METACOGNITIVE SKILLS OF SECOND YEAR EXTENDED AND MAIN STREAM UNIVERSITY MATHEMATICS STUDENTS: A CASE STUDY

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A thesis submitted to the Wits School of Education, Faculty of Humanities, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy

Johannesburg, 2017

ABSTRACT

Many universities have introduced so called extended degrees where students' first year workload is spread over two years to prevent the decline of graduates in mathematics and science. It has been put forward that extended degree courses should include the explicit training of mathematics students in the use of metacognitive skills. This is based on research that shows that successful students in mathematics are able to apply such metacognitive skills and that these skills play an important role in mathematical problem solving. Such skills are concerned with the actual regulation, coordination and control of one's own learning activities and cognitive processes. Given that extended degree students generally perform weakly in mathematics in comparison to main stream students (non-extended degree students) this research study sets out to consider the differences in the use of metacognitive skills of these two student groupings.

A qualitative case study was used to investigate collaborative solving of mathematical problems of one student pair. Students were trained in the use of metacognitive skills by using the metacognitive intervention method called IMPROVE. The student pair was video-recorded during talk-aloud protocols twice before explicit training in the IMPROVE method, and after instruction in order to evaluate students' development in the use of metacognitive skills. Video recordings were transcribed noting students' verbal and non-verbal actions and the coding of transcriptions in conjunction with content analysis was used in determining differences in students' metacognitive skills. Since students worked collaboratively, instances where students acted as so-called social triggers of each other's metacognitive skills, were also investigated. With student-researcher interaction during observations, the researcher was also regarded as a social trigger of students' metacognitive behaviour. Apart from these social triggers, environmental triggers of students' metacognitive skills were also scrutinised. Environmental triggers included the effect of task difficulty and the intervention of the IMPROVE method on students' metacognitive skills. This study on the social and environmental triggers of individual's metacognitive skills contributes to the relatively young field in viewing metacognition as cognitive activity that operates on multiple levels during collaborative problem solving, and that metacognition cannot solely be explained in terms of individualistic conceptions but also by social and environmental triggers. Results from the study show that, in general, the main stream student exhibited a greater number of metacognitive skills compared to the extended degree student. Furthermore, it seems that the IMPROVE method as an environmental trigger, had an effect on the development of both students' metacognitive behaviour. Research findings of the study also reveal that the researcher's intervention mainly resulted in the students acting as social triggers for each other's metacognitive behaviour. Furthermore, it was found that there were a greater number of occurrences in which the main stream student acted as social trigger for the extended degree student' metacognitive behaviour. The level of task difficulty also seems to have acted as environmental trigger for students' metacognitive behaviour. As an exploratory study, the findings of this study are not generalizable.

Keywords:

Metacognition, metacognitive skills, extended degree, main stream degree, IMPROVE method, social and environmental triggers of students' metacognitive skills

DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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ACKNOWLEDGEMENTS

I wish to acknowledge several people and institutions who have all contributed in significant and different ways to the doing of my research and the writing of this thesis:

My supervisors, Professor Margot Berger and Doctor Pete van Jaarsveld, for their invaluable guidance and expert assistance throughout the course of this research study.

Staff from the Department of Mathematics and Applied Mathematics and Faculty of Science, University of Johannesburg for their support, guidance, insightful discussions. In particular, Professor Kinta Burger who awarded me funds for 2011 – 2013; I used these funds in the administration and execution of this thesis and I was most grateful for them.

Margaret Whitley and Amanda Koekemoer for their help in the editing of the thesis. For this I am most grateful.

Doctor David Erwin and Professor Peter Dunsby from the Department of Mathematics and Applied Mathematics, University of Cape Town for their support.

My family and friends for their support and encouragement throughout my research journey.

My mother, grandmother and Donavan for their love, support, encouragement and who have contributed to my love for learning and reading.

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Chapter 1: Introduction

1.1 Introduction

A crucial role of higher education is the development of life-long learners of different kinds of knowledge and skills. This form of education requires much more than the traditional methods of teaching and instruction (Boud, 2004). Higher education should also be about the development of students who can think in a critical way, can formulate and solve problems, and can become autonomous learners (Boud, 2004; Dierick & Dochy, 2001). When considering mathematics at tertiary level there needs to be a shift from the development of learners who can merely memorise formulas and apply procedures mechanically, to that of students who can form deep and interrelated connections between mathematical concepts, procedures and principles. Mathematics should not be regarded as merely a subject regarding the acquisition of knowledge of mathematical concepts and procedures. To fully and completely understand mathematics, students need to engage in the process of mathematical thinking and need to do what makers and users of mathematics do; that is, students need to 'mirror' the activities of mathematicians within the boundaries of their own classroom community (Lampert & Blunk, 1998; Schoenfeld, 1992; Stein, Grover, & Henningsen, 1996).

It is my belief that in order for students to 'mirror' the activities of mathematicians, students need to develop metacognitive skills and be able to implement these skills. Students' engagement in such skills includes the monitoring of their own cognitive processes; engaging in self-regulation during task progress; ensuring that they are working accurately; deciding how to optimally use time and mental effort; as well as self-evaluation of their learning and academic performance. Similar sentiments are seen in the work of Larmar and Lodge (2014) who stress the importance of developing students' engagement in metacognitive activities and their use of metacognitive skills. These authors argue that students who are entering university and do not have sufficient metacognitive capital, are at increased risk of not completing their degrees. They add that higher education cannot assume that most students have the needed metacognitive capabilities to cope and adapt to learning activities at a tertiary level. Cassidy (2007) has also noted that students are not always able to evaluate accurately either their own level of performance or capabilities of learning.

A number of researchers in mathematics education agree that students' engagement in metacognitive skills plays an important role in mathematical problem solving. They argue that it is not enough for students to merely have mathematical content knowledge. Rather students need to know *when* and *how* to activate such knowledge and use it in efficient ways in new and complex situations (Focant, Grégoire & Desoete, 2006; Lucangeli & Cabrele, 2006; Mevarech, Tabuk & Sinai, 2006). While observing university students in her own mathematics course, Zan (2000) observed that mathematics students often did not realise the importance of asking *what* they do not understand and *why*. Schoenfeld (1987) made similar observations in his work on college students as they worked on geometry problems. He argued that it is not only what students know, but also how to use it (if at all) which is more important (ibid., p. 192). He argues that students have difficulty in solving mathematical problems since they do not make efficient use of self-monitoring and self-regulation strategies. Asking questions such as 'what' and 'why'; the 'activation' and 'efficient' use of knowledge; 'when' and 'how' to use such knowledge; as well as the efficient use of self-monitoring and self-regulation strategies, are related to notions of metacognition and metacognitive skills.

This research study focuses on furthering knowledge in the field of metacognitive skills. In particular, it explores what metacognitive skills tertiary mathematics students exhibit during collaborative problem solving, and the differences in their metacognitive skills. Furthermore, the study considers (i) the effect that explicit training in the use of metacognitive questioning techniques has on students' metacognitive skilfulness, and (ii) what factors, referred to as triggers, activate students' engagement in metacognitive activities.

This introductory chapter presents an overview of this research study. It starts by firstly giving an explanation of key concepts underpinning the study. This is followed by a background to the research study and a rationale for conducting the research. The author justifies the significance of the research in terms of the research questions that guide the study. A brief overview of the research setting, methodology employed, research methods used in collecting and analysing data to answer the research questions is provided. Lastly, the author's role as researcher in the study is discussed. This chapter concludes with a brief overview of each chapter of the thesis.

1.2 Explanation of Key Concepts

The following section gives a brief outline of the most important concepts that are used throughout this thesis: metacognition and metacognitive skills; metacognitive questioning techniques of the IMPROVE method; the 'paradox of metacognition'; and social and environmental triggers of metacognitive skills and their relation to the paradox of metacognition.

1.2.1 Metacognition and Metacognitive Skills

Although there are various conceptualisations of metacognition, it is generally perceived as the form of thinking an individual uses to understand and control his or her learning. The most commonly cited definition of metacognition is that of Flavell (1976, 1985). Flavell's (1976) initial definition of metacognition includes the idea that: "metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them…" (ibid., p. 323). This definition primarily focuses on what the individual knows. Flavell's (1985) later definition of metacognition focuses on the "knowledge of cognitive activity that takes as its object, or regulates any cognitive enterprise" (ibid., p. 104). This definition places more emphasis on the role of regulation of one's cognitive activities, which implicitly constitutes the monitoring and control of learning processes.

The concepts of monitoring and control of learning processes are related to the notion of metacognitive skills, which is the primary focus of this research study. Metacognitive skills refer to the control and coordination of one's own cognitive processes (Desoete, 2008; Focant, Grégoire & Desoete, 2006; Veenman, 2006)[1]. In mathematics, Mevarech, Tabuk and Sinai (2006) note that metacognitive skills involve self-regulatory strategies which students use in structuring mathematical problem solving processes, while Panaoura and Philippou (2007, p.150) contend that self-regulation refers to one's ability to select, combine and coordinate strategies in an effective way to overcome cognitive obstacles. Through observing university students in her own mathematics course, Zan (2000) noted that the students seemed to have problems implementing metacognitive skills during written tests. These metacognitive skills include regulating actions like: (a) the control and efficient use of time; (b) checking procedures; and (c) checking calculations. These observations led her to hypothesise that students fail the mathematics exam not necessarily due to a lack of content knowledge, but because of the insufficient and incorrect use of metacognitive skills in their work. The question then is: can students be taught the use of metacognitive skills? This is discussed further in the following section.

1.2.2 The IMPROVE Method and Metacognitive Questioning Techniques

In Chapter 2, an in-depth discussion is given on the different instructional programmes that have been used in developing students' metacognitive skills. Such instructional programmes have been designed in response to researchers' arguments that students need to be trained in the use of metacognitive skills since these skills do not tend to develop naturally (Desoete, 2007; Kuhn & Dean, 2004). Furthermore, a number of research studies have shown that metacognition can indeed be learned and that explicit training in the use of metacognitive skills improves students' learning (Mevarech & Fridkin, 2006; Michalski, Zion & Mevarech, 2007; Pressley & Gaskins, 2006).

One such intervention programme is the IMPROVE method. This method plays a pivotal role in this research study. As outlined in Chapter 2, the IMPROVE method

[1] In Chapter 2 (Literature Review), the concepts of metacognition and metacognitive skills are discussed in greater detail; in particular how these skills pertain to students' problem solving behaviour in mathematics.

as designed by Mevarech and Kramarski (1997) is used for the development of students' metacognitive skills as well as for training students in the use of these skills. IMPROVE is an acronym for the steps of which this instructional method consists: **I**ntroducing new concepts; **M**etacognitive questioning, **P**racticing; **R**eviewing and reducing difficulties; **O**btaining mastery; **V**erification; and **E**nrichment (Mevarech & Kramarski, 1997, p. 369). Instruction in the IMPROVE method use an approach of the explicit training of students in the use of metacognitive questioning techniques, as to be used during mathematical problem solving[2]. These questioning techniques enable students to engage in activities that promote the application of metacognitive skills. As a result of this, students are able to monitor and take control of the problem solving process. A number of research studies have reported on the positive effect of IMPROVE on students' mathematical reasoning and their mathematical learning performance, as well as the development of students' metacognitive skilfulness. The above findings are further discussed in Chapter 2.

In my research study, the potential effectiveness of the IMPROVE method on students' mathematical learning performance or problem solving behaviour is not a focus. Rather consideration is given to what metacognitive skills students display before and after explicit instruction in the use of the IMPROVE metacognitive questioning techniques. That is, the research study explores the contribution of IMPROVE to the development of students' metacognitive skilfulness. In this sense, IMPROVE is regarded as an external factor that activates or triggers metacognitive behaviour[3] within the individual. Such a view on IMPROVE ties in with the concepts of social and environmental triggers of students' metacognitive skills, as discussed below.

[2] Chapter 2 gives an in-depth discussion on these questioning techniques and the IMPROVE method.

[3] Throughout this thesis, the terms metacognitive skills, metacognitive activities and metacognitive behaviour will be used interchangeably.

1.2.3 The Paradox of Metacognition: Social and Environmental Triggers of Students' Metacognitive Skills

Although there are a diverse number of conceptualisations of metacognition, many researchers agree that metacognition is a mental activity residing within the individual. An argument supporting this idea is seen in the work of Kim, Park, Moore and Varma (2013). The authors believe that "the main agent of metacognition is still (the) individual, regardless of whether the individual engages in collaborative teams or works independently. Metacognition itself is a mental process within an individual drawing on the individuals' conceptual systems" (ibid., p. 381). Adding to such a view, these authors also address what they term the 'paradox of metacognition'. The principal idea behind this paradox is that although metacognition resides within the individual, metacognition cannot always be explained solely in terms of individualistic conceptions (ibid., p. 378). In essence, the above paradox is concerned with scenarios such as those where the student lacks the needed metacognitive skills in order to solve problems, but is still expected to engage in some form of metacognitive activity. As a result, the above authors propose that to solve this paradox of metacognition one also needs to consider external resources that activate or trigger students' metacognitive activities. Kim et al. (2013) note that these external sources consist of (a) social triggers, as emerge during interaction between the individual and others; and (b) environmental triggers, such as mathematical activities that stimulate metacognitive behaviour amongst learners, or mathematical tasks which differ in difficulty and complexity, as well as classroom activities and/or culture, all of which are situated in the learning environment.

A similar view is held throughout this research study: the researcher's focus is not only on what metacognitive skills the individual exhibits, but also what external factors trigger metacognitive behaviour of the student. Building on the views of Kim et al. (2013), the researcher was interested in what the possible social and environmental triggers of students' metacognitive behaviour are in the context of the study. The above concepts of the paradox of metacognition, as well as social and environmental triggers are discussed further in Section 2.6.

1.3 Background to the Study and Rationale

In Section 1.2, I briefly noted the importance of metacognitive skills, especially in students' learning and academic performance in mathematics. In this section, by referring to research conducted in the field of metacognition and metacognitive skills, I also stress the importance of these skills. I relate this to the purpose of the study, as well as give a justification for why the research study was conducted.

1.3.1 Importance and Training of Metacognitive Skills at University Level Mathematics

Research conducted in the field of metacognition in mathematics has shown that successful students in mathematics are able to apply metacognitive skills (Lucangeli & Cabrele, 2006; Schoenfeld, 1987); that such skills are an important aspect in the solving of mathematical tasks (Lucangeli & Cabrele, 2006); and that the use of metacognitive skills is an important facet in students' mathematical learning performance (Mevarech & Fridkin, 2006; Mevarech & Kramarski, 2003).

Quitadamo, Faiola, Johnson and Kurtz (2008) also note that the academic and personal benefits of critical thinking skills can be seen in students obtaining better grades, improvement of reasoning skills, and access to preferential employment. The authors (ibid., p. 328) regard critical thinking "as a process of purposeful *self-regulatory judgement* that drives problem-solving and decision making, or as the 'engine' that drives how we decide what to do or believe in a given context" (emphasis my own). Although the authors use the term 'critical thinking skills', their conceptualisation concurs with the prevalent conceptualisation of metacognitive skills. Apart from mentioning the benefits of metacognitive skills (here equated to critical thinking skills), the above authors also mention that although such skills are needed for academic and professional success, the large majority of U.S. college graduates lack these skills. Moreover, they cite a report by the Association of American Colleges and Universities (2005) in which it is noted that 93% of college faculty staff considers the development of critical thinking, hence the use of metacognitive skills, as essential amongst students. Still only 6% of graduates actually demonstrate such skills (Quitadamo et al., 2008, p. 327). It is entirely possible that these results are not unique to the U.S. and that similar trends may be observed world-wide, as well as in South Africa. The above results also raise another important issue: do students need to be trained in the use of metacognitive skills?

Lamar and Lodge (2014) argue that since metacognition is central to learning and plays a role in students' academic success, further research is needed on how to develop students' metacognitive behaviour within higher education. Grayson (2010) argues that the design and implementation of engineering courses at university level need to take into account different factors affecting student performance and that one such factor is students' metacognitive skills. A similar argument is encountered in the work of Loji (2010). He notes that numerous educators in the engineering field stress the importance of metacognition and believe that students need to be instructed in metacognitive skills in order for these students to take control of their own learning and to monitor their own progress during problem solving (ibid., p. 33). In the case of mathematics students, Desoete (2007) and Desoete, Roeyers and De Clercq (2003) both argue that metacognitive skills need to be explicitly taught to enhance students' mathematical skills; such skills do not necessarily develop spontaneously. I agree with Jacobs and de Bruin (2010) who believe that the approach toward the teaching of science and science related fields at university level will have to change so that first-year students are provided with more than content knowledge. Such provision should include the development of students' metacognitive skilfulness by training students in the use of such skills.

1.3.2 Concerns in Higher Education: the Case of Extended Degree Programmes

Although students benefit from the training in and use of metacognitive skills[4] there are other challenges concerning students entering higher education. Many universities have to address the challenge of the increase in the number of students entering universities, student retention rates and maintaining high standards in terms of content taught to students as well as the development of students as autonomous life-long learners. South African universities in particular, are faced with a high level

[4] This is elaborated on further in the Literature Review, Chapter 2.

of student failure accompanied by an increased drop-out rate. This is problematic in the higher education domain especially in mathematics and mathematics-related programmes. Worldwide, governments, educators and employees are concerned that students entering higher education lack the needed basic mathematical skills and knowledge to progress and succeed in their degrees (Harrison & Petrie, 2009). Consequently, programmes have been constructed in which students are given additional academic support such as the Sigma Project in the UK (ibid.), although such programmes are not unique to the UK. Varsavsky and Anaya (2009) note that many countries introduce pre-undergraduate programmes, also known as bridging programmes (courses), to support students academically and to help prevent the decline in the number of graduates in mathematics and fundamental sciences.

Bridging programmes are largely university preparation courses that students take as a means of preparing them for the intellectual challenges of a university education. They are specifically designed to equip students with the appropriate academic grounding in order to gain entry to a degree. Students who do not have the needed knowledge to gain entry to university degree courses and/or do not meet the university admission requirements are recommended to follow a bridging programme. Successful completion of such a programme is recognised as a basis of admission to the university and students are allowed to register for a formal degree. Examples of universities that offer bridging programmes are the University of Sydney and the University of Manchester (which refer to bridging programmes as foundation programmes). In South Africa, examples of bridging programmes are the University Preparation Programmes at the University of the Free State and the Foundation Programme at the University of Cape Town. At the University of Johannesburg, the Faculty of Science offers a one year bridging programme to students who want to improve their matric results, enabling them to gain entry to a science degree or diploma programmes. Only students intending to complete their tertiary qualification at the University of Johannesburg are accepted into the program. The University of KwaZula Natal (UKZN, South Africa) offers a similar programme, called the Science Foundation Programme (SFP) and is a one year access programme for applicants from disadvantaged schools who do not meet the entry requirements to go directly into any science degree. Applicants are also required to write an admission test to be accepted to the SFP and only after successfully completing the SFP are students offered entry at the faculties of Science, Agricultural or Health Sciences at UKZN. Grussendorf, Liebenberg and Houston (2004) note that “the vast majority of the SFP students who continued with a degree programme at the University of KwaZula Natal would not have been eligible for admission to a degree if they had not come through the SFP” (ibid., p. 266).

Other universities offer their first year courses extended over two years, where the bridging programme forms part of the students’ formal degree and the degree is referred to as an extended degree. Extended degree programmes are different to bridging programmes and the two should not to be confused. A bridging course, as referred to above, is usually a short course that allows students to redo school subjects to give them a solid pre-degree/diploma foundation and to upgrade their secondary education marks in order to gain entry to a university. An extended degree programme requires an extra year to complete a chosen degree. Eybers (2015) notes that the introduction of extended degree programmes within South African higher education institutions is not unique, but occur worldwide (for example, the Flinders University in Adelaide, Australia). Students who do not meet the university’s usual admissions requirements may be offered a place on an extended degree programme, where the initial entry requirements for such a programme are slightly less than the usual university entry requirements (Eybers, 2015; Jacobs, de Bruin, van Tonder & Viljoen, 2015).

Extended programmes within South Africa, are designed to equip students with the necessary skills which will help them to be successful in their studies (which is similar to the aim of bridging programmes). Apart from extending (spreading out) students’ first year courses over two years in an extended degree programme, students are also given foundational academic support and career guidance (Jacobs et al., 2015). McKay (2016) notes that extended degree programmes were also introduced to address the high failure/dropout rate amongst science students from higher education institutions in South Africa. She acknowledges that extended degree programmes enable “students with poor secondary school results (to) access a science degree, but within a context whereby academic support is provided to foster academic success” and that these programmes focus on “giving attention to the skills and knowledge that students will need to succeed in the subsequent years of their study programme" (ibid., pp. 191 – 192). Grayson (1996) and Jacobs et al. (2015) note that extended degree programmes within South Africa also allow for increased contact time between lecturer and students, and for students to have more time to appropriate and understand the course content better. Similar trends are seen at the University of Pretoria's (South Africa) extended degree programme which focuses on the promotion of the students' academic development (Du Preez, Steyn & Owen, 2008). Other South African universities that also offer extended degree programmes within the science field are the University of Cape Town, University of Johannesburg, the Nelson Mandela Metropolitan University, and Rhodes University, while in the UK similar extended programmes in science are offered at the University of Greenwich and the London Metropolitan University.

Since the present research study focused on students from the University of Johannesburg (UJ) it is important to mention the features of UJ's science extended degree programme.

1. Students who are either unable to meet the admission requirements for the three year main stream (non-extended) degree programme *or* who feel that they will have difficulty in completing the science degree in three years may opt for the extended degree programme.
2. In order to be admitted to the programme students need to fulfil the minimum entry requirements of the programme, although the requirements are not as stringent as the university entry requirements for non-extended degrees.
3. Apart from the entry requirements above, students also need to submit their National Benchmark Test (NBT)[5] results when applying for the programme.

[5] The National Benchmark Tests (NBTs) are a set of South African tests that measure an applicant's academic readiness for university. These tests complement and support, rather than replace or duplicate the National Senior Certificate (NSC). Universities can use the results in making decisions about an applicant's access to university. This means that the NBT results, in combination with the NSC results, are used to determine whether an applicant is ready for academic study at a university. Some universities use the results for placement within university. This means that the results are used to decide whether an applicant will need extra academic support after he/she has been admitted to university. Universities may also use the results to help develop curricula within their universities. The NCS is the main school-leaving certificate in South Africa more commonly known as a matriculation (matric) certificate where grade 12 is the final year of schooling.

4. The curriculum of the extended programme spreads the first year content over two years with foundational support in language and computer competence. Moreover, students are only allowed to join the non-extended degree programme after successfully completing courses from the extended programme.
5. Extended degree student groups are smaller allowing teaching staff to provide students with the fundamental content and skills to help promote success throughout the extended programme.
6. Students are also supported by help from tutors, and by additional contact periods and support from learning centres and mentors.

McKay (2013) says that “the main purpose of the extended time is to open up teaching and learning ‘space’ for academic support and scaffolding” (ibid., p. 684). McKay (2013) also notes that the extended degree programme focuses on supporting students to acquire skills and knowledge needed in their second and third years of study. “Academic literacy skills, such as essay writing, referencing, solving complex problems, statistical analysis and reading for understanding, are but a few areas of academic development covered in the first year” (ibid., 2013).

1.3.3 Metacognitive Skills Taught within Extended Degrees

Although extended degrees fundamentally give students the needed academic support, these courses focus mostly on content knowledge rather than on explicitly training students in metacognitive skills in a mathematical context. Kloot, Case and Marshall (2008), writing in the South African context, note that a central tenet behind the design of most foundation courses is that more contact time and tuition will lead to students’ successful university performance. The authors believe that this is not sufficient for student success, as it may inhibit development of student autonomy and students may merely pass subjects by means of rote-learning. When discussing the role and effectiveness of remedial mathematics programmes (bridging programmes) in the US, Bahr (2008) mentions that the fundamental principle of these programmes is that students who remediate successfully because of these courses, should have academic outcomes similar and comparable to students who do not require remediation. He also mentions that these programmes need to prepare students for success for all their course work. My position is that these academic outcomes should not only be about students' grades and marks obtained in courses, but also the development of their metacognitive skills. Moreover, since the aims of extended programmes are similar to remedial programmes in science, Bahr's (2008) sentiments certainly equally apply to extended degree programmes.

Grayson (1996, 1997, 2010) and Kloot et al. (2008) argue that metacognitive skills is one of the elements that affects students' performance in engineering extended degree programmes and that these skills need to be included in the design of effective foundation courses. Furthermore, Grayson (1996) argues that it is not students' lack of content knowledge that is the major cause of their poor performance, but students' lack of understanding and analysis of tasks and problems. That is, the problem lies with a lack of metacognitive skills. She also argues that many educational skills (such as metacognition) that form part of extended courses in South Africa should be integrated into main stream teaching as well (Grayson, 1997), especially since extended students also struggle with their studies in main stream degree courses. Similar arguments are found in du Preez et al. (2008) and Kloot et al. (2008) regarding extended degree programmes in South Africa.

1.4 Rationale

The literature referred to above pointed out that successful students in mathematics are able to apply metacognitive skills, and that metacognitive skills play an important role in mathematical problem solving and students' mathematical learning performance (Lucangeli & Cabrele, 2006; Mevarech & Fridkin, 2006; Mevarech & Kramarski, 2003; Schoenfeld, 1987). From the above, I hypothesise that extended degree students exhibit fewer and/or a lower quality of metacognitive skills compared to those of main stream degree students (students from non-extended degree courses), since extended degree course students mostly obtain low grades in mathematics and mathematics related subjects at high school level. Interestingly, Craig (2009) notes that weaknesses in South African students' mathematical knowledge and mathematical performance are not only observed in extended degree students, but also with main stream degree students. This raises the following concern: do mathematically weak main stream students also exhibit poor use of metacognitive skills? Should metacognitive skills be explicitly taught to main stream students? If extended degree courses are to include the instruction of metacognitive skills, this surely needs to be integrated into the main stream degree as well?

To integrate and include such skills in instructional programmes, Bahr (2008) notes that further research needs to be done in order to identify what obstacles hinder successful remediation of university (college) mathematics students. Also, in order to devise instructional methods that aid the development of metacognitive skills, Magno (2010) argues that we need to assess how the individual can successfully implement such skills. Ku and Ho (2010) also mention that there is very little research done in examining the individual differences between students' use of metacognitive strategies (skills). I agree with the above authors' views that we need to consider such differences in metacognitive skills of individuals, in order to design effective instructional programmes in developing these skills. Specifically if we want to consider students from the extended and main stream degree respectively, I believe that we need to assess whether these two groups of students exhibit different metacognitive skills.

Much of the aforementioned literature is concerned with:

(a) the role of metacognitive skills as an aid to improve mathematical learning performance (as seen from the IMPROVE research studies);
(b) the view that students need to be trained in the use of metacognitive skills; and
(c) the belief that in order to train students in these skills we need to ascertain what metacognitive skills students already exhibit.

With these three points in mind, this research study was conducted to investigate the metacognitive skilfulness of students from a one semester second year level calculus course (this course will be referred to as the Calculus 2 course) at the

University of Johannesburg, South Africa. Extended degree and main stream degree students who successfully completed the first year calculus course were allowed to register for the Calculus 2 course. Thus, the Calculus 2 course was composed of two subgroups: students from the main stream and extended degree respectively[6]. During formal Calculus 2 lecture time both student groups were instructed in the use of metacognitive questioning techniques and how to implement these techniques within collaborative problem settings. These techniques were based on the IMPROVE method. Consequently, this research study is an exploratory investigation into the:

(a) possible differences in metacognitive skilfulness between that of extended and main stream degree student within collaborative problem solving; and

(b) implementation of the IMPROVE method and its possible influence (effect) on students' metacognitive skilfulness. That is, did IMPROVE contribute to the development of students' metacognitive skilfulness?

Knowledge gained from the study may provide insights for the design of possible instructional method(s) to improve and develop future students' metacognitive skilfulness in the Calculus 2 course, as well as first year mathematics courses for both student groups; especially those from the extended degree Calculus 1 course. Furthermore, the development of these skills involves explicitly teaching metacognitive skills in both extended and main stream degree courses, in order for these two groups of students to ideally achieve similar and comparable academic outcomes.

1.5 Research Questions

As discussed earlier, in order to design the above instructional methods, one first needs to identify what metacognitive skills extended and main stream degree students exhibit. This thesis is a case study of one student pair, consisting of an extended and main stream degree student, as they work together in solving

[6] Students in the extended degree followed the first year calculus course content over a period of two years, while the main stream degree students studied the same content over one year only.

mathematical problems from the Calculus 2 course. I will discuss the research methodology, including why and how I selected the one student pair from four observed student pairs, in much more detail in Chapter 4. Section 1.6 also gives a brief discussion on the selection of the student pairs.

Apart from my role as researcher observing the students, I also assisted the students to solve the problems when needed. Although the number of student-researcher interactions was kept to a minimum, I was aware that my interventions during observations may have had an influence on students' metacognitive skilfulness during the period of observation. Although it was never the intention of the researcher to guide and/or assist students' during the observations, students lack of contribution and/or incorrect contributions led to such intervention. Moreover, since all students were trained in the use of the IMPROVE method's metacognitive questioning techniques, the researcher was interested in how these techniques acted as a trigger for students' metacognitive skills. The student pairs also solved problems of varying difficulty. I consequently became interested in the influence of varying task difficulty on the metacognitive skilfulness of the students. These observations, interests and curiosities led to the following research questions:

1. (a) What metacognitive skills do an extended and mainstream degree student respectively exhibit while working together as a pair on mathematical tasks, *before* explicit instruction in the IMPROVE method?

   (b) In particular, are there any differences in the metacognitive skills between the extended and main stream degree student before training?

2. (a) *After* explicit instruction in the IMPROVE method, what metacognitive skills do an extended and main stream degree student respectively exhibit while working together as a pair on mathematical tasks?

   (b) In particular, are there any differences in the metacognitive skills between the extended and main stream degree student after training?

3. What role does the researcher play in initiating the metacognitive skills of the students and how does the researcher influence the development of the metacognitive skills of the students over the period of the observation?

4. How does the level of task difficulty influence students' metacognitive behaviour (skills) over the period of observation?

Research question 2 implicitly refers to IMPROVE's effect (influence) on students' metacognitive skilfulness. Moreover, research questions 2 to 4 consider triggers of metacognitive skills of students. Relating the above research questions to the work done by Kim et al. (2013) (discussed in Section 1.2.3), I consider my role as a social trigger of students' metacognitive skills, as well as how IMPROVE and the level of task difficulty act as environmental triggers of students' metacognitive skills.

1.6 The Research

This research study is of a qualitative nature and employs a case study methodology in a constructivist and interpretivist paradigm. Its primary focus is the on-task metacognitive skills that a main stream and extended degree student exhibit respectively.

As mentioned earlier, four student pairs were observed over a time period of four observations. These students worked collaboratively to solve mathematical problems in a Calculus 2 course. The sample of students was constructed by both theoretical and random sampling methods. Only four extended degree students voluntarily agreed to be part of the research study. Four main stream students were then chosen randomly from the students who volunteered to form the four student pairs. All four student pairs were video-taped while working on the given problems. The video-tapes were transcribed and analysed in a draft version. However, this study reports on the data derived from only one student pair. This specific student pair was chosen since the data obtained from observations of their collaborative problem solving gave a rich, in-depth account of their metacognitive behaviour. The other three student pairs' minimal contributions and the low frequency of observed metacognitive skills, meant the data was not suitable for a fine-grained analysis. Accordingly this data does not form part of the final research findings.

Data was generated by means of video recordings and talk-aloud protocols in which all verbal and non-verbal actions of the students were transcribed. A taxonomy (analytic framework) was designed by the researcher. This taxonomy was applied through content analysis, by coding and analysing the transcribed data. Coding of data focused on specific instances in which students exhibited the use of metacognitive skills. The taxonomy was developed from existing researchers' work on the metacognitive skills of mathematics students. Chapter 4 discusses the methodology employed (including the reasons for one student pair being selected as the case study), as well as research methods used for the collection of data. Chapter 5 focuses on how the data was coded and analysed by using the designed taxonomy.

1.7 Situating the Researcher

When engaging in research, the possibility of power-relationships emerging between the researcher and the participants needs to be taken into account. In this study, although the researcher was not the lecturer of the participants, participants were still aware that the researcher had lectured the Calculus 2 course for a number of years. This fact may have altered students' behaviour during the researchers' presence in the observations. Still, no pre-existing relationship did exist between the researcher and the students. Moreover, observed students were not given special privileges in terms of receiving extra help or resources from the researcher, in assisting them in improving their academic and learning performance within the Calculus 2 course.

The research study was designed to collect data about students' naturalistic problem solving behaviour during collaborative working. Students were also informed that their performance within the Calculus 2 course was not being measured, and that there was no good or bad data to be collected.

The researcher acknowledges that when conducting research, researcher bias can be significant. Researcher bias cannot be removed entirely, although it can be reduced by acknowledging the position of the researcher and the observed participants.

1.8 Structure of the Thesis

This thesis is organised into nine chapters as outlined below.

Chapter 1: Introduction

This chapter introduces the research area that was investigated, the background to the study, and situates and explains the motivations behind undertaking the research.

Chapter 2: Literature Review

This chapter provides an in-depth review of research related specifically to metacognition and metacognitive skills in mathematics education. It discusses and examines how the literature defines and frames metacognition and metacognitive skills and its relation to education. This is followed by a discussion of metacognitive interventions relating to the teaching and learning of metacognitive skills, as well as an overview of the IMPROVE method as a metacognitive intervention in mathematics education. The chapter concludes with a discussion on metacognition within collaborative settings and social and environmental triggers of students' metacognitive behaviour.

Chapter 3: Research Paradigm and Theoretical Framework

This chapter outlines the paradigms in which the research is situated; that of a constructivist and interpretivist paradigm. The paradigms were used to position the researcher's views on the nature of knowledge, as well as how knowledge is acquired. The above 'knowledge' which the researcher considered in this study concerns mathematical concepts and practices students are introduced to in the learning of mathematics, as well as metacognitive skills within mathematical problem solving. The chapter also includes a discussion on how the above paradigms are related to the theoretical framework of the study. This framework outlines the researcher's views on the teaching and learning of mathematics, paying particular attention to the use of metacognitive skills as a tool to develop students' mathematical thinking in the hope of enabling them to mirror behaviour similar to that of mathematicians. The chapter concludes with the researcher's argument that in order for students to be part of a mathematical discourse, the use of metacognitive skills should be seen as a norm that should be adhered to. The researcher justifies the above argument by using the tenets of theories of symbolic interactionism, as well as the concept of socio-mathematical norms.

Chapter 4: Research Methodology and Research Methods

Chapter 4 discusses the research setting, sampling of participants, as well as a justification of the methods used for data collection. The researcher also discusses why a qualitative case study approach as methodology was used for this research study. The chapter concludes with a discussion on the important role of task difficulty as an environmental trigger of metacognitive skills and the implications it had for the research study.

Chapter 5: Analytical Framework, Data Analysis and Coding Procedures

This chapter provides an account of how the applied analytical framework, referred to as a taxonomy, developed chronologically and was designed. The taxonomy amounted to the construction of a set of codes organised into categories, which were adapted from different researchers' work. Chapter 5 also discusses how the taxonomy was used in the organisation and analysis of data in order to answer the research questions. The chapter concludes with a discussion of the codes that were used in the analysis of data, and that of metacognitive decision points. These decision points were the moments where students exhibited a particular use of metacognitive skills.

Chapter 6: Discussions and Results of the Four Observations

The chapter gives a detailed account of each of the four observations for the one case study student pair. The researcher discusses the students' collaborative problem solving, specifically focusing on what metacognitive skills students exhibited. Discussions on each of the observations also illustrate how codes of the taxonomy were applied to identify and code the metacognitive decision points. Discussions on each of the observations highlight instances where students acted as social triggers for each other's metacognitive behaviour, as well as the role of the researcher as social trigger for students' metacognitive behaviour.

Chapter 7: Comparison between Observations

This chapter compares the students' metacognitive behaviour across the four observations. It also identifies the differences between students' metacognitive skills. The researcher also considers the possible effect that the IMPROVE method as environmental trigger may have had on the development of each student's metacognitive behaviour over the period of observation. The possible effect of the level of task difficulty as environmental trigger of students' metacognitive skills is also given attention. The chapter concludes with a discussion on the role of the researcher as social trigger of students' metacognitive behaviour.

Chapter 8: Main Findings

Chapter 8 presents the main findings resulting from the data collection, and discussions from Chapter 6 and 7. Each of the four research questions are answered in terms of the findings.

Chapter 9: Conclusion

This chapter summarises the main findings of the study by discussing its significance in terms of the field of metacognition research. It focuses on the differences in the metacognitive skills of extended and non-extended tertiary mathematics students in this study, as well as the role of social and environmental triggers of students' metacognitive behaviour. The chapter also discusses what the findings mean for further research and educational applications, and provides recommendations for future research and practice.

Chapter 2: Literature Review

2.1 Introduction

The literature on metacognition is vast and has a great number of differing conceptualisations, operationalisations and definitions. This literature review tries to set out what metacognition is as it is used in this thesis; in particular metacognitive skills. I firstly discuss the difference between metacognition and cognition and their relationship as this plays an important role in understanding what metacognition entails and how it differs from cognition. A number of different definitions of metacognition will then be discussed, where the majority of them still refer back to the pioneering work of Flavell (1976, 1985). In defining metacognition I will discuss its different components: metacognitive knowledge and metacognitive skills. Since this thesis is concerned with the construct of metacognitive skills, an in-depth discussion will deal with this notion. Metacognitive knowledge is also discussed since it plays a role in students' metacognitive skilfulness. A rationale for and discussion of a number of researchers' views and studies which emphasise the importance of metacognition in educational research is provided. Furthermore, I discuss how students can be trained in the use of metacognitive skills in programmes such as that of IMPROVE (Mevarech & Kramarski, 1997). Since the research setting is within a collaborative problem solving environment, I delineate the different views of metacognition in collaborative settings and the relatively young field of research in which metacognition is seen as a social process. A number of researchers argue that metacognition in group work differs from that of individual metacognition, and hence metacognition in a social setting needs to be conceptualised in a different way. I therefore provide an outline of the construct of socially shared metacognition, in which metacognition is regarded as a social process within a group. Instead of using the notions and ideas peculiar to socially shared metacognition, I rather use the concept of metacognition in collaborative setting similar to that of Kim, Park, Moore and Varma (2013), where metacognition is seen to exist and operate on multiple levels; that of the individual, and the social and environmental levels.

2.2 Metacognition and Cognition

Although it is not the purpose of this thesis to clarify what cognition is, I believe that in order to understand what metacognition is, it is of importance that we first understand what the relationship is between them to be able to distinguish between these two concepts.

Using the term cognition in everyday life one may overlook the meaning of the concept. A useful definition of cognition can be found in Flavell, Miller and Miller (2002, p. 1) where it is defined as “what you know and think”. Veenman (2006) sees cognition as one’s intellectual abilities, a repertoire of cognitive skills contained within what he calls a cognitive toolbox consisting of basic cognitive operations. For me, cognition entails all of the above, but in this thesis I regard cognition mostly as one’s intellectual abilities. Furthermore, I use the above metaphor of the cognitive toolbox in my discussion on the relationship between metacognition and cognition.

Nelson (1999) regards metacognition as not detached from cognition, while Tarricone (2011, p. 1) regards metacognition as second-order cognition which entails knowledge and awareness, as well as the monitoring and control over the flow of one’s cognitive processing. Returning to Veenman’s (2006) cognitive toolbox, he notes that there are three mutually exclusive views (or models) that describe the relation between metacognition and one’s intellectual abilities. The first model regards metacognition (in particular metacognitive skills, which is the focus of this thesis) as a manifestation of intellectual abilities where metacognition forms an integral part of one’s cognitive toolbox (Veenman, Elshout & Meijer, 1997). The second model views metacognition as a separate concept to that of cognition where they are entirely two separated toolboxes (the one is not contained within the other). The third model, which I am in favour of, is the mixed model where metacognition is seen as related to cognition up to a point, and metacognition has the additional value and advantage of guiding learning processes. This view of the mixed model agrees with the views on cognition of Nelson (1999) and Tarricone (2011) above. Still it is important to note that the difference between metacognition and cognition is relational and not absolute (Nelson & Narens, 1994).

Although there is a relation between metacognition and cognition, the distinction between these two concepts may be clear at a conceptual/theoretical level, but not at an operational level. This is discussed in Artzt and Armour-Thomas (1992) where they noted that activities such as trying to understand a concept or operation, analysing, exploring, planning, implementing, and verifying can either be classified as cognitive or metacognitive. The authors argue that there is a conceptual difference between cognition and metacognition, but at the operational level the distinction is not that clear. They suggest that cognition is implicit in metacognitive activities, while on the other hand metacognition may be part of a cognitive act, although it may not be that apparent to the observer (ibid., p. 141). Because of this, Artzt and Armour-Thomas decided that none of the above activities can be classified as purely cognitive or purely metacognitive and that the distinction in classifications should be grounded on the predominant behaviour observed (ibid., p. 141). Their working distinction between cognition and metacognition is given as follows:

> Cognition is involved in the doing, whereas metacognition is involved in the choosing and planning what to do and monitoring what is being done.

The authors also mention that metacognition, in particular metacognitive skills are exhibited by:

1. statements made about the problem; and
2. statements made about the problem solving process.

Cognitive behaviours are revealed by verbal comments and non-verbal activities which have to do with the actual processing of information (ibid., p. 141). The above working distinction and views of Artzt and Armour-Thomas (1992) will be used extensively in this thesis and are discussed in Chapter 5.

2.3 Defining Metacognition

A number of researchers argue that it is not enough for students to merely have mathematical content knowledge, but that they need to know *when* and *how* to activate such knowledge and use it in an efficient way in new and complex situations, as well as be trained in how to activate and use such knowledge (Focant, Grégoire & Desoete, 2006; Lucangeli & Cabrele, 2006; Mevarech, Tabuk & Sinai, 2006). Baker and Brown (1984) argue that learners need to take control of their learning when doing intermediate or difficult tasks in order for learning to be effective. When solving mathematical problems there are a number of processes which students have to do in order to solve the problem successfully (or in part successfully). Students need to orientate themselves about what is asked and what needs to be done when solving problems, and they need to identify the goal of the problem. They also need to evaluate if they have the knowledge and the capability to complete the problem. They must also try to implement appropriate procedures and operations effectively. At the same time they need to monitor their progress in reaching the goal of the problem. After obtaining their solution, they must reflect and check if their solution does indeed answer the question. This amounts to students needing to orientate, analyse, plan, and monitor their execution of their solution, as well as finally discern if their solution is correct (Davidson & Sternberg, 1998; Metes, Pilot & Roossink, 1981). Hence, what is of importance, is that students are able to monitor their cognitive actions and take control over such actions. This includes asking themselves *what* they do not understand and *why* they do what they do during mathematical problem solving (Schoenfeld, 1987; Zan, 2000). Conceptions such as asking 'what' and 'why' during problem solving; the 'activation' and 'efficient' use of knowledge; as well as 'when' and 'how' to use such knowledge; and the efficient use of self-monitoring and self-regulation strategies, are all related and lead to the notions of metacognition and consequently that of metacognitive skills.

A number of definitions of metacognition are summarised in the Table 2.1 below:

Table 2.1: Definitions of Metacognition

Flavell (1976)	"Metacognition refers to one's knowledge, concerns one's own cognitive processes and products and anything related to them… Metacognition refers amongst other things, to the active monitoring and consequent regulation and orchestration of this processes in relation to the cognitive objects or data to which they bear, usually in service of some concrete goal or objective." (p. 232)
Flavell (1976)	"Knowledge and cognition about cognitive phenomena." (p. 906)
Flavell (1985)	"Knowledge or cognitive activity that takes as its object, or regulates, any aspect of cognitive enterprise." (p. 104)
Brown (1987)	"Metacognition refers loosely to one's knowledge and control of one's own cognitive system." (p. 66)
King (1999)	"Metacognition involves the ability to think about own cognitions, and to know how to analyse, to draw conclusions, to learn from, and to put into practice what has been learned." (in Rahman & Mazur, 2011, p. 135)
Zimmerman & Moylan (2009)	"Metacognition refers to the knowledge, awareness and regulation of one's thinking." (p. 299)

Although it is difficult to produce a simple definition of metacognition, the most basic definition of metacognition is "thinking about thinking" or "cognition of cognition" (Flavell, 1979; Brown, 1987). However such a simple definition does not take into account the weightiness and diverse meanings of metacognition, as well as the variations of what the term actually refers to (as seen in Table 2.1 above). Having a closer look into the different ways that metacognition is conceptualised and defined, one will notice that amongst researchers there is no consensus as to what metacognition actually entails. Although there are some similarities as to how metacognition is defined, closer inspection of the literature shows that a great number of definitions and conceptualisations of this term actually create more confusion than clarification. In Rahman and Masrur (2011,

p. 135) this confusion is discussed. They give a number of terms that apply to metacognition –

> metacognitive beliefs, metacognitive awareness, metacognitive experiences, metacognitive knowledge, feeling of knowing, judgment of learning, theory of mind, meta-memory, metacognitive skills, executive skills, higher-order skills, meta-components, comprehension monitoring, meta-learning, learning strategies, heuristic strategies, and self-regulation.

Because of such lack of clarity and consensus, one needs to go back to the original definitions of Flavell (1976, 1985). Flavell (1976) defined metacognition as the knowledge that one has about one's own cognitive processes or anything related to such processes, as well as how one regulates one's own cognitive processes – "(metacognition is) the active monitoring and consequent regulation and orchestration of these processes" (ibid., p. 232). In this way of defining metacognition we have that it refers to what individuals know about how they learn and how they control their cognitive processes. In 1985, Flavell (p. 104) defined metacognition as "knowledge or cognitive activity that takes as its object, or regulates, any aspect of cognitive enterprise". This definition of metacognition emphasises the role of regulation (monitoring and control) of learning.

Although the definitions of metacognition are diverse, there is still a general agreement that metacognition consists of at least two different, but related components (elements): metacognitive knowledge and metacognitive skills (which encompass metacognitive control and metacognitive monitoring).

2.3.1 Metacognitive Knowledge

Metacognitive knowledge refers to a base of knowledge which an individual draws from when making decisions about his/her own thinking. Metacognitive knowledge is seen as what one knows about one's own learning and cognitive processes (referred to as self-knowledge); how you solve problems and complete tasks (knowledge of tasks) and your own learning strategies (strategic knowledge) (Cotterall & Murray, 2009). Desoete (2008) notes that metacognitive knowledge can be seen as one's own knowledge, awareness and understanding of one's own cognitive processes and products. According to Flavell (1979; 1987) and Veenman (2006) metacognitive knowledge comprises the following elements:

- the actual knowledge one has about one's own learning methods;
- one's own personal beliefs and views as a problem solver;
- knowledge about the learning and problem solving strategies one possesses;
- the interplay between all of the above.

Although this thesis tries to uncover what metacognitive skills students' exhibit, it is still important to note that students' metacognitive skills are influenced by the above elements of metacognitive knowledge. For example, if the student has certain beliefs about himself, this will most certainly influence his approach as a problem solver. Moreover, not being aware and in control of skills on how to solve a problem will also influence what strategies he will use in order to solve the task at hand. If the student does indeed reflect after completing the task, the knowledge he gains during such reflection will possibly be used in his approach and solving of future tasks, and he will hopefully acquire new problem solving strategies. From the above, one gains the understanding that metacognitive knowledge is not stable. It changes over time as the student matures in his cognitive enterprises, as well as through individual learning and/or socialisation with peers and more competent peers in explicit and implicit learning situations (Wenden, 1999).

2.3.2 Metacognitive Skills

Metacognitive skills (also referred to as metacognitive strategies) are concerned with the actual regulation, coordination and control of one's own learning activities and one's own cognitive processes (Desoete, 2008; Focant, Grégoire & Desoete, 2006; Veenman, 2006). These skills enable one to evaluate and monitor one's own understanding and cognitive processes (Lucangeli & Cabrele, 2006).

Examples of such skills in mathematics are task analysis, planning, monitoring, checking and reflection (Veenman, 2006). As noted in Chapter 1, Mevarech et al. (2006) noted that such skills are concerned with self-regulatory strategies which the student uses in order to structure the problem solving process. The above concurs with the definition of self-regulation as given in Panaoura and Philippou (2007). According to them self-regulation refers to the processes that coordinate cognition and refers to one's ability to select, combine and coordinate strategies in an effective way to overcome cognitive obstacles (ibid., p.150). Artzt and Armour-Thomas (1992) argue that reflecting on one's own cognitive processes, monitoring of one's cognitive processes and making decisions in modifying one's cognitive processes during mathematical problem solving all form part of the notion of metacognitive skills. Drawing on the above literature I consider strategies (skills) that allow one to coordinate, plan, control, monitor and regulate one's cognitive processes as metacognitive skills.

It is important to note from the above discussion (and as seen from the literature) that the regulating task within the metacognitive skills range takes into account two concepts: metacognitive control and metacognitive monitoring. It is important to clarify the distinction between these two concepts, although they are related and subsumed under the same concept of metacognitive skills.

The relation between metacognitive control and metacognitive monitoring can be seen in the work done by Nelson and Narens (1990). This is illustrated in Figure 2.1 below, as adapted from Nelson and Narens (1990) and a combination of the research done by other researchers in the field. Figure 2.1 illustrates the flow of information between the cognitive object-level and the metacognitive meta-level. Cognitive activity means that the meta-level is informed by the object-level. The meta-level controls the object-level but not vice versa (Nelson & Narens, 1990). Metacognitive control and monitoring therefore perform two different functions even though they are related to one another. It is important to note that a student may monitor his progression and work accurately, but it may be the case that such monitoring is followed by ineffective control.

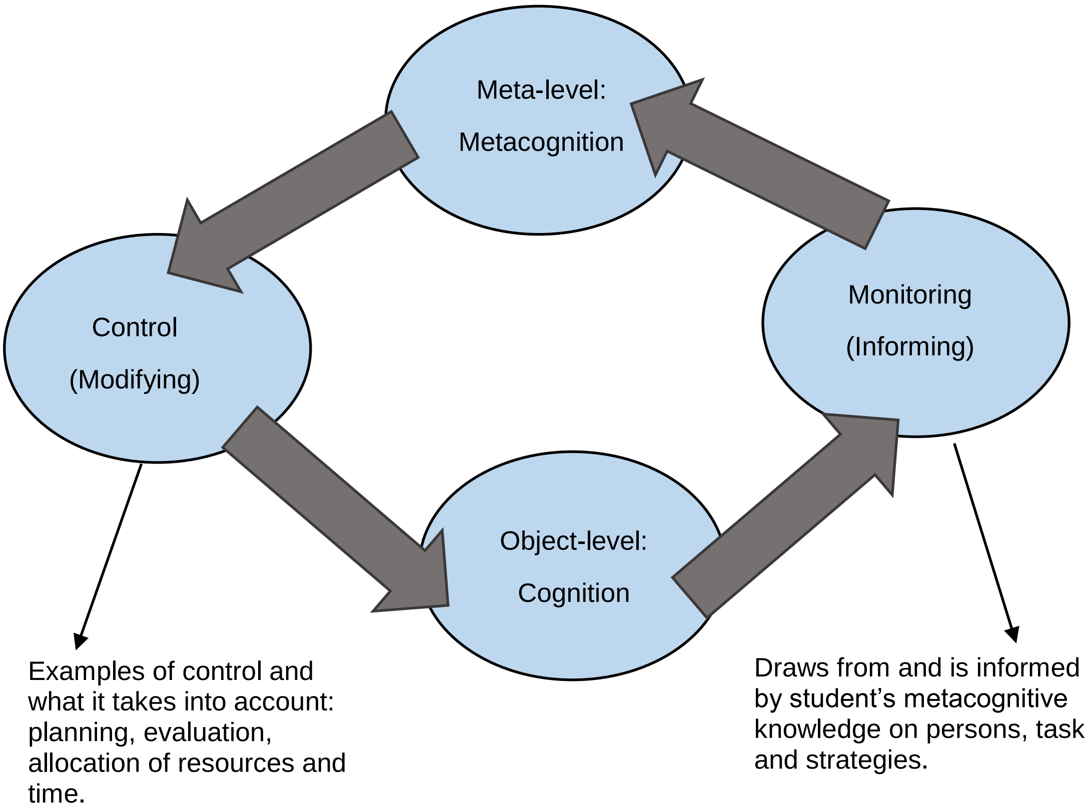


Figure 2.1: Metacognitive Control and Monitoring (adapted and combined from Brown, 1987; Efklides, 2006; Flavell, 1979; Nelson, 1996; Nelson & Narens, 1990, 1994)

Metacognitive monitoring refers to a student's assessment of the current state of his cognitive activity. Pintrich, Wolters and Baxter (2000) note that such monitoring takes into account:

- an assessment of how easy or difficult a task is and how able the student is to work on the task;
- the monitoring of one's progress in solving a task;
- one's comprehension of the operations in solving the problem and the (possible) learning that is taking place; and
- being conscious of being able to recall any information and knowledge during solving of the problem.

