle$ and $\langle \frac{\mathrm{Tr}M^{2k_1}}{2k_1}\frac{\mathrm{Tr}M^{2k_2}}{2k_2}\frac{\mathrm{Tr}M^{2k_3}}{k_3}\cdots\frac{\mathrm{Tr}M^{2k_n}}{k_n}\rangle_{\mathrm{conn},0}$ to be equivalent. The solutions for these correlators are

$$\langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle = \frac{d^{n-2}}{4}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!} \tag{8.1}$$

$$\langle \frac{\mathrm{Tr}M^{2k_1}}{2k_1}\frac{\mathrm{Tr}M^{2k_2}}{2k_2}\frac{\mathrm{Tr}M^{2k_3}}{k_3}\cdots\frac{\mathrm{Tr}M^{2k_n}}{k_n}\rangle_{\mathrm{conn},0} = \frac{(d-1)!}{4(d-n+2)!}\prod_{i=1}^{n}\frac{(2k_i)}{k_i!(k_i)!} \tag{8.2}$$

These two expressions have a common factor of $\prod_{i=1}^{n}\frac{(2k_i)}{k_i!(k_i)!}$. By using the following normalisation

$$\tilde{\sigma}_{2k}(Q) = \frac{(k!)^2}{(2k)!}\sigma_{2k}(Q) \qquad O^Q_{2k} = \frac{1}{k}\frac{(k!)^2}{(2k)!}\mathrm{Tr}(M^{2k}), \tag{8.3}$$

we have

$$\langle \tilde{\sigma}_{2k_1-1}(Q)\tilde{\sigma}_{2k_2-1}(Q)\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}(Q)\rangle = \frac{d^{n-2}}{4} \tag{8.4}$$

$$\left\langle \frac{O^Q_{2k_1}}{2}\frac{O^Q_{2k_2}}{2}\prod_{i=3}^{n}O^Q_{2k_i}\right\rangle_{\mathrm{conn},0} = \frac{1}{4}\frac{(d-1)!}{(d-n+2)!}. \tag{8.5}$$

Using the identity

$$d^{n-3} = \sum_{m=3}^{n}\tilde{S}^{(m-2)}_{n-2}\frac{(d-1)!}{(d-m+2)!}, \tag{8.6}$$

we see that the two correlators are related by

$$\langle \tilde{\sigma}_{2k_1-1}(Q)\tilde{\sigma}_{2k_2-1}(Q)\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}(Q)\rangle = \sum_{m=3}^{n}\tilde{S}^{(m-2)}_{n-2}\left\langle \frac{O^Q_{2k_1}}{2}\frac{O^Q_{2k_2}}{2}\prod_{i=3}^{m}O^Q_{2k_i}\right\rangle_{\mathrm{conn},0} \tag{8.7}$$

where the positive integer $\tilde{S}^{(m-2)}_{n-2}$ is known as a Stirling number of the second kind. This number counts the numbers of ways to partition $n-2$ elements into $m-2$ sets. The interpretation of (8.7) is clear as we now explain. The source of the disagreement lies in contact terms: there is no operator product expansion in the matrix model that describes what happens when two traces coincide. To get agreement one needs to add the separate contributions from the fusing of two matrix operators by hand. This is precisely what (8.7) is doing. Again, from this gauge-string duality, using the following normalization

$$\tilde{\sigma}_{2k+1}(P) = \frac{(k!)^2}{(2k+1)!}\sigma_{2k+1}(P) \qquad O^P_{2k+1} = \frac{k!(k-1)!}{(2k)!}\mathrm{Tr}\left(2M^{2k}\ln M - \left[\frac{1}{k}+2c_{2k}\right]M^{2k}\right)$$

we expect the correlators $\langle \tilde{\sigma}_{2k_1-1}(Q)\tilde{\sigma}_{2k_2-1}(Q)\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}(Q)\tilde{\sigma}_{2k+1}(P)\rangle$
and $\left\langle \frac{1}{4}\prod_{i=1}^{n}O^Q_{2k_i}O^P_{2k}\right\rangle_{\mathrm{conn},0}$ to be equivalent. The solutions for these normalised correlators are

$$\langle \tilde{\sigma}_{2k_1-1}(Q)\tilde{\sigma}_{2k_2-1}(Q)\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}(Q)\tilde{\sigma}_{2k+1}(P)\rangle = \frac{d^{n-2}}{4}\left(-2c_k+\frac{n-2}{d}\right) \tag{8.8}$$

$$\left\langle \frac{1}{4}\prod_{i=1}^{n}O^Q_{2k_i}O^P_{2k}\right\rangle_{\mathrm{conn},0} = \frac{1}{4}\frac{(d-1)!}{(d-n+1)!}\left(-2c_k+c_{d-1}-c_{d-n+1}\right) \tag{8.9}$$

Using the identity

$$(n-2)d^{n-3} = \sum_{m=3}^{n+1} \tilde{S}_{n-1}^{(m-2)} \frac{(d-1)!}{(d-m+2)!}(c_{d-1} - c_{d-m+2}) \tag{8.10}$$

we see that these two correlators are related by

$$\langle \tilde{\sigma}_{2k_1-1}(Q)\tilde{\sigma}_{2k_2-1}(Q)\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}(Q)\tilde{\sigma}_{2k+1}(P)\rangle = \sum_{m=3}^{n+1}\tilde{S}_{n-1}^{(m-2)}\left\langle \frac{1}{4}\prod_{i=1}^{m} O_{2k_i}^{Q} O_{2k}^{P}\right\rangle_{\text{conn},0} \tag{8.11}$$

Notice that (8.11) and (8.7) are identical in form. This implies that the same contact term argument that explains the discrepancy between correlators of the Kähler class, explains the disagreement between correlators of the puncture. Thus, in the end we have a complete confirmation of the duality at genus zero.

