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# Self-adjoint higher order differential operators with eigenvalue parameter dependent boundary conditions

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**Abstract**

Eigenvalue problems for even order regular quasi-differential equations with boundary conditions which depend linearly on the eigenvalue parameter  $\lambda$  can be represented by an operator polynomial  $L(\lambda) = \lambda^2 M - i\lambda K - A$ , where  $M$  is a self-adjoint operator. Necessary and sufficient conditions are given such that also  $K$  and  $A$  are self-adjoint.

**MSC:** Primary 34B07; secondary 34L99

**Keywords:** higher order differential operator; eigenvalue dependent boundary conditions; self-adjoint boundary conditions; quasi-derivative

## 1 Introduction

In order to solve linear partial differential equations of the form

$$\frac{\partial^2 u}{\partial t^2} + \mathcal{A}u = 0,$$

where  $\mathcal{A}$  is a linear differential operator with respect to the variable  $x$  on an interval  $I$ , the separation of variables method  $u(x, t) = y(x)e^{i\omega t}$  leads to

$$\omega^2 y = \mathcal{A}y.$$

For  $t$ -independent boundary conditions  $Bu = 0$ , setting  $\lambda = \omega^2$ , the operator theoretic realization leads to an eigenvalue problem for an operator  $A$  in the Lebesgue space  $L^2(I)$  with domain

$$\mathcal{D}(A) = \{y \in L^2(I) : \mathcal{A}y \in L^2(I), By = 0\}.$$

Such problems are well studied, and of particular importance is the case that  $A$  is self-adjoint. Many applications in physics and engineering can be represented by such self-adjoint operators.

However, problems like the Regge problem and the vibrating beam problem have boundary conditions with partial first order derivatives with respect to  $t$  or whose mathematical

model leads to an eigenvalue problem with the eigenvalue parameter  $\lambda = \omega$  occurring linearly in the boundary conditions. Such problems have an operator representation of the form

$$L(\lambda) = \lambda^2 M - i\lambda K - A \tag{1.1}$$

in a Hilbert space  $H = L^2(I) \oplus \mathbb{C}^k$ , where  $k$  is the number of eigenvalue dependent boundary conditions.

In general, the spectrum of  $L$  is no longer real but still has some particularly nice properties if  $K, M, A$  are self-adjoint with  $M \geq 0$  and  $K \geq 0$ , the resolvent set of  $L$  is nonempty, and  $L$  has a compact resolvent: it is symmetric with respect to the imaginary axis and eigenvalues with negative imaginary parts must lie on the imaginary axis. In this situation, the operators  $M$  and  $K$  are quite simple bounded self-adjoint operators. However, the operator  $A$  is determined by three ingredients: the differential equation  $\mathcal{A}$ , the parameter independent boundary conditions as homogeneous boundary conditions for  $A$ , and the parameter dependent boundary conditions as an inhomogeneous part of  $A$ . Hence one cannot make use of the criteria for self-adjointness in the case of parameter independent boundary conditions. Rather, the parameter dependent case is a proper extension of the parameter independent case.

For parameter independent boundary conditions, *i.e.*,  $k = 0$ , characterizations of self-adjointness for  $A$  in the case of formally symmetric even order quasi-differential expressions are known both for the regular and the singular cases, see [1] and in particular [1], Theorem 6 for the regular case. The simplest formulation of these self-adjointness conditions makes use of quasi-derivatives, and we will henceforth mostly use quasi-derivatives  $y^{[j]}$  rather than derivatives  $y^{(j)}$ . For the definition of the quasi-derivatives  $y^{[j]}$ , we refer the reader to (2.2)-(2.5), see also Remark 3.2.

Some special cases of self-adjoint boundary conditions for regular  $2n$ th order differential equations with  $k > 0$  are known. In [2], the second order problem related to the Regge problem was investigated, whereas the fourth order differential equation  $y^{(4)} - (gy)'$  related to a vibrating beam was dealt with in [3], where the boundary conditions are of the form

$$B_j(\lambda)y = y^{[p_j]}(a_j) + \lambda\beta_j y^{[q_j]}(a_j), \quad j = 1, \dots, 4, \tag{1.2}$$

with exactly one boundary condition depending on  $\lambda$ . A classification of all self-adjoint boundary conditions of the form (1.2) was obtained in [4]. A corresponding result for sixth order differential equations was given in [5].

In this paper we consider  $2n$ th order quasi-differential equations and derive necessary and sufficient conditions for  $2n$  boundary conditions of the form (1.2) to generate self-adjoint operators  $K$  and  $A$ .

