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Masters dissertation

Analysis of heat transfer in a hot body with non-constant internal heat generation and thermal conductivity

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Abstract

Heat transfer in a wall with temperature dependent thermal conductivity and internal heat generation is considered. We first focus on the steady state models followed by the transient heat transfer models. It turns out that the models considered are non-linear. We deliberately omit the group-classification of the arbitrary functions appearing in the models, but rather select forms of physical importance. In one case, thermal conductivity and internal heat generation are both given by the exponential function and in the other case they are given by the power law. We employ the classical Lie point symmetry analysis to determine the exact solutions, while also determining the optimal system for each case. The exact solutions for the transient models are difficult to construct. However, we first use the obtained exact solution for the steady state case as a benchmark for the 1D Differential Transform Method (DTM). Since confidence in DTM is established, we construct steady state approximate series solutions. We apply the 2D DTM to the transient problem. Lastly we determine the conservation laws using the direct method and the associated Lie point symmetries for the transient problem.

Declaration

I, Marcio Lourenco, hereby certify that the work done in this dissertation is wholly my own original work except where due references have been made. It is being submitted for the degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before at any other institution.

Signed:

Date:

Dedication

To my extraordinary parents Rosa and Evaristo, and amazing brother Mauro.

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I would first like to thank my supervisor Professor R. J. Moitsheki for his guidance and patience through my successes and struggles.

To my parents, my brother, my family and friends, your tremendous support, love and sacrifices for me are the reasons for making this achievement possible.

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Chapter 1

Introduction

In this dissertation, a focus is on the study of heat transfer in a hot body and in particular across a plane wall. Here, the restriction is the analysis of steady and transient heat transfer in a planar region. The similar analysis may be applied to cylindrical and spherical geometries. The study of heat transfer in hot bodies enjoys applications in heat transfer in slabs, solids or plane walls [1], heat transfer with application in biology such as heat transfer through the human head [2] and regulation of temperature through elephant ears [3] or through animal and human skin [4], and thermal energy storage [5]. The study of heat transfer in human body is also important in terms of development of new medical treatments [6].

Most models describing heat transfer in slabs assume constant thermal conductivity and internal heat generation is neglected or assumed to be constant (see e.g. [4]). Few exact solutions exist particularly when the thermal conductivity and internal heat generation are temperature dependent. This is because the resulting models are nonlinear and harder to solve exactly. In fact, only general solutions that do not satisfy the boundary conditions may be obtained.

1.1 Literature Review

The problem being considered is an investigation to determine the group invariant solutions for heat transfer in a hot body with internal heat generation, and to discover and explore possible physical events where these solutions will be useful.

We intend employing Lie point symmetry methods to construct exact solutions. Heat transfer problems have been considered by many symmetry analysts (for example [5,7,8]). In [8], group classification was performed to determine the form of a source or sink term, for which the non-linear heat equation admitted extra symmetries. Here, we aim to construct solutions which satisfy the boundary conditions; that is, we focus on boundary value problems. Asymmetric and symmetric cases are considered.

Exact solutions can be easily obtained for cases when the thermal conductivity and internal heat generation are both constant. Here, the cases where thermal conductivity and internal heat generation are temperature dependent will be investigated. This results in the equation becoming non-linear and difficult to solve exactly. However, numerical methods will be implemented to obtain approximate solutions, which will be compared to our exact solutions obtained for special cases.

The aim of [9] was to transform the equation governing the one-dimensional heat conduction with non-linear internal heat generation from a boundary value problem to an initial value problem by means of previously established methods. Here, we will attempt to reduce the partial differential equation to an ordinary differential equation, and hopefully solve the reduced ordinary differential equation analytically. Either the method of differential invariants or canonical coordinates will be employed to analyse the ordinary differential equations.

A number of techniques have been employed to solve non-linear problems associated with heat transfer. In [10], the 1D Differential Transform Method (DTM) was used to construct solutions of the non-linear ordinary differential equations (ODEs) arising in extended surface heat transfer. While the 2D DTM was applied to solve non-linear Gas Dynamic and Klein-Gordon equations, [11].

1.2 Aim and Objective

The aim of this dissertation is to obtain solutions for heat transfer in a hot body, in particular heat transfer through a plane wall, and to analyse these solutions so as to make a contribution to the understanding of heat transfer problems. First we consider the steady state problem, where thermal conductivity and internal heat generation are dependent on temperature, the problem is one-dimensional and non-linear. Then we obtain the Lie algebra of the governing equation by implementing the classical Lie symmetry approach. Thereafter, we classify the family of group invariant solutions by means of finding the one-dimensional optimal system of subalgebras for the obtained Lie algebra. It turned out that exact group invariant solution satisfied the boundary conditions only for special cases. In the case where exact solutions could not be obtained, we then apply the Differential Transform Method (DTM) to our problem. Firstly, the 1D DTM will be applied to the non-linear steady state problems, where its performance is compared to the known exact group invariant solutions for these cases. This is followed by applying the 2D DTM to the non-linear transient state problems. We then construct conservation laws using the direct method and derive the Lie point symmetries associated with the conserved vectors.

1.3 Outline

An outline of the dissertation is as follows

- Chapter 2 introduces the basic concepts and definitions of the mathematical tools used in this dissertation. We discuss the Lie point symmetry techniques, explain the procedure for the DTM, and we present the direct method used to construct conservation laws.

- In Chapter 3, we present mathematical models, describing steady and transient heat transfer in plane walls. We consider symmetric and asymmetric boundary conditions.

- The Lie point symmetries for all considered cases are obtained in Chapter 4.

- In Chapter 5 the construction of the one-dimensional optimal system of subalgebra for the algebras obtained in Chapter 4, is provided.

- In Chapter 6, we establish confidence in the Differential Transform Method by comparing its solutions to already known exact solutions for the steady state problem.

- The 2D DTM provides analytical solutions when applied to the transient state problem in Chapter 7. We only consider Power Law for thermal conductivity and internal heat generation since this was the only case in which group invariant solutions could not be obtained.

- In Chapter 8, conservation laws are constructed, using the Direct Method, as well as the associated Lie point symmetries.

- Chapter 9 presents some concluding remarks.

Chapter 2

Methods of solution

2.1 Introduction

In this chapter we introduce a brief theory of the Lie group analysis and the differential transformation method which will be used to solve the models presented in the next chapter. We also present a brief theory on one of the methods of constructing conservation laws.

2.2 Lie point symmetry techniques

We introduce in brief the theory of symmetry analysis of differential equations. The reader is referred to books such as [12, 13, 14, 15, 16] for a detailed account on this theory. This section deals with the implementation of the classical Lie symmetry theory to the governing partial differential equation (3.9). The objective of the classical Lie symmetry approach is to obtain a transformation which leaves the governing equation invariant. These transformations are then used to obtain their corresponding group invariant solutions. We restrict our discussion to second order partial differential equations. For the steady state, which are given by ordinary differential equations, one may drop one variable.

Given a second order partial differential equation (PDE)

$$F(t, x, \theta, \theta_t, \theta_{tt}, \theta_x, \theta_{xx}) = 0, \tag{2.1}$$

we seek transformations of the dependant variable θ and the dependant variables t and x that have the form

$$\begin{aligned}\bar{t} &= \bar{t}(t, x, \theta, \epsilon),\\ \bar{x} &= \bar{x}(t, x, \theta, \epsilon),\\ \bar{\theta} &= \bar{\theta}(t, x, \theta, \epsilon).\end{aligned} \tag{2.2}$$

This set of transformations characterize a one-parameter group with group parameter ϵ, chosen such that the governing equation (2.1) remains invariant. The infinitesimal transformations can be found by considering the Taylor series expansions of the Lie group of transformations (2.2) and can be written as

$$\begin{aligned}\bar{t} &\simeq t + \epsilon\xi^1(t, x, \theta),\\ \bar{x} &\simeq x + \epsilon\xi^2(t, x, \theta),\\ \bar{\theta} &\simeq \theta + \epsilon\eta(t, x, \theta),\end{aligned} \tag{2.3}$$

with the operator

$$X = \xi^1(t, x, \theta)\frac{\partial}{\partial t} + \xi^2(t, x, \theta)\frac{\partial}{\partial x} + \eta(t, x, \theta)\frac{\partial}{\partial \theta}. \tag{2.4}$$

This operator is a Lie point symmetry generator of the governing equation (2.1) if and only if the infinitesimal criterion for invariance holds:

$$X^{[2]}(\text{equation (2.1)}) \mid_{\text{equation (2.1)}} = 0. \tag{2.5}$$

We act on a PDE (2.1) with the second prolongation $X^{[2]}$ of the operator X since the equation under examination is of the second order. This given by

$$\begin{aligned} X^{[2]} =& \xi^1(t,x,\theta)\frac{\partial}{\partial t} + \xi^2(t,x,\theta)\frac{\partial}{\partial x} + \eta(t,x,\theta)\frac{\partial}{\partial \theta} + \zeta_x\frac{\partial}{\partial \theta_x} + \zeta_t\frac{\partial}{\partial \theta_t} \\ &+ \zeta_{xx}\frac{\partial}{\partial \theta_{xx}} + \zeta_{tt}\frac{\partial}{\partial \theta_{tt}} + \zeta_{xt}\frac{\partial}{\partial \theta_{xt}}, \end{aligned} \tag{2.6}$$

where the coefficient functions of the extended infinitesimals ζ are explicitly given by

$$\begin{aligned} \zeta_x(t,x,\theta) &= D_x[\eta(t,x,\theta)] - \theta_t D_x[\xi^1(t,x,\theta)] - \theta_x D_x[\xi^2(t,x,\theta)], \\ \zeta_t(t,x,\theta) &= D_t[\eta(t,x,\theta)] - \theta_t D_t[\xi^1(t,x,\theta)] - \theta_x D_t[\xi^2(t,x,\theta)], \\ \zeta_{xx}(t,x,\theta) &= D_x[\zeta_x(t,x,\theta)] - \theta_{xt} D_x[\xi^1(t,x,\theta)] - \theta_{xx} D_x[\xi^2(t,x,\theta)], \\ \zeta_{xt}(t,x,\theta) &= D_t[\zeta_x(t,x,\theta)] - \theta_{xt} D_t[\xi^1(t,x,\theta)] - \theta_{xx} D_t[\xi^2(t,x,\theta)], \\ \zeta_{tt}(t,x,\theta) &= D_t[\zeta_t(t,x,\theta)] - \theta_{tt} D_t[\xi^1(t,x,\theta)] - \theta_{xt} D_t[\xi^2(t,x,\theta)]. \end{aligned} \tag{2.7}$$

The total derivative operators are defined as:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + \theta_t\frac{\partial}{\partial \theta} + \theta_{tt}\frac{\partial}{\partial \theta_t} + \theta_{xt}\frac{\partial}{\partial \theta_x} + \ldots, \\ D_x &= \frac{\partial}{\partial x} + \theta_x\frac{\partial}{\partial \theta} + \theta_{xx}\frac{\partial}{\partial \theta_x} + \theta_{xt}\frac{\partial}{\partial \theta_t} + \theta_{xxx}\frac{\partial}{\partial \theta_{xx}} + \ldots \end{aligned} \tag{2.8}$$

Equation (2.5) results in an overdetermined system of determining equations, from which ξ^1, ξ^2 and η can be solved. By substituting these solutions for ξ^1, ξ^2 and η into (2.4), the symmetries admitted by the PDE can be found. These symmetries are called Lie point symmetries, which are local.

2.3 Differential Transform Method

2.3.1 One-Dimensional Differential Transform Method (1D-DTM)

Let $\phi(t)$ be an analytic function in a domain $\mathcal{D}$. The Taylor series expansion function of $\phi(t)$ with the center located at $t = t_j$ is given by [17]

$$\phi(t) = \sum_{\kappa=0}^{\infty} \frac{(t-t_j)^\kappa}{\kappa!} \left[\frac{d^\kappa \phi(t)}{dt^\kappa}\right]_{t=t_j}, \ \forall t \in \mathcal{D}. \tag{2.9}$$

The particular case of Equation (2.9) when $t_j = 0$ is referred to as the Maclaurin series expansion of $\phi(t)$ and is expressed as,

$$\phi(t) = \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \left[\frac{d^\kappa \phi(t)}{dt^\kappa}\right]_{t=0}, \ \forall t \in \mathcal{D}. \tag{2.10}$$

The differential transform of $\phi(t)$ is defined as follows;

$$\Phi(t) = \sum_{\kappa=0}^{\infty} \frac{\mathcal{H}^\kappa}{\kappa!} \left[\frac{d^\kappa \phi(t)}{dt^\kappa}\right]_{t=0}, \tag{2.11}$$

where $\phi(t)$ is the original analytic function and $\Phi(t)$ is the transformed function. The differential spectrum of $\Phi(t)$ is confined within the interval $t \in [0, \mathcal{H}]$, where $\mathcal{H}$ is a constant. From equations (2.10) and (2.11), the differential inverse transform of $\Phi(t)$ is defined as follows,

$$\phi(t) = \sum_{\kappa=0}^{\infty} \left(\frac{t}{\mathcal{H}}\right)^\kappa \Phi(t), \tag{2.12}$$

and if $\phi(t)$ is expressed by a finite series, then

$$\phi(t) = \sum_{\kappa=0}^{r} \left(\frac{t}{\mathcal{H}}\right)^\kappa \Phi(t). \tag{2.13}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of the function $\Phi(\kappa)$ are referred to as discrete, i.e., available, it is possible to restore the unknown function more precisely. The function $\phi(t)$ consists of the T-function $\Phi(\kappa)$, and its value is given by the sum of the T-function with $(t/\mathcal{H})^\kappa$ as its coefficient. In real applications, at the right choice of the constant $\mathcal{H}$, the discrete values of the spectrum reduce rapidly with larger values of argument κ [18].

Below are some of the theorems used when applying the one-dimensional differential transform method.

Theorem 2.1 If $\phi(t) = x(t) \pm z(t)$, then $\Phi(\kappa) = X(\kappa) \pm Z(\kappa)$.

Theorem 2.2 If $\phi(t) = \alpha x(t)$, then $\Phi(\kappa) = \alpha X(\kappa)$.

Theorem 2.3 If $\phi(t) = \dfrac{dx(t)}{dt}$, then $\Phi(\kappa) = (\kappa+1)\Phi(\kappa+1)$.

Theorem 2.4 If $\phi(t) = \dfrac{d^2x(t)}{dt^2}$, then $\Phi(\kappa) = (\kappa+1)(\kappa+2)\Phi(\kappa+2)$.

Theorem 2.5 If $\phi(t) = \dfrac{d^sx(t)}{dt^s}$, then $\Phi(\kappa) = (\kappa+1)(\kappa+2)\ldots(\kappa+s)\Phi(\kappa+s)$.

Theorem 2.6 If $\phi(t) = x(t)z(t)$, then $\Phi(\kappa) = \displaystyle\sum_{i=0}^{\kappa} X(i)Z(\kappa-i)$.

Theorem 2.7 If $\phi(t) = x(t)y(t)z(t)$, then $\Phi(\kappa) = \displaystyle\sum_{i=0}^{\kappa}\sum_{l=0}^{\kappa-i} X(l)Y(i)Z(\kappa-i-l)$.

Theorem 2.8 If $\phi(t) = t^s$, then $\Phi(\kappa) = \delta(\kappa-s)$.

Theorem 2.9 If $\phi(t) = \exp(\lambda t)$, then $\Phi(\kappa) = \dfrac{\lambda^\kappa}{\kappa!}$.

Theorem 2.10 If $\phi(t) = (1+t)^s$, then $\Phi(\kappa) = \dfrac{s(s-1)(s-2)\ldots(s-\kappa-1)}{\kappa!}$.

Theorem 2.11 If $\phi(t) = \sin(\omega t + \alpha)$, then $\Phi(\kappa) = \dfrac{\omega^\kappa}{\kappa!}\sin\left(\dfrac{\pi\kappa}{2!} + \alpha\right)$.

Theorem 2.12 If $\phi(t) = \cos(\omega t + \alpha)$, then $\Phi(\kappa) = \dfrac{\omega^\kappa}{\kappa!}\cos\left(\dfrac{\pi\kappa}{2!} + \alpha\right)$.

The Kronecker delta function $\delta(\kappa - s)$ is given by

$$\delta(\kappa - s) = \begin{cases} 1 & \text{if, } \kappa = s, \\ 0 & \text{if, } \kappa \neq s. \end{cases}$$

2.3.2 Two-Dimensional Differential Transform Method (2D-DTM)

Based on the one-dimensional differential transform method, the basic definitions of the two-dimensional transform are defined as follows

$$\Phi(\kappa, s) = \sum_{\kappa=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{\kappa!\, s!} \left[\frac{\partial^{\kappa+s} \phi(t,x)}{\partial t^{\kappa} \partial x^{s}} \right]_{(0,0)}, \tag{2.14}$$

where $\phi(t,x)$ is the original function and $\Phi(t,x)$ is the transformed function.

The differential inverse transform of $\Phi(t,x)$ is defined as

$$\phi(t,x) = \sum_{\kappa=0}^{\infty} \sum_{s=0}^{\infty} \Phi(\kappa, s) t^{\kappa} x^{s}. \tag{2.15}$$

From equations (2.14) and (2.15) it can be concluded that

$$\phi(t,x) = \sum_{\kappa=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{\kappa!\, s!} \left[\frac{\partial^{\kappa+s} \phi(t,x)}{\partial t^{\kappa} \partial x^{s}} \right]_{(0,0)} t^{\kappa} x^{s}. \tag{2.16}$$

In real applications, the function $\phi(t,x)$ is approximated by a finite series, and equation (2.15) can be written as

$$\phi(t,x) = \sum_{\kappa=0}^{m} \sum_{s=0}^{n} \Phi(\kappa, s) t^{\kappa} x^{s}. \tag{2.17}$$

Equation (2.17) implies that

$$\phi(t,x) = \sum_{\kappa=m+1}^{\infty} \sum_{s=n+1}^{\infty} \Phi(\kappa,s) t^{\kappa} x^{s}, \tag{2.18}$$

is negligibly small.

Below are some of the theorems used when applying the two-dimensional differential transform method.

Theorem 2.12 If $\phi(t,x) = x(t,x) \pm z(t,x)$, then $\Phi(\kappa,s) = X(\kappa,s) \pm Z(\kappa,s)$.

Theorem 2.13 If $\phi(t,x) = \alpha x(t,x)$, then $\Phi(\kappa,s) = \alpha X(\kappa,s)$.

Theorem 2.14 If $\phi(t,x) = \dfrac{\partial x(t,x)}{\partial t}$, then $\Phi(\kappa,s) = (\kappa+1)\Phi(\kappa+1,s)$.

Theorem 2.15 If $\phi(t,x) = \dfrac{\partial x(t,x)}{\partial x}$, then $\Phi(\kappa,s) = (s+1)\Phi(\kappa,s+1)$.

Theorem 2.16 If $\phi(t,x) = \dfrac{\partial^{r+q} x(t,x)}{\partial t^r \partial x^q}$, then

$\Phi(\kappa,s) = (\kappa+1)(\kappa+2)\ldots(\kappa+r)(s+1)(s+2)\ldots(s+q)\Phi(\kappa+r,s+q)$.

Theorem 2.17 If $\phi(t,x) = x(t,x)z(t,x)$, then

$\Phi(\kappa,s) = \displaystyle\sum_{i=0}^{\kappa} \sum_{j=0}^{s} X(i,s-j)Z(\kappa-i,j)$.

Theorem 2.18 If $\phi(t,x) = t^m x^n$, then $\Phi(\kappa,s) = \delta(\kappa-m)\delta(s-n)$.

Theorem 2.19 If $\phi(t,x) = \dfrac{\partial x(t,x)}{\partial x}\dfrac{\partial z(t,x)}{\partial x}$, then

$\Phi(\kappa,s) = \displaystyle\sum_{i=0}^{\kappa} \sum_{j=0}^{s} (s-j+1)(j+1)X(i,s-j+1)Z(\kappa-i,j+1)$.

Theorem 2.20 If $\phi(t,x) = \dfrac{\partial x(t,x)}{\partial t}\dfrac{\partial z(t,x)}{\partial x}$, then

$\Phi(\kappa,s) = \displaystyle\sum_{i=0}^{\kappa} \sum_{j=0}^{s} (\kappa-i+1)(s-j+1)X(\kappa-i+1,j)Z(i,s-j+1)$.

Theorem 2.21 If $\phi(t,x) = x(t,x)\ z(t,x)\ w(t,x)$, then

$$\Phi(\kappa,s) = \sum_{i=0}^{\kappa}\sum_{p=0}^{\kappa-i}\sum_{j=0}^{s}\sum_{q=0}^{s-j} X(i,s-j-q)\ Z(p,j)\ W(k-i-p,q).$$

Theorem 2.22 If $\phi(t,x) = x(t,x)\ \dfrac{\partial z(t,x)}{\partial x}\ \dfrac{\partial w(t,x)}{\partial x}$, then

$$\Phi(\kappa,s) = \sum_{i=0}^{\kappa}\sum_{p=0}^{\kappa-i}\sum_{j=0}^{s}\sum_{q=0}^{s-j}(j+1)(q+1)X(i,s-j-q)\ Z(p,j+1)\ W(k-i-p,q+1).$$

Theorem 2.23 If $\phi(t,x) = x(t,x)\ z(t,x)\ \dfrac{\partial^2 w(t,x)}{\partial x^2}$, then

$$\Phi(\kappa,s) = \sum_{i=0}^{\kappa}\sum_{p=0}^{\kappa-i}\sum_{j=0}^{s}\sum_{q=0}^{s-j}(q+1)(q+2)X(i,s-j-q)\ Z(p,j)\ W(k-i-p,q+2).$$

Theorem 2.24 If $\phi(t,x) = x(t,x)\ z(t,x)\ w(t,x)\ v(t,x)$, then

$$\Phi(\kappa,s) = \sum_{i=0}^{\kappa}\sum_{p=0}^{\kappa-i}\sum_{z=0}^{\kappa-i-p}\sum_{j=0}^{s}\sum_{q=0}^{s-j}\sum_{l=0}^{s-j-q} X(i,s-j-q-l)\ Z(p,j)\ W(z,q)\ V(k-i-p-z,l).$$

Theorem 2.25 If $\phi(t,x) = x^m \exp(\lambda t)$, then $\Phi(\kappa,s) = \dfrac{\lambda^s}{s!}\delta(\kappa-m)$.

