Arithmetic properties of Overpartition functions with combinatorial explorations of Partition inequalities and Partition Configurations

by

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A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy.

Johannesburg, 2017.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

Abdulaziz Alanazi

10th day of February 2017

Dedication

To my sons, Anas and Mohammed

Acknowledgements

Firstly, I would like to express my sincere appreciation to my supervisor Prof. A. O. Munagi for introducing me to Partition Theory and for his continuous support of my PhD studies, as well as his patience and motivation. His guidance assisted me during the complete period of my research and writing of this thesis. I was fortunate to work under his supervision.

In addition, my gratitude to all who contributed in the achievement of this thesis and helped me during my stay in Johannesburg. Particularly, I would like to thank Prof. A. Kara and Dr. D. Nyirenda who made themselves available to me whenever I needed them.

Most importantly I have to acknowledge my mother for her continuous support, encouragement and prayers throughout all of my studies. A special thanks to my wife Dalal for taking care of our family and being my pillar of strength during our stay in South Africa. Many thanks go to all my brothers, sisters and friends, without whom the culmination of this project would have been a much more difficult exercise.

Lastly, this thesis has been made possible through a financial grant conferred by the University of Tabuk in Saudi Arabia. Thank you to the University of the Witwatersrand for giving me the opportunity to pursue my studies.

Abstract

In this thesis, various partition functions with respect to ℓ-regular overpartitions, a special partition inequality and partition configurations are studied.

We explore new combinatorial properties of overpartitions which are natural generalizations of integer partitions. Building on recent work, we state general combinatorial identities between standard partition, overpartition and ℓ-regular partition functions. We provide both generating function and bijective proofs.

We then establish an infinite set of Ramanujan-type congruences for the ℓ-regular overpartitions. This significantly extends the recent work of Shen which focused solely on 3–regular overpartitions and 4–regular overpartitions. We also prove some of the congruences for ℓ-regular overpartition functions combinatorially.

We then provide a combinatorial proof of the inequality $p(a)p(b) > p(a+b)$, where $p(n)$ is the partition function and a, b are positive integers satisfying $a + b > 9$, $a > 1$ and $b > 1$. This problem was posed by Bessenrodt and Ono who used the inequality to study a maximal multiplicative property of an extended partition function.

Finally, we consider *partition configurations* introduced recently by Andrews and Deutsch in connection with the Stanley-Elder theorems. Using a variation of Stanley's original technique, we give a combinatorial proof of the equality of the number of times an integer k appears in all partitions and the number of partition configurations of length k. Then we establish new generalizations of the Elder and configuration theorems. We also consider a related result asserting the equality of the number of $2k$'s in partitions and the number of unrepeated multiples of k, providing a new proof and a generalization.

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Chapter 1

Introduction

A **partition** of a positive integer n is a representation of n as a sum of positive integers. A reordering of summands is not counted as a new partition; thus $2 + 1 + 1, 1 + 2 + 1$, and $1 + 1 + 2$ are considered the same partition of 4. The positive integers in the partition are called **parts**. The number of partitions of n is denoted by $p(n)$. For example, $p(5) = 7$ since the partitions of 5 are: 5, $4+1$, $3+2$, $3+1+1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$ and $1 + 1 + 1 + 1 + 1$. Conventionally, we set $p(0) = 1$ and $p(n) = 0$ for all negative integers n.

One of the most difficult challenges was to determine an explicit formula for $p(n)$. Hardy, Ramanujan and Rademacher answered this question quite completely [6]. Leonard Euler studied partitions intensively. He noted that the coefficient of $q^n z^m$ in the expression $(1+qz)(1+q^2z)(1+q^3z)\cdots$ represented the number of ways n can be written as a sum of m distinct parts. Euler proved one of the most important identities in the theory of integer partitions, which is: *For any positive integer n, the number of partitions of n using only odd parts equals the number of partitions of n into distinct parts.* This is called Euler's identity. For instance, the partitions of 5 into odd parts are: 5, $3 + 1 + 1$ and $1 + 1 + 1 + 1 + 1$ whereas, the partitions of 5 into distinct parts are: 5, $4 + 1$ and $3 + 2$. Euler's identity was generalized by Glaisher (as per [9]). Glaisher proved that: *The number of partitions of an integer n into parts not divisible by $d + 1$ equals the number of partitions of an integer n such that each part appears not more than d times.* In the special case, when $d = 1$, we obtain Euler's identity.

Generating functions were introduced in the theory of partitions by Euler in

1784 [6]. He used generating functions as a tool to discover a number of interesting properties about partitions. He was the first to state that the generating function for $p(n)$ is:

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= (1+q+q^2+\cdots)(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)\ldots \quad (1.1)\\ &= \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\cdots = \prod_{n=1}^{\infty}\frac{1}{1-q^n}.\end{aligned}$$

To explain how it works, we rewrite (1.1) as:

$$\begin{aligned}&(1+q^1+q^{1\cdot 2}+q^{1\cdot 3}+q^{1\cdot 4}+\cdots)\\&(1+q^2+q^{2\cdot 2}+q^{2\cdot 3}+q^{2\cdot 4}+\cdots)\\&(1+q^3+q^{3\cdot 2}+q^{3\cdot 3}+q^{3\cdot 4}+\cdots)\\&(1+q^4+q^{4\cdot 2}+q^{4\cdot 3}+q^{4\cdot 4}+\cdots)\cdots.\end{aligned} \tag{1.2}$$

If we multiply this out and choose one appropriate term from each bracket, we find that the term q^3 is obtained from the following:

$$\begin{aligned}&1\cdot 1\cdot q^3\cdot 1\cdots, q^{2+1}\cdot 1\cdot 1\cdots,\\&q^{1\cdot 3}\cdot 1\cdot 1\cdot 1\cdots.\end{aligned} \tag{1.3}$$

Each of the exponents in (1.3) corresponds to a partition of 3. Consequently, the coefficient of q^n in (1.2) is $p(n)$. Generating functions of restricted partition functions may be derived from (1.2). Let $p(n|\text{condition})$ be the number of partitions of n whose parts satisfy the stated condition. For example:

$$\sum_{n=0}^{\infty} p(n \mid \text{distinct parts})q^n = \prod_{n=1}^{\infty}(1+q^n) \tag{1.4}$$

$$= \sum_{n=0}^{\infty} p(n \mid \text{odd parts})q^n = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n+1}},$$

$$\sum_{n=0}^{\infty} p(n \mid \text{parts not exceeding } m)q^n = \prod_{n=1}^{m}\frac{1}{1-q^n}.$$

Ramanujan studied a table of $p(n)$ for $1 \le n \le 200$ which was computed by P. A. MacMahon [9] and observed that

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.5}$$

He then conjectured the following:

$$p(7n+5) \equiv 0 \pmod{7} \tag{1.6}$$

and

$$p(11n+6) \equiv 0 \pmod{11} \text{ for all } n \geq 1. \tag{1.7}$$

Ramanujan also proved the congruences (1.5), (1.6) and (1.7), see [34] and [35]. Generally, he made comparable conjectures for any modulus of the form $5^a\ 7^b\ 11^c$ for all $a, b, c > 0$. Subsequently, many mathematicians worked on this conjecture before G. N. Watson [39] and A. O. L. Atkin [10] settled the problem. In particular, Ahlgren and Boylan proved that the congruences (1.5), (1.6) and (1.7) are the only ones of the form

$$p(\rho n+\beta) \equiv 0 \pmod{\rho} \tag{1.8}$$

for all $n \in \mathbb{Z}$, ρ prime, and some fixed $\beta \in \mathbb{Z}$ (see [1]).

We now define the types of restricted partitions which are the most relevant for this project. Let ℓ be a positive integer. A partition is called **ℓ-regular** if no part is divisible by ℓ. Hence, Glaisher's generalization of Euler's identity may be rephrased as: *the number of ℓ-regular partitions of n equals the number of partitions of n such that each part appears not more than $\ell-1$ times.*

An *overpartition* of a positive integer n is a partition of n, where the first occurrence of each part-size may be overlined. Overpartitions generalize ordinary partitions. We denote the number of overpartitions of n by $\overline{p}(n)$, with $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ enumerates the following overpartitions:

$$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (\overline{1},1,1).$$

The three overpartitions with no overlined parts are the ordinary partitions of 3. The standard generating function for overpartitions is:

$$\sum_{n=0}^{\infty} \overline{p}(n) = \prod_{n\geq 1} \frac{1+q^n}{1-q^n}.$$

Corteel and Lovejoy [16] introduced overpartitions as a way to understand and interpret various q-series identities. In addition, overpartitions have been used as a tool in bijective proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation in [17] and [15]. The overpartition function has also been studied by several other mathematicians, see [16], [22] , [24], [31] and [32].

In a recent paper [13] Bessenrodt and Ono defined a multiplicative function $\theta(\lambda)$ on $p[n]$ as the product of the parts of λ, where $p[n]$ is the set of all partitions of n. Then they proved that the maximal value of this function is both unique and attainable at a unique partition of n. Their proofs rely on an "inequality for the partition function which seems not to have been noticed before". Later, Beckwith-Bessenrodt [11] proved a similar inequality dealing with ℓ-regular partitions for certain values of ℓ.

The last concept with which we will be concerned is the partition analogue of classical permutation patterns which were introduced by Andrews-Deutsch [8]. A partition λ is said to contain a *partition configuration* $A = (a_1, \ldots, a_k)$ if there is a distinct subsequence of parts of λ of the form $a_1 + j, a_2 + j, \ldots, a_k + j$ for some integer $j > 0$. For example, the partition $(1+2+2+4+4+5+8+9+9)$ contains an instance of $A = (0, 3, 6, 7)$ because the parts $2, 5, 8, 9$ exceed by 2 the successive entries of A.

In 1972, Stanley [38] proved the following identity:

Stanley's Theorem. *The number of occurrences of 1's among all partitions of n equals the number of different parts in all partitions of n.*

This was generalized later as:

Elder's Theorem. *The number of occurrence of k's among all partitions of n equals the number of different parts repeated k or more times in all partitions of n.*

This generalization is attributed to Paul Elder as was reported by Honsberger [26]. He named the theorem after Stanley for the case $k = 1$ and called the general case Elder's theorem.

The three major themes of this thesis are overpartitions, partition inequalities and partition configurations.

In Chapter Two, we provide a review of all the background results and tools that will be required in subsequent chapters.

In Chapter Three, we prove a general theorem which connects a restricted overpartition function with five other restricted ordinary partition functions. We also give an identity for colored partitions which extends the results of the main theorem.

In Chapter Four, our primary goal is to prove families of congruences satisfied by the functions defined in chapter three. The proof techniques used are classical, involving elementary generating function manipulation techniques as well as Ramanujan's theta functions. Then we give new combinatorial proofs of some of the congruence properties.

In Chapter Five, we give a combinatorial proof of the Bessenrodt-Ono [13] inequality. Then we indicate how to tackle the ℓ-regular partition version of the theorem which was formulated by Beckwith and Bessenrodt [11].

In Chapter Six, we give combinatorial proofs of the known major results related to partition configurations and establish new generalizations. The bijective proofs rely mostly on variations of Stanley's proof of Elder's theorem [38]. An extension of one of the results is proved using generating functions. Then we derive additional properties of the function which enumerates the parts appearing in all partitions of n.

In Chapter Seven, we will highlight the major results and state some open problems for further study related to our work.

Chapter 2

Preliminaries

In this chapter, we provide some background required to understand the concepts of generating function manipulation in proving partition identities. In addition, the concept of a bijective proof of partition identities will be introduced. We give a brief idea of proving partition congruences by using elementary generating function manipulation. Lastly, we describe a combinatorial proof of Stanley's theorem.

2.1 Generating function

Recall the generating function for the number of integer partitions

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The number of ℓ-regular partitions of n is denoted by $R_\ell(n)$. To derive the generating function we will use a similar technique to Euler's.

$$\begin{aligned}\sum_{n=0}^{\infty} R_\ell(n)q^n &= \frac{(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)\ldots}{(1+q^\ell+q^{2\ell}+\cdots)(1+q^{2\ell}+q^{4\ell}+\cdots)\ldots} \\ &= \left(\frac{1-q^\ell}{1-q}\right)\left(\frac{1-q^{2\ell}}{1-q^2}\right)\cdots = \prod_{n=1}^{\infty}\frac{1-q^{\ell n}}{1-q^n}. \qquad (2.1)\end{aligned}$$

Now we derive the generating function for partitions into distinct parts:

$$\sum_{n=0}^{\infty} p(n \mid \text{distinct parts})q^n = (1+q)(1+q^2)(1+q^3)\cdots = \prod_{n=1}^{\infty}(1+q^n).$$

Recall Euler's identity:

Theorem 2.1. *For any positive integer n, the number of partitions of n using only odd parts equals the number of partitions of n into distinct parts.*

Proof. Since partitions into odd parts are the same as 2-regular partitions we have

$$\begin{aligned}\sum_{n=0}^{\infty} R_2(n)q^n &= \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} \\ &= \prod_{n=1}^{\infty} \frac{(1+q^n)(1-q^n)}{1-q^n} = \prod_{n=1}^{\infty}(1+q^n) \\ &= \sum_{n=0}^{\infty} p(n \mid \text{distinct parts})q^n.\end{aligned}$$

□

The proof of Glaisher's generalization of Euler's identity is similar.

From the definition of an overpartition we see that it is a combination of an ordinary partition and a partition into distinct parts (as the overlined parts are distinct). Hence the generating function for the number of overpartitions is

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \times \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}. \tag{2.2}$$

The number of ℓ-regular overpartitions of n is denoted by $\overline{R_\ell}(n)$. The generating function for $\overline{R_\ell}(n)$ is

$$\sum_{n=0}^{\infty} \overline{R_\ell}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)(1-q^{\ell n})}{(1+q^{\ell n})(1-q^n)}. \tag{2.3}$$

2.2 Bijections for partition identities

Recall Euler's identity

$$p(n \mid \text{distinct parts}) = p(n \mid \text{odd parts}).$$

To prove this identity combinatorially, we need to find a bijection

$$p[n \mid \text{distinct parts}] \mapsto p[n \mid \text{odd parts}].$$

In this thesis square brackets are used to indicate corresponding enumerated sets, and exponents indicate multiplicities of the parts.

We will start with partitions into odd parts. We merge every two parts with the same size. We repeat this procedure until all parts are distinct. For example

$$\begin{aligned} 7+7+3+3+1+1+1+1 &\mapsto (7+7)+(3+3)+(1+1)+(1+1) \\ &\mapsto 14+6+(2+2) \\ &\mapsto 14+6+4. \end{aligned}$$

To invert the map, we start with partitions into distinct parts. We split every even part into two equal parts. This procedure shall be repeated until all parts are odd. For example

$$\begin{aligned} 14+6+4 &\mapsto 7+7+3+3+2+2 \\ &\mapsto 7+7+3+3+2+2 \\ &\mapsto 7+7+3+3+1+1+1+1. \end{aligned}$$

It is noticeable that the above combinatorial proof of this identity is generic. As we mentioned in Chapter 1 Euler proved this identity analytically. The underlying combinatorial proofs in general are often not difficult, however finding such proofs may require a great deal of ingenuity.