In Section 2 we give a precise definition of the boundary value problem and the quadratic operator pencil  $L$  associated with it. In Section 3 we derive necessary and sufficient conditions for  $K$  to be self-adjoint and for  $A$  to be symmetric. In Section 4 it is shown that  $A$  is self-adjoint if  $A$  is symmetric.

## 2 The eigenvalue problem

We first summarize some basic facts about quasi-differential equations for the convenience of the reader. For a more comprehensive discussion of quasi-differential equations,

the reader is referred to [6] and to [7] in the scalar case and to [8, 9] for the general case with matrix coefficients.

Let  $I = (a, b)$  be an interval with  $-\infty < a < b < \infty$ , and let  $m$  be a positive integer. For a given set  $S$ ,  $M_m(S)$  denotes the set of  $m \times m$  matrices with entries from  $S$ . Let

$$Z_m(I) := \left\{ G = (g_{r,s})_{r,s=1}^m \in M_m(L^1(I)), \right. \\ \left. g_{r,r+1} \text{ invertible a.e. for } 1 \leq r \leq m-1, g_{r,s} = 0 \text{ for } 2 \leq r+1 < s \leq m \right\}, \tag{2.1}$$

where  $L^1(I)$  denotes the complex-valued Lebesgue integrable functions on  $I$ .

For  $G \in Z_m(I)$ , define

$$Q_0 := \{y : I \rightarrow \mathbb{C}, y \text{ measurable}\} \tag{2.2}$$

and

$$y^{[0]} := y, \quad y \in Q_0. \tag{2.3}$$

Inductively, for  $r = 1, \dots, m$ , we define

$$Q_r = \{y \in Q_{r-1} : y^{[r-1]} \in AC(I)\}, \tag{2.4}$$

$$y^{[r]} = g_{r,r+1}^{-1} \left( y^{[r-1]'} - \sum_{s=1}^r g_{r,s} y^{[s-1]} \right), \quad y \in Q_r, \tag{2.5}$$

where  $g_{m,m+1} := 1$  and where  $AC(I)$  denotes the set of complex-valued functions which are absolutely continuous on  $I$ . Finally we set

$$\mathcal{A}y := i^m y^{[m]}, \quad y \in Q_m. \tag{2.6}$$

The expression  $\mathcal{A} = \mathcal{A}_G$  is called the quasi-differential expression associated with  $G$ , and the function  $y^{[r]}$ ,  $0 \leq r \leq m$ , is called the  $r$ th quasi-derivative of  $y$ . We also write  $\mathcal{D}(\mathcal{A})$  for  $Q_m$ .

Observe that the quasi-derivatives defined in (2.5) depend on  $G$ . However, since we are only going to deal with a single quasi-differential equation, we will not indicate this dependence explicitly.

In the remainder of the paper, we assume that  $m = 2n$  is an even positive integer, that  $G = (g_{r,s})_{r,s=1}^{2n} \in Z_{2n}(I)$ , and that  $w : I \rightarrow \mathbb{R}$  is positive a.e. and satisfies  $w \in L^1(I)$ .

Together with (2.6) we consider the boundary conditions  $B_j(\lambda)y = 0$ ,  $j = 1, \dots, 2n$ , taken at the endpoint  $a$  for  $j = 1, \dots, n$  and at the endpoint  $b$  for  $j = n + 1, \dots, 2n$ . We assume for simplicity that

$$B_j(\lambda)y = y^{[p_j]}(a_j) + i\lambda\beta_j y^{[q_j]}(a_j), \tag{2.7}$$

where  $a_j = a$  for  $j = 1, \dots, n$ ,  $a_j = b$  for  $j = n + 1, \dots, 2n$ ,  $\beta_j \in \mathbb{C}$  and  $0 \leq p_j, q_j \leq 2n - 1$ . Of course, the numbers  $q_j$  are ambiguous and irrelevant in case  $\beta_j = 0$ .

The differential expression (2.6) and the boundary conditions (2.7) define the eigenvalue problem

$$(-1)^n y^{[2n]} = \lambda^2 w y, \tag{2.8}$$

$$B_j(\lambda)y = 0, \quad j = 1, \dots, 2n. \tag{2.9}$$

We put

$$\Theta_1 = \{j \in \{1, \dots, 2n\} : \beta_j \neq 0\}, \quad \Theta_0 = \{1, \dots, 2n\} \setminus \Theta_1,$$

$$\Theta_r^a = \Theta_r \cap \{1, \dots, n\}, \quad \Theta_r^b = \Theta_r \cap \{n + 1, \dots, 2n\}, \quad \text{for } r = 0, 1,$$

and

$$k = |\Theta_1|. \tag{2.10}$$

**Assumption 2.1** We assume that the numbers  $p_1, \dots, p_n, q_j$  for  $j \in \Theta_1^a$  are distinct and that the numbers  $p_{n+1}, \dots, p_{2n}, q_j$  for  $j \in \Theta_1^b$  are distinct.