Theorem 2.26 If $\phi(t,x) = x^m \sin(\omega t + \alpha)$, then

$$\Phi(\kappa,s) = \frac{\omega^s}{s!}\delta(\kappa-m)\sin\left(\frac{s\pi}{2!}+\alpha\right).$$

Theorem 2.27 If $\phi(t,x) = x^m \cos(\omega t + \alpha)$, then

$$\Phi(\kappa,s) = \frac{\omega^s}{s!}\delta(\kappa-m)\cos\left(\frac{s\pi}{2!}+\alpha\right).$$

The Kronecker delta function $\delta(\kappa-m)$ is given by

$$\delta(\kappa-m) = \begin{cases} 1 & \text{if, } \kappa = m \text{ and } s = n, \\ 0 & \text{otherwise.} \end{cases}$$

2.4 Conservation Laws

Consider a kth order differential equation,

$$F(\mathbf{x}, \theta, \theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(k)}) = 0, \tag{2.19}$$

where $\mathbf{x}$ denotes n independent variables, u denotes the dependent variable and $\theta_{(i)}$ denotes all the partial derivatives of order i. For an arbitrary partial differential equation we write

$$D_i(T^i) = 0, \tag{2.20}$$

where T^i are differential functions of finite order. We define equation (2.20) as a conservation law for equation (2.19) if it satisfies the following equation

$$D_i\left[T^i(\mathbf{x}, \theta, \theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(l)})\right] = 0. \tag{2.21}$$

This can also be written as

$$D_i T^i \mid_{F=0} = 0. \tag{2.22}$$

The vector $T = (T^1, T^2, \ldots, T^n)$ is called a conserved vector.

A Lie Point symmetry generator

$$X = \xi^i(\mathbf{x}, \theta)\frac{\partial}{\partial x^i} + \eta(\mathbf{x}, \theta)\frac{\partial}{\partial u}, \tag{2.23}$$

is said to be associated with the conserved vector $T^i = (T^1, \ldots, T^n)$ for equation (2.19) if [19]

$$X(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0, \quad i = 1, \ldots, n. \tag{2.24}$$

In equation (2.24), X is prolonged appropriately. Using equation (2.24) we can determine the conserved vectors, [20].

We determine conservation laws using the Direct Method, which gives all local conservation laws. Equation (2.20) is a conservation law. The Direct Method uses equation (2.20) subject to equation (2.19) being satisfied as the determining equation for the conserved vectors. The components $T^1, \ldots, T^n$ are obtained by separating the resulting equation according to powers and products of the derivatives of θ.

2.5 Concluding remarks

In this chapter, we have provided brief accounts of the methods used in this dissertation. The reader is referred to text and citation as provided.

Chapter 3

Description of mathematical models

In this dissertation we consider heat transfer across a wall. Here thermal conductivity and internal heat generation of the wall are temperature dependent. We consider two scenarios, first the temperature being different at either end of the wall, for example in a heated house, temperature being higher inside the house than outside. And secondly, we consider the case where the temperature gradient at the center of the cross-sectional center of the wall is zero. More details will be provided on these scenarios in this chapter.

Variable	**Description**	**Unit**
k	Thermal conductivity	W/(m·K)
α	Thermal diffusivity	m^2/s
ρ	Density	kg/m^3
c_p	Specific heat capacity	J/K
Q	Internal heat generation	W/m^3
T	Temperature	K
x	Length	m
t	Time	s

Table 3.1: Nomenclature

We begin with the one-dimensional heat equation with constant internal heat generation and thermal conductivity given by

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{Q}{k} = 0, \tag{3.1}$$

where

$$\alpha = \frac{k}{\rho c_p}. \tag{3.2}$$

Equation (3.1) may be rewritten as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \rho c_p \frac{\partial T}{\partial t} + Q = 0. \tag{3.3}$$

Notice that k and Q are constant in (3.3). Now, we assume that thermal conductivity and the internal heat generation to be functions depending on T, which gives the non-linear partial differential equation

$$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) - \rho c_p \frac{\partial T}{\partial t} + Q(T) = 0. \tag{3.4}$$

The imposed boundary conditions are given by

$$\frac{\partial T}{\partial x}(t,0) = 0, \qquad T(t,-L) = T_s = T(t,L), \tag{3.5}$$

in one case and by

$$T(t,-L) = T_1, \qquad T(t,L) = T_2, \tag{3.6}$$

in the other case. Here $T_1 \neq T_2$, and the initial condition is given by

$$T(0,x) = f(x),$$

where $f(x)$ is some initial heat profile.

The boundary condition (3.5) is symmetric whilst (3.6) is the asymmetric condition.

In (3.5), studies may be carried out for $x \in [0, L]$. It can be seen in (3.6) that heat may flow from higher to lower temperatures as expected.

3.1 Dimensional Analysis

Introducing the non-dimensional variables and parameters

$$\bar{x} = \frac{x}{l}, \qquad \theta = \frac{T}{T_a},$$
$$q = \frac{Q}{q_a}, \qquad \bar{k} = \frac{k}{k_a},$$
$$\bar{t} = \frac{t\, k_a}{\rho c_p l^2}, \tag{3.7}$$

where l is the characteristic length in the x-direction, T_a is the characteristic temperature, q_a is the characteristic internal heat generation, k_a is the characteristic thermal conductivity and $\frac{\rho c_p l^2}{k_a}$ is the characteristic time, we obtain the dimensionless equation

$$\frac{\partial}{\partial \bar{x}}\left(\bar{k}(\theta)\frac{\partial \theta}{\partial \bar{x}}\right) - \frac{\partial \theta}{\partial \bar{t}} + \mathrm{Ng}\, q(\theta) = 0. \tag{3.8}$$

Here $\mathrm{Ng} = \dfrac{q_a l^2}{k_a T_a}$, the coefficient of the internal heat generation term. It is assumed that $\mathrm{Ng} > 0$. Notice that if $\mathrm{Ng} < 0$, then equation (3.9) represents heat transfer in a straight fin, whereby $q(\theta)$ is now interpreted as the heat transfer coefficient. We neglect the bars in further work for notational convenience, so the governing equation is given by

$$\frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - \frac{\partial \theta}{\partial t} + \mathrm{Ng}\, q(\theta) = 0. \tag{3.9}$$

Subject to the boundary conditions

$$\frac{\partial \theta}{\partial x}(t, 0) = 0, \qquad \theta(t, 1) = 1, \tag{3.10}$$

for a wall with identical temperatures at the boundaries, and

$$\theta(t,-1) = 0.1, \qquad \theta(t,1) = 1, \tag{3.11}$$

for a wall with varying temperatures at the boundaries.

Along with initial condition:

$$\theta(0,x) = f(x). \tag{3.12}$$

Figures 3.1 and 3.2 show heat transfer in a wall with internal heat generation.

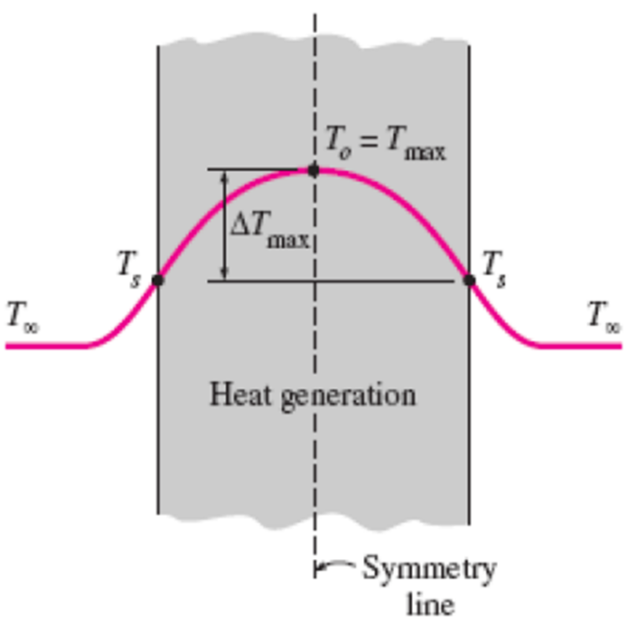

Figure 3.1: This figure shows heat transfer in a wall with internal heat generation. Here, the maximum temperature occurs at the center of the wall due to the symmetrical nature of the wall and uniformity of the initial heat generated. T_s is denoted to be the temperature at both boundaries of the wall, [21].

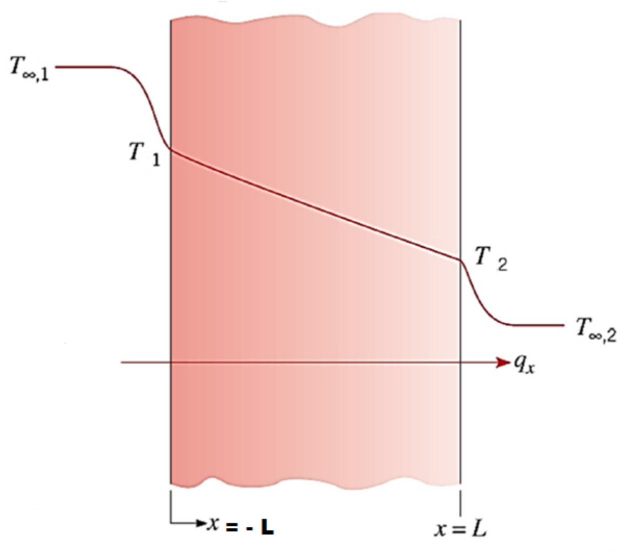

Figure 3.2: This figure shows heat transfer in a wall with different temperatures at each of the boundaries, [21].

3.2 Concluding remarks

In this chapter, we have presented the models for heat transfer across a wall. Thermal conductivity and internal heat generation are given as functions of temperature in this dissertation.

Chapter 4

Lie Point Symmetry Analysis

Using symmetry analysis it is possible to classify equation (3.9); that is, to determine all cases of the arbitrary functions for which this equation admits extra symmetries either by direct methods [12, 22] or preliminary group classification [23] or enhance group classification [24]. In this section we firstly consider obtaining ξ^1, ξ^2 and η for general $k(\theta)$ and $q(\theta)$. Later, we focus on the exponential and power law cases of $k(\theta)$ and $q(\theta)$. We do not claim group classification. However, in many engineering applications thermal conductivity is given by these cases. We thus concentrate on linearizable models (see e.g. [25]). The exact solutions obtained in this section will be used to benchmark the 1D DTM. We consider the cases listed below

	$k(\theta)$	$q(\theta)$
(a)	$\mathrm{e}^{m\theta}$	$\mathrm{e}^{n\theta}$
(b)	θ^m	θ^{n+1}

4.1 Case: General $k(\theta)$ and $q(\theta)$

Assuming the general form, namely

$$k = k(\theta), \qquad \text{and} \qquad q = q(\theta),$$

the employment of the second prolongation $X^{[2]}$ onto equation (3.9) yields the determining equation

$$\Big(\zeta_t - 2\theta_x \zeta_x k'(\theta) - k(\theta)\zeta_{xx} \\ + \eta \left(-\theta_x^2 k''(\theta) - \theta_{xx} k'(\theta) - \mathrm{Ng}\, q'(\theta) \right) \Big) |_{\text{equation(3.9)}} = 0. \tag{4.1}$$

The application of the operators in equation (2.8) onto the extended infinitesimals ζ_{ij}'s defined in equation (2.7) leads to

$$\zeta_t = \eta_t + \theta_t \left(\eta_\theta - \xi_t^1 \right) - \xi_t^2 \theta_x - \xi_\theta^1 \left(\theta_t\right)^2 - \xi_\theta^2 \theta_x \theta_t, \tag{4.2}$$

$$\zeta_x = \eta_x + \theta_x \left(\eta_\theta - \xi_x^2 \right) - \xi_x^1 \theta_t - \xi_\theta^2 \left(\theta_x\right)^2 - \xi_\theta^1 \theta_x \theta_t, \tag{4.3}$$

$$\zeta_{xx} = \eta_{xx} - \xi_{xx}^1 \theta_t + \left(2\eta_{x\theta} - \xi_{xx}^2 \right) \theta_x + \left(\eta_{\theta\theta} - \xi_{x\theta}^2 \right) \left(\theta_x\right)^2 - 2\xi_{x\theta}^1 \theta_t \theta_x \\ - \xi_{\theta\theta}^2 \left(\theta_x\right)^3 - \xi_{\theta\theta}^1 \theta_t \left(\theta_x\right)^2 + \left(\eta_\theta - 2\xi_x^2 \right) \theta_{xx} - 2\xi_x^1 \theta_{tx} - \xi_\theta^1 \theta_t \theta_{xx} \\ - 3\xi_\theta^2 \theta_x \theta_{xx} - 2\xi_\theta^1 \theta_x \theta_{tx}. \tag{4.4}$$

We then substitute the expansions for the coefficient functions obtained in equations (4.2) - (4.4) into the determining equation (4.1). The unknown functions ξ^1, ξ^2 and η are independent of the derivatives of θ as well as powers and products of their derivatives. Hence, we can separate the resulting determining equations with respect to the powers and products of the derivatives of θ. Applying this process and making further simplifications leads to an overdetermined system of linear homogeneous partial differential equations given by

$$\theta_{xx}(\theta_x)^2 : k(\theta)^2\xi^1_{\theta\theta} + k(\theta)\xi^1_\theta k'(\theta) = 0, \tag{4.5}$$

$$\theta_{xx}\theta_x : 2k(\theta)\xi^1_x k'(\theta) + 2k(\theta)\xi^2_\theta + 2k(\theta)^2\xi^1_{x\theta} = 0, \tag{4.6}$$

$$\begin{aligned}\theta_{xx} : &-\eta k'(\theta) - \mathrm{Ng}\, k(\theta)\xi^1_\theta q(\theta)\\ &- k(\theta)\xi^1_t + 2k(\theta)\xi^2_x + k(\theta)^2\xi^1_{xx} = 0,\end{aligned} \tag{4.7}$$

$$\theta_{tx}\theta_x : 2k(\theta)\xi^1_\theta = 0, \tag{4.8}$$

$$\theta_{tx} : 2k(\theta)\xi^1_x = 0, \tag{4.9}$$

$$(\theta_x)^4 : k(\theta)\xi^1_{\theta\theta}k'(\theta) + \xi^1_\theta k'(\theta)^2 = 0, \tag{4.10}$$

$$(\theta_x)^3 : k(\theta)\xi^2_{\theta\theta} + \xi^2_\theta k'(\theta) + 2\xi^1_x k'(\theta)^2 + 2k(\theta)\xi^1_{x\theta}k'(\theta) = 0, \tag{4.11}$$

$$\begin{aligned}(\theta_x)^2 : &-k(\theta)\eta_{\theta\theta} + \mathrm{Ng}\, k(\theta)q(\theta)\xi^1_{\theta\theta} - \eta k''(\theta) - \eta_\theta k'(\theta) - \xi^1_t k'(\theta)\\ &+ 2\xi^2_x k'(\theta) + k(\theta)\xi^1_{xx}k'(\theta) + 2k(\theta)\xi^2_{x\theta} = 0,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\theta_x : &\,2\mathrm{Ng}\, q(\theta)\xi^1_x k'(\theta) - 2\eta_x k'(\theta) + 2\mathrm{Ng}\, k(\theta)q(\theta)\xi^1_{x\theta} - 2k(\theta)\eta_{x\theta}\\ &+ k(\theta)\xi^2_{xx} - \mathrm{Ng}\, \xi^2_\theta q(\theta) - \xi^2_t = 0,\end{aligned} \tag{4.13}$$

$$\begin{aligned}1 : &\,\mathrm{Ng}\, k(\theta)q(\theta)\xi^1_{xx} - k(\theta)\eta_{xx} - (\mathrm{Ng})^2\, \xi^1_\theta q(\theta)^2 - \eta\mathrm{Ng}\, q'(\theta)\\ &+ \mathrm{Ng}\, \eta_\theta q(\theta) - \mathrm{Ng}\, q(\theta)\xi^1_t + \eta_t = 0.\end{aligned} \tag{4.14}$$

with the remainder of the equations being satisfied.

Equations (4.8) and (4.9) give the result

$$\xi^1 = \xi^1(t). \tag{4.15}$$

Using the result of equation (4.15) with equation (4.6) gives

$$\xi^2 = \xi^2(t, x). \tag{4.16}$$

Equations (4.15) and (4.16) satisfy equations (4.5), (4.10) and (4.11), and simplify equations (4.7), (4.12), (4.13) and (4.14) to the following:

$$-\eta k'(\theta) - k(\theta)\xi^1_t + 2k(\theta)\xi^2_x = 0, \tag{4.17}$$

$$-k(\theta)\eta_{\theta\theta} - \eta k''(\theta) - \eta_\theta k'(\theta) - \xi^1_t k'(\theta) + 2\xi^2_x k'(\theta) = 0, \tag{4.18}$$

$$-2\eta_x k'(\theta) - 2k(\theta)\eta_{x\theta} + k(\theta)\xi^2_{xx} - \xi^2_t = 0, \tag{4.19}$$

$$-k(\theta)\eta_{xx} - \eta \mathrm{Ng}\, q'(\theta) + \mathrm{Ng}\, \eta_\theta q(\theta) - \mathrm{Ng}\, q(\theta)\xi^1_t + \eta_t = 0. \tag{4.20}$$

We can solve for η using equation (4.17),

$$\eta = \frac{2k(\theta)\xi^2_x}{k'(\theta)} - \frac{k(\theta)\xi^1_t}{k'(\theta)}. \tag{4.21}$$

Substituting this result in equation (4.21) we have equations (4.18), (4.19) and (4.20) becoming:

$$P(k)(2\xi^2_x - \xi^1_t) = 0, \tag{4.22}$$

$$Q(k)\xi^2_{xx} - \xi^2_t = 0, \tag{4.23}$$

$$\begin{aligned} &\mathrm{Ng}\left[\frac{k}{k'}\, q' + \frac{k\, k''}{(k')^2}\, q - 2q\right]\xi^1_t - 2\mathrm{Ng}\left[\frac{k}{k'}\, q' + \frac{k\, k''}{(k')^2}\, q - q\right]\xi^2_x \\ &- \frac{k}{k'}\,\xi^1_{tt} + 2\frac{k}{k'}\,\xi^2_{tx} - 2\frac{k^2}{(k')^2}\,\xi^2_{xxx} = 0, \end{aligned} \tag{4.24}$$

where

$$P(k) = \frac{k^2 k'''}{(k')^2} - 2\frac{k^2 (k'')^2}{(k')^3} + \frac{k\,k''}{k'},$$
$$Q(k) = 4\frac{k^2 k''}{(k')^2} - 7k.$$

It can be shown that

$$P(k) = \frac{1}{4}\left[\frac{dQ}{d\theta} - \frac{k'}{k}Q\right]. \tag{4.25}$$

Thus when $Q(k) = 0$ then $P(k) = 0$ and (4.22) is identically satisfied. Equation (4.22) places no condition on $2\xi^2_x - \xi^1_t$.

Also from equation (4.25), $P(k) = 0$ if

$$\frac{dQ}{d\theta} = \frac{k'}{k}Q,$$

that is, if

$$\frac{dQ}{dk} = \frac{Q}{k},$$

that is, if

$$Q(k) = \alpha k,$$

where α is a constant. Now for ANY $m \neq 0$ (k not constant),

if	$k(\theta) = \theta^m$	then	$Q(k) = -\frac{(4+3m)}{m}k,$
if	$k(\theta) = e^{m\theta}$	then	$Q(k) = -3k.$

Hence if $k(\theta) = \theta^m$ or $k(\theta) = e^{m\theta}$ then for any $m \neq 0$, $P(k) = 0$ and (4.22) is identically satisfied. Equation (4.22) places no condition on $2\xi^2_x - \xi^1_t$.

It can be checked by direct substitution that if

$$k(\theta) = \theta^m \qquad \text{or} \qquad k(\theta) = e^{m\theta},$$

then for any $m \neq 0$, $P(k) = 0$.

If $\xi^2_{xx} \neq 0$ then (4.23) can be written as

$$\frac{4k^2k''}{(k')^2} - 7k = \frac{\xi^2_t}{\xi^2_{xx}},$$

and differentiating both sides by θ gives

$$\frac{4k^2k''}{(k')^2} - 7k = c,$$

where c is a constant. Thus

$$\xi^2_t = c\,\xi^2_{xx}.$$

There are two cases.

Case (i) $\quad \xi^2_{xx} \neq 0$:

$$Q(k) = \frac{4k^2k''}{(k')^2} - 7k = c,$$

$$\xi^2_t = c\,\xi^2_{xx}.$$

In this dissertation only the special case $c = 0$ is considered and

$$Q(k) = 0, \quad \xi^2_t = 0. \tag{4.26}$$

It follows that

$$Q(k) = \frac{a}{(\theta + b)^{\frac{4}{3}}}, \quad P(k) = 0. \tag{4.27}$$

Case (ii) $\quad \xi^2_{xx} = 0$:

$$\xi^2_t = 0. \quad \text{(from (4.23))} \tag{4.28}$$

Then $Q(k)$ and $P(k)$ are not determined without further assumptions.

Equations (4.26), (4.27) and (4.28) simplify (4.24) to

$$\mathrm{Ng}\left[q' + \frac{k''}{k'}q - 2\frac{k'}{k}q\right]\xi^1_t - 2\mathrm{Ng}\left[q' + \frac{k''}{k'}q - \frac{k'}{k}q\right]\xi^2_x - \xi^1_{tt} - 2k\xi^2_{xxx} = 0. \tag{4.29}$$

So from here, we will use equation (4.29) to solve for ξ^1 and ξ^2, and then use equation (4.21) to solve for η. Equation (4.22) needs to be considered for general $k(\theta)$. Thus, finding the symmetries of equation (3.9) for any given $k(\theta)$ and $q(\theta)$.

Note, these equations can only be used to find symmetries for cases where $k'(\theta) \neq 0$.

A complete group classification is not performed here. We may as an open question consider enhanced group classification (see for example [22, 24]).

4.2 Case: $k(\theta) = e^{m\theta}$ and $q(\theta) = e^{n\theta}$

We assume the exponential form of $k(\theta)$ and $q(\theta)$. Notice that $k(\theta)$ is not of the form in equation (4.27), so we consider case (ii), that is

$$\xi^2_{xx} = 0, \tag{4.30}$$

which gives

$$\xi^2 = c_1 x + c_2. \tag{4.31}$$

Now we substitute our choices for $k(\theta)$ and $q(\theta)$ into equation (4.29). This simplifies to

$$e^{n\theta}\,\mathrm{Ng}\left(n\xi^1_t - 2nc_1 - m\xi^1_t\right) - \xi^1_{tt} = 0 \tag{4.32}$$

Since the term ξ^1_{tt} is independent of θ we can split (4.32) as follows

$$\xi^1_{tt} = 0,$$

$$\text{that is,} \qquad \xi^1 = c_3 t + c_4, \tag{4.33}$$

$$\text{and} \qquad nc_3 - 2nc_1 - mc_3 = 0. \tag{4.34}$$

We can solve for c_1 in terms of c_3 using equation (4.34),

$$c_1 = \frac{c_3(n-m)}{2n}. \tag{4.35}$$

As a result, the infinitesimals become

$$\xi^1 = c_3 t + c_4, \tag{4.36}$$

$$\xi^2 = \frac{c_3(n-m)}{2n}x + c_2, \tag{4.37}$$

$$\eta = -\frac{c_3}{n}. \tag{4.38}$$

As such the Lie point symmetry generator admitted by equation (3.9) with exponential case is

$$X = (c_3 t + c_4)\frac{\partial}{\partial t} + \left[\left(\frac{n-m}{2n}\right)c_3 x + c_2\right]\frac{\partial}{\partial x} + \left(-\frac{c_3}{n}\right)\frac{\partial}{\partial \theta}. \tag{4.39}$$

By setting each constant $c_i = 1$ with $c_j = 0$ $(i, j = 1, 2, 3)$ in the infinitesimal generator in equation (4.39), we obtain a three-dimensional Lie algebra which is generated by the operators

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\\
X_2 &= \frac{\partial}{\partial t},\\
X_3 &= 2t\frac{\partial}{\partial t} + \left(\frac{n-m}{n}\right)x\frac{\partial}{\partial x} - \frac{2}{n}\frac{\partial}{\partial \theta}.
\end{aligned} \tag{4.40}
$$

Note that when $m = n$, we obtain the following operators

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\\
X_2 &= \frac{\partial}{\partial t},\\
X_3 &= t\frac{\partial}{\partial t} - \frac{1}{m}\frac{\partial}{\partial \theta}.
\end{aligned} \tag{4.41}
$$

4.3 Case: $k(\theta) = \theta^m$ and $q(\theta) = \theta^{n+1}$

In this section we will consider four sub-cases. Once again we will use equations (4.21) and (4.29) to determine the symmetries for each of the following sub-cases.