Metacognitive control takes into account the “regulation (of) some cognitive aspect” (Dunlosky & Metcalfe, 2009), and “the ability to use (metacognitive monitoring) judgements to alter behaviour” (Son & Schwartz, 2002). Thus, metacognitive monitoring focusses on the assessment/judgement of a cognitive enterprise, whereas metacognitive control has to do with what action(s) students make relating to their own cognitive enterprises during learning and how to handle or to take action to improve such enterprises.

Pintrich et al. (2000) noted that metacognitive control can be divided into four categories:

1. Planning activities – setting of goals for learning, time use, and performance.
2. Strategy selection and use – making decisions about which strategies to use for a task, or when to change strategies for performing a task.
3. Allocation of resources – control and regulation of time use, effort, pace of learning and performance.
4. Volitional control – control and regulation of motivation, emotion and environment.

My research focuses on students’ selection of strategies while working on solving the task at hand. Strategy selection is related to and subsumed under metacognitive skills such as planning, analysis, checking, controlling, evaluation and any behaviours that take into account the controlling and coordination of cognitive activities, which are intentionally (consciously) performed in controlling cognition (Brown & DeLoache, 1978; Brown, Bransford, Ferrara, & Campione, 1983; Efklides, 2006; Pintrich et al., 2000).

2.4 The Role of Metacognition in Education

University students need not only obtain a great deal of content knowledge during their degree, but also need to be trained in such a way that they will be life-long, autonomous learners. Sternberg (1998) notes that life-long learners increasingly need “not the textbook factoids, but rather, the learning to learn skills and the skills of accessing a knowledge base that form the heart of metacognition". Hacker, Dunlosky and Graesser (1998) also mention that "the promise of metacognitive theory is that it focusses precisely on those characteristics of thinking that can contribute to students' awareness and understanding of being self-regulatory organisms, that is, of being agents of their own thinking" (ibid., p. 7).

Apart from the above authors' views that metacognition is beneficial to student learning, research on metacognition in the educational domain has grown tremendously because of the beneficial learning outcomes that effective metacognition has on learning. Gavelek and Raphael (1985) contend that one of the reasons why investigating metacognition is of importance is that it helps learners control their own learning processes, and hence helps them to become autonomous. Metacognition also plays an important role in learning since it affects students' skills of acquisition, comprehension as well as the ability to retain information and knowledge. Also, the improved retention is presumed to result from the characteristic of metacognition as a volitional, conscious and intentional mental act, all of which have strengthening effects on the formation of meaning, in turn reinforcing its inclination to be remembered.

Metacognition also enables learners to apply what they have learned. It also has benefits for their learning efficiency, critical thinking and problem solving skills (Hartmann, 2002a). Pressley, Van Etten, Yokoi, Freebern and Van Meter (1998) found that successful learners have the knowledge of *how* to perform tasks, *when* to adopt certain appropriate strategies as well as being able to argue *why* a particular strategy will be useful (emphasis my own).

Evidence on the positive results (impact) that metacognition has on learning can be seen for example in the work of Veenman and Beishuizen (2004) who have shown that learners with a low aptitude for a subject were able to outperform those with a higher aptitude. It has also been shown that effective metacognitive functioning can improve students' learning outcomes (Adey & Shayer, 1994; Gunstone, 1991) for learning in general (Jackson, 1998, Ku & Ho, 2010; Muis & Franco, 2010; Paris & Winograd, 1990; Rezvan, Ahmadi & Abedi, 2006; Schraw, 2002; Sternberg, 1998; Veenman & Beishuizen, 2004; Vos & de Graaf, 2004; Weinert, 1987) but also in mathematics (Goos, Gailbraith & Renshaw, 2002; Hurme, Palonen & Jarvela, 2006; Martini & Shore, 2008) and in science education (Hartman 2002b; Yuruk, 2007).

It has been observed that successful students in mathematics are able to apply metacognitive skills (Lucangeli & Cabrele, 2006; Schoenfeld, 1987) and that such skills are an important aspect in the solving of mathematical tasks (Lucangeli & Cabrele, 2006). Similar findings can be found in Kramarski, Mevarech and Lieberman (2001); Kramarski, Mevarech and Arami (2002); Mevarech (1999); Mevarech and Amrany (2008); Mevarech and Fridkin (2006); and Mevarech and Kramarski (2003), who note that the use of metacognitive skills play an important role in mathematical learning performance of students. Veenman and Verhej (2001) also noted that metacognitive skilfulness is a strong predictor of a learner's task performance and successful study.

In the light of Hacker's et al. (1998) view of students as 'self-regulatory organisms', clarification on the concept of self-regulation and its relation to metacognition and metacognitive skills is needed. A number of researchers argue that learning how to learn is not only important, but in order for students to be successful in their academic learning they also need to be willing to learn. According to Zimmerman and Martinez-Pons (1990) this is what self-regulated learning is. It is defined as the willingness to learn and the knowing how to learn. Zimmerman (1986) notes that self-regulated learners are those who learn not only at a metacognitive level, but also learn at a motivational and behavioural level. According to Sternberg (1998), metacognition is merely one component needed to gain expertise in a certain domain. Motivation and affective factors (that form part of the concept of self-regulation) is also of importance in developing expertise. Motivation can have an effect on the development of metacognitive skills and awareness. Students who are motivated to understand a topic and not merely concerned with passing an exam are more likely to become metacognitively aware in their specific domain of study.

Self-regulatory learning is therefore much broader and encompasses the concept of metacognition. In fact, some researchers argue that metacognition is a subcomponent of self-regulation (Yilmaz-Tuzun & Topcu, 2001). Many researchers use the terms metacognition, self-regulation and self-regulated learning interchangeably (Dinsmore, Alexander & Loughlin, 2008). I do not adopt this approach but agree with Kaplan (2008) and Schunk (2008) that metacognition and self-regulation are related sharing the similar core notion of regulatory action, but that they still are two different entities.

For me 'the knowing how to learn' links to the domain of metacognitive skills which is the focus in this thesis. Still, in light of the above discussion, I do acknowledge the role that motivation and attitude play in academic performance of students. These two factors are taken into account when discussing the findings and conclusions emanating from the analyses of the data and in answering the research questions.

Metacognition on its own, is not a key factor that ensures success in academic performance and learning. Even if a student has a high level of metacognitive awareness, but delays being an active responsible learner his academic performance may be negatively influenced. Veenman et al. (2004) postulate that good learners need to have good metacognitive skills, but also need to motivate themselves to learn. For me students first need to be aware of and have metacognitive skills, as well as be instructed in how to adopt and use these skills.

2.5 Metacognitive Interventions: Training and Development of Metacognition

A number of research studies have been conducted on metacognitive interventions in which students were trained to develop their metacognitive skills and/or knowledge. Most of these studies have shown that such training produces higher educational attainment; that metacognition can be learned; and that it improves students' learning (Mevarech & Fridkin, 2006; Michalski, Zion & Mevarech, 2007; Pressley & Gaskins, 2006). Empirical research has been conducted, using both quantitative and/or qualitative methods, on metacognitive interventions in science education (Adey & Shayer, 1993; Georghiades, 2000; Zohar & David 2008), mathematics (Desoete, Roeyers & De Clercq 2003; Kramarski, 2004; Mevarech & Kramarski, 1997; Mevarech & Fridkin, 2006; Mevarech & Kramarski, 2003; Teong, 2003), chemical engineering (Case & Gunstone, 2006) and teacher education (Kramarski, 2008).

Sungur and Senlar (2009) note that in classrooms and social-cultural environments students need to be encouraged to use metacognitive skills with the support of teachers, instructors and fellow peers. The authors recommend that a student's autonomy also needs to be encouraged, which may help them to adopt and use metacognitive skills. Fisher (2002) notes that students need to engage with each other in sharing their experiences of their thought processes. In this way students may become more metacognitively aware and knowledgeable about their own learning processes. Another factor that needs to be taken into account is that metacognitive skills do not necessarily develop naturally in students – students need to be instructed and guided in developing such skills (Kuhn & Dean, 2004). Similar arguments are also made in the mathematical field in which Desoete, Roeyers and De Clercq (2003) and Desoete (2007) argue that metacognitive skills need to be explicitly taught to enhance students' mathematical learning performance and that such skills do not necessarily develop spontaneously. A number of South African studies also argue that the instruction in and the design of engineering and science related courses need to take into account that students need to be trained in metacognition for them to take control of their own learning and monitor their own progress during problem solving (Grayson, 2010; Jacobs & de Bruin, 2010; Loji, 2010).

The Cognitive Acceleration in Science Education (CASE) program (Adey, 2002) is an example of a study that has shown the effectiveness of metacognitive interventions on learning. The aim of this study was to accelerate students' thinking skills. Findings of the study show that students' learning improved not only in science, but also across the curriculum (Adey & Shayer, 1993). The CASE intervention did not only take into account the use of metacognition. Students were also introduced and instructed how to use concepts and terminologies that they would encounter in problems, and thinking strategies that enable them to understand where their thinking could be applied in different situations. Furthermore, problems were designed to produce cognitive conflict within a student's existing schemas of thinking. In the field of teacher training, Gillies and Khan (2009) conducted research in metacognitive intervention with teachers. They focused on how teachers can promote metacognitive thinking among their pupils. The students who were trained in metacognitive questioning outperformed students who did not receive any intervention, and trained students gave more information in the reasoning and justification of their answers. Table 2.2 below illustrates similar research done on metacognitive interventions in science education.

Table 2.2: Metacognitive Interventions in Science Education

Author(s)	Subjects	Intervention in and training of	Findings
Georghiades (2004)	Primary (elementary) school pupils	Metacognitive skills	Students who underwent intervention had higher achievement levels in class tests (when problems were straightforward and required minimal metacognitive ability, no advantage was shown)
Zohar and David (2008)	Pupils of age 13 – 14	Metacognitive knowledge about general and explicit thinking strategies	Intervention group students improved more in strategic and non-strategic knowledge, in comparison to the control group. Low achieving students also improved their skills to the same level as to that of the high achieving students

Michalsky, Mevarech and Haibi (2009)	Primary (elementary) school pupils	Understanding of scientific texts	Students receiving intervention outperformed all other groups who received no training

2.5.1 Metacognitive Interventions in Mathematics

One of the most prominent examples of metacognitive instructional methods, used in the intervention phase of my research, is that of IMPROVE. The IMPROVE method encourages mathematical reasoning by training students in the use of certain questions (referred to as metacognitive questioning techniques) to enhance their metacognitive skills. Another important component of the IMPROVE method is the use of cooperative settings. During these settings students work together in groups in which they are encouraged to use metacognitive questioning techniques. Mevarech, Tabuk and Sinia (2006) note that each of the questions derive from the literature on metacognition (for example the work done by Schoenfeld, 1987) and each of these questions has a unique contribution to metacognitive processes and mathematical achievement (Mevarech, 1999). Moreover, it seems that metacognitive instruction has the potential to enhance students' metacognitive skills and their learning of mathematics. For example, Mevarech and Kramarski (1997) examined junior high school students under the instruction of the IMPROVE method over a whole year. Quantitative analyses of their findings showed that the IMPROVE students significantly outperformed students with no metacognitive instruction on various measures of mathematics achievement. Similar findings on the positive effect of IMPROVE on mathematical reasoning and/or problem solving are discussed in Kramarski, Mevarech and Lieberman (2001); Kramarski, Mevarech and Arami (2002); Mevarech (1999); Mevarech and Amrany (2008); Mevarech and Fridkin (2006); and Mevarech and Kramarski (2003). The above studies incorporate aspects that may play a role in developing students' mathematical reasoning and contribute to the development of their metacognitive skilfulness. These include cooperative settings in which students work together, multi- and uni-level metacognitive training, students working on authentic tasks, and developing students' mathematical reasoning by means of worked-out examples and the effects of such examples.

IMPROVE was developed by Mevarech and Kramarski (1997) as a method for the development and use of metacognitive skills amongst students to enhance their mathematical thinking. IMPROVE is an acronym that refers to a method of instruction consisting of the following steps: **I**ntroducing new concepts; **M**etacognitive questioning, **P**racticing; **R**eviewing and reducing difficulties; **O**btaining mastery; **V**erification; and **E**nrichment (ibid., p. 369). The IMPROVE method guides and trains students in formulating specific questions when solving mathematical tasks. Through these questions the students determine and plan the structure of the problem; determine and make connections between new and existing knowledge; determine and apply appropriate strategies and principles in solving the new problem; and justify and explain their solution (ibid., p. 368).

The above authors' work is based on and develops the research findings of Schoenfeld (1985, 1987) in which he proposed that metacognitive questioning may help in teaching/instructing college students to regulate and control their problem-solving performance in mathematics. Schoenfeld (1987) found that training students in asking themselves questions such as "what am I doing right now?", "why am I doing it?", and "how does it help me?" periodically during problem solving, improved their self-control and self-regulating behaviours.

The IMPROVE method consists of three components:

(a) assisting and training students in metacognitive processes and questioning;
(b) students learning and working in a cooperative setting (referred to as COOP); and
(c) provision of feedback-corrective-enrichment.

Although the IMPROVE method compromises the three components (a) – (c), the present research study uses components (a) and (b) for the following reasons: Component (a) was used in training students in the use of metacognitive questions. In my study component (b), COOP-settings were used as a a tool in which

(i) students can develop metacognitive skills (by means of metacognitive questioning of the IMPROVE method and interaction with each other and the researcher),

(ii) students can verbalise their thought patterns, and

(iii) I can investigate how individuals' metacognitive activities emerge and operate within a collaborative setting.

In my work I will not refer to COOP-settings, but to the term 'collaborative setting'. 'Collaborative setting' in the context of the present study refers to two students engaging with each other and with me the researcher.

Component (c) feedback-corrective-enrichment took account of the learning time of *each individual* student in the classroom (emphasis my own). Corrective/enrichment activities are designed to meet the special needs of specific individuals and/or groups in the classroom. The teacher keeps track of individual learners' performance, where students who performed poorly in tests and tasks are given extra corrective activities (Mevarech & Kramarski, 1997). Because of the huge number of students in the Calculus 2 course (at least 300 students), the vast amount of mathematical content that students need to cover in a very short space of time, and students' busy time-tables, component (c) of the IMPROVE-method was not used.

During a lesson in which IMPROVE is implemented, the teacher first introduces the new concepts to the whole class. Students then work in small heterogeneous groups, in which they take turns in asking and answering four kinds of metacognitive questions (as developed by Mevarech and Kramarski, 1997). These are:

1. comprehension questions,
2. strategic questions,
3. connection questions, and
4. reflective questions.

Comprehension questions orient the student in articulating the main ideas of the problem (e.g. the student needs to describe the problem in his own words); classify the problem into an appropriate category (e.g. the problem is a rate of change problem or a simplification problem in which one uses factorisation); and elaborate the new concepts (e.g. to elaborate on the definition and meaning of a certain concept or terminology, or recognising and identifying what is given and what is the unknown). **Strategic questions** refer to strategies appropriate for solving the problem as well as the reasons for using that particular strategy. Students will select a certain strategy, justify their choice of strategy, and describe its application to the given problem as well. **Connection questions** are asked with a view to identify the similarities and differences between the problem at hand and the problems the students have previously solved and why this is the case (Mevarech & Kramarski, 1997; p. 369 – 370). **Reflective questions** are used in referring to those concepts in which the student reflects on the process and solution of the problem during and after problem-solving (e.g. asking what went wrong, why did I make certain errors or did I make any errors, and does the solution make sense?) (Kramarski, Mevarech & Arami, 2002, p. 228; Mevarech & Kramarski, 1997; p. 369 – 370). All of the above questions were used in my study.

During the cooperative settings of IMPROVE, small groups of four to six students work together on mathematical problems as they verbalise their thought patterns to each other and use metacognitive questions. Mevarech and Kramarski (1997) argue that peer interaction provides students with the opportunity to articulate their thoughts as well as explain their mathematical reasoning to others. Moreover, these COOP-settings play a role in enhancing and improving metacognitive skills and learning performance in mathematics. Hurme et al. (2006, p. 182) mention that metacognition appears to be part of a collaborative learning situation and that metacognitive regulation can also be considered a group level activity rather than an individual performance. Dillenbourg and Traum (2006) argue that in collaborative settings, one can expect learning to occur when students work together and make their thinking visible by asking questions, providing explanations, and discussing their differing viewpoints.

Another study that considered metacognitive intervention within mathematics is that of Cardelle-Elawar (1992) in which students were encouraged to use strategies for learning as well as remembering specific terminologies within a mathematics lesson. In contrast to IMPROVE, students worked individually but did receive feedback from the teacher. Pre- and post-test scores showed a significant difference between that of the experimental and control group. The experimental group significantly outperformed the control group in the use of appropriate strategies.

From the above the reader would note that collaborative settings play a prominent role in the literature and in my own study. I now examine the concept of socially shared metacognition and how it operates at different levels within these settings.

2.6 Metacognition in Collaborative Problem Solving: Socially Shared Metacognition, Socially Mediated Metacognition and Multiple Levels of Metacognition

A number of research studies have focused on the role and use of collaborative settings and social interaction as a platform for learning. Socio-cultural views as discussed in Vygotsky (1978) suggest that learning occurs through the process of social interaction. Schraw, Crippen and Hartley (2006) note that children's metacognitive awareness may be enhanced during collaborative problem solving, since they need to explain their reasoning to their fellow peers, as well as assess and critique suggestions made by other peers. The role of peers and more knowledgeable others can play a role in mediating and sharing metacognitive knowledge (Brown et al., 1983; Paris & Winograd, 1990) and it has been suggested that metacognitive activity is mediated among participants (Goos et al., 2002). There is evidence that group work is effective (Slavin, 1990) and during successful collaboration students make their individual thinking visible and this consequently enables them to make productive metacognitive decisions (Artz & Armour-Thomas, 1992; Forman, 1989; Kieran, 2001). In a similar vein and as discussed earlier, Dillenbourg and Traum (2006) argue that in collaborative settings one can expect learning to occur when students make their thinking visible by asking questions, providing explanations, and discussing their differing viewpoints. Moreover, it has been noted that peers and other participants during collaboration act as external regulators of each other's cognitive processes (Azevedo, 2005), and that not only do students monitor and regulate their own activity during collaboration, they also monitor and control their fellow peers working in the group (Jermann, 2004). As mentioned earlier, Mevarech and Kramarski (1997) argue that peer interaction provides students with the opportunity to articulate their thoughts as well as explain their mathematical reasoning to others. Collaborative settings play a role in enhancing and improving metacognitive skills and learning performance in mathematics. A number of researchers have also argued that metacognition should form an integral part of group work (Vauras, Iiskala, Kajamies, Kinnunen & Lehtinen, 2003; Salonen, Vauras & Efklides, 2005).

Apart from these findings and views, a relatively new field has developed recently in which researchers consider metacognition in a collaborative setting as a "social practice". Researchers started to consider what they call either socially shared metacognition (Iiskala, Vauras & Lehtinen, 2004), or socially mediated metacognition (Goos et al., 2002), or even collective metacognition (Hogan, 2001). Consequently, research on metacognition in group settings led to the emergence of a number of different terms which consider the relationships between group work, learning and metacognitive activities leading to the conceptualisation and operationalisation of these relationships. Very little consensus has emerged on the role and definition of metacognition in collaborative settings because metacognition as a construct focuses on the individual. Still, it is important to take into account the concepts of socially shared metacognition and socially mediated metacognition in order to discuss how these constructs and characteristics apply and do not apply to my own research, as well as what viewpoint I adopt when considering metacognitive skills of students in the collaborative setting of my study.

Hurme, Merenluoto and Jarvela (2009) note that "socially shared metacognition emerges when a group member regulated a group's problem-solving process and the other members react to the initiative" (ibid., p. 503). What is important to note here is that with socially shared metacognition one group member's metacognitive act and reaction to another member's metacognitive act and reaction are not considered as separate from each other – the act and reaction are regarded together as one entity. Researchers such as Iiskala et al. (2004) and Vauras et al. (2003) argue that metacognition which emerges between peers in group work (referred to as inter-individual metacognition) is different to that of an individual's metacognition, and hence such inter-individual metacognition needs to be conceptualised differently. Social processes have mostly been regarded as variables that influence and/or facilitate the individual's learning and metacognition, but now a movement has emerged in which the above authors argue that not only is the individual's self-regulation of importance, but that social forms of regulation (co-regulation and other-regulation) also need to be considered in order to make sense of regulation in collaborative settings. The reason for this is that the group is seen as a social system (Vauras, Salonen & Kinnunen, 2008) which is in contrast with members working next to each other, or solving the problem on their own. Furthermore, Volet, Vauras and Salonen (2009) argue that social regulation cannot be reduced to an individual member's self-regulation.

Consequently researchers introduced the concept of socially shared metacognition (or shared regulation) which refers to "the consensual monitoring and regulation of joint cognitive processes in (demanding) collaborative problem-solving situations" (Iiskala et al., 2004; Iiskala, Vauras, Lehtinen & Salonen, 2010; Vauras et al., 2003). Thus socially shared metacognition implies that individuals' metacognitive processes operate together to form a social entity where students engage in a joint, co-equal process of solving the problem and in which the students share complementary monitoring and regulation over the task (Iiskala et al., 2004, p. 4). In the words of Iiskala, et al. (2010, p.11) socially shared metacognition emerges when

> the pupils were able to jointly monitor and regulate a cognitive process towards a common goal. The dyad's learning process proceeded through both pupils' regulatory involvement so that the pupils' reciprocal turns, focusing on the problem and the mutual activity, together affected the course of the process. Hence, the

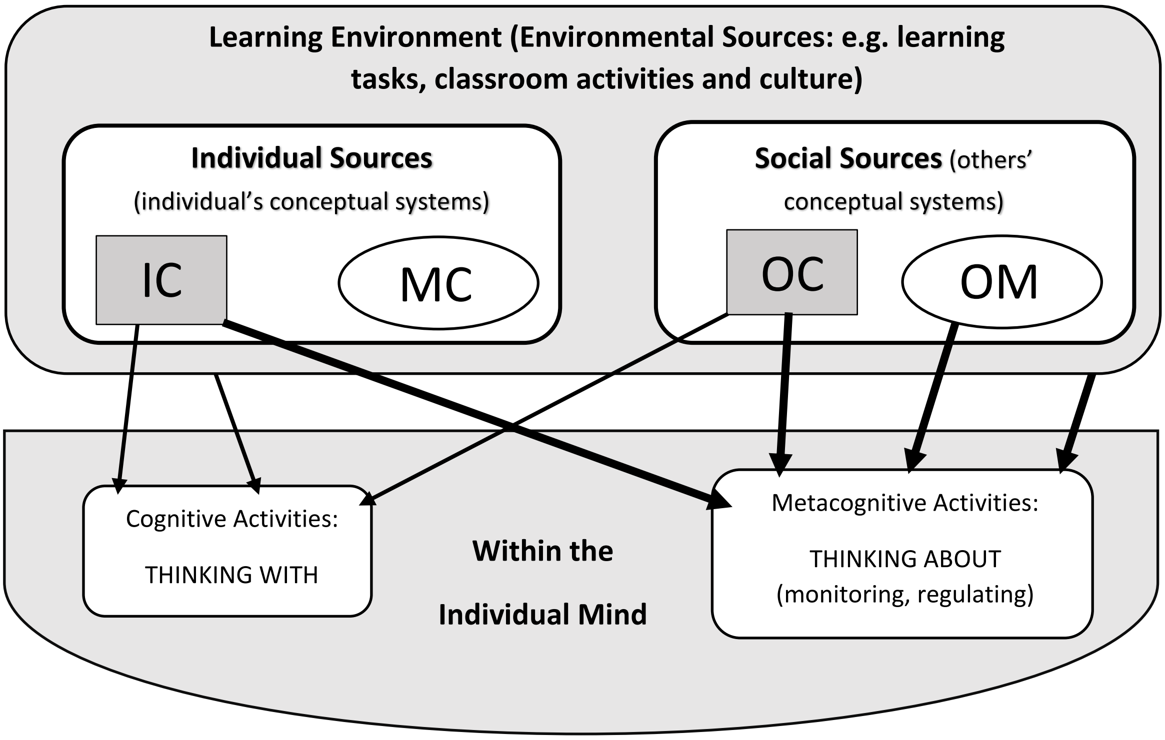
pupils' reciprocal turns together formed an entity which represents metacognitive regulation.

Kim, Park, Moore and Varma (2013) also explore how metacognitive activities operate and emerge during collaborative problem solving in mathematics. In contrast to above authors, Kim et al. (2013) disregard the notion of metacognition as a social process, as well as the notion of the group as a single, social entity. Instead, they consider social processes and interactions as contextual variables that influence and facilitate metacognitive processes of the individual. What remains is that "the main agent of metacognition is still (the) individual, regardless of whether the individual engages in collaborative teams or works independently. Metacognition itself is a mental process within an individual drawing on the individuals' conceptual systems" (Kim et al., 2013, p. 381). The authors' focus is on considering how an individual's metacognitive actions emerge and operate during collaborative problem solving. They consider what they termed 'the paradox of metacognition', in which they argue that although metacognition is a personal attribute of the individual, metacognition still cannot be explained solely in terms of individualistic conceptions. The above paradox essentially takes into account that a person is operating at a cognitive level, but at the same time also monitoring and/or controlling the cognitive operations, that is, the subject is also operating on a metacognitive level. A similar view was discussed in Section 2.3.2 and illustrated in Figure 2.1, explaining the relationship between the cognitive object-level and the metacognitive meta-level: cognitive activity focuses on how the meta-level is informed by the object-level, while the meta-level focuses on controlling the object-level but not vice versa (accordingly to the views of Nelson & Narens, 1990). Essentially, we have a higher-order mediator overlooking and governing the cognitive system, while simultaneously being part of it, which is a paradox. This paradox is very much related to Comte's paradox: one cannot split one's self in two, of whom one thinks whilst the other observes him thinking. As a result, one may ask the following: if an individual has difficulty overcoming a cognitive barrier, how is he/she able to overcome such a barrier by taking metacognitive control of the situation? Kim et al. (2013) notes similar conundrums in asking if a student reaches an obstacle during problem solving and the student has only access to his own internal thinking to help him to resolve the obstacle, how is it possible to make any progress if this same thinking caused the student to be stuck in the first place? Also, what happens in the case of a student lacking good self-regulation (metacognitive) skills? How will the student be able to realise if his own thinking is indeed correct or not when facing a situation where he is stuck? Moreover, will such a student be able to adjust his way of thinking? (ibid., p. 378).

The above authors propose that in order to solve this paradox of metacognition, one needs to consider not only the students' own internal psychological and cognitive resources (known as individual sources), but also take into account that there exist external resources which trigger metacognitive thinking. These external sources include (i) social triggers which are caused by the interaction between the individual and others; and (ii) environmental triggers such as specific types of mathematical activities, mathematical tasks of varying difficulty and complexity, and the classroom activities and culture (which are situated within the learning environment).

In shifting the focus back to the individual as sole agent of metacognition who has access to sources of metacognition at individual, social and environmental level, Kim et al. (2013) viewed metacognition in collaborative settings as *metacognition operating on three levels: individual, social and environmental* (emphasis my own). These three levels in essence refer to the three different sources or triggers that affect the individual's cognition and metacognition. Thus, instead of seeing the group as one single, social entity in which metacognition is regarded as a social process, Kim et al. (2013) regard metacognition as a construct that operates on multiple levels during collaborative work.

The above concepts are further illustrated in Figure 2.2 below, in which "thinking with cognitive components" refers to cognitive activities, and "thinking about cognitive components" to metacognitive activities (for e.g. monitoring, controlling, and regulating cognitive activities) (Kim et al., p. 379). The direction of the arrows indicates the influence of the triggers on cognition and metacognition.



IC = Individual Cognitive Components OC = Others' Cognitive Components

IM = Individual Metacognitive Components OM = Others' Metacognitive Components

Figure 2.2: Multiple Triggers of Individual Cognition and Metacognition (as adapted from Kim, Park, Moore & Varma, 2013, p. 379)

I adopt a similar view to that of Kim et al. (2013) in which I regard metacognition to reside within the individual, that is, the individual is still the sole agent of metacognition, but I also take into account that metacognition can operate on multiple levels during collaborative problem solving (according to the views of Kim et al., 2013). Moreover, the concepts of social and environmental triggers of metacognition in group work are for me similar to the findings of Goos et al. (2002) in which collaborative settings mediate metacognition amongst students. I have not used the term socially mediated cognition although I acknowledge the view that collaborative settings are capable of providing a platform in which students can regulate and control their own and others' mathematical thinking.

The construct of socially mediated metacognition was first noted in the work of Goos et al. (2002) in which they tried to uncover how collaborative problem solving could mediate metacognition and look beyond the individual; particularly the role students play in each other's metacognitive activities and development in solving mathematics problems. What was found was that students questioned their own and others' ideas, as well as monitored each other's cognitions during collaborative problem solving. Goos et al. (2002) concluded that "collaborative metacognitive activity proceeds through offering one's thoughts to others for inspection and acting as a critic of one's partner's thinking" (ibid., p. 207). The above research suggests that metacognition plays an important role in collaborative problem solving and that collaborative settings provide an environment in which students mediate each other's metacognition.

Hurme et al. (2006) investigated how students of different metacognitive awareness and activity levels may interact. Research was conducted with 13 year old students during a networked discussion in mathematics, in a computer supported collaborative setting. It was found that students with higher levels of metacognitive awareness and activity were more involved in the online discussions and solutions of the given problems. This suggested that students with a higher level of metacognitive awareness were more likely to interact during problem solving. Still, the study was only conducted over a time period of one week which made it difficult to determine if students with a lower metacognitive awareness would grow their awareness over a longer time while engaged with students with a higher metacognitive awareness.

A study by Sfard and Kieran (2001) has shown that this is not necessarily the case. The authors followed the collaborative problem solving between two students over a period of twenty months. Although the study did not take metacognition into account, it focused on the students' interactions when solving mathematical problems. They found that one boy was keener to solve the problem by himself. The other boy was more motivated to work collaboratively in solving the problem. Still, after working together over a period of twenty months, the boys' interactions did not change but remained the same as they were at the beginning of the research study. Findings like this may suggest that in the case of metacognitive activities and skills, metacognition does not naturally evolve to higher levels during collaborative settings, as well as that such settings do not always necessarily produce higher levels of metacognitive activity. As discussed earlier, students can be trained in the use of metacognitive skills as well as encouraged and motivated to use such skills in seeing the benefit of it in their academic performance. This concurs with Sfard and Kieran's (2001, p. 71) suggestion "if a conversation is to be effective and conducive to learning, the art of communication has to be taught".

Similar to the above view of Sfard and Kieran (2001), I argue that metacognitive skills form an integral part of mathematical activity; a factor that improves students' mathematical performance; and that these skills also form part of their mathematical activities. As mentioned earlier, in order for students to 'act' metacognitively, they need to be trained in the use of metacognitive skills, as well as be encouraged to adopt these skills. My belief is that metacognitive skills must be part of *'the art of communication that needs to be taught'* which links to my view that metacognitive skills are norms which students need to appropriate and develop in order to progress in their mathematical thinking and understanding. In Chapter 3, when discussing my epistemological and ontological views on mathematical learning, I elaborate on the concept of metacognitive skills as norms, as well as link it to the construct of socio-mathematical norms as discussed in the work of Yackel (2000, 2001) and Yackel and Cobb (1996).

2.7 Summary

There are a vast number of different definitions of metacognition, although metacognition is seen as second-order cognition and the most basic definition of metacognition is "thinking about thinking" or "cognition of cognition". Metacognition entails knowledge and awareness, as well as the monitoring and control over the flow of one's cognitive processing. Apart from a conceptual difference between cognition and metacognition, the distinction is not always that clear at the operational level. Still, there is a general agreement that metacognition consists of two different, but related components: metacognitive knowledge and metacognitive skills.

This thesis is concerned with students' metacognitive skills which takes into account the actual regulation, coordination and control of one's own learning activities and one's own cognitive processes. In mathematics such skills for example are task analysis, planning, monitoring, checking and reflection.

Research shows that metacognition helps learners control their own learning processes, and hence helps them to become autonomous. Moreover, it affects students' skills of acquisition, comprehension, as well as the ability to retain information and knowledge. It also has benefits for their learning efficiency and problem solving skills. There is also a great deal of evidence on the positive impact that metacognition has on students' learning and there is empirical evidence that metacognition is 'teachable'.

Within a collaborative setting, the individual differences in students' metacognitive ability influences and contributes to the group discussions. In particular, a new domain of research on metacognition within collaborative settings is that of socially shared metacognition. The emphasis in socially shared metacognition is on one group member's metacognitive reaction to another member's metacognitive act in which the act and reaction are regarded together as one entity and not considered as separate from each other. Researchers in the domain of socially shared metacognition argue that metacognition which emerges between peers in a group works differently to that of an individual's metacognition. With socially shared metacognition the emphasis is more on the social forms of regulation within collaborative settings.

Within this thesis, and in contrast to the above, social processes are regarded as variables that influence and/or facilitate the individual's learning and metacognition. The focus is still on the individual as sole agent of metacognition who has access to sources of metacognition at individual, social and environmental level. These sources can be seen as triggers that affect the individual's cognition and metacognition within a collaborative setting.

Chapter 3: Research Paradigm and Theoretical Framework

3.1 Introduction

This chapter begins in discussing the research paradigms involved in my study; that of a constructivist and interpretivist paradigm. With this, I expand on the relations between these two paradigms, my ontological and epistemological perspectives, as well as the applied theoretical framework of my study. The latter frames my perspective on teaching and learning of mathematics. The theoretical framework takes into account a number of theories, in particular theories from symbolic interactionism and the notion of socio-mathematical norms. These play a crucial role in the emergence of metacognitive skills in mathematical learning and activities, as well as the interrelated relationship between the IMPROVE metacognitive questions and that of socio-mathematical norms.

3.2 Research Paradigm

The purpose of this study was to investigate the metacognitive skills that students exhibit during collaborative problem solving in order to gain a better understanding and more insight into the possible differences in students' metacognitive skills. The study takes note of possible triggers of students' metacognitive skills which relate to metacognition as it operates at multiple levels.

During collaborative problem solving the possible levels[7] at which metacognitive skills may operate are:

1. Individual level: At this level the student is regarded as a single entity where metacognition resides within the individual. The focus is on the internal cognitive and metacognitive structures of the individual.
2. Social level: This level is characterised by a shift of the focus to peers and/or other more knowledgeable participants[8] who interact with the

[7] These different levels, as based on the work of Kim et al. (2013), were discussed in Chapter 2.
[8] In this study each student in the dyad was a potential source for triggering the other student's metacognitive behaviour. My role as researcher in interacting with the students was also seen as a social trigger. The main agent of metacognition, however, is still the individual.

student and who act as triggers for the individual's metacognitive behaviour[9].

3. Environmental level: The use and training of metacognitive questioning techniques of IMPROVE[10], and the tasks used for data collection and their level of difficulty are all triggers of metacognition at an environmental level.

It is evident from the above that both internal and external factors play a role in the execution of metacognitive skills, which also influence the appropriation of these skills. In viewing metacognitive skills as part of the cognitive toolbox (as discussed in Chapter 2 and used in the metaphor of Veenman, 2006), I regard these skills as part of a student's 'knowledge' base contained within their cognitive toolbox. The belief system that informs my view on how students appropriate metacognitive skills is discussed below.

The concept of a belief system links to the term paradigm. Researchers work within specific paradigms that describe or determine how the researcher perceives knowledge; how knowledge can be made known, investigated, gained or uncovered; and the nature of knowledge. Guba and Lincoln (1995) define 'research paradigm' as "the basic belief system or worldview that guides an investigation" (ibid., p. 105). In this study knowledge acquisition is considered within a constructivist paradigm in order to discuss how students' metacognitive behaviours may stem from and develop within collaborative settings.

The idea of constructivism is that through interaction, individuals construct or create their social understandings and meaning of reality (Savin-Baden & Howell Major, 2013). Knowledge is not simply transmitted but regarded as an internal construction within the individual. One of the primary tenets of constructivism is that the individual ascribes meanings to personal experiences and ideas by interacting with others in the same environment. Moreover, in a constructivist

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[9] As noted before, the terms metacognitive skills, metacognitive activities and metacognitive behaviour(s) are used interchangeably in this work.

[10] The IMPROVE method was discussed in Chapter 2, Section 2.5.1. IMPROVE is an acronym that refers to a method of instruction consisting of the following steps: Introducing new concepts; Metacognitive questioning, Practicing; Reviewing and reducing difficulties; Obtaining mastery; Verification; and Enrichment (Mevarech & Kramarski, 1997; p. 369).

paradigm truth and knowledge are regarded as intertwined. They cannot be separated; subjectivity and objectivity are therefore integrated (Savin-Baden & Howell Major, 2013). Furthermore, the individual tests and modifies these conceptual systems during new experiences (Schwandt, 2000). Consequently, researchers who adhere to the constructivist paradigm will investigate and collect data on how individuals construct and gain knowledge. The reason for this is that constructivists argue that the only thing they may come to know, is people's construction of their knowledge and realities (Jonassen, 1991). In my research I regard the 'knowledge' that students gain and make their own as the mathematical concepts and practices, as well as the appropriation and use of metacognitive skills.

Denzin and Lincoln (2011) note that any research paradigm takes into account the central concepts of epistemology and ontology. *Ontology* is concerned with the perceptions we have about the world and what exists in the world. In essence ontology is concerned with the nature of reality and knowledge. It takes into account such questions as 'What is real and what can be known about it?' (Savin-Baden & Howell Major, 2013). Within a constructivism paradigm, ontology is relativist (subjective) – one where reality is constructed and made up of our thoughts, rather than being static and unalterable (objective).

Epistemology is the theoretical perspective on knowledge (Crotty, 2003); it is the philosophy of knowledge and how it can be known (Honderich, 1995). Epistemology is one's view on knowledge: what it is; how it is obtained, and the relationship between the knower and the known (Savin-Baden & Howell Major, 2013). In a constructivist paradigm knowledge is constructed through dialogue and negotiation of meanings. Through interaction between elements, ourselves and others, meanings are created, felt and understood. From this, a subjective knowledge materialises (Crotty, 2003).

The above epistemological and ontological characteristics of constructivism correspond with some of the tenets of interpretivism. Interpretivism concerns itself with the measuring of behaviours. For the interpretivist researcher to understand the situation under study, the different understandings, experiences, and meanings of the individual need to be taken into account (Pring, 2000). Apart from the many and different complex situations which interact to form individuals' beliefs and behaviours, these beliefs and behaviours are still moulded by common generalised concepts, views and/or beliefs. An example of such a generalised concept is language (Pring, 2000). In my study I regard mathematics as a special form of language; a discourse with specific rules, ways of thinking (either tacit or explicit) and ways of writing in which students participate. In order for students to participate in this discourse they need to form a shared understanding of how to function within the discourse. The road that leads to this shared understanding starts from different points for each individual. Pring (2000) is of the view that the role of an interpretivist researcher is to investigate and/or understand these different points, as well as its influence on the construction of shared meaning and understanding.

Another characteristic of the interpretivist paradigm is that the understandings of such constructions and their origins cannot be reduced to basic, simplistic interpretations or summaries (Cohen, Manion & Morrison, 2007). Thick descriptions are needed in order to describe the complexity of an investigated situation. This concept of 'thick description' links to the applied research methodology used in my work – that of the case study which is discussed in Chapter 4.

3.3 Theoretical Framework: Teaching and Learning of Mathematics

Theoretical frameworks in educational research provide reference systems with which to view teaching and learning that are linked to specific ontological and epistemological views. Hodkinson and Macleod (2010) note that our perception of learning will influence the kind of research and the methods of conducting it. The theoretical framework should therefore be consistent with one's ontological and epistemological views.

Building on the above discussion of interpretivism I now turn to my assumptions on the teaching and learning of mathematics, and extend the discussion to the use and emergence of metacognitive skills in mathematical learning and activities. My assumptions are guided by the above paradigms and a number of theories. The theories include symbolic interactionism (Bauersfeld, 1992, 1993; Yackel, 2000, 2001; and Yackel & Cobb, 1996); the notion of socio-mathematical norms (Yackel, 2000, 2001; and Yackel & Cobb, 1996) as well as the views of Schoenfeld (1985, 1987, 1992); and the perspectives of David and da Penha Lopes (2005) and Sfard (1998, 2001, 2008). Before engaging with these theories I look at what mathematical learning and mathematical activity entail.

3.3.1 Mathematical Activity, Mathematicians and Students as Mathematists

I follow a train of thought similar to that of David and da Penha Lopes (2005) and Schoenfeld (1992) who see students' activity in mathematics as reflecting the behaviour and activities of mathematicians. The word *mathematician* is used here in the sense of the professional or research mathematician who solves mathematical problems by means of arguments within the unique and specialised discourse of mathematics. Such arguments use particular methods and steps of justification to obtain the solution to a mathematical problem.

In order for students to develop mathematical thinking and activity similar to that of mathematicians, students need to participate or be socialised into the modes of thinking and activities similar to that of mathematicians (Schoenfeld, 1992)[11]. I believe students only need to mirror the actions of mathematicians within the confines of their classroom community's knowledge and the level of mathematics they are being taught. This view is similar to Sfard's (2008) notion of *mathematist,* where students (school and undergraduate) are regarded as mathematists as they *participate* in mathematical discourse (emphasis my own). Again, the word *mathematician* is used exclusively to refer to professional or research mathematicians (ibid., pp. 128, 299).

[11] This is not to imply that students need to be trained to be at the same level of expertise and competency as that of (research) mathematicians (I do not believe this is always possible).

The following summarises my view on student engagement in mathematical activity/thinking during the solving of a mathematical task/problem, with the work of David and da Penha Lopes (2005, p. 18) as a strong point of departure:

(1) The student needs to be challenged by a problem to which the solution is not obvious.
(2) The student must be able to find and implement mathematically sound, logical and acceptable arguments or proofs; as well as justify and obtain a solution.
(3) Such an argument needs to be described, represented and communicated through the use of mathematical symbols.

Regarding the first point, some mathematical problems/tasks may have canonical solutions in which students already know how to solve the problem. Although these problems form a valuable and significant foundation in developing students' mathematical thinking, they do not always lead to the goal of developing student autonomy. The problems I use in my study do not always have standard, procedural solutions. Students need to analyse and explore the problem, and consider various strategies to solve the problem at hand. In point (2) the student needs to use 'proofs' to justify his solution. Proof here is used in a much broader sense than is traditionally meant in comparing it to the work of research mathematicians. By proof is meant a set of mathematical steps which logically demonstrate and explain how the student has reached a valid solution. Justification is synonymous with proof in this case. Furthermore, the student must be able to ground working steps in previously known work and mathematical facts. In order to achieve the above the student must be able to use mathematical symbols as in point (3). Such mathematical symbolism forms only a part of mathematical discourse. In order to develop students' mathematical thinking they need to be instructed how to participate in the discourse. This requires the use of the symbolic language of mathematics in an appropriate manner, as well as the translation of words or events into mathematical symbols and the ascribing of meaning to mathematical symbols (David & da Penha Lopes, 2005). Such a mathematical discourse compromises the use of words and/or symbols which (i) refer to quantities, shapes, numbers and (ii) operations on numbers and geometrical shapes and notions (Sfard, 2008). Having discussed what I regard as mathematical thinking I now consider processes that entail how students learn to act in a way similar to that of mathematicians.

3.3.2 Students' Learning of Mathematics

In mathematics education research there have been two major perspectives on how students learn: constructivist and sociocultural (Cobb, 1994). Since the 1990's there has been debates on which perspective is the most appropriate. The two perspectives have one main epistemological difference: the cognitive constructivist perspective (which should not be confused with a constructivist paradigm) views learning as taking place within the individual's mind, while the sociocultural view adheres to the tenet that learning and the gaining of knowledge is derived from social situations (Alexander, 2007; Greeno, Collins & Resnick, 1996). Sfard (1998) states that this difference in learning within education research can be explained by means of two metaphors on learning: acquisition and participation.

The acquisition perspective, which is related to the cognitive constructivist perspective, posits that learning and knowledge refer to something within the individual's mind that can be constructed, acquired and developed (Piaget, 1978). The acquisition metaphor, goes further in noting that knowledge exists outside the mind and learning entails the manner in which that knowledge is placed into the mind (Sfard, 1998). Although Piaget (1978) noted that social interactions are of importance in learning, it still occurs at an individual level. Sfard (1998) notes that Piaget regarded the individual as the unit of analysis. Hence the emphasis is not on the role of social interactions, but on the individual constructing knowledge because of these interactions.

The participation metaphor, which is related to sociocultural perspectives on learning, emphasises learning as participation in social and cultural activities (Lipponen, 2002; Vosniadau, 2007). Knowledge is created through participation in activities; that is, learning, is situated in the environment in which it is created. This knowledge is only relevant to the context of the learner's doing or being

(Sfard, 2008). Sociocultural theory, as developed by Vygotsky (1986), focuses on how social interaction plays a key role in the development of the individual student's skills and other learning attributes. Braten (1991) proposes that in Vygotsky's theory, metacognition can be regarded as an internalised skill which allows the individual to gain control and have access to internal representations of concepts. Thus from a Vygotskian perspective metacognition can be seen as a tool (skill) that can be developed during social interactions. I agree with this view of metacognition as a tool which may possibly emerge during and be developed through social interaction, although I also acknowledge the acquisition metaphor that metacognition and other intellectual abilities reside within the individual. This concurs with Kim et al.'s (2013) view that the sole agent of metacognition is still the individual (as discussed in Chapter 2, Section 2.6).

According to Vygotsky's theory, learning occurs through verbal interactions and an attempt to reach consensus. But what does such a consensus entail? I argue that such a consensus takes into account and/or is reach through negotiations of meanings and explaining one's reasoning to others. But where does this consensus originate and how does it tie in with metacognitive skills? In order to address these questions I use the theories of symbolic interactionism to further the notion of such negotiations where I regard metacognitive skills as a specific norm which students need to adopt while engaging in mathematical activities.

3.3.3 Symbolic Interactionism, Social Norms, Socio-mathematical Norms and their Relations to Metacognitive Skills in Mathematics

David and da Penha Lopes (2005) argue that metacognitive skills are a necessary condition in developing students' mathematical thinking. The authors argue that if students are to mirror the behaviour of mathematicians it needs to be remembered that being a mathematician entails more than just having mathematical content knowledge and manipulating mathematical symbols. Schoenfeld (1992) believes that metacognitive skills form part of the normal activity of mathematicians since mathematicians[12] are conscious of the processes of *'how to do'* mathematics and *when* it is appropriate to use such processes (emphasis my own).

By building on the above views, I suggest the following:

(a) Metacognitive behaviour (skills) are needed to develop students' mathematical thinking, and
(b) in order for students to mirror behaviour similar to that of mathematicians, students need to engage in metacognitive skills (strategies).

As noted before, I use the phrases 'metacognitive behaviour' and 'metacognitive skills' interchangeably. From the literature, metacognitive behaviour has a double role: (i) as a tool for developing mathematical thinking and (ii) as a manifestation of mathematical thinking/activity. In my work, the emphasis is strongly in regarding metacognitive skills as a tool for developing mathematical thinking. Besides this, I am of the view that such skills form part of mathematical discourse. With respect to the latter, I regard the use of metacognitive skills as a norm that students must adhere to in order to be part of the mathematical discourse. The question remains how do students develop such metacognitive skills and what processes are involved in developing such skills? I noted earlier that by using a Vygotskian perspective, metacognition can be developed through social interactions. Similar ideas were discussed in Chapter 2: collaborative settings serve as platforms for developing metacognition, and constitute a space in which metacognition can be mediated. I now supplement the above foundational perspectives with constructs from symbolic interactionism and the concept of socio-mathematical norms to address the above question. Moreover, I consider the link between socio-mathematical norms and metacognitive skills, and specifically how the metacognitive questions of the IMPROVE method are related to socio-mathematical norms.

Symbolic interactionism is a theoretical perspective which I use as a basis for describing how students' metacognitive behaviour and mathematical thinking are

[12] Again, the word "mathematician" is used here to refer to professional or research mathematicians.

‘formed’ or developed. Yackel (2000, 2001) and Yackel and Cobb (1996) use the tenets of symbolic interactionism (which I discuss below) in developing the notion of socio-mathematical norms in order to analyse and talk about students’ mathematical activity. Moreover, they illustrate how certain socio-mathematical norms regulate mathematical argumentation, explanation and justification and how these norms influence both students’ and teachers’ development of the mathematical discourse and culture during classroom interaction.

Symbolic interactionism emphasises the individual’s cognitive development as well as the cultural and social processes in which mathematical activity takes place (Yackel, 2000, 2001; Yackel & Cobb, 1996). That is, the individual’s learning is formed within the social setting, but in a reflexive manner whereby the individual also influences and creates the social setting. Thus, mathematical learning is a process of active construction from the individuals’ side as well as an acculturation to the practices and discourse of the mathematics culture (Yackel & Cobb, 1996, p. 460). The student’s actions are formed, in part, as he changes, abandons or revises his plans according to the actions of others. Thus social interaction is a process that moulds human conduct (Yackel, 2000, 2001). Social interaction in a mathematics community/culture is a means by which the student appropriates mathematical discourse. Moreover, the essence of a culture, *the core of the culture*, is not only the (content) knowledge which the culture carries with it, but learning *when* to do what and *how* to do it (emphasis my own). Having knowledge of the mathematical content is not sufficient if the student cannot identify in which situations to use this knowledge appropriately. The student needs to participate in the culture of the mathematics classroom in order to appropriate the core of the culture for himself (Bauersfeld, 1993, p. 4, as cited in Yackel & Cobb, 1996, p. 459). As noted above, the core of the culture is: *when* to do and *how* to, which is related to metacognitive skills (Schoenfeld 1985, 1987).

The emphasis on *symbolic* interactionism refers to the fact that during interaction there is the interpretation of others’ actions. Attempts to genuinely communicate involve understanding the meanings of another’s actions, hence *symbolic* interactionism. Also while interpreting the actions of others, individuals engaged in interaction attempt to indicate to others, through their actions, what their own intentions are. Thus, actions have meanings both for the person making them and for the person(s) to whom the action is directed (Yackel, 2001).

The above idea links to the second defining principle of symbolic interactionism: meaning is a social product. Meaning arises during the process of interaction between people; that is, meaning is not only constructed and formed by the individual, but through the actions of people as they interact (Yackel, 2000). So, if meanings are formed during social interaction, how do students develop the same understandings of what (appropriate) metacognitive behaviour is in a mathematics community? Remembering that symbolic interactionism places equal emphasis on both the individual and the social setting the individual constructs certain personal meanings/understandings from the setting whilst the setting is influenced by the meaning(s) of individual(s) – hence a reflexive relationship (similar views are discussed in Yackel & Cobb, 1996, pp. 459 – 460). As the individual forms these understandings, it also happens that normative understandings are constituted amongst students and teacher. Yackel (2000, 2001) argues that as these normative understandings develop, the individual also develops his own personal understandings; hence the individual's interpretations and understandings become compatible with that of the teacher and his fellow students. That is, interpretations, meanings and understandings become 'taken-as-shared', hence the use of the term normative and norm (evidence and illustrations of such taken-as-shared meanings are discussed in Yackel, 2000, 2001 and Yackel & Cobb, 1996).