Assumption 2.1 means that for any pair  $(r, a_j)$  the term  $y^{[r]}(a_j)$  occurs at most once in the boundary conditions (2.7).

For  $j \in \Theta_1$ , we choose  $\alpha_j \in \mathbb{R}$  and  $\varepsilon_j \in \mathbb{C}$  such that  $\beta_j = \alpha_j \varepsilon_j$ .

For  $y \in \mathcal{D}(\mathcal{A})$ , we define  $Y_R = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix}$  with  $Y = (y^{[0]}, y^{[1]}, \dots, y^{[2n-1]})^T$ . We denote the collection of the  $2n$  boundary conditions (2.9) by  $U$  and define the following matrices related to  $U$ :

$$U_r Y_R = (y^{[p_j]}(a_j))_{j \in \Theta_r}, \quad r = 0, 1,$$

$$V_1 Y_R = (\varepsilon_j y^{[q_j]}(a_j))_{j \in \Theta_1},$$

where  $y \in \mathcal{D}(\mathcal{A})$ . (2.11)

**Remark 2.2** In case that  $\Theta_r = \emptyset$  for  $r = 0$  or  $r = 1$ , the corresponding matrix  $U_r$  will be identified with the ‘zero’ operator from  $\mathbb{C}^{2n}$  into  $\{0\}$ .

The weighted Lebesgue space  $L^2(I, w)$  is the Hilbert space of all equivalence classes of complex-valued measurable functions  $f$  such that  $(f, f)_w := \int_I w(x)|f(x)|^2 dx < \infty$ . For convenience we define the operator  $\mathcal{A}_{\max}$  on  $L^2(I, w)$  by

$$\mathcal{D}(\mathcal{A}_{\max}) = \{y \in L^2(I, w) : w^{-1} \mathcal{A}y \in L^2(I, w)\}, \quad \mathcal{A}_{\max} y = w^{-1} \mathcal{A}y.$$

We will associate the quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\lambda K - A(U) \tag{2.12}$$

in the space  $L^2(I, w) \oplus \mathbb{C}^k$  with problem (2.8), (2.9), where

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ 0 & K_0 \end{pmatrix} \quad \text{with } K_0 = \text{diag}(\alpha_j : j \in \Theta_1).$$

The operator  $A(U)$  in  $L^2(I, w) \oplus \mathbb{C}^k$  is defined by

$$\mathcal{D}(A(U)) = \left\{ \tilde{y} = \begin{pmatrix} y \\ V_1 Y_R \end{pmatrix} : y \in \mathcal{D}(\mathcal{A}_{\max}), U_0 Y_R = 0 \right\},$$

$$(A(U))\tilde{y} = \begin{pmatrix} \mathcal{A}_{\max} y \\ U_1 Y_R \end{pmatrix}, \quad \tilde{y} \in \mathcal{D}(A(U)).$$

It is easy to see that a function  $y \in \mathcal{D}(\mathcal{A}_{\max})$  satisfies  $\mathcal{A}y = \lambda^2 w y$  and  $B_j(\lambda)y = 0$  for  $j = 1, \dots, 2n$  if and only if there is  $c \in \mathbb{C}^k$  such that  $(y, c)^T \in \mathcal{D}(A(U))$  such that  $L(\lambda)(y, c)^T = 0$ . In this case  $c$  is uniquely determined by  $y$ . Indeed, if  $y \in \mathcal{D}(\mathcal{A}_{\max})$  with  $\mathcal{A}y = \lambda^2 w y$  and  $B_j(\lambda)y = 0$  for  $j = 1, \dots, 2n$ , then  $U_0 Y_R = 0$  shows that  $(y, V_1 Y_R)^T \in \mathcal{D}(A(U))$  and

$$L(\lambda) \begin{pmatrix} y \\ V_1 Y_R \end{pmatrix} = \begin{pmatrix} \lambda^2 y - \mathcal{A}_{\max} y \\ -i\lambda K_0 V_1 Y_R - U_1 Y_R \end{pmatrix}.$$

Clearly, the first component is 0, and so is the second component since

$$i\lambda K_0 V_1 Y_R + U_1 Y_R = i\lambda K_0 (\varepsilon_j y^{[q_j]}(a_j))_{j \in \Theta_1} + (y^{[p_j]}(a_j))_{j \in \Theta_1} = (B_j(\lambda)y)_{j \in \Theta_1}.$$

Hence the operator pencil  $L$  is an operator realization of the eigenvalue problem (2.8), (2.9).