We conclude this section with an insightful bijection which was found by Stanley. Recall Elder's identity: *the total number of k's appearing in all partitions of n equals the number of different parts repeated k or more times in all partitions of n.*

For each partition λ of n and each part j of λ occurring at least k times, we will map

$$\underbrace{j,\ldots,j}_{k \text{ copies}} \longmapsto \underbrace{k,\ldots,k}_{j \text{ copies}}.$$

Hence, the number of times a given μ occurs (as an image of the map) is equal to the multiplicity (number of occurrence) of k in μ. See the example in Table 2.1

λ	parts repeated 3 times	μ
(1,1,1,1,1,1,1,1)	(1,1,1)	(3,1,1,1,1,1)
(2,1,1,1,1,1,1)	(1,1,1)	(3,2,1,1,1)
(2,2,1,1,1,1)	(1,1,1)	(3,2,2,1)
(2,2,2,1,1)	(2,2,2)	(3,3,1,1)
(2,2,2,2)	(2,2,2)	(3,3,2)
(3,1,1,1,1,1)	(1,1,1)	(3,3,1,1)
(3,2,1,1,1)	(1,1,1)	(3,3,2)
(4,1,1,1,1)	(1,1,1)	(4,3,1)
(5,1,1,1)	(1,1,1)	(5,3)

Table 2.1: The bijection $\lambda \to \mu$ for $k = 3$, $n = 8$.

Conversely, consider each partition μ of n and each part k of μ occurring r times. Then for each $i \in \{1, 2, \ldots, r\}$

$$\underbrace{k, \ldots, k}_{i \text{ copies}} \longmapsto \underbrace{i, \ldots, i}_{k \text{ copies}}.$$

This bijection was found by Richard Stanley [38]. Stanley Submitted this bijection to the Problems and Solutions section of *American Mathematical Monthly.* However, it was rejected for being "a bit on the easy side, and using only a standard argument" [38].

2.3 Ramanujan theta-functions

Ramanujan's general theta function, is denoted by $f(a, b)$ and defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.4}$$

The three most important special cases of (2.4) are given by

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{2.5}$$

$$\psi(q) := f(q, q^3) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{2.6}$$

$$f(q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2.7}$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

The three product representations in (2.5)-(2.7) are special cases of the Jacobi triple product identity

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The Jacobi triple product identity is said to be the most useful and celebrated theorem in the theory of theta-functions, which was first studied by Gauss (see [12]). The identity obtained for $f(-q)$ is called Euler's pentagonal number theorem.

The theory of Ramanujan theta-functions plays an important role in proving congruence properties of partition functions. We conclude this section by introducing the following lemma which is derived from the binomial theorem.

Lemma 2.1. *For any prime p,*

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}. \tag{2.8}$$

Proof. From the binomial theorem,

$$(1-q^n)^p = \sum_{j=0}^{p} \binom{p}{j} (-q^n)^j$$

$$= \binom{p}{0} - \binom{p}{1} q^n + \binom{p}{2} q^{2n} - \binom{p}{3} q^{3n} + \cdots + (-1)^p \binom{p}{p} q^{pn}$$
$$\equiv 1 + q^{pn} \equiv 1 - q^{pn} \pmod{p}.$$

Hence,

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}.$$

□

From this lemma, we may prove the the following corollary:

Corollary 2.1. *The number of overpartitions of $n > 0$ is always even.*

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}(n)q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty} \qquad (2.9)\\ &= \frac{(-q;q)_\infty(q;q)_\infty}{(q;q)_\infty(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\\ &\equiv \frac{(q;q)_\infty^2}{(q;q)_\infty^2} \equiv 1 \pmod 2.\end{aligned}$$

□

Chapter 3

Combinatorial identities for ℓ-regular overpartitions

3.1 Introduction

In 2004 Cortell and Lovejoy [16] studied overpartitions and found an identity between $\overline{R_\ell}(n)$ and a class of overpartitions. George Andrews [7] considered the enumeration of *singular overpartitions* which correspond to ℓ-regular overpartitions in which the parts satisfy prescribed congruences. Subsequently, Chen, Hirschhorn and Sellers [14] developed the arithmetic properties of these singular overpartition functions.

In a recent work Munagi and Sellers [33] proved new identities between sets of restricted partitions and certain overpartitions in which the overlined parts belong to specified residue classes.

In section 3.2, we prove a general theorem (Main Theorem) which connects $\overline{R_\ell}(n)$ with five other restricted partition functions. In section 3.3, we give an identity for colored partitions which extends the results of the main theorem.

The content of this chapter is largely taken from Alanazi-Munagi article [2]. I am appreciative to Prof. Munagi for his permission to include our joint work in this thesis.

3.2 A general partition theorem

This section is devoted to the statement and proof of a sequence of related partition identities connecting $\overline{R_\ell}(n)$ with different classes of restricted partitions and overpartitions.

We first establish a simple identity between overpartitions and ordinary partitions.

Proposition 3.1. *The number of overpartitions of n equals the number of partitions of $2n$ in which odd parts occur with even multiplicity.*

Proof. For a generating function proof let $E(n)$ denote the number of partitions of n in which odd parts occur with even multiplicity. Then

$$\begin{aligned}\sum_{n=0}^{\infty} E(2n)q^{2n} &= \prod_{n=1}^{\infty}(1+q^{2n}+q^{4n}+\cdots)(1+q^{2(2n-1)}+q^{4(2n-1)}+\cdots)\\ &= \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})(1-q^{2(2n-1)})}.\end{aligned}$$

On replacing q^2 by q the equation becomes

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-q^{2n-1})} = \prod_{n=1}^{\infty}\frac{(1+q^n)}{(1-q^n)} = \sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

□

The bijective proof is more insightful.

Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \dots) \in E[2n]$, $c_1 > c_2 > \cdots, u_j \geq 1\,\forall\, j$, and define the map $f : E[2n] \to \overline{p}[n]$ by $\lambda \mapsto f(\lambda) = \cup_{c\in\lambda} f_c(c^k)$ (multiset union), where

$$f_c : c^k \mapsto \begin{cases} c^{\frac{k}{2}} & \text{if } k \equiv 0 \pmod 4,\\ \frac{c}{2}, c^{\frac{k-1}{2}} & \text{if } k \equiv 1 \pmod 4,\\ \overline{c}, c^{\frac{k-2}{2}} & \text{if } k \equiv 2 \pmod 4\end{cases}$$

such that if $k \equiv 3 \mod 4$ then $f_c(c^k) = f_c(c), f_c(c^{k-1}) = \frac{c}{2}, \overline{c}, c^{\frac{k-2}{2}}$. (Note that the cases $k = 1, 3$ refer to even parts only since odd parts occur with even multiplicities). The inverse map $f^{-1} : \overline{p}[n] \to E[2n]$ is analogously given by:

$$f_c^{-1}(\overline{c}) = c^2;$$

$$f_c^{-1}(c^k) = \begin{cases} 2c, c^{2(k-1)} & \text{if } k \equiv 1 \pmod 2, \\ c^{2k} & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

□

To prove that the map f is bijective, we need to show that $f_c^{-1} f_c(c^k) = f_c f_c^{-1}(c^k) = c^k$ for all c^k. First case, let $\lambda \in E[2n]$, we choose c^k from λ for $k \equiv 0 \pmod 4$, $k \equiv 1 \pmod 4$ or $k \equiv 2 \pmod 4$:

(i) if $k \equiv 0 \pmod 4$ then

$$f_c^{-1} f_c(c^k) = f_c^{-1}(c^{\frac{k}{2}}) = c^k$$

(ii) if $k \equiv 1 \pmod 4$ then

$$f_c^{-1} f_c(c^k) = f_c^{-1}(\frac{c}{2}, c^{\frac{k-1}{2}}) = f_c^{-1}(\frac{c}{2}), f_c^{-1}(c^{\frac{k-1}{2}}) = c, c^{k-1} = c^k$$

(iii) if $k \equiv 2 \pmod 4$ then

$$f_c^{-1} f_c(c^k) = f_c^{-1}(\overline{c}, c^{\frac{k-2}{2}}) = f_c^{-1}(\overline{c}), f_c^{-1}(c^{\frac{k-2}{2}}) = c^2, c^{k-2} = c^k$$

Second case, let $\beta \in \overline{p}[n]$, we choose c from β. There are three possible choices for c which are overlined c, c occurs with even multiplicities or c occurs with odd multiplicities:

(i) if c is overlined, then

$$f_c f_c^{-1}(\overline{c}) = f_c(c^2) = \overline{c}$$

(ii) if the multiplicity of c is even, then

$$f_c f_c^{-1}(c^k) = f_c(c^{2k}) = c^k$$

(iii) if the multiplicity of c is odd, then

$$f_c f_c^{-1}(c^k) = f_c(2c, c^{2(k-1)}) = f_c(2c), f_c(c^{2(k-1)}) = c, c^{k-1} = c^k$$

These bijections are illustrated in Table 3.1 when $n = 3$; the lists under respective enumerators correspond one-to-one under the bijection.

$E[6]$	$\xrightarrow{f}$	$\overline{p}[3]$
(6)	$\to$	(3)
(4,2)	$\to$	(2,1)
(4,1,1)	$\to$	$(2,\overline{1})$
(3,3)	$\to$	$(\overline{3})$
(2,2,2)	$\to$	$(\overline{2},1)$
(2,2,1,1)	$\to$	$(\overline{2},\overline{1})$
(2,1,1,1,1)	$\to$	(1,1,1)
(1,1,1,1,1,1)	$\to$	$(\overline{1},1,1)$

Table 3.1: The bijections of Proposition 3.1 for $n = 3$.

Remark 1. *The action of the map f on a part of a partition λ is to halve the part (if it is even) or halve its multiplicity (up to possible overlining). The inverse of f reverses these operations. Thus f preserves ℓ-regularity provided that ℓ is odd, that is, if λ is ℓ-regular, then so is $f(\lambda)$, and conversely.*

We now state our main result.

Theorem 3.1. *Main Theorem:*
Let ℓ and n be positive integers with $\ell, n > 1$.
Let $B_\ell(n)$ denote the number of partitions of n in which odd parts occur with multiplicity $2, 4, ...,$ or $2(\ell-1)$ and even parts appear at most $\ell - 1$ times.
Let $Q_\ell(n)$ denote the number of ℓ^2-regular partitions of n in which parts not divisible by ℓ appear 0 or ℓ times. Then

$$B_\ell(2n) = Q_\ell(\ell n) = \overline{R_\ell}(n); \tag{3.1}$$

Let $\ell \equiv 1 \pmod 2$ and let $G_\ell(n)$ denote the number of ℓ-regular partitions of n in which odd parts occur with even multiplicities. Then

$$G_\ell(2n) = \overline{R_\ell}(n); \tag{3.2}$$

Let $\ell \equiv 0 \pmod 2$ and let $H_\ell(n)$ denote the number of 2ℓ-regular partitions of n in which odd parts occur with even multiplicities and each part $\equiv \ell \pmod{2\ell}$ appears at most once. Then

$$H_\ell(2n) = \overline{R_\ell}(n). \tag{3.3}$$

We present both a generating function proof and a bijective proof of the main theorem.

3.2.1 A generating function proof of Theorem 3.1

The generating functions for $B_\ell(2n)$ is given by

$$\sum_{n=0}^{\infty} B_\ell(2n)q^{2n} = \prod_{n=1}^{\infty}(1+q^{1\cdot 2n}+\cdots+q^{(\ell-1)\cdot 2n})(1+q^{2(2n-1)}+\cdots+q^{2(\ell-1)(2n-1)})$$

$$= \prod_{n=1}^{\infty}\frac{(1-q^{2\ell n})}{(1-q^{2n})}\frac{(1-q^{2\ell(2n-1)})}{(1-q^{2(2n-1)})}. \tag{3.4}$$

The generating function for $Q_\ell(\ell n)$ could be derived by using the same method that we used in (2.1)

$$\begin{aligned}\sum_{n=0}^{\infty} Q_\ell(\ell n)q^{\ell n} &= \prod_{n=1}^{\infty}\frac{(1+q^{\ell n}+q^{2\ell n}+\cdots)(1+q^{\ell n})}{(1+q^{\ell^2 n}+q^{2\ell^2 n}+\cdots)(1+q^{\ell^2 n})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{\ell^2 n})(1+q^{\ell n})}{(1-q^{\ell n})(1+q^{\ell^2 n})}\times\frac{(1-q^{\ell n})(1-q^{\ell^2 n})}{(1-q^{\ell n})(1-q^{\ell^2 n})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2\ell n})(1-q^{\ell^2 n})^2}{(1-q^{\ell n})^2(1-q^{2\ell^2 n})}.\end{aligned} \tag{3.5}$$

On the other hand, recall from (2.3)

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{R_\ell}(n)q^n &= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})}\times\frac{(1-q^n)(1-q^{\ell n})}{(1-q^n)(1-q^{\ell n})}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{\ell n})^2}{(1-q^n)^2(1-q^{2\ell n})}.\end{aligned} \tag{3.6}$$

Replacing q by q^ℓ yields (cf. (3.5))

$$\prod_{n=1}^{\infty}\frac{(1-q^{2\ell n})(1-q^{\ell^2 n})^2}{(1-q^{\ell n})^2(1-q^{2\ell^2 n})} = \sum_{n=0}^{\infty} Q_\ell(\ell n)q^{\ell n}.$$

To complete the proof of (3.1) we note that

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})}$$

and

$$\prod_{n=1}^{\infty}\frac{(1-q^{\ell n})}{(1-q^{2\ell n})}=\prod_{n=1}^{\infty}(1-q^{\ell(2n-1)}),$$

then (3.6) implies

$$\sum_{n=0}^{\infty}\overline{R_\ell}(n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{\ell(2n-1)})(1-q^{\ell n})}{(1-q^n)(1-q^{2n-1})}.$$

Replacing q by q^2 gives (cf. (3.4))

$$\prod_{n=1}^{\infty}\frac{(1-q^{2\ell(2n-1)})(1-q^{2\ell n})}{(1-q^{2n})(1-q^{2(2n-1)})}=\sum_{n=0}^{\infty}B_\ell(2n)q^{2n}.$$

In order to prove (3.2) we assume that ℓ is odd and consider the generating function

$$\begin{aligned}\sum_{n=0}^{\infty}G_\ell(2n)q^{2n}&=\prod_{n=1}^{\infty}\frac{(1+q^{2n}+q^{4n}+\cdots)(1+q^{2(2n-1)}+q^{4(2n-1)}+\cdots)}{(1+q^{\ell(2n)}+q^{2\ell(2n)}+\cdots)(1+q^{\ell\cdot 2(2n-1)}+q^{2\ell\cdot 2(2n-1)}+\cdots)}\\&=\prod_{n=1}^{\infty}\frac{(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})}\\&=\sum_{n=0}^{\infty}B_\ell(2n)q^{2n}.\end{aligned}$$

Thus (3.2) is established.