9 Appendix

A Examples of solving the recursion equation for the matrix model correlators using the Schwinger-Dyson equation

Using the Schwinger-Dyson equations, we can determine any correlator by solving a recursion relation. To test if (5.12) is really a valid expression for the correlators, we can verify that they solve the Schwinger-Dyson equations.

A.1 $\gamma^2(\mathbf{n_1}, \mathbf{n_2}, \cdots, \mathbf{n_{q-1}}, \mathbf{n_q}, \mathbf{0}, \mathbf{0})$

This is one of the correlators that were computed in [26]. The solution for this correlator was computed to be

$$\langle (Tr \ln M)^2 \prod_{i=1}^{q} Tr(M^{2n_i}) \rangle_{conn,0} = \frac{(n-1)!}{4(n-q)!} \prod_{i=1}^{q} \frac{(2n_i)}{n_i!(n_i-1)!}. \qquad \text{(A.1)}$$

Now, we want to check if our recursion relation and its solution produces the same results as the one calculated using (A.1). Consider the following correlators; $\gamma^2(1,1,0,0)$, $\gamma^2(1,2,0,0)$ and $\gamma^2(2,2,0,0)$. using the formula derived in [26], the numerical values of these correlators are

$$\begin{aligned} \gamma^2(1,1,0,0) &= 1. \\ \gamma^2(1,2,0,0) &= 12. \\ \gamma^2(2,2,0,0) &= 108. \end{aligned}$$

Now, we compute these correlators using equation (5.12) and see if the same results are obtained.

(i). $\gamma^2(1,1,0,0)$

$$\begin{aligned} \gamma^2(1,1,0,0) &= \frac{1}{2^2} \cdot \frac{(n-1)!}{(n-k+2)!} \cdot \mathcal{A} \cdot (1) \\ &= \frac{1}{4} \frac{(2-1)!}{(2-4+2)!} \frac{(2(1))!(2(1))!}{1!0!1!0!} \\ &= \frac{1}{4} \frac{4}{1} \\ &= 1 \end{aligned}$$

(ii). $\gamma^2(1,2,0,0)$

$$\begin{aligned}\gamma^2(1,2,0,0) &= \frac{1}{2^2}\cdot\frac{(n-1)!}{(n-k+2)!}\cdot\mathcal{A}\cdot(1)\\ &= \frac{1}{4}\frac{(3-1)!}{(3-4+2)!}\frac{(2(1))!(2(2))!}{1!0!2!1!}\\ &= \frac{1}{4}\frac{2}{1}\frac{2\cdot 24}{2}\\ &= 12\end{aligned}$$

(iii). $\gamma^2(2,2,0,0)$

$$\begin{aligned}\gamma^2(2,2,0,0) &= \frac{1}{2^2}\cdot\frac{(n-1)!}{(n-k+2)!}\cdot\mathcal{A}\cdot(1)\\ &= \frac{1}{4}\frac{(4-1)!}{(4-4+2)!}\frac{(2(2))!(2(2))!}{2!1!2!1!}\\ &= \frac{1}{4}\frac{6}{2}\frac{24\cdot 24}{2\cdot 2}\\ &= 108\end{aligned}$$

These values match perfectly with the values calculated using (A.1). To be general, use (5.12) for the correlator $\gamma(n_1,n_2,\cdots,n_q,\overline{0},\overline{0})$ and see if we can re-derive (A.1).

Evaluate $\gamma(n_1,n_2,\cdots,n_q,\overline{0},\overline{0})$ using (5.12) to find

$$\begin{aligned}\gamma^2(n_1,n_2,\cdots,n_q,0,0) &= \frac{1}{4}\cdot\frac{(n-1)!}{(n-k+2)!}\mathcal{AB}\\ &= \frac{1}{4}\frac{(n-1)!}{(n-k+2)!}\prod_{i=1}^{q}\frac{(2n_i)!}{n_i!(n_i-1)!}\prod_{j=1}^{2}A(\overline{n}_{q+j})\\ &= \frac{1}{4}\frac{(n-1)!}{(n-k+2)!}\prod_{i=1}^{q}\frac{(2n_i)!}{n_i!(n_i-1)!}\cdot 1.\end{aligned}$$

where $n=\sum_i n_i + 0$ and $k=q+2$. The above equation becomes

$$\gamma^2(n_1,n_2,\cdots,n_q,0,0) = \frac{1}{4}\frac{(n-1)!}{(n-q)!}\prod_{i=1}^{q}\frac{(2n_i)!}{n_i!(n_i-1)!}.$$

This is in perfect agreement with (A.1).

Recall the generalized solution to the correlator $\gamma^m(n_1, n_2, \cdots, n_q, n_{q+1}, n_{q+2}, \cdots, n_{q+m})$, where $q + m = k$

For $\mathbf{k = m}$:

$$\gamma^m(n_1, n_2, \cdots, n_q, n_{q+1}, n_{q+2}, \cdots, n_{q+m}) = \sum_P \sum_{r=0}^{n_1-1} \gamma(r, P) \quad \gamma(n_1 - r - 1, \tilde{P})$$
$$+ \sum_{r=2}^{q} 2n_r \gamma(n_1 + n_r - 1, S'_r) + \sum_{r=q+1}^{q+m} 2n_r \gamma(\overline{\overline{n_1 + n_r - 1}}, S'_r) + \sum_{r=q+1}^{q+m} \gamma(\overline{n_1 + n_r - 1}, S'_r). \tag{A.2}$$

For $\mathbf{k > m}$:

$$\gamma^m(n_1, n_2, \cdots, n_q, n_{q+1}, n_{q+2}, \cdots, n_{q+m}) = \sum_P \sum_{r=0}^{n_1-1} \gamma(r, P) \quad \gamma(n_1 - r - 1, \tilde{P})$$
$$+ \sum_{r=2}^{q} 2n_r \gamma(n_1 + n_r - 1, S'_r) + \sum_{r=q+1}^{q+m} 2n_r \gamma(\overline{n_1 + n_r - 1}, S'_r) + \sum_{r=q+1}^{q+m} \gamma^{m-1}(n_1 + n_r - 1, S'_r). \tag{A.3}$$

By picking any of the correlators we have studied and expanding them using the recursion relation (A.2) and (A.3), we can verify if (5.12) is the solution to these recursion relations. We choose correlators based on the value of k and the numbers of $\ln M$ insertions m.