Norms are social constructs which are understandings or interpretations that become normative or taken-as-shared by a cultural group (Yackel, 2000, 2001). Hence norms are a collective rather than individual notion. Yackel (2000, 2001) and Yackel and Cobb (1996) through their research on episodes of classroom interactions, note that these norms are socially constructed by explicit and implicit negotiations between teacher and students, and between the students themselves. Moreover, these norms are continuously modified by the students and the teacher during interaction.

In my research, these norms or understandings form part of metacognitive skills. The lecturer will introduce and encourage students to adopt metacognitive skills by using metacognitive questioning of the IMPROVE method. Thus as the lecturer uses metacognitive questioning he is modelling the norm of asking and engaging in such questioning – a norm which is implicitly linked to the behaviour of mathematicians. By demonstrating the use of metacognitive questioning (action), the lecturer has the intention of making students aware (meaning) of the usefulness of such questioning in order to develop mathematical thinking, as well as metacognitive skilfulness. Also, as students work in collaboration asking metacognitive questions and monitoring their own progress, other students may appropriate similar actions. In both of the above cases, the individual will ascribe meaning to the actions and behaviour of others. Hence a student may use similar metacognitive questions and skills as he interacts with the lecturer and his peers. During such participation, the individual develops personal understandings of and assigns meaning to the core of the mathematics culture: metacognitive skills (as adapted from Yackel & Cobb, 1996, p. 460). As discussed in the Literature Review, there is some evidence that such behaviour can be explicitly learned/adopted, as illustrated and evidenced in the IMPROVE studies; or be 'learned' indirectly as the individual participates in metacognitive questioning and self-reflection which is implicitly part of the classroom culture (Blumer, 1993, as cited in Yackel & Cobb, 1996, p. 459).

It must be noted that the above norms are not unique since metacognitive questioning, as well as justification and explanation are social norms that appear in other subjects apart from mathematics (Yackel & Cobb, 1996). In contrast, the metacognitive questioning strategy of the IMPROVE method is designed specifically for the purposes of regulating and improving students' mathematical thinking and learning performance. Thus I will refer to such metacognitive questioning as a socio-mathematical norm.

Yackel (2000, 2001) and Yackel and Cobb (1996) use the notion of socio-mathematical norms in describing those norms that are specific to mathematics. In this manner they differentiate between social norms that can be observed in other subject fields and norms specific to the mathematics culture. During their research on classroom interactions they determined the 'characteristics' of such socio-mathematical norms. In agreement with the tenets of symbolic interactionism, the authors were able to discern how socio-mathematical norms were constructed during classroom interactions. Yackel and Cobb (1996, p. 461) state that

> Normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant… are socio-mathematical norms.

To clear this subtle distinction between social norms and socio-mathematical norms they also give some examples (p. 461):

> The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a socio- mathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a socio-mathematical norm.

A closer look into the metacognitive questioning technique of IMPROVE implicitly involves the notions of explanation and justification, as well as seeking different mathematical solutions. I will regard these metacognitive questions as questions that may lead to the development of other socio-mathematical norms such as acceptable mathematical explanations or justifications. Moreover, these metacognitive questions are also regarded as desirable socio-mathematical norms that are to be instilled during formal lectures. Metacognitive skills therefore qualify as a form of socio-mathematical norms in this study.

Socio-mathematical norms as used according to the understanding of Yackel (2000, 2001) and Yackel and Cobb (1996) are not a set of fixed or pre-defined norms that already 'exist' within a classroom, but rather evolve and develop during interaction as well as in the negotiation of meanings during such interactions. Moreover, with respect to the present study, since the primary source of data will come from observations outside of the lecture environment I have to take into account that students' use of the above metacognitive questioning norms outside formal lecture time may be different from the intended metacognitive questioning during formal lectures. That is, students' enacted norms during these observations may be different from the intended socio-mathematical norms of the lecture. I will look to see if students appropriate these intended socio-mathematical norms and 'carry them over' to the observations outside formal lecture times, and what socio-mathematical norms may develop during the observations. Table 3.1 below illustrates the interrelated relationship between the IMPROVE metacognitive questions and that of socio-mathematical norms:

Table 3.1: Relationship between Metacognitive Questions and Socio-mathematical Norms (as based on and adapted from Kramarski, Mevarech & Armani, 2002; Mevarech & Kramarski, 1997; Yackel, 2000, 2001; and Yackel and Cobb, 1996)

Metacognitive Question from IMPROVE	Related Socio-mathematical Norm
Comprehension Questions	Student needs to articulate the main mathematical ideas of the problem; classify the problem into an appropriate mathematical category; elaborate new concepts; and understand the meaning of concepts.
Strategic Questions	Student needs to refer and select strategies appropriate to solving the problem; justify his decision(s); and explain the application of his chosen strategy.
Connection Questions	Student needs to compare the differences and similarities of the problem to that of other problems; and/or compare different solutions for the same problem.
Reflection Questions	Student needs to reflect on the processes and the solution: for example where did he go wrong; does his solution make sense when he explains to others?

Students cannot always be left on their own to devise mathematical ways and processes of gaining knowledge which are compatible with that of the wider mathematics society (Cobb, Yackel & Wood, 1992). Thus the role of the teacher in the classroom is that of being a representative of the mathematics community in introducing and engaging students in the use of metacognitive behaviour and socio-mathematical norms (Yackel & Cobb, 1996), where the establishment of such norms in the classroom may either be in an explicit or implicit way. The role of the teacher is to guide students in acculturation of the mathematical discourse. I suggest that the quality of constructing mathematical dispositions and mathematical ways of knowing is important. The quality and nature of such constructions lie in students' implementation and development of metacognitive skills as well as in the use and/or development of metacognitive questioning techniques. Moreover, I believe that these questions and skills can be used as tools to pave the way of students' acculturation into the mathematical discourse.

An example of the lecturer's role can be seen in a lesson on differential equations at university level, as described in Yackel (2000, 2001). The lecturer began the class by stating his expectations of the students. He then slowly engaged them in the use of socio-mathematical norms. Yackel observed that the lecturer gave explicit attention to the negotiation of the acceptable use of these norms in the classroom. As the lecture progressed, students contributed to the meaning and negotiation of these norms, according to the lecturer's expectations. Eventually it became routine for students to respond to questions by justifying and explaining their solutions and reasoning in a spontaneous way, without any prompting from the lecturer.

3.4 Summary

This chapter discussed the paradigms in which the research is situated; that of a constructivist and interpretivist theory. These paradigms are used to position the views on what is the nature of knowledge and how it is acquired. Within this study, the 'knowledge' that students acquire and make their own encompasses mathematical concepts and practices as well as the appropriation and use of metacognitive skills. According to a constructivist paradigm knowledge is created through dialogue and negotiation of meaning. Also, research conducted within an interpretivist paradigm explores the different understandings, experiences, and meanings of the individual. Still, the individual's beliefs and behaviours are moulded by common generalised concepts, views and/or beliefs. Mathematics can be regarded as such generalised concept or as a discourse (the term used in this thesis). This discourse includes specific rules, ways of thinking and ways of writing in which students participate. Moreover, the students also need to form a shared understanding of how to function within the mathematical discourse.

The ontological and epistemological perspectives of the above two paradigms are linked to the theoretical framework of the study which focuses on the researcher's perception of teaching in and learning of mathematics. In particular, the framework regards metacognitive skills (behaviour) as a tool to develop students' mathematical thinking; and suggests that students need to engage in metacognitive skills (strategies) in order to mirror behaviour similar to that of mathematicians and to participate in the mathematical discourse.

This chapter also discussed the acquisition and participation metaphor on how students learn. The metaphors respectively focus on the individual's attributes and social factors that influence and develop the individual's learning. These two metaphors in conjunction with the tenets of symbolic interactionism, were used to argue that metacognitive skills form part of the mathematical discourse. Furthermore, the use of these skills is seen as a norm that students must adhere to in order to be part of the mathematical discourse. The chapter concludes on how symbolic interactionism is used as a framework to guide the researcher's arguments and views that metacognition can be developed through social interactions such as collaborative settings that serve as platforms for developing and mediating metacognition.

Chapter 4: Research Methodology and Methods of Research

4.1 Introduction

The chapter starts in giving an outline on the research setting and the sampling of participants by means of the techniques of theoretical and random sampling. Although four student pairs were observed, only one pair was used in the research project to answer the research questions. A brief summary is also given on the ethical considerations within the study.

A case is made for a qualitative research approach on metacognition and how such an approach is used in my study. With this I discuss case study methodology and its characteristics as linked and applied to my own research.

The chapter then concludes with a discussion on how data was collected by means of observations and talk-aloud protocols, and an argument is made in favour of the important role of tasks used in the study and the difficulty of tasks as environmental triggers of metacognitive skills.

4.2 Research Setting and Participants

The research was conducted at a South African university, in the first semester of a second year course (referred to as Calculus 2 course in my study). The course dealt with the theory and problems of sequences and series, as well as vector calculus. The research setting comprised of one student pair as they worked together in solving mathematical problems over the course of the semester. This study may apply and be similar to other student pairs (or groups) within the above course. However, since this is a small qualitative exploratory project, it is not my intention as researcher to make such claims. Rather it is hoped that any results and theory generated from this study could be used by other researchers in other and/or similar contexts, as a starting point to develop further research in the investigated topic.

Theoretical sampling, also referred to as purposeful sampling, was used in choosing participants of the study. This sampling technique entails "selecting groups or categories to study on the basis of their relevance to one's research questions" (Mason, 1996, pp. 93 – 94). In the research study sampling of students took into account two possibly inter-related characteristics of each student in the pair: (a) whether the student was an extended or main stream degree student; and (b) the mathematical performance of the student (based on the marks they obtained in their first year calculus course). This decision was based on the belief that students' metacognitive skills may be linked to

(i) the student's prior academic experiences (which is reflected in the degree for which that student is enrolled: main stream or extended), and

(ii) his/her level of past mathematical performance in his/her first year calculus course.

The sampling procedure evolved over three different stages.

Stage 1: Students from the extended and main stream degree who enrolled for the Calculus 2 course were invited to partake in the research study. In particular, I focused on inviting students who obtained between 50% and 60%, as well as 70% and above in their final first year calculus examinations. The students who obtained marks in the above ranges were invited to a group meeting held at the beginning of the first semester of the Calculus 2 course. The number of students invited to the meeting was 234. During the meeting students were informed about the study. Participation and consent forms were handed out at the end of the meeting. After the meeting 121 students (51.71% of the 234 students) handed in their consent forms in which they voluntarily agreed to participate in the research study.

Stage 2: In stage 2, the 121 consent forms were scrutinised in order to determine which students qualified in terms of the required mark range. Table 4.1 below summarises the number of extended and main stream students who were then considered to form part of the study.

Table 4.1: Stage 2 of Sampling of Students

Students	Mark Range	Number of students	How many students chosen to form part of the study after stage 2
Extended	E1: 72 – 87% E2: 57 – 60% E3: 50 – 56%	7 7 6	7 students from range E1 and 6 students from range E3
Main stream	MS1: 75 – 92% MS2: 70 – 72% MS3: 56 – 60% MS4: 52 – 55% MS5: 50 – 51%	13 5 34 15 10	18 students from range MS1 and MS2 and 10 students from range MS5

Stage 3: Students who were chosen to form part of the study were contacted in order to confirm their participation in the study. Although extended degree students in range E7 were contacted to form part of the study, the main focus was still in forming student pairs that consisted of one extended degree student from E3 and one main stream student from ranges MS1 and MS2. This particular pairing of students were done in order to obtain student pairs consisting of a typical extended degree student with a very low mathematical performance, and a typical main stream student of a high mathematical performance. Above criteria in the forming of the student pairs, was led by the hypothesis that main stream students who have a high mathematical performance will most likely exhibit more metacognitive skills compared to low performing extended degree students.

Only four extended degree students from range E3 voluntarily agreed to form part of the research study. Four main stream students were then chosen randomly from the volunteering students in ranges MS1 and MS2. These four pairs were then observed over the course of the first semester. The problems used during the observations focused on questions on sequences and series.

At the start of the first semester students were divided into different classes with two different lecturers. This division was based on the students' decision on which lecturer's class they wanted to attend. In the first few weeks of the course, ‘traditional’ methods of instruction were implemented during formal lecture times. During this time, the different lecturers were using problems and followed lesson plans of their own choice. After about six to seven weeks into the Calculus 2 course the IMPROVE method was implemented during formal lecture times by both the lecturers. I refer to these lectures as IMPROVE lectures. All students who attended the IMPROVE lectures were instructed in the use of the metacognitive questioning technique as advocated by the IMPROVE method. Students and lecturers all used the same textbook and solved exactly the same problems during the IMPROVE lectures. The problems were designed by me as researcher. Furthermore, lecturers were trained by me in the use of the metacognitive techniques of the IMPROVE method.

In a ‘traditional’ lecture, the lecturer mostly stands in front of the class and writes notes on the board and/or makes use of the document camera and projector. During an IMPROVE lecture, the lecturer went further in modelling the form of metacognitive questioning that he/she wanted the students to emulate. Also during such a lecture, students were encouraged to work collaboratively as well as to engage in metacognitive questioning (both features of the IMPROVE method). While students worked in small groups, the lecturers were observing students’ questioning, as well as answering and helping students where needed. At the end of the lecture, the lecturer reviewed the new concepts and solved problems by modelling metacognitive skills and metacognitive questioning techniques at the same time. During IMPROVE lectures students were not assigned to work together in pairs. Students only worked as pairs if they volunteered to be observed as research participants outside of formal lectures. During IMPROVE lectures students were free to choose which peers they wanted to work with during problem solving. The reason for the above is that during the IMPROVE lectures the researcher wanted the students to interact with their chosen peers in a ‘natural’ setting in which they felt comfortable, in order to maximise the possibility for them to appropriate the metacognitive questioning techniques. Assigning students to specific groups or pairs during formal IMPROVE lectures may have inhibited metacognitive engagement, placed unnecessary strain on the students and/or contributed to student drop-out rates from the research study.

Apart from observing the student pairs outside their formal lecture time, the researcher also observed the IMPROVE lecturers. This was done in order to observe how the lecturers implemented the IMPROVE method. No field notes were taken while observing the IMPROVE lecturers.

Observations of research participant pairs commenced after three weeks into the semester and continued throughout the 14 week semester. Each student pair was observed twice before the IMPROVE method had been implemented during formal lectures. Another two observations took place after the IMPROVE method had been implemented by all the lecturers. Below Table 4.2 outlines the dates (according to week) of the observations and the length of the IMPROVE training.

Table 4.2: Timetable of Observation Dates and IMPROVE Training

Date	Observations and IMPROVE training
Week 4	Observation 1
Week 5	Observation 2
Week 7	IMPROVE implemented
Week 8	IMPROVE implemented
Week 10	Observation 3
Week 11	Observation 4

All verbal activities and non-verbal gestures of the students were video recorded during observations, as well as students' written solutions to the problems. Thereafter all observations were transcribed. Sections of the written solutions were included in the transcriptions where needed in order to make sense of the students' problem solving process. The data of one pair is the focus of the analysis since this pair demonstrated a higher frequency of metacognitive skills in comparison to the other student pairs. This student pair also consisted of a main stream student of a high mathematical performance and an extended degree student of a low mathematical performance. Moreover, and as noted previously, this specific student pair was chosen since the data obtained from observations of their collaborative problem solving gave a rich, in-depth account of their metacognitive behaviour. The other three student pairs' minimal contributions and the low frequency of observed metacognitive skills, meant the data was not suitable for a fine-grained analysis. Accordingly this data does not form part of the final research findings, and thus the results of this study are only concerned with the one student pair. By following Yin's (2003, p. 48) recommendation, the one student pair was selected as a representative or typical case in order to find (generate) new hypotheses and gain a deeper understanding of extended vs main stream degree students' metacognitive skills, that previous theory may have missed and/or not adequately explained. Moreover, the selection of this one student pair was chosen as an 'information rich' case that could address the research problem sufficiently. Furthermore, I agree with Gerring (2007, p. 40) in that "case studies may be more useful than cross-case studies when a subject is being encountered for the first time or is being considered in a fundamentally new way". Although there is a number of research investigations on students' metacognitive skills, this study is concerned with less understood phenomena, for example, researching metacognition as it operates on multiple levels to determine possible differences in tertiary mathematics students' metacognitive skills operating at these levels.

4.3 Ethics and Ethical Approval

Permission and ethical approval were granted by both the university at which the research study was undertaken (University of Johannesburg) and the institution where the researcher was enrolled for his PhD (University of the Witwatersrand). The research was subject to the rigorous ethical procedures employed by the University of the Witwatersrand. The research study was reviewed and approved by the University of the Witwatersrand's School of Education's Ethics Committee.

The purpose of the study was made clear to the lecturers and students who participated in the study. Students were informed that no extra work was required of them, other than being observed and video recorded while solving mathematics problems. As mentioned before, informed consent was sought from all participants. It was made clear that agreement to participate was voluntary, and would have no impact on assessment of the students' performance during the course or evaluation at the end of the course. Participants were also informed that they could withdraw from the study at any point during the data collection. It was also explained that any data used in the research would be done so anonymously and no identifying information about any student would be used. Pseudonyms were assigned to all participants of the study.

4.4 Research Approach and Methodology

In this section I make an argument why a qualitative research approach on metacognition is beneficial when researching a less understood phenomena such as metacognition as it operates on multiple levels. This is followed by a discussion on the tenets of a case study methodology as applied to my own research.

4.4.1 A Qualitative Research Approach on Metacognition

Creswell (2003) notes that when choosing a research approach, one needs to ensure that this approach ties in with the problem under investigation. A clear outline of the three main research approaches, namely, qualitative, quantitative and mixed method is given in Creswell (2009). In my work a qualitative approach was used in determining what differences there are in students' metacognitive skills, as these skills operate on three different levels; individual, social and environmental. Since my study is qualitative in nature, I will only note the underpinnings of this approach as used in my research and outlined in Table 4.3 and the discussion below. Terms placed in brackets are strategies, assumptions, methods etc. which were not used within my study, but which Creswell (2009) notes form part of a qualitative research approach.

Table 4.3: Characteristics of a Qualitative Research Approach (as adapted from Creswell, 2009)

Qualitative research approach characteristics	
Philosophical assumptions	Constructivist knowledge claims (advocacy/participatory knowledge claims)
Methodology: strategy of enquiry employed	Case Study (phenomenology, grounded theory, ethnography, and narrative)
Methods employed	Open-ended questions in conjunction with text and image data (emerging approaches)
Role of researcher in the study	Positions him/herself Collects participant meaning Focus on a single concept/phenomenon Studies the context and/or settings of the participant Makes interpretations of the data Brings in personal values into the study (Validates the accuracy of the findings, creates an agenda for change or reform, collaborates with the participants)

A case study methodology was used in my research, which I will discuss later in greater detail in Section 4.4.2. When considering Table 4.3 on the methods used in my study, students were video recorded during observations (image data was gathered) and then all verbal and non-verbal actions were transcribed, where this 'text' data was the main data source for answering my research questions.

My role as researcher also played an important and significant part in the study. By asking students questions during the observations, I tried to 'guide' the students on how to solve the problems when such guidance was needed. As noted previously, it was never my intention to guide and/or assist students during the observations, but students lack of contribution and/or incorrect contributions led to such intervention. In engaging with the students during observations, I formed part of the collaborative setting. I was not a detached observer (an outsider of the group). My role was 'observer as participant' (Cohen et al., 2007). In this position, I was primarily an observer of the student dyad. My role as participant was kept to a minimum. I only intervened when (a) I believed that the students' problem solving process was at risk; and (b) I needed clarification on students' behaviour and actions during observations. With this, I tried to 'capture' students' personal ways and means of interaction during collaborative problem solving. Moreover, as participant I was able to observe how students' metacognitive behaviour operated on a social level where the researcher/observer was a social trigger of students' metacognitive engagement. The researcher as the primary data collector was aware of subjective interpretations that could play a prominent role especially in analysing data. Some would argue that I participated on a level similar to that of a teacher/lecturer. This is not the case, although I do acknowledge that the students may have viewed me as an extra helper, almost like an assistant. Apart from how 'much' I did or did not form part of the participation, it still raises an important issue of reactivity effects in which students' behaviour may have changed due to my presence (Cohen et al., 2007). For example, students may have been less attentive in solving the problems since I was not their official lecturer. Another example is that during the observations I noticed that there were instances in which students started preparing for the next observation in order not to be seen as irresponsible or not dedicated to their work. These issues may have influenced students' metacognitive behaviour.

The above notion of 'subjective interpretations' is just one of the significant issues in undertaking research in metacognition. Data analysis into a 'fuzzy' concept such as metacognition tends to be very subjective. Georghiades (2004, p. 378) notes that "both identifying and 'measuring' metacognition rely on a researcher's subjective interpretation in assessing what is cognitive, and what is metacognitive". An example of this can be seen in Artz and Armour-Thomas (1992) in which the authors tried to obtain a clear distinction between cognition and metacognition at an operational level. Subjectivity in data analysis is also linked to another issue in research on metacognition: the significant number of diverse, different understandings and definitions of what metacognition entails (as discussed in my Literature Review). Such disparities in understanding metacognition and theories on how it is measured play a significant role in the analysis of data. With such a lack of clarity and/or agreement on what metacognition entails, it leads to different interpretations of data. Moreover, operationalisation of metacognition will most likely vary between different researchers and research approaches. Another issue in undertaking research in metacognition is that of gathering data. Trying to 'capture' a meta-physical entity such as metacognition relies heavily on the participants' ability to express their thoughts during problem solving. When the student is not able to word (express) his/her thoughts it may possibly give an incomplete picture of what metacognitive skills a student possesses.

Given the above issues in qualitative research in metacognition, I still believe that such an approach is of value to the research field. A great body of research into metacognition has been done in a quantitative or empirical way but these do not always consider the explorative possibilities that qualitative research offers (Pressley et al., 1998; Schraw, 2000). I agree with Creswell (2003) who argues that

> if a concept or phenomenon needs to be understood because little research has been done on it, then it merits a qualitative approach. Qualitative research is exploratory and useful when the researcher does not know the important variables to examine (ibid., p. 22).

Although it is not correct to say that 'little' research has been done on metacognition, it is still a fairly young field. The absence of a unifying theory of what metacognition entails, as well as the breach between theory and educational practice makes it difficult to discern, understand and isolate the variables within metacognition and their relationships to each other. Because of such difficulties I agree with Creswell (2003) that a qualitative research design is warranted. Qualitative approaches allow for examination of less understood phenomena, for example, researching metacognition as it operates on multiple levels to determine possible differences in tertiary mathematics students' metacognitive skills operating at these levels. An explorative qualitative approach may encourage a research foundation in the above areas which are somewhat incomplete or inconclusive. Further support for a qualitative approach in research on metacognition is seen in Pressley (2000) who notes that "qualitative analysis of complex cognitive and metacognitive processes makes a great deal of sense before attempting quantitative analyses of these processes" (p. 261). Nuckels et al. (2008) also expand on the potential benefits of a qualitative research approach on metacognition. Efklides and Misailidi (2010) support the view of moving more towards qualitative approaches in research on metacognition noting that "these developments promise a bright future for metacognition research, owing particularly to the development of new methodologies [exploratory qualitative methodologies] which allow deeper insight into the nature of metacognitive phenomena" (p. 1). For me such a richer examination is possible by means of a qualitative approach. Moreover, the above notions of 'richer examination' and 'deeper insight' are closely linked to the characteristics of case study methodology. Case studies are concerned with rich, in-depth descriptions and discussions, as well as chronological narratives of events (Cohen et al., 2007, p. 253). I now discuss my choice of a case study methodology.

4.4.2 Case Study Methodology

A number of different methodologies are used within the qualitative and quantitative research domains. Some of the prominent qualitative research methodologies are pragmatic research, grounded theory, ethnography, phenomenology, narrative and collaborative approaches, action research, and evaluations (Savin-Baden & Howell Major, 2013). Of these approaches, that of the case study was chosen for my research purposes. One of the main reasons for my choice is that, as the name suggests, a case refers to a specific instance of a phenomenon. A case study (by definition) focuses on a specific entity or unit of analysis, which is referred to as the case. Such a case is intrinsically bounded, or demarcated. To be a case, it needs to be bounded; otherwise it is not a case (Merriam, 2009). The one student pair which this study reports on formed my bounded system.

Other defining characteristics of case study methodology which apply to my research (as discussed in Hitchcock & Hughes, 1995) is that case studies:

(a) are concerned with rich, vivid descriptions of the events of the investigated case;
(b) take into account the sequential descriptions of the events which are relevant to the case;
(c) combine not only descriptions of events, but also the analysis of them;
(d) place importance on specific events that are relevant to the case; and
(e) regard the researcher as an integral part of the case.

The principal method of data collection was the observation of the student pairs during talk-aloud protocols. Observations were done outside formal lecture times. During the observations students were continuously asked and urged to verbalise their thoughts and engage with each other. Merriam (2009) argues (citing Bromley, 1986, p. 23) that case studies by means of observation have an advantage over other qualitative research designs since the observer can

> get as close to the subject of interest as (they) possibly can, partly by means of direct observations in natural settings, partly by their access to subjective factors (such as thoughts…), whereas experiments and surveys often use convenient derivative data, e.g. test results and official records.

Thus the method of research used in the present research study can also be referred to as an observational case (Bogdan & Biklen, 2007).

Students were video recorded during all observations in order to have a permanent/fixed record as reference data for the analysis and coding of students' metacognitive skills and mathematical activities. Having audio-visual material can overcome the researcher's subjective views if he/she only focuses on specific and/or frequent events. Erickson (1992, pp. 209 – 210, as cited in Cohen et al., 2007, p. 407) argues that

> audio-visual data collection has the capacity for completeness of analysis and comprehensiveness of material, reducing the dependence on prior interpretations by the researcher.

Video recordings were used in my study not only to capture students' verbal and non-verbal behaviours but also students' written solutions of the problems for the sake of completeness. It is precisely this completeness of analysis and the wide net of material/data needed in giving a 'thick' description of the observations, as a case study is characterised as a thick, rich and a complete description of the investigated entities (Merriam, 2009, p. 43). Robson (2002) also notes that one of the aims of a case study is to provide a rich description of the *real* life situation under investigation (emphasis my own). Although my student pair was observed outside formal lecture time, the students did solve problems with fellow peers during formal lecture and tutorial times too.

As noted earlier, I have referred to this case study as an observational case study. In fact there are a number of different types of case studies. Yin (2003) classifies case studies in terms of their *outcomes*. Such outcomes can be explorative, descriptive or explanatory. *Explanatory* case studies are generally used to test theories. *Descriptive* case studies provide a narrative in which a detailed account of the subject of study is given. *Exploratory* case studies may possibly serve as a pilot study to generate hypotheses in order to be used and tested in larger surveys or experiments. Moreover, explorative case studies are also used to gain an initial and/or better insight into a subject not well understood (Yin, 1993).

Merriam (1988) also considers three types of case studies which are similar to that of Yin (2003). The *descriptive* case study has the purpose of giving a narrative account of the studied phenomenon. *Interpretative* case studies are almost similar to that of the exploratory case study, but are mostly used to develop concepts which may be used to confirm hypotheses. Merriam (1998) also notes that interpretive case studies focus on analysing and interpreting the situation and hence generating theories. Here we have a move from merely describing the situation to the development of a set of concepts or theories that can help to explain the situation. *Evaluative* case studies are used when explaining situations. Yin (1993) notes that evaluative case studies are used to judge the merits of worth of the subject of study. With evaluative case studies the focus shifts from the description to judgement of the subject/case.

Three other types of case studies are also mentioned in Stake (1994) which are intrinsic, instrumental and collective. *Intrinsic* case studies focus on the understanding of the particular case under study. The *instrumenta*l case study considers a particular case in order to provide better insight into a certain theory or theories. This type of case study is used to support the researcher's understanding and exploration of the case, helping in confirming or refining an existing theory. With *collective* case studies, a number of smaller, individual case studies are combined in order to provide a more complete understanding of a situation.

I believe that my case study does not fit into one specific category but rather relates to a number of the above mentioned types. My case study has been used in a *descriptive* sense in order to provide a narrative of the situation; a thick, rich and in-depth description of the investigated subjects and events. Since I was investigating a situation that is not well understood, my case study can also be seen as *explorative* in which I tried to gain more insight into the relationships between the triggers of metacognitive behaviour and the metacognitive skills each student exhibits. In this sense, my case study can also be regarded as *instrumental* since it may provide more insight into metacognition in collaborative situations, in particular how metacognition operates on different levels during such situations. Results of this investigation may reveal how metacognition operates at different levels, or contribute to the existing literature of metacognition in collaborative settings.

Apart from a case study methodology being used in providing a rich, in-depth account of naturally occurring situations, it still has its pitfalls. As mentioned before, with case studies the researcher is closely involved in the research situation. This links to my previous discussion of the impact that the researcher's subjective views can have on the interpretation of the data, especially in considering what instances and behaviours of students can be regarded as either cognitive or metacognitive. Nisbet and Watt (1984, as cited in Cohen et al., 2007, pp. 256 – 257) note that since case studies are not that easily open for cross-checking, this may lead to the studies being selective and biased because of the researcher's personal and subjective views. Despite trying to address reflexivity within the researched case, bias of the researcher as observer can still affect the results, especially when the researcher is selective on what information needs to form part of the study. Another issue concerning case study research is that it is rarely generalizable. If a case study is carried out in a very specific situation it creates difficulties in generating generalisations. Because of the dependence on a single case and the boundedness of the case, such a generalisation can lead to a simplistic and incorrect worldview (Savin-Baden & Howell Major, 2013).

4.5 Data Collection Tools: Observations and Think-aloud Protocols

Most of the research done on metacognition in the education and the educational psychology field focuses on empirical studies (Schraw & Impara, 2000). Assessment of metacognition can be either measured by means of off-line or on-line methods. Off-line methods are used either before or after a task, while on-line methods examine processes and activities as they occur during the actual task (Van Hout-Wolters, 2000). These methods are also classified as prospective or predictive and as pre-task and during-task assessments respectively. Methods used *after* a task are referred to as retrospective (Veenman, 2005). A number of different tools are used in collecting data on students' metacognition. Amongst these, closed questionnaires are mostly used in assessing students' strategy use and metacognitive skills.

In the literature, the three main methods used to assess metacognition are questionnaires, interviews and inventories. Assessment of metacognitive knowledge can be either through interviews (e.g. Zimmerman & Martinez-Pons, 1990) and self-report questionnaires like the Knowledge Monitoring Assessment (KMA) (Tobias and Everson, 2000). When measuring metacognitive regulation (monitoring and control) examples of assessment include the Motivated Strategies for Learning Questionnaire (MSLQ) (Pintrich, Smith, Garcia & McKeachie, 1993) and the Learning and Study Strategy Inventory (LASSI) (Weinstein, Schulte & Palmer, 1987). Other methods include thinking-aloud protocols, eye movements, computer registrations of activities, as well as self-explanations, note taking, and stimulated recall. Desoete and Veenman (2006) list and discuss a great number of the above and other techniques, also illustrating the use of multi-method designs of metacognitive research. They also note the importance of using the appropriate measurement techniques in assessing a particular metacognitive component. Veenman (2005) argues that in order to advance methodological solutions, more multi-method designs are needed in understanding the number of diverse assessment techniques.

Retrospective methods, such as self-report measures (questionnaires) of metacognition, are used to get the student to reflect on his metacognitive activity after task-completion (Everson, Hartman, Tobias & Gourgey, 1991; Koch & Eckstein, 1995; Pintrich et al., 1993; Schraw & Dennison, 1994). A problem encountered in these methods is that the participants struggle to recall their cognitive processes in retrospect or they are not always aware of how their cognitive processing relates to their thought products (Garner & Alexander, 1989). Such problems can hinder the validity of the data, as metacognitive activities are assessed *after or before* the thinking process rather than *during* the process (emphasis my own). In fact, any measurement that requires participants to recall their cognition before or after task completion has the potential of giving an incomplete picture of the actual thinking process (Ku & Ho, 2010).

In my research study the use of observations with think-aloud protocols (also referred to as Verbal Protocol Analysis, VPA or talk-aloud protocol analysis) as an online assessment, were used in collecting information on what metacognitive skills the students exhibited. As noted before, all spoken words and all possible physical actions of the students were recorded. Although time-consuming, think-aloud protocols for assessing metacognitive skills are generally regarded as the most accurate and have drawn positive comment (Desoete, 2007; Focant, Grégoire & Desoete, 2006; Ku & Ho, 2010). Compared to other verbal reports, asking participants to verbalise their thoughts during a task reveals thinking processes more directly (Ericsson & Simon, 1993). A similar view is seen in Pressley (2000, p. 291) in which he notes that some researchers state that think-aloud protocols "offers a much more direct window on the processing than other forms of comprehension measurement", thus we have Ku and Ho (2010) argue that it is one of the most direct methods in gaining insight into the knowledge and methods of human problem-solving, and that it offers a way of accessing rich information that is unattainable through other means.

Concerns have been raised regarding whether the process of thinking would be interrupted, altered or incompletely reviewed under the think-aloud procedures (Garner & Alexander, 1989; Nisbett & Wilson, 1977). Ward and Traweek (1993) also note that talk-aloud protocols as a data collection method may not be able to access deeper level cognitive processes since these reports take place at a conscious level. Such issues raise concerns in terms of reliability and validity of the data, as well as any interpretations that can be drawn from the data.

Ericsson and Simon (1993) argue that verbalising one's thought does not alter the course of thinking nor affect the nature of on-going cognitive activities since the procedures do not involve participants interpreting their own thinking. Hence, think-aloud protocols are considered reliable because thinking aloud takes place almost simultaneously with the thinking process, which allows thinking activities to be closely followed while keeping the risk of losing information minimal (Schellings, Aarnoutse & van Leeuwe, 2006). Bannert and Mengelkamp (2008) have found that students' learning performance was not affected by being asked to think-aloud and similar results were found by Veenman, Elshout and Groen (1993). Apart from the above, it is still of great importance that the participating students should not analyse their thinking and learning in trying to interpret or reflect on what they are thinking and doing. Such reflection and understanding can take place during talk-aloud protocols. Hence, concurrent talk-aloud protocols (or concurrent VPA) was used in my study, where students merely verbalise their thoughts. In simply doing the task and verbalising their thoughts, there are no disruptions of the natural progression and sequence of the student's thoughts. In this way the think-aloud protocol is not a reactive tool and may avoid possible changes in the student's thought processes. I do agree that reactive tools are beneficial when trying to understand and uncover thought processes, but for the purpose of considering metacognition as it exactly occurs, a non-reactive form was needed and used in the present study. Still, one should not overlook the natural and spontaneous regulating effect that think-aloud protocols will inevitable have, resulting in alterations, changes and modification of the course of thinking.

Although talk-aloud (think-aloud) protocols mostly consist of one student at a time, in my study talk-aloud protocols consider student pairs as mentioned before. Schoenfeld (1985) argues that metacognitive skills are much easier to observe in group problem solving and that if one wants to elucidate decision making behaviour, two-person protocols may be the most appropriate. It must be stressed that although students were working in pairs, the focus was still on each individual (as much as possible). I consider the use of the social settings as a 'tool' in order to uncover the metacognitive skills that students exhibit. This second point also links to the view of metacognition as it operates on a social level, and that the students and the researcher are possible social triggers of metacognitive behaviour.

4.6 Tasks and Task Difficulty

Although tasks are mainly used as a platform to stimulate students' mathematical activities and thought processes, the additional factors of the type and level of difficulty of a task play a significant role when investigating metacognition. As noted in Chapter 2, the type and level of difficulty of tasks are regarded as environmental triggers of metacognitive behaviour. Each of the activities involved a different focus of problem solving as determined by specific questions contained within a task, which in turn directly affects the focus of metacognition (similar arguments can be seen in Lesh, Lester & Hjalmarson, 2003 and Stacey, 1992). I hold the view that the interactions within the learning environment can be seen as potentially stimulating and developing students' metacognitive ability. It is through these interactions that students may possibly become aware of their misconceptions and repair them through metacognitive activities that operate at an individual and/or social level.

Lesh et al. (2003) note that problems at different levels of conceptual and cognitive demand within problem solving processes can produce different metacognitive behaviours within problem solvers. So the complexity (or difficulty) of a task is an important variable that contributes to the elicitation of metacognition. Metacognition is triggered more when solving difficult problems and research has suggested that metacognition tends to emerge more frequently in difficult versus easy tasks (Helms-Lorenz & Jacobse, 2008; Iiskala et al., 2004, 2011; Prins, Veenman & Elshout, 2006; Vauras et al., 2003). Kim et al. (2013) argue that tasks that do not require high-order thinking do not always encourage and elicit metacognitive behaviour. According to them, task complexity involves both the conceptual and cognitive demands of a task. Efklides (2006, p. 6) notes that cognitive demands are more related to the context of the task, while conceptual demands of a task are "a function of one's developmental level and/or of domain-specific knowledge" and hence draw on the individual's conceptual systems. Iiskala et al. (2011) classify the complexity of tasks in terms of their processing complexity: how many steps are involved to get to a goal state, for example, one-step versus four-step addition, subtraction, multiplication and division problems.

In my study I considered task complexity in terms of conceptual demands on students and the processing complexity within each task (as advocated by the above authors). Tasks mainly focussed on the concepts of convergence and divergence of sequences and series and other mathematical concepts related to it. In order for students to solve problems on sequences and series they must have a good domain-specific knowledge base of the definitions and properties of convergence and divergence; the difference in meaning of convergence and divergence for that of a series and sequence respectively; the diverse number of tests (strategies) used in determining convergence/divergence; how these tests are related to similar concepts and techniques of differentiation and integration; how integration and differentiation are used in determining convergence and divergence of sequences and series; as well as what it means for a series of terms in a given variable to converge to a function in terms of that variable. These are just a few examples of what conceptual demands there are on a student's knowledge base, as well as prior knowledge gained in their first year calculus course on differentiation and integration that are used in problems on sequences and series. The view of Iiskala et al. (2011) relating to the processing complexity of a task is also a feature of the present study. Problems which focus on determining the convergence/divergence of series and sequences involve a number of steps, apart from the different ways to determine such convergence/divergence. More specifically, problems that focus on the construction of a Taylor or Maclaurin series[13] expansion for certain functions also entail a number of steps related to the domain-specific knowledge of sequences and series. Strategies in calculating derivatives and integrals, integration techniques and the use of factorial notation[14] contribute to the level of complexity of the task.

Solving problems on sequences and series is therefore quite dependent on students' knowledge of numerous concepts as well as properties of sequences and series. Prins et al. (2006) showed that metacognitive skills are activated by advanced learners in complex tasks, but in order for this to happen students still need to operate within the boundaries of their knowledge. Prins et al. (2006) note that while metacognitive skills may not be activated during easy tasks, such a phenomenon equally holds true when students are faced with extremely complicated tasks. Although the tasks used in my study are challenging, they were still similar to those the students were likely to meet in most of their mathematics lectures, as well as problems they had to solve outside formal lecture time. My reason for this selection of tasks is that extremely complicated tasks may just confuse the student and therefore not activate the student's metacognitive skills. The study focused on determining and qualitatively describing what metacognitive strategies two students in a pair used. It did not investigate the connection between task difficulty and metacognitive skills. Tasks were used to illustrate how students engage with mathematical problems and served as an environmental trigger for eliciting metacognitive behaviour in a collaborative setting. This is in accordance with Goos et al. (2002) and Volet and Mansfield (2007) who claim that task features and task difficulty both play a role in influencing group processes and group performance.

[13] The concepts of Taylor and Maclaurin series are discussed and dealt with in Chapter 6.
[14] The concept of factorial notation is discussed and dealt with in Chapter 6.

4.7 Summary

In this chapter I have outlined how theoretical and random sampling techniques were mainly used in the sampling of participants and how the sampling procedure evolved over three stages. In particular, the sampling of students took into account whether a student was an extended or main stream degree student; and the mathematical performance of the student. This decision was based on the researcher's belief that students' metacognitive skills may be linked to the above two criteria. Four student pairs were observed and video recorded twice before the IMPROVE method was implemented and twice after this teaching intervention. The study resulted in only considering one student pair's collaborative problem solving. An overview of the role of the researcher was also provided with an emphasis on the researcher as an integral part of the study.

Detailed information on a qualitative research approach was discussed and why such an approach is of benefit to an explorative study when researching less understood domains in the field of metacognition. I also discussed how the research is situated within a case study methodology in order to give rich, in-depth descriptions of the investigated student pair (the case), and sequential descriptions of the events that are relevant to the case.

An in-depth overview was given on the merit of talk-aloud protocols as data collection tool in order to give complete accounts of the investigated case. The chapter concluded with a discussion on the type and level of difficulty of tasks as additional factors that need to be taken into account when investigating metacognition. In particular, the type and difficulty of tasks are regarded as environmental triggers of metacognitive behaviour in my research study.

Chapter 5: Analytical Framework, Data Analysis and Coding Procedures

5.1 Introduction

In qualitative research, particularly in the interpretative research tradition, researchers have been paying attention towards developing new analytical frameworks to guide their data generation techniques and its analysis. In this chapter I discuss the construction and design of my own analytical framework as based on and adapted from a number of different researchers' work. The design of the framework was developed through the construction of a set of codes organised into categories. These codes and categories were used in the managing of and organisation of the data. Moreover, the framework created a new structure in guiding the analysis of data in answering the research questions.

In this chapter I will refer to my analytical framework as a taxonomy. The design of the taxonomy evolved in two developing phases. These two phases are discussed in depth in which I outline how the work of Artzt and Armour-Thomas (1992) and Meijer, Veenman and van Hout-Wolters (2006) played an integral part in the first phase; in particular in the designing of categories and the operationalising of metacognitive skills. The second phase largely had to do with the refinement of indicators as guided by the work of Goos (1994).

Furthermore, I also discuss what codes were used in indicating instances in which the researcher acted as a social trigger for students' metacognitive behaviour[15]. The chapter concludes with a discussion on the two most prominent types of codes that were used in the analysis of data. These codes are referred to as metacognitive decision points which indicate points in time where students exemplified metacognitive behaviour. Examples, as taken from the transcripts, of each of the codes are discussed in Appendix A.

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[15] As mentioned in previous chapters, the terms metacognitive skills, metacognitive behaviour and metacognitive activities are used interchangeably.

5.2 Coding and Content Analysis of Data

Coding is not a simple linear process and it is therefore difficult to represent as a chronological process. This section provides some insight on how the coding process developed in my study, but more importantly how codes were 'generated' as adaptations from other coding taxonomies. The taxonomy was applied in content analysis and in the coding and analyses of data. Content analysis in its most simple form concerns the process of summarising and reporting written data (Cohen et al., 2007). The method of content analysis is primarily concerned with the reduction, as well as the quantification of qualitative data through the use of a predefined coding system. Although predefined taxonomies are generally used, this study uses a taxonomy developed from the data. The textual data is derived from the transcriptions of students' verbal and non-verbal activities from the video data (Neuendorf, 2002). According to Krippendorp (2004, p. 30) texts can be seen as any written materials which are intended to be read, interpreted and understood by other people, apart from the analysts. Apart from the coding of such texts, Weber (1990) notes that content analysis is a research method that entails systematic procedures in order to make valid inferences from text. Thus, content analysis as a method is used to analyse texts, as well as condense and cross-examine them in summary form using a predefined taxonomy. Cohen et al. (2007, p. 476) outlines that content analysis at its core

> involves coding, categorising (creating meaningful categories in which the units of analysis – words, phrases, sentences etc. – can be placed), comparing (categories and making links between them) and concluding – drawing theoretical conclusions from the texts.

The design of content analysis is not always straightforward. A number of issues need to be addressed before data collection, in particular the development of a predefined taxonomy. Cohen et al. (2007) note that content analysis does not only make use of a predefined coding scheme, but also emergent categories (codes). The taxonomy for this study was developed alongside the collection of data and was continuously revised and modified, also during analysis of data. The reason for this is that existing taxonomies and codes of other researchers' contained metacognitive activities which did not apply to my study. Only selected codes from other taxonomies were used in my work. In some cases, codes were adjusted to fit my data. Later in this chapter, I give an in-depth rationale for my choices, as well as explanations and justifications of how my taxonomy was constructed from other researchers' work.

Some researchers distinguish between quantitative and qualitative content analysis, while others argue that content analysis in essence can only be quantitative. According to Neuendorf (2002) content analysis is not a qualitative form of analysis since it is concerned with producing counts and frequencies of categories and measurements. In contrast to this, Hurme et al. (2006) used content analysis as a means of quantifying qualitative material and they refer to the use of 'qualitative content analysis' in their work. Their research involved the coding of data which emerged during network discussions between students. By using predefined categories, frequencies of these categories were then produced as data. Strijbos, Martens, Prins and Jochems (2006) also regard content analysis as either being quantitative or qualitative. Quantitative analysis uses the coding and summarising of data to produce frequencies and percentages for statistical testing. With quantitative or prospective analysis, hypotheses are generated from theory which has been used. Qualitative content analysis on the other hand also employs the method of coding and the production of frequencies without statistical testing.

In my study, qualitative content analysis was used. No hypotheses were generated and no statistical testing was used. According to Strijbos et al. (2006) the purpose of such analysis is to understand a specific phenomenon. This aligns with a case study methodology which was used in trying to understand and uncover what metacognitive skills students exhibited during collaborative problem solving, and the levels at which these skills operated. It is with this that I believed that qualitative content analysis was therefore better suited to the study. Students' verbal and non-verbal activities were coded and counted for quantifying qualitative data without statistical testing (De Laat & Lally, 2003; Hurme et al., 2006). Rich, in-depth descriptions of the chronological research events and their special cases formed the core of the case study.

5.3 Construction of Taxonomy: Phase 1

This section sets out the first phase of the designing of my taxonomy consisting of categories and indicators of metacognitive skills applicable to my study. Two taxonomies (coding schemes), one by Artzt and Armour-Thomas (1992) and the other by Meijer, Veenman and van Hout-Wolters (2006) played an integral role in my work. Categories of each taxonomy, as well as the differences between the above authors' taxonomies were used in the design of my taxonomy (as outlined below). Furthermore, the operationalisation of metacognitive skills in my research is based on the above authors' views. Apart from considering the above authors' categories of metacognitive skills, I also relied on the work of Polya (1945), Garofalo and Lester (1985), Schoenfeld (1985) and a group of Dutch and Belgian researchers' work on the categories of metacognition. The result is that the taxonomy of my study only consisted of four categories as discussed below.

5.3.1 Foundational Works in the Development of the Taxonomy

Apart from the coding of metacognitive skills in the studies of Artzt and Armour-Thomas (1992) and Meijer, Veenman and van Hout-Wolters (2006) there are differences between the two taxonomies. The first distinction is that Artzt and Armour-Thomas focus on metacognitive skills of students in a group setting while the taxonomy of Meijer et al. was constructed for purposes of categorising the individual's metacognitive skills as he/she works alone on a problem. Because of this distinction I had to (i) be aware of the different ways that the authors used the concept of metacognitive skills, and (ii) modify the indicators of Meijer et al. to speak to those of Artzt and Armour-Thomas (and vice versa) to develop a consistent framework. This in turn influenced the design of my own taxonomy when considering metacognitive skills in a collaborative setting. The second distinction is that Artzt and Armour-Thomas focused exclusively on mathematical problem solving, while the taxonomy of Meijer et al. is not domain-specific. With this, Meijer's et al.'s focus was on how metacognitive skills applied to a range of different tasks or domains and not exclusively to that of mathematics (ibid., p. 216). In particular, their taxonomy addressed metacognitive skills in the social sciences and text studying (specifically to the study of a history text) as well as problem solving in physics (although the problem used in their work was still largely mathematics related) (Meijer et al., 2006, p. 216). Because the taxonomy of Meijer et al. is not domain-specific to mathematics and focuses on the individual's metacognitive skills, I examined other taxonomies used in mathematics. These additional taxonomies were used in order to enhance my own taxonomy by including indicators of metacognitive skills which did not appear in Meijer et al. and/or Artzt and Armour-Thomas. Possible parallels between the categories of the two above research groups and that of other authors' taxonomies were also considered. The additional taxonomies will be discussed later in depth.

5.3.2 Conceptualisation, Operationalisation and Categories of Metacognitive Skills

In order to examine the different categories and indicators of metacognitive skills as outlined in Artzt and Armour-Thomas, and Meijer et al., I examined the authors' conceptualisation and/or operationalisation of metacognitive skills. For Artzt and Armour-Thomas (1992, p. 139), metacognitive skills conceptualise the reflection on, and the modification, regulation and the monitoring of cognitive activities during problem solving. Since their taxonomy was used to categorise students' metacognitive behaviour in a group, metacognitive skills also concerned the interpersonal monitoring and regulation of members' goal-directed behaviour (ibid., pp. 149 – 155). Hence regulation and monitoring were regarded as being applicable to either one's own cognitive activities or those of others.

In the case of Meijer et al. (2006, p. 221) their taxonomy of metacognitive skills focused on the individual working alone on a problem (as noted previously). Within this individual setting, metacognitive skills were regarded as the strategic application of metacognitive knowledge[16] to achieve cognitive goals. In particular,

[16] The concept of metacognitive knowledge was discussed in the Literature Review, Chapter 2.

these skills involved the active monitoring, regulation and control of one's own cognitive processes (ibid., p. 217).

By combining the above authors' conceptualisations of metacognitive skills, I applied the following conceptualisation of metacognitive skills for the present study's taxonomy:

> Metacognitive skills in mathematical problem solving are those activities in which the student regulates, controls, monitors and/or reflects on his/her own or others' cognitive activities.

The above conceptualisation was broad and therefore needed a finer operationalisation of metacognitive skills, which defined them in terms of concrete activities, for purposes of coding students' verbal and non-verbal behaviour during talk-aloud protocols. Because metacognitive skills (and also metacognition implicitly) is such a broad concept it is linked to the notion that metacognitive skills can be distinguished at hierarchical levels, as seen in Meijer et al. (2006, p. 210). At the highest level, metacognitive skills constitute activities such as planning, monitoring and evaluation. The intermediate level is characterised by activities such as reflection and recapitulation (that falls under evaluation). At the lowest level, concrete task-level activities such as deciding to reread a passage in order to obtain more clarity, or examining a special case of the problem at hand (in order to possibly design a better plan to solve the problem) are characteristic activities.

In Meijer et al. (2006) the highest levels are referred to as categories of metacognitive skills. Activities at the lowest level are indicators of metacognitive skills. The six categories of the Meijer et al. taxonomy are:

1. Orientating
2. Planning
3. Executing
4. Monitoring
5. Evaluation
6. Elaboration

Within these categories Meijer et al. delineates the different concrete indicators of metacognitive skills.

In Artzt and Armour-Thomas (1992) there are similar categories referred to by the authors as episodes instead of categories. These episodes are as follows:

1. Read
2. Understand
3. Analyse
4. Explore
5. Plan
6. Implement
7. Verify
8. Watch and listen

Artzt and Armour-Thomas (1992) classified each of the eight episodes in their taxonomy as either cognitive, metacognitive, or neither cognitive nor metacognitive. The reason for this is that the authors argue that there is a conceptual difference between cognition and metacognition, but at the operational level the distinction is not that clear. They go further in saying that cognition is implicit in metacognitive activities, while on the other hand metacognition may be part of a cognitive act, although it may not be that apparent to the observer (ibid., p. 141). Because of this Artzt and Armour-Thomas decided that none of the episodes can be classified as purely cognitive or purely metacognitive and that the distinction in classifications is to be grounded on the predominant behaviour observed (ibid., p. 141). Their working distinction between cognition and metacognition, and hence their operationalisation of metacognitive skills is given as follows:

> Cognition is involved in the doing, whereas metacognition is involved in the choosing and planning what to do and monitoring what is being done.
>
> (Artzt & Armour-Thomas, 1992, p. 141)

The authors also mention that metacognitive skills are exhibited by:

1. statements made about the problem, and
2. statements made about the problem solving process.

Artzt and Armour-Thomas (1992) further note that cognitive behaviours are revealed by verbal comments and non-verbal activities which have to do with the actual processing of information (ibid., p. 141). These views of Artzt and Armour-Thomas are similar to Kim et al.'s (2013) distinction between that of cognition and metacognition, as well as their operationalisation of metacognitive skills. For them, cognitive activities have to do with 'thinking with' cognitive workings, while metacognition has to do with the 'thinking about' cognitive workings, in controlling, monitoring and regulation of such cognitive enterprises (ibid., pp. 379 – 381, 386).