Lastly for (3.3) we assume that ℓ is even and consider the generating function

$$\sum_{n=0}^{\infty}H_\ell(2n)q^{2n}=\prod_{n=1}^{\infty}\frac{1-q^{\ell n}}{1-q^{2n}}\times\frac{1+q^{\ell n}}{1+q^{2\ell n}}\times\frac{1}{1-q^{2(2n-1)}}.$$

Indeed a partition enumerated by $H_\ell(2n)$ is 2ℓ-regular and contains at most one distinct copy of each part $\equiv \ell \pmod{2\ell}$. This is enumerated by the function

$$\frac{(1-q^{\ell})(1-q^{2\ell})\cdots}{(1-q^2)(1-q^4)\cdots}(1+q^{\ell})(1+q^{\ell+2\ell})(1+q^{\ell+4\ell})\cdots=\prod_{n=1}^{\infty}\frac{1-q^{\ell n}}{1-q^{2n}}\times\frac{1+q^{\ell n}}{1+q^{2\ell n}}.$$

Since odd parts occur with even multiplicities, we have the contribution

$$\prod_{n=1}^{\infty}(1+q^{2(2n-1)}+q^{4(2n-1)}+q^{6(2n-1)}+\cdots)=\prod_{n=1}^{\infty}\frac{1}{1-q^{2(2n-1)}}.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} H_\ell(2n)q^{2n} &= \prod_{n=1}^{\infty} \frac{1-q^{2\ell n}}{(1-q^{2n})(1+q^{2\ell n})(1-q^{2(2n-1)})} \times \frac{1-q^{2\ell n}}{1-q^{2\ell n}} \\
&= \prod_{n=1}^{\infty} \frac{1-q^{2\ell n}}{(1-q^{2n})(1-q^{2(2n-1)})} \times \frac{1-q^{2\ell n}}{1-q^{4\ell n}} \\
&= \prod_{n=1}^{\infty} \frac{1-q^{2\ell n}}{(1-q^{2n})(1-q^{2(2n-1)})} \times (1-q^{2\ell(2n-1)}) \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})(1-q^{2(2n-1)})} \\
&= \sum_{n=1}^{\infty} B_\ell(2n)q^{2n}.
\end{aligned}$$

This completes the generating function proof of Theorem 3.1. □

3.2.2 A combinatorial proof of Theorem 3.1

We provide combinatorial proofs of the three parts of the theorem in the following order.

First we establish the the bijection $Q_\ell[\ell n] \iff \overline{R_\ell}[n]$. Then we prove the remaining parts according to the schemes

$B_\ell[2n] \iff G_\ell[2n] \iff \overline{R_\ell}[n]$ and $B_\ell[2n] \iff H_\ell[2n] \iff \overline{R_\ell}[n]$, corresponding to odd and even ℓ respectively.

Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots) \in Q_\ell[\ell n]$. Define the map $w : Q_\ell[\ell n] \to \overline{R_\ell}[n]$ by $\lambda \mapsto \cup_{c\in\lambda} w_c(c)$, where

$$w_c : c^{u_j} \mapsto \begin{cases} \left(\frac{c}{\ell}\right)^{u_j} & \text{if } \ell | c, \\ \overline{c} & \text{if } \ell \nmid c. \end{cases}$$

In other words, each multiple of ℓ is divided by ℓ and each non-multiple (which occurs exactly ℓ times) is replaced by one overlined copy. The inverse map is

$$w_c^{-1} : x \mapsto \begin{cases} c^\ell & \text{if } x = \overline{c}, \\ c\ell & \text{if } x = c. \end{cases}$$

This proves the bijection $Q_\ell[\ell n] \iff \overline{R_\ell}[n]$.

Next we define a new bijection θ to compose with f which was defined in the proof of Proposition 3.1:

$$B_\ell[2n] \xrightarrow{\theta} G_\ell[2n] \xrightarrow{f} \overline{R_\ell}[n].$$

If $\lambda = (c_1 \geq c_2 \geq \cdots) \in B_\ell[2n]$, then each $c_i = c$ can be expressed uniquely in the form $c = \ell^r m$ with $r \geq 0$ such that $\ell \nmid m$. Define $\theta : B_\ell[2n] \to G_\ell[2n]$ by setting $\theta(\lambda) = \cup_{c\in\lambda}\theta_c(c)$, with

$$\theta_c(c) = \theta_c(\ell^r m) = m^{\ell^r}.$$

It may be verified that θ is invertible. Note that θ is similar to the classical bijection of Glaisher between odd and strict ordinary partitions, see [6, 9]. To insure that the image is not divisible by ℓ, each part c is mapped to x copies of c/x, where x is the highest power of ℓ dividing c.

The fact that f is the required bijection between $G_\ell[2n]$ and $\overline{R}_\ell[n]$ follows from the proof of Proposition 3.1 and Remark 1.

The second part of the proof, $B_\ell[2n] \iff H_\ell[2n] \iff \overline{R_\ell}[n]$, also relies on the composition of two bijections ϕ and f:

$$B_\ell[2n] \xrightarrow{\phi} H_\ell[2n] \xrightarrow{f} \overline{R_\ell}[n].$$

We define $\phi : B_\ell[2n] \to H_\ell[2n]$ by $\phi(\lambda) = \cup_{c\in\lambda}\phi_c(c)$, where ϕ_c is explained below. Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots) \in B_\ell[2n]$ and consider $c^{u_j} \in \lambda$. Then one can write $c = \ell^r m$ where $0 \leq r \leq 1$ and $\ell \nmid m$. If m is odd and u_j is odd, then there are two cases:

$$\phi_c : c^{u_j} \mapsto \begin{cases} c & \text{if } u_j = 1, \\ c, m^{\ell^r(u_j-1)} & \text{if } u_j > 1. \end{cases}$$

Note that when $u_j > 1$, ϕ_c fixes one copy of c but assigns the other copies to m^{ℓ^r} apiece.

For all other cases apply the following transformation to each $c^k \in \lambda$:

$$\phi_c : c^k \mapsto m^{\ell^r k}.$$

To complete the proof, we give the inverse of ϕ. Let $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots) \in H_\ell[2n]$. In order to assign each $c^u \in \lambda$, we first obtain the ℓ-adic expansion of u: $u = m_0 + m_1\ell + \cdots + m_r\ell^r$, $m_i \in \{0, \ldots, \ell-1\}$. Thus each $c^u \in \lambda$ is equivalent to $c^u = c^{m_1\ell}, c^{m_2\ell^2}, \ldots, c^{m_r\ell^r}$. Then if $\lambda_i = c^{m_i\ell^i}$, we have $\phi^{-1}(\lambda) = \cup_{\lambda_i\in\lambda}\phi^{-1}_{\lambda_i}(\lambda_i)$ with

$$\phi_{\lambda_i}^{-1} : \lambda_i = c^{m_i \ell^i} \mapsto \begin{cases} c^{\ell^i}, (\ell^i c)^{m_i - 1} & \text{if } c \equiv 1 \pmod 2 \text{ and } m_i \equiv 1 \pmod 2 \text{ and } 0 \le i \le 1; \\ (\ell^i c)^{m_i} & \text{otherwise.} \end{cases}$$

□

Illustrations of the bijections $B_\ell[2n] \iff G_\ell[2n] \iff \overline{R_\ell}[n]$, $B_\ell[2n] \iff H_\ell[2n] \iff \overline{R_\ell}[n]$ and $Q_\ell[2n] \iff \overline{R_\ell}[n]$ are given for some of the partitions when $\ell = 3, 4$ and 5 and $n = 25$ in Tables 3.2, 3.3 and 3.4 respectively.

$B_3[50]$	$\xrightarrow{\theta}$	$G_3[50]$	$\xrightarrow{f}$	$\overline{R}_3[2])$
$(48, 2)$	$\rightarrow$	$(16^3, 2)$	$\rightarrow$	$(\overline{16}, 8, 1)$
$(32, 12, 4, 2)$	$\rightarrow$	$(32, 4^4, 2)$	$\rightarrow$	$(16, 4^2, 1)$
$(24^2, 2)$	$\rightarrow$	$(8^6, 2)$	$\rightarrow$	$(8^2, \overline{8}, 1)$
$(24, 16, 8, 1^2)$	$\rightarrow$	$(16, 8^4, 1^2)$	$\rightarrow$	$(8^3, \overline{1})$
$(16^2, 8^2, 2)$	$\rightarrow$	$(16^2, 8^2, 2)$	$\rightarrow$	$(\overline{16}, \overline{8}, 1)$
$(16^2, 12, 4, 2)$	$\rightarrow$	$(16^2, 4^4, 2)$	$\rightarrow$	$(\overline{16}, 4^2, 1)$
$(16, 12^2, 8, 1^2)$	$\rightarrow$	$(16, 8, 4^6, 1^2)$	$\rightarrow$	$(8, 4^3, \overline{4}, \overline{1})$
$(12^2, 8, 6^2, 2, 1^4)$	$\rightarrow$	$(8, 4^6, 2^7, 1^4)$	$\rightarrow$	$(4^3, \overline{4}, 2^2, \overline{2}, 1^3)$
$(8^2, 7^2, 6^2, 4^2)$	$\rightarrow$	$(8^2, 7^2, 4^2, 2^6)$	$\rightarrow$	$(\overline{8}, \overline{7}, \overline{4}, 2^2, \overline{2})$
$(8, 7^2, 4^2, 3^4, 2^2, 1^4)$	$\rightarrow$	$(8, 7^2, 4^2, 2^2, 1^{16})$	$\rightarrow$	$(\overline{7}, 4, \overline{4}, \overline{2}, 1^8)$

Table 3.2: An illustration of the bijections of Theorem 3.1 for $n = 25$, $\ell = 3$.

$B_4[50]$	$\xrightarrow{\phi}$	$H_4[50]$	$\xrightarrow{f}$	$\overline{R}_4[25]$
$(48, 2)$	$\rightarrow$	$(3^{16}, 2)$	$\rightarrow$	$(3^8, 1)$
$(32, 12, 4, 2)$	$\rightarrow$	$(12, 4, 2^{17})$	$\rightarrow$	$(6, 2^9, 1)$
$(24^2, 2)$	$\rightarrow$	$(6^8, 2)$	$\rightarrow$	$(6^4, 1)$
$(24, 16, 8, 1^2)$	$\rightarrow$	$(6^4, 2^4, 1^{18})$	$\rightarrow$	$(6^2, 2^2, 1^8, \overline{1})$
$(16^3, 2)$	$\rightarrow$	$(2, 1^{48})$	$\rightarrow$	(1^{25})
$(16^2, 12, 4, 2)$	$\rightarrow$	$(12, 4, 2, 1^{32})$	$\rightarrow$	$(6, 2, 1^{17})$
$(16, 12^2, 8, 1^2)$	$\rightarrow$	$(12, 3^4, 2^4, 1^{18})$	$\rightarrow$	$(6, 3^2, 2^2, 1^8, \overline{1})$
$(12^3, 8, 6)$	$\rightarrow$	$(12, 6, 3^8, 2^4)$	$\rightarrow$	$(6, 3^5, 2^2)$
$(8^3, 7^2, 4^3)$	$\rightarrow$	$(7^2, 4, 2^{12}, 1^8)$	$\rightarrow$	$(\overline{7}, 2^7, 1^4)$
$(8, 4^3, 3^6, 2^3, 1^6)$	$\rightarrow$	$(4, 3^6, 2^7, 1^{14})$	$\rightarrow$	$(3^2, \overline{3}, 2^3, \overline{2}, 1^7, \overline{1})$

Table 3.3: An illustration of the bijections of Theorem 3.1 for $n = 25$, $\ell = 4$.

$Q_5[125]$	$\xrightarrow{w}$	$\overline{R}_4[25]$
$(24^5, 5)$	$\rightarrow$	$(\overline{24}, 1)$
$(24^5, 1^5)$	$\rightarrow$	$(\overline{24}, \overline{1})$
$(19^5, 10, 4^5)$	$\rightarrow$	$(\overline{19}, \overline{4}, 2)$
$(15^8, 1^5)$	$\rightarrow$	$(3^8, \overline{1})$
$(15^7, 10^2)$	$\rightarrow$	$(3^7, 2^2)$
$(15^6, 10^3, 5)$	$\rightarrow$	$(3^6, 2^3, 1)$
$(15^5, 10^4, 5, 1^5)$	$\rightarrow$	$(3^5, 2^4, 1, \overline{1})$
$(10^{12}, 1^5)$	$\rightarrow$	$(1^{12}, \overline{1})$
$(10^{10}, 5^5)$	$\rightarrow$	$(2^{10}, 1^5)$
$(10^2, 5^{13}, 4^5, 3^5, 1^5)$	$\rightarrow$	$(\overline{4}, \overline{3}, 2^2, 1^{13}, \overline{1})$

Table 3.4: An illustration of the bijections of Theorem 3.1 for $n = 25$, $\ell = 5$.

3.3 A partial identity for colored partitions

We state a partition identity involving 2-color partitions.

Theorem 3.2. *Let $T_4(n)$ denote the number of partitions of n in which even parts are of two kinds and distinct, and odd parts occur with multiplicity* 4. *Then*

$$\overline{R_4}(n) = T_4(2n). \tag{3.7}$$

We remark that one part-size with two different colors are treated as distinct parts in Theorem 3.2. It is a special case ($\ell = 4$) of the following generalization to every even integer $\ell > 0$.

Theorem 3.3. *Let ℓ be an even positive integer and let $T_\ell(2n)$ denote the number of partitions of $2n$ in which odd parts occur with multiplicity ℓ and even parts are of two different kinds such that even parts of one kind are distinct and each even part of the other kind appears at most $\frac{\ell-2}{2}$ times. Then*

$$\overline{R_\ell}(n) = T_\ell(2n). \tag{3.8}$$

Remark 2. *If we combine Theorem 3.3 with the compatible functions defined in the main theorem (Theorem 3.1) we obtain the following five-way identity for every even integer $\ell > 0$:*

$$B_\ell(2n) = Q_\ell(\ell n) = H_\ell(2n) = T_\ell(2n) = \overline{R_\ell}(n).$$

Proof of Theorem 3.3

Since $\overline{R_\ell}(n) = B_\ell(2n)$ from Theorem 3.1 it will suffice to prove $B_\ell(2n) = T_\ell(2n)$. The generating function for $T_\ell(2n)$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} T_\ell(2n)q^{2n} &= \prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n}+\cdots+q^{(\frac{\ell-2}{2})2n})(1+q^{\ell(2n-1)}) \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^{2n})}\frac{(1-q^{\ell n})}{(1-q^{2n})}\frac{(1-q^{2\ell(2n-1)})}{(1-q^{\ell(2n-1)})}.
\end{aligned}$$

From Equation (3.4) we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} B_\ell(2n)q^{2n} &= \prod_{n=1}^{\infty}\frac{(1-q^{2\ell n})}{(1-q^{2n})}\frac{(1-q^{2\ell(2n-1)})}{(1-q^{2(2n-1)})} \times \frac{(1-q^{2(2n)})}{(1-q^{2(2n)})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1+q^{\ell n})}{(1-q^{2n})}\frac{(1-q^{2\ell(2n-1)})(1-q^{4n})}{(1-q^{2n})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1-q^{4n})(1-q^{2\ell n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})^2(1-q^{\ell n})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{\ell n})(1-q^{4n})(1-q^{2\ell(2n-1)})}{(1-q^{2n})^2(1-q^{\ell(2n-1)})} \\
&= \sum_{n=0}^{\infty} T_\ell(2n)q^{2n}.
\end{aligned}$$

We now give a bijection $g : B_\ell[2n] \to T_\ell[2n]$ as usual, according to parities. Let the two kinds or colors in the theorem be distinguished by subscripting with "a" and "b". Thus each even part-size $2r$ has either the form $(2r)_a$ or $(2r)_b$ with $(2r)_a \neq (2r)_b$ such that $(2r)_a$ is distinct while $(2r)_b$ may also occur in the same partition at most $(\ell-2)/2$ times. Since an odd part in $\lambda \in B_\ell[2n]$ has multiplicity $2, 4, \ldots, 2(\ell-1)$, we have

$$\text{If } c \equiv 1 \pmod 2, \text{ then } \; g_c : c^k \mapsto \begin{cases} (2c)_b^{\frac{k}{2}} & \text{if } 2 \leq k < \ell; \\ c^\ell & \text{if } k = \ell; \\ (2c)_b^{\frac{k-\ell}{2}}, c^\ell & \text{if } k > \ell. \end{cases}$$

$$\text{If } c \equiv 0 \pmod 2, \text{ then } \; g_c : c^k \mapsto \begin{cases} c_a & \text{if } k = 1. \\ (2c)_b^{\frac{k-1}{2}}, c_a & \text{if } k > 1 \text{ is odd.} \\ (2c)_b^{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

The inverse map is immediately seen to be $g^{-1} : T_\ell[2n] \to B_\ell[2n]$ with

$$g_c^{-1} : \begin{cases} c^\ell \mapsto c^\ell & \text{if } c \equiv 1 \pmod{2}; \\ c_a \mapsto c & \text{if } c \equiv 0 \pmod{2}; \\ c_b \mapsto (c/2)^2 & \text{if } c \equiv 0 \pmod{2}. \end{cases}$$

□

An illustration of the bijection $B_\ell[2n] \iff T_\ell[2n]$ is given for some partitions of $\ell = 6$ and $n = 6$ in Table 3.5.