4.3.1 Case (i): Ng $\neq 0$, $m = -\frac{4}{3}$ and $n = 0$

Here, $k(\theta)$ is of the form in equation (4.27), so we cannot assume $\xi^2_{xx} = 0$.

We get the following equation after substituting $k(\theta)$ and $q(\theta)$ into (4.29)

$$
-\xi^1_{tt} - 2\theta^{-4/3}\xi^2_{xxx} + \frac{4}{3}\mathrm{Ng}\,\xi^1_t = 0. \tag{4.42}
$$

We can now split equation (4.42) as follows

$$\xi^2_{xxx} = 0, \qquad \text{and} \tag{4.43}$$

$$-\,\xi^1_{tt} + \frac{4}{3}\mathrm{Ng}\,\xi^1_t = 0. \tag{4.44}$$

From equations (4.43) and (4.44) we get

$$\xi^1 = \frac{3e^{\frac{4}{3}\mathrm{Ng}t}c_1}{4\mathrm{Ng}} + c_2, \tag{4.45}$$

$$\xi^2 = x^2 c_5 + x c_4 + c_3, \tag{4.46}$$

$$\eta = -\frac{3\theta}{4}\left(4xc_5 + 2c_4 - e^{\frac{4}{3}\mathrm{Ng}t}c_1\right). \tag{4.47}$$

The coefficients of the infinitesimals ξ^1, ξ^2 and η above are substituted into the Lie point symmetry generator defined in equation (2.4) to obtain

$$X = \left(\frac{3e^{\frac{4}{3}\mathrm{Ng}t}c_1}{4\mathrm{Ng}} + c_2\right)\frac{\partial}{\partial t} + \left(x^2c_5 + xc_4 + c_3\right)\frac{\partial}{\partial x} - \frac{3\theta}{4}\left(4xc_5 + 2c_4 - e^{\frac{4}{3}\mathrm{Ng}t}c_1\right)\frac{\partial}{\partial \theta}. \tag{4.48}$$

By setting each constant $c_i = 1$ with $c_j = 0$ $(i, j = 1, 2, 3)$ in the infinitesimal generator in equation (4.48), we obtain a five-dimensional Lie algebra which is generated by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x},\\ X_2 &= \frac{\partial}{\partial t},\\ X_3 &= 2x\frac{\partial}{\partial x} - 3\theta\frac{\partial}{\partial \theta},\\ X_4 &= x^2\frac{\partial}{\partial x} - 3\theta x\frac{\partial}{\partial \theta},\\ X_5 &= \frac{e^{\frac{4\mathrm{Ng}t}{3}}}{\mathrm{Ng}}\frac{\partial}{\partial t} + \theta e^{\frac{4\mathrm{Ng}t}{3}}\frac{\partial}{\partial \theta} \end{aligned} \tag{4.49}$$

4.3.2 Case (ii): Ng $\neq 0$, $m = -\frac{4}{3}$, $n \neq -\frac{4}{3}$ and $n \neq 0$

Again, in this case $k(\theta)$ is of the form in equation (4.27), so we cannot assume $\xi^2_{xx} = 0$.

We get the following simplified equation after substituting $k(\theta)$ and $q(\theta)$ into (4.29)

$$\text{Ng}\,\theta^n \left(n\xi^1_t - 2n\xi^2_x + \frac{4}{3}\xi^1_t \right) - \xi^1_{tt} - 2\theta^{-\frac{4}{3}}\xi^2_{xxx} = 0. \tag{4.50}$$

We can now split equation (4.50) as follows

$$\xi^1_{tt} = 0, \tag{4.51}$$

$$n\xi^1_t - 2n\xi^2_x + \frac{4}{3}\xi^1_t = 0, \tag{4.52}$$

$$\xi^2_{xxx} = 0. \tag{4.53}$$

Equations (4.51) and (4.53) give

$$\xi^1 = c_1 t + c_2, \tag{4.54}$$

$$\xi^2 = c_5 x^2 + c_4 x + c_3, \tag{4.55}$$

respectively, and substituting this result into equation (4.52) gives

$$\left(\frac{3n+4}{3} \right) c_1 - 4nc_5 x - 2nc_4 = 0. \tag{4.56}$$

We can split equation (4.56) with respect to x, since all the constants are independent of x, it follows that

$$c_5 = 0, \qquad \text{and} \tag{4.57}$$

$$\left(\frac{3n+4}{3} \right) c_1 - 2nc_4 = 0. \tag{4.58}$$

We can solve for c_4 in terms of c_1 to get $c_4 = \dfrac{3n+4}{6n}c_1$.

So in summary, we get the following:

$$\xi^1 = c_1 t + c_2, \tag{4.59}$$

$$\xi^2 = \frac{3n+4}{6n}c_1\, x + c_3, \tag{4.60}$$

$$\eta = -\frac{\theta c_1}{n}. \tag{4.61}$$

The coefficients of the infinitesimals ξ^1, ξ^2 and η above are substituted into the Lie point symmetry generator defined in equation (2.4) to obtain

$$X = (c_1 t + c_2)\frac{\partial}{\partial t} + \left(\frac{3n+4}{6n}c_1\, x + c_3\right)\frac{\partial}{\partial x} - \frac{\theta c_1}{n}\frac{\partial}{\partial \theta}. \tag{4.62}$$

By setting each constant $c_i = 1$ with $c_j = 0$ $(i, j = 1, 2, 3)$ in the infinitesimal generator in equation (4.62), we obtain a three-dimensional Lie algebra which is generated by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= t\frac{\partial}{\partial t} + \frac{3n+4}{6n}\, x\frac{\partial}{\partial x} - \frac{\theta}{n}\frac{\partial}{\partial \theta}. \end{aligned} \tag{4.63}$$

4.3.3 Case (iii): Ng $\neq 0$, $m \neq -\frac{4}{3}$, $m \neq 0$ and $n = 0$

We notice that $k(\theta)$ is not of the form in equation (4.27), so we consider case (ii),

$$\xi^2_{xx} = 0. \tag{4.64}$$

Equation (4.64) gives

$$\xi^2 = c_1 x + c_2. \tag{4.65}$$

Now we substitute our choices for $k(\theta)$ and $q(\theta)$ into equation (4.29). This simplifies to

$$-\mathrm{Ng}\xi^1_t m - \xi^1_{tt} = 0. \tag{4.66}$$

From equation (4.66) we get

$$\xi^1 = -\frac{e^{-m\mathrm{Ng}t} c_3}{m\mathrm{Ng}} + c_4. \tag{4.67}$$

Using equations (4.21), (4.65) and (4.67) we get

$$\eta = \frac{2\theta c_1}{m} - \frac{e^{-m\mathrm{Ng}t}\theta c_3}{m}. \tag{4.68}$$

The coefficients of the infinitesimals ξ^1, ξ^2 and η above are substituted into the Lie point symmetry generator defined in equation (2.4) to obtain

$$X = \left(-\frac{e^{-m\mathrm{Ng}t} c_3}{m\mathrm{Ng}} + c_4\right)\frac{\partial}{\partial t} + (c_1 x + c_2)\frac{\partial}{\partial x} + \left(\frac{2\theta c_1}{m} - \frac{e^{-m\mathrm{Ng}t}\theta c_3}{m}\right)\frac{\partial}{\partial \theta}. \tag{4.69}$$

By setting each constant $c_i = 1$ with $c_j = 0$ $(i, j = 1, 2, 3)$ in the infinitesimal generator in equation (4.69), we obtain a four-dimensional Lie algebra which is generated by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= x\frac{\partial}{\partial x} + \frac{2\theta}{m}\frac{\partial}{\partial \theta}, \\ X_4 &= \frac{e^{-m\mathrm{Ng}t}}{\mathrm{Ng}}\frac{\partial}{\partial t} + \theta e^{-m\mathrm{Ng}t}\frac{\partial}{\partial \theta} \end{aligned} \tag{4.70}$$

4.3.4 Case (iv): Ng $\neq 0$, $m \neq -\frac{4}{3}$, $m \neq 0$ and $n \neq 0$

Again, we notice that $k(\theta)$ is not of the form in equation (4.27), so we consider case (ii),

$$\xi^2_{xx} = 0. \tag{4.71}$$

Equation (4.71) gives

$$\xi^2 = c_1 x + c_2. \tag{4.72}$$

Now we substitute our choices for $k(\theta)$ and $q(\theta)$ into equation (4.29). This simplifies to

$$\mathrm{Ng}\theta^n \left(\xi^1_t n - 2c_1 n - \xi^1_t m\right) - \xi^1_{tt} = 0. \tag{4.73}$$

Since the term ξ^1_{tt} is independent of θ we can split (4.73) as follows

$$\xi^1_{tt} = 0,$$

$$\text{that is,} \qquad \xi^1 = c_3 t + c_4., \tag{4.74}$$

$$\text{and} \qquad c_3 n - 2c_1 n - c_3 m = 0. \tag{4.75}$$

We can also solve for c_1 in terms of c_3 using equation (4.75),

$$c_1 = \frac{c_3(n-m)}{2n}. \tag{4.76}$$

So we have our infinitesimals reduce to

$$\xi^1 = c_3 t + c_4, \tag{4.77}$$

$$\xi^2 = \frac{c_3(n-m)}{2n} x + c_2, \tag{4.78}$$

$$\eta = -\frac{\theta c_3}{n}. \tag{4.79}$$

The coefficients of the infinitesimals ξ^1, ξ^2 and η above are substituted into the Lie point symmetry generator defined in equation (2.4) to obtain

$$X = (c_3 t + c_4)\frac{\partial}{\partial t} + \left(\frac{c_3(n-m)}{2n}x + c_2\right)\frac{\partial}{\partial x} - \left(\frac{\theta c_3}{n}\right)\frac{\partial}{\partial \theta}. \tag{4.80}$$

By setting each constant $c_i = 1$ with $c_j = 0$ $(i, j = 1, 2, 3)$ in the infinitesimal generator in equation (4.80), we obtain a three-dimensional Lie algebra which is generated by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= t\frac{\partial}{\partial t} + \left(\frac{n-m}{2n}\right)x\frac{\partial}{\partial x} - \frac{\theta}{n}\frac{\partial}{\partial \theta}. \end{aligned} \tag{4.81}$$

Note that when $m = n$, we obtain the following operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= t\frac{\partial}{\partial t} - \frac{\theta}{m}\frac{\partial}{\partial \theta}. \end{aligned} \tag{4.82}$$

4.4 Concluding remarks

In this chapter a number of cases have been observed for which the equation in question admits Lie point symmetries. Two cases of the thermal conductivity and the internal heat generation have been chosen, namely the exponential and power law depending on temperature by this thermal parameter.

Chapter 5

Classification of group invariant solutions

In this chapter we construct the one-dimensional optimal systems of subalgebras of the algebras admitted in Chapter 4. This technique helps us in classifying group invariant solutions of differential equations and it was first considered by Ovsiannikov [26]. We will follow the approach as set out by Olver in [15].

In theory, a family of group invariant solutions that correspond to a subgroup of the symmetries admitted by a differential equation can be constructed [27]. However, the thought of listing all the group invariant solutions seems impractical since the possible number of those subgroups is infinite in most cases. In view of that, we seek to obtain optimal systems of subalgebras of the Lie algebra which provide a useful and methodical way of classifying these group invariant solutions.

5.1 Optimal System

5.1.1 Case: $k(\theta) = e^{m\theta}$ and $q(\theta) = e^{n\theta}$

We begin by computing the commutators of the three-dimensional Lie algebra spanned by the base vectors X_1, X_2 and X_3 specified in equation (4.40). These are obtained by applying the commutation relation

$$[X_i, X_j] = X_i X_j - X_j X_i, \quad i, j = 1, ..., n, \tag{5.1}$$

where

$$[X_i, X_j] = -[X_j, X_i], \tag{5.2}$$

and

$$[X_i, X_i] = 0. \tag{5.3}$$

This results in

$$\begin{aligned} [X_1, X_2] &= X_1 X_2 - X_2 X_1 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right) \\ &= 0, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} [X_2, X_3] &= X_2 X_3 - X_3 X_2 \\ &= \frac{\partial}{\partial t}\left\{2t\frac{\partial}{\partial t} + \frac{(n-m)x}{n}\frac{\partial}{\partial x} - \frac{2}{n}\frac{\partial}{\partial \theta}\right\} \\ &\quad - \left\{2t\frac{\partial}{\partial t} + \frac{(n-m)x}{n}\frac{\partial}{\partial x} - \frac{2}{n}\frac{\partial}{\partial \theta}\right\}\left(\frac{\partial}{\partial t}\right) \\ &= 2X_2, \end{aligned} \tag{5.5}$$

and

$$
\begin{aligned}
[X_1, X_3] &= X_1X_3 - X_3X_1 \\
&= \frac{\partial}{\partial x}\left\{2t\frac{\partial}{\partial t} + \frac{(n-m)x}{n}\frac{\partial}{\partial x} - \frac{2}{n}\frac{\partial}{\partial \theta}\right\} \\
&\quad - \left\{2t\frac{\partial}{\partial t} + \frac{(n-m)x}{n}\frac{\partial}{\partial x} - \frac{2}{n}\frac{\partial}{\partial \theta}\right\}\left(\frac{\partial}{\partial x}\right) \\
&= \frac{(n-m)}{n}X_1. \qquad (5.6)
\end{aligned}
$$

For convenience, we will display the commutators of the Lie algebra in the form of a commutator table where the (i,j)th entry is $[X_i, X_j]$. This is shown in Table 5.1 below:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{(n-m)}{n}X_1$
X_2	0	0	$2X_2$
X_3	$\frac{(m-n)}{n}X_1$	$-2X_2$	0

Table 5.1: Commutator table of the three-dimensional Lie algebra (4.40).

Next, we calculate the adjoint representation using the commutator table given in Table 5.1 as well as the Lie series [15] given by

$$
\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_i})X_j &= X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i,[X_i,X_j]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_i,[X_i,[X_i,X_j]]] + ..., \qquad (5.7)
\end{aligned}
$$

where

$$i, j = 1, ..., n.$$

Note that

$$\mathrm{Ad}(e^{\epsilon X_i})X_i = X_i. \qquad (5.8)$$

As examples,

$$
\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_1})X_3 &= X_3 - \epsilon[X_1, X_3] + \frac{1}{2!}\epsilon^2[X_1,[X_1,X_3]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_1,[X_1,[X_1,X_3]]] + ..., \\
&= X_3 - \epsilon\left(\frac{n-m}{n}\right)X_1,
\end{aligned} \tag{5.9}
$$

and

$$
\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_3})X_1 &= X_1 - \epsilon[X_3, X_1] + \frac{1}{2!}\epsilon^2[X_3,[X_3,X_1]] \\
&\quad - \frac{1}{3!}\epsilon^3[X_3,[X_3,[X_3,X_1]]] + ..., \\
&= \left(1 - \epsilon\left(\frac{m-n}{n}\right) + \frac{\epsilon^2}{2!}\left(\frac{m-n}{n}\right)^2\right. \\
&\quad \left. -\frac{\epsilon^3}{3!}\left(\frac{m-n}{n}\right)^3 + \dots\right)X_1, \\
&= \exp\left(\epsilon\left(\frac{n-m}{n}\right)\right)X_1.
\end{aligned} \tag{5.10}
$$

The remainder of the adjoint representations are found in a similar fashion. We construct an adjoint table for the symmetry generators (4.40) where the (i,j)th entry indicates $\mathrm{Ad}(e^{\epsilon X_i})X_j$. As a result, we have the adjoint representation given by Table 5.2 below:

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon\left(\frac{n-m}{n}\right)X_1$
X_2	X_1	X_2	$X_3 - 2\epsilon X_2$
X_3	$\exp\left(\epsilon\left(\frac{n-m}{n}\right)\right)X_1$	$e^{2\epsilon}X_2$	X_3

Table 5.2: Adjoint table for the symmetry generators (4.40).

This is followed by considering the non-zero vector

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3, \quad a_i \in \mathbb{R}. \tag{5.11}$$

We are tasked with simplifying the coefficients a_i as much as we can through a number of well-judged applications of adjoint maps to X. To start with, we consider the scenario when $a_3 \neq 0$. We can choose $a_3 = 1$ for the purposes of scaling and so X is now

$$X = a_1 X_1 + a_2 X_2 + X_3. \tag{5.12}$$

With reference to Table 5.2, we can act on X by $\mathrm{Ad}(e^{\epsilon X_2})$ which returns

$$X = a_1 X_1 + (a_2 - 2\epsilon) X_2 + X_3. \tag{5.13}$$

Further, we can make the coefficient a_2 disappear for the distinct value of $\epsilon = a_2/2$. Accordingly,

$$X = a_1 X_1 + X_3. \tag{5.14}$$

Now, if we act on X by $\mathrm{Ad}(e^{\epsilon X_1})$ which returns

$$X = \left(a_1 - \epsilon \left(\frac{n-m}{n} \right) \right) X_1 + X_3 \tag{5.15}$$

Again, we can make the coefficient a_1 disappear for the distinct value of $\epsilon = a_1 \left(\dfrac{n}{n-m} \right)$. Thus it results in

$$X = X_3. \tag{5.16}$$

On the other hand, choosing $a_3 = 0$ in the vector (5.11) gives that

$$X = a_1 X_1 + a_2 X_2. \tag{5.17}$$

We now consider all possible situations for the coefficient a_2. We can first set $a_2 \neq 0$ and again, by means of scaling $a_2 = 1$, the vector X directly above becomes

$$X = a_1 X_1 + X_2. \tag{5.18}$$

Using Table 5.2, we can now act on this vector X by $\mathrm{Ad}(e^{\epsilon X_3})$ to get that

$$X = a_1 \exp\left(\epsilon\left(\frac{n-m}{n}\right)\right) X_1 + e^{2\epsilon} X_2,$$

and with manipulation X can be written as

$$X = a_1 \exp\left(\epsilon\left(\frac{n-m}{n}\right) - 2\epsilon\right) X_1 + X_2. \tag{5.19}$$

Specifying that $a_1 \neq 0$ gives rise to two cases: $a_1 > 0$ or $a_1 < 0$ and as such,

$$X = X_2 \pm X_1. \tag{5.20}$$

However, if we had chosen $a_1 = 0$, then the vector X in equation (5.17) would be

$$X = X_2. \tag{5.21}$$

The last choice is $a_2 = 0$ and this results in

$$X = X_1. \tag{5.22}$$

All constants simplified, an optimal system of one-dimensional subalgebras of the Lie algebra (4.40) is given by

$$\begin{aligned} &X_1, \\ &X_2, \\ &X_3, \\ X_2 \pm &X_1. \end{aligned} \tag{5.23}$$

The same approach is performed for the rest of the cases. Only the Commutator table, Adjoint table and Optimal system will be shown for the remaining cases.

5.1.2 Case: $k(\theta) = \theta^m$ and $q(\theta) = \theta^{n+1}$

5.1.2.1 Case (i): Ng $\neq 0$, $m = -\dfrac{4}{3}$ and $n = 0$

We will not consider the optimal system for this case because this case was found to have 5 symmetries and the optimal system of one-dimensional subalgebras of the Lie algebra (4.49) would include too many combinations to consider.

5.1.2.2 Case (ii): Ng $\neq 0$, $m = -\dfrac{4}{3}$ and $n \neq 0$

We will display the commutators of the Lie algebra in the form of a commutator table where the (i, j)th entry is $[X_i, X_j]$. This is shown in Table 5.3 below:
We construct an adjoint table for the symmetry generators (4.63) where the (i, j)th entry indicates $\text{Ad}(e^{\epsilon X_i})X_j$. As a result, we have the adjoint representation given by Table 5.4 below:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{4+3n}{6n}X_1$
X_2	0	0	X_2
X_3	$-\frac{4+3n}{6n}X_1$	$-X_2$	0

Table 5.3: Commutator table of the three-dimensional Lie algebra (4.63).

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon\left(\frac{4+3n}{6n}\right)X_1$
X_2	X_1	X_2	$X_3 - \epsilon X_2$
X_3	$\exp\left(\epsilon\left(\frac{4+3n}{6n}\right)\right)X_1$	$e^{\epsilon}X_2$	X_3

Table 5.4: Adjoint table for the symmetry generators (4.63).

With reference to Table 5.4, an optimal system of one-dimensional subalgebras of the Lie algebra (4.63) is given by those generated by

$$
\begin{aligned}
&X_1,\\
&X_2,\\
&X_3,\\
&X_2 \pm X_1. \qquad (5.24)
\end{aligned}
$$

5.1.2.3 Case (iii): Ng $\neq 0$, $m \neq -\frac{4}{3}$, $m \neq 0$ and $n = 0$

We will display the commutators of the Lie algebra in the form of a commutator table where the (i, j)th entry is $[X_i, X_j]$. This is shown in Table 5.5 below:

We construct an adjoint table for the symmetry generators (4.70) where the (i, j)th entry indicates $\mathrm{Ad}(e^{\epsilon X_i})X_j$. As a result, we have the adjoint representation given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_1	0
X_2	0	0	0	$-m\text{Ng}X_4$
X_3	$-X_1$	0	0	0
X_4	0	$m\text{Ng}X_4$	0	0

Table 5.5: Commutator table of the four-dimensional Lie algebra (4.70).

Table 5.6 below:

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$X_3 - \epsilon X_1$	X_4
X_2	X_1	X_2	X_3	$e^{\epsilon m\text{Ng}}X_4$
X_3	$e^{\epsilon}X_1$	X_2	X_3	X_4
X_4	X_1	$X_2 - \epsilon m\text{Ng}X_4$	X_3	X_4

Table 5.6: Adjoint table for the symmetry generators (4.70).