Meijer et al. (2006, pp. 210 – 211) argue that many activities during talk-aloud protocols will seem to be more cognitive in nature instead of metacognitive and that it is not uncommon to deduce metacognitive actions from cognitive actions. According to them some overt cognitive activities are sometimes taken to denote covert metacognitive activities, but that such inferences should be based on certain indicators (clues) in the talk-aloud protocol (ibid., pp. 211 – 212). To identify indicators applicable to my own talk-aloud protocols, I adopted a similar distinction between cognition and metacognition as that of Artzt and Armour-Thomas (1992) as well as the above views of Meijer et al. (2006). Using these criteria as guidelines, metacognitive skills were operationalised in the present research study as

> Any statement made about, or activity that addresses the problem and/or the problem solving process, in order to solve the mathematical problem and or to structure the problem solving process may be regarded as manifestations of metacognitive skills. Furthermore, they are only regarded as manifestations of metacognitive skills if such statements and activities indicate some form of control, monitoring, regulation of and/or reflection on one's and/or others' cognitive enterprises.

The above operationalisation (working definition) was used to determine the indicators that formed part of the present research study's taxonomy.

5.3.3 Episodes of Metacognitive Skills: Artzt and Armour-Thomas (1992)

As mentioned earlier, Artzt and Armour-Thomas did not classify episodes[17] in their taxonomy as purely metacognitive or purely cognitive. The different categories, with the corresponding cognitive levels are shown below in Table 5.1.

Table 5.1: Framework of Episodes classified by Predominant Cognitive Level (From Artzt & Armour-Thomas, 1992, p. 142)

Category	Predominant Cognitive Level
Read	Cognitive
Understand (*trying to understand*)	Metacognitive
Analyse	Metacognitive
Explore	Cognitive and Metacognitive
Plan	Metacognitive
Implement	Cognitive and Metacognitive
Verify	Cognitive and Metacognitive
Watch and listen	Level not assigned

The authors gave a rationale for the above classifications based on their working definition between cognition and metacognition and what activities exemplify metacognitive skills (as discussed earlier). In their view 'Analyse' and 'Plan' are predominantly metacognitive skills. During an episode of 'Analyse' and 'Plan' the student attempts to make sense and understand what the problem is about, build an appropriate perspective of the problem and try to reformulate the problem into that perspective as well as selecting possible strategies in solving the

[17] As noted earlier, Artzt and Armour-Thomas (1992) refer to categories of metacognitive skills as episodes.

mathematical problem (Artzt & Armour-Thomas, 1992, pp. 141 – 141). 'Understand' is classified as predominantly metacognitive, as any attempt in *trying* to understand what the problem is about or reflecting about the problem or meaning(s) of the problem are comments about the problem. Hence, 'Understanding' is classified as a category of metacognitive skills. Artzt and Armour-Thomas (1992 p. 142) state that

> although it is true that some of the things one does to understand a problem are cognitive, in a coding scheme that relies on the verbal comments of students, it is impossible to decipher the understanding that is being derived during the actual doing of the problem.

'Read' is categorised as predominantly cognitive, because it exemplifies an instance of doing. Further, when 'Exploring' the authors argue that if such exploration is guided by the monitoring of either oneself or one's group-mate, that behaviour can be categorised as exploration with monitoring, or exploration with metacognition. Because of such monitoring, it concerns either self- or group-regulation for control and focus. A similar argument holds true when applied to both 'Implement' and 'Verify', which can occur with or without monitoring and regulation. Lastly, the authors note because of the lack of verbalisation during the categories of 'Watch' and 'Listen', it made it difficult to deduce the level of cognition. Therefore, these categories were not typified as either cognitive or metacognitive (ibid., p. 142). Since the focus of my study was on metacognitive skills, only activities on a metacognitive level were reported and discussed. Hence, the above authors' categories of 'Explore', 'Implement' and 'Verify' only at a metacognitive level were considered for my research purposes and in the design of my taxonomy.

5.3.4 Categories of Metacognitive Skills: Meijer, Veenman and van Hout-Wolters (2006)

As noted previously, the Meijer et al. (2006) taxonomy is not domain-specific to that of mathematics. I therefore consulted other taxonomies in order to enhance my own taxonomy. This concept of enhancing is also pointed out by Meijer et al., in which they argue that the design of a new taxonomy should relate to other known taxonomies of metacognitive skills in order to have sufficient convergence between taxonomies. Some of the works that were used in constructing the Meijer et al. (2006) taxonomy are: Veenman (1993); Veenman, Elshout and Meijer (1997); Veenman, Prins and Verheij (2003); Veenman and Verheij (2003); and Veenman, Wilhelm and Beshuizen (2004).

A closer look at the work of the first four authors revealed that most of them were not domain-specific to mathematics. I then reviewed other works by Veenman and his colleagues which revealed a number of works on metacognitive skills in mathematics. These were used in extending my taxonomy: Prins, Veenman and Elshout (2006); Van der Stel, Veenman, Deleen and Haenen (2010); Van Der Stel and Veenman (2008); Van der Stel and Veenman (2010); Veenman (2006); and Veenman and Verheij (2003). The six studies relate to metacognitive skills of students in mathematics and/or mathematics related subjects in the Netherlands and Belgium. I refer to these works as those of the Dutch-Belgian School because of their frequent use in my study. I examined how these six studies were related to each other and hence implicitly also to the taxonomy of Meijer et al. (2006). This was done in order to eliminate any discrepancies between the taxonomies of the Dutch-Belgian School and Meijer et al. (2006), as well as to ensure possible overlapping in indicators of metacognitive skills between these different taxonomies and the taxonomy of Meijer et al.

The Dutch-Belgian School played an integral part in the design of the categories of the Meijer et al. (2006) taxonomy; a reason for using them in the development of the present research study's taxonomy. In particular, Veenman and Verheij (2003) studied the performance of university students in tasks on mathematical modelling and differential equations both of which were concepts in the curriculum content of the Calculus 2 course, the research site. Further, the majority of the taxonomies of the Dutch-Belgian School were based on the work of Veenman and Verheij (2003). Hence there was a greater alignment of the categories of metacognitive skills between these taxonomies. Moreover, most of the authors of the above school explicitly addressed metacognitive skills in mathematics. This was another reason why I considered these six articles, since they were domain-specific in contrast to that of Meijer et al.'s taxonomy which is domain-general. Furthermore, categories of metacognitive skills in Prins et al. (2006) overlapped with the categories in the other taxonomies as well.

All the taxonomies of the Dutch-Belgian School have four categories of metacognitive skills, with the exception of Veenman and Verheij (2003) which has five categories. These four categories are: Orientation, Planning, Evaluation and Elaboration. In the case of Prins et al. (2006) and Veenman and Verheij (2003) 'Planning' is just referred to as 'Systematical Orderliness', although they are the same in conceptualisation. In Veenman and Verheij (2003) the extra category 'Accuracy' was introduced before 'Evaluation', but this category can be subsumed under 'Evaluation'. In Veenman (2006), 'Elaboration' is just referred to as 'Reflection', although they are similar in conceptualisation.

The category 'Orientation' occurs at the onset of the problem solving procedure. 'Planning' refers to activities or strategies chosen by the student to devise a plan in order to solve the problem. The 'Evaluation' category concerns the implementation of the proposed plan. What is of importance is that such implementation needs to be accompanied by the monitoring and control of one's cognitive actions in order to consider 'Evaluation' as a category of metacognitive skills. The 'Elaboration' category is typically at the end of the problem solving procedure. Table 5.2 lists some examples of indicators of the Dutch-Belgian School categories.

Table 5.2: Examples of Indicators from the Dutch-Belgian School

Dutch-Belgian School Categories	**Examples of Indicators**
Orientation	Analysing the problem (Prins et al. (2006); and Veenman and Verheij (2003))
	Building a mental model of the task (Prins et al. (2006); and Veenman and Verheij (2003))

	Paraphrasing the problem statement (Veenman (2006))
	Activating prior knowledge that possibly may be useful in solving the problem (Van der Stel et al. (2010); Van Der Stel and Veenman (2008); and Van der Stel and Veenman (2010))
	Estimating and/or predicting the answer (Van der Stel et al. (2010); Van Der Stel and Veenman (2008); and Van der Stel and Veenman (2010))
Planning	Subgoaling (Van der Stel et al. (2010); Van Der Stel and Veenman (2008); and Van der Stel and Veenman (2010))
	Designing a step-by-step action plan (Van der Stel et al. (2010); Van Der Stel and Veenman (2008); Van der Stel and Veenman (2010); Veenman (2006))
Evaluation	Checking steps of the solution where needed (all articles of the Dutch-Belgian School)
	Commenting on activities that may possibly lead to the solution of the problem (Van der Stel et al. (2010); Van Der Stel and Veenman (2008); and Van der Stel and Veenman (2010))
	Error detection and precision of calculations (Prins et al. (2006); and Veenman and Verheij (2003))
	Monitoring one's progress towards the goal(s) of the problem and the on-going problem solving process (Prins et al. (2006); and Veenman (2006))
Elaboration	Recapitulating and drawing conclusions from the solution (all articles of the Dutch-Belgian Schools, excluding Veenman (2006))
	Relating one's answer(s) to the question of the problem (all articles of the Dutch-Belgian Schools, excluding Veenman and Verheij (2003))
	Paraphrasing of and reflection on the problem solving process (all articles of the Dutch-Belgian Schools, excluding Prins et al. (2006); and Veenman (2006))

	Drawing conclusions while referring to the problem statement (all articles of the Dutch-Belgian Schools, excluding Prins et al. (2006); and Veenman and Verheij (2003))

The reader would have noticed that the number of categories between Artzt & Armour-Thomas, Meijer et al, and Dutch-Belgian School are different, as well as differ in naming. This is illustrated in the Table 5.3.

Table 5.3: Categories of Metacognitive Skills from Artzt and Armour-Thomas (1992); Meijer, Veenman and van Hout-Wolters (2006); and the Dutch-Belgian School

	Artzt & Armour-Thomas	Meijer, Veenman & van Hout-Wolters	Dutch-Belgian School
Categories/Episodes	Understand Analyse Explore Plan Implement Verify	Orientating Planning Executing Monitoring Evaluation Elaboration	Orientation Planning Evaluation Elaboration

In Artzt and Armour-Thomas (1992) there are two categories ‘Read’ and ‘Watch and Listen’ which do not appear in Table 5.3. The reason for omitting these categories is that Artzt and Armour-Thomas regarded ‘Read’ as a category at a cognitive level, while for ‘Watch and Listen’ no cognitive or metacognitive level was assigned. A similar view was adopted for my taxonomy and hence I omitted these two categories in Table 5.3.

In developing the categories of my own taxonomy, I considered how the above categories as listed in Table 5.3 were (i) similar in conceptualisation but just differed in name, and (ii) which categories could be suitably subsumed under other categories.

In order to achieve an amalgamation of categories the foundational works which were used in the development of the taxonomies of Artzt and Armour-Thomas, and Meijer et al. (2006) were reviewed. The taxonomy of Artzt and Armour-Thomas was mostly constructed using the work of Polya (1945) on mathematical problem solving; Schoenfeld's (1985) extensive research on college students' mathematical problem solving behaviours; and a taxonomy of metacognitive skills in mathematics developed by Garofalo and Lester (1985). Meijer et al.'s taxonomy was constructed using the above works of Schoenfeld, and Garofalo and Lester (Meijer et al., 2006, p. 220). These three taxonomies are outlined below.

5.3.5 Categories of Metacognitive Skills: Polya (1945)

Polya (1945) regarded mathematical problem solving as a process consisting of four categories, namely 'Understanding', 'Planning', 'Carrying out the plan' and 'Looking back' as outlined in Table 5.4 (and as discussed in Artzt and Armour-Thomas, 1992, p. 138; and Schoenfeld, 1985, p. 24).

The above four categories are similar to that of the Dutch-Belgian School; where 'Evaluation' of the Dutch-Belgian School is the same as 'Carrying out the plan' of Polya, and 'Looking back' coincides with 'Elaboration' of the Dutch-Belgian School.

Table 5.4: Polya's (1945) Categories of Metacognitive Skills

Categories of metacognitive skills	Actions the category describes
Understanding	The student tries to comprehend and orientate him/herself to what the problem is about; as well as considering conditions, givens and unknowns of the problem
Planning	The student tries to devise a plan, that is, a sequence of steps in order to solve the problem at hand

Carrying out the plan	The student executes the proposed plan while monitoring (checking) if his/her calculations and solution steps are correct, and adheres to the proposed plan
Looking back	The student examines and evaluates what has been done by checking his/her results and solution and if the solution relates to the question and given conditions of the problem

5.3.6 Categories of Metacognitive Skills: Garofalo and Lester (1985)

A similar taxonomy of mathematical problem solving can be found in Garofalo and Lester (1985) and consists of four categories: Orientation, Organisation, Execution and Verification (ibid., p. 171). Table 5.5 describes the focus of category as discussed in Garofalo and Lester (1985, p. 171).

The four categories of Garofalo and Lester (1985) are similar to Polya's (1945) categories of mathematical problem solving. Moreover, these categories also strongly correlate to the four categories of the Dutch-Belgian School. The only differences are that 'Organisation' in Garofalo and Lester is referred to as 'Planning' in the Dutch-Belgian School; 'Execution' of Garofalo and Lester is called 'Evaluation' in the Dutch-Belgian School; and 'Verification' in Garofalo and Lester is known as 'Elaboration' in the Dutch-Belgian School, although they are similar in conceptualisation.

Table 5.5: Categories of Metacognitive Skills by Garofalo and Lester (1985)

Categories of metacognitive skills	**What the category takes into account**
Orientation	Trying to understand what the problem is about Analysing given information and conditions of the problem
Organisation	Devising a plan consisting of steps and strategies in order to solve the problem

	Devising a global plan in order to solve the problem, as well as local plans that make up and are used to implement the global plan
Execution	Implementation of the proposed plan with regulation and monitoring of one's own activities in adhering to the proposed plan, locally and globally
Verification	Evaluation on correctness of decisions and strategies used in the problem solving process Verifying the outcomes of the executed plan Evaluation and reflection of behaviours in the 'Orientation', 'Organisation' and 'Execution' categories

5.3.7 Categories of Metacognitive Skills: Schoenfeld (1985)

Schoenfeld (1985) has similar categories to those of the previous authors in which the categories are: Reading, Analysis, Exploring, Planning/Implement and Verification. 'Analysis' here is similar to 'Orientation' in the Dutch-Belgian School and Garofalo and Lester (1985), while 'Planning' in Schoenfeld corresponds to that of 'Planning' in the Dutch-Belgian School and 'Organisation' of Garofalo and Lester. 'Implement' of Schoenfeld is similar to 'Execution' of Garofalo and Lester, 'Evaluation' of the Dutch-Belgian School, and 'Carrying out the plan' of Polya (1945). 'Verification' is used in the same sense as in Garofalo and Lester and is similar to the 'Elaboration' category of the Dutch-Belgian School, and 'Looking back' of Polya.

The categories of Schoenfeld (1985) are also closely related to those of the Artzt and Armour-Thomas (1992). 'Planning' in Schoenfeld and Artzt and Armour-Thomas are the same in conceptualisation, where 'Planning' as defined by Artzt and Armour-Thomas is regarded as an approach consisting of steps or strategies to be used in order to solve the problem (Artzt and Armour-Thomas, 1992, p. 173). 'Implement' of Schoenfeld is similar to that of Artzt and Armour-Thomas while 'Verification' in Schoenfeld is also similar to that of 'Verify' in Artzt and Armour-Thomas. 'Reading' is regarded as cognitive, similar to Artzt and Armour-Thomas. 'Planning/Implement' of Schoenfeld was separated into two distinct categories in Artzt and Armour-Thomas (1992, p. 141) since the authors argue that these categories do not always follow each other consecutively in the problem solving process. 'Exploring' in Schoenfeld (1985, p. 298) and 'Explore' in the Artzt and Armour-Thomas (1992, pp. 173 – 174) are similar in definition. Schoenfeld regards 'Exploring' as a category that overlaps with 'Analysis' and 'Implement' – like a transmitting phase between 'Analysis' and 'Implement'. Schoenfeld (1985, p. 298) himself notes that 'Exploring' is like 'a broad tour' through the problem solving process, in that it can be assimilated into the Analysis-Plan-Implementation sequence – a similar viewpoint can be seen in Artzt and Armour-Thomas (1992, pp. 173 – 174). Schoenfeld further notes that if students do not monitor their actions/activities during 'Exploring', they can start a so-called 'wild goose chase' which can lead to disaster in solving the problem. Both Artzt and Armour-Thomas (1992, p. 173) and Schoenfeld (1985, p. 298) note that 'Exploring' is less well-structured than 'Analysis' and further removed from the original problem.

Having considered the above three taxonomies of Polya (1945), Schoenfeld (1985), and Garofalo and Lester (1985), the categories of the present research study taxonomy were constructed. The construction of the four categories with their corresponding indicators are discussed in Section 5.3.8, while examples on the construction and development of some indicators with their corresponding codes are discussed Section 5.3.9.

5.3.8 Categories of the Study's Taxonomy

Since 'Explore' (Artzt & Armour-Thomas, 1992) and 'Exploring' (Schoenfeld, 1985) are not so well-structured and form part of the greater Analysis-Planning-Implementation sequence, I decided not to consider 'Exploring'/'Explore' as a category on its own, but to be subsumed under 'Implement' of Artzt and Armour-Thomas. I also separated 'Planning/Implementation' into two distinct categories (similar to that of Artzt and Armour-Thomas as discussed above). Consequently, the Schoenfeld taxonomy (after my modifications) consists of four categories – similar to that of the Dutch-Belgian School, Garofalo and Lester (1985) and Polya

(1945). During the initial phase of analysis and coding of my video recordings I decided that my aim should be that my taxonomy consists of four categories. In order to accomplish this goal, I had to reduce the number of categories in the Artzt and Armour-Thomas (1992) and Meijer et al. (2006). How I approached this process is discussed below.

In Artzt and Armour-Thomas the category 'Understand' was introduced. The reason for this was that frequent comments and activities of students indicated to the authors that 'Understand' needs to be introduced as a separate category (ibid., p. 141). Both the categories 'Understand' and 'Analysis' of Artzt and Armour-Thomas captured the way in which the student tried to orientate him/herself around the problem: the student breaks the problem up into its basic parts, tries to gain an appropriate perspective of the problem, and reformulates the problem into that perspective. During my own observations of students in this study, I noticed that there is not such a clear distinction between 'Understand' and 'Analysis'. Hence I decided to place 'Understanding' and 'Analysis' together under one category, which I called 'Orientation'. The 'Orientation' category of my taxonomy corresponded to that of 'Orientating' in the Meijer et al., 'Orientation' of the Dutch-Belgian School, and Garofalo and Lester (1985), and 'Understanding' of Polya (1945).

The categories 'Orientating' and 'Planning' as used in Meijer et al. (2006, pp. 229, 235) are similar to that of 'Orientating' and 'Planning' in Schoenfeld, the Dutch-Belgian School and Garofalo and Lester, whilst the 'Planning' of Meijer et al. and Artzt and Armour-Thomas is also similar in conceptualisation.

In analysing the taxonomies of Meijer et al. and Artzt and Armour-Thomas, the indicators of the categories 'Executing', 'Monitoring' and 'Evaluation' of Meijer et al. strongly correlated to the indicators of the category of 'Implement' of Artzt and Armour-Thomas. Hence, I subsumed these three categories of Meijer et al. under 'Implement' of Artzt and Armour-Thomas, in creating a unique category 'Execution' in my taxonomy.

The category of 'Elaboration' in Meijer et al. (pp. 226, 229, 237) is similar to that of 'Verifying' of Artzt and Armour-Thomas; they only differ in the use of indicators.

Consequently, I decided to combine 'Elaboration' (of Meijer et al.) and 'Verify' (of Artzt and Armour-Thomas) as one category in my taxonomy, namely 'Verification'. The category 'Verification' in my taxonomy is a similar conceptualisation as that of 'Verify' of Artzt and Armour-Thomas (p. 175). 'Verify' according to Artzt and Armour-Thomas indicates that after the student has decided that the solution or part of the solution has been obtained, he/she reviews/evaluates the work.

In conclusion, my taxonomy consisted of four main categories namely: Orientation, Planning, Execution and Verification.

5.3.9 Indicators of the Study's Taxonomy

Having determined the four main categories in my taxonomy I then determined the indicators of each category, as obtained and/or adapted from of Artzt and Armour-Thomas (1992), Meijer et al. (2006), and the Dutch-Belgian School. Not all indicators from the above authors' taxonomies were used in my study. Indicators with their corresponding codes, as used in my study are summarised in Table 5.6 below. A short elaboration on what certain indicators entail is given as well.

Table 5.6: Categories and Indicators of Metacognitive Skills used in the Research Study

ORIENTATION (O)
Activating prior knowledge (APK) Student considers/uses domain specific knowledge relevant to the problem
Building a mental model of the task (BMM**)** Student tries to represent problem in own words, trying to represent problem in own way to make sense of the problem Students engages in an attempt to reformulate/simplify the problem Student engages in an attempt to get an appropriate perspective of the problem and reformulate the problem into that perspective Student makes a sketch, diagram, table to represent the problem Student paraphrases what is asked for

Identifying and repeating important information (to be remembered) (IMP) Writing down key facts Selection of relevant information needed to solve the problem
Using external source to get explanations (UES)
Rereading at orientation (RRO)
PLANNING (P)
Formulate an action plan (FAP) Selecting steps/strategies to solve the problem or describing an approach (steps/strategies) to be used or intended to be used to solve the problem Subgoaling Estimating the answer Designing a step-by-step action plan instead of working by trial-and-error Setting up a sequence of steps in order to solve the problem Designing an action plan before actually solving the problem Writing down calculations step-by-step
Considering different ways of solving the problem (CDWS)
Organising thought by questioning oneself (OT)
Decision to change strategy on basis of former interim outcomes (DCS)
EXECUTION (E)
Executing action plan (EAP)
Monitoring action plan Keeping track of progress being made, verifying that results obtained provide an answer to solution statement
Error detection (plus correction) and keeping track (EDKT) Error detection (plus correction), keeping track Checking answers by recalculating (checking calculations) Precision in calculation Avoidance of negligent mistakes
Evaluate current situation (ECS) Control of learning process / problem solving process Checking Drawing away from the problem to see what has been done and/or where the solution is leading to. Comments and questions for e.g. are: “what are you doing?”, “what am I doing?”, “this is not getting us anywhere”, “I think that it is the answer”, “I have used all the given conditions, now I will start…”, “wait, we forgot to use…”

Giving suggestions to others (GSO)
Checking memory capacity (CMC)
Claiming progress in understanding (CLU)
Finding similarities, analogies (FSA)
Note-taking, underlining, circling, highlighting, writing out of work in an orderly manner (NUL)
VERIFICATION (V)
Student reviews work in general (VG) Student evaluates the outcomes/solutions whether the outcomes reflect the adequate problem understanding, analysis, planning and/or implementation Student checks if the solution satisfies the conditions of the problem Student checks if the solution process makes sense
Concluding (CON) Relating answer to the question Recapitulating and drawing conclusions Drawing conclusions while referring to the problem statement Relating conclusions to the subject matter
Reflection on the learning process (REF)
Drawing conclusions beyond the information given
Commenting on personal habits (CPH)

As noted earlier, my operationalisation (working definition) of metacognitive skills is:

> Any statement made about, or activity that addresses the problem and/or the problem solving process, in order to solve the mathematical problem and to structure the problem solving process may be regarded as manifestations of metacognitive skills. Moreover, they are only regarded as manifestations of metacognitive skills if such statements and activities indicate some form of control, monitoring, regulation of and/or reflection on one’s and/or others cognitive enterprises.

All indicators from the taxonomies of Artzt and Armour-Thomas (1992), Meijer et al. (2006), and the Dutch-Belgian School were examined for their agreement with indicators of my taxonomy. Indicators which were similar in description were placed under one common code. As an example consider the following indicators:

1. 'Consider/use domain specific knowledge relevant to the problem from 'Understanding' of Artzt and Armour-Thomas,
2. 'Activating prior knowledge' from 'Orientating' of Meijer et al., and
3. 'Activating prior knowledge' of from 'Orientation' of Van der Stel, Veenman, Deleen and Haenen (2010); Van Der Stel and Veenman (2008); and Van der Stel and Veenman (2010) of 'Orientation' of the Dutch-Belgian School.

These three indicators were joined together as one indicator namely 'activating prior knowledge' with the corresponding code APK, as adopted from Meijer et al. (2006, p. 235).

Considering the following indicators:

1. 'Try to represent/restate problem in own words' ('Understanding', Artzt and Armour-Thomas).
2. 'Trying to represent problem in own way to make sense of the problem' ('Understanding', Artzt and Armour-Thomas).
3. 'Engage in an attempt to reformulate/simplify the problem' ('Analysing', Artzt and Armour-Thomas).
4. 'Make a diagram/list' ('Understanding', Artzt and Armour-Thomas).
5. 'Engage in an attempt to get an appropriate perspective of the problem and reformulate the problem into that perspective' ('Analysing', Artzt and Armour-Thomas).
6. 'Paraphrasing what is asked for' ('Orientation', Dutch-Belgian School; Veenman (2006)).
7. 'Building a mental model of the task' ('Orientation', Dutch-Belgian School; Prins et al. (2006); Veenman (2006); and Veenman and Verheij (2003)).

8. ‘Making a sketch, diagram, table to represent the problem’ (‘Orientation’, Dutch-Belgian School; Van der Stel, Veenman, Deleen and Haenen (2010); Van Der Stel and Veenman (2008); Van der Stel and Veenman (2010); Veenman (2006); and Veenman and Verheij (2003)).

All of the above indicators are linked to the same concept: the student trying to represent the task in his/her own way; trying to build a mental image of the task accordingly to his/her view of the task. These indicators were combined into one indicator, namely ‘building a mental model of the task’ as adopted from the Dutch-Belgian School (Prins et al., 2006; Veenman & Verheij, 2003). Since no code was given by the Dutch-Belgian School or Artzt and Armour-Thomas, I created my own code for the above indicator namely ‘building a mental model’ (BMM). A similar procedure was followed throughout my taxonomy.

There were also indicators that were not pertinent to my taxonomy and hence not included. For example the indicator ‘fill in values establish givens’ (FV) from the category ‘Orientating’ of Meijer et al. had no relevance to the student tasks used during observations of my study, and thus this indicator was excluded. Some indicators, such as ‘entirely reading the problem’ (from ‘Orientation’, Dutch-Belgian School) in my view is not an example of metacognitive skills, and therefore were also excluded from my taxonomy. For this particular case, I would introduce my own new indicator such as ‘rereading at orientation’ with code RRO, only if such reading took place on a metacognitive level, that is, the student *intentionally decided* to reread a certain passage that was not completely understood, i.e. the concrete activity of rereading had to adhere to the my working definition of what metacognitive skills entail. A similar viewpoint is discussed in Meijer et al. (2006, p. 210).

Some indicators were closely related but differed in description. For example, in the ‘Execution’ category of my taxonomy, ‘evaluate current situation’ (ECS) differs from ‘executing action plan’ (EAP) in that ECS is the local monitoring of *particular* instances of the problem solving process, while EAP applies to the *global* monitoring of adhering to the proposed plan. Verbal and non-verbal behaviours concerned with the monitoring and checking of activities had to be related to the proposed plan in order for such behaviour to be coded EAP.

Some codes appearing in one category were transferred to another category, as found appropriate. An example of this can be seen in the code 'UES' (using an external source) which formed part of the 'Planning' category of Meijer et al., and was placed under the 'Orientation' category of my taxonomy. The reason for this was because I noted during recoding and reanalysis of transcripts that 'UES' predominantly occurred during the 'Orientation' of the problem solving process when students consulted the textbook and/or personal notes to assist them on how to solve a problem.

The indicator 'reaction to question of experimenter/observer' (RE) from the category 'Executing' of the Meijer et al. was excluded from my taxonomy. Rather it was adapted in indicating students' reactions to my questions and/or interventions to indicate instances of the social triggers of students' metacognitive behaviour. When I noted during observations that a student was on a wild goose chase then I would bring it to the student's attention. If the student reacted to my question (intervention), reflected on and/or monitored his/her activities, then such an instance was regarded as metacognitive behaviour brought forth by a social trigger, and the indicator RE applied. I regarded RE as an indicator that could only be used in conjunction with other indicators and not as a stand-alone indicator. Thus RE was always linked to a specific activity and situation, and hence to other indicators. Each occurrence of RE depended on the context in which it occurred. To explain this further, let us say the student checked his/her memory capacity (CMC) because of interrogation from the observer. Then the indicator is not purely CMC, but CMC because of reaction to the observer. Thus the code which was used and assigned to the student's metacognitive behaviour was CMC with RE. This was denoted as CMC-RE in the coding of transcripts. Coding that included reaction to observer occurred in the majority of the categories of my taxonomy.

5.4 Construction of Taxonomy: Phase 2

As mentioned earlier in the discussion on the method of content analysis (Section 5.2) the construction of my taxonomy was an ongoing, cyclical process of refinement and improvement and took place concurrently with the coding and analysis of transcripts. Phase 2 saw a further adaption and refining of my taxonomy during the recoding and reanalysis of transcripts. Coding of transcripts was done by applying the codes as listed in Table 5.6. Frequency tables were constructed to represent the first counts of codes, in which the majority of counts were quite sparse. Because of this, the codes in Table 5.6 were again collapsed and/or adapted. A second round of frequency tables was compiled to capture counts. Although the result was that there were a smaller number of counts that were sparse, there were still code counts that were scant. With this it was decided to introduce and construct a new set of codes by amalgamating the codes of Table 5.6. More taxonomies were reviewed to accomplish this and the second phase of the construction of my taxonomy was based on adaptions of the work of Goos (1994).

The ideas of Goos were used to address the issue of scant counts of codes in my study. Furthermore, her study also had a similar concern to mine in that my categories of metacognitive skills were too broad to serve as an alternative to the counts of codes in Table 5.6. This is seen in the work of Goos where she mentions that in Schoenfeld's work (1985) the use of categories "was useful for labelling macroscopic structural elements of the students' problem-solving attempts," but that Schoenfeld's original scheme was designed to allow generalisations of students' metacognitive behaviour. Goos argued that more detailed information was needed in determining instances where students exemplified the use of metacognitive skills, so-called 'metacognitive decision points' as she called them. Goos goes further in noting that the identification of metacognitive decision points is beneficial in revealing "the unique contributions made by two individual students, and the pattern of interactions between them."

The metacognitive decision points of Goos was adapted and used in my study, since as a technique it could be used to identify the interactions between students and students' contributions to the problem solving process. Codes of Table 5.6 were subsumed under these decision points, where metacognitive decision points would serve as new indicators for each of the four categories of metacognitive skills of my taxonomy.

By using the concept of metacognitive decision points, the effect of categories being too broad and macroscopic to represent instances of students' metacognitive behaviour was minimised. These decision points also increased the frequency of codes. Although the codes from Table 5.6 were of importance in identifying specific and particular instances of metacognitive skills, they were too limited for the purpose of counting instances of students' metacognitive behaviour. In this sense the codes/indicators of Table 5.6 were regarded as the 'building blocks' of my taxonomy.

According to Goos (1994, p. 147) the construct of metacognitive decision points includes:

1. Instances 'where new information was recognised', or
2. 'local assessments of specific aspects of the solution were made.'

The above definition was adapted in order to concur with my operationalisation of metacognitive skills (as discussed in Section 5.3.2). Hence, in my study

> Metacognitive decision points are points in the observations at which students exhibited the use of metacognitive skills in which they made local assessments of the problem solving process, identified new information to be used in the solving the problem, and/or applied the use of new ideas (strategies) in order to solve the problem. Instances of metacognitive decision points had to bear witness of signs the active control, monitoring, regulation of and/or reflection on one's and/or others cognitive enterprises.

The two main metacognitive decision points Goos used in her work were *New Idea (NI)* and *Local Assessment (LA)*. These constructs are discussed below from Goos' point of view and how I have adapted these metacognitive decision points, as well as constructed my own in the process of identifying new metacognitive decision points in my data. Goos (1994, pp. 147 – 148) explains that the 'decision point, New Idea (NI), occurred where previously overlooked or unrecognised information came to light or the possibility of taking a new approach was mentioned.' In my work NI was adapted and used in a different sense as to that of Goos. I used the construct of NI exclusively when the student changed the original plan/strategy on how to solve the problem, or proposed a new strategy/plan in solving the problem. Hence NI falls exclusively under the Planning category of my taxonomy.

Goos' concept of 'previously overlooked or unrecognised information came to light', was adapted and used for other instances of metacognitive behaviour within my taxonomy. Examples of this are 'checking memory capacity' (CMC); noticing unfamiliar words or terms' (NUT); and/or 'finding similarities, analogies' (FSA); all indicators/codes of which fall under the Execution category of my taxonomy and not that of the Planning category. As noted above, NI as used in my taxonomy exclusively forms part the of Planning category of my taxonomy and is associated with the codes (building blocks) 'considering different ways of solving the problem' (CDWS) and 'decision to change strategy on basis of former interim outcomes' (DCS), although the code CDWS may also form part of the metacognitive decision point, 'Proposed Idea' (PI) as discussed below.

The remaining two codes/indicators of my Planning category namely 'formulate an action plan' (FAP) and 'organising thought by questioning oneself' (OT) did not form part of the indicator NI. Goos' construct of NI did not concur with the way that I used FAP and OT in my work. Hence a new indicator called 'proposed idea (PI)' was introduced in which the codes FAP and OT were subsumed under PI. The indicator 'proposed idea' used to denote metacognitive instances in which the student organises his thoughts in considering a plan (strategy) to solve the problem at hand and/or formulates a plan (strategy) to solve the problem. Thus NI was only used when the proposed idea (PI) was discarded; that is if the student realised a strategy (procedure) did not work and he implemented a new strategy on how to solve the problem. Also, the code CDWS does not exclusively fall under NI. It also applies to instances where the student considered a number of different strategies (plans) on how to solve a problem, before execution of one of these strategies. Hence, CDWS is also subsumed under PI. In order to place the concepts of NI and PI under the category of Planning, I also introduced a new 'umbrella indicator' or 'umbrella' metacognitive decision point: 'Formulating Plan' (FP), where FP consists of the adapted Goos NI and my own PI which is illustrated in Figure 5.1 below.

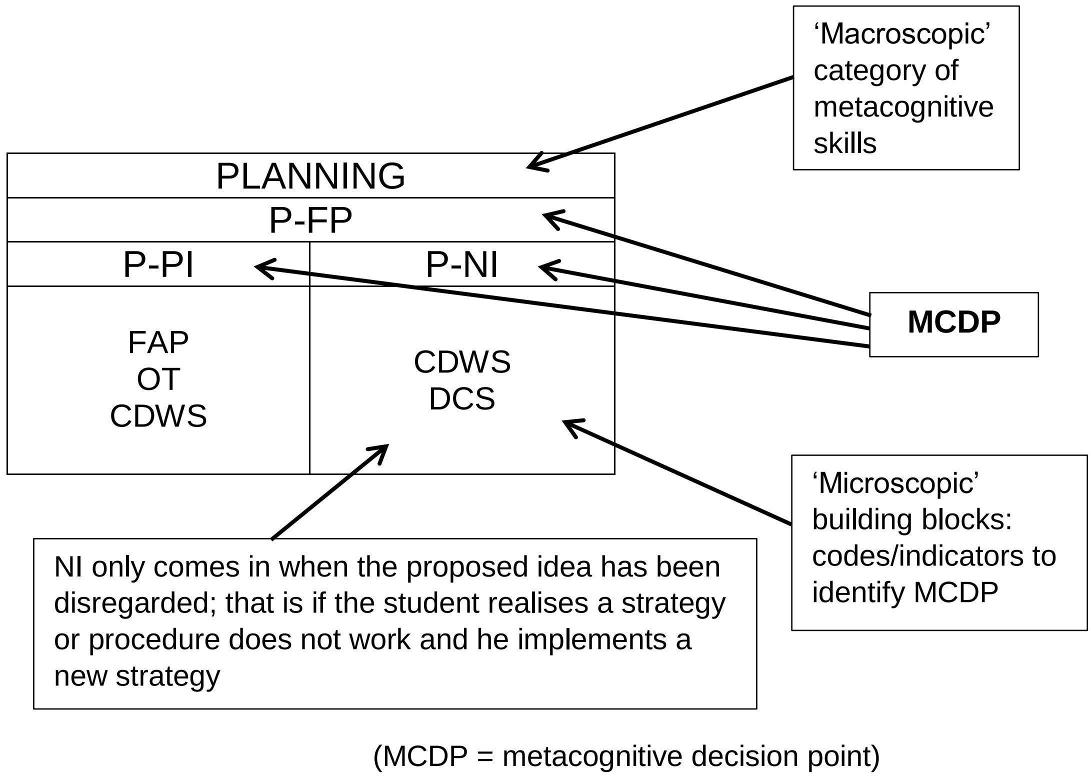

Figure 5.1: Metacognitive Decision Points of 'Formulating Plan' (FP), 'Proposed Idea' (PI) and 'New Idea' (NI) as adapted from Goos (1994)

The other metacognitive decision point Goos (1994, p. 148) uses is that of 'Local Assessment' (LA), in which she considers four different types:

1. procedure (checking accuracy of execution, or assessing the reasonableness /relevance or usefulness of the strategy);
2. result (assessing accuracy or reasonableness);
3. knowledge (identifying what is known or unknown); and
4. task difficulty.

The construct Local Assessment says precisely what it means and how it was used in my work: the student makes an assessment which is a statement and/or activity that indicates some form of control, monitoring, regulation of and/or reflection on one's own and/or others cognitive enterprises during the problem solving process.

The first LA of Goos (1994), 'Local Assessment of Procedure', was placed under my Execution category. The reason for this decision is that during LA of Procedure, the student checks (makes a local assessment) on the accuracy of the execution or assesses the reasonableness/relevance/usefulness of the procedure and/or of what they as student pair are doing. Thus, Local Assessment of Procedure as adapted from Goos and as used in my study compromised:

1. E-LAPR = local assessment of reasonableness/relevance/usefulness of procedure, and
2. E-LAPA = local assessment of accuracy of execution.

Both E-LAPR and E-LAPA fall under the Execution category of my work (hence the E at the beginning of each code).

The second local assessment of Goos, namely 'Local Assessment of Result' was placed under my Verification category. My argument for this is that LA of Result requires that the student

(i) considers what has been done after the problem has been solved;
(ii) he verifies what has been done;
(iii) checks the accuracy of his (or the student pair's) calculations; or
(iv) reflects on the problem (task) difficulty, where such reflection may include commenting on his own personal habits.

All of these activities take place after the student executed the problem. Goos' construct of LA of Result is an instance where the student assesses accuracy, or the reasonableness of the result is considered. I named these two constructs for my own coding purposes as:

1. V-LARA = local assessment of result, with focus on accuracy.
2. V-LARR = local assessment of result, with the focus on the reasonableness of the result.

Both V-LARA and V-LARR fall under the Verification category of my taxonomy (hence the V at the beginning of each code). V-LARA focusses exclusively on the accuracy of the students' solution and calculations. Goos' construct of LA of Task Difficulty did not form part of my taxonomy. Instead it was subsumed under my indicator V-LARR since it included students' (i) comments on task difficulty; (ii) comments on their personal habits; and (iii) reflection on the learning process, as well as (iv) he reasonableness of their solution and/or usefulness of their implemented strategy. V-LARR also took account of the microscopic indicators CPH (commenting on personal habits) and REF (reflection on the learning process) of Table 5.6. Hence Goos' LA of Task Difficulty does not explicitly occur in my work, but is built into V-LARR.

In Goos' (1994, p. 148) work, Local Assessment of Knowledge is predominantly concerned with the students identifying what is known or unknown. I used this local assessment of Goos in a similar way in relation to the codes/indicators of my Orientation category for example 'identifying and repeating important information (to be remembered)' (IMP); and 'using an external source' (UES). I expanded and adapted Goos' construct of Local Assessment of Knowledge which concerns the student merely 'identifying what is known or unknown' (ibid., p. 148). In my study, Local Assessment of Knowledge (coded as O-LAK) also recognised instances where the student used past/prior knowledge to build a mental model of the task, which corresponded to the codes BMM and APK of Table 5.6. For example, O-LAK was used in my taxonomy where students either asked themselves or others if they had done similar problems in the past to the task at hand. Hence O-LAK was placed under the Orientation category of my taxonomy, where O-LAK denotes 'local assessment of knowledge or knowledge building' (since Goos does not have the construct of knowledge building in her work).

In summary, my taxonomy recognises two main metacognitive decision points:

1. P-FP, which can either be P-PI or P-NI; and
2. LA, which can be O-LAK, E-LAPR, E-LAPA, V-LARA or V-LARR (depending on which one of the main four categories the students find themselves in the problem solving process).

The finalised taxonomy that was used in the coding of transcripts, consisting of the indicators of metacognitive decision points, is outlined in Table 5.7 below (which includes the indicators of Table 5.6).

Table 5.7: Final Taxonomy with Metacognitive Decision Points as Indicators of Metacognitive Skills

Category	ORIENTATION	PLANNING		EXECUTION	VERIFICATION
Codes for metacognitive decision points	O-LA	P-FP		E-LA	V-LA
	O-LAK	P-PI	P-NI	E-LAPR/E-LAPA	V-LARA/V-LARR
Indicators of metacognitive decision points	APK BMM IMP RRO UES	FAP CDWS OT	CDWS DCS	EAP ECS NUL EDKT GSO CLU CMC FSA	CON VG CPH REF

- O-LAK = local assessment of knowledge or knowledge building (falling under Orientation)
- P-PI = proposed idea and P-NI = new idea (falling under Planning)
- E-LAPR = local assessment of usefulness/reasonableness of procedure (falling under Execution)
- E-LAPA = local assessment of accuracy of execution (falling under Execution)
- V-LARA = local assessment of the accuracy of result (falling under Verification)
- V-LARR = local assessment of the reasonableness of result (falling under Verification)

Figure 5.2 below illustrates the hierarchy between all the categories, metacognitive decision points and indicators at a microscopic level and how they are related to one other.

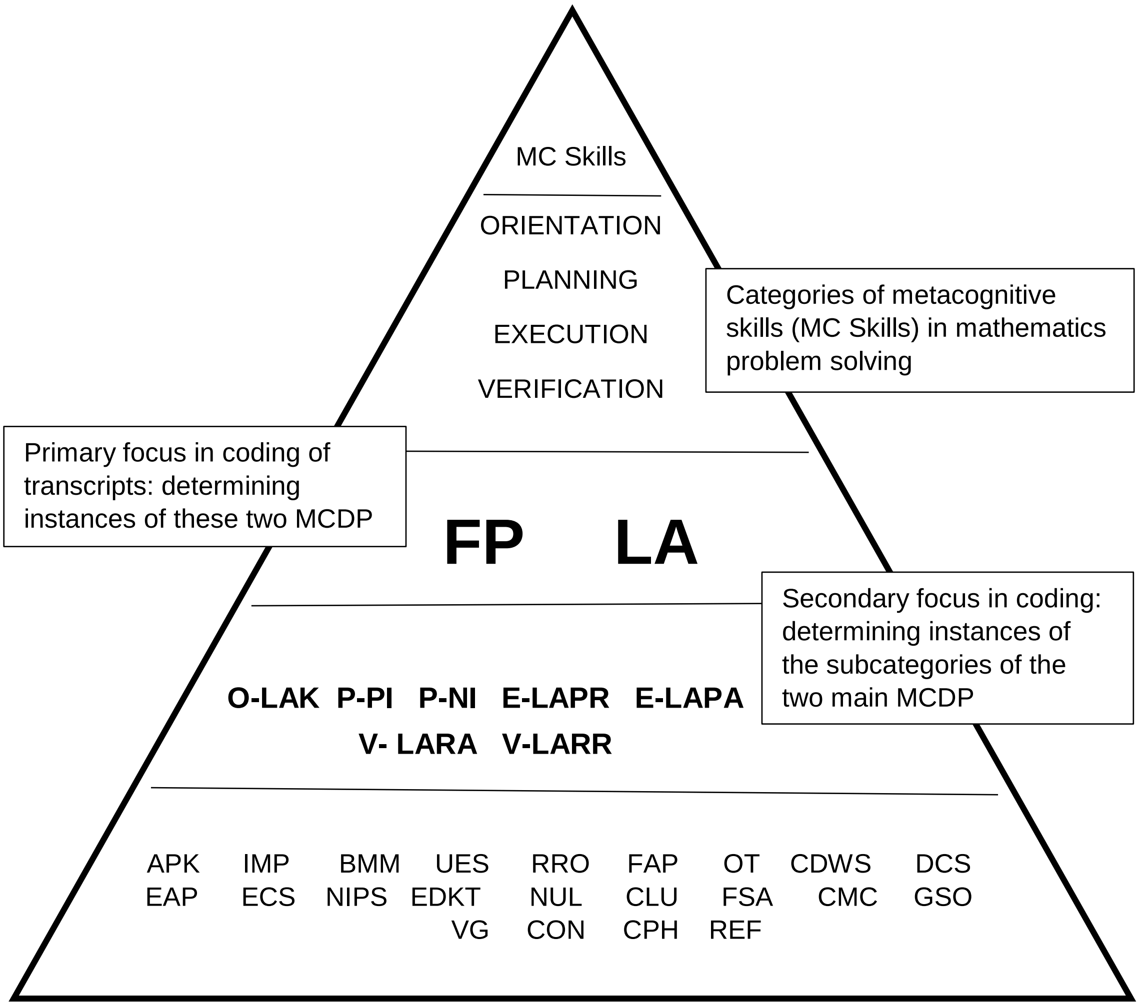

Figure 5.2: Hierarchical Levels of Categories, Metacognitive Decision Points (MCDP) and Microscopic Indicators within the Study's Taxonomy (as adapted from Goos, 1994 and indicators from Table 5.6)

Figure 5.2 also illustrates how the second stage in the construction of my taxonomy identified the

1. two main metacognitive decision points (MCDP), LA and FP; and

2. subcategories of these two metacognitive decision points (in order to have a richer, in-depth picture of students' behaviour during the observations).

The above metacognitive decision points were used in the recoding and reanalysis of the transcripts. Indicators at microscopic level were not discarded and still formed part of the study.

5.5. Summary

This chapter focused on the design of my taxonomy which was used in order to code the transcripts of the four observations of the study. By applying qualitative content analysis in conjunction with the use of the taxonomy, transcripts were analysed in order to answer the research questions. The taxonomy consists of four 'macroscopic' indicators, referred to as categories, of metacognitive skills namely Orientation, Planning, Execution and Verification. These categories consist of metacognitive decision points 'Proposed Idea', 'New Idea' and five types of 'Local Assessments'. These metacognitive decision points on their own consist of 'microscopic' building blocks of the taxonomy; the codes which were used to indicate instances of students' metacognitive behaviour. The construction of codes also took into account instances where the researcher acted as a social trigger of students' metacognitive behaviour. The complete list of codes is given in Table 5.6 and Table 5.7.

The design of the taxonomy developed over two stages which took place during the collection of data, as well as during the analysis of the transcripts. The taxonomy was designed by considering and adapting a number of taxonomies from different author's work such as Artzt and Armour-Thomas (1992); Meijer, Veenman and van Hout-Wolters (2006); a group of researchers which I referred to as the Dutch-Belgian School; as well as Goos (1994). Furthermore, the design of the taxonomy also required the creation of an operationalisation definition of metacognitive skills in order to develop indicators (codes) of metacognitive skills.

Examples of each of these codes are given in Appendix A.

Chapter 6: Discussions and Results of the Four Observations

6.1 Introduction

This chapter gives a detailed description of the students' problem solving behaviour for each of the four observations. In particular, it discusses the metacognitive skills each student exhibited. Attention is also given to students' interaction between each other as well as student-researcher interaction. The chapter also contains vignettes from the transcribed protocols, illustrating students' verbal and non-verbal activity, as well as how the study's applied taxonomy (as outlined in Chapter 5 and Tables 5.6 and 5.7) was used in the coding of data. The correct and complete and correct solutions to the questions of each observation is outlined in Appendix C, as well as each student's solution.

In transcribing the protocols, the following conventions were adopted: completed turns of each speaker have been numbered sequentially; the symbols [ ] are used for non-verbal actions; … indicates either pause, interruption in the speech, or a jump from one turn to another which do not follow sequentially upon each other; ( ) is used to indicate the codes of instances where metacognitive skills are exhibited. The main stream degree student is Dean, while Will is the extended degree student (pseudonyms are used throughout). The symbol * is used for codes where Will is exhibiting metacognitive behaviour; while no * is used for instances where Dean exhibits metacognitive behaviour[18]. The initials D, W and R respectively refer to Dean, Will and the researcher.

As examples, the code (i) E-EAP-RE denotes metacognitive behaviour exhibited by Dean, during Execution in which he is executing the action plan, but in reaction to the researcher's intervention; and (ii) O-UES* denotes metacognitive behaviour exhibited by Will, during Orientation in using an external source.

[18] As noted previously, the terms metacognitive skills, metacognitive behaviour and metacognitive activity are used interchangeably.

6.2 Observation 1

In observation 1, the objective of the task was to determine the convergence/divergence of each of three different series.

6.2.1 Observation 1: Question 1

The first question dealt with the following series:

$$\sum_{n=2}^{\infty} \frac{n^2}{n^3 + 1}$$

Both students started solving the problem without any clear indication of trying to understand the question or orientate themselves. Will was the first to state what approach needed to be taken to solve the problem.

10. W: Yes… looking at it… I think we should actually divide by the highest power. (P-PI*; P-FAP*)

Initially it was not clear from the students' statements what strategy they were employing, but it became evident that both of them were using the nth term test for divergence[19]. In order to obtain a solution to the question they had to evaluate the limit of the sequence of the corresponding series:

$$\lim_{n \to \infty} \frac{n^2}{n^3 + 1}$$

Dean's approach was slightly different to that of Will in that he converted all n's into x's and considered the limit of $f(x)$ as $x \to \infty$.

[19] The nth term test for divergence: If the limit of a sequence $\{a_n\}$ is not zero or does not exist, then the corresponding series $\sum a_n$ is divergent. The test is inconclusive if the limit of the sequence is zero. The series can then either be convergent or divergent, and then another test needs to be used to determine the convergence/divergence of the series.

Quite early in the observation I had to urge the students to verbalise all their thoughts about what tests/strategies they were using to determine the convergence/divergence of series. This was a common trend in the observation, but it was also valuable in that it showed that the students struggled with (i) the terminology used, and (ii) recalling tests and methods used in problems on sequences and series (examples of this are discussed below). As Dean was evaluating the limit he was still not sure what strategy he was applying. I intervened and asked Dean to make it clear what tests (methods) he was using.

21, 22. D: And if the limit of f of x exists and then the series will converge there… If I remember correctly… (E-LAPR; E-ECS)

24. R: Dean, can you tell us what methods are you applying here? What test or…

25. D: I forgot the proper names sir, but…

Dean explains that if the limit of the sequence exists, (or in his case that of the function f) the corresponding series will then be convergent, although his reasoning is wrong in terms of what the nth term test states. Will on the other hand was working quietly on his own. The researcher engaged Will to comment on what Dean did. Will followed a similar train of thought to that of Dean and merely stated that he thought they were on the 'right track'.

Although both students' reasoning behind the use of the nth term test was faulty, they still proceeded with using this test in calculating the limit. Dean worked ahead of Will and found that the value of the limit is zero and concluded that the series converges (which is incorrect, since the series actually diverges and needs to be confirmed by using other test(s)).

Again I intervened in order to make sure that both students were (i) considering the convergence/divergence of the sequence or the series, and (ii) if they were applying the nth term test for divergence correctly. Dean replied that he kept getting confused between the terms sequence and series. In emphasising to the students that they needed to focus on determining the convergence/divergence of the series; and that they should explain why they considered the limit of the corresponding sequence, I hoped they would correct any misconceptions. In response, both Dean and Will chuckled and Dean explained to Will how he applied the nth term test for divergence (which he refers to as the diverging test/divergence test). Will was mostly quiet during Dean's explanation.