$B_6[2n]$	$\xrightarrow{g}$	$T_6[2n]$
(12)	$\to$	(12_a)
(6,6)	$\to$	(12_b)
(4,4,4)	$\to$	$(8_b,4_a)$
(8,4)	$\to$	$(8_a,4_a)$
(8,2,2)	$\to$	$(8_a,4_b)$
(4,4,2,2)	$\to$	$(8_b,4_b)$
(8,2,1,1)	$\to$	$(8_a,2_a,2_b)$
(2,2,2,2,2,1,1)	$\to$	$(4_b,4_b,2_a,2_b)$
(2,2,2,2,1,1,1,1)	$\to$	$(4_b,4_b,2_b,2_b)$
(2,2,2,1,1,1,1,1,1)	$\to$	$(4_b,2_a,1,1,1,1,1,1)$

Table 3.5: The bijections of Theorem 3.3 for $n = 6,\ \ell = 6$.

Chapter 4

Congruence Properties of ℓ-regular overpartitions

4.1 Introduction

Andrews [7] noted that one of his functions is the same as $\overline{R_3}(n)$, and he proved that, for all $n \geq 0$,

$$\overline{R_3}(9n+3) \equiv \overline{R_3}(9n+6) \equiv 0 \pmod{3} \tag{4.1}$$

using elementary generating function manipulations. Motivated by this congruence result, Chen, Hirschhorn and Sellers [14] extensively studied the arithmetic properties of these singular overpartition functions.

In recent days, Shen [36] returned to the functions of Lovejoy and proved a finite set of congruences satisfied by $\overline{R_3}$ and $\overline{R_4}$. In particular, Shen proved the following eight congruence results:

Theorem 4.1. *For all $n \geq 0$,*

$$\begin{aligned}
\overline{R_3}(4n+1) &\equiv 0 \pmod{2}, \\
\overline{R_3}(4n+3) &\equiv 0 \pmod{6}, \\
\overline{R_3}(9n+3) &\equiv 0 \pmod{6}, \quad \textit{and} \\
\overline{R_3}(9n+6) &\equiv 0 \pmod{24}.
\end{aligned}$$

Theorem 4.2. *For all $n \geq 0$,*

$$\begin{aligned}
\overline{R_4}(12n+4) &\equiv 0 \pmod{3}, \\
\overline{R_4}(12n+8) &\equiv 0 \pmod{3}, \\
\overline{R_4}(12n+7) &\equiv 0 \pmod{24}, \text{ and} \\
\overline{R_4}(12n+11) &\equiv 0 \pmod{24}.
\end{aligned}$$

Our primary goal in this chapter is to prove families of congruences satisfied by the functions $\overline{R_\ell}$ for infinitely many values of ℓ. The proof techniques used are classical, involving elementary generating function manipulation techniques as well as Ramanujan's theta functions. In addition, we will give combinatorial proofs for some of the congruences.

Throughout this work, we will utilize the generating functions (2.3), which is

$$\sum_{n=0}^{\infty} \overline{R_\ell}(n) = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})(1+q^n)}{(1-q^n)(1+q^{\ell n})}.$$

We will also make use of Ramanujan's theta function (2.5), which is

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2n}).$$

(See Berndt's book [12] for a detailed discussion of the function $\varphi(q)$ and its relatives.)

The content of this chapter is largely taken from the articles [2] and [5]. I am appreciative to Prof. Munagi and Prof. Sellers for their permissions to include our joint work in this thesis.

4.2 New Congruence Results

Motivated by Andrews' congruences (4.1), Chen, Hirschhorn, and Sellers [14] have already provided an infinite family of congruences satisfied by $\overline{R_3}(n)$ modulo 3 and small powers of 2. Our first goal in this chapter is to show that $\overline{R_\ell}$ satisfies at least one congruence modulo 3 for an infinite set of values ℓ.

Theorem 4.3. *For all $n \geq 0$ and all $j \geq 3$, $\overline{R_{3^j}}(27n+18) \equiv 0 \pmod 3$.*

Proof. As per recent work of Munagi and Sellers [33], we define the function $A_\ell(n)$ to be the number of overpartitions of n in which only parts not divisible by ℓ may be overlined. We find from [33] that, for all $n \geq 0$ and all $j \geq 3$, $A_{3^j}(27n+18) \equiv 0 \pmod 3$ where

$$\sum_{n\geq 0} A_{3^j}(n)q^n = \prod_{n\geq 1} \frac{(1-q^{3^j n})}{(1-q^{2\cdot 3^j n})} \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^n)^2}. \tag{4.2}$$

Next, note that from (3.6)

$$\begin{aligned}
\sum_{n\geq 0} \overline{R_{3^j}}(n)q^n &= \prod_{n\geq 1} \frac{(1-q^{2n})(1-q^{3^j n})^2}{(1-q^n)^2(1-q^{2\cdot 3^j n})} \\
&= \prod_{n\geq 1} \frac{(1-q^{3^j n})^2}{(1-q^{3^j n})(1+q^{3^j n})} \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^n)^2} \\
&= \prod_{n\geq 1} \frac{(1-q^{3^j n})}{(1+q^{3^j n})} \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^n)^2}.
\end{aligned} \tag{4.3}$$

Via elementary manipulations, it is then clear from (4.2) and (4.3) that

$$\sum_{n\geq 0} \overline{R_{3^j}}(n)q^n = \left(\prod_{n\geq 1}(1-q^{3^j n})\right) \sum_{n\geq 0} A_{3^j}(n)q^n.$$

Moreover,

$$\prod_{n\geq 1}(1-q^{3^j n})$$

is a function of q^{27}, which means that $\overline{R_{3^j}}(27n+18)$ is simply a linear combination of values of $A_{3^j}(27m+18)$ (no other terms can enter this sum). Therefore, thanks to the corresponding congruence result for A_{3^j} from [33], the theorem follows. $\square$

Interestingly enough, it is also the case that $\overline{R_9}(n)$ satisfies congruences modulo 3. However, they appear to be of a different nature than those satisfied by $\overline{R_3}$ (as stated in [14]) and $\overline{R_{3^j}}$ for $j \geq 3$ (as given in Theorem 4.3). Thus we need to discuss $\overline{R_9}(n)$ separately.

In order to consider $\overline{R_9}(n)$ modulo 3, we will utilize a number of results of Hirschhorn and Sellers [23]. In particular, we will consider the two functions

$$D(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$$

and

$$Y(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty},$$

where $(q;q)_\infty = (q)_\infty$.

It is worth noting that $D(q) = \varphi(-q)$ where $\varphi(q)$ is defined in (2.5).

In [23, Lemma 3.1] the following three identities are proved (where $\omega = e^{2\pi i/3}$):

$$D(q) = D(q^9) - 2qY(q^3),$$

$$D(q)D(\omega q)D(\omega^2 q) = \frac{D(q^3)^4}{D(q^9)}, \text{ and}$$

$$D(q^3)^3 - 8qY(q)^3 = \frac{D(q)^4}{D(q^3)}.$$

Now note the following from (3.6):

$$\begin{aligned}
\sum_{n\geq 0} \overline{R_9}(n) q^n &= \frac{(1-q^{2n})(1-q^{9n})^2}{(1-q^n)^2(1-q^{2\cdot 9n})} \\
&= \frac{(q^2;q^2)_\infty (q^9;q^9)_\infty^2}{(q;q)_\infty^2 (q^{2\cdot 9};q^{2\cdot 9})_\infty} \\
&= \frac{D(q^9)}{D(q)} \\
&= \frac{D(q^9)}{D(q)} \frac{D(\omega q)}{D(\omega q)} \frac{D(\omega^2 q)}{D(\omega^2 q)} \\
&= \frac{D(q^9)D(\omega q)D(\omega^2 q)}{\frac{D(q^3)^4}{D(q^9)}} \\
&= \frac{D(q^9)}{D(q^3)^4}(D(q^{81}) - 2q^9Y(q^{27}))(D(q^9) - 2\omega qY(q^3))(D(q^9) - 2\omega^2 qY(q^3)) \\
&= \frac{D(q^9)}{D(q^3)^4}(D(q^{81}) - 2q^9Y(q^{27}))(D(q^9)^2 + 2qD(q^9)Y(q^3) + 4q^2Y(q^3)^2).
\end{aligned}$$

Thus, we can 3–dissect the generating function for $\overline{R_9}$ to obtain

$$\sum_{n\geq 0} \overline{R_9}(3n+2)q^n = \frac{D(q^3)}{D(q)^4}(D(q^{27}) - 2q^3Y(q^9))(4Y(q)^2).$$

Next, we simplify this generating function modulo 3, utilizing the three identities mentioned above:

$$
\begin{aligned}
\sum_{n\geq 0}\overline{R_9}(3n+2)q^n &= \frac{D(q^3)}{D(q)^4}(D(q^{27})-2q^3Y(q^9))(4Y(q)^2)\\
&\equiv \frac{D(q^3)}{D(q)^4}(D(q^9)^3-2q^3Y(q^3)^3)(4Y(q)^2) \pmod 3\\
&\equiv \frac{D(q^3)}{D(q)^4}(4D(q^9)^3Y(q)^2-8q^3Y(q^3)^3Y(q)^2) \pmod 3\\
&\equiv \frac{D(q^3)}{D(q)^4}(4D(q^9)^3Y(q)^2-(D(q^9)^3-\frac{D(q^3)^4}{D(q^9)})Y(q)^2) \pmod 3\\
&\equiv \frac{D(q^3)}{D(q)^4}(3D(q^9)^3Y(q)^2+\frac{D(q^3)^4}{D(q^3)^3}Y(q)^2) \pmod 3\\
&\equiv \frac{D(q^3)}{D(q)^4}(D(q)^3)(Y(q)^2) \pmod 3\\
&= \frac{D(q^3)}{D(q)}Y(q)^2\\
&= \frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\frac{(q;q)_\infty^2(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2}\\
&= \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty}.
\end{aligned}
$$

Therefore, we know that

$$\sum_{n\geq 0}\overline{R_9}(3n+2)q^n \equiv \sum_{n\geq 0} a_3(n)q^{2n} \pmod 3$$

where $a_3(n)$ is the number of 3–cores of n. Note that a t-core partition of n is a partition whose Ferrers graph has no hook numbers divisible by t (see [27] for more details). This leads to two congruence results for $\overline{R_9}$.

Theorem 4.4. *For all* $n \geq 0$, $\overline{R_9}(6n+5) \equiv 0 \pmod 3$.

Proof. This result follows immediately from the fact that

$$\sum_{n\geq 0}\overline{R_9}(3n+2)q^n \equiv \sum_{n\geq 0} a_3(n)q^{2n} \pmod 3$$

and the fact that the series on the right–hand side is an even function of q. Therefore, for all $n \geq 0$,

$$\overline{R_9}(3(2n+1)+2) = \overline{R_9}(6n+5) \equiv 0 \pmod 3.$$

□

In a similar vein, the generating function work above also proves that, for all $n \geq 0$,

$$\overline{R_9}(6n+2) \equiv a_3(n) \pmod 3. \tag{4.4}$$

This is truly significant as it provides infinitely many Ramanujan–like congruences modulo 3 satisfied by $\overline{R_9}$. One way to see this is to note that $a_3(n)$ is infinitely often identical to zero (see the work of Hirschhorn and Sellers [25] for elementary proofs of some of the arithmetic properties of $a_3(n)$). Indeed, we can easily prove the following result.

Theorem 4.5. *Let $p \equiv 2 \pmod 3$ be prime. For each $1 \leq k \leq p-1$, let r be the least nonnegative integer such that*

$$r \equiv \frac{p^2-1}{3} + kp \pmod{p^2}.$$

Then, for all $n \geq 0$,

$$\overline{R_9}(6(p^2n+r)+2) \equiv 0 \pmod 3.$$

Proof. The proof relies on a result found in Hirschhorn and Sellers [25]. Namely, under the hypothesis of this theorem, it is the case that $a_3(p^2n+r) = 0$. Thanks to this fact and (4.4), the proof is complete. $\square$

We now turn our attention to congruences satisfied by $\overline{R_\ell}$ modulo small powers of 2. As with numerous other overpartition functions, it is clear that, for each ℓ, $\overline{R_\ell}(n)$ satisfies many congruences modulo small powers of 2. (See, for example, [22, 24, 23, 28, 32] where this phenomenon is also noted.)

With the goal of proving such congruences modulo small powers of 2, we develop an extremely beneficial way to rewrite the generating function for $\overline{R_\ell}(n)$ in terms of Ramanujan's theta function $\varphi(q)$.