A.2 $\mathbf{m = 1}$: $\gamma(\mathbf{n_1}, \mathbf{n_2}, \cdots, \mathbf{n_q}, \overline{\mathbf{n}}_{\mathbf{q+1}})$.

We start by considering correlators that contains single $\ln M$ insertion and we will only consider $k = 2$ and $k = 3$

A.2.1 $\mathbf{k = 2}$: $\gamma(\mathbf{n_1}, \overline{\mathbf{n_2}})$.

The solution for this correlator is

$$\gamma(n_1, \overline{n_2}) = \frac{(n_1 + n_2 - 1)!}{2^1(n_1 + n_2)!} \frac{(2n_1)!}{n_1!(n_1 - 1)!} \frac{(2n_2)!}{n_2!(n_2 - 1)!}$$
$$\Big(\sum_{j=1}^{n_1+n_2-1} \frac{1}{j} - \sum_{j=1}^{n_1+n_2-k+2} \frac{1}{j} + 2 \sum_{j=1}^{2n_2} \frac{1}{j} - \sum_{j=1}^{n_2} \frac{1}{j} - \sum_{j=1}^{n_2-1} \frac{1}{j} \Big). \tag{A.4}$$

Now, lets evaluate two examples that are of type of this correlator and see if this is the solution to the recursion relation. We consider the following two examples; $\gamma(2,\overline{3})$ and $\gamma(1,3,\overline{5})$.

Example 1: $\gamma(2,\overline{3})$.

Expanding (A.3)

$$\begin{aligned}\gamma(2,\overline{3}) &= \gamma(0)\gamma(1,\overline{3}) + \gamma(1)\gamma(0,\overline{3}) + \gamma(0,\overline{3})\gamma(1) + \gamma(1,\overline{3})\gamma(0) + 2(3)\gamma(\overline{4}) + \gamma(4)\\ &= 2\gamma(0)\gamma(1,\overline{3}) + 6\gamma(\overline{4}) + \gamma(4)\end{aligned} \tag{A.5}$$

Evaluate $\gamma(2,\overline{3})$, $\gamma(1,\overline{3})$ and $\gamma(\overline{4})$ using (5.12)

$$\gamma(2,\overline{3}) = \frac{492}{5}, \quad \gamma(1,\overline{3}) = \frac{79}{4}, \quad \gamma(\overline{4}) = \frac{449}{60} \quad \text{and} \quad \gamma(4) = 14.$$

Putting everything together

$$\begin{aligned}\gamma(2,\overline{3}) &= 2(1)\frac{79}{4} + 6(\frac{449}{60}) + 14\\ &= \frac{492}{5}.\end{aligned}$$

Therefore (A.5) is satisfied.

A.2.2 $k = 3$: $\gamma(n_1, n_2, \overline{n_3})$

The solution for this type of correlator is

$$\gamma(n_1,n_2,\overline{n_3}) = \frac{(n_1+n_2+n_3-1)!}{2^1(n_1+n_2+n_3-1)!}\frac{(2n_1)!}{n_1!(n_1-1)!}\frac{(2n_2)!}{n_2!(n_2-1)!}\frac{(2n_3)!}{n_3!(n_3-1)!}$$
$$\left(\sum_{j=1}^{n_1+n_2+n_3-1}\frac{1}{j} - \sum_{j=1}^{n_1+n_2+n_3-k+2}\frac{1}{j} + 2\sum_{j=1}^{2n_3}\frac{1}{j} - \sum_{j=1}^{n_3}\frac{1}{j} - \sum_{j=1}^{n_3-1}\frac{1}{j}\right). \tag{A.6}$$

Example 2: $\gamma(1,3,\overline{5})$

Expanding $\gamma(1,3,\overline{5})$

$$\gamma(1,3,\overline{5}) = 6\gamma(3,\overline{5}) + 10\gamma(\overline{5},3) + \gamma(5,3). \tag{A.7}$$

Again, using (5.12), we evaluate the following

$$\gamma(5,3) = 9450, \quad \gamma(3,\overline{5}) = \frac{51645}{8} \quad \gamma(\overline{5},3) = \frac{51645}{8} \quad \text{and} \quad \gamma(1,3,\overline{5}) = 112740.$$

Substitute this value to (A.7)

$$\gamma(1,3,\overline{5}) = 6 \times \frac{51645}{8} + 10 \times \frac{51645}{8} + 9450$$
$$= 112740.$$

(A.7) is satisfied.

A.3 $\mathbf{m = 2}$: $\gamma(\mathbf{n_1}, \mathbf{n_2}, \cdots, \mathbf{n_q}, \overline{\mathbf{n}}_{\mathbf{q+1}}, \overline{\mathbf{n}}_{\mathbf{q+2}})$.

For this type of correlator, we consider two cases, that is $k = 2$ case and $k = 3$ case .

A.3.1 $\mathbf{k = 3}$: $\gamma(\mathbf{n_1}, \overline{\mathbf{n}}_{\mathbf{2}}, \overline{\mathbf{n}}_{\mathbf{3}})$.