At the end Dean stated the nth term test correctly. He also noted that the test was inconclusive if the sequence of the corresponding series has a limit of zero but forgot what to do if this was the case. This is important to note, since Dean still repeatedly applied the nth term test incorrectly throughout the observation. Will reflected on Dean's explanation and more than once wanted confirmation that the series was indeed convergent, since he believed that the series was divergent. Once again, Dean explained to Will why the series is convergent (although his argument was still wrong). Apart from the above, Dean was still uncertain about his result. Will also remained unsure and continued to reflect on what was done and discussed.

93. R: Will? Are you still suspicious?

94. W: [*chuckles*] No, I'm just thinking about that, um…

(V-LARR-RE*; V-VG-RE*)

95. R: Ok, what are you thinking? Tell us. Please.

96. W: …I think it is convergent and yes… I just… No, I just had the reasoning wrong behind it. [*chuckles*] (V-LARR-RE*; V-VG-RE*)

Although both students seemed unsure about their reasoning and conclusion, they still proceeded to the next question (even though they both got the answer wrong by saying that the series converged).

6.2.2 Observation 1: Question 2

In question 2, the students again had to determine the convergence/divergence of the following series:

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

Different from what was observed in the first question, both students orientated themselves in giving information that was associated with the given series. Moreover, the students also had different approaches on how to solve the problem as seen below.

106: D: Ah, let's see. So but if we can… Because if we can… because we can always like start at the divergence test because it's the easiest to apply… (O-LAK; O-APK)

So if this… so if you can apply that limit again like we did on top there [*refers to question 1*] and then if that limit is not zero or approaches it or does not exist then this will be divergent. So let's try that… (P-PI; P-FAP)

107. R: Will is something going on in your head? ...

112. W: I'm thinking this will eventually end up as that one all over n to the power r [*writes* $1/n^r$ *in noting that the given series will 'eventually' become* $1/n^r$] (O-LAK*; O-APK*) …

114. W: I don't really…

115. D: geometric series

116. W: Right. I think it's eventually going to lead to that. And then we know that that tends to zero, I think. [*points to* $1/n^r$ *in saying that it tends to* 0 *as* $n \to \infty$] (O-LAK*; O-APK*, P-PI*; P-FAP*)

117. D: Wasn't that a geometric series what…? A geometric series is like a… [*writes* ar^{n-1} *when referring to 'geometric series'*] (E-LAPR; E-EAP)…

119. D: It's a times r to the n minus one. a times r to the n minus one, that's a geometric series. (E-LAPR; E-EAP)

120. W: Mmm…

121. D: Then only if r, this r, is between zero and one *[points to the* r *in* ar^{n-1}] (E-LAPR; E-EAP)

122. W: Is greater than zero.

123. D: Yes, between minus one and one and then this will converge in a geometric series. (E-LAPR; E-EAP)

124. W: Oh… [*shakes his head*]

Dean followed a similar approach as in question 1 in applying the nth term test for divergence (as seen from line 106). Moreover, as seen from the above situation, Dean also considered what Will was doing and realised that Will wanted to use the concept of a geometric series[20]. Dean started executing Will's strategy to compare the given series with the general geometric series and the conditions under which a geometric series converges. Here Dean pointed out that they could not apply Will's idea and then disbanded Will's strategy. They both continued to execute Dean's proposed strategy in considering the limit

$$\lim_{n\to\infty} \frac{e^{1/n}}{n^2}.$$

[20] A geometric series is of the form $\sum_{n=1}^{\infty} ar^{n-1}$ and is convergent for $-1 < r < 1$.

The students collaborated but were unsure about the method they followed in evaluating the above limit. Moreover, Dean turned to me and said he forgot his 'rules'.

146. D: Sir I'm forgetting now. Sir you forget all your rules. (E-LAPR, E-CMC)…

148. D: I forget all my rules. (E-LAPR, E-CMC)

Will commented that since they do not have the indeterminate form ∞/∞ they could not apply L'Hospital's Rule[21] in order to evaluate the limit. Dean approved of Will's evaluation for solving the limit, while despondent that the rule could not be applied. Dean reverted to the given series

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

and considered the numerator and denominator the fraction respectively. During Dean's explanation he addressed the researcher, while Will followed and understood Dean's explanation. Both students were evaluating the situation by letting go of their initial thinking and concluded that the limit of the sequence is zero. Dean concluded that the series is convergent by giving a similar argument as in the first question. Will agreed that the series is convergent but gave no justification. He merely noted that because the limit exists, the series is convergent.

Again I intervened in asking the students if they were sure that the series is convergent. The students then started commenting on their own personal habits and how they regarded themselves as problem solvers.

[21] L'Hospital's Rule is a rule used in evaluating limits which have to do with indeterminate forms $0/0$ or ∞/∞.

210. R: Ok, so you both agree the series is convergent, and you're finished with the problem now. Do you both feel satisfied that this is the answer?

211. D: Well sir I don't know, I'm forgetting let's prove it mathematically, but logically if I look at it like that, as n increases the terms of each successive term does get smaller. (V-LARR-RE; V-CPH-RE, V-CON-RE)

Dean reflected on the problem and discussed his own attributes as a problem solver. Dean, as before acknowledged that he is confused. He mentioned again that he forgot how to do the problems in a 'mathematical' way and how to apply the correct procedures – this is similar to what he mentioned in lines 146 and 148 earlier.

219. D: It's just about I'm confused. Like I forget… I need to go and do more maths, ok, like after this I'll go and study it. [*laughs*] (V-LARR-RE; V-CPH-RE)…

222. D: [*laughs*] No, because as I said before, so logically when I look at it it seems like the terms are getting smaller and smaller and smaller so it will eventually reach one number. But I just can't prove it mathematically, not yet. (V-LARR-RE; V-CPH-RE, V-CON-RE)…

225. R: Do you mean you're having difficulty to write it down mathematically? Is that it?

226. D: I think like the… Last week's lectures were on this stuff. I wasn't actually paying attention, but what our… But when I sit down and I ook at it I'll remember it. (V-LARR-RE; V-CPH-RE)

227. R: Ok.

228. D: But now it's like I'm thinking about what I remember from class, I forgot how to like apply these things properly. (V-LARR-RE; V-CPH-RE)

The researcher turned to Will to ask him if he had similar experiences to that of Dean. Will said that he had difficulty making deductions after each 'step' of the problem. He also commented that it was easy doing the procedures, but that the reasoning behind why and what he was doing was difficult.

229. R: Ok. How do you feel about that, Will? Do you have similar experiences?

230. W: I do but then my problem is with the deduction. I can get the mathematical proof, it's just like I can do the steps but then the reasoning behind it, and I think I kind of need more, a deeper understanding.

231. W: Because it's very easy to carry out and just follow the procedure without knowing what you're actually doing.

232. R: What do you mean by 'deeper understanding'? More practise, or what?

233. W: More practise. More practise and with the practise actually making sure you know why when it's a specific series you have to use a specific formula.

It was not clear what Will meant by 'deeper understanding' but line 233 suggests that the 'why' and 'when' of the application of the test is difficult. Such procedures correspond to the operational definition of what metacognitive skills entails (even though Will might not have been aware of these skills himself). Will also recognised that through practice one can gain such a ‘deep understanding’ and agreed with Dean’s sentiment that practise is needed when doing mathematics.

Neither student considered whether their solution was indeed correct. Although the series is convergent, on which they both agreed, neither of them applied the correct strategy. The students merely went on to the next question.

6.2.3 Observation 1: Question 3

The third question the students had to do was similar to question 1:

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

Neither student orientated themselves at the start to the question. Will was first to suggest that they follow a similar method as in question 1. Dean agreed with Will’s proposed idea and both students started evaluating the limit of the sequence of the corresponding series

$$\lim_{n \to \infty} \frac{n}{n^4 + 1} .$$

After evaluating the limit, both obtained an answer of zero. This is similar to their answers to the previous two questions, but in this case the students did not immediately conclude that the series is convergent. Both students were uncomfortable with their answers and asked if they could use their textbooks to help them. It seemed that both students were upset to realise they made a mistake – this is indeed the case as seen later in the observation.

Dean paged through his textbook to look for a particular theorem[22] and acknowledged that it was the theorem he had used throughout the observation. He realised that he applied the theorem incorrectly noting that the inverse of the theorem is not true. He also recognises that his ‘logic’ may have been wrong.

[22] The theorem Dean is referring to is ‘Theorem 6’ in their textbook. It states that if a series $\sum a_n$ is convergent, then the limit of the corresponding sequence $\{a_n\}$ is zero. The inverse of this theorem is not generally true: if the limit is zero, the test is inconclusive. The converse of Theorem 6 is referred to as nth term test for divergence. This test states that if the limit of the sequence $\{a_n\}$ is not zero or does not exist, then the series $\sum a_n$ is divergent.

This was not the first time that Dean referred to ‘logic’. This was seen in question 2 where Dean distinguished between doing a problem ‘mathematically’ and looking at it logically (lines 211 and 222).

322. D: Like there that’s what I… You see, this is what I was talking about, like this Theorem 6. If a series oh, that a_n is convergent then the limit is zero.

323. R: Oh, ok.

324. D: But now it says that the, that the inverse[23] is not true, if the limit equals zero you cannot conclude that a_n is convergent. Ah! (V-LARR; V-REF)

325. R: Why are you making ‘Ah!’?

326. D: That’s my, my, my voice in my head when I get something wrong. *[laughs*] (V-LARR; V-REF, taken in conjunction with line 324 above)…

330. D: …I’m thinking because like maybe my logic is flawed in that sense, because now I’m applying the wrong theorem because we’ve been assuming, as it says here, the converse.

In the light of the above discussion the researcher asked the students if their solutions to all three questions were wrong. Dean replied that he wanted to redo the questions, while Will thought their solutions were not wrong. A similar disagreement between the two students was seen earlier in question 1, where Will did not agree with Dean that the series was convergent. During that incident Dean explained his reasoning and how he applied the tests while Will was still unsure if the series was convergent or divergent. Dean went further in explaining

[23] Dean actually used the wrong terminology here, since he had to use the term converse instead of inverse. In line 330 he used the correct term.

to Will the correct application of Theorem 6 and the nth term test for divergence upon which Will saw his misconception.

339. W: …reading Theorem 6, we've proved that the limit a_n is equal to zero, so the series is convergent. If we had found that the limit is not equal to zero…

340. D: What about there, the converse?

341. W: …we would have not proved…

342. D: [*reads from W's textbook*] You see, the converse of theorem 6 is not true. In general if limits of the series is zero it has to be proved then we cannot conclude that the sequence, the series is convergent. That's what I was talking about…

347. R: Will, do you agree with him (Dean)?

348. W: [*after having looked at the textbook*] Now I get it, yes. (V-LARR*)**…**

351. W: Yes, now I get that we were actually proving that it is equal to zero, but that doesn't tell us that it is convergent. (V-LARR*)

In realising his flawed reasoning of Theorem 6 and the nth term test for divergence, Dean paged through his textbook to find a different way to solve the questions. Again, as seen previously in the observation, Dean admitted that he keeps forgetting the content of the work. He turned to the definition of the convergence of a series in using the definition of partial sums[24] of the series (although his understanding of the definition was still flawed). In doing this he was considering a new strategy on how to solve the problem. By referring to the

[24] The definition of the convergence of a series $\sum a_n$ uses the sequence of partial sums $\{S_n\}$, with $S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, \ldots$ and $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. If the sequence $\{S_n\}$ converges to a value S, then the series $\sum a_n$ is convergent with sum S.

textbook he was using an external source to guide him toward a better option of partial sums for testing the convergence/divergence of the series.

Will consulted his personal notebook, instead of the textbook to try to understand Dean's discussion on partial sums. After reading his notes Will turned to Dean to acknowledge Dean's correct definition of convergence. This was the only instance in which Will tried to orientate himself around Dean's discussion.

378. W: [*reads from his notebook*] It says convergence of a series is defined by the series of partial sums and so on. You are actually right… (O-LAK; O-APK)

379. D: That was true.

380. W: Yes.

381. D: We wrote those notes in class. [*refers to Will's notes*]

382. W: In the lecture.

383. D: That's a geometric series that example, this one, yes, sorry. [*D points to an example of a geometric series in W's notes*] (O-LAK; O-BMM, O-UES)

384. W: And our first one wasn't geometric was it? [*pages through his work done during the observation*] (V-LARA*; V-VG*)

385. D: It was geometric? *[D looking through his textbook and at W's notes, while W is looking at his notes from lecture, while both students consider question 1 of the task*] (V-LARA, V-VG)…

388. D: No, we can't do that one.

389. R: Which one? Tell us?

390. D: I was just thinking of the test for divergence shows that if the limit's not equal to zero or if it doesn't exist then the whole series is divergent, so… But none of our limits were not equals to zero, didn't exist, so we can't use the divergence theory there. (V-LARA, V-VG)

…

399. W: No, he's right. Um, well my notes are saying [*smiles*] if the limit as n to infinity of a series a_n is not equal to zero or the limit doesn't exist then the sum of a_n is divergent. And that's the divergence test. And we can't actually do the divergence test because as you [*refers to Dean*] say they are all equal to zero. (V-LARR*; V-REF*)

It is interesting to note here that both students acted as social triggers for each other's metacognitive behaviour. Starting with Will referring to his lecture notes a 'chain reaction' occurred where one student acted as a social trigger for the other and vice versa. This is illustrated in Figure 6.1 below.

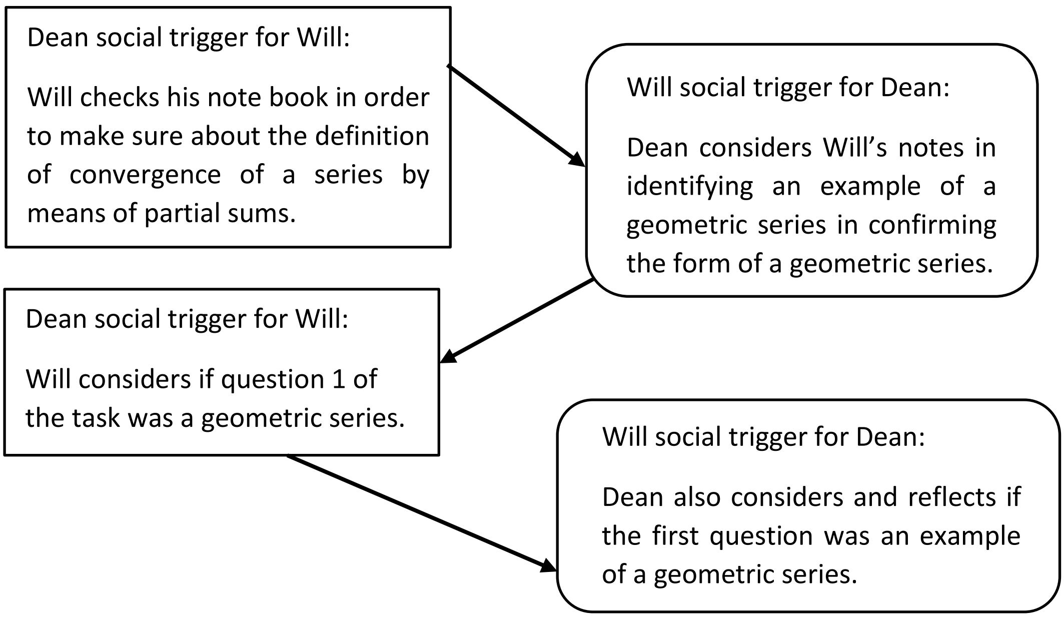

Figure 6.1: Students as Social Triggers of Each Other's Metacognitive Behaviour in Observation 1

Dean suggested that they then proceed with his proposed plan of partial sums. Unfortunately the researcher had to stop the observation at this point since the next observation with another pair of students was about to start.

6.2.4 Summary of Observation 1

It became evident quite early in the observation that the researcher played a prominent role in urging the students to make it clear what tests and methods they were using; the difference between a sequence and series; and if the students were applying the nth term test for divergence correctly. Prompting by the researcher revealed that students struggled with the terminology of sequences and series and that they had difficulty in recalling what the tests entailed. Throughout the observation Dean was more verbal and explained his reasoning while Will more often than not agreed with Dean and did not always justify his answers. The researcher had to urge Will several times to collaborate in the problem solving process. Eventually Will collaborated more with Dean as they started working on question 2.

Throughout the observation Dean repeatedly noted that he (i) forgot the names of the strategies (tests) and methods they were applying; (ii) kept confusing the concepts of sequences and series; and (iii) forgot how to apply the correct procedures to solve the problem (as discussed with the students during formal lecture times). In question 2 the students spoke about their personal attributes as problem solvers and beliefs on mathematics. Dean reiterated that he forgot the ‘rules’ for solving the problems ‘mathematically’ instead of approaching the problem in a ‘logical’ way. Will admitted that he found it easy to do the procedures (the ‘steps’) for solving the problem, but that he had difficulty with reasons for deciding what steps to apply. Both students agreed that practice is needed in order to do mathematics. Will felt that through practice he would gain a ‘deeper understanding’ of the work while Dean argued that through practising he would understand the work better and be able to solve the problems.

During the task both students used the nth term test for divergence in questions 1 to 3. Although their reasoning and application of the test was incorrect they still progressed to show that the first two series were convergent. The students did not make any conclusion on the convergence/divergence of the third series because Dean started questioning himself and realised that they were applying the nth term test incorrectly. Dean realised their mistake only when they turned to their textbooks for guidance. He commented that his 'logic' was flawed.

The students also acted as social triggers for each other's metacognitive behaviour. In question 2 Will considered using the definition of a geometric series and L'Hospital's Rule to work on. Dean intervened when Will wanted to use a geometric series which would not lead to an answer. Will therefore acted as a social trigger for Dean's metacognitive behaviour in deliberating the implementation of a geometric series as a strategy. When Will realised that L'Hospital's Rule was unsuitable for evaluating the limit Dean concurred. In this instance Will acted as a social trigger for Dean's metacognitive behaviour.

Dean acting as a social trigger for Will's metacognitive behaviour mostly occurred where he explained the application of the nth term test for divergence to Will. This occurred in question 1 and twice in question 3. Moreover, different to the dynamics in the beginning of the observation Will justified why he agreed with Dean. This may be due to Will starting to understand the work because of Dean's assistance and explanations.

Furthermore, in question 3 and as illustrated in Figure 6.1, a 'chain reaction' occurred where one student acted as a social trigger for the other and vice versa.

Not all the metacognitive behaviours of the students manifested in observation 1 have been discussed. Table 6.1 below, however shows the indicators of the metacognitive behaviour of each student. The table shows the extent to which Dean (D) and Will (W) exploited their knowledge and the manner in which they monitored their progress. The types and frequency of Formulating Plan (FP) and Local Assessment (LA) metacognitive decision points initiated by each student across the observation were recorded. This is discussed below Table 6.1, where the differences in the students' metacognitive behaviours are compared using the frequency and types of metacognitive decision points.

Table 6.1: Metacognitive Skills of Each Student during Observation 1

MCDP	Dean	Will		MCDP	Dean	Will
P-PI	2	3		**P-PI-RE**	0	0
P-NI	2	0		**P-NI-RE**	0	0
Total P-FP	4	3		**Total P-FP-RE**	0	0
O-LAK	1	2		**O-LAK-RE**	0	0
Total O-LAK	1	2		**Total O-LAK-RE**	0	0
E-LAPA	2	4		**E-LAPA-RE**	0	0
E-LAPR	5	1		**E-LAPR-RE**	0	1
Total E-LA	7	5		**Total E-LA-RE**	0	1
V-LARA	0	2		**V-LARA-RE**	0	0
V-LARR	5	3		**V-LARR-RE**	6	4
Total V-LA	5	5		**Total V-LA-RE**	6	4

Codes for metacognitive decision points (MCDP)

O-LAK = local assessment of knowledge or knowledge building (Orientation)

P-PI = proposed idea/plan (Planning)

P-NI = new idea/plan (Planning)

P-FP = formulate plan (Planning)

E-LAPA = local assessment of accuracy of execution (Execution)

E-LAPR = local assessment of usefulness/reasonableness of procedure (Execution)

V-LARA = local assessment of accuracy of result (Verification)

V-LARR = local assessment of reasonableness/usefulness of result (Verification)

As pointed out in the observation, students did not orientate themselves to question 1 before considering what strategy to use to solve the problem. The only exceptions were with Will in question 2 and question 3. Both students contributed equally in formulating plans (strategies) to solve the problems. Neither of them needed any assistance regarding the test or strategy to solve the problems.

Will focused mostly on monitoring the accuracy of his calculations and working (E-LAPA), while Dean took the lead in making local assessments in monitoring the usefulness of the strategies and procedures they applied during Execution (E-LAPR). Will made fewer local assessments on the reasonableness of their procedure since he merely agreed with Dean without giving a justification. There were also a number of times that Will was uncertain about how the tests should be applied during execution. Dean also had to explain and correct Will on how to apply the nth term test for divergence. This may explain why Will made fewer local assessments on the usefulness of the executed strategies.

During Verification Will was more concerned with verifying the accuracy of their calculations than Dean. During question 3 when the students turned to their textbook, Will checked if the first question was not a geometric series. Discussions and explanations of the nth term test (during Verification) were mostly driven by Dean who was the main agent in realising that they applied the test incorrectly. It was only at the end of the observation that Will finally realised their misunderstanding of the nth term test for divergence.

Students' local assessments made because of intervention from the researcher, occurred most frequently during Verification. This was because the researcher questioned the students multiple times as to whether their application of the nth term test was correct. Other instances that contributed to the higher frequency of this local assessment are because of the students' confusion around terminology, and forgetting the content and/or theory of the work (as seen for example with Dean).

6.3 Observation 2

In observation 2, the objective of the task was similar to that of observation 1. Students had to determine the convergence/divergence of the following three series:

(1)

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$$

(2)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$$

(3)

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos(n\pi)$$

All three series were alternating series in which the students only had to apply one test to determine the convergence/divergence of the given series: the alternating series test.

Since this test pervades this observation, it is important to discuss what it entails. In this test, two properties need to be satisfied in order for an alternating series to be convergent. Note that all three the series of this observation were convergent.

Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ ...$$

with $b_n > 0$*, satisfies the following two conditions*

(i) $b_{n+1} \leq b_n$ *for all* $n \geq k$ *for some positive integer* k*, and*

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Condition (i) means that the sequence $\{b_n\}$ needs to be decreasing. We say further that a sequence is monotone if it is either increasing or decreasing (to be monotone it cannot be both increasing and decreasing). For example, if we have $b_{n+1} \geq b_n$ for all $n \geq k$, for some k a positive integer, then we say that the sequence $\{b_n\}$ is monotone increasing (or just increasing). For property (ii), the sequence $\{b_n\}$ must only converge to the value 0.

Since sequences played such an important role in each of the questions, I first started the observation by asking both students what they had studied on sequences and series so far on their own, including the alternating series test. My reason for this was because in observation 1 the students had some confusion about the differences between sequences and series. Moreover, they focused on using only one test in solving the questions of observation 1 instead of considering a number of other tests that they could have used. In asking the students what they had studied I was able to obtain a clearer picture of the knowledge and practice students had had on the content before they started with the questions of observation 2. Information gathered from this enquiry was beneficial in considering the students' problem solving progress and the metacognitive skills they exhibited during the observation.

Dean answered my question with much confidence. He listed the sections he had studied from their prescribed textbook by counting them on his fingers. This was the first instance that showed that Dean prepared for observation 2, evident in the dialogue below.

5. D: No, so far since last week's disaster I made sure I knew 11.3, 11.4 and 11.5 now. Hopefully 11.2 will come before next week.[25]

Will did not study all the sections as in much depth as Dean did. He focused only on section 11.1 on sequences[26]. He found section 11.5 quite easy although it seemed he had difficulty when doing the problems on his own. Furthermore, as observed later in the observation, it seemed that Will had difficulty in using and applying the theory on the alternating series test when trying to solve the problems. His difficulty was already evident in his words in line 25 below.

18. W: I studied 11.1 in detail and the others I sort of went through. Like I did a few of the examples, but not really exam type study. [*pulls his face*]…

23. W: … And 11.5, ah, that, that seemed so simple.…

25. W: It seemed really simple until I started doing it. [*pulls his face*]

6.3.1 Observation 2: Question 1

After my enquiry, both students considered the series of the first question:

$$\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$$

---

[25] Sections 11.2 – 11.5 dealt with sequences and series in the students' prescribed textbook. Section 11.2 addressed the foundations and first definitions on the convergence/divergence of series, as well as geometric series. Section 11.2 was the section which Dean had difficulty with in observation 1, especially on the nth term test for divergence. Sections 11.3 – 11.4 dealt with other tests on the convergence/divergence of series which the students could have used in solving the questions of observation 1, but which they did not apply. Section 11.5 focused only on the alternating series test.

[26] Section 11.1 dealt exclusively with the properties of the convergence/divergence of sequences.

Will wanted to start with the second question that contained a factorial[27], while Dean wanted to start with the first question. It appeared that Dean was uncomfortable with problems involving factorials (cf. in line 32 below). This was confirmed later in the observation where Dean had difficulty in working with factorials. Different to what was seen in observation 1, both students orientated themselves about the first two questions regarding which was the easier to work with first.

29. W: Can we start with the second one instead of the first one? (O-LAK*; O-BMM*)

30. D: Why, what's wrong with the first one?

31. W: Um…

32. D: But the second one $2n$ plus 1 factorial that will like… be whoah! (O-LAK*; O-BMM*)

33. W: Well, ok, you can start with the first one.

34. D & W: [*D & W laugh*] …

40. W: Um, I actually think the second one would be easier than the first one. (O-LAK*; O-BMM*)

Apart from their differences the students agreed to start with question 1. Dean commented that he remembered doing this same question on his own as it formed part of the exercises from their textbook. This was another instance that confirmed that Dean did indeed practice beforehand for observation 2.

I reminded the students again of the difference between a sequence and a series. I did this intentionally to help myself have a better understanding of their verbal

[27] The factorial is a function that computes the product of the first n natural numbers, written as $n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 4 \cdot 3 \cdot 2 \cdot 1$. For example $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. By convention $0! = 1$.

and non-verbal actions when analysing the video transcripts. Moreover, I also reminded the students about this difference since in observation 1 the students were confused about sequences and series, as well as the tests used to determine convergence/divergence of a series. Dean however was still confused about the difference between sequences and series (similar behaviour was seen later in the observation).

Apart from the students orientating themselves about the question, the students also started interacting with each other much earlier in observation 2 than in observation 1. During this interaction both Dean and Will acted as social triggers for each other's metacognitive behaviours.

58. D: So, um, Will, you said it's like that alternating series, right? (O-LAK; O-BMM)

59. W: Yes

60. D: It's minus one to the power of n. So…

…

62. W: Yes, it is. Yes, it is. We can begin with this one.

63. D: So but what does the…

64. W: First thing we have to do? (P-FAP*; P-PI*)

65. D: … the properties of the alternating series they said that b_{n+1}… (P-FAP; P-PI)

66. W: b_n must be positive. (P-FAP*; P-PI*)

67. D: Must be greater than b_n so…

68. W: Or greater than zero. Yes. (O-LAK*; O-APK*)

69. D: And the limits of b_n must be, equal zero.

70. W: No.

71. D: Those are the two alternating series properties. (P-FAP; P-PI)

72. W: Um, the limit of b_n must be greater than zero mustn't it? It has to be greater than zero. It has to be… No, it has to be greater than…

As seen from line 72 we note that Will was not that sure of the properties of the alternating series test. Will then turned to his personal notes even though Dean alerted Will to the two properties of the alternating series test. In this sense Dean acted as a social source trigger of Will's metacognitive behaviour. A similar occurrence was seen in observation 1, question 3 where Will also turned to his notes to check and understand Dean's discussion of the definition of the convergence of a series by means of partial sums.

The students first started checking if the corresponding sequence $\{b_n\} = \left\{\frac{\ln n}{n}\right\}$ of the given series $\sum \frac{(-1)^n \ln n}{n}$ was indeed decreasing[28]. Both students had difficulty proving this. Dean wrote down the following in which he stated that the sequence is increasing (which is not the case) and explained and justified his reasoning:

91. D:

$$b_n = \frac{\ln n}{n} < b_{n+1} = \frac{\ln(n+1)}{n+1}$$

I then intervened to make sure that Dean was clear about what he was trying to prove. Again, Dean revealed that he is confused between a sequence and a series (similar to what was observed in observation 1).

Dean returned to the inequality he had written down (cf. line 91) and scratched out the inequality sign without giving a reason for his action. The reason for his action is also not clear. In having difficulty showing that the sequence was decreasing, he started by trying to evaluate the limit of the sequence. Will on the

[28] A sequence $\{b_n\}$ is decreasing if $b_n \geq b_{n+1}$ for some positive integer k such that $k \geq n$.

other hand was mostly quiet with Dean working ahead of him. This was a reoccurring theme in this observation. Will wrote down the following:

127. W:

$$b_n = \frac{ln\, n}{n}$$

$$b_{n+1} = \frac{ln(n+1)}{n+1}$$

$$b_{n+1} \leq b_n$$

It is important to note that Will did not explain his reasoning or justify why the above inequality was true. Since it was not clear if Will was able to prove that the sequence was decreasing, I encouraged the students to work together to compare their work. Will started explaining his work to Dean, while Dean monitored and checked what Will had done.

129. W: Ok. [*points to his work*] We just have to prove that, so now I'm going to plug these two in just to show, just to show that the first part is actually true because n plus one is, well it makes the denominator bigger which makes the whole thing smaller in turn.

130. D: But $\ln n$ is also increasing. [*points to the* $\ln n$ *in the numerator of* b_n] (E-LAPA)

131. W: Mmm

132. D: And that's what I was…

133. R: …What is he showing about this sequence. [*points to the inequality on W's page*] What does this say about the sequence?

134. D: That it's decreasing.

135. R: Ok

136. D: Sorry, we want to find that… I think since we're like undecided about this we should just move on to… Because I always I get stuck trying to prove…

137. D: … this [*points to the inequality in his work*], like just directly from looking at the terms, then you try finding the derivative of the associated function to see if it's decreasing or increasing. (E-LAPR)

It is interesting that Dean was able to recognise that the sequence is decreasing when considering the inequality in Will's work (line 134 above), but he did not realise that the inequality in his own work actually stated that the sequence is increasing (cf. line 91 above). As noted earlier, Dean also scratched out the inequality sign in his work (I was not sure what the meaning behind his action was). This scenario may lead to the conclusion that Dean had difficulty in proving that the sequence $\{b_n\}$ is decreasing – evident from his statements in lines 136 and 137 above. Dean made a local assessment by noting that the method they were using to show that the sequence is decreasing could have been replaced by another method. This is an instance where Dean ruminated on the usefulness of their procedure of considering an alternative method of showing that the sequence is decreasing. This method required that Dean consider the associated function $y = f(x) = \ln x / x$ of the sequence $\{b_n\}$ where $f(n) = b_n$. Dean noted that by showing that f was a decreasing function, the sequence $\{b_n\}$ would also be decreasing[29].

Dean started determining the derivative of f in which Will acted as a social trigger for Dean's metacognitive behaviour by confirming that they had to use the quotient rule to determine the derivative of f. Dean predominantly led the problem solving process by writing and calculating the derivative, while Will mostly watched what Dean was doing. When Will did make comments on Dean's working, it was still not that clear if he understood what Dean was doing, or

[29] A function f is said to be decreasing if $f'(x) < 0$ for all x in that interval. Also, a sequence $\{a_n\}$, with $a_n = f(n)$, will also be decreasing for all $x \geq k$ for $k \in \mathbb{Z}^+$ if $f'(x) < 0$ for all $x \geq k$.

monitoring the problem solving process and merely copying Dean's work. Moreover, Will was also behind Dean in his working – similar to what we have seen before in this observation.

The students went on to determine the limit of $\{b_n\}$ in considering the limit $\lim_{n\to\infty} \frac{\ln n}{n}$. Will was still confused and had difficulty in understanding Dean's working.

187. W: No [*laughs*] I went through it. I'm thinking it's actually finding how when I am using L'Hospital's Rule [*points to Dean's limit*] the alternating series test like links back to another test (E-LAPA*; E-FSA*), I forgot what it actually was where we actually do, do this and we find the derivative. It was in my notes but, yes, remembering notes… (E-LAPA*; E-CMC*)

Line 187 and the discussion after it, illustrates that Will was reflecting on how the properties and calculations to do with sequences played an important role in problems on alternating series. This was quite in contrast to what was seen in the beginning of the observation where Will mentioned that he studied the textbook section on sequences in depth (line 18). Apart from this, Will also mentioned that he had difficulty when doing the problems on the alternating series test on his own (lines 23 and 25). In view of the above it seems that Will still had difficulty working with sequences compared to Dean's approach to solving the problem.

In contrast to questions 1 and 2 of observation 1, where both students did not verify their work or made any conclusions on their own after solving the problems, Dean did reflect on and checked his solution of question 1 in observation 2.

Will was still catching up with Dean and also wrote down that the given series was convergent. Still there was no clarity that Will understood what Dean had done and that he had checked his own solution to ensure that the series was indeed convergent. With this, I decided to intervene and ask Will

(i) if he had done similar work and problems before;

(ii) that while solving the problem at hand if he had been reflecting on past work they had done;

(iii) what was it he was trying to recall during the problem solving process; and

(iv) did some of the methods they were using in solving the problem look familiar to work he had done in the past?

Will mostly answered yes to my questions without elaborating or giving any reasons. When Will started explaining to me what he did understand from Dean, Dean was also monitoring what Will was saying since he corrected Will in line 241 below. Here Dean acted as a social trigger for Will's metacognitive behaviour.

235. R: Just say it as it is, what comes into your mind. What makes sense?

236. W: [*laughs*]

237. D: This stuff [*points to his own solution of question 1*]

238. W: Exactly – that's what comes into my mind. This stuff [*refers to Dean's solution of question 1*]. Um, ok, um, in the second part we were using L'Hospital's Rule to like find the integral… (V-LARR*-RE; V-REF*-RE)

239. R: Mmm.

240. W: But…

241. D: The derivative. (V-LARA-RE)

242. W: The derivative, not the integral, of this whole function to prove that it is equal to zero in the end. So it's just… (V-LARA*-RE)

243. R: It's just what?

244. W: [*laughs*]

245. R: It reminds you of something in the past? …

249. W: And I think it's… I don't know.

During the above discussion Will had difficulty voicing his thoughts. Dean interjected and started discussing his personal views on mathematics. This is another case of Dean taking a leading role in the observation. Dean reflected on the mathematical procedures they used in question 1, which were based on and included work they had previously studied and had been tested on (the codes V-LARR-RE, V-REF-RE applied here). Dean also commented on his personal habits by declaring that even though he did not remember all the content from his first year calculus course, he was still able to recall some of the work (the codes V-LARR-RE, V-CPH-RE were applicable here as well). He went further to mention his fondness for mathematics and that he enjoyed doing mathematical problems. Will also contributed to the discussion in agreeing with Dean that mathematics plays an important role in their other subjects. In contrast to Dean, Will does not enjoy mathematics as much as Dean does. Moreover, while Will was discussing his above views on mathematics, Dean had started to write down the second question of the task.

6.3.2 Observation 2: Question 2

In question 2 there were a number of instances in which I noted Dean's uncertainty when working with the factorial in the given series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$$

Line 32 earlier reflects Dean's uncomfortableness about how to approach question 2. Dean's comment below was an indication on how he tried to orientate himself to the problem, as well as identifying the most important and striking feature of the series – that of the factorial function[30].

[30] The factorial is a function that computes the product of the first n natural numbers, written as $n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 4 \cdot 3 \cdot 2 \cdot 1$. For example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. By convention $0! = 1$.

285. D: How do I solve the factorial? (O-LAK; O-IMP, O-BMM)

Recognising that Dean's uncertainty about the factorial function is considerable I encouraged the students to consider how the above series was related to the definition of a general alternating series. By doing this I was actually suggesting to the students what strategy (plan) they had to implement to solve the problem – that of the alternating series test. Because of this neither student exhibited any form of planning for question 2. Will did not orientate himself about the question whereas Dean did. Both students started implementing the alternating series test, with Dean again taking the leading role in monitoring the problem solving process.

In trying to evaluate the limit on the sequence $\{b_n\} = \left\{\frac{1}{(2n+1)!}\right\}$ Dean was uncertain whether it was possible find the limit of a factorial. As in observation 1 Dean tried to solve the problem using his 'logic' rather than prove it 'mathematically' (cf. line 295 below).

290. D: I see this as, the one over that factorial is the b_n part, but I don't know, is it possible… I don't think it's possible to take the limits over the factorial series because they are not continuous terms. So… (E-LAPR; E-ECS)

293. D: ….but I haven't seen a factorial function, I don't, I never saw…. (E-LAPR; E-ECS)

294. R: Have your lecturers not done any factorials with you this whole semester so far?

295. D: I think they done it, but I can't recall it, but I think logically if I look at this if it's 1 over the factorial series it will keep on being 1 over 2, 1 over 6, 1 over 24, then it will go the whole time, so it's decreasing logically, but I can't prove it mathematically. That's what I think. (E-LAPA; E-EAP)

296. R: Did you write it down? Write it down, work with each other.

297. W: Ok, yes.

298. R: What's wrong Will? Why are you…

299. W: No, I get what he's saying [*refers to Dean*]. It's decreasing because it's 1 over the factorial and when you're doing factorials you go from the largest to the smallest in the whole… You go from the largest to 1, basically. So it's going be 1 over 1, ok, that's 1…(E-LAPR*; E-CLU*)

In line 296 above I urged the students to write down what they were thinking and to interact with each other. This is just one of multiple cases in which I had to urge the students repeatedly to write down their work in order to structure the problem solving process, and to work together in solving the problem. In line 299 Will understood Dean's argument that the sequence is decreasing. This was regarded as an example of metacognitive behaviour since Will justified his understanding in terms of the definition of the factorial function by noting that the denominator of $b_n = \frac{1}{(2n+1)!}$ becomes bigger and hence b_n decreases in value. Furthermore, for the first time in this observation, Will also started contributing to the problem solving process of his own accord without any urging from me – he started drawing a sketch in order to visualise how the above sequence was indeed decreasing. Will was checking (verifying) for himself that the sequence was decreasing (by using a sketch). Dean monitored and evaluated what Will was drawing.

308. W: Now, I feel like drawing it now. Ok, not drawing it, but like drawing what I think… (E-LAPA*; E-ECS*)…

313. W: No, um, ok, that's basically going to be one, let's say… [*drawing a sketch in a* xy*-coordinate system he has sketched*]

314. D: One over one so it's… [*watches while Will is drawing*]

315. W: One over one, right. And then it's going to be one over three…

318. W: Over six?

319. D: It's one over… so it's becoming like it's over here. [*points to Will's work*]

320. W: One over six, oh!

321. D: It's making like a hyperbola there. [*points to W's work*] (E-LAPA; E-ECS)…

323. W: But then at the end it has to get to one over…

324. D: But one over infinity is equals to… to zero because it's going to be infinity factorial which is infinity times infinity.

Line 324 is significant because Dean recognised that the limit of the sequence $\{b_n\}$ was zero as n tends to infinity, although later in the observation. Dean was unsure if it was possible to apply limits to sequences containing factorials. Although the above scenario illustrated how the students interacted to justify their argument that the sequence was decreasing, they still had not formally proved it analytically. Again, I urged the students to work in a more structured manner by writing down their arguments. Both students started determining the expressions for b_n and b_{n+1} although their results differed. Dean only wrote out what the expressions were for b_n and b_{n+1}, while Will placed an inequality sign between b_n and b_{n+1}:

341. W:

$$b_n \geq b_{n+1}$$

$$\frac{1}{(2n+1)!} \geq \frac{1}{\big(2(n+1)\big)!}$$

It is not that clear if Will understood why the inequality held true. His action here was similar to what was observed earlier in question 1 (cf. the discussion around line 127) in which he merely wrote down the inequality. In asking the students to consider each other's work Dean did not raise a discussion on whether Will's inequality was indeed true. He also did not ask Will why his inequality was true. Dean was more concerned about the inclusion of the equal sign. Dean commented that Will's inequality

$$\frac{1}{(2n+1)!} \geq \frac{1}{\big(2(n+1)\big)!}$$

should actually have read

$$\frac{1}{(2n+1)!} > \frac{1}{\big(2(n+1)\big)!}.$$

365. W: I didn't know we were supposed to think about that. [*laughs, and with the word 'that' he refers to the equality sign that forms part of the inequality sign*] (E-LAPA; E-ECS)

366. W: I just thought we were supposed to put it there to show that at some point it could be equal to. [*with the word 'it' he refers to the equality sign that forms part of the inequality sign*] …

368. D: Maybe we could… Ok, you're right. You're right [*addresses Will*]. On this side, looking at the left side, but if you look at the right side as n approaches infinity, both of them do actually equal zero in the end [*points to the inequality Will has written*]. (E-LAPA; E-ECS)

I wanted Dean to consider whether Will's inequality in line 341(b) was indeed true. Although this did not happen, my interrogation created a space for student interaction where Dean acted as a social trigger for Will's metacognitive behaviour (line 365) and vice versa (line 368). Line 368 is a second instance in which Dean again implicitly considered the limits of the factorials as n tends to infinity (similar to line 324). This is important to bear in mind because earlier in the observation Dean was not sure how to evaluate limits containing factorials. Moreover, in considering if Will's inequality was indeed true Dean again used the terms sequence and series interchangeably in saying that the series was decreasing (which is not the case). This is similar to what was observed in question 1 earlier, as well as in observation 1.

When the students started to prove the second property of the alternating series test, Dean was again bothered by the factorial and had difficulty evaluating the following limit:

$$\lim_{n\to\infty} \frac{1}{(2n+1)!}$$

As noted previously, Dean was unsure on how to evaluate the limits of factorials. Realising Dean's difficulty and Will's minimal input in solving the problem, I asked Dean if he had not already evaluated the limit of the sequence earlier as was the case in lines 324 and 368.

391. R: Now what was very interesting is what you said here [*points to the inequality in Will's work, in line 341*] and more specifically when we looked at Will's work, as n tends to infinity both these factorials become bigger and bigger and what happens with all the fractions there?

392. D: Equal to zero…

395. D: Yeah, but logically I thought that, that was like in words… but to write it down in maths is a different thing…(E-LAPA-RE; ECS-RE)

Again, we note that Dean had difficulty in representing his results 'mathematically'. He could logically argue about how to obtain the solution, but found it difficult to work within the mathematical discourse and symbolic notation.

Similar instances were frequently seen in observation 1, and earlier in line 295 of question 2.

The students started writing out their final solution while working independently. Will still had difficulty knowing where to start writing down his solution whereas Dean realised that it was indeed possible to evaluate limits containing factorials. It also seemed that Will had a similar realisation, although there was not enough evidence to confirm that Will indeed understood. Neither of the students reflected on their solution or the problem solving process, or checked if their solutions were correct. Instead, they started with the last question of the observation.

6.3.3 Observation 2: Question 3

The series of this question was not given in an explicit form of an alternating series $\sum(-1)^n b_n$, but instead as

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos(n\pi) .$$

The crux of the question was that the students had to take into account that for n even $\cos(n\pi) = 1$ and for n odd $\cos(n\pi) = -1$, hence simplifying the given series as:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} .$$

It is then easy to show that the above series is convergent by the alternating series test in considering the sequence $\{b_n\}$ to be $\left\{\frac{1}{n}\right\}$.

Both students did not mention what plan (strategy) to follow to solve the problem. It may have been the case that both students assumed that the third question would automatically also be an alternating series. What was not apparent at this stage was that Dean had done a similar problem before (this is discussed later in the observation, in lines 536 and 538). This may have resulted in Dean realising that he was able to immediately apply the alternating series test, and that he had no difficulty in solving this question. Moreover, Dean also mentioned that this question is easy.

Noting that the value of $\cos(n\pi)$ alternates between minus one and one, Dean explained his reasoning for being able to write the given series as an alternating series. Will was uncertain about $\cos(n\pi)$ in not understanding that the n in conjunction with π produced the alternation between minus one and one. Will's confusion over this continued to occur throughout the remainder of the observation. Furthermore, Dean also acted as a social trigger for Will's metacognitive behaviour in assisting Will to reflect on how $\cos(n\pi)$ can be regarded as the $(-1)^n$ factor of an alternating series.

Will also made a mistake in writing $\cos(n\pi) = (-1)^{n+1}$. Asking Dean to consider what Will had written, the students interacted for a quite a long time. Dean eventually pointed out Will's mistake which Will corrected. Once again, Dean served as a social trigger for Will's metacognitive behaviour which led to Will checking his calculations and reasoning. Had I not intervened this interaction may not have occurred.

While Dean started to write out the terms b_n and b_{n+1} in order to prove that the sequence $\{b_n\}$ was decreasing, Will was still unsure about the factor n within $\cos(n\pi)$ and turned to Dean to ask for assistance. In obtaining more clarity, it appeared that Will was finally understanding that the n in conjunction with π produced the alternating signs.

When starting with question 3, Dean mentioned it was easy. The reason behind this was because he had already done a similar question, as seen from lines 536 and 538 below. This also confirmed that he needed to apply the alternating series test although he did not verbalise what strategy he was going to implement at the start of question 3.

536. D: Well there's also a question like this in our exercises and it looks hard for us, but then you see this again and then…

537. R: You see a pattern, as you said a few moments ago.

538. D: The alternating series and you're like 'yay' and they become easy because then you're just left with one term over another…
(E-LAPR; E-FSA)

Dean moved on to prove that the sequence $\{b_n\}$ was indeed decreasing and gave a clear argument for his proof. Will on the other hand, wrote out the following

546. W:

$$b_n = \frac{1}{n} \qquad b_n \geq b_{n+1} \qquad b_{n+1} = \frac{1}{n+1} \qquad \frac{1}{n} > \frac{1}{n+1}$$

From what Will had written above it was not that clear that he necessarily understood how to show that $\{b_n\}$ was decreasing and his procedure was seemingly rote. It seemed that he merely wrote down his solution without understanding or justifying why the sequence was indeed decreasing. He seemed not to engage with the 'why', 'what' and/or 'how' of what he was doing. Line 547 below seems to affirm this. Moreover, as in question 2 he found the reasoning behind the question difficult.

547. W: This is always how I think of studying it because the, the first part of the alternating series test proof, well not proof, but the test, is to show that b_n is greater than, or it's equal to, b_{n+1}. So that, that just shows that it is decreasing and while it kind of shows that we can use the alternating series test. And then the second part will be checking the limit.

Neither of the students had difficulty in showing that the limit of the sequence $\{b_n\}$ was indeed zero. Dean checked if both properties of the alternating series were indeed met, while Will was still busy calculating the limit. Only Dean verified his work. Will merely wrote down that the series was convergent. At the end of the task, I again asked the students if the problems were difficult. For Dean it was not that difficult in comparison to the previous observation. Dean's preparation for this observation may have contributed to him finding the questions of this task much easier than those in the previous observation.

6.3.4 Summary of Observation 2

Table 6.2 below illustrates the number of different metacognitive decision points for each student respectively.

Table 6.2: Metacognitive Skills of Each Student during Observation 2

MCDP	Dean	Will		MCDP	Dean	Will
P-PI	1	1		**P-PI-RE**	0	0
P-NI	0	0		**P-NI-RE**	0	0
Total P-FP	1	1		**Total P-FP-RE**	0	0
O-LAK	8	4		**O-LAK-RE**	0	0
Total O-LAK	8	4		**Total O-LAK-RE**	0	0
E-LAPA	11	7		**E-LAPA-RE**	5	1
E-LAPR	7	1		**E-LAPR-RE**	1	0
Total E-LA	18	8		**Total E-LA-RE**	6	1
V-LARA	2	0		**V-LARA-RE**	1	1
V-LARR	0	0		**V-LARR-RE**	1	1
Total V-LA	2	0		**Total V-LA-RE**	2	2

Codes for metacognitive decision points:

O-LAK = local assessment of knowledge or knowledge building (Orientation)

P-PI = proposed idea/plan (Planning)

P-NI = new idea/plan (Planning)

P-FP = formulate plan (Planning)

E-LAPA = local assessment of accuracy of execution (Execution)

E-LAPR = local assessment of usefulness/reasonableness of procedure (Execution)

V-LARA = local assessment of accuracy of result (Verification)

V-LARR = local assessment of reasonableness/usefulness of result (Verification)

The only instance in which the students formulated a plan (strategy) (P-FAP, P-PI) occurred in question 1. The relatively small frequency of this metacognitive decision point was because Dean had prepared beforehand for this observation and mostly took the lead. This was evident in question 3 and may be the reason why Dean did not explicitly mention what strategy to use. With question 2, Dean had difficulty working with factorials. In assisting the students with this question, I might have led them to automatically implement the alternating series test. Thus I told the students what strategy to follow. Hence we have no occurrence of the metacognitive decision point P-PI for questions 2 and 3.

Since the alternating series test was the only correct strategy to use throughout the observation, students did not need to apply any alternative strategies. This is the reason for no occurrences of the metacognitive decision point P-NI. Moreover, Will mostly observed what Dean was doing and rarely suggested what approach to follow. This was seen repeatedly throughout the observation in which Dean took a leading role in the problem solving process.

From Table 6.2 we note that Dean produced more local assessments in orientating himself about the problem. The majority of instances where students orientated themselves to the problems were before and at the start of question 1. This was because the students argued about which question to start with and outlined the properties of the alternating series test. Furthermore, during Orientation of question 1, both Dean and Will acted as a social trigger for each other's metacognitive behaviour. In question 3 Will's difficulty and confusion about the n in $\cos(n\pi)$ prevailed throughout question 3. Dean did not experience this difficulty and was able to tell Will that that $\cos(n\pi)$ could be reduced to $(-1)^n$. In this case, Dean acted mostly as a social trigger for Will's metacognitive behaviour.

When implementing the alternating series test Dean made more local assessments on the accuracy of his calculations and/or procedures (E-LAPA and E-LAPA-RE) of his solutions. These local assessments indicated the points where Dean was:

(i) monitoring his progress in proving that the sequence was decreasing when comparing terms b_n and b_{b+1} (question 1) and calculations around the derivative (question 2);
(ii) checking the value of the index at which the series started (question 1);
(iii) checking the accuracy of his notation (questions 1 to 3); and
(iv) monitoring what properties of the alternating series tests he applied to obtain his solutions (questions 1 and 2).

Other instances of local assessments of calculations and procedures were where Dean interacted with Will. Student interaction mostly occurred because of my intervention in urging the students to work together; in particular when I asked Dean to check and comment on Will's work. In these cases, I acted as a social trigger for Dean's metacognitive behaviour.

Throughout questions 1 to 3, Dean checked Will's work. Question 2 was of particular interest since instead of Dean checking that Will did prove that the sequence $\{b_n\}$ was decreasing, Dean was more preoccupied about the correctness of what inequality sign to use. Here Dean focused on a minor detail within the inequality, instead of being concerned that the inequality as a whole was true. The other possibility was that Dean was regulating for accuracy, whereas he habitually regulates for reasonableness. Moreover, question 2 also played a significant role because of Dean's difficulty working with factorials. Although Dean was able to argue and justify in his own logical manner that the sequence $\{b_n\} = \left\{\frac{1}{(2n+1)!}\right\}$ was decreasing and had a limit of zero, he had difficulty proving it mathematically. This is similar to what was observed in observation 1.

Will mostly checked and monitored the accuracy of his calculations and mathematical procedures when interacting with Dean. This occurred throughout questions 1 to 3. In particular, Will had difficulty in showing that the sequences were decreasing in all three questions. Apart from this, Will did play an important role in question 2 by drawing a sketch to illustrate that the sequence $\{b_n\} = \left\{\frac{1}{(2n+1)!}\right\}$ was decreasing.

The students' local assessments around the reasonableness/usefulness of their mathematical procedures (E-LAPR) mostly focused on proving that the conditions of the alternating series held true. This is exemplified by Dean who was mostly preoccupied with checking different methods that could be used to show that the conditions held true. Will only once considered the viability of the solutions when the students were working on question 2 whilst Dean did so more often.