It is clear that  $M$  and  $K$  are bounded self-adjoint operators and that  $M$  is non-negative. The operator  $A(U)$  is not self-adjoint, in general, and we will give necessary and sufficient conditions for the operator  $A(U)$  to be self-adjoint.

### 3 Symmetry conditions for $A(U)$

We will denote the canonical inner product in  $L^2(I, w) \oplus \mathbb{C}^k$  by  $\langle \cdot, \cdot \rangle$ .

The Lagrange form of  $A(U)$  is defined by

$$F_U(\tilde{y}, \tilde{z}) = \langle A(U)\tilde{y}, \tilde{z} \rangle - \langle \tilde{y}, A(U)\tilde{z} \rangle, \quad \tilde{y}, \tilde{z} \in \mathcal{D}(A(U)).$$

The operator  $A(U)$  is symmetric if and only if its Lagrange form is identically zero. For this it is necessary that  $\mathcal{A}$  is formally symmetric, and for the remainder of this paper we make therefore the following assumption.

**Assumption 3.1** We assume that

$$G = -CG^*C,$$

where

$$C = ((-1)^r \delta_{r, 2n+1-s})_{r,s=1}^{2n} \tag{3.1}$$

and  $\delta$  is the Kronecker delta.

It is easy to verify that Assumption 3.1 holds if and only if

$$g_{r,s} = (-1)^{r+s+1} \overline{g}_{2n+1-s, 2n+1-r}, \quad r, s = 1, \dots, 2n. \tag{3.2}$$

**Remark 3.2** Classical formally self-adjoint differential expressions are of the form

$$(-1)^n \sum_{j=0}^n (g_j y^{(j)})^{(j)}$$

with  $g_j \in C^j[0, a]$  for  $j = 0, \dots, n$  and invertible  $g_n$ . It is easy to verify that this is a quasi-differential equation with quasi-derivatives

$$\begin{aligned} y^{[r]} &= y^{(r)}, \quad r = 0, \dots, n-1, \\ y^{[n]} &= g_n y^{(n)}, \\ y^{[r]} &= y^{[r-1]'} + g_{2n-r} y^{[2n-r]}, \quad r = n+1, \dots, 2n. \end{aligned}$$

The corresponding matrix  $G = (g_{r,s})_{r,s=1}^{2n}$  has the entries  $g_{r,r+1} = 1$  for  $r = 1, \dots, n-1$  and  $r = n+1, \dots, 2n-1$ ,  $g_{n,n+1} = g_n^{-1}$ ,  $g_{r,2n-r+1} = -g_{2n-r}$  for  $r = n+1, \dots, 2n$ , while all other entries are zero. It is easy to see that Assumption 3.1 holds in this case if and only if  $g_j = \overline{g_j}$  for  $j = 0, \dots, n$ , so that the formal self-adjointness condition reduces to the well-known condition that all  $g_j$ ,  $j = 0, \dots, n$ , are real-valued functions.

From [10], Lemma 3.3 we know that the Lagrange identity

$$(w^{-1}Ay, z)_w - (y, w^{-1}Az)_w = Z_R^*DY_R, \quad y, z \in \mathcal{D}(A_{\max}) \tag{3.3}$$

holds, where

$$D = (-1)^n \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}. \tag{3.4}$$

**Proposition 3.3** *The Lagrange form  $F_U$  of  $A(U)$  has the representation*

$$F_U(\tilde{y}, \tilde{z}) = Z_R^*WY_R, \quad \tilde{y}, \tilde{z} \in \mathcal{D}(A(U)),$$

where

$$W = D + (V_1^*U_1 - U_1^*V_1). \tag{3.5}$$

*Proof* Let  $\tilde{y}, \tilde{z} \in \mathcal{D}(A(U))$ . Then

$$F_U(\tilde{y}, \tilde{z}) = (w^{-1}Ay, z)_w + (V_1Z_R)^*U_1Y_R - (y, w^{-1}Az)_w - (U_1Z_R)^*V_1Y_R,$$

and an application of the Lagrange identity (3.3) completes the proof of the lemma.  $\square$

By definition, an operator in a Hilbert space is symmetric if and only if its Lagrange form is identically zero. Hence we have the following.

**Corollary 3.4** *The differential operator  $A(U)$  is symmetric if and only if  $Z_R^* W Y_R = 0$  for all  $\tilde{y}, \tilde{z} \in \mathcal{D}(A(U))$ .*

The nullspace and range of a matrix  $M$  are denoted by  $N(M)$  and  $R(M)$ , respectively.