The solution for this correlator is given by

$$\begin{aligned}\gamma(n_1,\overline{n}_2,\overline{n}_3) &= \frac{(n-1)!}{(n-k+2)!}\frac{\mathcal{AB}}{2^m} \\ &= \frac{(n_1+n_2+n_3-1)!}{2^2(n_1+n_2+n_3-1)!}\frac{(2n_1)!}{n_1!(n_1-1)!}\frac{(2n_2)!}{n_2!(n_2-1)!}\frac{(2n_3)!}{n_3!(n_3-1)!} \\ &\left(2\sum_{j=1}^{2n_2}\frac{1}{j}-\sum_{j=1}^{n_2}\frac{1}{j}-\sum_{j=1}^{n_2-1}\frac{1}{j}\right)\left(2\sum_{j=1}^{2n_3}\frac{1}{j}-\sum_{j=1}^{n_3}\frac{1}{j}-\sum_{j=1}^{n_3-1}\frac{1}{j}\right)\end{aligned} \tag{A.8}$$

Example 1: $\gamma(1,\overline{2},\overline{1})$

Evaluating this correlator using (A.8) we find

$$\begin{aligned}\gamma(1,\overline{2},\overline{1}) &= \frac{1}{4}\cdot 2\cdot 12\cdot 2\cdot\frac{10}{3} \\ &= 40.\end{aligned} \tag{A.9}$$

Now, let see if the same solution is obtained by using recursion equation

$$\begin{aligned}\gamma(1,\overline{2},\overline{1}) &= 4\gamma(\overline{2},\overline{1}) + 2\gamma(\overline{1},\overline{2}) + \gamma(2,\overline{1}) + \gamma(1,\overline{2}) \\ &= 6\gamma(\overline{2},\overline{1}) + \gamma(2,\overline{1}) + \gamma(1,\overline{2}) \\ &= 6(\frac{14}{3}) + \frac{1}{3}\cdot\frac{1}{2}\cdot 12\cdot\frac{10}{3} + \frac{1}{3}\cdot\frac{1}{2}\cdot 2\cdot 16 \\ &= 28 + \frac{20}{3} + \frac{16}{3} \\ &= 40.\end{aligned} \tag{A.10}$$

In the third line, we used equation (A.8) and (5.12) to evaluate the numerical values of this correlator. The results from (A.9) agrees with the results from (A.10).

Example 2: $\gamma(1,\overline{5},\overline{3})$

Evaluating this correlator using (A.8) we find

$$\begin{aligned}\gamma(1,\overline{5},\overline{3}) &= \frac{1}{4}\cdot 2\cdot 1260\cdot 60\cdot\frac{88313}{37800}\\ &= 88131.\end{aligned} \tag{A.11}$$

Computing the same correlator, using the recursion relation, gives

$$\begin{aligned}\gamma(1,\overline{5},\overline{3}) &= 10\gamma(\overline{5},\overline{3})+6\gamma(\overline{3},\overline{5})+\gamma(5,\overline{3})+\gamma(3,\overline{5})\\ &= 16\gamma(\overline{5},\overline{3})+\gamma(5,\overline{3})+\gamma(3,\overline{5})\\ &= 10(\frac{150091}{32})+\frac{1}{8}\cdot\frac{1}{2}\cdot 1260\cdot\frac{173}{2}+\frac{1}{8}\cdot\frac{1}{2}\cdot 60\cdot\frac{3443}{2}\\ &= \frac{150091}{2}+\frac{51645}{8}+\frac{54495}{8}\\ &= 88131.\end{aligned} \tag{A.12}$$

Again, the results from (A.11) agrees with the one in (A.12). Therefore, we conclude that (A.8) is indeed the solution for correlators of form $\gamma(n_1,\overline{n}_2,\overline{n}_3)$

A.3.2 $\mathbf{k=2}:\gamma(\overline{\mathbf{n}}_{\mathbf{1}},\overline{\mathbf{n}}_{\mathbf{2}})$

The solution for this correlator is given by

$$\begin{aligned}\gamma(\overline{n}_1,\overline{n}_2) = &\frac{(n_1+n_2+-1)!}{2^2(n_1+n_2)!}\frac{(2n_1)!}{n_1!(n_1-1)!}\frac{(2n_2)!}{n_2!(n_2-1)!}\\ &\Bigg[\Bigg(2\sum_{j=1}^{2n_1}\frac{1}{j}-\sum_{j=1}^{n_1}\frac{1}{j}-\sum_{j=1}^{n_1-1}\frac{1}{j}+\sum_{j=1}^{n_1+n_2-1}\frac{1}{j}-\sum_{j=1}^{n_1+n_2}\frac{1}{j}\Bigg)\\ &\Bigg(2\sum_{j=1}^{2n_2}\frac{1}{j}-\sum_{j=1}^{n_2}\frac{1}{j}-\sum_{j=1}^{n_2-1}\frac{1}{j}+\sum_{j=1}^{n_1+n_2-1}\frac{1}{j}-\sum_{j=1}^{n_1+n_2}\frac{1}{j}\Bigg)+(\psi^1(n_1+n_2)-\psi^1(n_1+n_2-k+3))\Bigg].\end{aligned} \tag{A.13}$$

where $\psi^1(n_1+n_2)$ is a polygamma function. Lets evaluate the following correlators

Example 1: $\gamma(\overline{1},\overline{2})$

Using equation (A.13) to evaluate $\gamma(\overline{1},\overline{2})$, we get

$$\gamma(\overline{1},\overline{2}) = \frac{14}{3} \tag{A.14}$$

Expanding this equation using (A.2) we obtain

$$\begin{aligned}\gamma(\overline{1},\overline{2}) &= \gamma(0)\gamma^2(0,2) + \gamma^1(0,2)\gamma^1(0) + 4\gamma^2(2) + \gamma^1(2)\\ &= 1\cdot\gamma^2(0,2) + 4\gamma^2(2) + \gamma^1(2).\end{aligned} \tag{A.15}$$

To evaluate (A.15) directly is challenging. The difficulties in evaluating (A.15) comes from the term $\gamma^2(2)/\gamma(\overline{\overline{2}})$. When we derived the recursion relations of correlators with $\ln M$ insertion using the Schwinger-Dyson equations, we never consider correlators that took a form $\langle Tr(M^{2n_k}(\ln M)^2)\rangle$. Instead, we only consider correlator which took a form $\langle Tr(M^{2n_k}\ln M)\rangle$. To avoid this problem of deal with terms of such form , we use the following trick to evaluate $\gamma(\overline{n}_1,\overline{n}_2)$.