A crucial factor that may have influenced the problem solving process of this observation was that Dean practised beforehand. This was likely to have played a role in Dean not having difficulty in doing the questions, especially since he mostly led the problem solving processes. Will was not that prepared for the observation, since he had only practised questions on sequences in depth. He also acknowledged that he had found doing the problems on alternating series difficult, resulting in him following what Dean was doing. Furthermore, Dean was mostly the social trigger of Will's metacognitive behaviour.

Neither of the students out of their own verified and/or reflected on the reasonableness of their solutions (V-LARR) after solving the questions. Verification only occurred because of my interrogation (as seen from Table 6.2). Only Dean verified the accuracy of his work after completion of the questions (V-LARA). Will only once checked the accuracy of his work.

Furthermore, both students agreed that mathematics is of importance in their other courses. Also, Dean mentioned that he enjoys mathematics and finds it an amazing subject, while Will noted that it 'haunts' him and not doing well in mathematics affects the performance in one's other subjects as well.

6.4 Observation 3

Before observation 3, students were trained during formal lectures on the use of metacognitive questions as advocated by the IMPROVE[31] method. During the IMPROVE lectures, the lecturer modelled the form of metacognitive questioning that he wanted the students to emulate. Also during a lecture, students were encouraged to work collaboratively as well as to engage in metacognitive questioning (both features of the IMPROVE method). While students worked in small groups, the lecturer was observing students' questioning, as well as answering and helping students where needed. At the end of the lecture, the lecturer reviewed the new concepts and solved problems by modelling metacognitive skills and metacognitive questioning techniques at the same time.

In particular, during their first introduction to the IMPROVE method, students did examples and exercises on how to apply this method to problems on determining power series of functions[32].

In observation 3, the students had to do similar problems which were the following:

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[31] As discussed in Chapter 2, IMPROVE was developed by Mevarech and Kramarski (1997) as a method for the development and use of metacognitive skills amongst students to enhance their mathematical thinking. IMPROVE is an acronym that refers to a method of instruction consisting of the following steps: **I**ntroducing new concepts; **M**etacognitive questioning, **P**racticing; **R**eviewing and reducing difficulties; **O**btaining mastery; **V**erification; and **E**nrichment (Mevarech & Kramarski, 1997, p. 369). The IMPROVE method guides and trains students in formulating specific questions when solving mathematical tasks. The four kinds of metacognitive questions used in IMPROVE are: comprehension, strategic, connection, and reflective questions (as discussed in Chapter 2).

[32] A power series in $(x-a)$ or a power series centred around $x=a$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

where $c_1, c_2, c_3, \ldots \in \mathbb{R}$ are referred to as the coefficients of the series.

(1) Determine a power series centred around $a = 0$ for $f(x) = \sqrt[4]{1 + x^3}$.

(2) Determine a power series centred around $a = 1$ for $f(x) = \ln x$.

(3) Determine a power series centred around $a = 0$ for $f(x) = \sin^2 x$.
(hint: given $\sin^2 x = (1 - \cos 2x)/2$)

Before solving the above three problems, I asked the students their opinion on the IMPROVE metacognitive questions. In particular, I was interested if they would employ the four metacognitive questions[33] of IMPROVE when working on mathematical problems on their own.

Dean responded that the use of the four metacognitive questions and the process of applying them in solving problems is quite lengthy. Moreover, he also did not like to solve problems by following a set process of a sequence of steps involving the asking and answering of the four metacognitive questions.

Dean did reveal that in general he does ask himself what needs to be done to solve a problem, and had he done similar problems before? Dean admitted that he does apply metacognitive questioning techniques in his own manner, but that he does not apply all the metacognitive questions from the IMPROVE method.

Will, similar to Dean, also thought the questions were not that useful. Later he contradicted himself in admitting that he did find comprehension and strategic questions beneficial. Will noted further that when given a mathematical problem he considers what information is given, the goal (objective) of the problem, and whether he had done similar problems before. Furthermore, Will acknowledged

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[33] The four kinds of metacognitive questions used in IMPROVE are: comprehension, strategic, connection, and reflective questions (as discussed in Chapter 2)**. Comprehension questions** orient the student in articulating the main ideas of the problem; classify the problem into an appropriate category; and elaborate on new concepts. **Strategic questions** refer to strategies appropriate for solving the problem as well as the reasons for using that particular strategy. Students will select a certain strategy, justify their choice of strategy, and describe its application to the given problem as well. **Connection questions** take into account the similarities and differences between the problem at hand and the problems the students have previously solved and why this is case. **Reflective questions** are used in referring to those concepts where the student reflects on the process and solution of the problem during and after problem-solving – asking what went wrong, why did I make certain errors or did I make any errors, and does the solution make sense?

that asking themselves metacognitive questions forms part of how they solve mathematical problems.

Will revealed that he was not comfortable with reflection questions and while doing exercises and practice questions he did not reflect on the solution during and after the problem solving process.

In noting that both students did not comment on connection questions, I asked them if they had ever compared problems in terms of their differences and similarities. By illustrating two examples, Dean explained that he had done this. Although Will also had compared questions he did not give examples.

6.4.1 Observation 3: Question 1

After the discussion, the students turned to the first question of the task:

(1) Determine a power series centred around $a = 0$ for $f(x) = \sqrt[4]{1 + x^3}$.

The students had little difficulty with this question but produced an incorrect answer fairly quickly.

Students were allowed to use their textbooks and immediately referred to them as an external source in orientating themselves about the problem. Dean used his textbook to find out the different types of power series they could use. Since the power series was centred around $a = 0$, Dean realised that they could discard the method of determining a Taylor series[34], and considered either using a Maclaurin series[35] or binomial series[36] expansion. Dean decided to apply the

[34] A Taylor series for a function $y = f(x)$ centred around $x = a$, is a power series of the form:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where the coefficients of the series are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

[35] A Maclaurin series is a Taylor series centred around $a = 0$, of the form:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

[36] A binomial series is a special case of the Maclaurin series and is of the form:

method of a binomial series in which he outlined how the given function $f(x) = \sqrt[4]{1+x^3}$ could be written as a binomial series. From early in this observation Dean structured his solution and took control by ensuring that they implemented the correct strategy.

99. D: Because it says determine a power series centred around a equal to zero. (O-LAK; O-APK, O-IMP, O-UES)…
So then this [*reads the question*] when it says a is equal to zero, you can already eliminate the Taylor series. (P-PI; PCDWS) …

101. D: And you can look at the Maclaurin and the binomial series, and then I'm thinking like this root four you can write it as. (P-PI; P-CDWS)

102. W: Brackets.

103. D: Brackets one plus x to three to the power of one over four… [*notes that he can write* $\sqrt[4]{1+x^3}$ *as* $(1+x^3)^{1/4}$] …

105. D: And then this [*points to* $f(x) = \sqrt[4]{1+x^3}$] is like the one plus x in the binomial series [*refers to* $(1+x)^k$ *in the definition of a binomial series*] and you have the k as quarter and then x will be a x^3 over there so you can use the binomial series to write that [referring to $f(x) = \sqrt[4]{1+x^3}$] as a power series. (P-PI; P-FAP)

Will agreed with Dean's explanation of why they were able to apply the proposed strategy (plan). Contrary to the previous two observations Will now gave a clear

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$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k \cdot (k-1)}{2!}x^2 + \frac{k \cdot (k-1) \cdot (k-2)}{3!}x^3 + \cdots$$

where nth term of the series is

$$\frac{k \cdot (k-1) \cdot (k-2) \cdots \big(k-(n-1)\big)}{n!}x^n.$$

Furthermore, $\binom{k}{n}$ is referred to as the binomial coefficient with

$$\binom{k}{n} = \frac{k!}{n!\,(k-n)!}.$$

explanation and justification of why he agreed. He also revealed his knowledge of the different types of power series. This was progress for Will because in the previous two observations he was mostly fixed on using one strategy only and did not suggest what strategies to use to solve a question. The above instance regarded as a possible form or progression in Will's metacognitive behaviour.

109. R: You need to verbalise, Will.

110. W: That's just what I thought because when you look at, when you look at it, there's nothing you really can do to it. You… If you try and express it as a fraction it wouldn't really work, you wouldn't be able to. (P-PI*; P-CDWS*) … So they've told you it's centred around a is equal to zero, and you know that that's only a Maclaurin and a binomial. And the binomial's from the Maclaurin, so it's just a special type. (O-LAK*; O-APK*, O-IMP*, and P-PI*; P-CDWS*)

Although it is not that clear from line 110 above, that by saying 'If you try and express it as a fraction it wouldn't really work' Will was actually referring to the concept of a geometric power series[37]. Will was thinking about how to write the given function $f(x) = \sqrt[4]{1 + x^3}$ in the geometric form of $\frac{1}{1-x}$. Will did not realise this strategy would have been incorrect. Later in question 3, Will makes a similar mistake.

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[37] A Geometric power series is one of the first power series the students studied. They were taught how to convert radical functions of the form $1/(1 - x)$ into a power series in noting that for $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

In general for $\left|\frac{x}{k}\right| < 1$, we have

$$\frac{a}{k-x} = \sum_{n=0}^{\infty} \frac{a}{k}\left(\frac{x}{k}\right)^n.$$

While implementing the proposed plan the students worked independently. Will was using his textbook to assist him to solve the problem. Dean was writing out the general form[38] of the binomial series in finding similarities between the given function and how the binomial series is defined. In this question, Dean was much more in control of the problem solving process compared to observations 1 and 2. In particular, Dean repeatedly checked if he was adhering to the executed plan of following all the required steps and also worked in a more structured way compared to the first two observations. This can be regarded as a possible form or progression in Dean's metacognitive behaviour.

133. D: So I'm just writing out the general form of this equation here. [*writes out the general form of the binomial series*] (E-LAPA; E-EAP)

…

135. D: We can like see the similarities between this function here. [*points to* f] (E-LAPA; E-EAP)

…

137. D: And the general formula, because there was just one plus x to the k and if you put this like a bracket there, make that x, then it's one plus x to the power k. (E-LAPA; E-EAP)

…

139. D: So x is a cubed there, so it will be x to the $3n$ and this k is a quarter. So then the power series is…(E-LAPA; E-EAP)

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[38] According to the textbook the students were using, the general form of the Binomial series was given by

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Dean mostly verbalised what he was doing while Will worked quietly on his own. Metacognitive behaviour was not always evident in the case of Will. Although both students worked independently they obtained the same answer. Furthermore, since the students only focused on the form of the binomial series given by $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, they merely substituted a $1/4$ in the place of k and replaced x with x^3, in obtaining their answer as

$$(1+x^3)^{1/4} = \sum_{n=0}^{\infty} \binom{1/4}{n} x^{3n}.$$

Neither of the students verified their solution or checked if it was correct, but realised that they could not have a non-integer value in the binomial coefficient $\binom{k}{n}$. Although their answer was wrong, both students were happy with their solution and proceeded to question 2. The only metacognitive form of Verification activity demonstrated by the students was that they both found the question easy.

6.4.2 Observation 3: Question 2

In question 2 the students also had to determine a power series, but this time centred around $a = 1$ for the function $f(x) = \ln x$. Both students orientated themselves about the question. Will was the first to note that they had to determine a Taylor series for f.

156. W: So we know that that is a Taylor. (O-LAK*; O-APK*) …

159. R: Is it a Taylor series?

160. W: Yes.

161. D: Because it's like any value.

162. W: Any value....

…

164. D: …because like you can, as I said, if they give you a equals something then you have to look at Taylor series because the other series are all derived when a is zero. (O-LAK; O-APK)

165. R: What other series?

166. D: Um, the Maclaurin series and the binomial series…

…

168. D: They [*refers to Maclaurin and binomial series*] all come from the fact that we assume that a is equal to zero and then that's how we get there. So when a equals another integer value, you can only look at the Taylor series because that's the only type you can use.

From the above we note the students' use of naïve and/or imprecise mathematical language. What they meant by 'any value' of a is that a can be any integer value, as long as $a \neq 0$. This conversation is very important because although the students mentioned that the power series is not centred around zero, they still made the mistake in expanding the series around zero (as discussed later in the observation).

Similar to observation 2 and the previous question, Will again turned to his personal notes for guidance, while Dean was writing out the question. Both students were exercising control over the problem solving process, although they were working independently. I asked Dean why he was working in a more structured manner in comparison with the previous two observations. He pointed out that the questions of the previous observations were much simpler and easier and continued to say:

186. D: But now like this, you have to write down that they're looking for that [*underlines 'power series' and a = 1*]. And you have to put this here because if I don't write that down I might forget that a is a number and I might just continue working on some other things. (E-LAPA; E-NUL)

From the above, Dean seems to find that determining the power series of a function is not that 'simple' in relation to the problems of observations 1 and 2. Dean not only exercised control at the start of the problem solving process but throughout implementation of the proposed plan.

Since there was very little interaction between the students I urged them to collaborate on the problem (this happened repeatedly in question 2). Dean explained to Will that they had to determine the nth derivative of f. In doing this Dean was actually guiding Will on how to apply the proposed strategy: that by determining a Taylor series[39] one needs to determine the nth derivative of the function. Both students started determining the derivatives of f and Will noted from his calculated derivatives that the series will be alternating. Although Will was monitoring where the solution was leading to he still did not realise that his derivatives were incorrect evidenced by:

201. W:

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{1}{x^3},\ f''''(x) = -\frac{1}{x^4}$$

[39] A Taylor series for a function $y = f(x)$ centred around $x = a$, is a power series of the form:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where the coefficients, c_n of the series are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

In order for the students to obtain $f^{(n)}(a)$ they had to determine the first few derivatives, in deducing from these the expression of $f^{(n)}(a)$.

It is only later in the observation, with Dean's assistance, that Will corrected his mistake.

Although both students pointed out earlier that the series was not centred around zero, both Dean and Will still substituted the value $x = 0$ into their calculated derivatives. Dean realised and corrected his mistake, while Will did not. With me intervening and encouraging the students to work together, Dean pointed out Will's mistake and acted as a social trigger for Will's metacognitive behaviour. However neither realised that Will differentiated incorrectly (cf. line 201 above). In encouraging them to compare answers again, Dean pointed out Will's errors in his derivatives. Dean once again acted as a social trigger for Will's metacognitive behaviour, to which Will reacted:

243. W: I made a mistake with my derivatives. (E-LAPA*-RE; E-EDKT*-RE)

244. R: What?

245. W: I wasn't actually thinking. Because it's minus two. Well, when you get here it becomes minus two. (E-LAPA*-RE; E-EDKT*-RE)

246. R: You said you weren't thinking. What does that mean if you were not thinking?

247. W: I wasn't paying attention to what I was doing.

What is evident from the above is that Will was not monitoring his progress and taking control of his problem solving when determining the derivatives of f. Similar behaviour occurred when Will evaluated the derivatives at $a = 0$ and not $a = 1$. In both cases, the researcher had to intervene to ask the students to collaborate. Will only realised his mistakes because of Dean acting as a social trigger for him.

Dean returned to his solution to try to determine a general expression for $f^{(n)}(1)$ from the first five derivatives of $f(x) = \ln x$, that he evaluated in the point $a = 1$.

Dean was monitoring his behaviour and verbalising his thoughts while Will was quiet and busy recalculating the derivatives of f. Similar to previous observations Dean was working ahead of Will.

Having difficulty with his work, Will again turned to his textbook for guidance. Dean had no difficulty with executing the steps of the proposed strategy (the procedure for calculating a Taylor series for the given function) and was able to obtain an expression for $f^{(n)}(1)$, namely $f^{(n)}(1) = (-1)^{n+1}(n-1)!$.

Dean mostly struggled with how the value $f(1) = \ln 1 = 0$ was related to his expression of $f^{(n)}(1)$. After some time he realised that $f(1) = 0$ did not affect the power series representation and that it could be ignored (for most of his work Dean exhibited some form of control by monitoring his progress). For the most part, Will quietly observed Dean working. Will, in discussion, revealed that the procedure behind determining a Taylor series was more difficult than he thought.

329. W: I don't know, I was thinking of it's always simpler…

330. W: [*paging through his textbook*] …when I look at it in the way that the power series was represented at first when we were first taught how to do it [*refers to geometric power series*[40]]. (E-LAPR*; E-ECS*)…

332. W: It's somewhere here. Like this. [*points to the geometric power series in his textbook to* $\sum x^n = 1 + x + x^2 + x^3 + \cdots$] When you actually stretch it out [*meaning expanding the series*]. I always rather stretch it out and then look at what the difference is between

[40] A Geometric power series is one of the first power series the students studied. They were taught how to convert radical functions of the form $1/(1-x)$ into a power series in noting that for $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

In general for $\left|\frac{x}{k}\right| < 1$, we have

$$\frac{a}{k-x} = \sum_{n=0}^{\infty} \frac{a}{k}\left(\frac{x}{k}\right)^n.$$

that and that, you know… It doesn't make much sense. I'm also confused, I'm also going to look at it.

During the discussion above, Dean tried to simplify the coefficients of his Taylor series expansion. In turning to his textbook for guidance, Dean was able to resolve this problem demonstrating that he was monitoring his progress by checking that he was on the right track. Although Will observed what Dean was doing, he did not understand Dean's explanation and calculations. Dean explained his solution to Will who remained confused after Dean corrected two of Will's misconceptions. Dean again acted as a social trigger for Will's metacognitive behaviour.

Dean concluded that the solution to the question is

$$\frac{(-1)^{n+1}}{n}(x-1)^n$$

but also mentioned that he 'forgot something'. He remembered that they had to determine a series and corrected the above expression by including a sigma sign. As a final answer Dean obtained

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$$

Dean's correction, inserting the sigma sign, is evidence of progress since in the previous observations he had difficulty distinguishing between sequences and series. Progress in mathematical understanding may have occurred because of a heightened awareness brought about by Dean's implementation of the IMPROVE method.

Dean also noticed that the starting index $n = 0$ of the series was incorrect and changed it to $n = 1$ (because the term corresponding to $n = 0$ fell away since $f(1) = \ln 1 = 0$). Furthermore, Dean also did not verify the accuracy of his answer, but mentioned that 'it looks mathematical' and 'I shall leave it like that'.

In this observation, similar to observations 1 and 2, Dean made comments about problem solving and/or his solution as being ‘mathematical’ versus ‘logical’.

In asking him to clarify what he meant by ‘it looks mathematical’, Dean said that his answer did not contain ‘too many terms’. It was not initially clear what Dean meant by this, but by relating the graph of $f(x) = \ln x$ to his series representation of f, Dean argued that by adding smaller and smaller terms to the series, the series representation agreed with the graph of f. He continued to say that as n increases ($n \to \infty$), the factor $(-1)^{n+1}/n$ will become smaller and smaller (tend to zero) and consequently the terms $\frac{(-1)^{n+1}(x-1)^n}{n}$ of the series become negligible. For Dean this correlated to the graph of the ln -function in becoming ‘flatter and flatter’ confirmed in the conversation below. (This is an example of the code –RE because of my questioning). Such verification may not have occurred if there was no intervention from my side. Although Dean tried to verify the validity of his answer graphically his argument was still wrong.

386. D: Some answers you know it looks wrong and you have too many terms for a simple function. When you see like some functions, they look simple, like just say sine of x and then you like end up with a whole page of terms and you know you’ve done something wrong because it’s a simple function to represent by adding, to keep adding something [*meaning adding terms of the series together*]. (V-LARR-RE)

…

388. D: But even like with $\ln x$, it’s a simple function, it’s not like some e to the power $\ln x$ minus $7x$ over 50 or something. So it’s $\ln x$, it’s a simple function and the graph just goes like… like if you draw the graph of $\ln x$ [*draws the graph*] it starts like that, goes like that, ok, this here is 1. So you can see that every time this factor [*refers to* $(-1)^{n+1}/n$] here is going to get smaller and smaller like this is going to get… because this is going to be 1 over 1, 1 over 2, 1 over 3…(V-LARR-RE)

…

390. D: And you keep decreasing that [*refers to* $(-1)^{n+1}/n$]. Then you can see as you keep adding, the ln-graph becomes flatter and flatter. (V-LARR-RE)

In reflecting on Dean's solution, Will revealed that he found it difficult to obtain the nth derivative from the first few derivatives of the function. In this instance Will verified the problem solving process for himself with Dean's solution acting as a social trigger for Will's metacognitive behaviour (this was coded as V-LARR*; V-REF*). It was surprising that Will claimed he understood Dean's justification in lines 386 – 390 above which may suggest that

1. Will actually did not understand or follow Dean's argument (but there was no hard evidence that this was the case), or
2. Will may have had difficulty knowing how to explain and/or justify a mathematical argument. That is, he had difficulty in mathematical reasoning and so agreed with Dean's flawed argumentation.

Option 2 seems to be the more realistic possibility, since Will demonstrated difficulty with mathematical reasoning in observation 1. Will said that he had difficulty making deductions after each 'step' of the problem, and that it was easy doing the procedures but that the reasoning behind 'why' and 'what' he was doing was difficult. Moreover in observation 2, Will repeatedly did not explain his reasoning or justify his solutions. When Dean acted as a social trigger for Will's metacognitive behaviour, Will only then checked his calculations and reasoning. Furthermore, Will's procedures in observation 2 were rote. He seemed not to engage with the 'why', 'what' and/or 'how' of what he was doing.

When Will revealed that he had difficulty obtaining the nth derivative, it spurred Dean on to explain to Will how he obtained his answer. While explaining, Dean also verified the accuracy of his working – this was the only occurrence in which

Will acted as a social trigger for Dean's metacognitive behaviour. Although the students worked together in verifying the accuracy of Dean's solution, neither of them recognised the incorrect index of the series (as discussed earlier). In drawing the students' attention to this, Dean checked his calculations again and realised that starting at index $n = 0$ was not possible since the first term of the series would be undefined. Only Dean corrected this error while Will was mostly confused and unsure what the mistake was.

Dean acknowledged that if his mistake was not pointed out to him he would not have checked for the series having to start at the correct index value. Further to this discussion, Dean mentioned that he merely applied the definition of the Taylor series (as stated in the textbook), without reflecting on the possibility that a series could start at an index other than zero. This can be regarded as an instance of metacognitive behaviour. Furthermore, only Dean realised the importance of reflection questions during implementation and verification of the problem solving process while Will did not contribute to the discussion on reflection questions.

6.4.3 Observation 3: Question 3

In question 3 the students had to obtain a power series for $f(x) = \sin^2 x$ centred around $a = 0$, given the hint $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Neither of the students thought of the easiest and quickest solution – to use the standard Maclaurin series representation of $\cos x$ and some algebra in conjunction with the above identity.

Different to the previous two questions, both students considered strategies which were not helpful in solving the problem (similar to observation 1). It seems that the students may have regarded the question as more difficult than what it was. Dean confirmed this when he mentioned that the question may be easier than suggested by the form in which it was given.

Similar to the previous two questions, both students orientated themselves about the question. Will focused only on using a Maclaurin series whereas Dean again considered a number of different possible strategies on how to solve the question.

Dean proposed using a geometric power series to solve the problem but soon realised that this strategy could not be used. Dean decided to use the binomial series expansion instead which was also not helpful. Will evaluated what Dean was doing and explained that he agreed on Dean's chosen strategy.

Since this was the second time that the students decided to apply an incorrect strategy I decided to intervene by asking them if they were sure about using a binomial series expansion. Will remembered an example of a geometric power series as an approach to solving question 3. That Dean had to point out to Will that a geometric power series was not useful in solving the problem is evidence that Will either did not recall Dean's advice on not using the geometric power series, or forgot Dean's advice, or possibly did not understand what Dean explained earlier on. The students had to leave for a lecture and were not able to solve the third question.

6.4.4 Summary of Observation 3

Observation 3 was the first observation in which the students had the opportunity to exhibit what metacognitive skills they may have possibly acquired, after having been introduced to the IMPROVE method.

Although the students were not that enthusiastic about the metacognitive questioning techniques, Dean and Will did point out that they asked themselves what plan (strategy) to follow in solving a problem, which is related to strategic questions of the IMPROVE method. Moreover, both students also mentioned that before solving a problem they did ask themselves if they had seen and done similar problems before. In this sense the students actually applied comprehension and connection questions. Considering Table 6.3 below and what was observed during this observation, both students orientated themselves about the three questions. This is different to observation 1, in which the students exemplified very little activities related to Orientation behaviour. In observation 2, the frequency of Orientation activities was much higher than in observation 3. This was mostly due to the students deciding which question to do first; Dean struggling with the factorial notation in question 2; and Will's difficulty with how the series of question 3 can be written in the form of an alternating series[41]. The frequency of orientation activities being much higher in this observation compared to observation 1 may be due to the students' training in the IMPROVE method and their use of some of the metacognitive questioning techniques. Another factor that could contribute to this increase in frequency is the increase in task difficulty. This was evident from Dean mentioning that the questions of observations 1 and 2 were easier in comparison to the questions of observation 3.

There was also an increase in the number of Proposed Ideas (P-PI) and New Ideas (P-NI) from observation 2 to 3. Factors similar to those that could have played a role in the Orientation activities may also have influenced the increase in number of Planning activities. Dean proposed more strategies for solving the problems than Will, and similar behaviour between the two students were seen when the students considered a number of different strategies (P-NI, P-CDWS) for solving question 3.

Table 6.3: Metacognitive Skills of Each Student during Observation 3

MCDP	Dean	Will		MCDP	Dean	Will
P-PI	5	2		**P-PI-RE**	0	0
P-NI	2	0		**P-NI-RE**	0	1
Total P-FP	7	2		**Total P-FP-RE**	0	1
O-LAK	5	3		**O-LAK-RE**	1	1
Total O-LAK	5	3		**Total O-LAK-RE**	1	1

[41] In Chapter 7 an in-depth analysis (account) is given between the different observations' metacognitive decision points.

E-LAPA **E-LAPR** **Total E-LA**	22 8 30	14 1 15		**E-LAPA-RE** **E-LAPR-RE** **Total E-LA-RE**	1 0 1	1 0 1
V-LARA **V-LARR** **Total V-LA**	1 0 1	0 3 3		**V-LARA-RE** **V-LARR-RE** **Total V-LA-RE**	1 2 3	0 2 2

Codes for metacognitive decision points (metacognitive skills):

O-LAK = local assessment of knowledge or knowledge building (Orientation)

P-PI = proposed idea/plan (Planning)

P-NI = new idea/plan (Planning)

P-FP = formulate plan (Planning)

E-LAPA = local assessment of accuracy of execution (Execution)

E-LAPR = local assessment of usefulness/reasonableness of procedure (Execution)

V-LARA = local assessment of accuracy of result (Verification)

V-LARR = local assessment of reasonableness/usefulness of result (Verification)

When the students implemented the proposed strategies, Dean made more local assessments in both the accuracy and reasonableness/usefulness of procedures than Will (E-LAPA and E-LAPR). In this observation and the previous one Dean acted as a social trigger for Will's metacognitive behaviour.

During this observation Dean in particular focused on accurately performing each step of his executed strategies and was monitoring his work more, compared to previous observations. Dean's self-monitoring influenced Will to check the required steps of an executed strategy. In the cases of correcting and explaining Will's mistakes, Dean acted as a social trigger for Will's metacognitive behaviour which led to Will's revision of his work.

That task difficulty had increased relative to the previous two observations was acknowledged by Dean. The frequency of Dean acting as a social trigger for Will's metacognitive behaviour in explaining the solution process may indicate that Will also found the questions of observation 3 more difficult, supported by Will's comment in question 2 that he found calculating Taylor series expansions more difficult than he thought they were.

Table 6.3 shows that students' activities during Verification was much lower in comparison to other metacognitive decision points. That the students rarely reflected on their solution after problem solving is evident from the students admitting that they did not always review their work after solving a problem. This has been consistent throughout observations 1 to 3. It is possible that the IMPROVE method had no effect in creating an awareness amongst the students to review their work, given their lack of enthusiasm for IMPROVE. Students only admitted to the use of connection, comprehension and strategic questions of IMPROVE. Moreover, that students mostly reflected on their solution in question 2 and did not complete question 3, would have contributed to the fewer metacognitive decision points during Verification. One particular case of note in this observation was Dean's local assessments on the reasonableness of his result in question 2. However his reasoning for believing his answer to be correct was faulty.

6.5 Observation 4

In observation 4, the objective of the task was similar to that of observation 3 – determining the power series[42] of functions. One of the differences between these two observations is that observation 4 also contained a question on series either being conditionally convergent, absolutely convergent or divergent (which is discussed later). Moreover, the questions of observation 4 were also more difficult in comparison to that of observation 3, as will be explained in Section 6.5.1.

[42] A power series in $(x-a)$ or a power series centred around $x=a$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

where $c_1, c_2, c_3, \ldots \in \mathbb{R}$ are referred to as the coefficients of the series.

6.5.1 Observation 4: Question 1

In question 1 the students only had to determine the first four terms of the power series of $f(x) = \arcsin x$. It was not expected of the students to determine the power series representation of the function which facilitates determining the general nth term of the series. This is important to note since both students first determined the general power series representation but had difficulty in determining the first four terms of the series.

In comparison to the questions of observation 3, question 1 of observation 4 was more advanced. It required a number of important steps as well as a great deal of calculation. Below I outline the basic structure of the solution to assist the reader in understanding the students' work.

Using the method of determining a Maclaurin series for f (since $a = 0$) would create difficulty since repeated differentiation would result in f having more complex derivatives. The students should have realised that it was easier to first consider the derivative of $f(x) = \arcsin x$ in relating it to a function of which the power series was easier to obtain. This was possible by considering that

$$\arcsin x = \int f'(x)\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx.$$

Using the above and determining a power series for $f'(x) = \frac{1}{\sqrt{1-x^2}}$ the students would then be able to determine a power series for f.

Noting that the power series had to be centred at $= 0$, the students could then determine a binomial series[43] expansion for f'. Since the students only had to

[43] A binomial power series is a power series, centred around the point $a = 0$ and of the form:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k \cdot (k-1)}{2!}x^2 + \frac{k \cdot (k-1) \cdot (k-2)}{3!}x^3 + \cdots$$

where nth term of the series is

$$\frac{k \cdot (k-1) \cdot (k-2) \cdots \big(k-(n-1)\big)}{n!}x^n.$$

Furthermore, $\binom{k}{n}$ is referred to as the binomial coefficient with

$$\binom{k}{n} = \frac{k!}{n!\,(k-n)!}.$$

determine the first four terms of the power series of f, they only had to determine the first four terms of the power series of f'. The terms, according to the binomial series expansion are as follows:

$$f'(x) = (1 - x^2)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \cdots$$
$$= 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2^2 \cdot 2!}x^4 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}x^6 + \cdots$$

Taking into account that

$$\arcsin x = \int \frac{1}{\sqrt{1 - x^2}}\, dx$$

it follows from the above that

$$\arcsin x = \int 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2^2 \cdot 2!}x^4 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}x^6 + \cdots\ dx$$
$$= x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2^2 \cdot 2!} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} \cdot \frac{x^7}{7} + \cdots$$

The students were able to obtain a similar a similar (but not the exact correct) answer to the above. Furthermore, it was Dean who guided the problem solving process that resulted in both students having a similar solution structure.

Dean was initially quite surprised by the question and its perceived difficulty, confirmed by the fact that later in the observation he struggled and was not keen to continue. Moreover, he also revealed that he found the second question even more difficult than the first one. In orientating himself about the question he turned to his textbook. Though it was not initially clear what was behind this action, Dean was actually looking for the derivative of $\arcsin x$. Apart from the above, it was evident quite early in the problem that Dean already had a clear structure for solving the problem. Later in the observation, while explaining to Will, he outlined the proposed strategy step by step. Will also orientated himself about the question. By rereading it, he gained a better understanding of what was expected of them, as well as placing emphasis on the series being centred around zero.

While Dean was paging through the textbook looking at examples that would help, Will was uncertain if $\arcsin x$ is the reciprocal of $\sin x$[44]. Dean acted as a social trigger for Will's metacognitive behaviour and then corrected his mistake. Although Will was held back by a misconception, he was still trying to recall prior knowledge. This is illustrated below:

21. W: But isn't the arcsine one all over the sine of x? (O-LAK*, O-APK*)

22. D: One over sine x?

23. W: [*nods*]

24. D: What you going to do with it? (O-LAK, BMM)

25. W: You put one over…

26. D: arcsine x… One over sine x is cosec x. One over sine x… (O-LAK, O-APK)

27. W: Ah my mistake. Ah, so ok.

I noticed Dean's frustration in having to look up the derivative of $\arcsin x$ in his textbook and I decided to give the students the derivative. Furthermore, because of Will's confusion around Dean's proposed strategy on how to answer the question, I asked the students to collaborate on the problem. I was actually imposing on Dean to act as a social trigger for Will's metacognitive behaviour. Dean gave a lengthy, but clear explanation to Will by first considering an example from the textbook and relating its structure to that of question 1. Moreover, Dean's outline of the steps of his strategy were similar to the strategy I have explained

[44] Note that $\arcsin x = \sin^{-1} x$, is the inverse of $\sin x$, while $\frac{1}{\sin x} = (\sin x)^{-1}$ is the reciprocal of $\sin x$ and is equal to $\csc x$.

earlier. Will was able to follow and understand Dean's outline of the proposed strategy, as well as assist Dean in correcting algebraic errors. During this interaction both students were exhibiting metacognitive behaviour in jointly constructing a proposed plan on how to solve the problem and being social triggers for each other's metacognitive behaviours.

Similar collaboration was observed while the students were implementing the proposed strategy. Dean being able to obtain only a partial solution turned to Will for assistance. This was quite a rare situation compared to the previous three observations in which Will mostly turned to Dean for assistance.

100. D: [*Looks at what he wrote down*]

$$\int (1-x^2)^{-\frac{1}{2}}\,dx = \int \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}\,dx$$

How the… How do I integrate a binomial series? How do I…? [*chuckles*] (E -LAPA, E-ECS)

101. W: Hmm? [*looks at D's work*]

…

103. D: How do I integrate a binomial series? … (E-LAPA, E-ECS)

…

105. W: [*points to D's sheet of paper*] It's the integral with d over dx, so you bring this inside and this one all over n doesn't have the x so, it doesn't change in this. (E-LAPA*, E-GSO*)

…

111. W: …the whole thing all over $2n$ plus x to the power $2n+1$. [*points to x^{2n} in D's work*] (E-LAPA*, E-GSO*)

…

122. D: So this whole series here over $2n+1$, right? (E-LAPA, E-ECS)

…

127. W: Yes, it's this whole thing over $2n+1$ and the x^{2n+1}. (E-LAPA*, E-GSO*)

With reference to lines 100 – 127, in conjunction with what was observed in the previous three observations, Dean helped Will in solving the problems. However, in observation 4 now, Will played a more prominent role in assisting Dean. It may be the case that because of the effect of the IMPROVE method Will was not only able to follow and understand how to solve the problem, but was also able to collaborate more with Dean in the problem solving process and assist him. This was not the case in observations 1 to 3.

After integrating the series Dean obtained the following:

143. D:

$$\int \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n\, x^{2n+1}}{2n+1}$$

Reminding himself that he only had to determine the first four terms of the power series, Dean realised that he was unable to determine the binomial coefficient[45] of each term, since he could not have negative factorials (instance coded as E-LAPA, E-EAP). Will lagged behind Dean (as in previous observations) but caught up with Dean and had the same realisation about negative factorials. Dean turned to his textbook for help in considering the expanded form[46] of the binomial series

[45] The general binomial coefficient $\binom{k}{n}$ of the binomial series is given by

$$\binom{k}{n} = \frac{k!}{n!\,(k-n)!}.$$

[46] The expanded form of the binomial series is:

(instance coded as E-LAPA, E-CMC). He then pursued using this expanded form to determine the first four terms of his power series in line 143.

Collaborating with each other on how to determine the first four terms, Dean more than once had to assist and explain to Will how to use the expanded form of the binomial series. Dean again acted as a social trigger for Will's metacognitive behaviour since Will was not always sure of how to proceed to solve the problem.

The students spent the bulk of the time determining the first four terms of the series, while interacting with each other but also working independently. Both students were monitoring their working and controlling the problem solving process. As the students were working Dean eventually revealed his frustration with the calculation of the first four terms. He did not want to continue with the question and turned to me to help him with the solution. In the end, only Dean obtained the first four terms of the series since Will was still behind in completing the question. Similar to previous observations, Will copied Dean's solution, but only partially, while Dean moved on to question 2 (although his solution was not entirely correct). Neither of them verified their solution or reflected on the problem solving process.

6.5.2 Observation 4: Question 2

Similar to question 1, the students had to determine the power series centred around $a = 0$ for $f(x) = \sinh x$, which is pronounced 'sine-h' or 'sine hyperbolic x'.

At the start of the question, Dean mentioned that it was more difficult than question 1. Will revealed that they did not frequently use the above function in

---

$$(1+x)^k = 1 + kx + \frac{k \cdot (k-1)}{2!}x^2 + \frac{k \cdot (k-1) \cdot (k-2)}{3!}x^3 + \cdots$$

where nth term of the series is

$$\frac{k \cdot (k-1) \cdot (k-2) \cdots \big(k-(n-1)\big)}{n!}x^n.$$

their studies. Despite of this, both students did find the question much easier in comparison to question 1 and were able to obtain a solution quickly.

The students collaborated with each other in orientating themselves about the definition of $\sinh x$ which is

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

After considering a number of different strategies and emphasising that the series had to be centred around $a = 0$, Dean then explained the plan they were to implement. In discussing the above strategy, Dean also revealed that it was easier to determine a Maclaurin series expansion[47] in finding the nth derivative of f, than using the above definition of $\sinh x$ [48]. This is illustrated below.

396. D: Wait, we're being stupid.

397. W: Why?

396. D: Instead of like writing this [*refers to* $\sinh x$] out as it's proper e to the x, how we define sine-h x [*referring to the definition of* $\sinh x$], why don't we just derive this the whole time [*refers to* $\sinh x$], because $\sinh x$… the derivative of this is cos-h. (P-FAP; P-PI, P-CDWS)

…

[47] A Maclaurin series is a power series centred around $a = 0$, of the form:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

[48] In using the definition of $\sinh x$, Dean was explaining to Will that the power series of $\sinh x$ could be obtained from the sum of the power series expansions of e^x and e^{-x}, where

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

398. D: … and the derivative of this is hyperbolic cos. The derivative of that is hyperbolic sine, this keeps going up and down. (P-FAP; P-PI)

…

406. R: Ok, what are you guys doing? What are saying?

407. D: No, Sir, we were, we thought it would be hard to like, because when you write out sine-h x and the definition of e to the x minus e to the minus x over two [*refers to the definition of* $\sinh x$]… (P-FAP; P-CDWS)

…

409. D: Then to think of how to write it as a power series was hard, but if you take this hint here: a equals zero, and then you just derive that, so you get $\sinh x$, $\cosh x$, $\sinh x$, $\cosh x$. And at zero it's just zero, then one, zero, one and then there's a pattern there, so… (P-FAP; P-PI)

Will was brought into the conversation and he revealed that he understood Dean's approach on how to solve the problem and was able to explain it in his own words (this instance was coded as E-LAPR*-RE; E-CLU*-RE). Dean being happy with the proposed plan mentioned that he thought the problem was much easier than question 1 – this is different to what was observed in the beginning of question 2.

As Dean implemented the proposed strategy he focused on checking what the correct values of $\sinh x$ and $\cosh x$ were when evaluating it for $x = 0$. Dean showed control of the problem solving process throughout the implementation of their plan, but he also acted (again) as a social trigger for Will's metacognitive behaviour. In trying to relate the procedure of how they would determine the power series for $\sin x$ to that of $\sinh x$, Will turned to Dean for help. Will was unsure if the derivative of $\cosh x$ was $-\sinh x$ (as is the case with $\cos x$). Dean clarified Will's misconception, after which Will also agreed that the question was much easier after Dean's assistance.

During the implementation of the proposed plan Will did not write any work down and mostly observed what Dean was doing. This is similar to previous observations in which Dean was taking a leading role in the problem solving process, although Will now in question 2 also monitored their progress. Will mostly monitored his behaviour in terms of trying to make sense of the problem solving process, by making local assessments on the reasonableness and/or usefulness of the procedures Dean was implementing. Dean mostly made local assessments on the accuracy of their working. Once more Dean acted as a social trigger for Will's metacognitive behaviour when Will had difficulty understanding his solution. Similar to question 1, Dean gave a lengthy, but clear explanation to Will by relating the definition of a Maclaurin series from the textbook to his solution as well as why the terms having zero derivatives could be ignored. Moreover, Dean also tried to engage Will in reflecting and monitoring where the solution was leading to (as seen from lines 472, 491 and 497 below).

467. D: [*points to definition of Maclaurin series*[49] *in the textbook and looks at W*] So you see that power series is just f zero and then it's plus the first derivative x plus the second derivative of that… (E-LAPA; E-CMC, E-ECS)

…

472. D: …And I want to expand it using this [*refers to the above definition*], so you can see f of zero is just zero, right? [*addressing Will*] And then plus the first derivative at zero… (E-LAPA; E-ECS)

---

[49] The definition Dean was pointing to was:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

473. W: Mmm

474. D: …which is just one over one factorial, that's just one there. And this plus, the second derivative's going to be zero. So this is going to be zero over two factorial times x^2… [*relating his solution to the definition in the textbook, while explaining to W*] (E-LAPA; E-ECS)

…

490. W: [*watches D working*]

491. D: You see that pattern there? [*addressing Will*] So this will fall away, that will fall away [*referring to the derivatives which are equal to zero*] (E-LAPA; E-FSA, E-ECS)

…

497. D: What are we left with? [*addressing Will*]… We're left with x's and cubes so it will be x^{2n+1} because you need the odd numbers. (E-LAPA; E-ECS)

…

505. W: [*points to the zero terms in D's working*] Um, aren't these still terms and the ones that follow, hey, aren't these still terms? (E-LAPA*; E-ECS*)

…

508. D: They don't make any difference… Because you're adding zero and zero doesn't, you know… [*D gives a lengthy explanation in helping W to understand that derivatives which are zero are negligible in determining the power series of* $\sinh x$]

…

520. W: Yes, now I understand why the other terms are falling away, can actually fall away.

Neither of the students verified the accuracy of their work and checked for mistakes. Only when I asked them what the difference was between the power series expansion of $\sin x$ and $\sinh x$, did they reflect on the solution (these instances were coded as V-LARR-RE; V-REF-RE and V-LARR*-RE; V-REF*-RE respectively). Moreover, Will also elaborated and reflected on the problem solving process of Dean's explanation on the zero terms being negligible in the power series expansion of $\sinh x$ (also coded as V-LARR*-RE; V-REF*-RE).

6.5.3 Observation 4: Question 3

The students did not immediately start with question 3, but first had a short discussion on and orientated themselves about question 4. Dean mentioned that the questions asked in question 4 were easy and similar to those of their tests. Will on the other hand was not that comfortable with question 4 and noted that it involved a lot of work.

Dean persisted on wanting to start with question 4, but after persuading them they started question 3, which read:

Determine a power series centred around $a = 0$ for $g(x) = \sinh^{-1} x$, if given

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

The students could have determined a Maclaurin series for g by repeated differentiation of the above definition. Unfortunately this would not have been the most elegant approach since the derivatives would become more and more complex. The ideal method of solving the problem would have used the following relationship from their first year calculus course:

$$\sinh^{-1} x = \int \frac{1}{\sqrt{1 + x^2}}\, dx.$$

Using the above, students could then follow a similar strategy as in question 1 in solving question 3.

The students started question 3 by orientating themselves about the given definition of $\sinh^{-1} x$. Line 604 below is one of a number of occurrences in which Dean showed his lack of interest in solving question 3. It may be the case that he already realised before starting with question 3 that it was a difficult problem and he did not know how to approach it. This was probably the case since he rather wanted to start with question 4 which he mentioned as being ‘easy’[50].

595. D: [*points to the given definition of* $\sinh^{-1} x$] What does that mean? That this is equivalent to that? (O-LAK; O-BMM)

596. W: Um, this means we can use this [*points to the definition*] as our $g(x)$ instead of the sine-h. (O-LAK*; O-BMM*, O-IMP*)

…

599. W: And then find the relationship with that. (O-LAK*; O-BMM*)

600. D: Yo! [*looks suprised*]

…

604. D: There’s so much deriving to do, I don’t want to do this. (O-LAK; O-BMM)

In line 596 Will referred to $\sinh^{-1} x$ as ‘sine-h’ instead of correctly calling it ‘sine-h inverse’. This occurred throughout this observation which shows his naïve and/or imprecise use of mathematical language. The occurrence of such problematic mathematical language is not unique to observation 4; both students exhibited similar behaviour in observation 1.

Dean started proposing a number of different strategies for solving the problem but did not make it clear what approach he wanted to follow (coded as P-PI; P-

[50] Similar behaviour was observed in observation 2 in which Dean wanted to do question 1 first instead of question 2 (Will wanted to start with question 2). The reason for this was because Dean had difficulty in working with the factorials within question 2.

CDWS). For the first time in all four observations Dean was at a complete loss on how to solve the problem and turned to Will for help. Will proposed using the strategy of a Maclaurin series in repeatedly differentiating the given definition of $g(x) = \sinh^{-1} x$ to obtain the nth derivative of g. Dean, not being at ease with Will's suggestion tried to manipulate the definition of $\sinh^{-1} x$ but created more difficulty for himself. Realising that Dean was at a loss, I urged the students to work together and consider Will's proposed plan.

Both students took quite a long time to determine the derivatives of g, but monitored their behaviour by making local assessments in which they

(i) considered the usefulness of the procedures they were carrying out (coded as E-LAPR; E-ECS and E-LAPR*; E-ECS*);

(ii) took note of mistakes they had made and correcting it (coded as E-LAPA; E-EDKT and E-LAPA*; E-EDKT*);

(iii) took control of the problem solving process by implementing more structure and working in an orderly manner (E-LAPA; E-NUL and E-LAPA*; E-NUL*);

(iv) evaluated where the solution was leading to, for example if it was possible to simplify mathematical expressions (E-LAPA; E-ECS and E-LAPA*; E-ECS*); and

(v) assessed whether their process was adhering to the proposed plan (E-LAPA; E-EAP and E-LAPA*; E-EAP*).

Noticing that the students' strategy was not ideal, I intervened and asked if they were finding the problem difficult and if the repeated differentiation of $\sinh^{-1} x$ was cumbersome. In reflecting on the usefulness of the implemented plan both students agreed that there might have been an easier way of answering the question. Contrary to the observation in line 604, Dean wanted to carry on with the problem. He also mentioned that it was fun answering the question and that 'maths is fun'. Dean made similar comments before in previous observations.

With me intervening by giving the students the derivative of $g(x) = \sinh^{-1} x$, the students were able to devise a strategy for solving the problem as shown below. The coding –RE applied because of my intervention.

785. D: Ah man [*chuckles and seems surprised and pleased when looking at the given derivative*]

…

788. D: Because now you have the derivative and then like maybe what we've done the first time you have a, a one over the root of one plus x^2 [*referring to the derivative* $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$ *I gave them*] (O-LAK-RE; O-BMM-RE, O-APK-RE and P-NI-RE; P-DCS-RE)

…

790. D: And then you can just…

791. W: [*smiles broadly and seems surprised and pleased when looking at the given derivative*]

…

793. D: Do what you've done the first time [*refers to strategy in the first question of the observation*]. (P-NI-RE; P-DCS-RE)

…

795. W: In the first question. (P-NI*-RE; P-DCS*-RE)

…

801. W: I feel like it will now be easier to represent this as a Maclaurin again because I found the derivative of… (P-NI*-RE; P-DCS*-RE)

…

803. W: …of sine-h x looks. (P-NI*-RE; P-DCS*-RE)

From the above it is clear that Will understood what the new strategy involved – this is evident from him referring to the first question in line 795. Will again used incorrect terminology when referring to $\sinh^{-1} x$ (cf. line 803).

Dean elaborated on how the new strategy was similar to that of questions 1 and 2. Furthermore, he also gave a much more detailed outline of the proposed strategy after being given the derivative of $\sinh^{-1} x$. Will also contributed to the discussion and it was obvious that he understood Dean's reasoning. However, the students were confused about and queried the use of the given definition of $(x) = \sinh^{-1} x$. When reflecting on the usefulness of the given definition in trying to solve the problem both students commented that they would need to revise the work from the first year calculus course. Similar comments were made in previous observations. The students also felt that not knowing the derivative of $\sinh^{-1} x$ created more difficulty in solving the problem since it was taking much longer (the codes E-LAPR; E-ECS applied to both Dean and Will in this instance).

At the start of the implementation of the proposed plan Will began to write their solution. Dean later wanted to write down the solution on his own. He seemed to feel more in control of the problem solving process when ordering his 'thoughts' and writing out the solution 'neatly'. Dean's metacognitive behaviour of structuring and working in an orderly manner also occurred in observation 3. This behaviour was not prominent during observations 1 and 2. It emerged only after the implementation of the IMPROVE method. As noted in observation 3, it is possibly the case that the IMPROVE method acted as an environmental trigger in influencing Dean's metacognitive behaviour of working in a more structured way than before[51].

The students worked independently to obtain the result. After completing the problem, the students were asked to compare their solutions. Only Dean verified the accuracy of his work. He also mentioned that he found the question difficult.

[51] The level of task difficulty as environmental trigger must also be taken into account. Dean himself mentioned that the questions of observation 3 were more complex than those of the previous observations, hence him focusing more on structuring his work. The level of task difficulty in observation 4 might also then spurred similar behaviour by Dean.

Will did not comment on the level of difficulty of the problem and also did not reflect on the problem. Both students moved on to question 4.

6.5.4 Observation 4: Question 4

As noted earlier the students had a short discussion on and orientated themselves about question 4 before starting question 3. Dean mentioned that the questions asked in question 4 were easy and similar to questions in one of their tests. Will on the other hand was not that comfortable with question 4 and mentioned that it involved a lot of work. In Question 4 the students had to determine if the three given series were absolutely convergent, conditionally convergent or divergent. These questions were similar to those of observations 1 and 2, although the procedure for solving the questions was much longer. Since the students only focused on question 4.1 and did not obtain a solution, I decided not to include a discussion on the concepts of conditional and absolute convergence. The reader will still be able to follow the discussion on this question without a knowledge of these forms of convergence.

When starting formally with question 4, Will orientated himself about the number of different convergence/divergence tests that could be used. He also said that in general he had difficulty knowing what test to apply. Dean also orientated himself about question 4 as a whole, by discussing when and where certain tests were more applicable than others.

Since question 4.1 was similar to a question the students had done in one of their tests, they mostly reflected on this question and did not write down the complete solution (this was different to what was observed in questions 1 to 3).

Although Will proposed a strategy for answering question 4.1, Dean rejected it. Dean mostly led the problem solving process which he also monitored and assessed while explaining it to Will. He considered a number of different strategies in evaluating their usefulness on how to solve the problem. Will mostly monitored their progress and also commented on the usefulness of the implemented strategies. In explaining to Will what strategies were more useful

Dean acted as a social trigger for Will's metacognitive behaviour. Will suggested he had a better understanding about how to approach questions on convergence/divergence of series with Dean's assistance.

Unfortunately the students did not complete the question since they had to go to their next lecture. Neither of the students wrote down a solution to question 4.1.

6.5.5 Summary of Observation 4

Observation 4 was the second observation after the students had been introduced to the metacognitive questioning techniques of the IMPROVE method[52]. Furthermore, the problems of observation 4 were similar to examples the students had done during formal lectures while using the above metacognitive questioning techniques[53].

As discussed before, the students revealed in observation 3 that they were not that positive about the use of the metacognitive questioning. However, it seemed that their metacognitive behaviour manifested activities related to that of comprehension, connection and strategic questions. Orientation activities were mostly related to comprehension and connection questions, while Planning activities were related to that of strategic questions. These metacognitive activities occurred more in observation 3 than in the previous two observations attributable possibly as previously suggested to the IMPROVE method, and to task difficulty as an environmental triggers for students' metacognitive behaviour.