With reference to Table 5.6, an optimal system of one-dimensional subalgebras of the Lie algebra (4.70) is given by those generated by

$$\begin{aligned} &X_1 + aX_4, \\ &X_2 + aX_1, \\ &X_3 + aX_2, \\ &X_4. \end{aligned} \tag{5.25}$$

5.1.2.4 Case (iv): Ng $\neq 0$, $m \neq -\frac{4}{3}$, $m \neq 0$ and $n \neq 0$

We will display the commutators of the Lie algebra in the form of a commutator table where the (i,j)th entry is $[X_i, X_j]$. This is shown in Table 5.7 below:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{n-m}{2n}X_1$
X_2	0	0	X_2
X_3	-$\frac{n-m}{2n}X_1$	-X_2	0

Table 5.7: Commutator table of the three-dimensional Lie algebra (4.81).

We construct an adjoint table for the symmetry generators (4.81) where the (i,j)th entry indicates $\mathrm{Ad}(e^{\epsilon X_i})X_j$. As a result, we have the adjoint representation given by Table 5.8 below:

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon\frac{n-m}{2n}X_1$
X_2	X_1	X_2	$X_3 - \epsilon X_1$
X_3	$\exp\left(\epsilon\left(\frac{n-m}{2n}\right)\right)X_1$	$\exp(\epsilon)X_2$	X_3

Table 5.8: Adjoint table for the symmetry generators (4.81).

With reference to Table 5.8, an optimal system of one-dimensional subalgebras of the Lie algebra (4.81) is given by those generated by

$$\begin{aligned} &X_1 + aX_2, \\ &X_2, \\ &X_3 + aX_2. \end{aligned} \tag{5.26}$$

5.2 Group invariant solutions

In this section we construct the group invariant solutions for the governing equation (3.9). It turns out that the exact solutions obtained are more general. One may seek approximate solutions.

5.2.1 Case: $k(\theta) = e^{m\theta}$ and $q(\theta) = e^{n\theta}$

5.2.1.1 Steady state solution

We begin by considering the group invariant solution under the time-translational symmetry generator

$$X_2 = \frac{\partial}{\partial t}, \tag{5.27}$$

of the optimal system (5.23), with $n = m$.

The Lagrangian system provides the corresponding differential equations of characteristic curves which are given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\theta}{0}. \tag{5.28}$$

We see that both x and θ are invariant and as a result

$$x = I_1 \quad \text{and} \quad \theta = I_2, \tag{5.29}$$

where I_1 and I_2 are constants.

Note: The terms $\frac{dx}{0}$ and $\frac{d\theta}{0}$ in equation (5.28) do not represent division by 0. In fact, they are written to note that the variables x and θ are invariant for the characteristic system under consideration.

We write $I_2 = F(I_1)$, and find that the invariant solution admitted by the generator X_2 is the steady state solution

$$\theta(t,x) = F(x). \tag{5.30}$$

Substituting the invariant solution in equation (5.30) into the governing equation (3.9) results in the following second-order non-linear ordinary differential equation in terms of $F(x)$

$$m\, e^{F(x)m}(F'(x))^2 + e^{F(x)m}F''(x) + \mathrm{Ng}\, e^{F(x)m} = 0. \tag{5.31}$$

Solving this ODE gives following solution

$$F(x) = \frac{\ln\left(\cos\left(\sqrt{m}\sqrt{\mathrm{Ng}}\,(x-c_1)\right)\right)}{m} + c_2. \tag{5.32}$$

The resulting invariant boundary conditions are obtained by applying the invariant solution in equation (5.30) onto the two sets of boundary conditions, namely equations (3.10) and (3.11).

We obtain $F'(0) = 0$ and $F(1) = 1$ when equation (5.30) is applied to equation (3.10), which gives the exact solution for the steady case

$$F(x) = \frac{\ln\left(\cos\left(\sqrt{m}\sqrt{\mathrm{Ng}}\,x\right)\right) - \ln\left(\cos\left(\sqrt{m}\sqrt{\mathrm{Ng}}\right)\right) + m}{m}. \tag{5.33}$$

The solution in equation (5.33) is depicted in Figures 5.1, 5.2, 5.3 and 5.4.

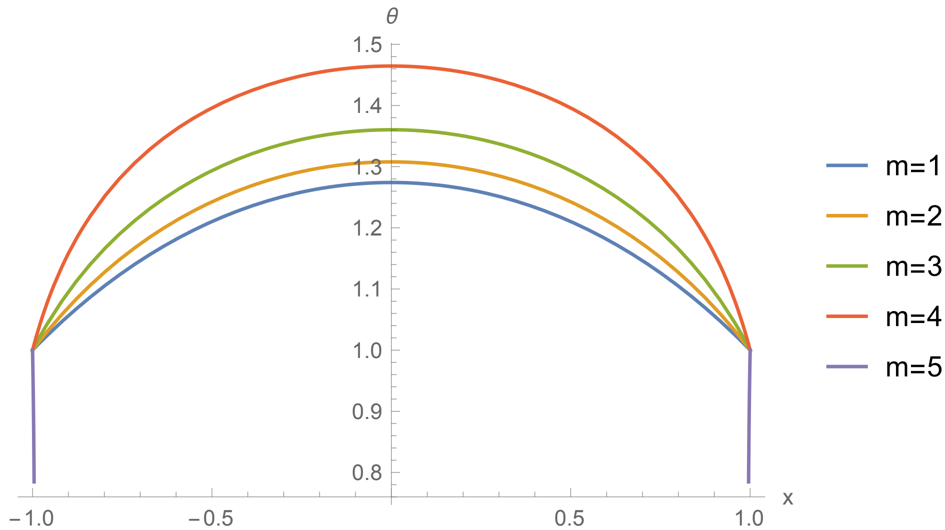

Figure 5.1: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of m, for when $m > 0$, using boundary condition (3.10). Here $\mathrm{Ng} = 0.5$ is kept fixed.

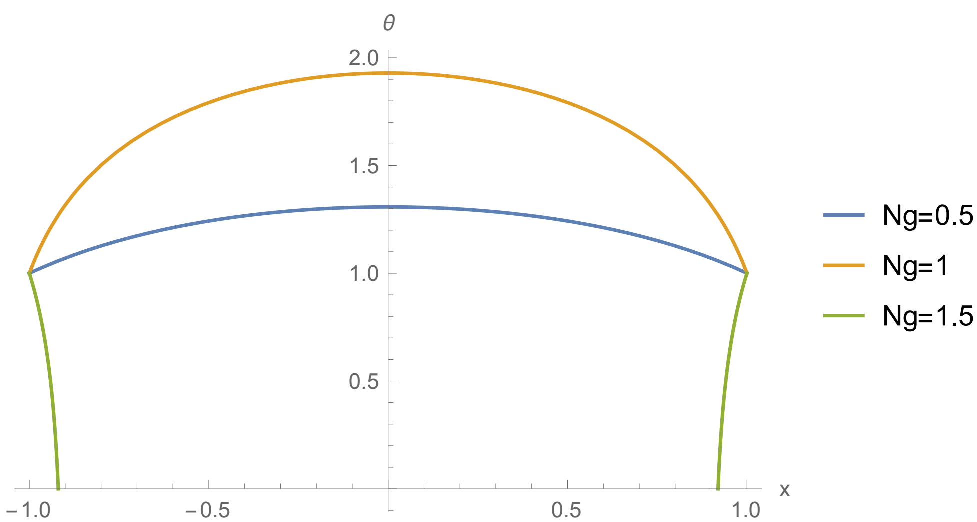

Figure 5.2: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = 2$ is kept fixed.

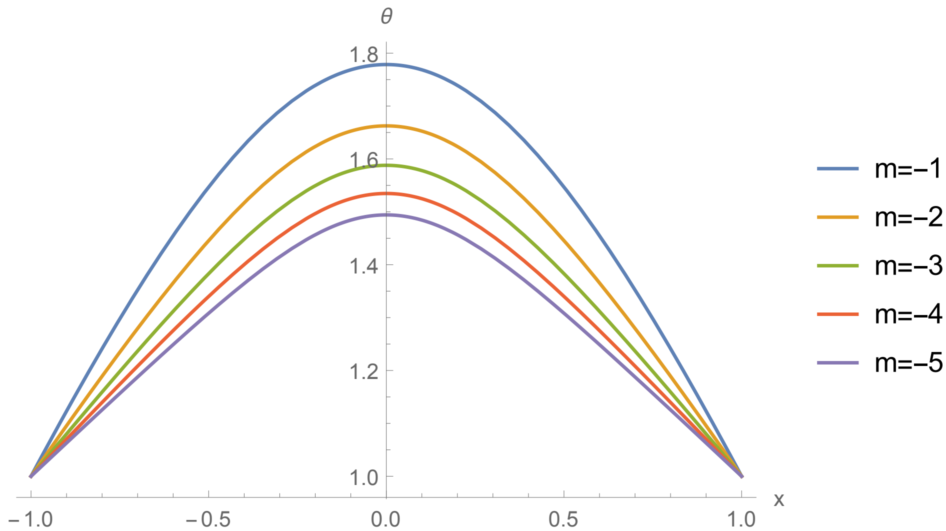

Figure 5.3: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of m, for when $m < 0$, using boundary condition (3.10). Here $\mathrm{Ng} = 2$ is kept fixed.

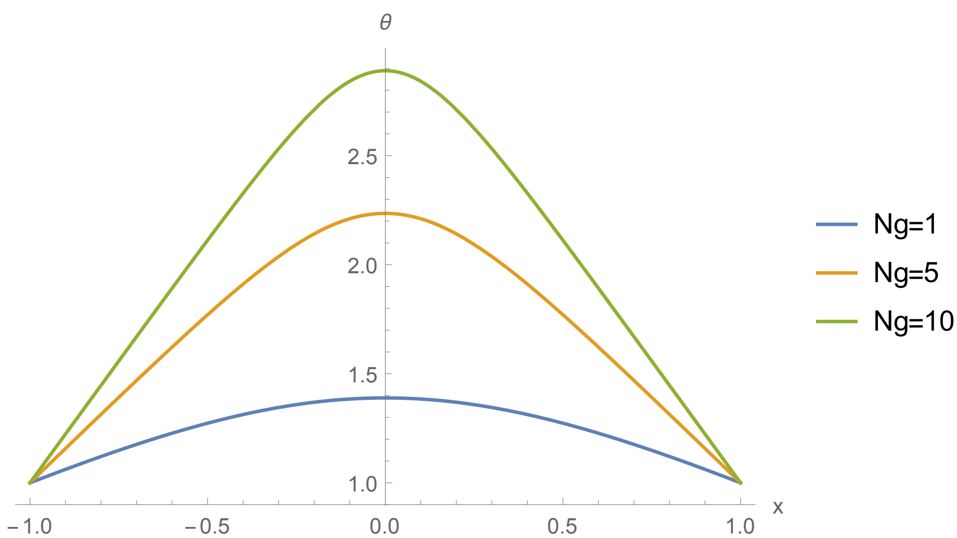

Figure 5.4: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = -2$ is kept fixed.

The exact solution using equation (3.11) could not be found.

5.2.1.2 Group invariant solution via subgroup generated by X_3

Here, we consider the group invariant solution corresponding to the operator X_3 of the optimal system (5.23), with $n = m$, which is given by

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{m}\frac{\partial}{\partial \theta}. \tag{5.34}$$

The characteristic system is given by

$$\frac{dt}{t} = \frac{dx}{0} = -\frac{m\ d\theta}{1}. \tag{5.35}$$

We see that x is invariant and so we write

$$x = I_1, \tag{5.36}$$

where I_1 is a constant.

Now, the first and last terms of the characteristic system (5.35) can be written as a variable separable ordinary differential equation given by

$$\frac{dt}{t} = -m\ d\theta. \tag{5.37}$$

Integrating both sides gives

$$\begin{aligned} \ln(t) &= -m\ \theta + \ln(I_2), \\ \ln(t) + m\ \theta &= \ln(I_2), \\ t\ e^{m\ \theta} &= I_2. \end{aligned} \tag{5.38}$$

We write $I_2 = F(I_1)$, and find that the invariant solution admitted by the generator X_3 is given by

$$\theta(t,x) = \frac{\ln\left(\frac{F(x)}{t}\right)}{m}. \tag{5.39}$$

Substituting the invariant solution in equation (5.39) into the governing equation (3.9) results in the following second-order non-linear ordinary differential equation in terms of $F(x)$

$$\frac{F''(x) + m\mathrm{Ng}F(x) + 1}{mt} = 0, \tag{5.40}$$

which has general solution

$$F(x) = c_2 \sin\left(\sqrt{m}\sqrt{\mathrm{Ng}}x\right) + c_1 \cos\left(\sqrt{m}\sqrt{\mathrm{Ng}}x\right) - \frac{1}{m\mathrm{Ng}}. \tag{5.41}$$

So now we have

$$\theta(t,x) = \frac{\ln\left(\frac{c_2 \sin\left(\sqrt{m}\sqrt{\mathrm{Ng}}x\right) + c_1 \cos\left(\sqrt{m}\sqrt{\mathrm{Ng}}x\right) - \frac{1}{m\mathrm{Ng}}}{t}\right)}{m}. \tag{5.42}$$

The solution in equation (5.42) using boundary condition (3.10) is depicted in Figures 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, while the solution in equation (5.42) using boundary condition (3.11) is depicted in Figures 5.11, 5.12, 5.13, 5.14, 5.15 and 5.16.

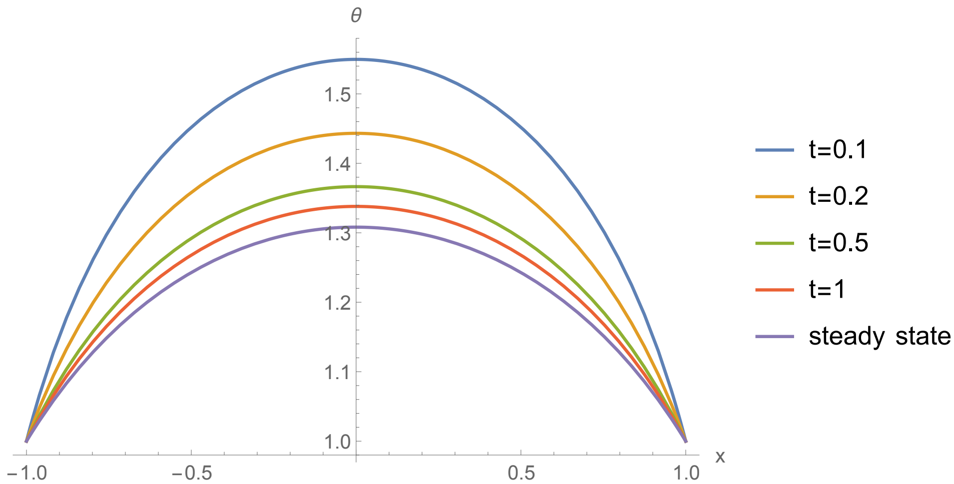

Figure 5.5: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of t, using boundary condition (3.10). Here $\mathrm{Ng} = 0.5$ and $m = 2$ are kept fixed.

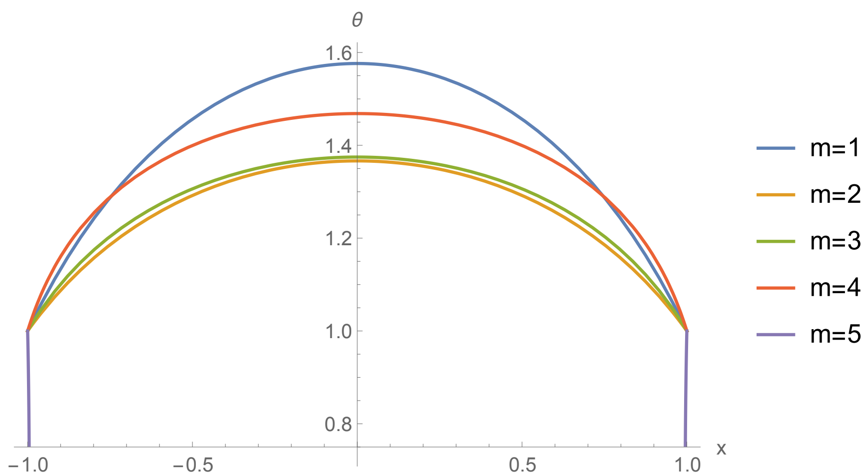

Figure 5.6: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of m, for when $m > 0$, using boundary condition (3.10). Here $\mathrm{Ng} = 0.5$ and $t = 0.5$ are kept fixed.

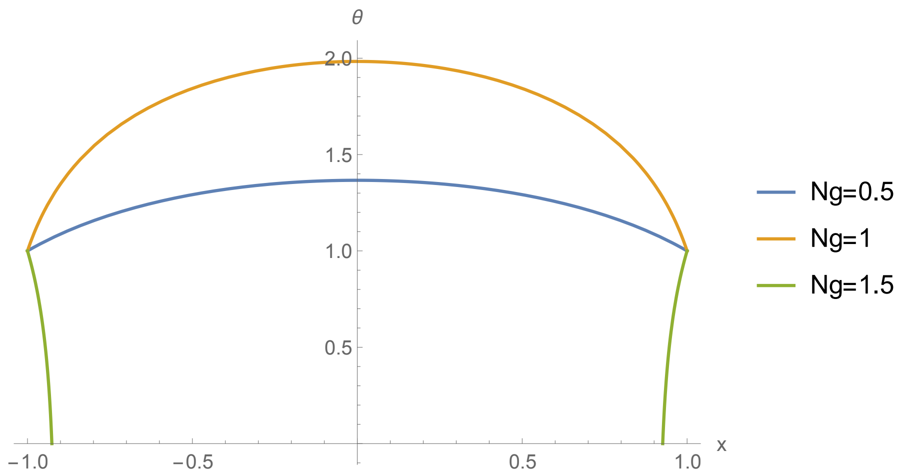

Figure 5.7: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = 2$ and $t = 0.5$ are kept fixed.

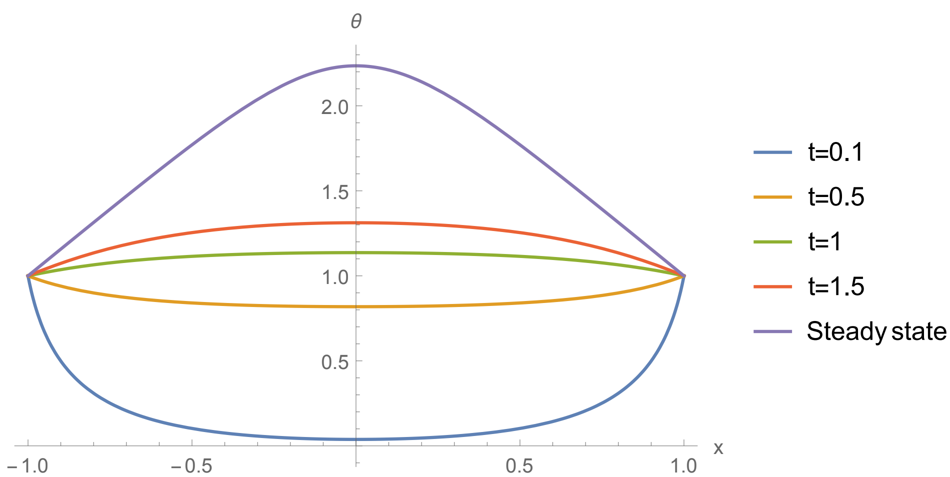

Figure 5.8: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of t, using boundary condition (3.10). Here $\mathrm{Ng} = 5$ and $m = -2$ are kept fixed.

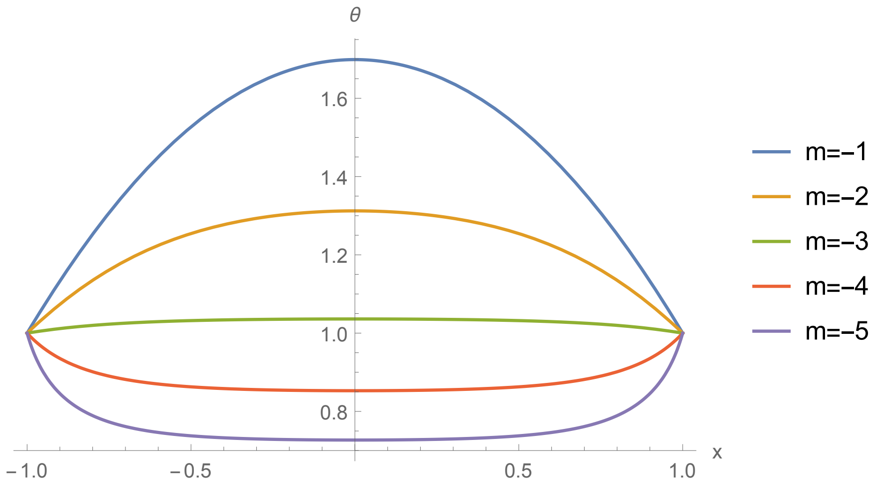

Figure 5.9: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of m, for $m < 0$, using boundary condition (3.10). Here $\mathrm{Ng} = 5$ and $t = 1.5$ are kept fixed.

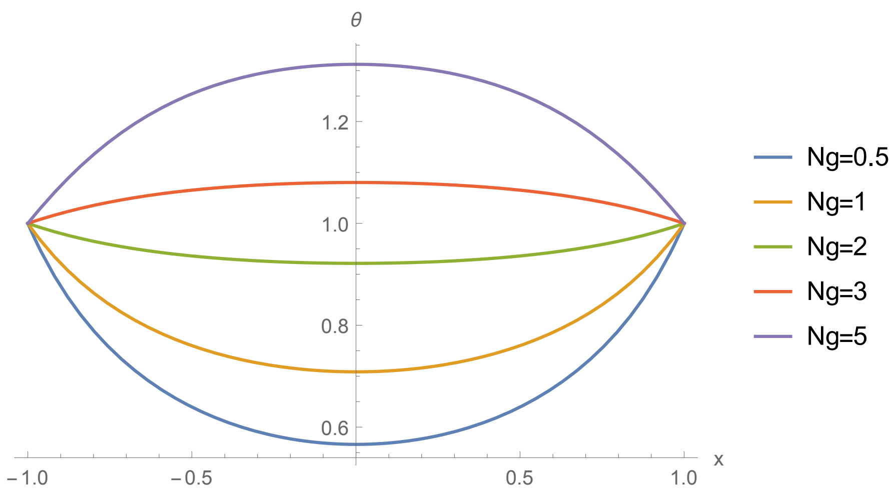

Figure 5.10: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = -2$ and $t = 1.5$ are kept fixed.