Expanding the correlator $\gamma(n_1,\overline{n}_2,\overline{n}_3)$

$$\begin{aligned}\gamma(n_1,\overline{n}_2,\overline{n}_3) = &\sum_{r=0}^{n_1-1}\gamma(r,P)\gamma(n_1-r-1,\overline{P}) + \sum_{r=0}^{n_1-1}\gamma(r,\overline{P})\gamma(n_1-r-1,P)\\ &+ 2n_2\gamma(\overline{n_1+n_2-1},\overline{n}_3) + 2n_3\gamma(\overline{n_1+n_3-1},\overline{n}_2) + \gamma(n_1+n_2-1,\overline{n}_3) + \gamma(n_1+n_3-1,\overline{n}_2).\end{aligned}$$

Let $n_1 = 1$,

$$\begin{aligned}\gamma(1,\overline{n}_2,\overline{n}_3) &= 2n_2\gamma(\overline{n}_2,\overline{n}_3) + 2n_3\gamma(\overline{n}_3,\overline{n}_2) + \gamma(n_2,\overline{n}_3) + \gamma(n_3,\overline{n}_2)\\ &= \gamma(\overline{n}_2,\overline{n}_3)(2n_2+2n_3) + \gamma(n_2,\overline{n}_3) + \gamma(n_3,\overline{n}_2)\\ \Rightarrow \quad \gamma(\overline{n}_2,\overline{n}_3) &= \frac{\gamma(,\overline{n}_2,\overline{n}_3) - \gamma(n_2,\overline{n}_3) - \gamma(n_3,\overline{n}_2)}{2(n_2+n_3)}\end{aligned} \tag{A.16}$$

Re-labelling the subscripts of n in this last expression

$$\gamma(\overline{n}_1,\overline{n}_2) = \frac{\gamma(1,\overline{n}_1,\overline{n}_2) - \gamma(n_1,\overline{n}_2) - \gamma(n_2,\overline{n}_1)}{2(n_1+n_2)} \tag{A.17}$$

To see if this equation gives the same results as (A.14) when choose $n_1 = 1$ and $n_2 = 2$, we first evaluate the following correlators using the previous results

$$\gamma(1,\overline{1},\overline{2}) = 40, \quad \gamma(\overline{1},2) = \frac{20}{3} \quad \text{and} \quad \gamma(\overline{2},1) = \frac{16}{3}.$$

Then,

$$\gamma(\overline{1},\overline{2}) = \frac{40 - \frac{16}{3} - \frac{20}{3}}{2(1+2)}$$
$$= \frac{14}{3}.$$

This is the exact result found in (A.14) by using equation (A.8).

Example 2: $\gamma(\overline{2},\overline{3})$

Evaluating this correlator using (A.17) we find

$$\gamma(\overline{2},\overline{3}) = \frac{\gamma(1,2,\overline{3}) - \gamma(2,\overline{3})\gamma(\overline{2},3)}{2(5)}.$$

Evaluating the correlators appearing in this last equation we have

$$\gamma(\overline{2},\overline{3}) = \frac{368}{5} \quad \gamma(1,\overline{2},\overline{3}) = 940, \quad \gamma(2,\overline{3}) = \frac{492}{5} \quad \text{and} \quad \gamma(\overline{2},3) = \frac{528}{2}$$

Using these results we now obtain

$$\gamma(\overline{2},\overline{3}) = \frac{940 - \frac{492}{5} - \frac{528}{5}}{2(5)}$$
$$= \frac{368}{5}.$$

We see that (A.17) is satisfied. Therefore, the correct expression for correlators of form $\gamma(\overline{n}_1,\overline{n}_2)$ is given by (A.17).

A.3.3 $\mathbf{k = 4} : \gamma(\mathbf{n_1}, \mathbf{n_2}, \overline{\mathbf{n}}_\mathbf{3}, \overline{\mathbf{n}}_\mathbf{4})$

The solution for this correlator is given by

$$\gamma(n_1,n_2,\overline{n}_3,\overline{n}_4) = \frac{(n_1+n_2+n_3+n_4-1)!}{2^2(n_1+n_2+n_3+n_4-k+2)!}\frac{(2n_1)!}{n_1!(n_1-1)!}\frac{(2n_2)!}{n_2!(n_2-1)!}\frac{(2n_3)!}{n_3!(n_3-1)!}\frac{(2n_4)!}{n_4!(n_4-1)!}$$
$$\Bigg[\left(2\sum_{j=1}^{2n_3}\frac{1}{j} - \sum_{j=1}^{n_3}\frac{1}{j} - \sum_{j=1}^{n_3-1}\frac{1}{j} + \sum_{j=1}^{n_1+n_2+n_3+n_4-1}\frac{1}{j} - \sum_{j=1}^{n_1+n_2+n_3+n_4}\frac{1}{j}\right)$$
$$\left(2\sum_{j=1}^{2n_4}\frac{1}{j} - \sum_{j=1}^{n_4}\frac{1}{j} - \sum_{j=1}^{n_4-1}\frac{1}{j} + \sum_{j=1}^{n_1+n_2+n_3+n_4-1}\frac{1}{j} - \sum_{j=1}^{n_1+n_2+n_3+n_4}\frac{1}{j}\right)$$
$$+ \left(\psi^1(n_1+n_2+n_3+n_4) - \psi^1(n_1+n_2+n_3+n_4-k+3)\right)\Bigg]. \qquad \text{(A.18)}$$

To confirm that this solution holds for correlators of form $\gamma(n_1, n_2, \overline{n}_3, \overline{n}_4)$, we consider two examples.