We state the following lemmas, the proofs of which may be found in [33]:

Lemma 4.1.

$$\varphi(-q^2)^2 = \varphi(q)\varphi(-q)$$

Lemma 4.2.

$$\frac{1}{\varphi(-q)} = \varphi(q)\varphi(q^2)^2\varphi(q^4)^4\dots$$

Combining (2.3) with Lemma 4.2, we have

$$\sum_{n\geq 0}\overline{R_\ell}(n)q^n = \frac{\varphi(q)\varphi(q^2)^2\varphi(q^4)^4\dots}{\varphi(q^\ell)\varphi(q^{2\ell})^2\varphi(q^{4\ell})^4\dots}. \tag{4.5}$$

Corollary 4.1. *For all $n\geq 1$, $\overline{R_\ell}(n)\equiv 0 \pmod 2$.*

Proof. Since $\varphi(q) = 1 + 2\sum_{n\geq 1} q^{n^2}$, we know that

$$\varphi(q)\equiv 1 \pmod 2.$$

So (4.5) gives

$$\sum_{n\geq 0}\overline{R_\ell}(n)q^n \equiv \frac{1\cdot 1\cdot 1\dots}{1\cdot 1\cdot 1\dots}\equiv 1 \pmod 2.$$

□

Corollary 4.2. *For all $n\geq 1$ and an integer $k>0$,*

- *if ℓ is a square , then*

$$\overline{R_\ell}(n)\equiv\begin{cases}2 \pmod 4 & \text{if } n=k^2, \text{ where } \ell \text{ does not divide } n;\\ 0 \pmod 4 & \text{otherwise,}\end{cases}$$

- *if ℓ is not a square, then*

$$\overline{R_\ell}(n)\equiv\begin{cases}2 \pmod 4 & \text{if } n=k^2 \text{ or } n=\ell k^2;\\ 0 \pmod 4 & \text{otherwise,}\end{cases}$$

Proof. Thanks to (4.5), we know

$$\sum_{n\geq 0}\overline{R_\ell}(n)q^n \equiv \frac{\varphi(q)}{\varphi(q^\ell)} \pmod 4$$

since $\varphi(q^i)^j\equiv 1 \pmod 4$ for any even $j\geq 2$. Next, we know, in view of Lemma 4.1, that

$$\varphi(q) = \frac{\varphi(-q^2)^2}{\varphi(-q)}.$$

Thus,

$$\sum_{n\geq 0}\overline{R_\ell}(n)q^n \equiv \frac{\varphi(q)}{\varphi(q^\ell)} \pmod 4$$

$$
\begin{aligned}
&\equiv \frac{\varphi(q)\varphi(-q^{\ell})}{\varphi(-q^{2\ell})^2} \pmod 4 \\
&\equiv \varphi(q)\varphi(-q^{\ell}) \pmod 4
\end{aligned}
$$

since $\varphi(-q^{2\ell})^2 \equiv 1 \pmod 4$.

Therefore,

$$
\begin{aligned}
\sum_{n\geq 0} \overline{R_\ell}(n)q^n &\equiv \varphi(q)\varphi(-q^{\ell}) \pmod 4 \\
&= (1+2\sum_{n\geq 1} q^{n^2})(1+2\sum_{n\geq 1}(-q^{\ell})^{n^2}) \\
&\equiv 1+2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1}(-q^{\ell})^{n^2} \pmod 4 \\
&\equiv 1+2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1} q^{\ell n^2} \pmod 4.
\end{aligned}
$$

The result follows. □

It is clear that Corollary 4.2 provides a framework from which we can write down infinitely many congruences modulo 4 satisfied by $\overline{R_\ell}$ for certain values of ℓ. We provide such an infinite family of results here.

Corollary 4.3. *Let ℓ be a square, p be a prime, and let r be a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $\overline{R_\ell}(pn+r) \equiv 0 \pmod 4$.*

Proof. Assume ℓ is a square. Thanks to Corollary 4.2, we know that $\overline{R_\ell}(n)$ is divisible by 4 unless n is a square. Thus, in order for this result to be false, we must have $pn+r = k^2$ for some k. But this implies that $r \equiv k^2 \pmod p$, and this cannot occur because r is assumed to be a quadratic nonresidue modulo p. The result follows.

□

Clearly, Corollary 4.3 provides $\frac{p-1}{2}$ congruences modulo 4 for each prime p and for each square value of ℓ. Thus, we have demonstrated infinitely many congruences modulo 4 which are satisfied by $\overline{R_\ell}$ (for a specific set of values of ℓ).

It is worth noting that the proof technique used in Corollary 4.2 could be extended to write down similar results for moduli which are higher powers of 2. However, the results will undoubtedly be less elegant than those above, so we refrain from doing so here.

4.3 Combinatorial proofs of Corollary 4.1 and Corollary 4.2

For Corollary 4.1, the proof goes as follows:

Proof. The number of overpartitions of $n > 0$ is always even. This is because an overpartition is obtained from an ordinary partition $\lambda = (c_1^{u_1}, c_2^{u_2}, \cdots, c_r^{u_r})$ by overlining the first occurrence of each part-size or not. Thus λ alone gives rise to 2^r overpartitions. $\square$

To prove Corollary 4.2, we need the following lemma:

Lemma 4.3. *For all* $n \geq 1$,

$$\overline{p}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2 \text{ for some integer } k \\ 0 \pmod 4 & \text{otherwise.} \end{cases}$$

Proof. Combining the generating functions of $\overline{p}(n)$ and $D(q)$ with Lemma 4.2, we have

$$\sum_{n\geq 0} \overline{p}(n)q^n = \frac{1}{D(q)} = \frac{1}{\varphi(-q)} = \varphi(q)\varphi(q^2)^2\varphi(q^4)^4 \cdots .$$

Since $\varphi(q^i)^j \equiv 1 \pmod 4$ for any even $j \geq 2$ and applying the Lemma 4.1, we have

$$\begin{aligned} \sum_{n\geq 0} \overline{p}(n)q^n &\equiv \varphi(q) \pmod 4 \\ &= (1 + 2\sum_{n\geq 1} q^{n^2}). \end{aligned}$$

The result follows.

Alternatively, Lemma 4.3 can be proved combinatorially as follows:
We decompose overpartitions of n into two sets: those containing a unique part-size and those containing two or more different part-sizes. Then we see that the latter set of overpartitions has cardinality 2^r, $r > 1$, that is, a cardinality divisible by 4. On the other hand, partitions with a single part-size arise from divisors of n. Each divisor d of n gives the partition $(d^{n/d})$ which in turn generates 2 overpartitions. Since a square has an odd number of divisors, $\tau(k^2) \equiv 1 \pmod 2$, we deduce that $\overline{p}(k^2) \equiv 2 \pmod 4$. $\square$

We can now give a combinatorial proof of Corollary 4.2:

Proof. Let $m(\ell|n)$ be the number of multiples ℓ dividing n. By the proof of Lemma 4.3, it will suffice to find the parity of $\tau(n)-m(\ell|n)$: a divisor d of n generates a single part-size ℓ-regular overpartition provided that ℓ does not divide d. In each case we exclude the divisors enumerated by $m(\ell|n)$ and compare the parity of $\tau(n)-m(\ell|n)$ with the the parity of $\tau(n)$, and conclude that $\overline{R_\ell}(n) \equiv 2 \pmod 4$ if $\tau(n)-m(\ell|n)$ is odd and $\overline{R_\ell}(n) \equiv 0 \pmod 4$ if $\tau(n)-m(\ell|n)$ is even.

Consider the first case of the second bullet point $n = k^2$ or $n = \ell k^2$ given that ℓ is not a square.
If $n = k^2$ and ℓ does not divide n, then $m(\ell|n) = 0$. So $\tau(n) - m(\ell|n)$ is odd. If $n = k^2$ and ℓ divides n, then $m(\ell|n) = \tau(n/\ell)$ which is even. So $\tau(n) - m(\ell|n)$ is still odd. But if $n = \ell k^2$, then $\tau(n)$ is even and $m(\ell|n) = \tau(k^2)$ which is odd. So $\tau(n) - m(\ell|n)$ is odd.

The second case of the second bullet point has two parts namely (i) $n = k^2$ with ℓ a square factor of n, (ii) $n \neq k^2$ and $n \neq \ell k^2$ and (iii) $n = k^2$ with ℓ is a square which is not a factor of n. In (i) we find that both $\tau(n)$ and $m(\ell|n)$ are odd; so $\tau(n) - m(\ell|n)$ is even. In (ii) it is clear that both $\tau(n)$ and $m(\ell|n)$ are even. In (iii) we find that $\tau(n)$ is odd and $m(\ell|n)$ is even; so $\tau(n) - m(\ell|n)$ is odd. $\square$

Chapter 5

Combinatorial proof of a partition inequality of Bessenrodt-Ono

5.1 Introduction

We consider a combinatorial question related to a multiplicative property of the partition function $p(n)$, the number of partitions of a positive integer n. In this chapter, integer partitions will be written in weakly decreasing order. Thus a partition λ of n into m parts will be expressed as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and $\sum_{i\geq 1} \lambda_i = n$.

Theorem 5.1. *If a, b are integers with $a, b > 1$ and $a + b > 9$, then*

$$p(a)p(b) > p(a + b).$$

Bessenrodt and Ono [13] proved Theorem 5.1 using a classical analytic result of D. H. Lehmer [30], and subsequently asked for a combinatorial argument for proving the theorem. The purpose of this chapter is to provide such a proof of Theorem 5.1.

The Cartesian product of two sets of partitions, P and Q, will be denoted by $P \oplus Q$, that is,

$$P \oplus Q = \{(\lambda; \pi) \mid \lambda \in P, \pi \in Q\}.$$

When $p = |P|$ and $q = |Q|$, we will associate the number pq with the cardinality of the set $P \oplus Q$.

We will give a combinatorial proof of Theorem 5.1 by cutting partitions and then showing inductively that it is sufficient to establish an injective map

$$f_0 : p[c+d \,|\, \text{no 1's and no 2's}] \longrightarrow p[c \,|\, \text{no 1's}] \oplus p[d \,|\, \text{no 2's}]$$

for integers $c, d > 1$ with $c+d > 4$. Finding such a decent map is possible since one can work with partitions of c that contain 2's and partitions of d that contain 1's.

This method is expected to carry over to other classes of partitions. For example, it is indicated in Section 5.3 how this combinatorial technique may be applied to 2-regular partitions.

The content of this chapter is largely taken from the article [4]. I am appreciative to Prof. Munagi and Dr. Gagola III for their permissions to include our joint work in this thesis.

5.2 Combinatorial Proof of Theorem 5.1

In order to prove the theorem we first establish a few lemmas.

Lemma 5.1. *If c, d are integers with $c \geq 2$, $d \geq 2$ and $c+d \geq 5$, then*

$$p(c \,|\, \text{no 1's})p(d \,|\, \text{no 2's}) \geq p(c+d \,|\, \text{no 1's and no 2's}).$$

Note that when $c+d = 4$, we have $(1)(1) \geq 1$ since $p(2 \,|\, \text{no 1's})p(2 \,|\, \text{no 2's}) \geq p(4 \,|\, \text{no 1's and no 2's})$.

Proof. For $\lambda = (\lambda_1, ..., \lambda_t) \in p[c+d]$ let

$$i = i(\lambda) = \max\{j \in \mathbb{N} \,|\, 1 \leq j \leq t, \lambda_j + \cdots + \lambda_t \geq d\}.$$

Furthermore, let $\lambda_i = x + y$ ($x = x(\lambda)$, $y = y(\lambda)$) such that

$$x + \lambda_{i+1} + \cdots + \lambda_t = d \quad \text{and} \quad y + \lambda_1 + \cdots + \lambda_{i-1} = c.$$

Note that $0 < x \leq \lambda_i$. Now define a map

$$f_0 : p[c+d \,|\, \text{no 1's and no 2's}] \rightarrow p[c \,|\, \text{no 1's}] \oplus p[d \,|\, \text{no 2's}]$$

as follows. For $\lambda = (\lambda_1, \ldots, \lambda_t) \in p[c+d \mid \text{no 1's and no 2's}]$,

$$f_0(\lambda) = \begin{cases} (\lambda_1, ..., \lambda_{i-1};\ \lambda_i, ..., \lambda_t) & \text{if } y = 0; \\ (\lambda_1, ..., \lambda_{i-1}, y;\ \lambda_{i+1}, ..., \lambda_t, 1^x) & \text{if } y \geq 2; \\ \left(\lambda_1, ..., \lambda_{i-2}, \left\lceil \frac{\lambda_{i-1}+1}{2} \right\rceil, \left\lfloor \frac{\lambda_{i-1}+1}{2} \right\rfloor;\ \lambda_{i+1}, ..., \lambda_t, 1^x\right) & \text{if } y = 1. \end{cases}$$

Here, f is clearly well-defined. Note that if $y = 1$, then $x \geq 2$ and $\lambda_{i-1} - y \geq 2$ whereas $\left\lceil \frac{\lambda_{i-1}+1}{2} \right\rceil - \left\lfloor \frac{\lambda_{i-1}+1}{2} \right\rfloor \in \{0, 1\}$. Hence, f_0 is one-to-one and the result follows. $\square$

An illustration of the injections f_0 is given in Table 5.1.

$p[6+8 \mid \text{no 1's and no 2's}]$	$\xrightarrow{f_0}$	$p[6 \mid \text{no 1's}] \oplus p[8 \mid \text{no 2's}]$
(14)	$\to$	(6; 8)
(11,3)	$\to$	(6; 3,1^5)
(10,4)	$\to$	(6; 4,1^4)
(9,5)	$\to$	(6; 5,1^3)
(8,6)	$\to$	(6; 6,1^2)
(8,3,3)	$\to$	(6; 3,3,1^2)
(7,7)	$\to$	(6; 7,1)
(7,4,3)	$\to$	(6; 4,3,1)
(6,5,3)	$\to$	(6; 5,3)
(6,4,4)	$\to$	(6; 4,4)
(5,5,4)	$\to$	(3,3; 4,1^4)
(5,3,3,3)	$\to$	(3,3; 3,3,1^2)
(4,4,3,3)	$\to$	(4,2; 3,3,1^2)

Table 5.1: The injections of Lemma 5.1 for $c = 6$ and $d = 8$.

Lemma 5.2. *If a is an integer with $a > 2$, then*

$$p(a \mid \text{no 2's})p(2) \geq p(a+2 \mid \text{no 2's}).$$

Moreover, if $a > 5$, then

$$p(a \mid \text{no 2's})p(2) > p(a+2 \mid \text{no 2's}).$$

Proof. Denote a partition λ of $a+2$ by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t, 1^s)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 1$. Define a map

$$f_1 : p[a+2 \mid \text{no 2's}] \to p[a \mid \text{no 2's}] \oplus p[2]$$

by

$$f_1(\lambda) = \begin{cases} (\lambda_1, \ldots, \lambda_{t-1} - 1, \lambda_t - 1, 1^s;\ 1,\ 1) & \text{if } t \geq 2 \text{ and } \lambda_t > 3; \\ (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - 2, 1^s;\ 2) & \text{if } t \geq 2 \text{ and } \lambda_t = 3; \\ (\lambda_1, \ldots, \lambda_t, 1^{s-2};\ 1,\ 1) & \text{if } t < 2 \text{ and } s \geq 2; \\ (1^{\lambda_1 - 1};\ 2) & \text{if } t = 1 \text{ and } s = 1; \\ (\lambda_1 - 2;\ 2) & \text{if } t = 1 \text{ and } s = 0. \end{cases}$$

Since $a > 2$, f_1 is well-defined. Therefore, since f_1 is one-to-one,

$$p(a+2 \mid \text{no 2's}) \leq p(a \mid \text{no 2's})p(2).$$

Furthermore, if $a > 5$, then $(a-3, 3; 2)$ is not contained in the image of f_1. Hence, for $a > 5$,

$$p(a+2 \mid \text{no 2's}) < p(a \mid \text{no 2's})p(2).$$

□

The injections f_1 are illustrated in Table 5.2 when $a = 6$.

$p[6+2 \mid \text{no 2's}]$	$\xrightarrow{f_1}$	$p[6 \mid \text{no 2's}] \oplus p[2]$
(8)	$\to$	(6; 2)
(7,1)	$\to$	(1^6; 2)
(6,1,1)	$\to$	(6; 1,1)
(5,3)	$\to$	(5,1; 2)
(5,1,1,1)	$\to$	(5,1; 1,1)
(4,4)	$\to$	(3,3; 1,1)
(4,3,1)	$\to$	(4,1^2; 2)
(4,1,1,1,1)	$\to$	(4,1^2; 1,1)
(3,3,1,1)	$\to$	(3,1^3; 2)
(3,1,1,1,1,1)	$\to$	(3,1^3; 1,1)
(1,1,1,1,1,1,1,1)	$\to$	(1^6; 2)

Table 5.2: The injections of Lemma 5.2 for $a = 6$.