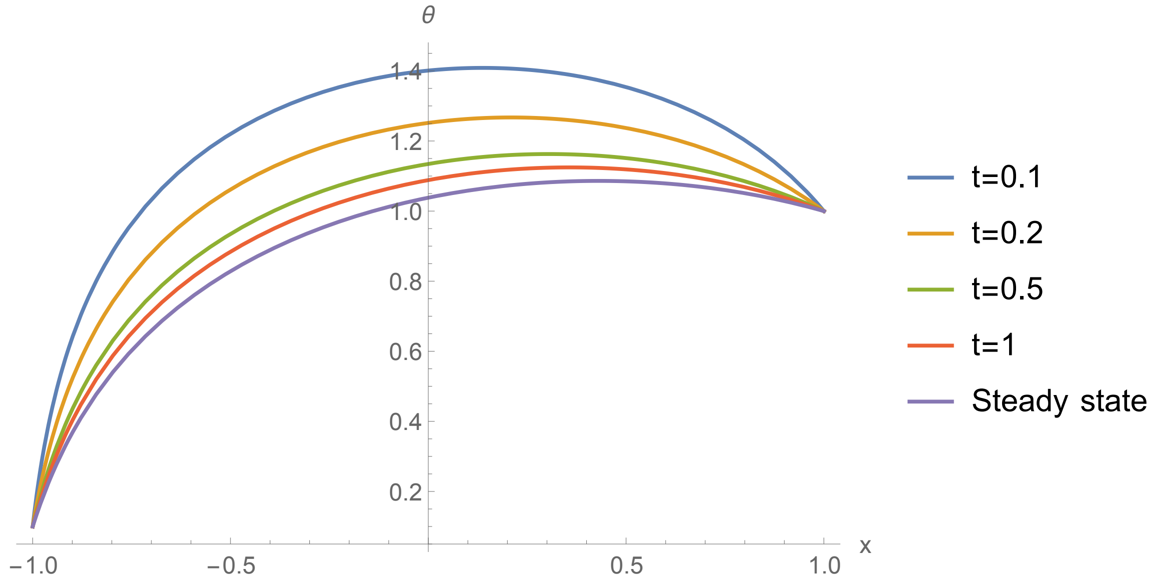

Figure 5.11: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of t, using boundary condition (3.11). Here $\mathrm{Ng} = 0.5$ and $m = 2$ are kept fixed.

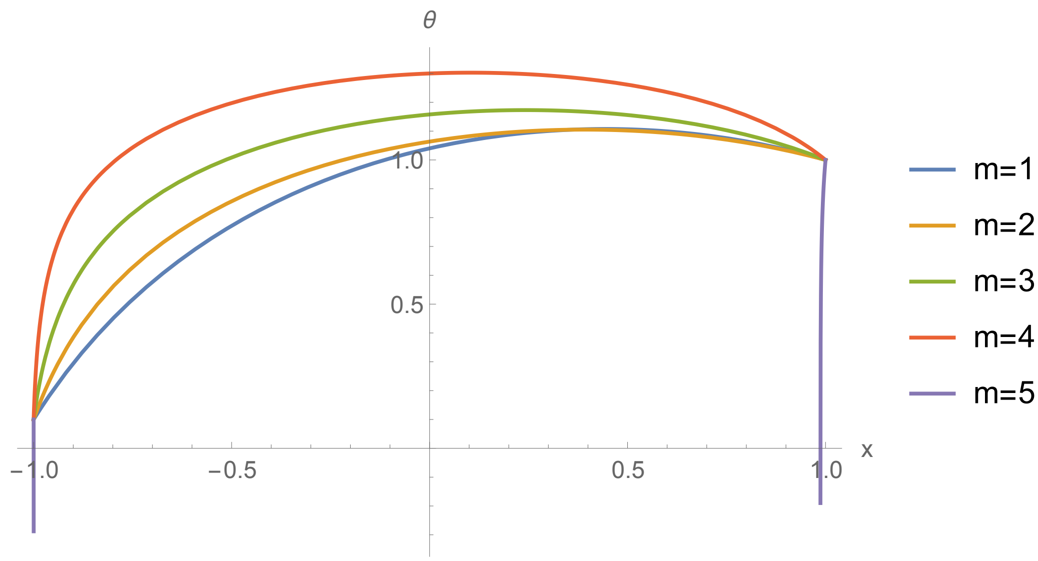

Figure 5.12: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of m, when $m > 0$, using boundary condition (3.11). Here $\mathrm{Ng} = 2$ and $t = 0.5$ are kept fixed.

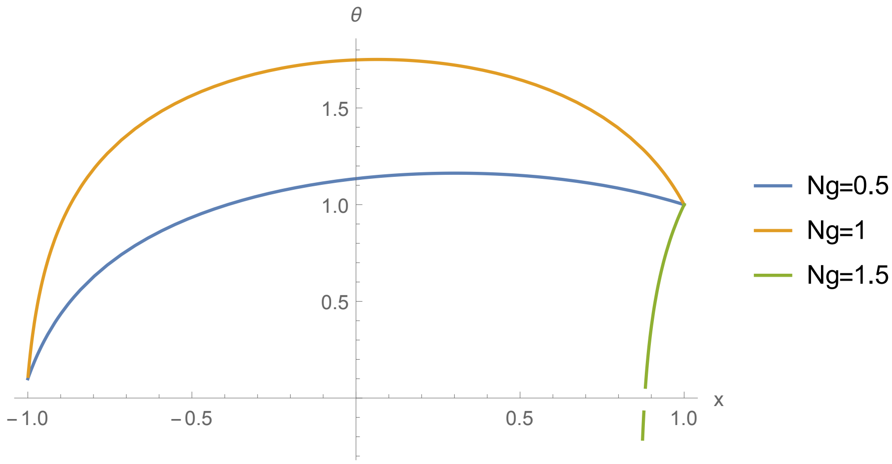

Figure 5.13: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of Ng, using boundary condition (3.11). Here $t = 0.5$ and $m = 2$ are kept fixed.

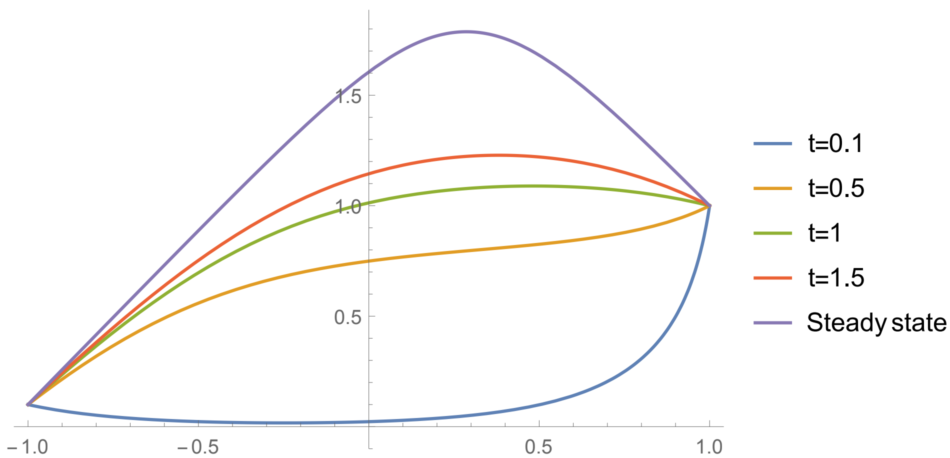

Figure 5.14: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of t, using boundary condition (3.11). Here $\mathrm{Ng} = 5$ and $m = -2$ are kept fixed.

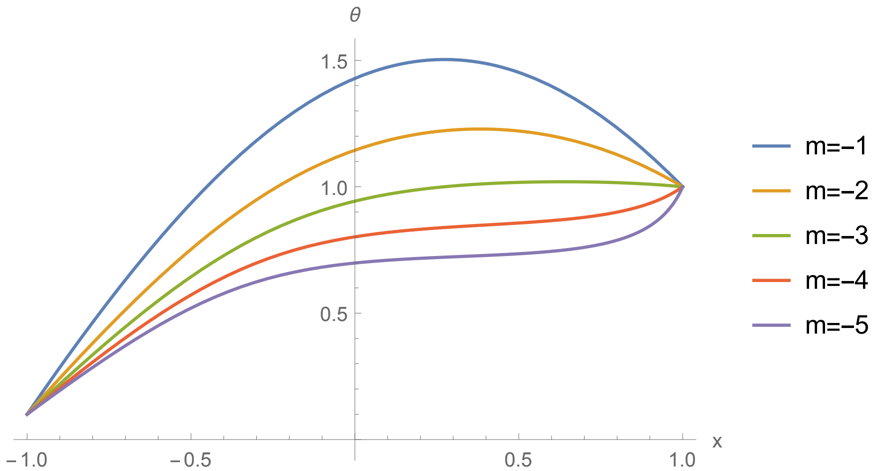

Figure 5.15: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of m, when $m < 0$, using boundary condition (3.11). Here $t = 1.5$ and $\mathrm{Ng} = 5$ are kept fixed.

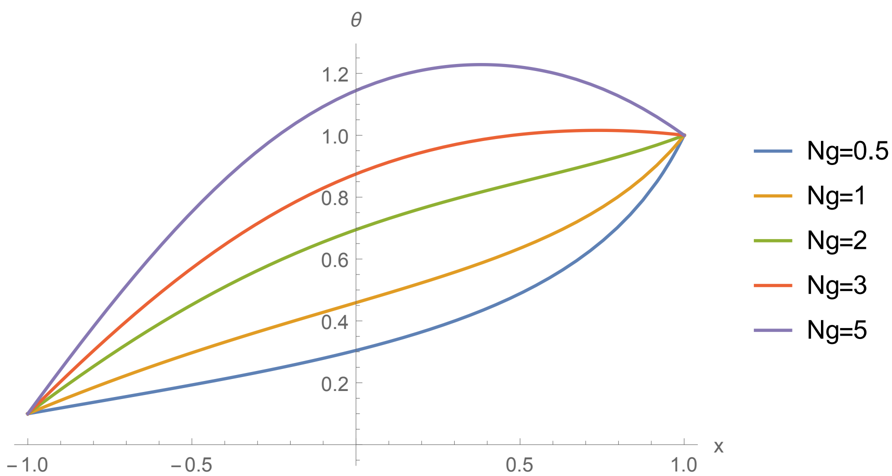

Figure 5.16: Temperature in a wall with exponential thermal conductivity and exponential internal heat generation and varying values of Ng, using boundary condition (3.11). Here $t = 1.5$ and $m = -2$ are kept fixed.

5.2.2 Case: $k(\theta) = \theta^m$ and $q(\theta) = \theta^{n+1}$

5.2.2.1 Steady state solution

We begin by considering the group invariant solution under the time-translational symmetry generator

$$X_2 = \frac{\partial}{\partial t}, \tag{5.43}$$

of the optimal system (5.26), with $n = m$.

The Lagrangian system provides the corresponding differential equations of characteristic curves which are given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\theta}{0}. \tag{5.44}$$

We see that both x and θ are invariant and as a result

$$x = I_1 \quad \text{and} \quad \theta = I_2, \tag{5.45}$$

where I_1 and I_2 are constants.

We write $I_2 = F(I_1)$, and find that the invariant solution admitted by the generator X_2 is the steady state solution

$$\theta(t, x) = F(x). \tag{5.46}$$

Substituting the invariant solution in equation (5.46) into the governing equation (3.9) results in the following second-order non-linear ordinary differential equation in terms of $F(x)$

$$F(x)^{m-1}\left(F(x)\left(F''(x)+\mathrm{Ng}F(x)\right)+mF'(x)^2\right)=0. \tag{5.47}$$

Solving this ODE using DSolve in Mathematica gives the following solution

$$F(x)=c_2\left(\cos\left(\sqrt{m+1}\sqrt{\mathrm{Ng}}\,(x-c_1)\right)\right)^{\frac{1}{m+1}}. \tag{5.48}$$

The resulting invariant boundary conditions are obtained by applying the invariant solution in equation (5.46) onto the two sets of boundary conditions, namely equations (3.10) and (3.11).

We obtain $F'(0)=0$ and $F(1)=1$ when equation (5.46) is applied to equation (3.10), which gives the exact solution for the steady case

$$F(x)=\left(\cos\left(\sqrt{m+1}\sqrt{\mathrm{Ng}}\right)\right)^{-\frac{1}{m+1}}\left(\cos\left(\sqrt{m+1}\sqrt{\mathrm{Ng}}x\right)\right)^{\frac{1}{m+1}}. \tag{5.49}$$

The solution in equation (5.49) is depicted in Figures 5.17, 5.18, 5.19 and 5.20.

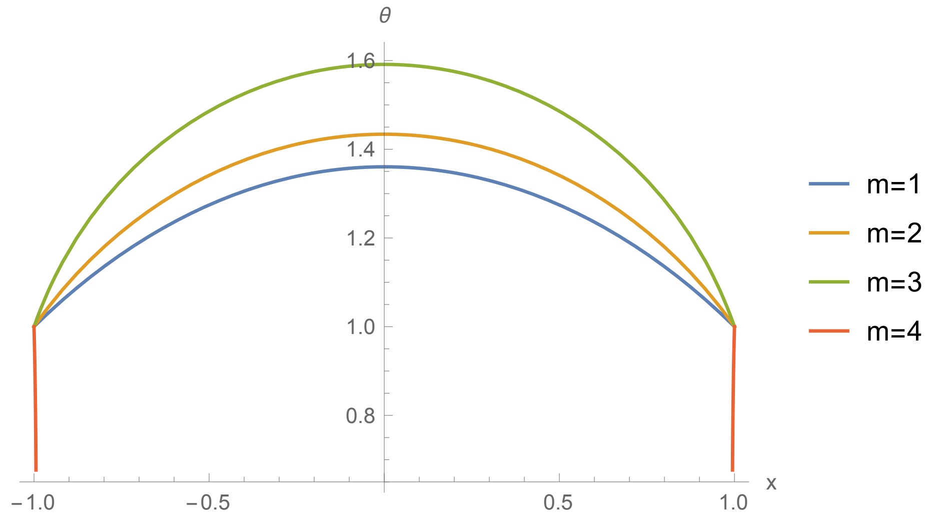

Figure 5.17: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of m, for when $m > -1$, using boundary condition (3.10). Here $\text{Ng} = 0.5$ is kept fixed.

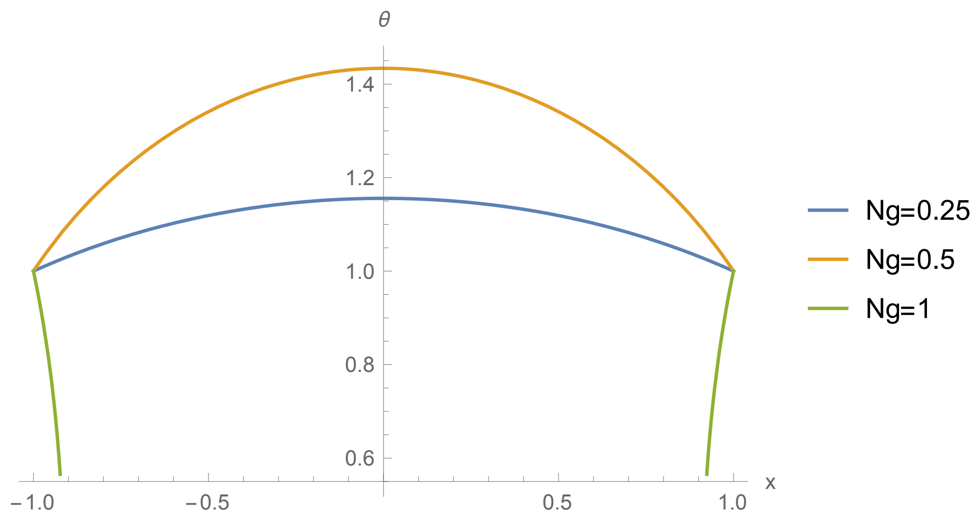

Figure 5.18: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = 2$ is kept fixed.

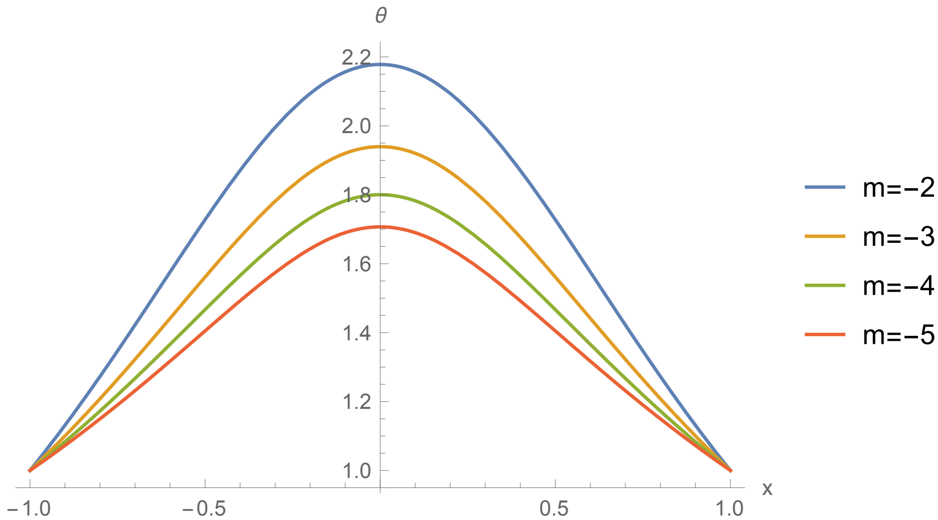

Figure 5.19: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of m, for $m < -1$, using boundary condition (3.10). Here $\mathrm{Ng} = 2$ is kept fixed.

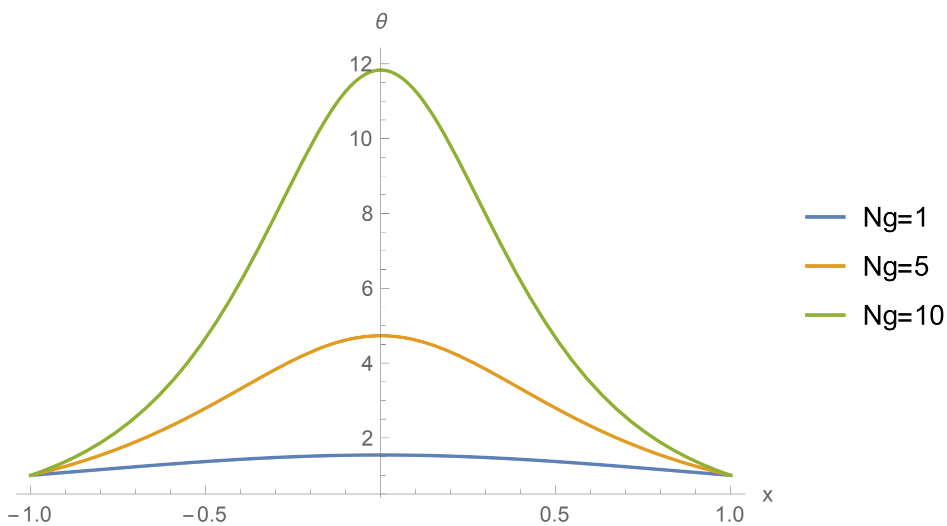

Figure 5.20: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = -2$ is kept fixed.

The exact solution using equation (3.11) could not be found.

5.2.2.2 Group invariant solution via subgroup generated by X_3

Here, we consider the group invariant solution corresponding to the operator X_3 of the optimal system (5.26), with $n = m$ and $a = 0$, which is given by

$$X_3 = t\frac{\partial}{\partial t} - \frac{\theta}{m}\frac{\partial}{\partial \theta}. \tag{5.50}$$

The characteristic system is given by

$$\frac{dt}{t} = \frac{dx}{0} = -\frac{m\, d\theta}{\theta}. \tag{5.51}$$

We see that x is invariant and so we write

$$x = I_1, \tag{5.52}$$

where I_1 is a constant.

Now, the first and last terms of the characteristic system (5.51) can be written as a variable separable ordinary differential equation given by

$$\frac{dt}{t} = -m\frac{d\theta}{\theta}. \tag{5.53}$$

Integrating both sides gives

$$\begin{aligned}
&\ln(t) = -m\ln(\theta) + \ln(I_2),\\
&\ln(t) + \ln(\theta^m) = \ln(I_2),\\
&t\,\theta^m = I_2.
\end{aligned} \tag{5.54}$$

We write $I_2 = F(I_1)$, and find that the invariant solution admitted by the generator X_3 is given by

$$\theta(t, x) = \left(\frac{F(x)}{t}\right)^{\frac{1}{m}}. \tag{5.55}$$

Substituting the invariant solution in equation (5.55) into the governing equation (3.9) results in the following second-order non-linear ordinary differential equation in terms of $F(x)$

$$\frac{\left(\frac{F(x)}{t}\right)^{\frac{1}{m}-1}\left(F(x)\left(mF(x)\left(F''(x) + m\mathrm{Ng}F(x)\right) + F'(x)^2\right) + mF(x)^2\right)}{mt^2} = 0, \tag{5.56}$$

which has no exact solution. Since it is hard to construct group invariant (exact) solutions which satisfy the boundary conditions, we resort to approximate methods for solution.

5.3 Discussion of results

In Figures 5.1 and 5.2, we see that the temperature increases with increasing values of m and Ng. However, it seems that for certain values of m and Ng the temperature in the wall is too high and the solutions do not have meaning. We found there to be a threshold for Ng in terms of m that prevent excessive temperatures. The threshold value was found to be

$$\mathrm{Ng} = \frac{\left(\frac{\pi}{2}\right)^2}{m}, \tag{5.57}$$

for $m > 0$.

Figures 5.3 and 5.4 show the temperature profile for when we consider $m < 0$. Here, no threshold value for Ng was needed as the temperature in the wall does not get excessively high in these cases. Again, an increase in m and Ng results in an increase in temperature.

Figures 5.5, 5.6 and 5.7 show the transient temperature profile for when we consider $m > 0$. In Figure 5.5, we see that temperature decreases with time. This transient temperature profile approaches the steady state solution as time evolves. Figures 5.6 and 5.7 behave identically to Figures 5.1 and 5.2, where the temperature increases with increasing values of m and Ng. The threshold value for Ng is the same as in equation (5.57).

Figures 5.8, 5.9 and 5.10 show the transient temperature profile for when we consider $m < 0$. We see that the behaviour of the graphs change, depending on the values for t, m and Ng. We found this relation to be

$$t_{critical} = \frac{e^{-m}}{\mathrm{Ng}(-m)}, \tag{5.58}$$

for $m < 0$. This implies that for a given choice of Ng and m, any t value which is less than the $t_{critical}$ value from equation (5.58) will result in the graph representing heat transfer in a fin, while any t value which is greater than the $t_{critical}$ value from equation (5.58) will result in the graph representing heat transfer with internal heat generation. We also notice that by considering $m < 0$, temperature now increases with increasing time, while an increase in m and Ng also result in an increase in temperature.

Figures 5.11 - 5.16 show the solutions using boundary condition (3.11). As in Figure 5.5, Figure 5.11 shows that the temperature decreases with time, and that the transient temperature profile approaches the steady state as time evolves. Figures 5.12 and 5.13 behave similarly to Figures 5.1, 5.2, 5.6 and 5.7, with the threshold value for Ng being the same as in equation (5.57).