Example 1: $\gamma(1, 1, \overline{1}, \overline{1})$

Expanding this correlator

$$\begin{aligned}\gamma(1,1,\overline{1},\overline{1}) &= 2\gamma(1,1,\overline{1}) + 2\gamma(\overline{1},1,\overline{1}) + 2\gamma(\overline{1},1,\overline{1}) + \gamma(1,1,\overline{1}) + \gamma(1,1,\overline{1}) \\ &= 4\gamma(\overline{1},1,\overline{1}) + 4\gamma(1,1,\overline{1}) \end{aligned} \tag{A.19}$$

Evaluating $\gamma(1, 1, \overline{1}, \overline{1})$ using (A.18) and the other correlators in the expansion using the results we have derived before, we get

$$\gamma(1,1,\overline{1},\overline{1}) = 64, \quad \gamma(\overline{1},1,\overline{1}) = 8 \quad \text{and} \quad \gamma(1,1,\overline{1}) = 8.$$

Using these results, the recursion relation becomes

$$\begin{aligned}\gamma(1,1,\overline{1},\overline{1}) &= 4(8) + 4(8) \\ &= 64,\end{aligned}$$

and equation (A.19) is satisfied.

Example 2: $\gamma(1, 3, \overline{2}, \overline{4})$

Expanding this correlator

$$\gamma(1,3,\overline{2},\overline{4}) = 6\gamma(3,\overline{2},\overline{4}) + 4\gamma(\overline{2},3,\overline{4}) + 8\gamma(\overline{4},3,\overline{2}) + \gamma(2,3,\overline{4}) + \gamma(4,3,\overline{2}) \tag{A.20}$$

Evaluating $\gamma(1, 3, \overline{2}, \overline{4})$ using (A.18) and the other correlators in this expansion using the results we have derived before

$$\gamma(1,3,\overline{2},\overline{4}) = 2617920, \quad \gamma(3,\overline{2},\overline{4}) = 127600 \quad \gamma(2,3,\overline{4}) = 153120 \quad \text{and} \quad \gamma(4,3,\overline{2}) = 168000$$

Using these results, the recursion equation becomes

$$\begin{aligned}\gamma(1,3,\overline{2},\overline{4}) &= 6(127600) + 4(127600) + 8(127600) + 153120 + 168000 \\ &= 2617920\end{aligned}$$

so that equation (A.20) is satisfied.

A.4 $\mathbf{m=3}:\gamma(\mathbf{n_1},\mathbf{n_2},\cdots,\mathbf{n_q},\overline{\mathbf{n}}_{\mathbf{q+1}},\overline{\mathbf{n}}_{\mathbf{q+2}},\overline{\mathbf{n}}_{\mathbf{q+3}})$.

Here, we write down the solution for this type of correlator ($m=3$) for the $k=4$ and $k=3$ cases.

A.4.1 $\mathbf{k=4}:\gamma(\mathbf{n_1},\overline{\mathbf{n}}_{\mathbf{2}},\overline{\mathbf{n}}_{\mathbf{3}},\overline{\mathbf{n}}_{\mathbf{4}})$.

The solution for these correlators is

$$\begin{aligned}\gamma(n_1,\overline{n}_2,\overline{n}_3,\overline{n}_4)=&\frac{(n-1)!}{2^3(n-1)!}\frac{(2n_1)!}{n_1!(n_1-1)!}\frac{(2n_2)!}{n_2!(n_2-1)!}\frac{(2n_3)!}{n_3!(n_3-1)!}\frac{(2n_4)!}{n_4!(n_4-1)!}\\&\Bigg[\Bigg(2\sum_{j=1}^{2n_4}\frac{1}{j}-\sum_{j=1}^{n_4}\frac{1}{j}-\sum_{j=1}^{n_4-1}\frac{1}{j}\Bigg)\Bigg(\Bigg(-2\sum_{j=1}^{2n_3}\frac{1}{j}+\sum_{j=1}^{n_3}\frac{1}{j}+\sum_{j=1}^{n_3-1}+\sum_{j=1}^{n-k+2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\Bigg)\\&\Bigg(-2\sum_{j=1}^{2n_2}\frac{1}{j}+\sum_{j=1}^{n_2}\frac{1}{j}+\sum_{j=1}^{n_2-1}+\sum_{j=1}^{n-k+2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\Bigg)-\psi^{(1)}(n-k-3)+\psi^{(1)}(n)\Bigg)\\&+\Bigg(-2\sum_{j=1}^{2n_2}\frac{1}{j}+\sum_{j=1}^{n_2}\frac{1}{j}+\sum_{j=1}^{n_2-1}+\sum_{j=1}^{n-k+2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\Bigg)\Bigg(\psi^{(1)}(n-k-3)-\psi^{(1)}(n)\Bigg)\\&-\Bigg(\Bigg(-2\sum_{j=1}^{2n_3}\frac{1}{j}+\sum_{j=1}^{n_3}\frac{1}{j}+\sum_{j=1}^{n_3-1}+\sum_{j=1}^{n-k+2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\Bigg)\\&\Bigg(-2\sum_{j=1}^{n-k+2}\frac{1}{j}\sum_{j=1}^{2n_2}\frac{1}{j}+\sum_{j=1}^{n-k+2}\frac{1}{j}\sum_{j=1}^{n_2}\frac{1}{j}+\sum_{j=1}^{n-k+2}\frac{1}{j}\sum_{j=1}^{n_2-1}+\sum_{j=1}^{n-k+2}\frac{1}{j}\sum_{j=1}^{n-k+2}\frac{1}{j}\\&+2\sum_{j=1}^{n-1}\frac{1}{j}\sum_{j=1}^{2n_2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\sum_{j=1}^{n_2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\sum_{j=1}^{n_2-1}-2\sum_{j=1}^{n-1}\frac{1}{j}\sum_{j=1}^{n-k+2}\frac{1}{j}+\sum_{j=1}^{n-1}\frac{1}{j}\sum_{j=1}^{n-1}\frac{1}{j}+\sum_{j=1}^{n-1}\frac{1}{j}\\&-\sum_{j=1}^{n-k+2}\frac{1}{j}\Bigg)-\psi^{(1)}(n-k-3)+\psi^{(1)}(n)\Bigg)\Bigg(\sum_{j=1}^{n-k+2}\frac{1}{j}-\sum_{j=1}^{n-1}\frac{1}{j}\Bigg)\Bigg(\psi^{(1)}(n-k-3)-\psi^{(1)}(n)\Bigg)\\&-\psi^{(1)}(n-k-3)+\psi^{(1)}(n)\Bigg].\end{aligned}\tag{A.21}$$