Lemma 5.3. *If a, b are integers with $a \geq b \geq 3$ and $a + b > 7$, then*

$$p(a \mid \text{no 2's})p(b) > p(a + b \mid \text{no 2's}).$$

Proof. Let $n = a + b$. We will apply induction on n.
Base case: It can easily be checked that the inequality holds for $n = 8$.
Inductive step: Suppose that $n > 8$ and that the inequality holds for $n - 1$. Thus, if $b \geq 4$, then $p(a \mid \text{no 2's})p(b-1) > p(a+b-1 \mid \text{no 2's})$ by the inductive hypothesis. Likewise, if $b = 3$, then $p(a \mid \text{no 2's})p(b-1) > p(a+b-1 \mid \text{no 2's})$ by Lemma 5.2. Hence, by Lemma 5.1,

$$\begin{aligned}
&p(a + b \mid \text{no 2's}) = \\
&\quad = p(a + b \mid \text{no 2's and at least one 1}) + p(a + b \mid \text{no 2's and no 1's}) \\
&\quad = p(a + b - 1 \mid \text{no 2's}) + p(a + b \mid \text{no 2's and no 1's}) \\
&\quad < p(a \mid \text{no 2's})p(b - 1) + p(a + b \mid \text{no 2's and no 1's}) \\
&\quad = p(a \mid \text{no 2's})p(b \mid \text{at least one 1}) + p(a + b \mid \text{no 2's and no 1's}) \\
&\quad = p(a \mid \text{no 2's}) \left[p(b) - p(b \mid \text{no 1's})\right] + p(a + b \mid \text{no 2's and no 1's}) \\
&\quad \leq p(a \mid \text{no 2's}) \left[p(b) - p(b \mid \text{no 1's})\right] + p(a \mid \text{no 2's})p(b \mid \text{no 1's}) \\
&\quad = p(a \mid \text{no 2's})p(b).
\end{aligned}$$

Therefore, by the principle of mathematical induction, the inequality holds for $n \geq 8$. □

Proof of Theorem 5.1

Let $n = a + b$. We apply induction on n.
Base case: It can easily be checked and shown that the inequality holds for $n \in \{10, 11\}$.
Inductive step: Suppose that $n \geq 12$ and that the inequality holds for $n - 1$. Without loss of generality, assume that $a \geq b$. Thus, by the inductive hypothesis, $p(a+b-2) < p(a-2)p(b)$. By Lemmas 5.2 and 5.3, $p(a+b|\text{no 2's}) < p(a|\text{no 2's})p(b)$. Hence,

$$\begin{aligned}
p(a + b) &= p(a + b \mid \text{at least one 2}) + p(a + b \mid \text{no 2's}) \\
&= p(a + b - 2) + p(a + b \mid \text{no 2's})
\end{aligned}$$

$$
\begin{aligned}
&< p(a+b-2) + p(a \mid \text{no 2's})p(b) \\
&< p(a-2)p(b) + p(a \mid \text{no 2's})p(b) \\
&= p(a-2)p(b) + [p(a) - p(a \mid \text{at least one 2})]\, p(b) \\
&= p(a-2)p(b) + [p(a) - p(a-2)]\, p(b) \\
&= p(a)p(b).
\end{aligned}
$$

Therefore, by the principle of mathematical induction, the inequality holds for $n \geq 10$. □

5.3 An Inequality for ℓ-Regular Partitions

The number of ℓ-regular partitions (that is, partitions with no part divisible by ℓ) is commonly denoted by $p_\ell(n)$. An analogous theorem dealing with ℓ-regular partitions is stated by Beckwith-Bessenrodt in [11].

Theorem 5.2. *For an integer ℓ with $2 \leq \ell \leq 6$ define n_ℓ, m_ℓ by the following table*

ℓ	2	3	4	5	6
n_ℓ	3	2	2	2	2
m_ℓ	22	17	9	9	9

Then for any positive integers a, b with $a, b \geq n_\ell$ and $a + b \geq m_\ell$ we have

$$p_\ell(a)p_\ell(b) > p_\ell(a+b).$$

The proof of Theorem 5.2 uses a result by Hagis which is an analogue of a classical analytic result of D. H. Lehmer [30]. A combinatorial proof may be approached like the proof of Theorem 5.1. We present a combinatorial proof for $\ell = 2$ below. The proof method for $3 \leq \ell \leq 6$ is similar to that of $\ell = 2$, but the number of cases increases rapidly for larger ℓ so we omit those for $3 \leq \ell \leq 6$.

Notations for the number and set of such partitions that fulfill prescribed conditions are analogous to those for $p(n)$.

5.3.1 Proof of Theorem 5.2 for $\ell = 2$

Theorem 5.3. *If a, b are integers with $a \geq b \geq 3$ and $a + b > 21$, then*

$$p_2(a)p_2(b) > p_2(a+b).$$

We give a best-possible adaptation of the results and proofs in Section 5.2.

Lemma 5.4. *If c, d are integers with $c \geq 5$ and $d \geq 2$, then*

$$p_2(c \mid \text{no 1's})p_2(d \mid \text{no 3's}) \geq p_2(c+d \mid \text{no 1's and no 3's}).$$

Proof. We adopt the notations introduced in the proof of Lemma 5.1 only wary of the fact that all parts in all partitions are now odd. Define a map

$$f_2 : p_2[c+d \mid \text{no 1's and no 3's}] \to p_2[c \mid \text{no 1's}] \oplus p_2[d \mid \text{no 3's}]$$

as follows. For $\lambda = (\lambda_1, \ldots, \lambda_t) \in p_2[c+d \mid \text{no 1's and no 3's}]$,

$$f_2(\lambda) = \begin{cases} (\lambda_1, ..., \lambda_{i-1};\ \lambda_i, ..., \lambda_t) & \text{if } y = 0; \\ (\lambda_1, ..., \lambda_{i-1}, y;\ \lambda_{i+1}, ..., \lambda_t, 1^x) & \text{if } y \equiv 1 \pmod 2,\ y \geq 3; \\ \Big(\lambda_1, ..., \lambda_{i-2}, 2\left\lceil \frac{\lambda_{i-1}-1}{4} \right\rceil + 1, & \\ \quad 2\left\lfloor \frac{\lambda_{i-1}-1}{4} \right\rfloor + 1;\ \lambda_{i+1}, ..., \lambda_t, 1^x\Big) & \text{if } y = 1; \\ (\lambda_1, ..., \lambda_{i-1} - 2, y-1, 3;\ \lambda_{i+1}, ..., \lambda_t, 1^x) & \text{if } y \equiv 0 \pmod 2,\ y \geq 4,\ i \neq 1; \\ (y-3, 3;\ \lambda_{i+1}, ..., \lambda_t, 1^x) & \text{if } y \equiv 0 \pmod 2,\ i = 1; \\ (\lambda_1 + 2, \lambda_2, ..., \lambda_{i-1};\ \lambda_{i+1}, ..., \lambda_t, 1^x) & \text{if } y = 2. \end{cases}$$

Note that if $y = 1$, then $x \geq 4$ and $\lambda_{i-1} - y \geq 4$. Hence, since f_2 is well-defined and one-to-one, the result follows. $\square$

Lemma 5.5. *If a is an integer with $a > 8$, then*

$$p_2(a \mid \text{no 3's})p_2(3) > p_2(a+3 \mid \text{no 3's}).$$

Proof. Denote a partition λ of $n = a+3$ by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$. Define a map

$$f_2 : p_2[a+3 \mid \text{no 3's}] \to p_2[a \mid \text{no 3's}] \oplus p_2[3]$$

by

$$f_2 : \lambda \mapsto \begin{cases} (\lambda_1, \ldots, \lambda_{t-3};\ 3) & \text{if } \lambda_{t-2} = \lambda_{t-1} = \lambda_t = 1; \\ (\lambda_1, \ldots, \lambda_{t-2} - 2, 1;\ 1^3) & \text{if } \lambda_{t-1} = \lambda_t = 1 \text{ and } \lambda_{t-2} \geq 7; \\ (\lambda_2, \ldots, \lambda_{t-2}, 1^{\lambda_1 - 1};\ 1^3) & \text{if } \lambda_{t-1} = \lambda_t = 1 \text{ and } \lambda_{t-2} = 5; \\ (\lambda_1, \ldots, \lambda_{t-1} - 2;\ 1^3) & \text{if } \lambda_t = 1 \text{ and } \lambda_{t-1} \geq 7; \\ (\lambda_1, \ldots, \lambda_{t-2}, 1^3;\ 1^3) & \text{if } \lambda_t = 1 \text{ and } \lambda_{t-1} = 5; \\ (\lambda_1, \ldots, \lambda_{t-1}, 1^{\lambda_t - 3};\ 1^3) & \text{if } \lambda_t \geq 5. \end{cases}$$

Since $a > 4$, f_2 is well-defined. Note that, in order for the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{t-3}, 5, 1, 1)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_s)$, with $\lambda'_s \geq 5$, to have the same image under f_2, $\lambda'_{s-1} = \lambda_{t-2} = 5$. In this case, $\lambda'_s = 5$ and $\lambda_1 \geq \lambda_{t-2} = 5$. But then $\lambda_1 - 1 \geq 4 > 2 = \lambda'_s - 3$ implying that $f_2(\lambda) \neq f_2(\lambda')$. Thus, f_2 is one-to-one and $p_2(a+3 \mid \text{no 3's}) \leq p_2(a \mid \text{no 3's}) p_2(3)$. Furthermore, since $a > 8$, $(1^a; 1^3)$ is not contained in the image of f_2 when a is odd and $(5, 1^{a-5}; 1^3)$ is not contained in the image of f_2 when a is even. Hence, for $a > 8$, $p_2(a{+}3|\text{no 3's}) < p_2(a|\text{no 3's})p_2(3)$. □

Lemma 5.6. *If a is an integer with $a > 13$, then*

$$p_2(a \mid \text{no 3's}) p_2(4) > p_2(a+4 \mid \text{no 3's}).$$

Proof. The inequality can easily be checked for $14 \leq a \leq 23$. Suppose now that $a > 23$. Denote a partition λ of $n = a+4$ by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t, 1^s)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 1$. Define a map

$$f_2 : p_2[a+4 \mid \text{no 3's}] \to p_2[a \mid \text{no 3's}] \oplus p_2[4]$$

by

$$f_2 : \lambda \mapsto \begin{cases} (\lambda_1, \ldots, \lambda_{t-1} - 2, \lambda_t - 2, 1^s;\ 1^4) & \text{if } t \geq 2 \text{ and } \lambda_t > 5; \\ (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - 4, 1^s;\ 3, 1) & \text{if } t \geq 2 \text{ and } \lambda_t = 5;; \\ (\lambda_1, \ldots, \lambda_t, 1^{s-4};\ 1^4) & \text{if } t < 2 \text{ and } s \geq 4; \\ (\lambda_1 - 20, 5, 5, 5, 1^{s+1};\ 1^4) & \text{if } t = 1 \text{ and } s < 4. \end{cases}$$

Since $a > 23$, $n > 27$, f_2 is well-defined. Furthermore, f_2 is one-to-one and $p_2(a+4 \mid \text{no 3's}) \leq p_2(a \mid \text{no 3's}) p_2(4)$. Since $a > 23$, $(a;\ 3, 1)$ is not contained in the image of f_2 when a is odd and $(a-5, 5;\ 3, 1)$ is not contained in the image of f_2 when a is even. Hence, $p_2(a+4 \mid \text{no 3's}) < p_2(a \mid \text{no 3's}) p_2(4)$. □

Lemma 5.7. *If a, b are integers with $a \geq b \geq 5$ and $a + b > 17$, then*

$$p_2(a \mid \text{no 3's})p_2(b) > p_2(a + b \mid \text{no 3's}).$$

Proof. Let $n = a + b$. We will apply induction on n.
Base case: It can easily be checked that the inequality holds for $n = 18$.
Inductive step: Suppose that $n > 18$ and that the inequality holds for $n-1$. Thus, if $b \geq 6$, then $p_2(a \mid \text{no 3's})p_2(b-1) > p_2(a+b-1 \mid \text{no 3's})$ by the inductive hypothesis. Likewise, if $b = 5$, then $p_2(a \mid \text{no 3's})p_2(b-1) > p_2(a+b-1 \mid \text{no 3's})$ by Lemma 5.6. Hence, by Lemma 5.4,

$$\begin{aligned}
&p_2(a+b \mid \text{no 3's}) = \\
&\quad = p_2(a+b \mid \text{no 3's and at least one 1}) + p_2(a+b \mid \text{no 3's and no 1}) \\
&\quad = p_2(a+b-1 \mid \text{no 3's}) + p_2(a+b \mid \text{no 3's and no 1's}) \\
&\quad < p_2(a \mid \text{no 3's})p_2(b-1) + p_2(a+b \mid \text{no 3's and no 1's}) \\
&\quad = p_2(a \mid \text{no 3's})p_2(b \mid \text{at least one 1}) + p_2(a+b \mid \text{no 3's and no 1's}) \\
&\quad = p_2(a \mid \text{no 3's})\left[p_2(b) - p_2(b \mid \text{no 1's})\right] + p_2(a+b \mid \text{no 3's and no 1's}) \\
&\quad \leq p_2(a \mid \text{no 3's})\left[p_2(b) - p_2(b \mid \text{no 1's})\right] + p_2(a \mid \text{no 3's})p_2(b \mid \text{no 1's}) \\
&\quad = p_2(a \mid \text{no 3's})p_2(b).
\end{aligned}$$

Therefore, by the principle of mathematical induction, the inequality holds for $n \geq 18$. $\square$

Proof of Theorem 5.3

Let $n = a + b$. We apply induction on n.
Base case: It can easily be checked and shown that the inequality holds for $n \in \{22, 23, 24\}$.
Inductive step: Suppose that $n \geq 25$ and that the inequality holds for $n - 1$ and $n - 2$. Without loss, assume that $a \geq b$ and therefore $a - 3 \geq 10$. Thus, by the inductive hypothesis, $p_2(a + b - 3) < p_2(a - 3)p_2(b)$. By Lemmas 5.5, 5.6 and 5.7, $p_2(a + b \mid \text{no 3's}) < p_2(a \mid \text{no 3's})p_2(b)$. Hence,

$$\begin{aligned}
p_2(a+b) &= p_2(a+b \mid \text{at least one 3}) + p_2(a+b \mid \text{no 3's}) \\
&= p_2(a+b-3) + p_2(a+b \mid \text{no 3's})
\end{aligned}$$

$$
\begin{aligned}
&< p_2(a+b-3) + p_2(a \mid \text{no 3's})p_2(b) \\
&< p_2(a-3)p_2(b) + p_2(a \mid \text{no 3's})p_2(b) \\
&= p_2(a-3)p_2(b) + [p_2(a) - p_2(a \mid \text{at least one 3})]\, p_2(b) \\
&= p_2(a-3)p_2(b) + [p_2(a) - p_2(a-3)]\, p_2(b) \\
&= p_2(a)p_2(b).
\end{aligned}
$$

Therefore, by the principle of mathematical induction, the inequality holds for $n \geq 22$. $\square$

We have the following conjecture in relation to overpartitions.