**Proposition 3.5** *The differential operator  $A(U)$  is symmetric if and only if  $W(N(U_0)) \subset (N(U_0))^\perp$ .*

*Proof* From [10], Corollary 5.5 we know that

$$\{Y_R : y \in \mathcal{D}(\mathcal{A}_{\max})\} = \mathbb{C}^{4n}. \tag{3.6}$$

Hence  $\{Y_R : \tilde{y} \in \mathcal{D}(A(U))\} = N(U_0)$ . An application of Proposition 3.4 completes the proof. □

**Corollary 3.6** *If  $A(U)$  is symmetric, then  $\text{rank } W = 2(2n - k)$  and  $W(N(U_0)) = (N(U_0))^\perp$ .*

*Proof* Since  $\dim(N(U_0))^\perp = \text{rank } U_0 = 2n - k$ , we have

$$2n - k \geq \dim W(N(U_0)) \geq \dim N(U_0) - (4n - \text{rank } W) = -2n + k + \text{rank } W. \tag{3.7}$$

Hence  $\text{rank } W \leq 2(2n - k)$ . Since  $V_1^* U_1 - U_1^* V_1$  has  $2k$  non-zero entries and  $D$  is invertible,  $\text{rank } W \geq 2(2n - k)$  and  $\text{rank } W = 2(2n - k)$  follows. In this case, all the inequalities of (3.7) are equalities and  $\dim W(N(U_0)) = \dim(N(U_0))^\perp$  holds. Thus it follows from Proposition 3.5 that  $W(N(U_0)) = (N(U_0))^\perp$ . □

In view of Corollary 3.6, we may assume that  $\text{rank } W = 2(2n - k)$  when investigating the symmetry of  $A(U)$ . Since  $(N(U_0))^\perp = R(U_0^*)$ , see [11], Theorem IV.5.13, Proposition 3.5 and Corollary 3.6 lead to the following.

**Corollary 3.7** *Let  $\text{rank } W = 2(2n - k)$ . Then the differential operator  $A(U)$  is symmetric if and only if  $W(N(U_0)) = R(U_0^*)$ .*

We now give an explicit description for the condition  $\text{rank } W = 2(2n - k)$ .

**Proposition 3.8**  *$\text{rank } W = 2(2n - k)$  if and only if the following conditions hold:*

1. For  $s \in \Theta_1$ ,  $p_s + q_s = 2n - 1$ ;
2. For  $s \in \Theta_1^{(a)}$ ,  $\varepsilon_s = (-1)^{q_s+n}$ ;
3. For  $s \in \Theta_1^{(b)}$ ,  $\varepsilon_s = (-1)^{q_s+n+1}$ .

*Proof* Note that

$$V_1^* U_1 - U_1^* V_1 = \begin{pmatrix} V_2 & 0 \\ 0 & V_3 \end{pmatrix}, \tag{3.8}$$

where

$$V_2 = \sum_{s \in \Theta_1^{(a)}} (\overline{\varepsilon_s} \delta_{i,q_s+1} \delta_{j,p_s+1} - \varepsilon_s \delta_{i,p_s+1} \delta_{j,q_s+1})_{i,j=1}^{2n},$$

$$V_3 = \sum_{s \in \Theta_1^{(b)}} (\overline{\varepsilon_s} \delta_{i,q_s+1} \delta_{j,p_s+1} - \varepsilon_s \delta_{i,p_s+1} \delta_{j,q_s+1})_{i,j=1}^{2n}.$$

Since  $D$  has exactly one non-zero entry in each row and column and  $V_1^* V_0 - V_0^* V_1$  has exactly  $2k$  non-zero entries, it follows that  $\text{rank } W = 2(2n - k)$  if and only if each non-zero entry of  $V_2$  cancels a non-zero entry of  $(-1)^{n-1} C$  and each non-zero entry of  $V_3$  cancels a non-zero entry of  $(-1)^n C$ . Since the non-zero entries of  $C$  are in rows  $i$  and columns  $j$  such that  $i + j = 2n + 1$ , we obtain that  $\text{rank } W = 2(2n - k)$  if and only if conditions 1, 2, and 3 are satisfied.  $\square$

**Corollary 3.9** *The boundary eigenvalue problem (2.8), (2.9) has an operator pencil representation (2.12) with self-adjoint operator  $K$  and symmetric operator  $A(U)$  if and only if*

1.  $\beta_j \in \mathbb{R}$  and  $p_j + q_j = 2n - 1$  for all  $j \in \Theta_1$ ;
2.  $W(N(U_0)) = R(U_0^*)$ .