Figures 5.14, 5.15 and 5.16 show the transient temperature profile for when we consider $m < 0$, while also using boundary condition (3.11). These graphs behave similarly to Figures 5.8, 5.9 and 5.10, however we were unable to find a relation for this case.

Figures 5.17, 5.18, 5.19 and 5.20 look at the case where thermal conductivity and internal heat generation are represented by the power law. We see that Figures 5.17 and 5.18 behave similarly to Figures 5.1 and 5.2, however the threshold for Ng was found to be

$$\mathrm{Ng} = \frac{\left(\frac{\pi}{2}\right)^2}{m+1}, \tag{5.59}$$

for $m > -1$.

Figures 5.19 and 5.20 show the temperature profile for when we consider $m < -1$. Here, no threshold value for Ng was needed as the temperature in the wall does not get excessively high in these cases. Again, an increase in m and Ng results in an increase in temperature, similar to Figures 5.3 and 5.4.

5.4 Concluding remarks

In this chapter we have constructed the one-dimensional optimal systems for the exponential and power law cases and sub-cases within. Group invariant solutions were found for both cases by using the obtained optimal systems. Firstly, the steady state solution was found followed by the transient solution. Graphs were depicted to show the solution behaviour as variables changed. It can be seen that we were able to obtain the steady state solutions for both cases although only using the symmetric boundary conditions. We also managed to find the transient solution for the exponential case, using both sets of boundary conditions. We were unable to obtain the transient solution for the power law case. We resort to another method of solution in the following chapters to solve for the transient solution for the power law case.

Chapter 6

1D DTM solutions for steady heat transfer given the power law thermal conductivity and internal heat generation

6.1 Introduction

In this chapter we consider the steady state model describing the temperature profile in a hot body such as across a wall with both thermal conductivity and internal heat generation being functions of temperature given by the power law, that is

$$\frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) + \mathrm{Ng}\, \theta^{n+1} = 0, \tag{6.1}$$

subject to

$$\frac{d\theta}{dx}\mid_{x=0}=0, \tag{6.2}$$

$$\theta(1)=1. \tag{6.3}$$

Note that exact solutions exist only if $m=n$. A simple transformation $w=\theta^{m+1}$ will linearize equation (6.1) as such we may construct an exact solution. This solution is special, since it is possible when $m=n$ only.

6.2 Comparison of DTM and exact solutions

The exact solution of equation (6.1) subject to equations (6.2) and (6.3), when $m=n$ is given by

$$\theta(x)=\left[\frac{\cos\left(\sqrt{\text{Ng}\ (m+1)}x\right)}{\cos\left(\sqrt{\text{Ng}\ (m+1)}\right)}\right]^{\frac{1}{m+1}}, \quad m\neq -1. \tag{6.4}$$

Notice that we restrict $\text{Ng}>0$, otherwise the equation will represent heat transfer in a straight fin (see for example [25]). Furthermore, for the same reason we consider $m<-1$. This restriction implies that we consider heat transfer in a planar region (across a wall).

We use this exact solution as a benchmark for the DTM. The exact and approximate analytical solutions will be compared, using results for the cases $m=-2$ and $m=-3$, keeping Ng fixed.

6.2.1 Case $m = -2$ and $n = -2$

Applying the DTM to equation (6.1), given $\mathcal{H}$, we obtain the recurrence relation given by

$$\sum_{i=0}^{k} [-2(i+1)\Phi(i+1)(k-i+1)\Phi(k-i+1) \\ +(k-i+1)(k-i+2)\Phi(k-i+2)\Phi(i) + \Phi(i)\Phi(k-i)\ \mathrm{Ng}] = 0. \tag{6.5}$$

Applying DTM to the boundary condition (6.2) at a point $x = 0$ yields

$$\Phi(1) = 0, \tag{6.6}$$

and the other boundary condition (6.3) is considered as

$$\Phi(0) = c, \tag{6.7}$$

where c is a constant to be determined. We use the iterative equation (6.5) to construct the power series solution

$$\Phi(2) = -\frac{\mathrm{Ng}\ c}{2} \tag{6.8}$$

$$\Phi(3) = 0 \tag{6.9}$$

$$\Phi(4) = \frac{5\ \mathrm{Ng}^2\ c}{24} \tag{6.10}$$

$$\Phi(5) = 0 \tag{6.11}$$

$$\Phi(6) = -\frac{61\ \mathrm{Ng}^3\ c}{720} \tag{6.12}$$

$$\Phi(7) = 0 \tag{6.13}$$

$$\Phi(8) = \frac{277\ \mathrm{Ng}^4\ c}{8\,064} \tag{6.14}$$

$$\Phi(9) = 0 \tag{6.15}$$

$$\Phi(10) = -\frac{50\ 521\ \mathrm{Ng}^5\ c}{3\ 628\ 800} \tag{6.16}$$

$$\Phi(11) = 0 \tag{6.17}$$

$$\Phi(12) = \frac{540\ 553\ \mathrm{Ng}^6\ c}{95\ 800\ 320} \tag{6.18}$$

$$\Phi(13) = 0 \tag{6.19}$$

$$\Phi(14) = -\frac{199\ 360\ 981\ \mathrm{Ng}^7\ c}{87\ 178\ 291\ 200} \tag{6.20}$$

$$\vdots$$

These terms may be taken as far as desired. Substituting equations (6.6) to (6.20) into equation (2.12), we obtain the following series solution

$$\begin{aligned}\theta(x) = c - \frac{\mathrm{Ng}\ c}{2}\ x^2 + \frac{5\mathrm{Ng}^2\ c}{24}\ x^4 - \frac{61\ \mathrm{Ng}^3\ c}{720}\ x^6 + \frac{277\ \mathrm{Ng}^4\ c}{8\ 064}\ x^8 \\ - \frac{50\ 521\ \mathrm{Ng}^5\ c}{3\ 628\ 800}\ x^{10} + \frac{540\ 553\ \mathrm{Ng}^6\ c}{95\ 800\ 320}\ x^{12} - \frac{199\ 360\ 981\ \mathrm{Ng}^7\ c}{87\ 178\ 291\ 200}\ x^{14} + \ldots\end{aligned} \tag{6.21}$$

We need to obtain the value of c, so we use the boundary condition (6.3) at the point $x = 1$. Thus, we get

$$\begin{aligned}\theta(1) = c - \frac{\mathrm{Ng}\ c}{2} + \frac{5\mathrm{Ng}^2\ c}{24} - \frac{61\ \mathrm{Ng}^3\ c}{720} + \frac{277\ \mathrm{Ng}^4\ c}{8\ 064} \\ - \frac{50\ 521\ \mathrm{Ng}^5\ c}{3\ 628\ 800} + \frac{540\ 553\ \mathrm{Ng}^6\ c}{95\ 800\ 320} - \frac{199\ 360\ 981\ \mathrm{Ng}^7\ c}{87\ 178\ 291\ 200} + \ldots = 1.\end{aligned} \tag{6.22}$$

We then substitute the obtained value of c into equation (6.21) to obtain the expression for $\theta(x)$.

We compare the above obtained solution for $\theta(x)$ to the exact solution, given by equation (6.4) with $m = -2$

$$\theta(x) = \frac{\cosh\left(\sqrt{\mathrm{Ng}}\right)}{\cosh\left(\sqrt{\mathrm{Ng}}\ x\right)}. \tag{6.23}$$

We present Table 6.1 which shows a comparison between the exact solution and solution given by DTM for the case $m = -2$. We see that the absolute errors between the results are very slight. Additionally, Figure 6.1 displays the comparison of the exact solution and the solution given by DTM.

6.2.2 Case $m = -3$ and $n = -3$

Applying the DTM to equation (6.1), given $\mathcal{H}$, we obtain the recurrence relation given by

$$\sum_{i=0}^{k}[-3(i+1)\Phi(i+1)(k-i+1)\Phi(k-i+1) \\ +(k-i+1)(k-i+2)\Phi(k-i+2)\Phi(i)+\Phi(i)\Phi(k-i)\ \mathrm{Ng}] = 0. \tag{6.24}$$

Substituting equations (6.6) and (6.7) into equation (6.24), and solving for the rest of the terms we get the following series solution

$$\theta(x) = c - \frac{\mathrm{Ng}\ c}{2}\,x^2 + \frac{7\ \mathrm{Ng}^2\ c}{24}\,x^4 - \frac{139\ \mathrm{Ng}^3\ c}{720}\,x^6 + \frac{5\ 473\ \mathrm{Ng}^4\ c}{40\ 320}\,x^8 \\ - \frac{51\ 103\ \mathrm{Ng}^5\ c}{518\ 400}\,x^{10} + \frac{34\ 988\ 647\ \mathrm{Ng}^6\ c}{479\ 001\ 600}\,x^{12} - \frac{4\ 784\ 061\ 619\ \mathrm{Ng}^7\ c}{87\ 178\ 291\ 200}\,x^{14} + \ldots \tag{6.25}$$

Again, we can solve for c using equation (6.3).

We compare the above obtained solution for $\theta(x)$ to the exact solution, given by equation (6.4) with $m = -3$

$$\theta(x) = \left(\frac{\cosh\left(\sqrt{2}\ \mathrm{Ng}\right)}{\cosh\left(\sqrt{2}\ \mathrm{Ng}\ x\right)}\right)^{\frac{1}{2}}. \tag{6.26}$$

We present Table 6.2 which shows a comparison between the exact solution and solution given by DTM for the case $m = -3$. We see that the absolute errors between the results are again very slight. Additionally, Figure 6.2 displays the comparison of the exact solution and the solution given by DTM.

x	DTM	Exact	Error
0	1.2605908671	1.2605918365	0.0000009694
0.1	1.2574459422	1.2574469092	0.0000009670
0.2	1.2480891602	1.2480901199	0.0000009598
0.3	1.2327498266	1.2327507746	0.0000009480
0.4	1.2117950566	1.2117959884	0.0000009319
0.5	1.1857088919	1.1857098037	0.0000009118
0.6	1.1550661851	1.1550670732	0.0000008881
0.7	1.1205037463	1.1205046066	0.0000008603
0.8	1.0826912454	1.0826920632	0.0000008178
0.9	1.0423040737	1.0423047560	0.0000006823
1	1	1	0

Table 6.1: Results of the DTM and Exact solutions for $m = -2$ and $n = -2$, with $\mathrm{Ng} = 0.5$.

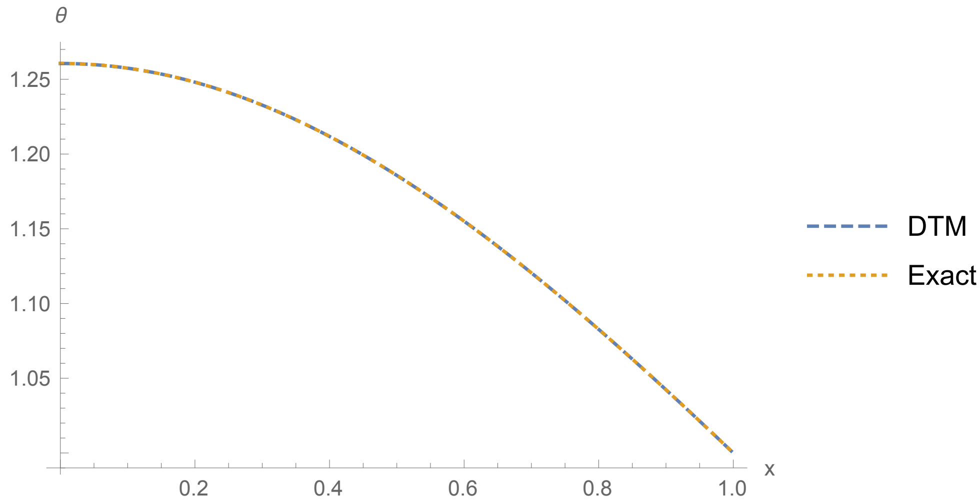

Figure 6.1: Comparison of the DTM and Exact solutions with $m = -2$, $n = -2$ and $\mathrm{Ng} = 0.5$.

x	DTM	Exact	Error
0	1.1708512740	1.1708539860	0.0000027117
0.1	1.1688064584	1.1688091654	0.0000027070
0.2	1.1627216350	1.1627243279	0.0000026929
0.3	1.1527423065	1.1527449763	0.0000026698
0.4	1.1391001833	1.1391028214	0.0000026382
0.5	1.1220984590	1.1221010578	0.0000025988
0.6	1.1020938127	1.1020963649	0.0000025522
0.7	1.0794771186	1.0794796145	0.0000024959
0.8	1.0546547291	1.0546571265	0.0000023974
0.9	1.0280319148	1.0280339335	0.0000020187
1	1	1	0

Table 6.2: Results of the DTM and Exact solutions for $m = -3$ and $n = -3$, with $\mathrm{Ng} = 0.35$.

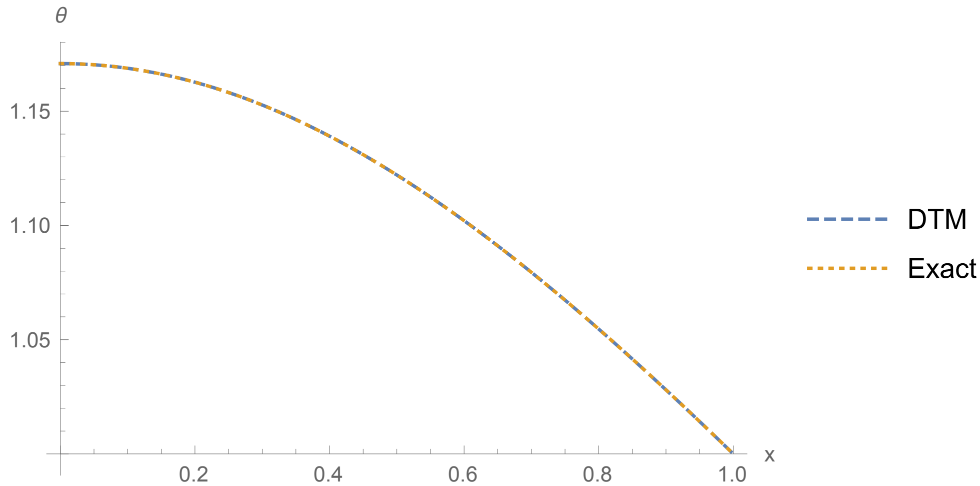

Figure 6.2: Comparison of the DTM and Exact solutions with $m = -3$, $n = -3$ and $\mathrm{Ng} = 0.35$.

6.3 Distinct exponents, $m \neq n$

6.3.1 Case $m = -2$ and $n = -3$

Applying the DTM to equation (6.1), given $\mathcal{H}$, we obtain the recurrence relation given by

$$\sum_{i=0}^{k} [-2(i+1)\Phi(i+1)(k-i+1)\Phi(k-i+1) \\ +(k-i+1)(k-i+2)\Phi(k-i+2)\Phi(i)] + \Phi(k)\,\mathrm{Ng} = 0. \tag{6.27}$$

Substituting equations (6.6) and (6.7) into equation (6.27), and solving for the rest of the terms we get the following series solution

$$\theta(x) = c - \frac{\text{Ng}}{2}\,x^2 + \frac{\text{Ng}^2}{6c}\,x^4 - \frac{\text{Ng}^3}{18c^2}\,x^6 + \frac{19\ \text{Ng}^4}{1\ 008c^3}\,x^8 - \frac{29\ \text{Ng}^5}{4\ 536c^4}\,x^{10} + \frac{59\ \text{Ng}^6}{27\ 216c^5}\,x^{12} - \frac{3\ 641\ \text{Ng}^7}{4\ 953\ 312c^6}\,x^{14} + \dots \tag{6.28}$$

We can solve for c using equation (6.3). Figure 6.3 shows the 1D DTM solution with varying values of Ng for when $m = -2$ and $n = -3$.

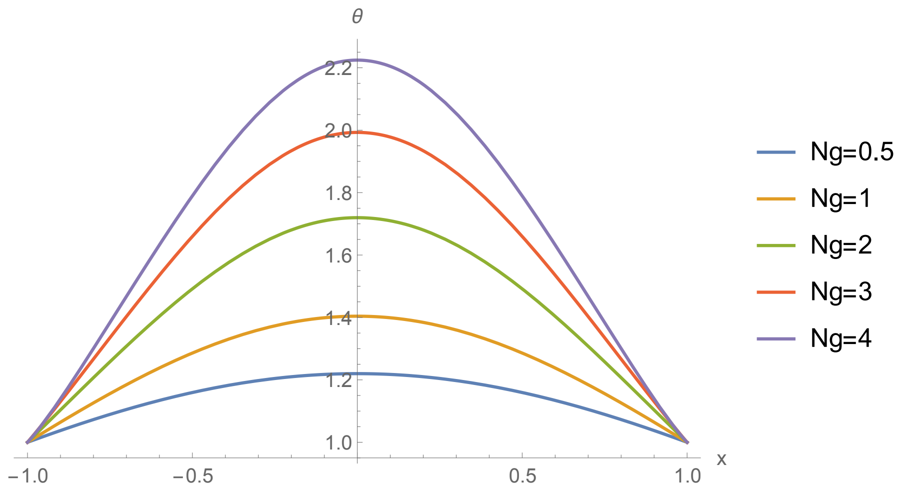

Figure 6.3: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = -2$ and $n = -3$.

6.3.2 Case $m = -3$ and $n = -4$

Applying the DTM to equation (6.1), given $\mathcal{H}$, we obtain the recurrence relation given by

$$\sum_{i=0}^{k} [-3(i+1)\Phi(i+1)(k-i+1)\Phi(k-i+1) + (k-i+1)(k-i+2)\Phi(k-i+2)\Phi(i)] + \Phi(k)\ \text{Ng} = 0. \tag{6.29}$$

Substituting equations (6.6) and (6.7) into equation (6.29), and solving for the rest of the terms we get the following series solution

$$\begin{aligned}\theta(x) = c - \frac{\mathrm{Ng}}{2}\,x^2 + \frac{\mathrm{Ng}^2}{4c}\,x^4 - \frac{3\,\mathrm{Ng}^3}{20c^2}\,x^6 + \frac{27\,\mathrm{Ng}^4}{280c^3}\,x^8 \\ - \frac{179\,\mathrm{Ng}^5}{2\,800c^4}\,x^{10} + \frac{5\,323\,\mathrm{Ng}^6}{123\,200c^5}\,x^{12} - \frac{6\,381\,\mathrm{Ng}^7}{215\,600c^6}\,x^{14} + \ldots\end{aligned} \tag{6.30}$$

Again, we can solve for c using equation (6.3). Figure 6.4 shows the 1D DTM solution with varying values of Ng for when $m = -3$ and $n = -4$.

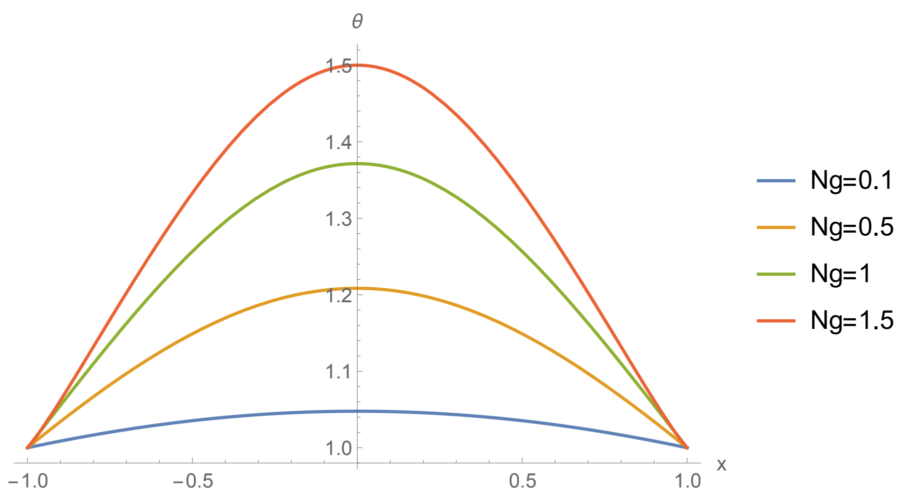

Figure 6.4: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of Ng, using boundary condition (3.10). Here $m = -3$ and $n = -4$.

6.4 Discussion of results

In this chapter, we have considered heat transfer in a hot body where the thermal conductivity and internal heat generation are given by the power law. We have seen from Figures 6.1 and 6.2 that the solutions obtained using the 1D DTM compare very well with the exact solutions, namely equations (6.23) and (6.26). Furthermore, from Table 6.1 we notice an absolute error of approximately $9.7e - 007$, while in Table 6.2

we see an absolute error of approximately $2.7e-006$. This confirms that the DTM can provide accurate results with little computational effort.

Figures 6.3 and 6.4 show the 1D DTM solution for distinct exponents. In both figures we see that temperature increases as we increase Ng. We noticed in Figure 6.3 that when we considered Ng > 5 the solution did not represent that of heat transfer in a wall. An identical observation was made in Figure 6.4, but with Ng > 1.5. This might be due to the order of our Taylor series solution not being high enough, or possibly a threshold for Ng also exists when we have $m \neq n$.

6.5 Concluding remarks

In this chapter we employed the 1D DTM to construct the approximate analytical solution, for the cases where exact solutions could not be obtained. First we established confidence in DTM by comparing the special exact solution to the DTM solution. Approximate solutions were constructed for the case $m \neq n$.

Chapter 7

2D DTM solution for transient heat transfer given the power law thermal conductivity and internal heat generation

7.1 Introduction

In this chapter we consider the transient model describing the temperature profile in a hot body such as across a wall with both thermal conductivity and internal heat generation being functions of temperature given by the power law, that is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) + \mathrm{Ng}\, \theta^{n+1}, \tag{7.1}$$

subject to symmetric boundary conditions given by

$$\frac{\partial\theta}{\partial x}\,|_{x=0} = 0, \tag{7.2}$$

$$\theta(t,1) = 1, \tag{7.3}$$

with initial condition

$$\theta(0,x) = 0,\ 0 \le x \le 1. \tag{7.4}$$

We have established confidence in the DTM although applied to steady state problem. A slight modification in the code allows solution for the transient problem.

7.2 Case: $m = -2$ and $n = -2$

Applying the 2D DTM on equation (7.1) with $m = -2$ and $n = -2$, we obtain the following recurrence relation

$$\begin{aligned}&\sum_{i=0}^{k}\sum_{p=0}^{k-i}\sum_{z=0}^{k-i-p}\sum_{j=0}^{s}\sum_{q=0}^{s-j}\sum_{l=0}^{s-j-q}\Phi(p,j)\Phi(z,q)(k-i-p-z+1)\Phi(i,s-j-q-l)\\&\times\Phi(k-i-p-z+1,l) = -2\sum_{i=0}^{k}\sum_{j=0}^{s}(j+1)(s-j+1)\Phi(k-i,j+1)\Phi(i,s-j+1)\\&+\sum_{i=0}^{k}\sum_{j=0}^{s}(j+1)(j+2)\Phi(k-i,j+2)\Phi(i,s-j)+\mathrm{Ng}\sum_{i=0}^{k}\sum_{j=0}^{s}\Phi(k-i,j)\Phi(i,s-j),\end{aligned} \tag{7.5}$$

where $\Phi(\kappa,s)$ is the differential transform of $\theta(t,x)$.