Conjecture 5.1. *If a, b are integer with $a, b \geq 1$, then*

$$\overline{p}(a)\overline{p}(b) \geq \overline{p}(a+b).$$

Chapter 6

Partition Configurations

Andrews and Deutsch [8] recently devised a proof technique for the Stanley-Elder identities using "partition configurations" (defined below), and stated a parallel result based on the divisibility of parts. Gilbert [21] explored the origins of the Stanley-Elder theorems and indicated that the theorems were originally discovered by N. J. Fine [20, 19]. Dastidar and Gupta [18] considered certain generalizations of the theorems and developed Ramanujan-type congruence properties for $g_k(n)$. Further relevant work on this problem may be found in Knopfmacher and Munagi [29].

In this chapter, we give combinatorial proofs of the main results in [8] and establish new generalizations. Our bijective proofs rely mostly on variations of Stanley's proof of Elder's theorem (see Section 2.2).

Definition 6.1. A *partition configuration*, A, is a finite nondecreasing sequence of non-negative integers containing 0. The weight of a partition configuration $A = (a_1, \ldots, a_k)$, of length k, is given by $w(A) = a_1 + a_2 + \cdots + a_k$.

Definition 6.2. A partition λ is said to contain a partition configuration $(a_1, \ldots, a_k)$ if there is a distinct subsequence of parts of λ of the form $a_1 + j, a_2 + j, \ldots, a_k + j$ for some integer $j > 0$.

For example, the partition $(1 + 2 + 2 + 4 + 4 + 5 + 8 + 9 + 9)$ contains an instance of $A = (0, 3, 6, 7)$ because the parts $2, 5, 8, 9$ exceed by 2 the successive entries of A.

The number of parts function $g_k(n)$ is defined as the total number of occurrences of k in all partitions of n. The first main result in [8] is the configuration theorem:

Theorem 6.1. *(Andrews-Deutsch)*
Let A be a partition configuration of length k. The total number of configurations A in all partitions of n is equal to $g_k(n - w(A))$.

The second main result concerns divisibility of parts:

Theorem 6.2. *(Andrews-Deutsch)*
Given $k \geq 1$, in each partition of n we count the number of times a part divisible by k appears uniquely (i.e. is not a repeated part); then sum these numbers over all the partitions of n. The result is equal to $g_{2k}(n + k)$.

In Section 6.1 we state a reformulation of Theorem 6.1 and discuss the consequences and proofs. In Section 6.2 we present generalizations of Elder's theorem and Theorem 6.1. Section 6.3 is devoted to a combinatorial proof of Theorem 6.2. An extension of the theorem is proved using generating functions. Lastly, Section 6.4 contains additional properties of the function $g_k(n)$.

The content of this chapter is largely taken from the article [3]. I am appreciative to Prof. Munagi for his permission to include our joint work in this thesis.

6.1 Combinatorial proof of the configuration theorem

Theorem 6.1 depends on the weight $w(A)$ and length of a partition configuration A but not on specification of the parts. Since $0 \in A$, we can recover A from any of its occurrences in a partition λ. Thus if $(b_1, \ldots, b_k) \subseteq \lambda$ represents an occurrence of A, then $A = (0, b_2 - b_1, \ldots, b_k - b_1)$ is an expression containing $k - \ell$ initial zeros and a partition of $w(A)$ into ℓ parts, $0 \leq \ell < k$. So Theorem 6.1 does not rely on the length k as a stringent defining property of A but rather as a preferred measure for traversing partition subsequences. Thus a configuration may be identified with the partition determined by its nonzero parts. The foregoing observations lead to the following definition.

Definition 6.3. Given a positive integer k and a partition $\beta = (\beta_1, \ldots, \beta_\ell)$, $0 \leq \ell < k$, a *translate* of β, of length k, is any k-part partition of the form

$$A(\beta, k)_j = (j, \ldots, j, \beta_1 + j \ldots, \beta_\ell + j),$$

where j is a positive integer and appears with multiplicity $k - \ell > 0$ as a part.

We see that a partition λ contains a configuration

$$A = A(\beta, k) = (0, \ldots, 0, \beta_1, \ldots, \beta_\ell)$$

if and only if the sequence of parts of λ contains a distinct translate of the underlying (possibly empty) partition $(\beta_1, \ldots, \beta_\ell)$.

Theorem 6.1 then implies the following more inclusive statement.

Theorem 6.3. *Let n, m, k be positive integers with $k \leq n$, $0 \leq m < n$, and let β be a partition of m into less than k parts. The number of distinct translates of β, of length k, in all partitions of n is equal to the number of k's in all partitions of $n - m$.*

Note that Theorem 6.1 may be obtained from Theorem 6.3 by specifying β with k. For example if $m = 5$ and $k = 4$, the translates of each

$$\beta \in \{(5), (1, 4), (2, 3), (1, 1, 3), (1, 2, 2)\}$$

give the same number of 4's in partitions of $n > 5$, where $\beta = (5) \implies A = (0, 0, 0, 5)$, $\beta = (1, 4) \implies A = (0, 0, 1, 4)$, and so forth.

We remark that the generating function proof of Theorem 6.1 given in [8] is sufficient to prove Theorem 6.3 since m is equal to the weight $w(A)$ of any partition configuration A with the given length. For completeness we reproduce the proof here. Let $T(n, A(\beta, k))$ denote the number of distinct translates of β, of length k, in all partitions of n. Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} T(n, A(\beta, k)) q^n &= \sum_{j=1}^{\infty} \frac{q^{(j+\beta_1)+(j+\beta_2)+\cdots+(j+\beta_k)}}{\prod_{n=1}^{\infty}(1-q^n)} \\
&= \frac{q^m \sum_{j=1}^{\infty} q^{kj}}{\prod_{n=1}^{\infty}(1-q^n)}, \text{ where } m = \textstyle\sum_{i=1}^{k} \beta_i \\
&= \frac{q^{m+k}}{(1-q^k)^2 \prod_{\substack{n=1 \\ n \neq k}}^{\infty} (1-q^n)} \\
&= \frac{q^{m+k}(1+q^k+q^{2k}+\cdots)^2}{\prod_{\substack{n=1 \\ n \neq k}}^{\infty} (1-q^n)} \\
&= \frac{q^m(q^k+q^{2k}+q^{3k}+\cdots)(1+q^k+q^{2k}+\cdots)}{\prod_{\substack{n=1 \\ n \neq k}}^{\infty} (1-q^n)}
\end{aligned}$$

$$= q^m(q^k + 2q^{2k} + 3q^{3k} + \cdots) \prod_{\substack{n=1 \\ n\neq k}}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \cdots)$$

$$= q^m \sum_{n=0}^{\infty} g_k(n) q^n.$$

In the next subsection, we present our bijective proof.

We first note some proof applications of Theorem 6.3. The set of translates of $\beta = (\beta_1, \cdots, \beta_\ell)$ will be denoted by $A[\beta, k]$:

$$A[\beta, k] = \{(1, \ldots, 1, \beta_1 + 1 \ldots, \beta_\ell + 1), (2, \ldots, 2, \beta_1 + 2 \ldots, \beta_\ell + 2), \ldots\}.$$

Stanley's Theorem: Take $\beta = \emptyset$ and $k = 1$; thus $m = 0$ and

$$A[\emptyset, 1] = \{(1), (2), \ldots\}.$$

Elder's Theorem: Take $\beta = \emptyset$ and $k \geq 1$; thus $m = 0$ and

$$A[\emptyset, k] = \{(1, \ldots, 1), (2, \ldots, 2), \ldots\}.$$

The following result was discovered independently by Knopfmacher and Munagi [29] and Andrews and Deutsch [8]:

The number of sequences of elements of a multiset of k consecutive integers in all partitions of n is equal to $g_k(n - \binom{k}{2})$.

To prove the statement take $\beta = (1, 2, \ldots, k-1)$ and $k \geq 1$; thus $m = \binom{k}{2}$ and

$A_k[\beta, k] = \{(1, 2, \ldots, k), (2, 3, \ldots, k+1), \ldots\}$.

6.1.1 Proof of Theorem 6.3

Let $T[n, A[\beta, k]]$ denote the multiset of translates of β of length k in partitions of n, and let $g_k[n]$ be the multiset of k's in partitions of n, that is $|g_k[n]| = g_k(n)$.
We describe a bijection $\theta : T[n, A[\beta, k]] \to g_k[n - m]$ as follows.

If $\beta = (\beta_1, \ldots, \beta_\ell) \vdash m$, where the notation $\beta \vdash m$ means that β is a partition of m, then $(j, \ldots, j, \beta_1 + j, \ldots, \beta_\ell + j) \in T(n, A(\beta, k))$, and

$$\theta : \; (j, \ldots, j, \beta_1 + j, \ldots, \beta_\ell + j) \longmapsto \underbrace{k, \ldots, k}_{j \text{ copies}}. \tag{6.1}$$

In practical terms, we write down all partitions λ of n containing a length k translate of β, writing down each partition λ of n as many times as there are length k translates of β within it. For each of these partitions take an instance of $(j, \ldots, j, \beta_1 + j, \ldots, \beta_\ell + j)$ within it, and remove these parts and replace them by j parts equal to k. This produces a list of partitions of $n - m$ in which each partition containing r parts equal to k occurs exactly r times.

Conversely, if a partition $\gamma \vdash n - m$ contains r copies of k, then for each $j \in \{1, \ldots, r\}$, we use (6.1) to map γ to a partition of n containing j translates of a fixed partition of m. So γ produces r partition pre-images.

This gives the asserted bijection. $\square$

The bijection is illustrated in Table 6.1 for $n = 11$, $\beta = (3)$ and $k = 3$. Using a larger value of n, say $n = 15$, then, for instance, $(4, 4, 7)$ with translate $(4, 4, 7)$ maps to $(3, 3, 3, 3)$ while $(1, 1, 2, 2, 4, 5)$ with translates $(1, 1, 4)$, $(2, 2, 5)$ maps to $(2, 2, 3, 5)$ and $(1, 1, 3, 3, 4)$ respectively. Conversely $(3, 3, 3, 3)$ has the following pre-images corresponding respectively to $1, 2, 3$ and 4 copies of 3: $(1, 1, 3, 3, 3, 4)$, $(2, 2, 3, 3, 5), (3, 3, 3, 6), (4, 4, 7)$, and so forth.

$T[11, A[(3), 3]]$	translates	$g_3[8]$
(1, 1, 4, 5)	(1, 1, 4)	(3, 5)
(2, 2, 2, 5)	(2, 2, 5)	(2, 3, 3)
(1, 1, 1, 4, 4)	(1, 1, 4)	(1, 3, 4)
(1, 1, 2, 2, 5)	(2, 2, 5)	(1, 1, 3, 3)
(1, 1, 2, 3, 4)	(1, 1, 4)	(2, 3, 3)
(1, 1, 1, 1, 3, 4)	(1, 1, 4)	(1, 1, 3, 3)
(1, 1, 1, 2, 2, 4)	(1, 1, 4)	(1, 2, 2, 3)
(1, 1, 1, 1, 1, 2, 4)	(1, 1, 4)	(1, 1, 1, 2, 3)
(1, 1, 1, 1, 1, 1, 1, 4)	(1, 1, 4)	(1, 1, 1, 1, 1, 3)

Table 6.1: The bijection $T[11, A[\beta], 3] \to g_3[8]$where $\beta = (3)$

Note the following property of Table 6.1 which is analogously shared by all such tables:

"*Each partition in the first column appears as many times as the number of β-translates it contains and each partition in the third column appears as many times* *as the number of k's it contains".*

Remark 3. *The map θ can be factored into a composition of two bijections as follows:*
A de-configuration or leveling map: $\rho : T[n, A[\beta, k]] \to T[n-m, A[\emptyset, k]]$, where

$$\rho : \ (a_1, \ldots, a_k) \longmapsto (a_1, \ldots, a_k) - A(\beta, k)_0 = (a_1, a_1, \ldots, a_1).$$

The Elder map: $\varepsilon : T[n, A[\emptyset, k]] \to g_k[n]$, where

$$\varepsilon : \ (a, \ldots, a) \longmapsto \underbrace{k, \ldots, k}_{a \text{ copies}}.$$

The bijection ε (strictly $\varepsilon = \theta_{|(j,\ldots,j)}$) was popularized by Richard Stanley [38] who used it to prove Elder's theorem.

Then we see that $\theta = \varepsilon\rho$.

6.2 Generalization of the Elder and Configuration Theorems

In this section we give natural extensions of Elder's theorem and Theorem 6.3.

Let $v_k(n, t)$ denote the number of multiples of k appearing at least t times in all partitions of n. Thus Elder's theorem takes the compact form

$$g_k(n) = v_1(n, k). \tag{6.2}$$

Theorem 6.4. *The number of multiples of k appearing at least t times in all partitions of n equals the number of tk's in all partitions of n:*

$$v_k(n, t) = g_{tk}(n), \quad t = 1, 2, \ldots.$$

Note that Theorem 6.4 becomes Elder's theorem when $k = 1$.

Proof. Let $v_k[n, t]$ be the set of objects enumerated by $v_k(n, t)$. Define the map $\varepsilon_{k,t} : v_k[n, t] \to g_{tk}[n]$ as follows. If $\lambda \vdash n$ contains $r \geq t$ copies of mk, replace t copies of mk by m copies of tk:

$$\varepsilon_{k,t} : \ \underbrace{mk, \ldots, mk}_{t \text{ copies}} \longmapsto \underbrace{tk, \ldots, tk}_{m \text{ copies}}.$$

Conversely, if $\lambda \vdash n$ contains r copies of tk, then for each $j \in \{1, 2, \ldots, r\}$ replace j copies of tk with t copies of jk:

$$\varepsilon_{k,t}^{-1} : \underbrace{tk, \ldots, tk}_{j \text{ copies}} \longmapsto \underbrace{jk, \ldots, jk}_{t \text{ copies}}.$$

Thus $\varepsilon_{k,t}$ is a bijection. Hence the result.

The bijection is illustrated in Table 6.2 for $n = 12, k = 3, t = 2$. □

$v_3[12, 2]$	multiples of 3	$g_6[12]$
(6, 6)	(6, 6)	(6, 6)
(3, 3, 6)	(3, 3)	(6, 6)
(1, 3, 3, 5)	(3, 3)	(1, 5, 6)
(2, 3, 3, 4)	(3, 3)	(2, 4, 6)
(1, 1, 3, 3, 4)	(3, 3)	(1, 1, 4, 6)
(3, 3, 3, 3)	(3, 3)	(3, 3, 6)
(1, 2, 3, 3, 3)	(3, 3)	(1, 2, 3, 6)
(1, 1, 1, 3, 3, 3)	(3, 3)	(1, 1, 1, 3, 6)
(2, 2, 2, 3, 3)	(3, 3)	(2, 2, 2, 6)
(1, 1, 2, 2, 3, 3)	(3, 3)	(1, 1, 2, 2, 6)
(1, 1, 1, 1, 2, 3, 3)	(3, 3)	(1, 1, 1, 1, 2, 6)
(1, 1, 1, 1, 1, 1, 3, 3)	(3, 3)	(1, 1, 1, 1, 1, 1, 6)

Table 6.2: The bijection $v_k[12, t] \to g_{tk}[12]$ for $k = 3$, $t = 2$.