*Proof* We have seen in Proposition 3.8 that three sets of conditions have to be satisfied in order that the necessary condition  $\text{rank } W = 2(2n - k)$  for symmetry of  $A(U)$  holds. Conditions 2 and 3 can always be satisfied if we put  $\alpha_j = \beta_j (-1)^{q_s+n}$  for  $j \in \Theta_1^a$  and  $\alpha_j = \beta_j (-1)^{q_s+n+1}$  for  $j \in \Theta_1^b$ , and for  $K$  to be self-adjoint it is therefore necessary and sufficient that  $\beta_j$  are real. The remaining conditions now follow easily from Proposition 3.8 and Corollary 3.7.  $\square$

We could now give explicit conditions for symmetry of  $A(U)$  in terms of the boundary conditions (2.7). However, we will see in the next section that  $A(U)$  is self-adjoint if and only if it is symmetric. In order to avoid duplication we will therefore postpone deriving these explicit conditions to the next section.

#### 4 Self-adjointness conditions for $A(U)$

From Corollary 3.9 we know that for self-adjointness of  $K$  and  $A(U)$  the condition  $\beta_j \in \mathbb{R}$  for all  $j \in \Theta_1$  is necessary. Hence we require without loss of generality that the numbers  $\varepsilon_s$  for  $s \in \Theta_1$  are chosen as in Proposition 3.8, conditions 2 and 3.

**Assumption 4.1** For  $s \in \Theta_1^{(a)}$ , let  $\varepsilon_s = (-1)^{q_s+n}$ , and for  $s \in \Theta_1^{(b)}$ , let  $\varepsilon_s = (-1)^{q_s+n+1}$ .

For convenience, we set

$$\begin{aligned} \tilde{p}_j &= p_j + 1, \tilde{q}_j = q_j + 1 \quad \text{for } j = 1, \dots, n, \\ \tilde{p}_j &= p_j + 2n + 1, \tilde{q}_j = q_j + 2n + 1 \quad \text{for } j = n + 1, \dots, 2n. \end{aligned}$$

The range  $R(U_r^*)$  of  $U_r^*$  for  $r = 0, 1$  is the span of all standard unit vectors  $e_{\tilde{p}_j}$  in  $\mathbb{C}^{4n}$  with  $j \in \Theta_r$ , and  $R(V_1^*)$  is the span of all standard unit vectors  $e_{\tilde{q}_j}$  in  $\mathbb{C}^{4n}$  with  $j \in \Theta_1$ . Hence it follows from Assumptions 2.1 and 4.1 that

$$U_0 U_0^* = \text{id}_{\mathbb{C}^{2n-k}}, \quad U_1 U_1^* = \text{id}_{\mathbb{C}^k}, \quad V_1 V_1^* = \text{id}_{\mathbb{C}^k}, \tag{4.1}$$

$$U_1 U_0^* = 0, \quad V_1 U_0^* = 0, \quad U_1 V_1^* = 0. \tag{4.2}$$



**Theorem 4.2** *The operator  $A(U)$  is densely defined, the domain  $\mathcal{D}((A(U))^*)$  of its adjoint  $(A(U))^*$  is the set of all  $\tilde{z} = \begin{pmatrix} z \\ d \end{pmatrix}$  in  $L^2(I, w) \oplus \mathbb{C}^k$  such that there is  $c \in \mathbb{C}^k$  such that  $z \in \mathcal{D}(\mathcal{A}_{\max})$  and*

$$D^*Z_R + U_1^*d - V_1^*c \in R(U_0^*). \tag{4.3}$$

For  $\tilde{z} = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathcal{D}((A(U))^*)$ , the vectors  $d$  and  $c$  are uniquely determined by  $z$ , namely,  $d = -U_1D^*Z_R$  and  $c = V_1D^*Z_R$ , and

$$(A(U))^* \tilde{z} = \begin{pmatrix} \mathcal{A}_{\max}z \\ V_1D^*Z_R \end{pmatrix}. \tag{4.4}$$

*Proof* By definition of the adjoint (possibly as a linear relation),  $\tilde{z} = \begin{pmatrix} z \\ d \end{pmatrix} \in L^2(I, w) \oplus \mathbb{C}^k$  belongs to  $\mathcal{D}((A(U))^*)$  if and only if there is  $\tilde{u} = \begin{pmatrix} u \\ c \end{pmatrix} \in L^2(I, w) \oplus \mathbb{C}^k$  such that for all  $\tilde{y} = \begin{pmatrix} y \\ V_1Y_R \end{pmatrix} \in \mathcal{D}(A(U))$  the identity

$$\langle A(U)\tilde{y}, \tilde{z} \rangle = \langle \tilde{y}, \tilde{u} \rangle \tag{4.5}$$

holds.