A.4.2 $\mathbf{k=3}:\gamma(\overline{\mathbf{n}}_{\mathbf{1}},\overline{\mathbf{n}}_{\mathbf{2}},\overline{\mathbf{n}}_{\mathbf{3}})$

The solution for these correlators is

$$
\begin{aligned}
\gamma(\overline{n}_1,\overline{n}_2,\overline{n}_3) &= \frac{(n-1)!}{(n-k+2)!}\frac{\mathcal{AB}}{2^m} \\
&= \frac{(n_1+n_2+n_3-1)!}{2^3(n_1+n_2+n_3-1)!}\frac{(2n_1)!}{n_1!(n_1-1)!}\frac{(2n_2)!}{n_2!(n_2-1)!}\frac{(2n_3)!}{n_3!(n_3-1)!} \\
&\left(2\sum_{j=1}^{2n_1}\frac{1}{j}-\sum_{j=1}^{n_1}\frac{1}{j}-\sum_{j=1}^{n_1-1}\frac{1}{j}\right)\left(2\sum_{j=1}^{2n_2}\frac{1}{j}-\sum_{j=1}^{n_2}\frac{1}{j}-\sum_{j=1}^{n_2-1}\frac{1}{j}\right)\left(2\sum_{j=1}^{2n_3}\frac{1}{j}-\sum_{j=1}^{n_3}\frac{1}{j}-\sum_{j=1}^{n_3-1}\frac{1}{j}\right)
\end{aligned} \tag{A.22}
$$

To check if this is the correct solution for the correlator $\gamma(\overline{n}_1,\overline{n}_2,\overline{n}_3)$, we compute a couple of examples of this correlator using (A.22) and compare these results with the ones we obtained by expanding the same correlator.

Computing the correlators; $\gamma(\overline{2},\overline{2},\overline{1})$ and $\gamma(\overline{4},\overline{3},\overline{5})$ using (A.22), we get

$$
\gamma(\overline{2},\overline{2},\overline{1}) = 200 \quad \text{and} \quad \gamma(\overline{4},\overline{3},\overline{5}) = \frac{28171847}{3}.
$$

To expand $\gamma(\overline{n}_1,\overline{n}_2,\overline{n}_3)$, we use the same trick as when we expand $\gamma(\overline{n}_1,\overline{n}_2)$

Expansion of $\gamma(\overline{n}_1,\overline{n}_2,\overline{n}_3)$

Consider the correlator $\gamma(\overline{n}_1,\overline{n}_2,\overline{n}_3)$. This correlator can be expanded using (A.3) as follows

$$
\begin{aligned}
\gamma(1,\overline{n}_2,\overline{n}_3,\overline{n}_4) &= 2n_2\gamma(\overline{1+n_2-1},\overline{n}_3,\overline{n}_4) + 2n_3\gamma(\overline{1+n_3-1},\overline{n}_2,\overline{n}_4) + 2n_4\gamma(\overline{1+n_4-1},\overline{n}_2,\overline{n}_3) \\
&+ \gamma(n_2,\overline{n}_3,\overline{n}_4) + \gamma(n_3,\overline{n}_2,\overline{n}_4) + \gamma(n_4,\overline{n}_2,\overline{n}_3) \\
&= 2n_2\gamma(\overline{n_2},\overline{n}_3,\overline{n}_4) + 2n_3\gamma(\overline{n_3},\overline{n}_2,\overline{n}_4) + 2n_4\gamma(\overline{n_4},\overline{n}_2,\overline{n}_3) \\
&+ \gamma(n_2,\overline{n}_3,\overline{n}_4) + \gamma(n_3,\overline{n}_2,\overline{n}_4) + \gamma(n_4,\overline{n}_2,\overline{n}_3)
\end{aligned} \tag{A.23}
$$

$$
\gamma(\overline{n}_2,\overline{n}_3,\overline{n}_4) = \frac{\gamma(1,\overline{n}_2,\overline{n}_3,\overline{n}_4) - \gamma(n_2,\overline{n}_3,\overline{n}_4) - \gamma(n_3,\overline{n}_2,\overline{n}_4) - \gamma(n_4,\overline{n}_2,\overline{n}_3)}{2(n_2+n_3+n_4)} \tag{A.24}
$$

Let's evaluate the correlators, $\gamma(\overline{2},\overline{2},\overline{1})$ and $\gamma(\overline{4},\overline{3},\overline{5})$ using (A.24) and see if the results are consistent with the one's in (A.23).

<u>Example 1: $\gamma(\overline{2},\overline{2},\overline{1})$</u>

$$
\begin{aligned}
\gamma(\overline{n}_1,\overline{n}_2,\overline{n}_3) &= \frac{\gamma(1,\overline{n}_1,\overline{n}_2,\overline{n}_3) - \gamma(n_1,\overline{n}_2,\overline{n}_3) - \gamma(n_2,\overline{n}_1,\overline{n}_3) - \gamma(n_3,\overline{n}_1,\overline{n}_2)}{2(n_1+n_2+n_3)} \\
\gamma(\overline{2},\overline{2},\overline{1}) &= \frac{\gamma(1,\overline{2},\overline{2},\overline{1}) - \gamma(2,\overline{2},\overline{1}) - \gamma(2,\overline{2},\overline{1}) - \gamma(1,\overline{2},\overline{2})}{2(5)}
\end{aligned}
$$

Using the previous results and (A.21) , the following correlators are found

$$\gamma(1,\overline{2},\overline{2},\overline{1}) = 2680, \quad \gamma(2,\overline{2},\overline{1}) = 240 \quad \text{and} \quad \gamma(2,\overline{2},\overline{1}) = 200.$$

Now, plugging these results to the expansion of $\gamma(\overline{2},\overline{2},\overline{1})$

$$\begin{aligned}\gamma(\overline{2},\overline{2},\overline{1}) &= \frac{2680 - 240 - 240 - 200}{2(5)} \\ &= 200\end{aligned}$$

This results agrees perfectly with (A.23).