Remark 4. *Theorem 6.4 implies the symmetry property:* $v_k(n, t) = v_t(n, k)$.

Definition 6.4. Given positive integers k, t and a partition $\beta = (\beta_1, \ldots, \beta_\ell)$, $0 \leq \ell < t$, a *k-translate* of β, of length t, is any t-part partition of the form

$$A_k(\beta, t)_{jk} = (jk, \ldots, jk, \beta_1 + jk \ldots, \beta_\ell + jk),$$

where j is a positive integer and jk appears with multiplicity $t - \ell > 0$ as a part.

Thus $A_1(\beta, t)_j = A(\beta, t)_j$. We now state a generalization of Theorem 6.3.

Theorem 6.5. *Let n, m, k, t be positive integers with $t \leq n$, $0 \leq m < n$, and let β be a partition of m into less than t parts. The number of distinct k-translates of β, of length t, in all partitions of n is equal to the number of tk's in all partitions of $n - m$.*

Note that Theorem 6.5 reduces to Theorem 6.3 when $k = 1$.

Proof. We define a bijection $\theta_{k,t} : T[n, A_k[\beta, t]] \to g_{tk}[n - m]$. Since $\varepsilon = \varepsilon_{1,t}$ and $\theta = \theta_{1,t}$, it is clear that Remark 3 and the proof of Theorem 6.4 imply the definition

$$\theta_{k,t} = \varepsilon_{k,t}\rho.$$

This shows that $\theta_{k,t}$ may be realized as a composition of two bijections. □

As an illustration suppose that $k = 2, t = 5$ and $\beta = (1, 1, 4)$, then
$A_2[\beta, 5] = \{(2, 2, 3, 3, 6), (4, 4, 5, 5, 8), (6, 6, 7, 7, 10), \dots\}$.
So if $(2, 2, 3, 3, 6) \subseteq \lambda \vdash n$ for $T[n, A_2[\beta, 5]]$, then

$$(2, 2, 3, 3, 6) \xrightarrow{\rho} (2, 2, 2, 2, 2) \xrightarrow{\varepsilon_{2,5}} (10),$$

where $(10) \subseteq \lambda \vdash n - 6$ for $g_{10}[n - 6]$. Similarly,

$$(4, 4, 5, 5, 8) \xrightarrow{\rho} (4, 4, 4, 4, 4) \xrightarrow{\varepsilon_{2,5}} (10, 10);$$

and so forth.

An alternative proof of Theorem 6.4 may be deduced from Theorem 6.5 as follows: take $\beta = \emptyset$, $k \geq 1$ and $t \geq 1$ so that $A_k[\beta, t] = \{(k, \dots, k), (2k, \dots, 2k), \dots\}$.

6.3 Divisibility of parts

This section is devoted to the proof of Theorem 6.2. First we establish a related result.

Theorem 6.6. *We have*

$$g_k(n) = g_{2k}(n) + g_{2k}(n + k).$$

Proof. The theorem is a special case ($t = 2$) of Theorem 6.8 below. But we give a full bijection here: $g_{2k}[n] \cup g_{2k}[n + k] \longrightarrow g_k[n]$.

If $2k \in \lambda$, then

$$2k \mapsto \begin{cases} k, k & \text{if } 2k \in g_{2k}[n], \\ k & \text{if } 2k \in g_{2k}[n + k]. \end{cases}$$

The rule for $r > 1$ copies of $2k$, denoted by $(2k)^r$, is as follows:
(I) If $(2k)^r \in g_{2k}[n]$, then for each $j \in \{1, \ldots, r\}$ replace j copies of $2k$ with $2j$ copies of k.
(II) If $(2k)^r \in g[n+k]$, then for each $j \in \{1, \ldots, r\}$ replace j copies of $2k$ with $2j-1$ copies of k.
Note that if $(2k)^r \in \lambda$, then λ begets r image partitions for $g_k[n]$ in either case.

The inverse map is obtained by:
Let λ have $r \geq 1$ parts equal to k. Then we map λ to $\lfloor \frac{r}{2} \rfloor$ partitions of n by replacing $2j$ parts k by j parts $2k$ ($1 \leq j \leq \frac{r}{2}$), and also to $\lceil \frac{r}{2} \rceil$ partitions of $n+k$ by replacing $2j+1$ parts k by $j+1$ parts $2k$ for $0 \leq j < \frac{r}{2}$.

Hence the bijection.

The bijection is illustrated in Table 6.3. Note that if a partition λ contains r copies of k then λ appears $\lfloor \frac{r}{2} \rfloor$ times as an image of a member of $g_{2k}[n]$, and $\lfloor \frac{r+1}{2} \rfloor$ times as an image of a member of $g_{2k}[n+k]$.

□

$g_4[7] \cup g_4[9]$	$\rightarrow$	$g_2[7]$
(3,4)	$\rightarrow$	(2,2,3)
(1,2,4)	$\rightarrow$	(1,2,2,2)
(1,1,1,4)	$\rightarrow$	(1,1,1,2,2)
(4, 5)	$\rightarrow$	(2,5)
(1, 4, 4)	$\rightarrow$	(1,2,4)
(1, 4, 4)	$\rightarrow$	(1,2,2,2)
(2,3,4)	$\rightarrow$	(2,2,3)
(1,1,3,4)	$\rightarrow$	(1,1,2,3)
(1,2,2,4)	$\rightarrow$	(1,2,2,2)
(1,1,1,2,4)	$\rightarrow$	(1,1,1,2,2)
(1,1,1,1,1,4)	$\rightarrow$	(1,1,1,1,1,2)

Table 6.3: The bijection $g_{2k}[n] \cup g_{2k}[n+k] \rightarrow g_k[n]$ for $n = 7,\ k = 2$.

6.3.1 Proof and extension of Theorem 6.2

Define $f_k(n)$ as the number of times a multiple of k appears uniquely in all partitions of n. Then Theorem 6.2 takes the form

$$f_k(n) = g_{2k}(n+k). \tag{6.3}$$

The proof is deduced from Theorems 6.6 and 6.4:

$$g_{2k}(n+k) = g_k(n) - g_{2k}(n) = v_k(n,1) - v_k(n,2) = f_k(n).$$

$\square$

Now consider the function

$f_k(n,s) :=$ number of multiples of k appearing exactly s times in all partitions of n.

Thus $f_k(n,1) = f_k(n)$. By definition we have

$$f_k(n,s) = v_k(n,s) - v_k(n,s+1).$$

Hence from Theorem 6.4 we obtain:

Theorem 6.7. *We have*

$$f_k(n,s) = g_{sk}(n) - g_{(s+1)k}(n).$$

Note that Theorem 6.7 is a generalization of Theorem 6.2 since Equation (6.3) may be stated as

$$f_k(n,1) = g_k(n) - g_{2k}(n).$$

Proof. The generating function for $g_{sk}(n)$ is given by

$$\sum_{n=0}^{\infty} g_{sk}(n)q^n = (q^{sk} + 2q^{2sk} + 3q^{3sk} + \cdots) \prod_{\substack{n=1 \\ n\neq sk}}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \cdots)$$

$$= \frac{q^{sk}}{(1-q^{sk})^2} \prod_{\substack{n=1 \\ n\neq sk}}^{\infty} \frac{1}{1-q^n} = \frac{q^{sk}}{1-q^{sk}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} (g_{sk}(n) - g_{(s+1)k}(n))q^n &= \left(\frac{q^{sk}}{1-q^{sk}} - \frac{q^{(s+1)k}}{1-q^{(s+1)k}} \right) \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{j=1}^{\infty} q^{skj}(1-q^{kj}) \\
&= \sum_{j=1}^{\infty} \frac{q^{skj}}{\prod_{\substack{n=1 \\ n\neq kj}}^{\infty} (1-q^n)}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} f_k(n,s)q^n.$$

Equating the coefficients of q^n on both sides gives the theorem. □

Remark 5. *The function $f_1(n,s)$ has been tabulated in Sloane [37, A197126] under the description "number of cliques of size s in all partitions of $n \geq 1$" (where a "clique" refers to all parts in a partition with the same value). If we designate this as the sequence of 1-cliques, then $f_k(n,s)$ assumes the definition: "number of k-cliques of size s in all partitions of $n \geq 1$" (where a "k-clique" refers to all copies of a fixed multiple of k in a partition). Some of the sequences $f_k(n,s)$ are shown in Table 6.4.*

$f_1(n,s)$

n \ s	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	3	0	1	0	0
4	4	2	0	1	0
5	8	2	1	0	1
6	11	4	2	1	0
7	19	5	3	1	1
8	26	10	3	3	1
9	41	11	7	3	2
10	56	20	8	5	3

$f_2(n,s)$

n \ s	1	2	3	4	5
1	0	0	0	0	0
2	1	0	0	0	0
3	1	0	0	0	0
4	2	1	0	0	0
5	3	1	0	0	0
6	6	1	1	0	0
7	8	2	1	0	0
8	13	4	1	1	0
9	18	5	2	1	0
10	28	8	3	1	1

$f_3(n,s)$

n \ s	1	2	3
2	0	0	0
3	1	0	0
4	1	0	0
5	2	0	0
6	3	1	0
7	5	1	0
8	7	2	0
9	12	2	1
10	16	4	1
11	24	5	2

$f_4(n,s)$

n \ s	1	2	3
3	0	0	0
4	1	0	0
5	1	0	0
6	2	0	0
7	3	0	0
8	5	1	0
9	7	1	0
10	11	2	0
11	15	3	0
12	23	4	1

$f_5(n,s)$

n \ s	1	2	3
4	0	0	0
5	1	0	0
6	1	0	0
7	2	0	0
8	3	0	0
9	5	0	0
10	7	1	0
11	11	1	0
12	15	2	0
13	22	3	0

Table 6.4: Small values of $f_k(n,s)$ for $k = 1, 2, 3, 4$ and 5

6.4 Further properties of $g_k(n)$

The summation of Theorem 6.7 over s results in the following identity.

Corollary 6.1. *The number of k-cliques in all partitions of n equals the number of k's in all partitions of n:*

$$\sum_{s\geq 1} f_k(n,s) = g_k(n).$$

A bijective proof of Corollary 6.1 is given by $(mk)^s \longleftrightarrow (k)^{sm}$. This bijection is equivalent to $\varepsilon\sigma_k$, where $\sigma_k : mk \mapsto (m)^k$.

If an integer k occurs as a part of $\lambda \vdash n$, we can delete k from λ to obtain an arbitrary partition of $n-k$. So the number of partitions of n containing at least j copies of k is $p(n-jk)$; the number containing exactly j copies of k is $p(n-jk) - p(n-(j+1)k)$. Therefore

$$g_k(n) = \sum_{j\geq 1} j(p(n-jk) - p(n-(j+1)k)) = \sum_{j\geq 1} p(n-jk). \tag{6.4}$$

An immediate consequence is

$$p(n) = g_k(n+k) - g_k(n), \tag{6.5}$$

since the right-hand side of (6.5) is equal to $\sum\limits_{j\geq 0} p(n-jk) - \sum\limits_{j\geq 1} p(n-jk) = p(n)$.

We will need the following extension of Equation (6.5) (replace k by tk, then n by $n-ik$):

$$p(n-ik) = g_{tk}(n+(t-i)k) - g_{tk}(n-ik), \tag{6.6}$$

where i,t are integers. Thus (6.5) may be obtained by setting $i=0$ in (6.6).

From (6.4) and (6.6) we have

$$g_k(n) = \sum_{j\geq 1} g_{tk}(n+(t-j)k) - \sum_{j\geq 1} g_{tk}(n-jk).$$

Group the summations into pairs of t summands, then isolate the first pair:

$$\begin{aligned} g_k(n) &= \sum_{i\geq 0}\left(\sum_{j=it+1}^{(i+1)t} g_{tk}(n+(t-j)k) - \sum_{j=it+1}^{(i+1)t} g_{tk}(n-jk)\right) \\ &= \sum_{j=1}^{t} g_{tk}(n+(t-j)k) - \sum_{j=1}^{t} g_{tk}(n-jk) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\geq 0}\sum_{j=(i+1)t+1}^{(i+2)t} g_{tk}(n+(t-j)k) - \sum_{i\geq 1}\sum_{j=it+1}^{(i+1)t} g_{tk}(n-jk) \\
&= \sum_{j=1}^{t} g_{tk}(n+(t-j)k) + \left(-\sum_{j=1}^{t} g_{tk}(n-jk) + \sum_{j=t+1}^{2t} g_{tk}(n+(t-j)k)\right) \\
&\quad + \sum_{i\geq 1}\left(\sum_{j=(i+1)t+1}^{(i+2)t} g_{tk}(n+(t-j)k) - \sum_{j=it+1}^{(i+1)t} g_{tk}(n-jk)\right) \qquad (6.7)
\end{aligned}
$$

The two summations inside either pair of parentheses are identical with opposite signs. So only the first summation survives. Reversing the order of summation in the latter we obtain the next result (also stated in [18]).

Theorem 6.8. *The following identity holds for all integers $n, k, t > 0$:*

$$g_k(n) = \sum_{j=0}^{t-1} g_{tk}(n+jk).$$

We remark that Theorem 6.8 may be proved bijectively for any $t > 1$ by extending the proof of Theorem 6.6. The relevant bijection $\bigcup_j g_{tk}[n+jk] \longrightarrow g_k[n]$ is obtained as follows. If $(tk)^v \in g_{tk}[n+jk]$, then for each $i \in \{1,\ldots,v\}$ replace i copies of tk with $it-j$ copies of k. The reverse transformation may be deduced analogously.

Chapter 7

Conclusion

In this thesis, special emphasis was placed on ℓ-regular overpartitions, a special partition inequality and partition configurations. We have utilized the Ramanujan theta-functions and bijections to prove certain of their arithmetic and combinatorial properties.

In Chapter Three, we proved the main theorem, Theorem 3.1, and Theorem 3.3 which connect ℓ-regular overpartition function with certain restricted ordinary and color partition functions by utilizing both generating functions and bijective proofs. Theorem 3.1 as well as Theorem 3.3 contain seemingly incomplete partition identities because some of these are given only for selective parities of ℓ. It will be of interest to obtain extensions of the identities to all integers $\ell > 0$.

In Chapter Four, an infinite set of Ramanujan-type congruences for the ℓ-regular overpartitions was found and the congruences were proved by using elementary generating function manipulation. However, the congruences for ℓ-regular overpartition functions modulo 2 and 4 were proved combinatorially. Nevertheless, it will be interesting to extend the congruences for ℓ-regular overpartition functions to modulo higher power of 2.

In Chapter Five, we gave a combinatorial proof of Bessenrodt and Ono inequality, $p(a)p(b) > p(a+b)$ where a, b are positive integers satisfying $a+b > 9$, $a > 1$ and $b > 1$. We then utilized the same proof method to prove the same inequality for 2-regular partitions. Furthermore, despite strong experimental verification evidence of the conjectures 5.1 which has been done by using Maple, the proof has not been found, and it remains an open problem.

In Chapter Six, a new generalization of Elder's theorem was given. We proved combinatorially the equality of the number of times an integer k appears in all partitions and the number of partition configurations of length k by utilizing a variation of Stanley's original bijection. We also provided a new proof and a generalization for the equality of the number of $2k$'s in partitions and the number of unrepeated multiples of k.

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