Hence let  $\tilde{z}, \tilde{u} \in L^2(I, w) \oplus \mathbb{C}^k$  such that (4.5) holds for all  $\tilde{y} \in \mathcal{D}(A(U))$ . If  $y$  has compact support in  $I$ , then (4.5) reduces to

$$(\mathcal{A}_{\max}y, z)_w = (y, u)_w.$$

This, the formal symmetry Assumption 3.1 and [10], Theorem 4.2 show that  $z \in \mathcal{D}(\mathcal{A}_{\max})$  and  $\mathcal{A}_{\max}z = u$ . We can now conclude that (4.5) holds if and only if

$$(\mathcal{A}_{\max}y, z)_w + d^*U_1Y_R = (y, \mathcal{A}_{\max}z)_w + c^*V_1Y_R.$$

In view of the Lagrange identity (3.3), the above is equivalent to

$$Z_R^*DY_R + d^*U_1Y_R = c^*V_1Y_R.$$

Since the range of all  $Y_R$  with  $y \in \mathcal{D}(A(U))$  is  $N(U_0)$ , it follows that (4.5) is equivalent to  $z \in \mathcal{D}(\mathcal{A}_{\max})$ ,  $u = \mathcal{A}_{\max}z$  and

$$D^*Z_R + U_1^*d - V_1^*c \in N(U_0)^\perp = R(U_0^*). \tag{4.6}$$

Applying  $U_1$  and  $V_1$ , respectively, to (4.6) and observing (4.1) and (4.2) it follows that  $d$  and  $c$  are uniquely given by  $d = -U_1D^*Z_R$  and  $c = V_1D^*Z_R$ . From the uniqueness of  $u$  and  $c$  we see that  $(A(U))^*$  is not only a linear relation but a linear operator, so that  $A(U)$  is densely defined. □

**Remark 4.3** The matrix  $D$  is invertible and

$$D^{-1} = -D = D^*, \tag{4.7}$$

see [8], (2.7).

**Proposition 4.4** *Assume that  $\text{rank } W = 2(2n - k)$ . Then  $U_1D = V_1$  and  $V_1D = -U_1$ .*

*Proof* By definition of  $U_1$  and  $D$  we can write

$$U_1D = (-1)^n \begin{pmatrix} U_1^a C & 0 \\ 0 & -U_1^b C \end{pmatrix},$$

where  $U_1^\alpha = (\delta_{j,p_i+1})_{i \in \Theta_1^\alpha, j=1, \dots, 2n}$  for  $\alpha = a, b$ . In view of Proposition 3.8 we conclude that

$$\begin{aligned} U_1^\alpha C &= (\delta_{2n+1-j,p_i+1} (-1)^{p_i+1})_{i \in \Theta_1^\alpha, j=1, \dots, 2n} \\ &= (\delta_{j,q_i+1} (-1)^{q_i})_{i \in \Theta_1^\alpha, j=1, \dots, 2n}. \end{aligned}$$

Hence  $U_1D = V_1$ , and (4.7) gives  $V_1D = U_1D^2 = -U_1$ . □

**Proposition 4.5** *If  $A(U)$  is symmetric, then  $A(U)$  is self-adjoint.*

*Proof* We have to show that  $\mathcal{D}((A(U))^*) \subset \mathcal{D}(A(U))$ . By Theorem 4.2,  $\mathcal{D}((A(U))^*)$  is the set of all  $\begin{pmatrix} z \\ V_1 Z_R \end{pmatrix}$  such that  $z \in \mathcal{D}(\mathcal{A}_{\max})$  and  $D^*Z_R + U_1^*d - V_1c \in R(U_0^*)$ . But Theorem 4.2, Proposition 4.4 and (4.7) imply

$$\begin{aligned} D^*Z_R - V_1^*c + U_1^*d &= D^*Z_R - V_1^*V_1D^*Z_R - U_1^*U_1D^*Z_R \\ &= -DZ_R - V_1^*U_1Z_R + U_1^*V_1Z_R \\ &= -WZ_R, \end{aligned}$$

so that  $\mathcal{D}((A(U))^*) \subset \mathcal{D}(A(U))$  if and only if  $W^{-1}(R(U_0^*)) \subset N(U_0)$ .

We know that  $\text{rank } U_0 = 2n - k$  and  $\dim N(U_0) = 4n - \text{rank } U_0 = 2n + k$ , whereas  $\dim N(W) = 4n - \text{rank } W = 2k$  by Corollary 3.6. Altogether, we conclude

$$\dim W^{-1}(R(U_0^*)) \leq \dim N(W) + \text{rank } U_0 = 2n + k = \dim N(U_0).$$

But from Corollary 3.7 we conclude that  $N(U_0) \subset W^{-1}(R(U_0^*))$ , and it follows that  $N(U_0) = W^{-1}(R(U_0^*))$ . □

**Proposition 4.6** *Assume  $\text{rank } W = 2(2n - k)$ . Then  $W(N(U_0)) = R(U_0^*)$  if and only if*

- (i)  $p_s + p_r \neq 2n - 1$  for all  $r, s \in \Theta_0^a$ ,
- (ii)  $p_s + p_r \neq 2n - 1$  for all  $r, s \in \Theta_0^b$ .