<u>Example 2: $\gamma(\overline{4},\overline{3},\overline{5})$</u>

The expansion of this correlator is

$$\gamma(\overline{4},\overline{3},\overline{5}) = \frac{\gamma(1,\overline{4},\overline{3},\overline{5}) - \gamma(4,\overline{3},\overline{5}) - \gamma(3,\overline{4},\overline{5}) - \gamma(5,\overline{4},\overline{3})}{2(12)} \tag{A.25}$$

The numerical values for the above correlators are

$\gamma(1,\overline{4},\overline{3},\overline{5}) = 262320736, \quad \gamma(4,\overline{3},\overline{5}) = 12363820, \quad \gamma(3,\overline{4},\overline{5}) = 11988020 \quad \text{and} \quad \gamma(5,\overline{4},\overline{3} = 12594120$

Substituting these numerical values into (A.25)

$$\begin{aligned}\gamma(\overline{4},\overline{3},\overline{5}) &= \frac{262320736 - 12363820 - 11988020 - 12594120}{24} \\ &= \frac{28171847}{3}\end{aligned}$$

This results agree with (A.23).

References

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[4] R. d. M. Koch and S. Ramgoolam, "From Matrix Models and Quantum Fields to Hurwitz Space and the absolute Galois Group," arXiv:1002.1634 [hep-th].

[5] R. de Mello Koch and S. Ramgoolam, "Strings from Feynman Graph counting : without large N," Phys. Rev. D **85**, 026007 (2012) [arXiv:1110.4858 [hep-th]].

[6] R. de Mello Koch and S. Ramgoolam, "A double coset ansatz for integrability in AdS/CFT," JHEP **1206**, 083 (2012) [arXiv:1204.2153 [hep-th]].

[7] R. de Mello Koch, S. Ramgoolam and C. Wen, "On the refined counting of graphs on surfaces," Nucl. Phys. B **870**, 530 (2013) [arXiv:1209.0334 [hep-th]].

[8] J. Pasukonis and S. Ramgoolam, "Quivers as Calculators: Counting, Correlators and Riemann Surfaces," JHEP **1304**, 094 (2013) [arXiv:1301.1980 [hep-th]].

[9] J. B. Geloun and S. Ramgoolam, "Counting Tensor Model Observables and Branched Covers of the 2-Sphere," arXiv:1307.6490 [hep-th].

[10] R. d. M. Koch and S. Ramgoolam, "CFT4 as SO(4,2)-invariant TFT2," arXiv:1403.6646 [hep-th].

[11] D. J. Gross, "Two-dimensional QCD as a string theory," Nucl. Phys. B **400**, 161 (1993) [hep-th/9212149].

[12] D. J. Gross and W. Taylor, "Two-dimensional QCD is a string theory," Nucl. Phys. B **400**, 181 (1993) [hep-th/9301068].

[13] D. J. Gross and W. Taylor, "Twists and Wilson loops in the string theory of two-dimensional QCD," Nucl. Phys. B **403**, 395 (1993) [hep-th/9303046].

[14] S. Cordes, G. W. Moore and S. Ramgoolam, "Large N 2-D Yang-Mills theory and topological string theory," Commun. Math. Phys. **185**, 543 (1997) [hep-th/9402107].

[15] R. Gopakumar, "What is the Simplest Gauge-String Duality?," arXiv:1104.2386 [hep-th].

[16] R. Gopakumar, "From free fields to AdS," Phys. Rev. D **70**, 025009 (2004) [hep-th/0308184].

[17] R. Gopakumar, "From free fields to AdS. 2.," Phys. Rev. D **70**, 025010 (2004) [hep-th/0402063].

[18] R. Gopakumar, "From free fields to AdS: III," Phys. Rev. D **72**, 066008 (2005) [hep-th/0504229].

[19] S. S. Razamat, "On a worldsheet dual of the Gaussian matrix model," JHEP **0807**, 026 (2008) [arXiv:0803.2681 [hep-th]].

[20] S. S. Razamat, "From Matrices to Strings and Back," JHEP **1003**, 049 (2010) [arXiv:0911.0658 [hep-th]].

[21] M. Marino, "Les Houches lectures on matrix models and topological strings," hep-th/0410165.

[22] A. Okounkov and R. Pandharipande, "The Equivariant Gromov-Witten theory of P^1," math/0207233 [math-ag].

[23] T. Eguchi, K. Hori and S. K. Yang, "Topological sigma models and large N matrix integral," Int. J. Mod. Phys. A **10**, 4203 (1995) [hep-th/9503017].

[24] T. Eguchi and S. K. Yang, "The Topological CP1 model and the large N matrix integral," Mod. Phys. Lett. A **9**, 2893 (1994) [hep-th/9407134].

[25] T. Eguchi, K. Hori and C. S. Xiong, "Gravitational quantum cohomology," Int. J. Mod. Phys. A **12**, 1743 (1997) [hep-th/9605225].

[26] R. Gopakumar and R. Pius, "Correlators in the Simplest Gauge-String Duality," JHEP **1303**, 175 (2013) [arXiv:1212.1236].

[27] W. T. Tutte, "A Census of slicings", Can. J. Math. 14, 708-722 (1962).

[28] K. Hori, "Constraints for topological strings in D $\geq$ 1," Nucl. Phys. B **439**, 395 (1995) [hep-th/9411135].

[29] M. Mulase, M. Penkava, "Ribbon Graphs, Quadratic Differentials on Riemann Surfaces, and Algebraic Curves Defined over $\bar{Q}$," [arXiv:math-ph/9811024].

[30] R. de Mello Koch and L. Nkumane, " Topological String Correlators from Matrix Models" [arXiv:1411.5226], accepted for publication in JHEP.