Applying the 2D differential transform on the initial condition (7.4) and boundary conditions (7.2) and (7.3) we obtain the following transformations respectively

$$\Phi(0,s) = 0,\ s = 0,1,2,\dots \tag{7.6}$$

$$\Phi(\kappa,1) = 0,\ \kappa = 0,1,2,\dots \tag{7.7}$$

$$\Phi(\kappa,0) = c,\ c \in \mathbb{R},\ \kappa = 1,2,3,\dots, \tag{7.8}$$

where c is a constant. We use equations (7.6) - (7.8) and the iterative equation (7.5) to construct the power series solution

$$\Phi(1,2) = -\frac{\text{Ng}\ c}{2} \tag{7.9}$$

$$\Phi(2,2) = \frac{c^3 - \text{Ng}\ c}{2} \tag{7.10}$$

$$\Phi(3,2) = \frac{4c^3 - \text{Ng}\ c}{2} \tag{7.11}$$

$$\Phi(1,4) = \frac{5\ \text{Ng}^2\ c}{24} \tag{7.12}$$

$$\Phi(2,4) = -\frac{12\ \text{Ng}\ c^3 - 5\ \text{Ng}^2\ c}{24} \tag{7.13}$$

$$\Phi(3,4) = \frac{8c^5 - 48\ \text{Ng}\ c^3 + 5\ \text{Ng}^2\ c}{24} \tag{7.14}$$

$$\vdots$$

These terms may be taken as far as desired. Substituting equations (7.6) to (7.14) into equation (2.17), we obtain the following series solution

$$\begin{aligned}\theta(t,x) =& ct + ct^2 + ct^3 + ct^4 - \frac{\text{Ng}\ c}{2}tx^2 + \frac{c^3 - \text{Ng}\ c}{2}t^2x^2 + \frac{4c^3 - \text{Ng}\ c}{2}t^3x^2 \\ &+ \frac{5\ \text{Ng}^2\ c}{24}tx^4 - \frac{12\ \text{Ng}\ c^3 - 5\ \text{Ng}^2\ c}{24}t^2x^4 + \frac{8c^5 - 48\ \text{Ng}\ c^3 + 5\ \text{Ng}^2\ c}{24}t^3x^4 + \dots\end{aligned} \tag{7.15}$$

We need to obtain the value of c, so we use the boundary condition (7.3) at the point $x = 1$. Thus, we get

$$\begin{aligned}\theta(t,1) =& ct + ct^2 + ct^3 + ct^4 - \frac{\text{Ng } c}{2}t + \frac{c^3 - \text{Ng } c}{2}t^2 + \frac{4c^3 - \text{Ng } c}{2}t^3 \\ &+ \frac{5 \text{ Ng}^2 c}{24}t - \frac{12 \text{ Ng } c^3 - 5 \text{ Ng}^2 c}{24}t^2 + \frac{8c^5 - 48 \text{ Ng } c^3 + 5 \text{ Ng}^2 c}{24}t^3 + \ldots = 1.\end{aligned} \tag{7.16}$$

We then substitute the obtained value of c into equation (7.15) to obtain the expression for $\theta(t, x)$. Using the first 24 terms of the power series solution we plot the solution for equation (7.15) for various parameters as shown in Figures 7.1 and 7.2.

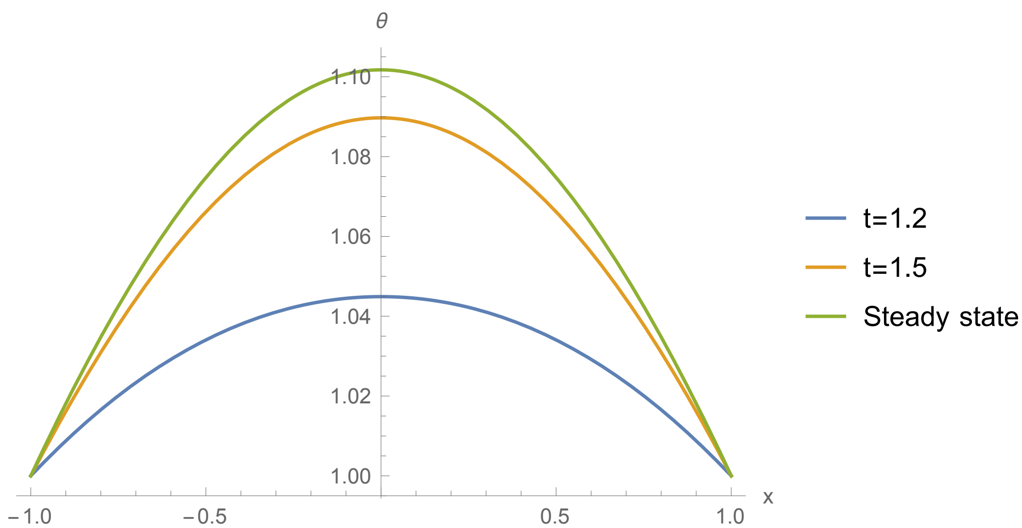

Figure 7.1: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of t, using boundary condition (3.10). Here $\text{Ng} = 0.2$ is kept fixed.

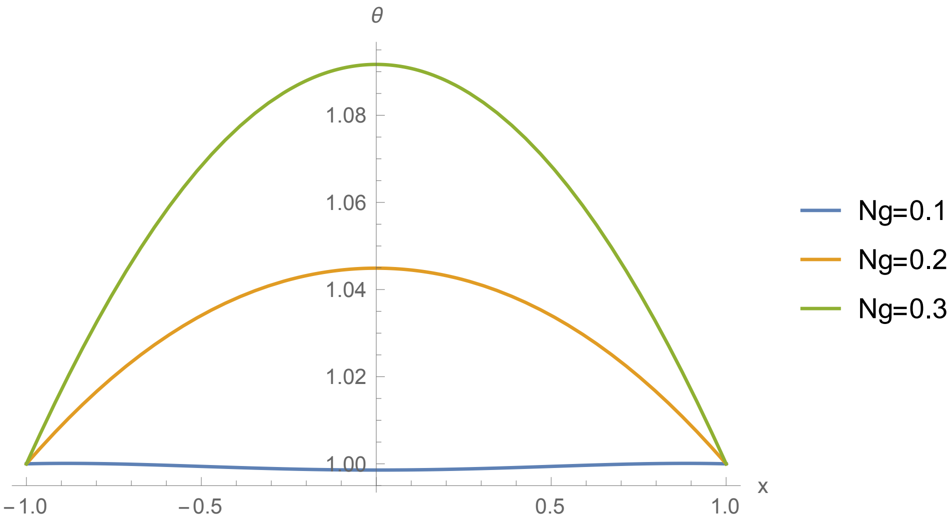

Figure 7.2: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of Ng, using boundary condition (3.10). Here $t = 1.2$ is kept fixed.

7.3 Case: $m = -3$ and $n = -3$

Applying the 2D DTM on equation (7.1) with $m = -3$ and $n = -3$, we obtain the following recurrence relation

$$\sum_{i=0}^{k}\sum_{p=0}^{k-i}\sum_{z=0}^{k-i-p}\sum_{w=0}^{k-i-p-z}\sum_{j=0}^{s}\sum_{q=0}^{s-j}\sum_{l=0}^{s-j-q}\sum_{v=0}^{s-j-q-l}\Phi(p,j)\Phi(z,q)\Phi(w,l)(k-i-p-z-w+1)$$
$$\times\,\Phi(i,s-j-q-l-v)\Phi(k-i-p-z-w+1,v) = -3\sum_{i=0}^{k}\sum_{j=0}^{s}(j+1)(s-j+1)$$
$$\times\,\Phi(k-i,j+1)\Phi(i,s-j+1) + \sum_{i=0}^{k}\sum_{j=0}^{s}(j+1)(j+2)\Phi(k-i,j+2)\Phi(i,s-j)$$
$$+\,\mathrm{Ng}\sum_{i=0}^{k}\sum_{j=0}^{s}\Phi(k-i,j)\Phi(i,s-j), \tag{7.17}$$

where $\Phi(\kappa, s)$ is the differential transform of $\theta(t, x)$.

Applying the 2D DTM on the initial condition (7.4) and boundary conditions (7.2) and (7.3) we obtain the following transformations respectively

$$\Phi(0,s)=0,\ s=0,1,2,\dots \tag{7.18}$$

$$\Phi(\kappa,1)=0,\ \kappa=0,1,2,\dots \tag{7.19}$$

$$\Phi(\kappa,0)=c,\ c\in\mathbb{R},\ \kappa=1,2,3,\dots, \tag{7.20}$$

where c is a constant. We use equations (7.18) - (7.20) and the iterative equation (7.17) to construct the power series solution

$$\Phi(1,2)=-\frac{\mathrm{Ng}\ c}{2} \tag{7.21}$$

$$\Phi(2,2)=\frac{c^3-\mathrm{Ng}\ c}{2} \tag{7.22}$$

$$\Phi(3,2)=\frac{4c^3-\mathrm{Ng}\ c}{2} \tag{7.23}$$

$$\Phi(1,4)=\frac{5\ \mathrm{Ng}^2\ c}{24} \tag{7.24}$$

$$\Phi(2,4)=-\frac{12\ \mathrm{Ng}\ c^3-5\ \mathrm{Ng}^2\ c}{24} \tag{7.25}$$

$$\Phi(3,4)=\frac{8\ c^5-48\ \mathrm{Ng}\ c^3+5\ \mathrm{Ng}^2\ c}{24} \tag{7.26}$$

$$\vdots$$

These terms may be taken as far as desired. Substituting equations (7.18) to (7.26) into equation (2.17), we obtain the following series solution

$$\begin{aligned}\theta(t,x)=&ct+ct^2+ct^3+ct^4-\frac{\mathrm{Ng}\ c}{2}tx^2+\frac{c^3-\mathrm{Ng}\ c}{2}t^2x^2+\frac{4c^3-\mathrm{Ng}\ c}{2}t^3x^2\\&+\frac{5\ \mathrm{Ng}^2\ c}{24}tx^4-\frac{12\ \mathrm{Ng}\ c^3-5\ \mathrm{Ng}^2\ c}{24}t^2x^4+\frac{8\ c^5-48\ \mathrm{Ng}\ c^3+5\ \mathrm{Ng}^2\ c}{24}t^3x^4+\dots\end{aligned} \tag{7.27}$$

We need to obtain the value of c, so we use the boundary condition (7.3) at the point $x = 1$. Thus, we get

$$\begin{aligned}\theta(t,1) =& ct + ct^2 + ct^3 + ct^4 - \frac{\text{Ng } c}{2}t + \frac{c^3 - \text{Ng } c}{2}t^2 + \frac{4c^3 - \text{Ng } c}{2}t^3 \\ &+ \frac{5 \text{ Ng}^2 c}{24}t - \frac{12 \text{ Ng } c^3 - 5 \text{ Ng}^2 c}{24}t^2 + \frac{8 c^5 - 48 \text{ Ng } c^3 + 5 \text{ Ng}^2 c}{24}t^3 + \ldots = 1.\end{aligned} \tag{7.28}$$

We then substitute the obtained value of c into equation (7.27) to obtain the expression for $\theta(t,x)$. Using the first 24 terms of the power series solution we plot the solution for equation (7.27) for various parameters as shown in Figures 7.5 and 7.6.

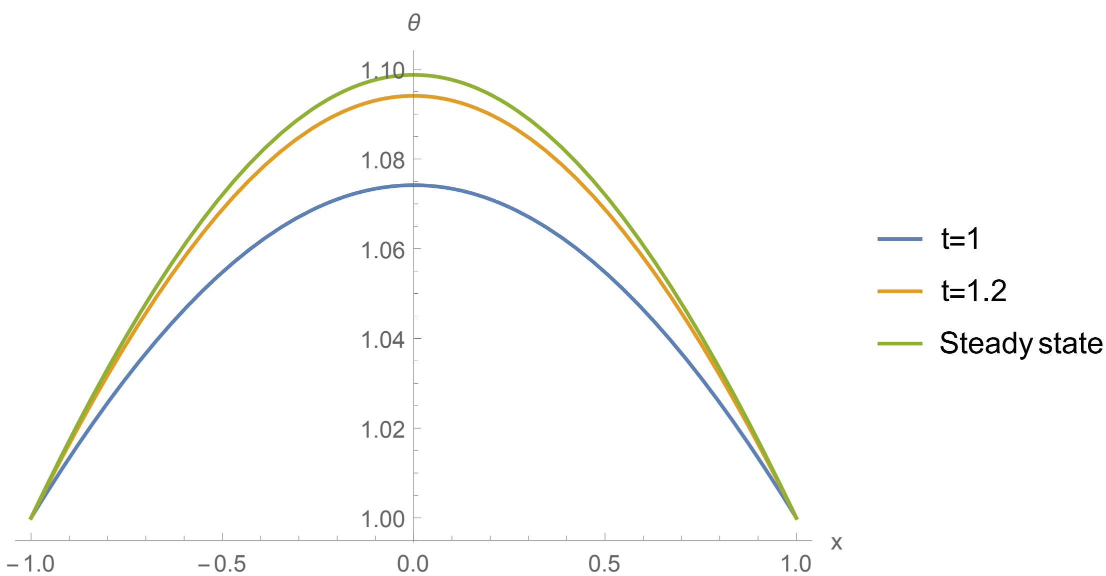

Figure 7.3: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of t, using boundary condition (3.10). Here $\text{Ng} = 0.2$ is kept fixed.

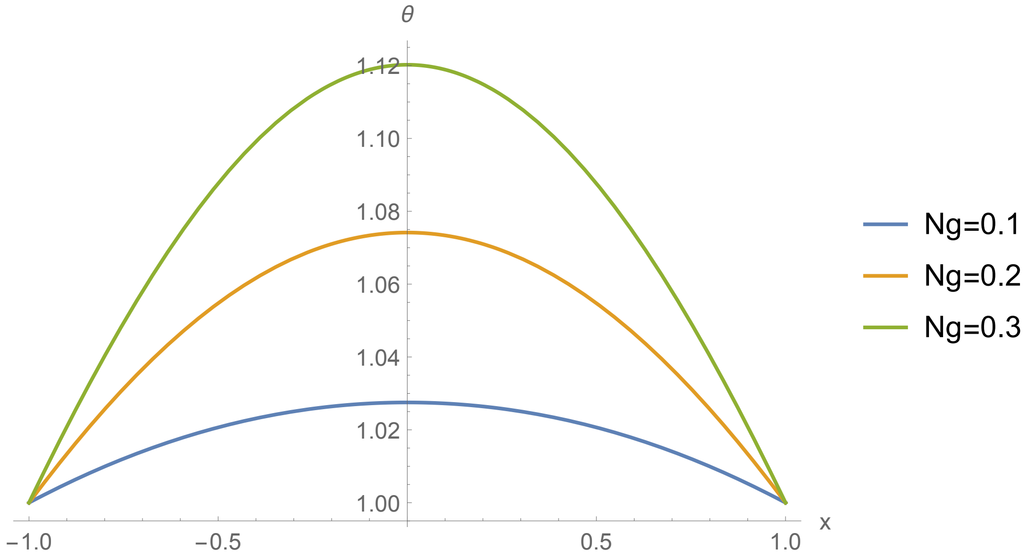

Figure 7.4: Temperature in a wall with power law thermal conductivity and power law internal heat generation and varying values of Ng, using boundary condition (3.10). Here $t = 1$ is kept fixed.

7.4 Discussion of results

In Figures 7.1 and 7.2, we see that the temperature increases with increasing time and increasing Ng values, as expected from what we obtained in Chapter 5 when considering $m < 0$. We compared this DTM solution to that of the steady state found in Chapter 5, and noticed that when evaluating the solution using Ng ≈ 0.5 the results varied slightly, hence we focused on Ng < 0.5 for the DTM solution. We also noticed that the DTM solution becomes the steady state at around $t \approx 2$.

Figures 7.3 and 7.4 behave similarly to Figures 7.1 and 7.2, however, the steady state was reached quicker in Figure 7.3 than in Figure 7.1. Additionally, the steady state temperature found in Figures 7.1 and 7.3 are almost identical.

A possible reason for similar temperatures and for reaching the steady state in a very short time could be the result of choosing a small Ng value. We mentioned above that when focusing on the steady state 2D DTM solution and the steady state solution found in section 5.2.2.1, we noticed a value of Ng > 0.5 gave us some differences. These differences might be the result of the order of our Taylor series expansion being too low, and therefore we would need to find more terms in order to investigate solutions using larger Ng values.

A problem arises when we consider the asymmetric boundary conditions as we need to solve for two constants. In most cases we found that if the order of our Taylor series solution exceeded 6 then no real values for the constants were obtained. So we limited our Taylor series solution to have an order of 5. However, these solutions represented heat transfer in a longitudinal fin.

7.5 Concluding remarks

In this chapter, we have successfully employed the 2D DTM to transient heat conduction problems for heat transfer across a wall where thermal conductivity and internal heat generation are given by the power law. We presented results which approached the steady state as time evolves.

Chapter 8

Conservation laws and associated Lie point symmetries

8.1 Conservation laws

In this chapter we construct conservation laws for the transient heat equation with internal heat generation. A number of methods to construct conservation laws exist, namely the direct method, the multiplier method and the characteristic method. A good comparison of these methods was given in [28]. Since these methods yield the same results, we shall just use one, the direct method.

8.1.1 Case: General $k(\theta)$ and $q(\theta)$

Consider the governing equation (3.9),

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) + \mathrm{Ng}\, q(\theta). \tag{8.1}$$

A conservation law for equation (8.1) satisfies

$$D_tT^1 + D_xT^2 \mid_{\text{equation(8.1)}} = 0, \tag{8.2}$$

where D_t and D_x are the total derivatives given in (2.8).

For simplicity and convenience we seek conserved vectors of the form

$$T^1 = T^1(t, x, \theta, \theta_x), \qquad T^2 = T^2(t, x, \theta, \theta_x). \tag{8.3}$$

Substituting equation (8.3) into equation (8.2) we obtain the following partial differential equation

$$\frac{\partial T^1}{\partial t} + \theta_t \frac{\partial T^1}{\partial \theta} + \theta_{tx} \frac{\partial T^1}{\partial \theta_x} + \frac{\partial T^2}{\partial x} + \theta_x \frac{\partial T^2}{\partial \theta} + \theta_{xx} \frac{\partial T^2}{\partial \theta_x} \mid_{\text{equation(8.1)}} = 0. \tag{8.4}$$

Expanding equation (8.4) we get

$$\begin{aligned} &\frac{\partial T^1}{\partial t} + \left[k'(\theta)(\theta_x)^2 + k(\theta)\theta_{xx} + \text{Ng}\, q(\theta)\right] \frac{\partial T^1}{\partial \theta} + \theta_{tx} \frac{\partial T^1}{\partial \theta_x} \\ &+ \frac{\partial T^2}{\partial x} + \theta_x \frac{\partial T^2}{\partial \theta} + \theta_{xx} \frac{\partial T^2}{\partial \theta_x} = 0. \end{aligned} \tag{8.5}$$

Since T^1 and T^2 are independent of the second derivatives of θ, we can separate equation (8.5) by second derivatives of θ as follows

$$\theta_{xx} : k(\theta)\frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial \theta_x} = 0, \tag{8.6}$$

$$\theta_{tx} : \frac{\partial T^1}{\partial \theta_x} = 0, \tag{8.7}$$

$$1 : \frac{\partial T^1}{\partial t} + k'(\theta)(\theta_x)^2 \frac{\partial T^1}{\partial \theta} + \text{Ng}\, q(\theta)\frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial x} + \theta_x \frac{\partial T^2}{\partial \theta} = 0. \tag{8.8}$$

From equation (8.7) we get $T^1 = T^1(t,x,\theta)$. We now integrate equation (8.6) with respect to θ_x to get an explicit expression for T^2

$$T^2 = -k(\theta)\frac{\partial T^1}{\partial \theta}\theta_x + A(t,x,\theta). \tag{8.9}$$

Substituting T^2 into equation (8.8) we get

$$\frac{\partial T^1}{\partial t} + \mathrm{Ng}\, q(\theta)\frac{\partial T^1}{\partial \theta} - k(\theta)\frac{\partial^2 T^1}{\partial x \partial \theta}\theta_x + \frac{\partial A}{\partial x} - k(\theta)\frac{\partial^2 T^1}{\partial \theta^2}(\theta_x)^2 + \theta_x\frac{\partial A}{\partial \theta} = 0. \tag{8.10}$$

We now separate equation (8.10) by θ_x as follows

$$(\theta_x)^2 : -k(\theta)\frac{\partial^2 T^1}{\partial \theta^2} = 0, \tag{8.11}$$

$$\theta_x : -k(\theta)\frac{\partial^2 T^1}{\partial x \partial \theta} + \frac{\partial A}{\partial \theta} = 0, \tag{8.12}$$

$$1 : \frac{\partial T^1}{\partial t} + \mathrm{Ng}\, q(\theta)\frac{\partial T^1}{\partial \theta} + \frac{\partial A}{\partial x} = 0. \tag{8.13}$$

We get an explicit expression for T^1 by integrating equation (8.11) twice with respect to θ

$$T^1 = B(t,x)\theta + C(t,x), \tag{8.14}$$

assuming $k(\theta) \neq 0$.

From this, equation (8.12) becomes

$$-k(\theta)\frac{\partial B}{\partial x} + \frac{\partial A}{\partial \theta} = 0. \tag{8.15}$$

We get an explicit expression for A by integrating equation (8.15) with respect to θ

$$A(t,x,\theta) = \frac{\partial B}{\partial x}\int_1^{\theta} k(\theta^*)\, d\theta^* + D(t,x), \tag{8.16}$$

where $D(t,x) = A(t,x,1)$.

Substituting A and T^1 into equation (8.13) we get

$$0 = \theta\frac{\partial B}{\partial t} + \frac{\partial C}{\partial t} + \mathrm{Ng}\, q(\theta)B(t,x) + \frac{\partial^2 B}{\partial x^2}\int_1^{\theta} k(\theta^*)\, d\theta^* + \frac{\partial D}{\partial x}. \tag{8.17}$$

We will use equation (8.17) and T^1 and T^2

$$T^1 = B(t,x)\theta + C(t,x), \tag{8.18}$$

$$T^2 = -k(\theta)B(t,x)\theta_x + \frac{\partial B}{\partial x}\int_1^{\theta} k(\theta^*)\, d\theta^* + D(t,x), \tag{8.19}$$

to solve for the conserved vectors for different cases of $k(\theta)$ and $q(\theta)$. When substituting our choices for $k(\theta)$ and $q(\theta)$ into equation (8.17) we will be able to split the equation with respect to powers of θ, allowing us to find $B(t,x)$, $C(t,x)$ and $D(t,x)$.