*Proof* Defining for  $c = a, b$ ,

$$\begin{aligned} M_c &= \text{span}\{e_{p_j+1} : j \in \Theta_0^c\} \subset \mathbb{C}^{2n}, \quad c = a, b, \\ N_c &= \mathbb{C}^{2n} \ominus M_c = \text{span}\{e_j : j \in \{1, \dots, 2n\} \setminus \{p_s + 1 : s \in \Theta_0^c\}\} \subset \mathbb{C}^{2n}, \\ W_a &= (-1)^n C + V_2, \quad W_b = (-1)^{n+1} C + V_3, \end{aligned}$$

where  $V_2$  and  $V_3$  are as in (3.7), it follows that

$$R(U_0^*) = \left\{ \begin{pmatrix} u_a \\ u_b \end{pmatrix} : u_a \in M_a, u_b \in M_b \right\}, \quad N(U_0) = \left\{ \begin{pmatrix} u_a \\ u_b \end{pmatrix} : u_a \in N_a, u_b \in N_b \right\},$$

and

$$W = D + V_1^* U_1 - U_1^* V_1 = \begin{pmatrix} W_a & 0 \\ 0 & W_b \end{pmatrix}$$

in view of (3.5) and (3.8). Therefore  $W(N(U_0)) = R(U_0^*)$  if and only if  $W_c(N_c) = M_c$  for  $c = a, b$ . Now let  $c \in \{a, b\}$ . From Proposition 3.8 and its proof we find for  $j \in \{1, \dots, 2n\}$  that

$$W_c(e_j) = \begin{cases} \pm e_{2n+1-j} & \text{if } j \in \{1, \dots, 2n\} \setminus \{p_s + 1, q_s + 1 : s \in \Theta_1^c\}, \\ 0 & \text{if } j \in \{p_s + 1, q_s + 1 : s \in \Theta_1^c\}. \end{cases}$$

Observing condition 1 in Proposition 3.8 it follows that

$$\begin{aligned} W_c(N_c) &= \text{span}\{e_{2n+1-j} : j \in \{1, \dots, 2n\} \\ &\quad \setminus (\{p_s + 1, q_s + 1 : s \in \Theta_1^c\} \cup \{p_s + 1 : s \in \Theta_0^c\})\} \\ &= \text{span}\{e_j : j \in \{1, \dots, 2n\} \\ &\quad \setminus (\{p_s + 1, q_s + 1 : s \in \Theta_1^c\} \cup \{2n - p_s : s \in \Theta_0^c\})\}. \end{aligned}$$

Hence  $W_c(N_c) = M_c$  holds if and only if the sets

$$\Psi_1^c := \{p_s + 1, q_s + 1 : s \in \Theta_1^c\} \cup \{2n - p_s : s \in \Theta_0^c\} \quad \text{and} \quad \Psi_0^c := \{p_s + 1 : s \in \Theta_0^c\}$$

are complementary subsets of  $\{1, \dots, 2n\}$ . But by Assumption 2.1 and condition 1 in Proposition 3.8 the listed elements in  $\Psi_0^c$  as well as in  $\Psi_1^c$  are mutually distinct, so that the sets  $\Psi_0^c$  and  $\Psi_1^c$  are complementary if and only if they are disjoint. It is clear that this latter property holds if and only if  $2n - p_j \notin \Psi_0^c$  for all  $j \in \Theta_0^c$ . This completes the proof of the proposition. □

**Theorem 4.7** *The boundary eigenvalue problem (2.8), (2.9) has an operator pencil representation (2.12) with self-adjoint operators  $K$  and  $A(U)$  if and only if*

1.  $\beta_j \in \mathbb{R}$  and  $p_j + q_j = 2n - 1$  for all  $j \in \Theta_1$ ;
2.  $p_s + p_r \neq 2n - 1$  for all  $r, s \in \Theta_0^a$ ,
3.  $p_s + p_r \neq 2n - 1$  for all  $r, s \in \Theta_0^b$ .

*Proof* This theorem is an immediate consequence of Corollary 3.9 and Propositions 4.5 and 4.6. □

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

Both authors contributed equally to the manuscript and read and approved the final submitted version.

**Acknowledgements**

This research was partially supported by a grant from the NRF of South Africa, grant number 80956.

Received: 22 January 2015 Accepted: 30 April 2015 Published online: 16 May 2015

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