In observation 4 a similar trend occurred in students' metacognitive activities. Compared to observations 1 and 2, Table 6.4 below shows that the number of Orientation and Planning metacognitive activities for observation 4 had increased[54]. Moreover, for observation 4 the number of local assessments made during Execution increased compared to observation 3. The number of local

[52] The IMPROVE method was discussed in Chapter 2.
[53] The four kinds of metacognitive questions used in IMPROVE are: comprehension, strategic, connection, and reflective questions (as discussed in Chapter 2).
[54] Except in the case of the Orientation category in observation 2 because of students disagreement on which question to start with.

assessments on the usefulness of a procedure during execution (E-LAPR*) made by Will was much higher when compared to all the previous observations. The frequency was also greater in comparison to those made by Dean in observation 4. Hence it seems that observation 4 provides the strongest indication that the IMPROVE method may have played a role as an environmental trigger for both students' metacognitive behaviour.

Dean played a significant role as a social trigger for Will's metacognitive behaviour. This occurred mostly with Will making local assessments of the reasonableness and/or usefulness of Dean's mathematical procedures; the implemented strategies; and Dean's explanations of how to solve the problems. Apart from this, there were a number of instances in which Will also acted as a social trigger for Dean, as evident in questions 1 and 3. This scenario is also different to previous observations, where Dean acted mostly as a social trigger for Will's metacognitive behaviour.

Table 6.4: Metacognitive Skills of Each Student during Observation 4

MCDP	Dean	Will		MCDP	Dean	Will
P-PI	5	2		**P-PI-RE**	0	0
P-NI	0	0		**P-NI-RE**	1	1
Total P-FP	5	2		**Total P-FP-RE**	1	1
O-LAK	13	5		**O-LAK-RE**	1	2
Total O-LAK	13	5		**Total O-LAK-RE**	1	2
E-LAPA	25	17		**E-LAPA-RE**	0	0
E-LAPR	10	15		**E-LAPR-RE**	2	1
Total E-LA	35	32		**Total E-LA-RE**	2	1
V-LARA	0	0		**V-LARA-RE**	1	0
V-LARR	0	0		**V-LARR-RE**	1	2
Total V-LA	0	0		**Total V-LA-RE**	2	2

Codes for metacognitive decision points (metacognitive skills):

O-LAK = local assessment of knowledge or knowledge building (Orientation)

P-PI = proposed idea/plan (Planning)

P-NI = new idea/plan (Planning)

P-FP = formulate plan (Planning)

E-LAPA = local assessment of accuracy of execution (Execution)

E-LAPR = local assessment of usefulness/reasonableness of procedure (Execution)

V-LARA = local assessment of accuracy of result (Verification)

V-LARR = local assessment of reasonableness/usefulness of result (Verification)

My role as social trigger occurred mostly during question 3 where the students battled with deciding what strategy to use. Interrogation by me of the students to evaluate the usefulness of the strategy they were implementing, led to me giving them a hint which enabled them to solve the problem.

It seems that task difficulty as environmental trigger played an important role in question 3 where it was a primary cause for the students' difficulty in deciding which strategy to use. As revealed by students themselves, their lack of content knowledge from their first year calculus course also contributed to the difficulty of question 3. The students' lack of content knowledge was also evident in observation 1 where the students continued trying to answer all the questions by using the same strategy which they applied incorrectly. Another example was seen in question 4 of observation 4. Will, not knowing what strategy to apply found this question difficult. It seems that one can then infer that if students do not work within the boundaries of their knowledge their metacognitive activities in terms of strategy development can lead them astray and create more difficulty in solving the problems. Moreover, if their knowledge field is narrow and restricted, there is less base for metacognition to manifest upon (this is discussed further in Chapters 7 and 8). It was also observed that although the students monitored their behaviour during the execution (implementation) of a faulty strategy, they still had difficulty in obtaining the correct result.

Throughout observation 4 students interacted with each other, as well as worked independently. During these instances students mostly monitored the problem solving process, as well as assessed where the problems' solutions were leading to with Dean mostly leading (guiding) the problem solving process. Similar to previous observations, the students did not verify their work. Because of my intervention, only after question 2 did students reflect on the usefulness of results (V-LARR-RE).

6.6 Summary

This chapter gave an in-depth account of the students' metacognitive activities and problem solving behaviour during collaborative problem solving, for each of the four observations. Discussions on each of the observations also focused on how students acted as social triggers for each other's metacognitive behaviour. The possible effect of IMPROVE on the development of students' metacognitive behaviour was also considered, as well as the effect of researcher's intervention on students' metacognitive behaviour.

In chapter 7 I will compare the four different observations, focusing specifically on the differences between the students' metacognitive skills; the effect of IMPROVE and task difficulty as environmental triggers; as well my role as researcher in the students' metacognitive behaviour.

Chapter 7: Comparisons between Observations

7.1 Introduction

This chapter compares students' metacognitive behaviour across the four observations. In particular, it focuses on the possible effect that IMPROVE as environmental trigger may have had on the development of each student's metacognitive behaviour over the observed time period. Another environmental trigger of students' metacognitive skills, namely level of task difficulty, is also considered within this chapter. As mentioned in previous chapters, between observations 2 and 3, students were explicitly trained in the use of metacognitive questioning techniques from the IMPROVE method.

Students' metacognitive behaviour is discussed for each of the four metacognitive categories, Planning, Orientation, Execution and Verification respectively when considering comparisons between the four observations. This is dealt with in sections 7.2 to 7.5. The focus is on the two main metacognitive decision points used in this study, namely 'formulating plan' (FP) and 'local assessments' (LA)[55]. In order to address the research questions of this study, particular focus is on the comparisons at each of the above decision points

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[55] The concept of metacognitive decision point was discussed in Chapter 5. As used in this study, and as adapted from the work of Goos (1994), these decision points indicate points in time where students exemplified metacognitive behaviour.

between the main stream student (Dean) and the extended degree student (Will)[56]. The chapter concludes with Section 7.6, which discusses the role of the researcher as social trigger of students' metacognitive behaviour within each of the above four categories.

7.2 Planning: Comparisons between Observations 1 to 4

The metacognitive decision point 'formulating plan' (FP) consisted of the two sub-metacognitive decision points, namely 'proposed idea' (P-PI) and 'new idea' (P-NI) which formed part of the Planning category. 'Proposed idea' as indicator was used to denote metacognitive instances in which the student considered a plan (strategy) or formulated a plan on how to solve the problem. The code, P-NI was only used when the proposed idea (P-PI) was discarded; that is if the student realised a strategy (procedure) was not that usefull and a new strategy was considered. Below I discuss each metacognitive decision point respectively and compare its occurrences between the different observations, as well as differences between students' metacognitive behaviour during Planning.

7.2.1 Proposed idea (P-PI): Comparisons between Observations 1 to 4

As illustrated in Figure 7.1 below, the number of strategies proposed by the students for solving the problem only started to increase from observation 2 to observation 3; thereafter the frequency remained the same. The level of task difficulty as environmental trigger was a likely factor to have caused this increase, although there is not evidence to support this conjecture.

[56] For the remainder of the chapter Dean and Will are referred to as the 'main stream student' and 'extended degree student' respectively. This is done for discussion purposes in order to address the research questions of the study.

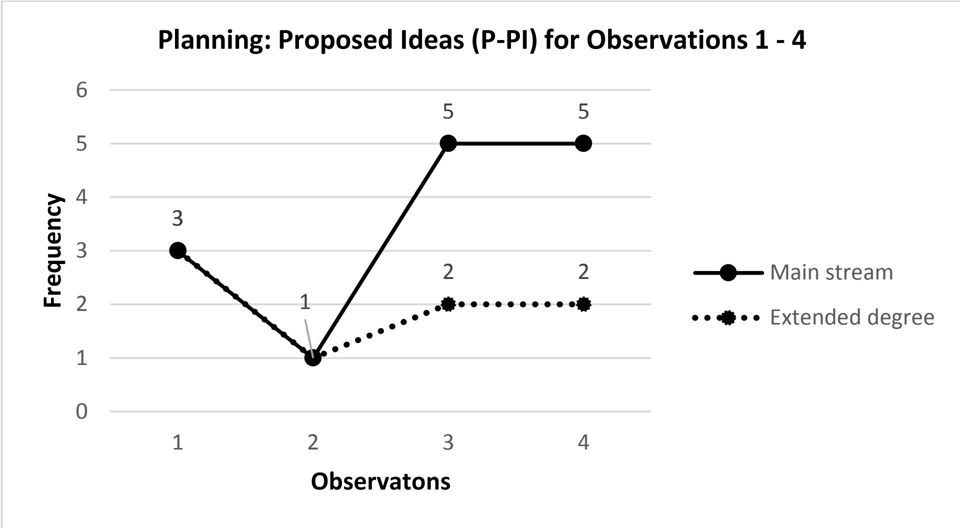

Figure 7.1: Students' 'Proposed Ideas' (during Planning) over Observations 1 to 4

Disregarding observation 2 and comparing observation 1 to observations 3 and 4, there is only an increase in the 'proposed ideas' generated by the main stream student. Both IMPROVE and the level of task difficulty may have contributed to this increase, although it is not clear if this is indeed the case. The frequency of strategies proposed by the extended degree student in observation 1 was higher compared to observations 3 and 4. It is not clear what the reason was for the decrease in frequency, although possible factors that could have contributed to this were (i) the extended degree student's lack of engagement throughout most of the observations and/or (ii) the main stream student's role in mostly leading the problem solving process. The latter was evident from observations 3 and 4, in which the main stream student generated a greater number of 'proposed ideas' compared to the extended degree student.

In observations 3 and 4, only the main stream student considered different ways of solving some of the questions. This also contributed to the increase in the frequency of 'proposed idea' from observation 1 to observations 3 and 4 (disregarding observation 2). It was noted that during observation 1 the main stream student was mostly fixated on using one strategy only. It may be the case that IMPROVE contributed to the main stream student considering a number of different strategies in observations 3 and 4, different to observation 1, and hence contributing to the above increase[57].

7.2.2 New idea (P-NI): Comparisons between Observations 1 to 4

The metacognitive decision point 'new idea' (P-NI) only applied to instances where the student recognised that a strategy was not beneficial and devised a new strategy on how to solve the problem.

From Figure 7.2 below we note that the main stream student only proposed new strategies on how to solve problems during observations 1 and 3. As noted earlier, the extended degree student never proposed any new strategies ('new ideas') on how to solve a problem. This may possibly be due to the main stream student mostly guiding the problem solving process in proposing what strategies to employ. Moreover, when the extended degree student did propose a strategy, it was mostly rejected by the main stream student. This was because the proposed strategies of the extended degree student were inappropriate to the task. Examples of the above were in question 2 of observation 1, and question 4.1 of observation 4, in which the main stream student explained to the extended degree student why his strategies were not useful.

[57] The main stream student's lack of content knowledge also may have contributed in him not being able to consider different strategies, as seen from observation 1. This was evident when he turned to his textbook for help on what strategies to use. After consulting his textbook he realised his faulty application of the inappropriate strategy. Similar behaviour was also observed for the extended degree student during observation 1, when the student turned to his notebook for assistance.

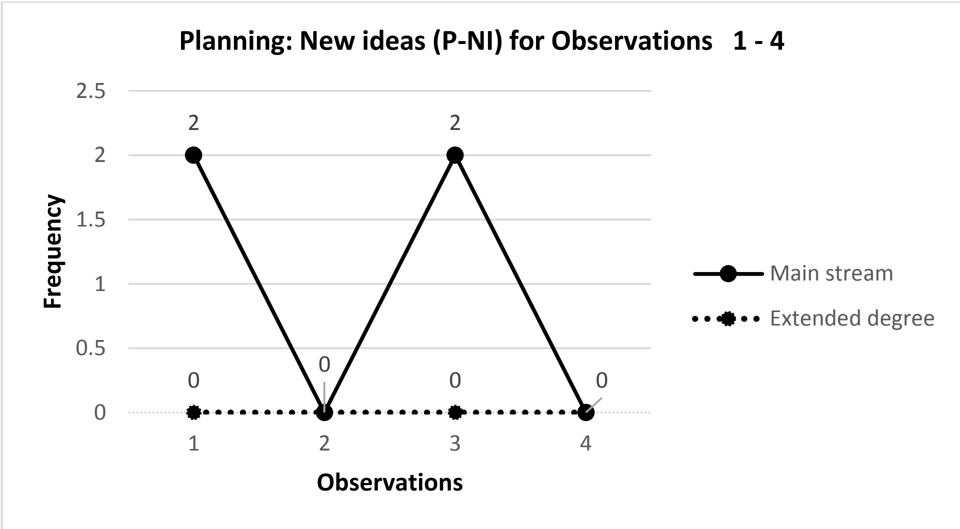

Figure 7.2: Students' 'New Ideas' (during Planning) over Observations 1 to 4

In observation 1, the two instances where the main stream student considered a new strategy was only towards the end of the observation. Observation 2 only had one possible strategy to apply in all three questions. Hence the occurrence of new and different strategies was not possible. Moreover, the main stream student only considered new strategies in question 3 of observation 3. Students also had difficulty with this question and were not able to complete it.

The increase in frequency of P-NI from observation 2 to 3 is most likely because of the influence of the IMPROVE method on the main stream student's metacognitive behaviour. The possible effect of IMPROVE can be seen from the main stream student's ability to realise early enough in question 3 that the proposed strategies were not appropriate, and hence to propose new strategies. This is different to observation 1: it was only after completing all three questions in observation 1 that the main stream student proposed a new strategy on how to solve the questions.

The zero occurrence of the indicator P-NI in observation 4 was because the main stream student proposed the correct strategies for questions 1 and 2. The only occurrence of both students applying a new strategy was in question 3, but this was due to the researcher's intervention, guiding the students to abandon their initial proposed strategy.

7.3 Orientation: Comparisons between Observations 1 to 4

The metacognitive decision point 'local assessment of knowledge or knowledge building' (O-LAK) was assigned to the Orientation category. It was used to denote metacognitive instances in which the student organised his thoughts about the problem before proposing a strategy on how to solve a problem.

Chapter 5 gave an in-depth discussion on what this local assessment took into account, as well as its indicators. Examples of these indicators during Orientation included instances where the student identified and repeated important information; used past/prior knowledge to build a mental model of the task; or asked either himself or others if they had done similar problems before.

7.3.1 Local Assessment of Knowledge or Knowledge Building (O-LAK): Comparisons between Observations 1 to 4

Figure 7.3 below outlines the two students' local assessments during Orientation over the four observations.

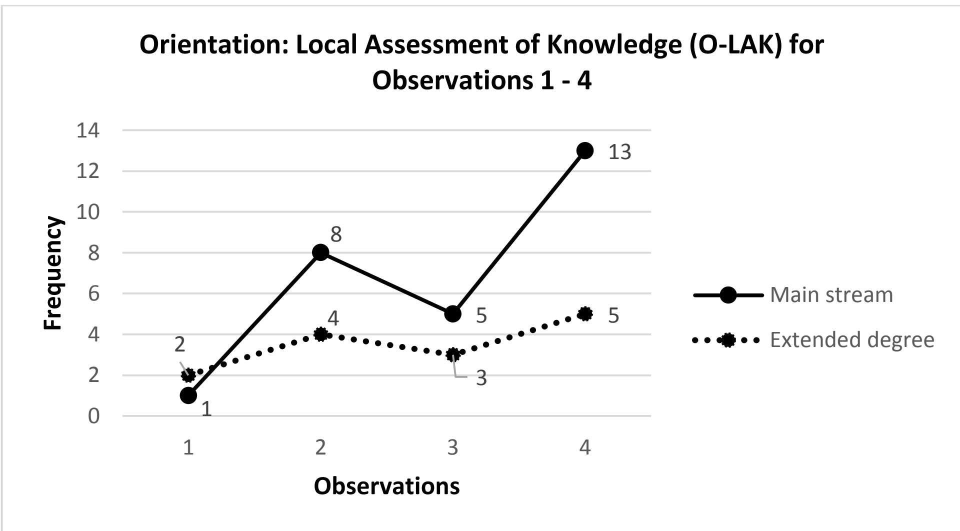

Figure 7.3: Students' Local Assessments of Knowledge (during Orientation) over Observations 1 to 4

During observation 1 the students orientated themselves only during the second question of the given task – hence the reason for the small frequency of local assessments. Although the frequency of local assessments made by the extended degree student was higher compared to the main stream student, he still considered content knowledge which was inappropriate in solving the question. Only with the assistance from the main stream student was the extended degree student able to realise his mistake.

The increase in the frequency in O-LAK from observation 1 to 2 was mainly due to students arguing about which question to start with. Different to observation 1, the students orientated themselves about all three questions in observation 2. This also contributed to the increase in local assessments. Moreover, the main stream student also had difficulty with question 2 which stimulated the increase in number of local assessments he made. This also contributed to the big increase from observation 1 to 2. Also, the main stream student prepared for observation 2 and was able to identify two questions he had done before. This also contributed to the number of Orientation activities of the main stream student.

Disregarding observation 2, there was an increase in Orientation activities from observation 1 to observations 3 and 4. This was because both students orientated themselves about *all* the questions of observations 3 and 4. It may be the case that the IMPROVE method influenced students' metacognitive behaviour and hence produced this increase in Orientation activities, although there is not enough evidence to support this inference.

The increase in students' metacognitive behaviour from observation 3 to 4 seems most likely due to the increase in level of task difficulty experienced by the students. Finding the questions more difficult was pointed out more than once by the main stream student during observation 4. In particular, the sharp increase in the orientation activities of the main stream student was also due to the following factors:

(a) comparisons he made between different problems within the same observation;
(b) orientating himself around activities of the extended degree student in assisting him to gain a better perspective of the questions; and
(c) the number of questions in observation 4 were more than in observation 3.

In relation to point (b) above, we also had in observation 4 that the extended degree student collaborated more with the main stream student during orientation, in assisting the main stream student to gain a better perspective on two out of the four questions. Such behaviour also contributed to an increase in the extended degree student's local assessments during Orientation. Furthermore, the extended degree student did not exhibit similar behaviour in previous observations.

7.4 Execution: Comparisons between Observations 1 to 4

Students' metacognitive behaviour during execution (implementing) of a proposed strategy were either indicated as:

1. E-LAPA, local assessment of the accuracy of procedure; or
2. E-LAPR, assessment of the usefulness, relevance or reasonableness of a procedure.

The frequency of students' metacognitive activities during Execution was in general much higher compared to those during Orientation and Planning. This is since students spent most of their time executing the problem which led to a higher frequency of local assessments made during Execution. Furthermore, the execution category of the taxonomy used in this study has the greatest number of metacognitive indicators compared to the other three categories.

7.4.1 Local Assessment of Accuracy of Procedure (E-LAPA): Comparisons between Observations 1 to 4

The metacognitive decision point E-LAPA entailed the student's monitoring of his working, that is, checking if the working agreed accurately (correctly) with the steps of the implemented plan. Other examples were where the student wrote out his work in an orderly manner to structure it and hence the problem solving process; or where the student checked the accuracy/precision of his calculations[58]. Figure 7.4 below illustrates the differences in students' metacognitive activities during execution.

Only in observation 1 did the extended degree student make more local assessments than the main stream student. This was mostly due to him giving suggestions to and assisting the main stream student. This was not always the case, since the main stream student was mostly a social trigger for the extended degree student's metacognitive behaviour[59]. The main stream student mostly gave advice by explaining procedures of the problem solving process to the extended degree student. Similar to the categories Orientation and Planning, the main stream student's role of leading the problem solving process may also have contributed to him exhibiting a higher number of metacognitive activities during Execution as compared to the extended degree student.

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[58] More examples of this metacognitive decision point are discussed in Chapter 5 and Appendix A.

[59] This was not always the case in observation 1, as was seen from the 'chain reaction' where one student acted as a social trigger for the other's metacognitive behaviour and vice versa (cf. Figure 6.1).

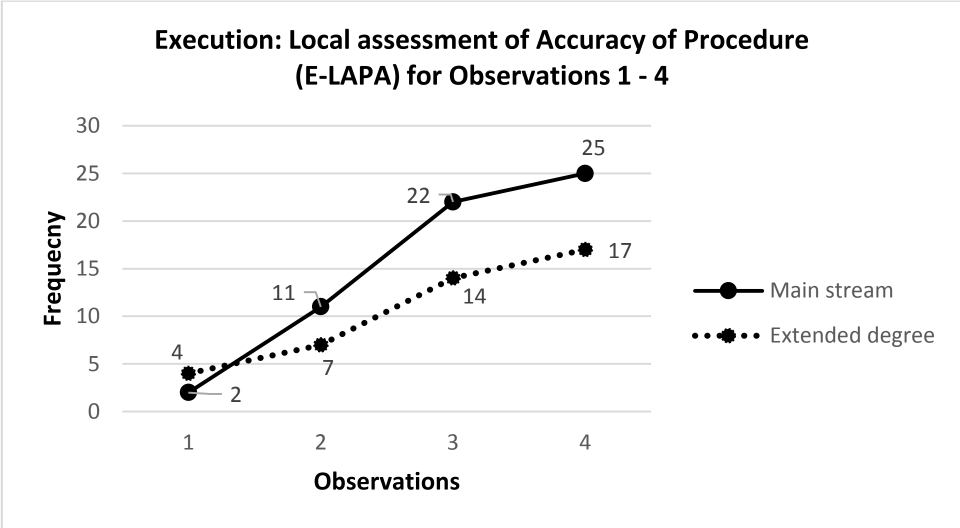

Figure 7.4: Students' Local Assessments of the Accuracy of Procedure (during Execution) over Observations 1 to 4

There are other possible reasons for the greater number of local assessments made by the main stream student. As discussed in the Literature Review, being more mathematically competent and successful in mathematics[60], the main stream student was more able to apply metacognitive skills[61]. Being more mathematically competent, may also have contributed to the frequency of E-LAPA of the main stream student being greater than that of the extended degree student. Moreover, it was the main stream student who was able to solve the majority of the problems. Similar behaviour was discussed in the Literature Review, in which it was pointed out that being able to apply metacognitive skills is of importance to mathematical problem solving (cf. Lucangeli & Cabrele, 2006).

The smaller number of local assessments E-LAPA made by the extended degree student may be attributed to his poor content knowledge. An example of this was seen in observation 2. The extended degree student only studied one section

[60] The main stream student is considered as being more mathematical competent and successful in his mathematical learning performance. This is based on the degree he was enrolled for, as well as his academic results for his first year calculus course.

[61] Similar findings can be found in Kramarski, Mevarech & Arami, 2002; Mevarech & Amrany, 2008; Mevarech & Fridkin, 2006; Mevarech & Kramarski, 2003; and Schoenfeld, 1987 which showed that successful students in mathematics were able to apply metacognitive skills and these skills were an important aspect in the solving of mathematical tasks (as discussed in the Literature Review, Chapter 2).

whereas the main stream student prepared for the observation and studied a number of sections from his textbook before observation 2. Having a broader scope of knowledge on the content most probably gave the main stream student an advantage in being able to monitor and keep control over the problem solving process. Because of not understanding the work and/or having limited content knowledge, the extended degree student mostly turned to the main stream student for help. Also, mostly observing the main stream student's working as well as lagging behind him, may have contributed to a lower number of local assessments made by the extended degree student.

As seen from Figure 7.4, the frequency of E-LAPA increased for both students after explicit training in IMPROVE. Even though the students were not that positive about the metacognitive questioning techniques of IMPROVE, it may be the case that these techniques did contribute to an increase in the students' metacognitive skills. An example of this was seen from observations 3 and 4. The main stream student took more control over the problem solving process by working in a more orderly and structured manner compared to previous observations. Also, the extended degree student contributed more to the problem solving progress as was seen from observations 3 and 4, and with the main stream student turning to the extended degree student for help. This was not always the case for observations 1 and 2. This change in student activities may be because of the possible effect of IMPROVE on the student's metacognitive behaviour.

Although the IMPROVE method may have contributed to the increase in the students' metacognitive skills, the possibility of the level of task difficulty should not be ignored, even though the students were able to solve the majority of questions from observations 3 and 4.

7.4.2 Local Assessment of Reasonableness/Usefulness of Procedure (E-LAPR): Comparisons between Observations 1 to 4

This local assessment occurred when the students evaluated (monitored) the usefulness/reasonableness (relevance) of a procedure, a method or an implemented proposed strategy. As a metacognitive decision point it also took into account the student considering the reasonableness (practicality) of a procedure or an answer during Execution.

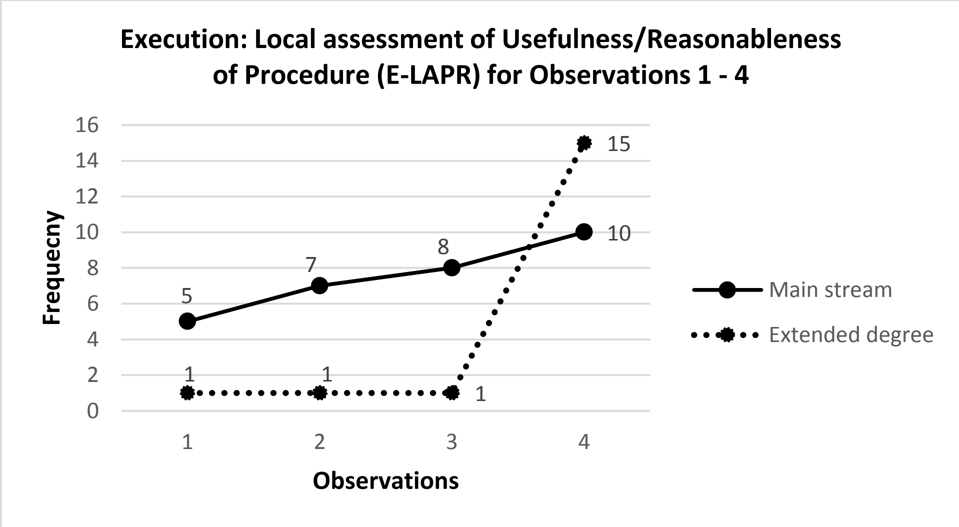

Figure 7.5: Students' Local Assessments of the Usefulness/Reasonableness of Procedure (during Execution) over Observations 1 to 4

As seen from Figure 7.5, the main stream student made more local assessments compared to the extended degree student[62]. The only exception was in observation 4. The higher number of local assessments by the main stream student can be attributed to reasons similar as to that of E-LAPA (cf. Section 7.4.1)[63].

The sharp increase from observation 3 to 4 in the frequency of number of local assessments made by extended degree student may be due to a number of

[62] From Figure 7.5 there is a steady increase in the number of local assessments made by the main stream student, different to that of E-LAPA (cf. Figure 7.4). The reason for this was not clear and it was difficult to identify what factors may have contributed to this.

[63] The main stream student was more mathematically competent than the extended degree student, hence the higher frequency of E-LAPR of the main stream student compared to the extended degree student. Moreover, the poor content knowledge of the extended degree student may have contributed to the lower number of local assessments E-LAPR made by him. The main stream student had a broader scope of knowledge on the content and it may have been to his advantage in being able to monitor and keep control over the problem solving process.

possible reasons[64]. While interacting, the extended degree student mostly commented on the usefulness of the methods and/or procedures implemented by the main stream student. Moreover, in voicing his difficulties and turning to the main stream student for assistance, he was taking control over his actions. It was by engaging with the main stream student that he was trying to understand the 'why' and 'how' behind the procedures implemented by the main stream student. Such behaviour from the extended degree student did not always occur in the first two observations, as seen from the consistency in frequency of E-LAPR in Figure 7.5. Other factors that may have also contributed to this sharp increase in frequency from observation 3 to 4 are the effect of IMPROVE on the extended degree student's metacognitive behaviour and/or an increase in the level of task difficulty. Still, there remains uncertainty as to what was the case.

7.5 Verification: Comparisons between Observations 1 to 4

As was seen from Chapter 6 on the discussion of observations 1 to 4, the students rarely exhibited metacognitive behaviour during Verification.

When verifying their work, students either made local assessments on the accuracy of their solution (coded V-LARA), or assessed the reasonableness of their solution and/or reflected on the problem solving process (coded as V-LARR). Furthermore, the students' metacognitive behaviour during Verification did not follow a particular pattern as was the case for Orientation and Execution. Moreover, students mostly started to reflect on their solution and/or the problem solving after intervention by the researcher. Students' behaviour with respect to the above two metacognitive decision points are discussed further below in Sections 7.5.1 and 7.5.2 respectively.

7.5.1 Local Assessment of the Accuracy of Result (V-LARA): Comparisons between Observations 1 to 4

From Figure 7.6 below we note that it was mostly the main stream student who considered the accuracy of his work during Verification. Apart from factors that

[64] It is important to note that most of the extended degree student's local assessments occurred during student interaction.

may have contributed to students verifying their work, the students in general exhibited very little metacognitive behaviour during Verification.

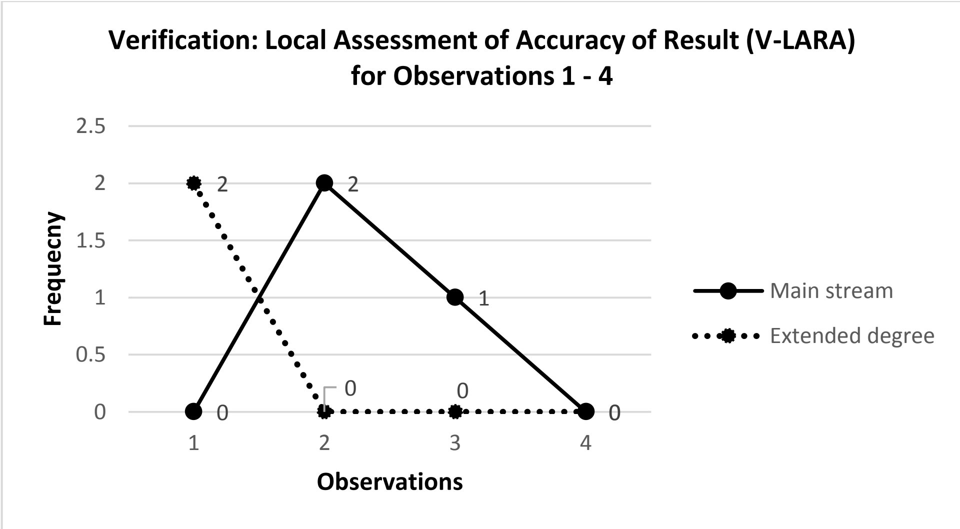

Figure 7.6: Students' Local Assessments of the Accuracy of a Result (during Verification) over Observations 1 to 4

Students' metacognitive behaviour during Verification was mostly irregular as seen from Figure 7.6. When comparing student's metacognitive behaviour over the four observations, it was difficult to find a common factor (or factors) that contributed to the differences in students' metacognitive behaviour between the four observations. There was also no consistency in what factors and/or triggers spurred students to verify their work. Two extraneous factors that did contribute to the students not always verifying their work were that (i) the students had to go to their next lecture, and (ii) the researcher had to start with another observation with a different pair of students.

In Observation 3 the students acknowledged that they did not always review their work after problem solving. Hence, it seems that IMPROVE had almost no effect on the students' metacognitive activities pertaining to Verification. Furthermore, the students' lack of enthusiasm for using the metacognitive questioning techniques of IMPROVE may have contributed to the low frequency, as well as to the decrease and/or inconsistency in V-LARA.

7.5.2 Local Assessment of the Reasonableness/Usefulness of Result (V-LARR): Comparisons between Observations 1 to 4

The code V-LARR applied to instances where students reflected on the problem solving process and/or their solution in considering either (i) the reasonableness (soundness) of a result; (ii) or the usefulness of certain procedures, strategies or methods within their solution. As pointed out in Chapter 6, V-LARR also applied to instances in which students commented on their own personal characteristics as problem solvers. Figure 7.7 below outlines each student's metacognitive behaviour with respect to V-LARR over the four observations.

Again the main stream student made more local assessments during Verification compared to the extended degree student. Students' comments on their personal characteristics contributed mostly to the high frequency in V-LARR of observation 1.

Similar to the discussions on V-LARA, it was also difficult to determine a common factor (or factors) that possibly caused the differences between observations. Again, there was no regularity in what factors and/or triggers impelled students to assess the usefulness and/or reasonableness of their results[65]. From Figure 7.7, it also seems that IMPROVE had little influence (effect) on the students' metacognitive activities during Verification. Again, students' lack of enthusiasm about IMPROVE may have contributed to the low frequency of V-LARR in observations 3 and 4.

[65] Similar to what was discussed in Section 7.5.1, two factors that did contribute to the small number of occurrences of V-LARR were that (i) the students had to go to their next lecture, and (ii) the researcher had to start with an observation of a different pair of students.

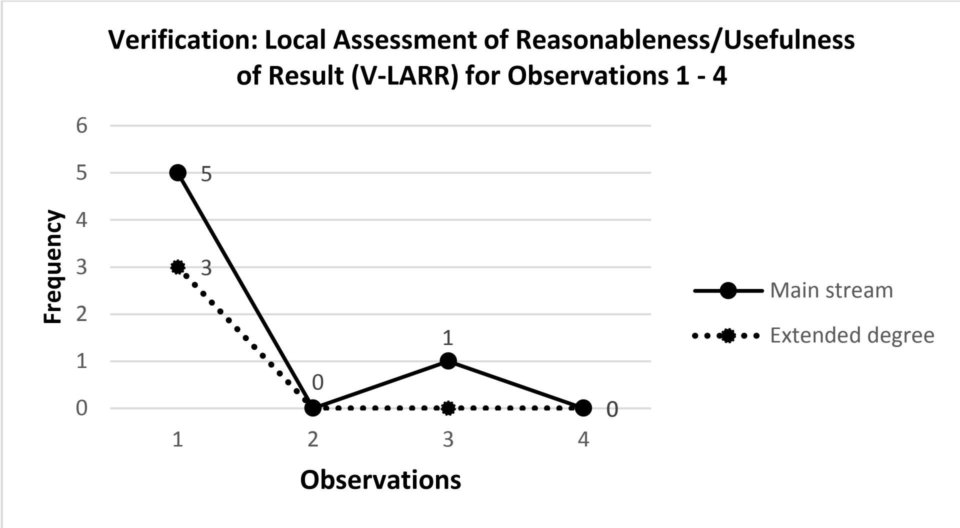

Figure 7.7: Students' Local Assessments of the Reasonableness/Usefulness of a Result (during Verification) over Observations 1 to 4

7.6 Students' Metacognitive Behaviour as Result of the Researcher's Intervention

During the four observations there were also instances of student(s)-researcher interactions. As noted previously, it was never the researcher's intention to guide and/or assist students during the observations, but students' lack of contribution and/or incorrect contributions led to such intervention. Mostly, the researcher urged the students to work together on solving the problems, but at times also assisted the students. In intervening with the students' problem solving process, the researcher acted as a social trigger for their metacognitive behaviour. The purpose of this section is to outline the different ways in which the researcher acted as a social trigger, and the effect it had on students' metacognitive behaviour and their problem solving skills over the observed time period.

7.6.1 Impact of the Researcher as Social Trigger for Students' Metacognitive Behaviour

Apart from the researcher encouraging students to collaborate during problem solving, the researcher's contribution to the students' metacognitive behaviour was relatively small. This is seen from Figure 7.8 and 7.9.

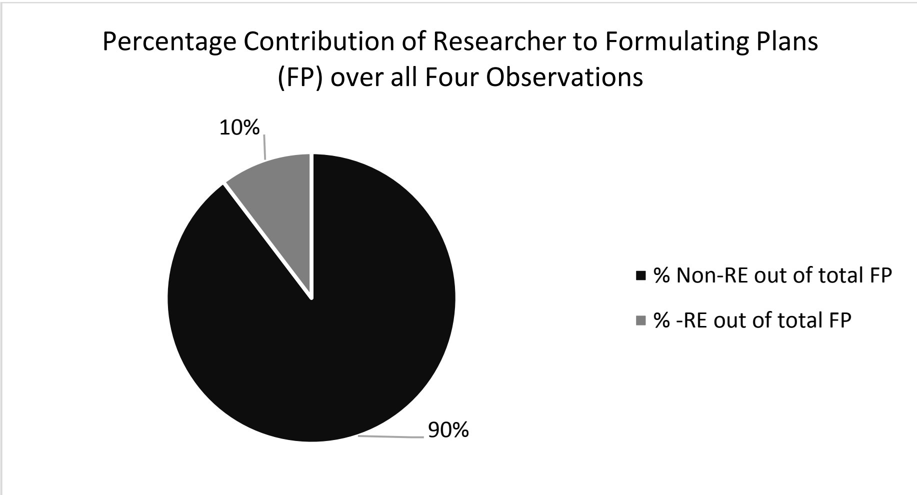

Figure 7.8: Percentage Contribution of the Researcher (% -RE) as Social Trigger to Students' Formulating of Plans, over all Four Observations

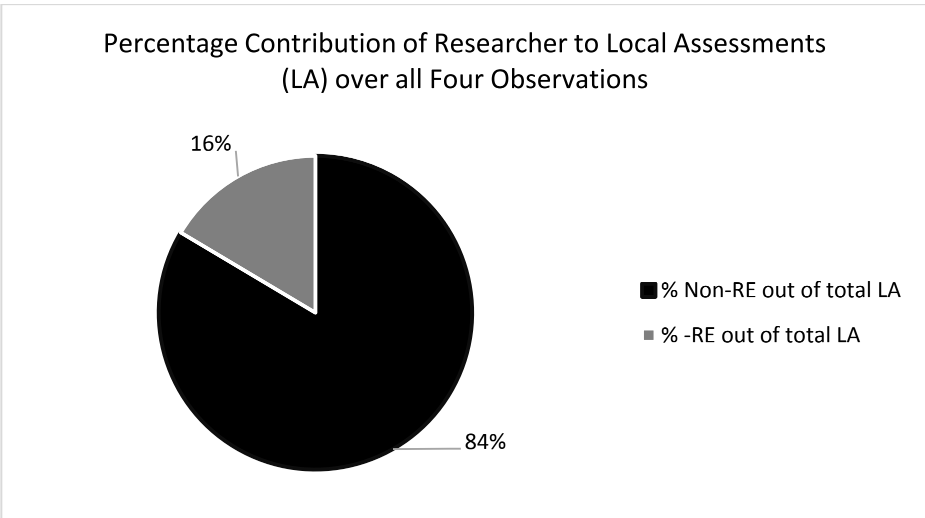

Figure 7.9: Percentage Contribution of the Researcher (% -RE) as Social Trigger to Students' Local Assessments, over all Four Observations

A closer inspection into each respective student's metacognitive skills, reveals that the number of metacognitive activities due to the researcher's intervention was much lower than those made by students without intervention. This is illustrated in Table 7.1 and 7.2 below.

Table 7.1: Main Stream Student's Metacognitive Decision Points (MCDP) consequent upon Researcher's Intervention versus no Intervention

Main stream student MCDP: Total over all four observations			
MCDP (no intervention)	Frequency	MCDP-RE (with intervention)	Frequency
P-PI	14	P-PI-RE	0
P-NI	4	P-NI-RE	1
Total FP	18	Total FP-RE	1
O-LAK	27	O-LAK-RE	2
E-LAPR	30	E-LAPR-RE	3
E-LAPA	60	E-LAPA-RE	6
V-LARA	3	V-LARA-RE	3
V-LARR	6	V-LARR-RE	10
Total LA	126	Total LA-RE	24

Table 7.2: Extended Degree Student's Metacognitive Decision Points (MCDP) consequent upon Researcher's Intervention versus no Intervention

Extended degree student MCDP: Total over all four observations			
MCDP (no intervention)	Frequency	MCDP-RE (with intervention)	Frequency
P-PI	8	P-PI-RE	0
P-NI	0	P-NI-RE	2
Total FP	8	Total FP-RE	2
O-LAK	14	O-LAK-RE	3
E-LAPR	18	E-LAPR-RE	2
E-LAPA	40	E-LAPA-RE	1
V-LARA	2	V-LARA-RE	1
V-LARR	3	V-LARR-RE	11
Total LA	77	Total LA-RE	18

It was only during Verification that the main stream student exhibited more local assessments due to the researcher's intervention (in the case of V-LARR-RE). Similar findings were also true for the extended degree student (both V-LARA-RE and V-LARR-RE). A more in-depth discussion on the above findings regarding students' metacognitive behaviour during Verification is presented later in the chapter, in Section 7.6.5.

7.6.2 The Researcher as Social Trigger during Planning

As seen from Table 7.1 and 7.2 the number of instances in which the researcher acted as a social trigger during Planning was relatively low. Moreover, and as discussed in Chapter 6, the researcher only intervened with the students' proposed strategies in the following three cases:

1. *Observation 2, question 2:*

   The students were explicitly told what strategy to use. Students were able to implement the strategy but had difficulty in solving the problem because of poor content knowledge.
2. *Observation 3, question 3:*

   The researcher intervened since the students were considering a strategy that created much difficulty in solving the question. Unfortunately the students did not solve the problem due to time constraints.
3. *Observation 4, question 3*:

   After struggling to execute the proposed strategy the researcher gave the students a hint on how to solve this question. This enabled them to devise a new strategy on their own. They were able to solve the question successfully without further assistance from the researcher.

Considering Figure 7.8, in conjunction with Table 7.1 and 7.2 and the above discussion, it follows that the researcher's overall contribution to the students' development of metacognitive skills with regards to Planning was small. Moreover, it was generally the case that the main stream student was able to propose strategies on how to solve the problems without any assistance from the researcher. Furthermore, with the main stream student mostly leading the problem solving process, the students were able to successfully solve the majority of the questions over the four observations[66], mostly without any assistance from the researcher.

7.6.3 The Researcher as Social Trigger during Orientation

Intervention by the researcher during Orientation only occurred in observations 3 and 4. Table 7.3 summarises the instances of intervention, the outcome of an intervention, and the effect it had on the students' metacognitive behaviour.

Table 7.3: Instances of Students' Metacognitive Behaviour consequent upon Researcher's Intervention during Orientation

Occurrence of intervention	**Researcher's action**
Observation 3, question 1	Announced to the students that they are allowed to use their textbooks.
Student's / Students' reaction	**Impact of intervention on students' metacognitive behaviour**
The main stream student immediately turned to his textbook in orientating himself about the different forms of power series.	With the help of his textbook, the main stream student was able to propose a strategy on how to solve the question. Apart from this, both students were able to implement the proposed strategy successfully and also solve the problem, without further assistance from the researcher.
Occurrence of intervention	**Researcher's action**
Observation 4, question 3	Students were given a hint.

[66] The only exception to this was in observation 1. Both students applied a strategy which was not useful in solving the questions. Moreover, as pointed out before, the students also applied the strategy incorrectly and did not successfully solve the questions.

Student’s / Students’ reaction	Impact of intervention on students’ metacognitive behaviour
By considering the given hint, students orientated themselves about the question by relating it to a similar question from the same observation.	Because of the given hint, the main stream student was able to propose a strategy on how to solve the problem. Still, both students were able to solve the problem without any further assistance from the researcher.
Occurrence of intervention	**Researcher’s action**
Observation 4, question 4.1	Before solving the problems the researcher asked the extended degree student why he had difficulty with the question.
Student’s / Students’ reaction	**Impact of intervention on students’ metacognitive behaviour**
In reacting to the researcher, the extended degree student orientated himself about the question.	No further assistance was needed from the researcher. The main stream student had no difficulty with the question and was able to assist the extended degree student in solving the problem.

During observation 3, the students admitted that they did usually orientate themselves about mathematical problems (thus they were implicitly using comprehension and connection questions from the IMPROVE method). Also, as discussed in Chapter 6 and Section 7.3.1, students mostly orientated themselves about the questions without any intervention from the researcher during observations 1 and 2. Hence it seems most likely that metacognitive skills pertaining to Orientation already formed part of the students’ metacognitive repertoire. Furthermore, the IMPROVE method may have played some role in developing these skills (although there is not enough evidence to show that this is indeed the case, as discussed in Section 7.3.1). In conclusion, it appears from the above that the researcher played an insignificant role in students’ development of metacognitive skills during Orientation.

7.6.4 The Researcher as Social Trigger during Execution

In general, the researcher intervened very little during students' implementation of the proposed plans, as can be seen from Tables 7.1 and 7.2. The following section discusses the researcher's contribution to students' metacognitive activities in terms of their assessments of the accuracy of their solutions and the usefulness/reasonableness of procedures during Execution.

7.6.4.1 Local Assessment of Accuracy of Procedure in Reaction to the Researcher (E-LAPA-RE) for Observations 1 to 4

From Figure 7.10 we note that intervention from the researcher occurred mostly during observation 2. The high frequency of E-LAPA-RE of the main stream student was mostly due to him having difficulty with the calculations around factorials in question 2 and hence the researcher assisting the student.

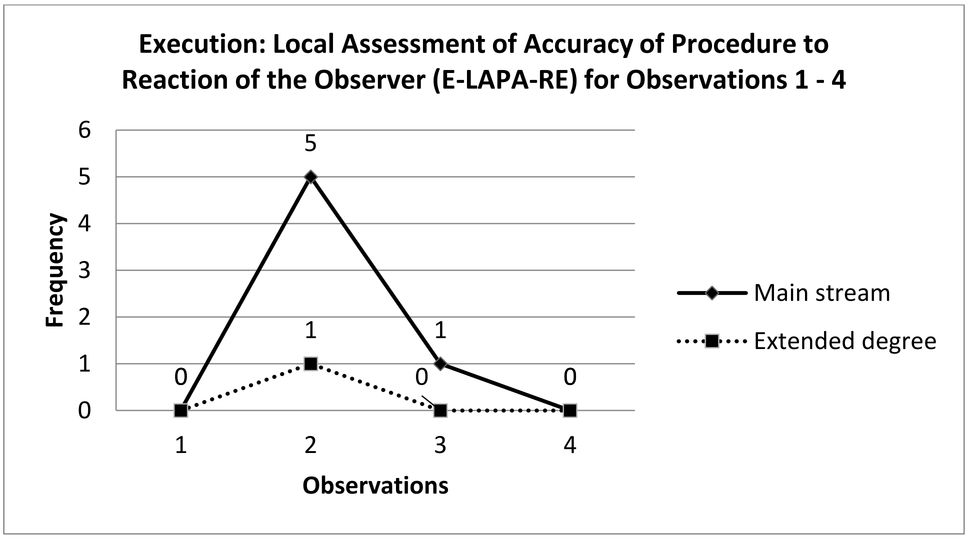

Figure 7.10: Students' Local Assessments of the Accuracy of Procedure due to Reaction to the Researcher (during Execution) over Observations 1 to 4

From Figure 7.10 we also note that even after IMPROVE was implemented, the number of instances of the researcher's intervention decreased. This is different to what was discussed in Section 7.4.1 in which it was noted that there was an increase in the frequency of E-LAPA for both students (without the researcher's intervention). Hence, after IMPROVE it seems that the researcher had very little contribution in the development of the students' metacognitive behaviour with regards to the accuracy of their work.

Apart from the above, and as pointed out earlier, the researcher mostly urged the students to work together. This is also regarded as a form of intervention, and most certainly had an effect on both students' metacognitive activities. Working together, the students acted as social triggers for each other's metacognitive behaviour. In particular, the main stream student was mostly a social trigger for the extended degree student's metacognitive behaviour (as mentioned before). In working together the students started monitoring their own and each other's work and/or thought processes. Hence the researcher's role in urging the students to collaborate seems to have had a substantial effect on students' metacognitive behaviour over the four observations.

7.6.4.2 Local Assessment of Usefulness/Reasonableness of Procedure in Reaction to the Researcher (E-LAPR-RE) for Observations 1 to 4

Similar to the case of E-LAPA, the researcher provided little assistance to the students during their evaluations on the usefulness/reasonableness of procedures (as can be seen from Tables 7.1 and 7.2). The higher frequency in both students' metacognitive behaviour during observation 4 was mostly due to the researcher's given hints[67]. This is seen from the Figure 7.11 below.

[67] As pointed out before: the students were still able, without any assistance, to devise a new strategy on how to solve the question. Moreover, they were able solve the question successfully without further assistance from the researcher.

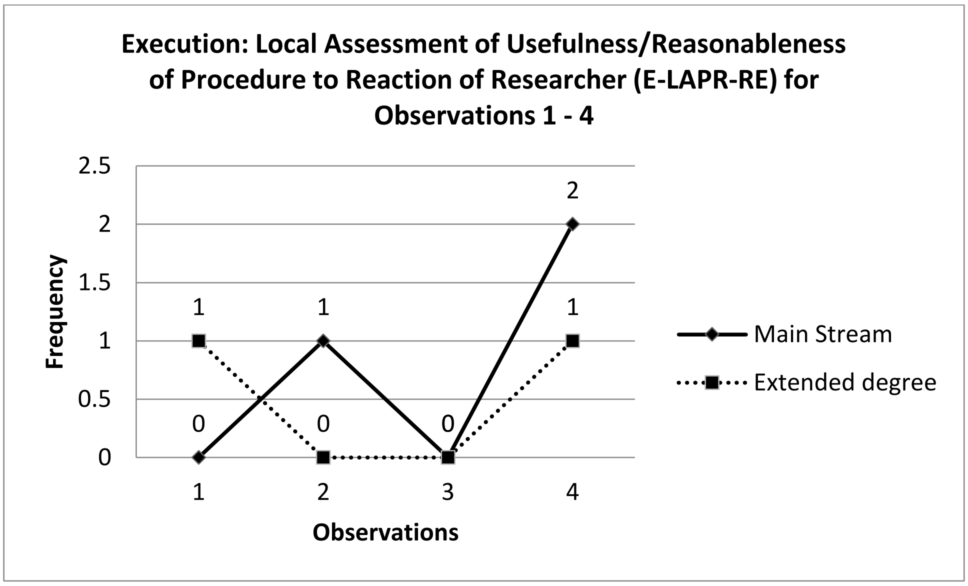

Figure 7.11: Students' Local Assessments of the Usefulness/Reasonableness of Procedures due to Reaction to the Researcher (during Execution) over Observations 1 to 4

The main stream student's response to the researcher's intervention during observation 2 was related to the student's confusion around the terminology of sequences and series. The local assessment made by the extended degree student during observation 1, was due to the researcher urging him to evaluate the main stream student's working. This was the first of many instances in which the researcher urged the students to collaborate in solving the problems. As pointed out in the previous section, urging the students to work together may have had an effect on both students' metacognitive activities, since they acted as social triggers for each other's metacognitive behaviour.

In conclusion, the researcher's role in urging the students to collaborate seemed to have had a substantial effect on the development of students' metacognitive behaviour; in particular the behaviour of the extended degree student.

7.6.5 The Researcher as Social Trigger during Verification

As mentioned before, the students exhibited very little metacognitive activity during Verification, and students mostly started to reflect on their solution and/or the problem solving process because of intervention by the researcher.

The purpose of this section is to emphasise the above results by means of quantitative representations of these findings, as well as to discuss students' local assessments as a result of the researcher's intervention. Section 7.6.5.1 deals with students local assessments on the accuracy of their work, while Section 7.6.5.2 discusses students' evaluations (reflections) on the reasonableness and/or usefulness of procedures which were used in their solutions.

When considering local assessments made by the students during Verification in reaction to the researcher, Figure 7.12 below shows that the percentage of local assessments because of reaction to the researcher was greater than the percentage of local assessments made without intervention from the researcher. This emphasises the finding that students seldom verified (reviewed) their work after problem solving of their own account.

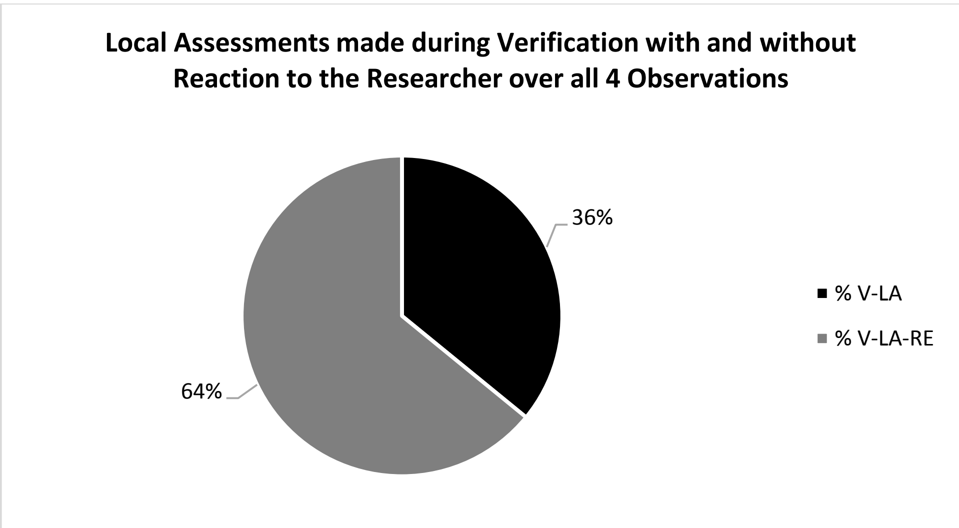

Figure 7.12: Local Assessments made during Verification without Reaction to the Researcher (V-LA) versus Local Assessments made because of Reaction to the Researcher (V-LA-RE), over all four Observations

The number of local assessments of the reasonableness/usefulness of results (V-LARR-RE) were also greater than the number of local assessments of the accuracy of results (V-LARA-RE) because of the researcher's intervention. This is illustrated in Figure 7.13 below.