8.1.2 Case: $k(\theta) = e^{m\theta}$ and $q(\theta) = e^{n\theta}$

Consider the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(e^{m\theta}\frac{\partial \theta}{\partial x}\right) + \mathrm{Ng}\, e^{n\theta}. \tag{8.20}$$

Substituting our choices for $k(\theta)$ and $q(\theta)$ into equation (8.17) we get

$$0 = \theta\frac{\partial B}{\partial t} + \frac{\partial C}{\partial t} + \mathrm{Ng}\, e^{n\theta}B(t,x) + \frac{\partial^2 B}{\partial x^2}\left(\frac{e^{m\theta}}{m} - \frac{1}{m}e^{m}\right) + \frac{\partial D}{\partial x}. \tag{8.21}$$

We now consider two cases to separate equation (8.21).

8.1.2.1 Case: $m \neq n$, $m \neq 0$

$$e^{m\theta} : \frac{\partial^2 B}{\partial x^2}\frac{1}{m} = 0, \tag{8.22}$$

$$e^{n\theta} : \text{Ng } B(t,x) = 0, \tag{8.23}$$

$$\theta : \frac{\partial B}{\partial t} = 0, \tag{8.24}$$

$$1 : \frac{\partial C}{\partial t} + \frac{\partial}{\partial x}\left(D - \frac{1}{m}e^{m}\frac{\partial B}{\partial x}\right) = 0. \tag{8.25}$$

Equation (8.24) implies that $B = B(x)$ and equation (8.22) implies that $B = c_1 x + c_2$. Since equation (8.25) is satisfied independently of the PDE (8.20), it follows that

$$C(t,x) \quad \text{and} \quad D(t,x) - \frac{1}{m}e^{m}\frac{\partial B}{\partial x}(t,x)$$

are the components of a trivial conserved vector. We therefore set

$$C(t,x) = 0, \qquad D(t,x) - \frac{1}{m}e^{m}\frac{\partial B}{\partial x}(t,x) = 0.$$

Equation (8.23) implies that Ng $= 0$ or $B = 0$. If $B = 0$ then we get trivial conservation laws. So for non-trivial conservation laws given Ng $= 0$ we obtain

$$D_t\left[(c_1 x + c_2)\theta\right] + D_x\left[-e^{m\theta}(c_1 x + c_2)\theta_x + c_1\frac{e^{m\theta}}{m}\right] = 0. \tag{8.26}$$

A conserved vector of equation (8.20) with Ng $= 0$ is therefore a linear combination of the two conserved vectors

$$(i)\ T^1 = x\theta, \qquad T^2 = -e^{m\theta}x\theta_x + \frac{e^{m\theta}}{m}, \tag{8.27}$$

$$(ii)\ T^1 = \theta, \qquad T^2 = -e^{m\theta}\theta_x. \tag{8.28}$$

8.1.2.2 Case: $m = n$, $m \neq 0$

$$e^{m\theta} : \frac{\partial^2 B}{\partial x^2}\frac{1}{m} + \mathrm{Ng}\, B(t,x) = 0, \tag{8.29}$$

$$\theta : \frac{\partial B}{\partial t} = 0, \tag{8.30}$$

$$1 : \frac{\partial C}{\partial t} + \frac{\partial}{\partial x}\left(D - \frac{1}{m}e^{m}\frac{\partial B}{\partial x}\right) = 0. \tag{8.31}$$

Equation (8.30) implies that $B = B(x)$. Since equation (8.31) is satisfied independently of the PDE (8.20), it follows that

$$C(t,x) \quad \text{and} \quad D(t,x) - \frac{1}{m}e^{m}\frac{\partial B}{\partial x}(t,x)$$

are the components of a trivial conserved vector. We therefore set

$$C(t,x) = 0, \qquad D(t,x) - \frac{1}{m}e^{m}\frac{\partial B}{\partial x}(t,x) = 0.$$

Consider first $m > 0$. The general solution of equation (8.29) is

$$B(x) = c_1 \cos(\sqrt{m\mathrm{Ng}}\, x) + c_2 \sin(\sqrt{m\mathrm{Ng}}\, x). \tag{8.32}$$

For non-trivial conservation laws we get

$$\begin{aligned} &D_t\left[\left(c_1 \cos(\sqrt{m\mathrm{Ng}}\, x) + c_2 \sin(\sqrt{m\mathrm{Ng}}\, x)\right)\theta\right] \\ &+ D_x\left[- e^{m\theta}\left(c_1 \cos(\sqrt{m\mathrm{Ng}}\, x) + c_2 \sin(\sqrt{m\mathrm{Ng}}\, x)\right)\theta_x \right.\\ &\left. + e^{m\theta}\sqrt{\frac{\mathrm{Ng}}{m}}\left(c_2 \cos(\sqrt{m\mathrm{Ng}}\, x) - c_1 \sin(\sqrt{m\mathrm{Ng}}\, x)\right)\right] = 0. \end{aligned} \tag{8.33}$$

The non-trivial conserved vectors are

$$(i)\ T^1 = \cos(\omega x)\theta, \qquad T^2 = -e^{m\theta}\cos(\omega x)\theta_x - e^{m\theta}\frac{\omega}{m}\sin(\omega x), \tag{8.34}$$

$$(ii)\ T^1 = \sin(\omega x)\theta, \qquad T^2 = -e^{m\theta}\sin(\omega x)\theta_x + e^{m\theta}\frac{\omega}{m}\cos(\omega x), \tag{8.35}$$

where $\omega = \sqrt{m\,\mathrm{Ng}}$.

Consider next $m < 0$. Then

$$B(x) = c_1 e^{\sqrt{|m|\mathrm{Ng}}\,x} + c_2 e^{-\sqrt{|m|\mathrm{Ng}}\,x}, \tag{8.36}$$

and the corresponding conservation laws have the form

$$(i)\ T^1 = e^{\omega x}\theta, \qquad T^2 = -e^{m\theta+\omega x}\theta_x + \frac{\omega}{m}e^{m\theta+\omega x}, \tag{8.37}$$

$$(ii)\ T^1 = e^{-\omega x}\theta, \qquad T^2 = -e^{m\theta-\omega x}\theta_x - \frac{\omega}{m}e^{m\theta-\omega x}, \tag{8.38}$$

where $\omega = \sqrt{|m|\,\mathrm{Ng}}$.

8.1.3 Case: $k(\theta) = \theta^m$ and $q(\theta) = \theta^{n+1}$, where $m \neq -1$

Consider the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) + \mathrm{Ng}\,\theta^{n+1}. \tag{8.39}$$

Substituting our choices for $k(\theta)$ and $q(\theta)$ into equation (8.17) we get

$$0 = \theta\frac{\partial B}{\partial t} + \frac{\partial C}{\partial t} + \mathrm{Ng}\,\theta^{n+1}B(t,x) + \frac{\partial^2 B}{\partial x^2}\frac{\theta^{m+1}}{m+1} + \frac{\partial}{\partial x}\left(D - \frac{1}{m+1}\frac{\partial B}{\partial x}\right). \tag{8.40}$$

We now consider two cases to separate equation (8.40) by powers of θ.

8.1.3.1 Case: $m \neq n$, $m \neq -1$, $m \neq 0$ and $n \neq 0$

$$\theta^{m+1} : \frac{\partial^2 B}{\partial x^2}\frac{1}{m+1} = 0, \tag{8.41}$$

$$\theta^{n+1} : \mathrm{Ng}\,B(t,x) = 0, \tag{8.42}$$

$$\theta : \frac{\partial B}{\partial t} = 0, \tag{8.43}$$

$$1 : \frac{\partial C}{\partial t} + \frac{\partial}{\partial x}\left(D - \frac{1}{m+1}\frac{\partial B}{\partial x}\right) = 0. \tag{8.44}$$

Equation (8.43) implies that $B = B(x)$ and equation (8.41) implies that $B = c_1 x + c_2$. Since equation (8.44) is satisfied independently of the PDE (8.39) it follows that

$$C(t,x) \quad \text{and} \quad D(t,x) - \frac{1}{m+1}\frac{\partial B}{\partial x}(t,x)$$

are the components of a trivial conserved vector. We therefore set

$$C(t,x) = 0, \qquad D(t,x) - \frac{1}{m+1}\frac{\partial B}{\partial x}(t,x) = 0.$$

Equation (8.42) implies that $\mathrm{Ng}=0$ or $B=0$. If $B=0$ then we get trivial conservation laws. So for non-trivial conservation laws given $\mathrm{Ng}=0$ we obtain

$$D_t\left[(c_1x+c_2)\theta\right]+D_x\left[-\theta^m(c_1x+c_2)\theta_x+c_1\frac{\theta^{m+1}}{m+1}\right]=0. \tag{8.45}$$

A conserved vector of equation (8.39) with $\mathrm{Ng}=0$ is therefore a linear combination of the two conserved vectors

$$(i)\ T^1=x\theta, \qquad T^2=-\theta^m x\theta_x+\frac{\theta^{m+1}}{m+1}, \tag{8.46}$$

$$(ii)\ T^1=\theta, \qquad T^2=-\theta^m\theta_x. \tag{8.47}$$

8.1.3.2 Case: $m=n$, $n\neq-1$, $m\neq0$ and $m\neq-1$

$$\theta^{m+1}: \frac{\partial^2 B}{\partial x^2}\frac{1}{m+1}+\mathrm{Ng}\,B(t,x)=0, \tag{8.48}$$

$$\theta: \frac{\partial B}{\partial t}=0, \tag{8.49}$$

$$1: \frac{\partial C}{\partial t}+\frac{\partial}{\partial x}\left(D-\frac{1}{m+1}\frac{\partial B}{\partial x}\right)=0. \tag{8.50}$$

Equation (8.49) implies that $B=B(x)$. Since equation (8.50) is satisfied independently of the PDE (8.39) it follows that

$$C(t,x) \quad \text{and} \quad D(t,x)-\frac{1}{m+1}\frac{\partial B}{\partial x}(t,x)$$

are the components of a trivial conserved vector. We therefore set

$$C(t,x)=0, \qquad D(t,x)-\frac{1}{m+1}\frac{\partial B}{\partial x}(t,x)=0.$$

Consider first $m+1>0$. The general solution of equation (8.48) is

$$B(x) = c_1 \cos\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right) + c_2 \sin\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right). \tag{8.51}$$

For non-trivial conservation laws we get

$$\begin{aligned} &D_t\left[\left(c_1 \cos\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right) + c_2 \sin\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right)\right)\theta\right] \\ &+ D_x\Bigg[-\theta^m \left(c_1 \cos\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right) + c_2 \sin\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right)\right)\theta_x \\ &+ \theta^{m+1}\sqrt{\frac{\mathrm{Ng}}{m+1}}\left(c_2 \cos\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right) - c_1 \sin\left(\sqrt{(m+1)\mathrm{Ng}}\, x\right)\right)\Bigg] = 0. \end{aligned} \tag{8.52}$$

The non-trivial conserved vectors are

$$(i)\ T^1 = \cos(wx)\theta, \qquad T^2 = -\theta^m \cos(wx)\theta_x - \frac{\theta^{m+1}}{m+1} w \sin(wx), \tag{8.53}$$

$$(ii)\ T^1 = \sin(wx)\theta, \qquad T^2 = -\theta^m \sin(wx)\theta_x + \frac{\theta^{m+1}}{m+1} w \cos(wx), \tag{8.54}$$

where $\omega = \sqrt{(m+1)\ \mathrm{Ng}}$.

Consider next $m+1<0$. Then

$$B(x) = c_1 e^{\sqrt{|m+1|\mathrm{Ng}}\, x} + c_2 e^{-\sqrt{|m+1|\mathrm{Ng}}\, x}, \tag{8.55}$$

and the corresponding conservation laws have the form

$$(i)\; T^1 = e^{wx}\theta, \qquad T^2 = -\theta^m e^{wx}\theta_x + \frac{\theta^{m+1}}{m+1} w e^{wx}, \tag{8.56}$$

$$(ii)\; T^1 = e^{-wx}\theta, \qquad T^2 = -\theta^m e^{-wx}\theta_x - \frac{\theta^{m+1}}{m+1} w e^{-wx}, \tag{8.57}$$

where $w = \sqrt{|m+1|\mathrm{Ng}}$.

8.2 Associated Lie point symmetries

We have derived conserved vectors for both cases of the transient models, using the direct method. We now focus on finding Lie point symmetries associated with the conserved vectors from the previous section.

The determining equation for the Lie point symmetries X associated with the conserved vector $T = (T^1, T^2)$ is given by [20]

$$X(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0, \quad i = 1, \ldots, n, \tag{8.58}$$

where

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial \theta} + \zeta_x \frac{\partial}{\partial \theta_x} + \ldots. \tag{8.59}$$

Equation (8.58) consists of two components

$$X(T^1) + T^1 D_x(\xi^2) - T^2 D_x(\xi^1) = 0, \tag{8.60}$$

$$X(T^2) + T^2 D_t(\xi^1) - T^1 D_t(\xi^2) = 0. \tag{8.61}$$

We consider the exponential and power law cases as in the previous chapters. It turned out that the governing equation with the exponential case does not admit the associated Lie point symmetries. However, the power law case yielded some interesting results, as observed below.

Consider $n = m$ and $m + 1 < 0$.

Substituting the elementary conserved vector (8.56) into equations (8.60) and (8.61) gives

$$\begin{aligned} &\omega e^{\omega x}\xi^2\theta + \eta e^{\omega x} + e^{\omega x}\theta\frac{\partial \xi^2}{\partial x} + \theta e^{\omega x}\theta_x\frac{\partial \xi^2}{\partial \theta} - \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\frac{\partial \xi^1}{\partial x} - \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\theta_x\frac{\partial \xi^1}{\partial \theta} \\ &+ \theta^m\theta_x e^{\omega x}\frac{\partial \xi^1}{\partial x} + \theta^m(\theta_x)^2 e^{\omega x}\frac{\partial \xi^1}{\partial \theta} = 0, \end{aligned} \tag{8.62}$$

and

$$\begin{aligned} &\xi^2\frac{\omega^2}{m+1}\theta^{m+1}e^{\omega x} - \xi^2\omega\theta^m e^{\omega x}\theta_x + \eta\omega e^{\omega x}\theta^m - m\eta\theta^{m-1}e^{\omega x}\theta_x - \theta^m e^{\omega x}\frac{\partial \eta}{\partial x} \\ &- \theta^m e^{\omega x}\theta_x\frac{\partial \eta}{\partial \theta} + \theta^m e^{\omega x}\theta_t\frac{\partial \xi^1}{\partial x} + \theta^m e^{\omega x}\theta_t\theta_x\frac{\partial \xi^1}{\partial \theta} + \theta^m e^{\omega x}\theta_x\frac{\partial \xi^2}{\partial x} + \theta^m e^{\omega x}(\theta_x)^2\frac{\partial \xi^2}{\partial \theta} \\ &+ \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\frac{\partial \xi^1}{\partial t} + \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\theta_t\frac{\partial \xi^1}{\partial \theta} - \theta^m e^{\omega x}\theta_x\frac{\partial \xi^1}{\partial t} - \theta^m e^{\omega x}\theta_x\theta_t\frac{\partial \xi^1}{\partial \theta} - e^{\omega x}\theta\frac{\partial \xi^2}{\partial t} \\ &- \theta e^{\omega x}\theta_t\frac{\partial \xi^2}{\partial \theta} = 0. \end{aligned} \tag{8.63}$$

Separating equation (8.62) by derivatives of θ we find

$$(\theta_x)^2 : \theta^m e^{\omega x}\frac{\partial \xi^1}{\partial \theta} = 0, \tag{8.64}$$

$$\theta_x : e^{\omega x}\theta\frac{\partial \xi^2}{\partial \theta} - \omega e^{\omega x}\frac{\theta^{m+1}}{m+1}\frac{\partial \xi^1}{\partial \theta} + e^{\omega x}\theta^m\frac{\partial \xi^1}{\partial x} = 0, \tag{8.65}$$

$$1 : \xi^2\omega e^{\omega x}\theta + \eta e^{\omega x} + \theta e^{\omega x}\frac{\partial \xi^2}{\partial x} - \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\frac{\partial \xi^1}{\partial x} = 0. \tag{8.66}$$

Separating equation (8.63) by derivatives of θ we find

$$(\theta_x)^2 : \theta^m e^{\omega x}\frac{\partial \xi^2}{\partial \theta} = 0, \tag{8.67}$$

$$\theta_x : -\omega\theta^m e^{\omega x}\xi^2 - m\eta\theta^{m-1}e^{\omega x} - \theta^m e^{\omega x}\frac{\partial \eta}{\partial \theta} + \theta^m e^{\omega x}\frac{\partial \xi^2}{\partial x} - \theta^m e^{\omega x}\frac{\partial \xi^1}{\partial t} = 0, \tag{8.68}$$

$$\theta_t : \theta^m e^{\omega x}\frac{\partial \xi^1}{\partial x} + \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\frac{\partial \xi^1}{\partial \theta} - \theta e^{\omega x}\frac{\partial \xi^2}{\partial \theta} = 0, \tag{8.69}$$

$$1 : \frac{\omega^2}{m+1}\xi^2 e^{\omega x}\theta^{m+1} + \eta\omega e^{\omega x}\theta^m - \theta^m e^{\omega x}\frac{\partial \eta}{\partial x} + \frac{\theta^{m+1}}{m+1}\omega e^{\omega x}\frac{\partial \xi^1}{\partial t} - \theta e^{\omega x}\frac{\partial \xi^2}{\partial t} = 0. \tag{8.70}$$

From equations (8.64), (8.65) and (8.67) we get

$$\xi^1 = \xi^1(t), \tag{8.71}$$

and

$$\xi^2 = \xi^2(t, x), \tag{8.72}$$

with equation (8.69) being satisfied.

By reducing equation (8.66) we get

$$\eta = -\xi^2\omega\theta - \theta\frac{\partial \xi^2}{\partial x}, \tag{8.73}$$

and

$$\frac{\partial \eta}{\partial x} = -\frac{\partial \xi^2}{\partial x}\omega\theta - \theta\frac{\partial^2 \xi^2}{\partial x^2}. \tag{8.74}$$

Substituting equations (8.73) and (8.74) into equation (8.70) we get

$$\begin{aligned} &\frac{\omega^2}{m+1}\xi^2\theta^{m+1} + \left(-\xi^2\omega\theta - \theta\frac{\partial \xi^2}{\partial x}\right)\omega\theta^m - \theta^m\left(-\frac{\partial \xi^2}{\partial x}\omega\theta - \theta\frac{\partial^2 \xi^2}{\partial x^2}\right) \\ &+ \frac{\theta^{m+1}}{m+1}\omega\frac{d\xi^1}{dt} - \theta\frac{\partial \xi^2}{\partial t} = 0. \end{aligned} \tag{8.75}$$

Separating equation (8.75) by powers of θ gives

$$\theta : \frac{\partial \xi^2}{\partial t} = 0, \qquad \Rightarrow \xi^2 = \xi^2(x), \tag{8.76}$$

$$\theta^{m+1} : \frac{\omega^2}{m+1}\xi^2 - \xi^2\omega^2 + \frac{d^2\xi^2}{dx^2} + \frac{\omega}{m+1}\frac{d\xi^1}{dt} = 0. \tag{8.77}$$

Differentiating equation (8.77) with respect to t and solving for ξ^1 we get

$$\xi^1(t) = c_1 t + c_2, \tag{8.78}$$

where c_1 and c_2 are arbitrary constants. Substituting ξ^1 into equation (8.68) and solving for ξ^2 gives

$$\xi^2(x) = \begin{cases} \frac{c_1}{m\omega} + c_3\exp\left(-\frac{m\omega}{2+m}x\right), & m \neq -2 \\ \frac{c_1}{m\omega}, & m = -2 \end{cases} \tag{8.79}$$

Substitute (8.79) into the remaining determining equation (8.77). Equation (8.77) is identically satisfied for $m = -2$ and $m = -4/3$. It is satisfied for $m \neq -2$ and $m \neq -4/3$ provided $c_3 = 0$.

Thus we are left with

$$\xi^1 = c_1 t + c_2, \tag{8.80}$$

$$\xi^2 = \begin{cases} \frac{c_1}{m\omega}, & m \neq -\frac{4}{3} \\ \frac{c_1}{m\omega} + c_3 \exp\left(-\frac{m\omega}{2+m}x\right), & m = -\frac{4}{3} \end{cases} \tag{8.81}$$

$$\eta = \begin{cases} -\frac{c_1}{m\omega}\theta, & m \neq -\frac{4}{3} \\ -\frac{c_1}{m}\theta - c_3\frac{2\omega}{2+m}\theta \exp\left(-\frac{m\omega}{2+m}x\right). & m = -\frac{4}{3} \end{cases} \tag{8.82}$$

The Lie point symmetries associated with the conserved vectors (8.56) are given by

$$X_1 = \frac{\partial}{\partial t}, \tag{8.83}$$

$$X_2 = t\frac{\partial}{\partial t} + \frac{1}{m\omega}\frac{\partial}{\partial x} - \frac{\theta}{m}\frac{\partial}{\partial \theta}, \tag{8.84}$$

$$X_3 = \exp(2\omega x)\frac{\partial}{\partial x} - 3\omega \exp(2\omega x)\theta\frac{\partial}{\partial \theta}. \tag{8.85}$$

The associated Lie point symmetry (8.84) applies for $n = m \neq -4/3$, $m + 1 < 0$. The associated Lie point symmetry (8.85) applies for $m = -4/3$.

8.3 Concluding remarks

In this chapter we have constructed the conservation laws for equation (8.1) with exponential and power law cases of thermal conductivity and internal heat generation. It turned out that conserved vectors may be constructed for the simple case of the nonlinear heat equation given the exponential case. Some interesting new conservation laws and associated Lie point symmetries are obtained when the internal heat generation and thermal conductivity are given as power law functions of temperature. There are a number of open questions that may be explored from this chapter. Also, one may use the double reduction method by Sjoberg [29] to construct exact solutions.

Chapter 9

Conclusions

This dissertation focused on obtaining solutions for heat transfer in a hot body with various functional forms of thermal conductivity and internal heat generation. We began by calculating the Lie point symmetries for when thermal conductivity and internal heat generation were given by exponential functions, firstly, and then given by the power law. Using these Lie point symmetries, we found the optimal system for various cases. From the optimal systems, we constructed group invariant solutions for each case, namely the steady state solution and the transient solution. For the exponential case, analytical solutions for both the steady state and the transient state were found. However, for the power law case the steady state solution was found but we were unable to find the transient solution analytically. So the Differential Transform Method (DTM) was employed to construct analytical series solutions for cases where our Lie point symmetries failed to obtain a solution, namely for the power law case. The 1D DTM was used to solve the steady state ordinary differential equations. We made a comparison with the DTM solution and the group invariant solution and found that these solutions agreed with one another, hence confidence was established in the DTM. The 2D DTM was then used for the transient state problem of the power law case. Lastly, we constructed conservation laws, using the direct method, and derived associated Lie point symmetries.

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