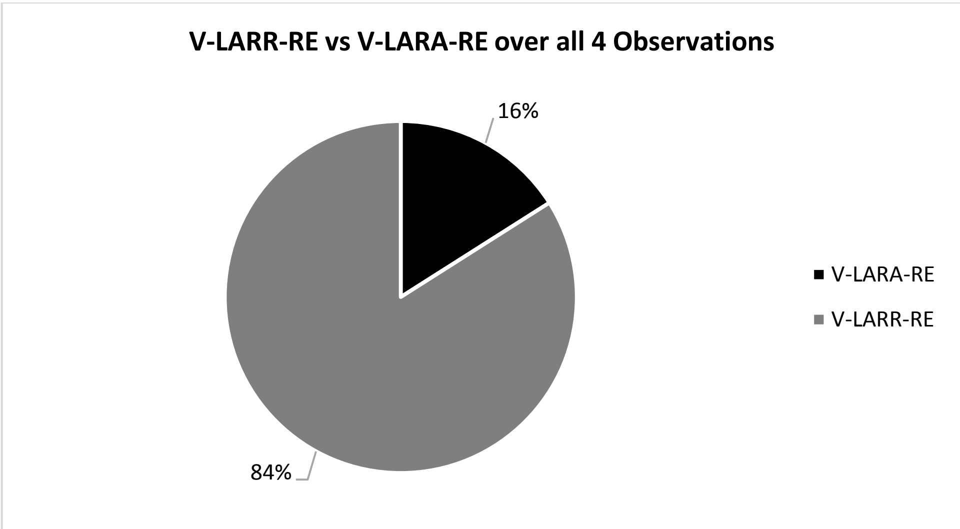

Figure 7.13: Percentage Local Assessments of the Reasonableness/Usefulness of Results (V-LARR-RE) versus Local Assessments of the Accuracy of Results (V-LARR-RE) because of Reaction to the Researcher, over all four Observations

7.6.5.1 Local Assessment of the Accuracy of Results in Reaction to the Researcher (V-LARA-RE) for Observations 1 to 4

Similar to the case of V-LARA and V-LARR, it was difficult to find a common factor (or factors) that contributed to the differences in the students' metacognitive behaviours with respect to V-LARA-RE between the four different observations. Again, as mentioned before, factors that may have strongly contributed to students not always verifying their work were that (i) the students had to go to their next lecture, and (ii) the researcher had to start with an observation of a different pair of students.

As seen from Figure 7.14 below, it seems that IMPROVE also had no effect on the students' metacognitive activities during Verification. Furthermore, the students' lack of enthusiasm for using the metacognitive questioning techniques of IMPROVE may also have contributed to the low frequency, as well as the decrease and/or inconsistency in the frequency of V-LARA-RE.

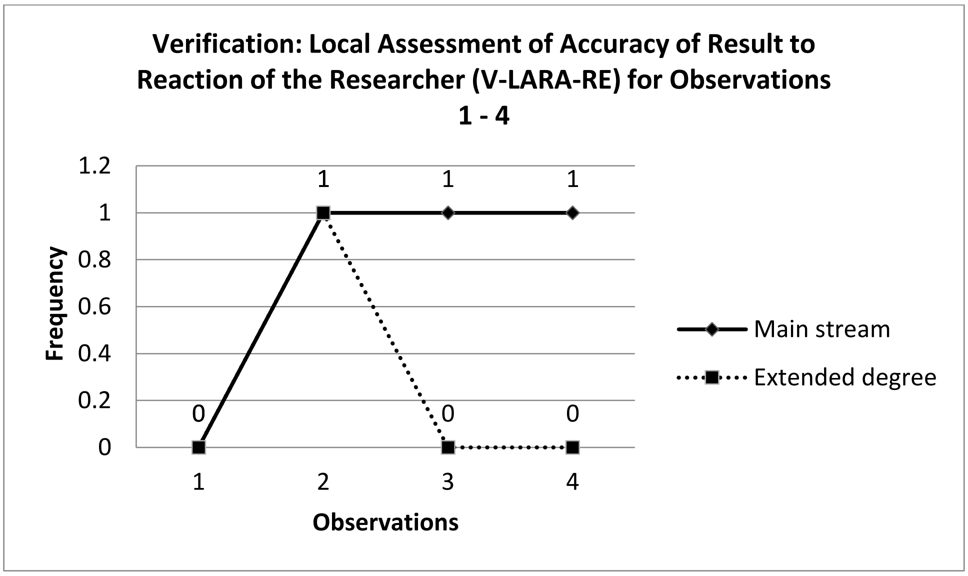

Figure 7.14: Students' Local Assessments of the Accuracy of a Result, due to Reaction to the Researcher (Verification) over Observations 1 to 4

From Figure 7.14 we note that the main stream student mostly exhibited metacognitive behaviour during Verification in terms of verifying accuracy of results, in reaction to the researcher[68]. Furthermore, the main stream student's verification of results mostly occurred when he:

(i) considered if the executed plan was accurately implemented;
(ii) considered if methods/procedures within the executed plan was correctly implemented;
(iii) had particular difficulty with solving a question; or
(iv) lacked the needed content knowledge on how to solve the problems.

[68] The frequency of V-LARA for the main stream student was also greater compared to that of the extended degree student, as discussed in Section 7.5.1.

A clearer distinction between the two students' metacognitive activities was seen in their local assessments on the reasonableness and/or usefulness of results. This is dealt with in the following section.

7.6.5.2 Local Assessment of the Reasonableness/Usefulness of Result in Reaction to the Researcher (V-LARR-RE) for Observations 1 to 4

Similar to the discussions on V-LARA, V-LARR and V-LARA-RE, it was also difficult to determine a common factor (or factors) that caused the differences between observations. Again, there was no regularity in the factors and/or triggers that impelled students to assess the usefulness and/or reasonableness of their results[69]. As seen from Figure 7.16 below, and from similar discussions in previous sections, it was not clear if IMPROVE did have any influence on students' metacognitive activities during Verification.

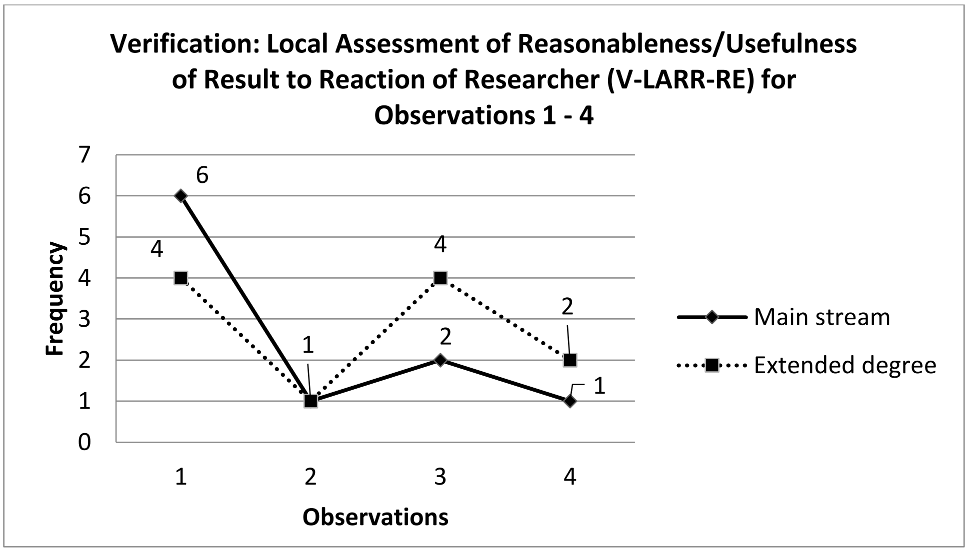

Figure 7.15: Students' Local Assessments of the Reasonableness/Usefulness of a Result, due to Reaction to the Researcher (Verification) over Observations 1 to 4

[69] Similar to what was discussed in Section 7.5.1, two factors that influenced the frequency of V-LARR and V-LARR-RE respectively, were that (i) the students had to go to their next lecture, and (ii) the researcher had to start with an observation of a different pair of students.

Students' comments on their personal characteristics in particular contributed to the high frequency of V-LARR-RE for observation 1 (cf. Figure 7.15). It was also in observation 1 that the main stream student reflected on the soundness of their application of the proposed strategy (which only occurred after completing all three questions). Students reflecting on their faulty application of the strategy also contributed to the high frequency of V-LARR-RE in the above observation.

Furthermore, in observations 2 to 4, students only reflected on and/or verified their work with respect to (i) questions they found particularly difficult and/or (ii) questions in which they lacked the needed content knowledge to solve.

As can be seen from Figures 7.7 and 7.15, the extended degree student omade local assessments on the reasonableness and/or usefulness of the results (V-LARR) only because of the researcher's intervention. Moreover, after observation 2, the extended degree student also generated more local assessments compared to the main stream student in the case of V-LARR-RE. This is different to what was generally observed in most of the metacognitive categories and the decision points made by the respective students. Furthermore, when the extended degree student reflected on the reasonableness and/or usefulness of procedures (methods), it was in commenting and reflecting on the solutions of the main stream student.

During observation 3, students faced the most difficulty with question 2. Because of this, the researcher engaged them in reviewing their solution and the problem solving process after completion of the question. This contributed to the increase in local assessments from observation 2 to 3. Furthermore, the extended degree student was the first in starting to reflect on the procedure(s) used in the main stream student's solution. This on its own spurred the main stream student to explain his solution to the extended degree student. Consequently, both students reviewed and reflected on the solution. One can only speculate if such reflection and reviewing of work would have occurred without intervention from the researcher.

7.7 Summary

This chapter discussed the differences between students' metacognitive behaviour between each of the four observations. In general, the main stream student exhibited a greater number of metacognitive activities compared to the extended degree student (in those instances where there was no intervention from the researcher). Furthermore, it seems that the IMPROVE method and the level of task difficulty may have possibly influenced students' metacognitive behaviours over the observed time period. This was seen from the increase in students' metacognitive activities during Orientation, Planning and Execution. Apart from the students acting as social triggers for each other's metacognitive behaviour, the researcher also acted as a social trigger for their behaviour. The researcher's influence on the students' metacognitive behaviour was minimal during Planning, Orientation and Execution. In general, the researcher acted as a social trigger for students' metacognitive behaviour during Verification. In particular, the researcher seemed to have contributed the most in spurring the students on to reflecting on the usefulness and/or reasonableness of their results after problem solving (V-LARR-RE). The main findings of this chapter are further discussed in Chapter 8, in answering the research questions of the study.

Chapter 8: Main Findings

8.1 Introduction

This chapter presents the main findings resulting from the data collection, and the discussions in Chapters 6 and 7. It provides information about what metacognitive skills each student exhibited during collaborative problem solving, as well as the differences in students' metacognitive skills. As an exploratory study, a major goal of the study is to generate hypotheses that provide opportunity for future research. In order to investigate the students' metacognitive behaviour, the study addressed the following research questions:

1. (a) What metacognitive skills do an extended and mainstream degree student respectively exhibit while working together as a pair on mathematical tasks, *before* explicit instruction in the IMPROVE method?

   (b) In particular, are there any differences in the metacognitive skills between the extended and main stream degree student before training?

2. (a) *After* explicit instruction in the IMPROVE method, what metacognitive skills do an extended and main stream degree student respectively exhibit while working together as a pair on mathematical tasks?

   (b) In particular, are there any differences in the metacognitive skills between the extended and main stream degree student after training?

3. What role does the researcher play in initiating the metacognitive skills of the students and how does the researcher influence the development of the metacognitive skills of the students over the period of the observation?

4. How does the level of task difficulty influence students' metacognitive behaviour (skills) over the period of observation?

Sections 8.2 to 8.5 address each of the above research questions respectively. Research question 2 implicitly takes into account the effect (influence) of IMPROVE as environmental trigger on students metacognitive skilfulness. In answering research questions 1 and 2, the role of the researcher was not taken into account. Moreover, answers to the above two questions focused on the general findings relating to students' metacognitive skilfulness (behaviour) over all four observations. The researcher as social trigger of students' metacognitive behaviour is dealt with separately in research question 3. Section 8.5 (focusing on research question 4) considers the possible effect that the level of task difficulty, as environmental trigger, may have had on students' metacognitive behaviour.

Students' roles as social triggers for each other's metacognitive behaviour is discussed in this chapter as well. As a final discussion, I consider students' lack of content knowledge, motivation and self-efficacy that may have affected their metacognitive behaviour.

8.2 Research Question 1

Before IMPROVE was implemented, both students exemplified some metacognitive behaviour during Planning, Orientation and Execution. Students exhibited very little metacognitive activity during Verification.

Both students tended to apply inappropriate strategies to questions where they lacked the needed content knowledge in order to solve the problems. Goldberg and Bush (2003) observed similar behaviour in Grade 3 mathematics learners. In that study, when solving novel tasks, most learners immediately started applying a strategy, without considering its usefulness during execution of the solution as well as after obtaining the solution. Although students from the present study solved tasks similar in structure to those they had seen before, it seems that their lack of content knowledge may have contributed to the use of inappropriate strategies.

Further findings of the present study show that the students did not always orientate themselves about all the questions of a task, whereas during Execution both students assessed and/or monitored the accuracy of their work, as well as the reasonableness and/or usefulness of their procedures. It was also found that during Orientation students turned to their textbooks and/or lecture notes for further assistance in considering examples and/or theory used in the questions of the tasks. In using these resources, students were taking control of the problem solving process, and hence exhibiting metacognitive behaviour. Efklides, Kiorpelidou and Kiosseoglou (2006) argue that students' use of worked-out examples can be seen as metacognitive activity, since students' self-explanations on how they understand an example and the use of such examples as an aid can be metacognitive in nature.

When considering the differences between the students' metacognitive skills in the present study, it was only the main stream student who was able to generate new strategies to solve problems after discarding unhelpful strategies. He was also the only one who realised when an incorrect strategy was applied. Findings also show that in general the main stream student exhibited more metacognitive skills during Orientation compared to the extended degree student. Similar results were true for students' metacognitive behaviour during Execution. As noted in Chapters 1 and 2, research has pointed out that successful students in mathematics are able to apply metacognitive skills (Lucangeli & Cabrele, 2006; Schoenfeld, 1987). This concurs with the present study's findings: the main stream student who was academically stronger and more successful in mathematical problem solving, also exhibited the greater number and variety of metacognitive skills as compared to the extended degree student.

As mentioned earlier, both students exhibited very little metacognitive activity during Verification. Similar results can be seen in the work of Muir and Beswick (2005). Conducting research on Grade 6 learners solving of non-routine mathematical problems, the authors found that the majority of learners did not verify (check) the reasonableness of their answers. What was also alarming is that learners had difficulty knowing how to verify their solutions. Apart from this, learners who were more successful in their mathematical problem solving, did show a greater interest in verifying their solutions. With respect to the latter result, similar findings were seen in the present study as well. The main stream student, who predominantly guided the students in successfully solving the problems, was also the one who exhibited the greatest number of metacognitive activities during Verification. Moreover, only the main stream student verified if the proposed strategies were implemented correctly. Also, the number of local assessments the main stream student made when considering the reasonableness and/or usefulness of the solutions, were more compared to the extended degree student.

8.3 Research Question 2

After completing observations 1 and 2 the students were explicitly trained in the use of the metacognitive questioning techniques as advocated by the IMPROVE method. After IMPROVE was implemented both students exhibited metacognitive behaviour during all four categories. Even though the IMPROVE method encourages students to reflect on their solutions after problem solving, both students still exhibited very little metacognitive behaviour during Verification. Furthermore, from the discussions in Chapter 6 and 7 it seems that the IMPROVE method may have played an important role as an environmental trigger in the development of students' metacognitive skills. Below, this inference is discussed further for each of the four metacognitive categories.

The number of metacognitive activities, during Orientation and Execution, of both students increased after explicit training in IMPROVE. Moreover, students also orientated themselves about *all* the questions of each task, after the implementation of IMPROVE. Students again consulted their textbooks for help to solve the problems. Since there was an increase in students' metacognitive activities during Orientation, it seems that the IMPROVE method may have contributed to the development of students' metacognitive behaviour. There is, however, not enough evidence to substantiate this conjecture.

An increase in both students' metacognitive activities during Execution was also observed after the implementation of IMPROVE. It is likely that IMPROVE may have played a role in the development of students' metacognitive behaviour indicated by (i) the main stream student working in a noticeably more orderly and structured manner, and (ii) the extended degree student contributing more and more to the problem solving process. That the above behaviour was not as evident before IMPROVE was implemented, highlights the positive effect of IMPROVE on the development of students' metacognitive skills in this study.

Also, it seemed that IMPROVE also served as a trigger for a conscious effort from the students in improving their cognitive structures themselves – being consciousnes about their lack of knowledge they turned to their textbooks and/or notebooks for assistance, which is an example metacognitive behaviour. Results of the present study are similar to findings from research conducted by Mevarech et al. (2006) and Mevarech and Fridkin (2006). Both research groups showed that the IMPROVE method facilitates the development of students' metacognitive skills.

Since students exhibited very little metacognitive behaviour during Verification it is difficult to infer that the IMPROVE method had any effect on their metacognitive skills in this category. As outlined in Chapter 3, the author regards the use of the IMPROVE metacognitive questioning techniques and the implementation of metacognitive skills as desirable socio-mathematical norms[70]. Moreover, the researcher also argues that in order for students to be part of the mathematical discourse, they need to adhere to these norms. It was also the researcher's hope that students would adhere to these socio-mathematical norms outside formal lecture times and 'carry them over' to the observations.That students' metacognitive behaviour was very low during Verification, implies that it may be the case that students' enacted norms during observations were different from the intended socio-mathematical norms students were encouraged to follow during formal lectures. Students also revealed that they did not always apply the metacognitive questioning techniques of IMPROVE; in particular reflection questions which are mostly used during Verification. Hence, it may be the case that students either (i) did not appropriate the intended socio-mathematical norms and 'carry them over' to the observations, or (ii) believed that these norms were not helpful in solving mathematical problems[71]. Other researchers such as Wilson, Fernandez and Hadaway (1993) observed similar behaviour in their work;

[70] The concept of socio-mathematical norms was discussed in Chapter 3, and its relation to tenets of symbolic interactionism, as outlined in Yackel (2000, 2001) and Yackel and Cobb (1996).
[71] This was evident from Observation 3, where both students noted that they thought the IMPROVE metacognitive questions techniques were not useful.

they argued that creating an awareness amongst high school learners in mathematics to reflect on their work is very difficult.

Other findings of the present research study also reveal that only in the case of the main stream student was there an increase in the number of Planning activities, after IMPROVE. In contrast, metacognitive activities (during Planning) of the extended degree student decreased after IMPROVE. Also, the number of instances in which the main stream student considered different possible strategies for solving a problem was more compared to the extended degree student. Furthermore, after explicit training in IMPROVE, the main stream student considered new strategies for solving some of the problems after discarding an unhelpful strategy. This was not the case with the extended degree student.

Differences between students' metacognitive skills were also seen during Orientation, where the number of metacognitive activities of the main stream student were greater than that of the extended degree student. During Execution, the number of local assessments relating to accuracy made by the main stream student was more than those made by the extended degree student. Furthermore, it was only at the end of the observed time period that there was a sharp increase in the number of local assessments on the reasonableness and/or usefulness of procedures made by the extended degree student. Still, the number of instances where the main stream student assessed the reasonableness and/or usefulness of procedures was more compared to the extended degree student.

During Verification there was a decrease in the number of local assessments made by the main stream student after implementation of IMPROVE. The extended degree student did not exhibit any form of metacognitive activity during Verification, after the explicit training in IMPROVE. Moreover, it was only during the third observation that the main stream student exhibited any metacognitive behaviour during Verification. In conclusion, since there was a decrease in metacognitive activity of the main stream, and no metacognitive behaviour by the extended degree student after IMPROVE in Verification phase, it seems that

IMPROVE may have had little or no effect on students' metacognitive skills during Verification[72].

8.4 Research Question 3

Although it was never the intention of the researcher to guide and/or assist students during the observations, students' lack of contribution and/or incorrect contributions led to such intervention. That the number of student metacognitive activities due to the researcher's intervention was much lower in comparison to those exhibited by students without intervention, indicates that the researcher tried to assist the students as little as possible. When the researcher did assist the students, it was either to give them hints, or to ask them leading questions. Researcher intervention also occurred when the students were applying an inappropriate strategy, or the students were at a loss as to how to solve a problem. Also, when students were confused about mathematical terms or had difficulty in using these terms, the researcher intervened.

The above interventions can be regarded as a form of scaffolding. Meyer and Turner (2002) note that scaffolding supports the development of students' self-regulatory skills[73] and can take on different forms such as the scaffolding of learner autonomy. In this research study, scaffolding due to researcher intervention, was mostly geared at supporting and developing student autonomy. This was done in order for students to take more and more responsibility in controlling and monitoring their problem solving behaviour without frequent researcher intervention. Still, there is not enough evidence to determine if the researcher's aim was indeed achieved and if there was a development in the students' autonomy.

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[72] These findings are only concerned with instances in which the researcher did not act as a social trigger for students' metacognitive behaviour.

[73] As pointed out in the Literature Review, metacognitive skills form part of self-regulatory skills. For e.g., Meyer and Turner's (2002) conception of self-regulated learning not only takes into account students' regulation of and control over their cognitive activities, but also their motivation and behaviour.

Apart from the above researcher interventions, students were able to propose strategies for solving the majority of questions without assistance from the researcher. Furthermore, students were able to successfully solve the majority of problems, without researcher intervention. The researcher also contributed very little to the students' metacognitive activity during Orientation and Execution, where intervention during Execution mostly concerned the researcher wanting the students to evaluate the usefulness and/or reasonableness of their mathematical procedures. In the case of Verification, students very seldom of their own volition verified (reviewed) their work after problem solving. Moreover, it was during Verification that researcher intervention occurred the most, with the researcher spurring students on to reflect on the usefulness and/or reasonableness of their results after problem solving.

Apart from the above, researcher intervention mostly took the form of the researcher urging the students to collaborate. This form of intervention seems to have had a greater influence on the students' metacognitive activities, as opposed to instances where the researcher assisted students. As a result of urging students to work together, students acted as social triggers for each other's metacognitive behaviour. An in-depth discussion on the effect students had on each other's metacognitive activity is given in Section 8.6.

In conclusion, since there was a small number of interventions in which the researcher assisted students, it is most likely the case that the researcher contributed very little to the development of the students' metacognitive skills over the period of observation. The only exception was during Verification where the researcher played a more prominent role in initiating students' metacognitive behaviour. This was particularly the case when students reviewed the usefulness and/or reasonableness of procedures. In comparison with the above, the researcher's role in urging the students to collaborate seems to have had a greater effect on students' metacognitive behaviour (skilfulness) over the four observations.

8.5 Research Question 4

The research study also focused on the level of task difficulty as environmental triggers of students' metacognitive behaviour. The concept of task difficulty as environmental trigger of students' metacognitive behaviour, originates from the work of Kim et al. (2013). The authors suggest that task difficulty influences and facilitates one's metacognitive processes. In Chapter 4, the effect of task difficulty on students' metacognitive behaviour was described at length. It was pointed out that problems which require different levels of conceptual and cognitive demand during problem solving processes can produce different metacognitive behaviours within problem solvers (Lesh et al., 2003). Moreover, metacognition is also activated more during the solving of difficult problems and metacognitive behaviour occurs more frequently in difficult than in easy tasks (Helms-Lorenz & Jacobse, 2008; Iiskala et al., 2004, 2011; Prins et al. 2006; Vauras et al., 2003).

Chapter 7 showed that the tasks increased in difficulty over the period of observation. Even the main stream student noted that he found the tasks of observations 3 and 4 more difficult than the tasks of observations 1 and 2. The increase in both students' metacognitive activities during Execution over the observation period leads one to conjecture that the perceived task difficulty may have contributed to an increased frequency of metacognitive activity. Moreover, students made more local assessments while answering questions that they found difficult. Hence the level of task difficulty is likely to have increased the frequency of metacognitive activities during Execution. Although students' metacognitive activity was very low during Verification, they still exhibited metacognitive behaviour while working on questions they found difficult. Hence throughout this study it seems that the level of task difficulty was one of the primary environmental triggers for students' metacognitive behaviour.

8.6 Students as Social Triggers of Metacognitive Behaviour

Both students acted as social triggers for each other's metacognitive behaviour which resulted in students monitoring their own and each other's work and/or thought processes.

During Orientation and Verification, there were few instances in which students acted as social triggers. It was mostly during Execution that students acted as social triggers for each other's metacognitive skills. During Planning the main stream student mostly proposed strategies to solve the problems with very little assistance from the extended degree student. In general, for all four metacognitive categories, it was the main stream student who acted predominantly as a social trigger for the extended degree student's metacognitive behaviour. It was only towards the end of the period of observation that the extended degree student became more of a social trigger for the main stream student's metacognitive behaviour.

When engaging with the extended degree student in his sequence of problem solving steps, the main stream student was actually scaffolding the extended degree student's approach to the solving of problems. Similar findings were also encountered in King (2007), with the more knowledgeable peer scaffolding the group's problem solving, and in the work of Azevedo (2005) who found that a more knowledgeable peer acted as an external regulator for the group. Hurme et al. (2006) also found that during students' collaborative network discussions in solving geometry problems, the more knowledgeable peers co-regulated other students' problem solving. Apart from this finding, the above authors could not explain how the peers benefited from each other's help and/or thinking during social interaction. In this study the extended degree student benefitted by interacting with the main stream student. While the main stream student took a leading role in solving the majority of problems, the extended degree student monitored and evaluated procedures implemented by the main stream student; corrected several misconceptions, as pointed out by the main stream student; as well as evaluated the main stream student's approach to solving the problems. Furthermore, the extended degree student reflected on the usefulness and/or reasonableness of procedures implemented by the main stream student. When facing difficulties, the extended degree student also consulted the main stream student and sought assistance. This is indicative of him taking control of his actions and the problem solving process and using his collaborator as an additional resource over and above his notes and the textbook, thus exhibiting metacognitive behaviour.

As seen from the above, metacognitive behaviour when initiated by one student triggers metacognitive activity in the other. Apart from the one student being the predominant social trigger, their collaborative dialogical exchanges served as triggers for each other's metacognitive behaviour respectively (as was seen in observation 1).

8.7 Students' Content Knowledge, Motivation and other Affective Factors Impacting Students' Metacognitive Behaviour

8.7.1 Poor Content Knowledge Affecting Metacognitive Behaviour

Apart from complex tasks serving as a catalyst for stimulating students to exhibit metacognitive skills, it was also pointed out in Chapter 4 that students who exhibit such skills operate from within the boundaries of their knowledge (cf. Prins et al., 2006).

The present research study recounts a number of instances of students' lack of content knowledge affecting their problem solving processes, and how this influenced the amount of metacognitive skills they exhibited. Furthermore, as discussed in Chapter 4, tasks used in this study were conceptually demanding. Tasks depended on students' domain-specific knowledge of concepts such as sequences and series, as well as that of power series expansions of functions and the related knowledge of differential and integral calculus. Many of the questions required a number of steps and/or procedures to solve a problem; this increased their complexity[74].

The affect of students' lack of content knowledge on their metacognitive behaviour was particularly evident during observation 1. Both students fixated on one strategy to solve all three of the questions of the task and applied the strategy

[74] Processing of complex tasks and level of task difficulty was dealt with in Chapter 4, Section 4.6.

incorrectly. Also, students' lack of knowledge of the variety of possible strategies to solve the three questions contributed to their difficulty in solving the problems. Observation 2 illustrates that the students did not have sufficient knowledge to solve the problems. The extended degree student's lack of content knowledge further contributed to his difficulty in taking control of the problem solving process. Consequently he asked the main stream student for assistance. During observation 2 the main stream student had difficulty recalling the content of their first year calculus course. This affected his problem solving process in which the researcher intervened to assist (as seen in question 2 of observation 2). Students also demonstrated a naïve use of mathematical language when speaking about sequences and series and their confusion about these two concepts made it difficult for them to solve the problems in observations 1 and 2.

Students' lack of content knowledge not only negatively impacted their problem solving potential but also influenced their interaction in a collaborative setting. Hurme et al. (2006, 2009) note that students' lack of content knowledge may lead to them not acknowledging the importance of the proposed and/or implemented procedures of other students[75]. This lack of content knowledge may further contribute to students not being able to understand explanations of peers. Similar findings emerge in the present research study, for example the extended degree student's reaction to the work of the main stream student. Particularly during observations 1 and 2 the extended degree student did not always seem to understand the mathematical procedures carried out by the main stream student. He also often could not justify or give an argument for agreeing with the main stream student's solution. The extended degree student's inability to do so relates to his lack of the required content knowledge.

[75] The above researchers' work dealt with students working in a computerised collaborative environment.

8.7.2 Students' Motivation Affecting their Metacognitive Behaviour

As described in the literature review, motivation is another factor that influences students' metacognitive behaviour. Sternberg (1998) noted that motivational and affective factors influence students' development of metacognitive skills. Lau and Chan (2003) pointed out the importance of motivating students to use metacognitive skills to improve their learning performance, while Veenman et al. (2004) suggested that students must be self-motivated to use metacognitive skills without being urged by others to use them. In this regard, Wong (2012) makes the point that if students have learnt such metacognitive skills but are not motivated to use them, their learning performance would not necessarily improve.

In this study, students' lack of motivation to use the IMPROVE method's metacognitive questioning techniques may have affected the development of their metacognitive skills, over the period of observation. This was seen in particular in Verification where students exhibited very little metacognitive activity. Both students admitted that they did not use the reflection questions from IMPROVE and that they usually did not reflect on and/or verify their solutions after completing a problem.

Although such lack of motivation may have negatively affected the development of students' metacognitive skills there is not enough evidence to make such a claim. This provides an opportunity for future research to investigate the influence of motivation on the development of metacognitive skills.

8.7.3 Affective Factors and Students' Metacognitive Behaviour

Although this research study did not focus on determining the possible effects of affective factors on metacognitive skills they may still have influenced the students' metacognitive behaviour. Affective factors like student beliefs and their disposition towards mathematics were revealed especially during observation 2. This is discussed further below.

Affective factors are also related to the concepts of self-efficacy and self-regulated learning[76]. Marcou and Philippou (2005) regard self-efficacy as one's belief in your competence to succeed in a task. Moreover, the above authors also note that:

> …students who view themselves as capable to solve mathematical problems will choose to perform (a) task, compared to low efficacious students who might attempt to avoid involvement in the task.

The authors' notion of self-efficacy, I suggest, includes students' disposition towards and beliefs about mathematics. Having a positive attitude and being enthusiastic about solving mathematical problems are surely linked to self-efficacy.

Tanner and Jones (2003) have noted that students with high self-efficacy are more likely to use self-regulated strategies than low efficacious students. Furthermore, there is a direct relationship between students' use of metacognitive skills (strategies) and having high self-efficacy (Pintrich & De Groot, 1990). Marcou and Philippou (2005) have also shown that as the self-efficacy of students increases they are most likely to employ self-regulation skills when solving mathematical problems, and vice versa. From the above it seems that self-regulated learning, the use of metacognitive skills (strategies) and self-efficacy are inter-related and exert an influence on one another.

In my research study it seemed that the main stream student had high self-efficacy; this was evident when he mentioned his fondness for mathematics and his enjoyment of mathematical problems. This was not the case with the extended degree student (as seen in observation 2). Furthermore, in question 3 of observation 4 which the students had difficulty with, the main stream student was

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[76] Self-regulated learning was discussed in Chapter 2, in which it was highlighted that such learning is much broader than and encompasses the concept of metacognition and that metacognition is a subcomponent of self-regulation (Yilmaz-Tuzun & Topcu, 2001). Furthermore, it was also mentioned that metacognition, motivation and affective factors all form part of the concept of self-regulation and that all of these components are needed to develop and gain expertise in a certain domain (Sternberg, 1998). In Chapter 2, I noted that I agree with Kaplan (2008) and Schunk (2008) that metacognition and self-regulation are related in sharing the similar core notion of regulatory action, but are two different entities.

motivated to continue with their proposed strategy even though the researcher wanted to give the students a hint. The main stream student also mentioned that it was ‘fun’ to solve the problem and that ‘mathematics is fun’. Concurrent with the research referred to above, having a high self-efficacy may have contributed to the main stream student’s problem solving performance and his higher frequency of metacognitive activity compared to that of the extended degree student.

The extended degree student’s negative disposition towards mathematics was revealed by his statement that mathematics ‘haunts’ him and that he had difficulty in solving mathematical problems. Having such low self-esteem may have contributed to him exhibiting few metacognitive skills. His poor self-image of himself as a problem solver may have affected his problem solving performance and his use of metacognitive skills. Such an argument seems to be validated by the research findings of Panaoura & Philippou (2007). They have shown that students with high self-image have a positive disposition towards using metacognitive skills (strategies) while those with a low self-image are most likely not to implement such skills.

8.8 Summary

This chapter outlines the general findings on the metacognitive skills that students manifested as well as differences in the metacognitive behaviour of the students. Over the period of observation students exhibited metacognitive skills in all four categories of metacognitive activity where the lowest activity occurred during Verification.

Before and after the IMPROVE method was implemented, the frequency of metacognitive activities of the main stream student was greater than that of the extended degree student. After the IMPROVE techniques were implemented there was an increase in both students’ metacognitive activities only during Orientation and Execution. It was only in the case of the main stream student that there was an increase in metacognitive activity during Planning. During

Verification however, there was a decrease in the metacognitive activity of both students.

The chapter also detailed the role of the researcher as social trigger when urging students to work together. This seems to have contributed to the development of the students' metacognitive skills. Students also acted as social triggers for each other's metacognitive behaviour where the main stream student contributed to the development of the extended degree student's metacognitive behaviour. Furthermore, it seems that the IMPROVE method and the level of task difficulty as environmental triggers contributed to the development of students' metacognitive behaviour.

Lastly, it was also suggested that students' lack of content knowledge, disposition towards mathematics and self-efficacy were also likely to have influenced the lesser than expected use of metacognitive skills.

Chapter 9: Conclusions

9.1 Introduction

This chapter provides a summary of the main research findings, contributing to the body of knowledge on students' metacognitive skills, and the role of social and environmental triggers of students' metacognitive skilfulness (behaviour). Recommendations for future research, taking into account the limitations of the present study are also dealt with. A hypothesis for future research is also outlined and discussed. The chapter concludes with a discussion on the implications of the study's findings for educational practice.

9.2 Summary of General Findings

The research in this thesis has resulted in the emergence of the following findings on students' metacognitive behaviour over the observed time period:

It was found that, in general, the main stream student exhibited a greater number of metacognitive skills compared to the extended degree student during collaborative problem solving. Furthermore, it seemed that the IMPROVE method as an environmental trigger, had an effect on the development of both students' metacognitive behaviour. Apart from the above findings, this research study also highlighted the low occurrence of students' metacognitive activity during Verification, also pointing out that the IMPROVE method seemed to have had almost no effect on students metacognitive behaviour in the above category. Research findings of the study also revealed the role of the researcher as social trigger for students' metacognitive behaviour, which mostly consisted of the researcher urging students to collaborate. Such intervention from the researcher mainly resulted in the students acting as social triggers for each other's metacognitive behaviour. Findings also show that the researcher acted mostly as social trigger for students' metacognitive behaviour during Verification. Furthermore, it was found that there were a greater number of occurrences in which the main stream student acted as social trigger for the extended degree student's metacognitive behaviour. The level of task difficulty also seemed to have acted as an environmental trigger for students' metacognitive behaviour. This was evident from the high number of occurrences of students' metacognitive activity during more difficult mathematical problems. Students' lack of content knowledge seemed to have negatively impacted their metacognitive behaviour; especially during the first two observations. Lastly, affective factors such as students' motivation in engaging with metacognitive activities; their disposition towards mathematics; and students' self-efficacy are factors that possibly impacted students' metacognitive behaviour.

9.3 Significance of the Research

Findings of this study have extended knowledge of what metacognitive skills tertiary mathematics students exhibit. In particular, it has shown that differences in the metacognitive skills between that of an extended degree and main stream student (in this case) do exist. The study also advanced knowledge in the field of regarding metacognition as operating on different levels. This was done in considering social and environmental triggers of students' metacognitive behaviour.

9.3.1 Qualitative Findings on the Possible Metacognitive Skills of Tertiary Mathematics Students

This study contributed knowledge to the small field of metacognitive skilfulness of tertiary mathematics students. In particular, the study advanced the body of knowledge on the different metacognitive skills and frequency of occurrence of these skills that a particular extended and main stream degree student respectively exhibit. As a result, the study showed in which of the four metacognitive categories, students exhibited the greatest number of metacognitive skills, before and after the IMPROVE method was implemented. This is significant, since it shows that apart from both students engaging in metacognitive activity, it still was in varying degree of frequency between the different metacognitive categories, for each student respectively and that

IMPROVE had different effects on the metacognitive behaviour of the main stream and extended degree student respectively.

Apart from results reported in a quantitative form, the study also highlighted the different types (forms) of metacognitive skills that students exhibited. An example of these different forms of metacognitive behaviour was seen in the mainstream student *either* assessing the accuracy of his working, *or* the usefulness/reasonableness of a mathematical procedure[77]. Similar results were noted during students' reflection on the problem solving process, after completing a problem. Moreover, this study illustrated what were the different types of metacognitive skills students exhibited before and after explicit training in IMPROVE, contributing to the research field on how with the use of metacognitive instructional programmes, skills can be developed further and used in training students in the different forms of metacognitive skills. The above findings of such a qualitative nature provide an example of the power of exploratory qualitative research, in uncovering how a theoretical construct such as metacognition, manifests in different forms (types) of metacognitive behaviour in individuals.

9.3.2 Triggers of Students' Metacognitive Skills and the Paradox of Metacognition

This research has shown that there may be several factors that affect students' metacognitive behaviour. In particular, this study considered factors that triggered students' metacognitive behaviour, which were referred to as social and environmental triggers of students' metacognitive skills (as based on the work of Kim et al., 2013). Moreover, as discussed in the Literature Review, these triggers are linked to the concept of the paradox of metacognition and hence contributed in furthering knowledge on the above paradox[78], as further discussed below.

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[77] These findings take into account students' metacognitive behaviour with and without intervention from the researcher.

[78] The paradox of metacognition notes that metacognition cannot be explained solely in terms of a student's individual attributes, but that external factors that serve as triggers for the individual's metacognitive behaviour also need to be considered (Kim et al., 2013 and as discussed in Chapter 2).

Chapters 6 to 8 discussed the effect of social and environmental triggers on students' metacognitive behaviour. It was noted that the researcher as social trigger of students' metacognitive behaviour seemed to have played a prominent role during Verification. Furthermore, the researcher as social trigger created a space in which the students acted as social triggers for each other's metacognitive behaviour (this was true in general for all four metacognitive categories). Students also acted as social triggers for each other's' metacognitive behaviour without intervention from the researcher. Moreover, it was the main stream student who acted predominantly as a social trigger for the extended degree student's metacognitive behaviour. Similar findings of students acting as social triggers for each other's metacognitive behaviour can be seen in Kim et al. (2013). While observing students' collaborative problem solving, the authors note how one student acted as a catalyst (trigger) for the other students' metacognitive behaviour. Goos (1994) reports similar findings, although she does not use the notion of social triggers in her work. While observing a student pair as they worked together in solving mathematical problems, Goos (1994) found that each student had different, but complementary, metacognitive strengths in which students monitored and controlled each other's problem solving activities, as well as spurring metacognitive behaviour in each other. Different to both Goos (1994) and Kim et al. (2013), the present study focussed on how students who differ in academic performance (that is, main steam vs. extended degree student) acted as social triggers for each other's metacognitive behaviour.

The present research study has also shown how the IMPROVE method and the level of task difficulty acted as environmental triggers that affected students' metacognitive activities, in increasing the number of metacognitive skills exhibited by the students. Similar findings are discussed in the work of Kim et al. (2013), noting that the frequency in students' exhibited metacognitive skills often varies across different types of problems, and even between the different stages of problem solving. Also, both Mevarech, Tabuk and Sinai (2006) and Mevarech and Fridkin (2006) showed that the IMPROVE method can be used in the development of students' metacognitive skills as well as in increasing students' use of these skills.

As pointed out in the Literature Review, Kim et al. (2013) argue that the paradox of metacognition can be solved in considering metacognition as operating on different levels, that is, acknowledging the existences of external resources which trigger an individual's metacognitive activity. The present research study contributes to the solving of this paradox, since it was found that when students lacked the needed metacognitive skills, they were still able to control (regulate) their cognitive activities either (i) with the support of others' conceptual systems; or (ii) by engaging in practices such as that of the IMPROVE method that serves as platform for potential metacognitive activity (note that this was not true in general for all instances in which students lacked the needed metacognitive skills).

Since the purpose of extended degrees are to equip students with the necessary skills which will help them to be successful in their studies, I argued that the training of metacognitive skills needs to be introduced in extended degree courses (similar arguments can be found in Grayson, 1996, 1997, 2010; du Preez et al., 2008; and Kloot et al., 2008, as discussed in Chapter 1)[79]. By taking into account the (i) differences in metacognitive skills between main stream and extended degree students, and (ii) triggers of the students' metacognitive skills, educational practitioners and researchers are able to incorporate training in the use of metacognitive skills, as one of the possible factors that can support both these student groups' ongoing learning performance throughout university.

In conclusion, findings of this research study are of significance since it implies that in order to have a clear (accurate) understanding of metacognition, researchers should also take into account the role of social and environmental triggers of students' metacognitive behaviour; its possible benefits in the development of students' metacognitive skills; and how the above triggers can be incorporated in the design of instructional programmes that focus on the training of metacognitive skills. Above arguments are discussed further in Sections 9.4 and 9.5.2.

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[79] In Chapter 2 it was also pointed out that a number of researchers argue that students need to be instructed in the use of metacognitive skills (Desoete, 2007; Desoete, Roeyers & De Clercq, 2003; Loji, 2010).

9.4 Recommendations and Hypothesis for Future Research

While the in-depth exploratory qualitative nature of this research study has its strengths, it also has its limitations. One of the main limitations is that of the small number of participants. That the study was a small explorative study of one student pair, implies that the findings cannot be considered widely applicable to a larger and more diverse population of students. As such, the researcher hopes that other educational practitioners and researchers would continue to build upon this study, either through replication of data, or through the use of other exploratory methodologies to uncover, as well as test more general results. Such replication of data needs to consider a significantly greater number of student pairs (each consisting of a main stream and extended degree student respectively). Moreover, the possibility of a pairing of an extended degree student with a higher academic performance than that of a main stream student, should also be researched in future research[80], as well as student pairs where both students are either academically low or academically high performers. Considering all possible different student pairings could lead to more comprehensive findings, in hopefully bringing forth possible similarities and/or differences between the different student pairs. This may give a much deeper and richer insight into the differences between main stream and extended degree students' metacognitive skills.

Considering the results of the present study as a starting point for further investigation I propose the following hypothesis for future study:

Differences in the metacognitive skilfulness exist between students of different mathematical learning performance and the type of degree they have enrolled for, that is, extended degree vs. main stream degree.

(a) Students of a low mathematical performance, such as those from extended degrees, may exhibit mathematical behaviour which mostly focuses on and includes procedural skills, algorithms and representation of answers. Given that these students have a small cognitive base to work upon may affect their

[80] As noted in previous chapters, a student's academic performance in mathematics is based on his/her marks obtained for a course.

metacognitive behaviour, in that such behaviour is limited and minimal compared to students of a high mathematical performance. Furthermore, developing the metacognitive skilfulness of students of a low mathematical performance may be hindered, since they have more difficulty in operating at a cognitive and metacognitive level at the same time, than compared to more capable peers, for e.g. students from a main stream degree.

(b) Students of a high mathematical performance, such as those from main stream degrees, may exhibit mathematical behaviour which mostly focuses on and includes the use and applications of mathematical concepts. That is, compared to students of a low mathematical performance, mathematically high high mathematical performers have a deeper and more advanced understanding of mathematical concepts and the properties of such concepts, as well as the concept of proof in mathematics. As a result, having a stronger conceptual cognitive base, students of a high mathematical performance are more able to exhibit metacognitive behaviour, compared to students of a low mathematical performance, for e.g. extended degree student. Apart from this, there is still room for improvement in and development of the metacognitive skilfulness of students of a high mathematical performance. Furthermore, developing the metacognitive skilfulness of high mathematical performers may be easier compared to to students of a low mathematical performance. This is most likely, since high mathematical performers may find it easier to operate at a cognitive and metacognitive level at the same time, in contrast to their less capable peers, for e.g. students from an extended degree.

Further recommendations for future research may also include the choice of data collection tools. Although the choice of such tools used in this research study have been discussed and justified in Chapter 4, the use of other tools can only add to and deepen the knowledge gained in this study. Tools such as stimulated recall interviews could have given more insight into the observed metacognitive activities of each of students, thereby increasing the reliability of the research findings. Further interviews into the students' views of the IMPROVE method may also have given a clearer picture on students' disposition towards each of the metacognitive questioning techniques, and/or the possibility of students being more receptive towards these questioning techniques over the observed time period. Furthermore, an increase of the observation time period may have resulted in a clearer and more complete assessment on the development of the students' metacognitive skills.

Future research based on this study, could also consider group settings, in which the researcher (if possible) plays a negligible role in the students' collaborative problem solving. This may result in different findings compared to the current research study. Future research should also consider increasing the reliability of the effect of IMPROVE; specifically its effect on the frequency of occurrence of students' metacognitive skills. Such an increase in reliability could possibly be achieved in developing a broader and more detailed analytical framework (compared to the current framework of the study). Such a framework needs to take into account an alignment (mapping) of each of the four metacognitive questioning techniques to the different metacognitive decision points of the applied taxonomy of this study.

As discussed in the Literature Review, research findings on the effect of the IMPROVE method have shown that students trained in this method significantly outperformed students with no metacognitive instruction on various measures of mathematics achievement[81]. Still, researchers and educational practitioners need to be aware of the neutral or negative impact of metacognitive activity on students' problem solving performance, as was seen from the current research study. This is discussed further in the following section.

[81] This can be seen in the findings of Kramarski, Mevarech and Arami (2002); Kramarski, Mevarech and Lieberman (2001); Mevarech (1999); Mevarech and Amrany (2008); Mevarech and Fridkin (2006); Mevarech and Kramarski (2003); Michalski, Zion and Mevarech (2007); and Pressley and Gaskins (2006).

9.5 Implications for Educational Practice

9.5.1 Improving the Quality of Metacognition

As noted in Chapters 6 to 8, there were instances in which students exhibited metacognitive monitoring, but did not take the necessary control action based on such monitoring[82]. As a result of this, students had difficulty either in solving the problems or successfully solving the problems[83]. In other instances students took control in either assessing the usefulness and/or reasonableness of a procedure (or result), but their reasoning behind such metacognitive assessments were faulty. That is, students metacognitively monitored their problem solving but made a control decision that did not improve their problem solving performance or worsened the performance[84]. As a consequence, educational practitioners need to be aware of how students' implementation of inappropriate or inaccurate metacognitive skills, or inappropriate activities consequent upon appropriate metacognitive skills may affect the development of these skills. The aim of educators (and researchers alike) should not only be on an increase in the frequency of metacognitive skills exhibited by students, but also on improving the actions taken as a result of these metacognitive activities. When encouraging metacognitive activity amongst students, educational practitioners should also make students aware of the quality of their metacognitive activities and how it will impact their learning performance and problem solving performance in mathematics.

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[82] Monitoring and control need to be understood here as forming part of metacognitive activities and not cognitive activities. The concepts of metacognitive monitoring and control were discussed in the Literature Review, Chapter 2.

[83] An example of this was seen in observation 1, where the students monitored the implementation of the proposed strategy (test), as well as questioned (assessed) if they were applying the strategy correctly. Still, both students did not take the needed control in checking if (i) the implemented strategy was appropriate in solving the questions, and (ii) how to correctly apply the strategy. It was only after completing all three questions that the students took the needed control in correcting their mistakes. Another example was seen in observation 3, question 2 in which the extended degree student monitored (assessed) where the solution process was leading to, but still did not realise or correct the mistakes he had made.

[84] An example of this was seen in observation 3, question 2. The main stream student considered if his solution to the question was plausible. After intervention from the researcher, the main stream student took control in verifying his result, although his justification behind the validity of his answer was still mathematically incorrect.

9.5.2 Training of Metacognitive Skills in Extended Degree Programmes

One of the main research findings of the study is the observed differences in the metacognitive skills between that of a main stream and extended degree student. This points to a possible lack of training of students' explicit metacognitive skills; particularly in the case of extended degree students. That extended degree students do not exhibit metacognitive behaviour almost similar to that of main stream students, may be due to educational practitioners not creating environments which focus on developing these students' metacognitive skills. This can be seen from an absence or shortage of instructional programmes which explicitly train university students in the use of metacognitive skills.

Because of such a lack, when joining their fellow main stream degree students in courses that do not form part of an extended degree programme, extended degree students often perform weakly in these courses or even do not pass these courses. Furthermore, extended degree students' learning and academic performance in non-extended degree courses is lower in comparison to their main stream degree peers. This is partly due to the extended degree students lacking the needed metacognitive skills and partly because of the students' poor content knowledge. Thus the aim of extended degree courses should not only focus on increasing contact time between lecturer and students and discussing course content at a slower pace, but also on the development of extended degree students' metacognitive skilfulness. Furthermore, researchers and educational practitioners may argue that similar metacognitive instructional practices are also needed in the case of main stream degree students. Such an argument seems reasonable, since there is the possibility of main stream students also exhibiting very little metacognitive behaviour or metacognitive skills of a low quality. Similar arguments can be seen in Craig (2009) and Grayson (1997), as discussed in Chapter 1.

9.5.3 Metacognition in Collaborative Learning Environments

When designing and developing group-learning environments (two or more participants), educational practitioners need to be aware of how metacognition develops as a result of an individual's metacognitive processing, *and* how metacognition also operates at a social level. Within this thesis, metacognition at a social level was regarded and observed as metacognitive activity resulting from student interaction. In particular, during such interaction students acted as social triggers for each other's metacognitive behaviour[85]. Thus, educational practitioners need to be aware of how metacognition when operating at a social level, impacts the learning experience of all students in the group. Metacognition at a social level may also be typified by instances in which students collaborate in a joint, sequential engagement of metacognitive activity. Although such engagement necessarily begins with one student, this triggers metacognitive behaviour amongst other learners as well. In this research study, such a joint sequential engagement of metacognitive activity was typified as a 'chain reaction' in which students acted as social triggers for each other's metacognitive behaviour[86]. In conclusion, it seems that both researchers and educators should investigate the possible benefits of such a reciprocated, sequential form of metacognitive activity that may play a vital role in the development of students' metacognitive skilfulness.

9.6 Summary

Findings of this study show that differences between the metacognitive skills of an extended degree and main stream degree student do exist, before and after explicit training in the IMPROVE method. The study also highlighted the role of social and environmental triggers of students' metacognitive skills and its possible benefits on the development of students' metacognitive skills. Still, it

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[85] This is not exclusive to students' metacognitive activities, but also holds true for cognitive processing, students' motivation and affective attributes, although this was not dealt with in this study.

[86] An example of this was seen in observation 1.

needs to be taken into account that the above findings are not true in general, since the study only focused on one student pair.

This study has also pointed out that if extended degree programmes are to be efficient in improving learning and academic performance of tertiary mathematics students, these programmes should consider the possible benefits of explicitly training students in the use of metacognitive skills, as well as emphasising the importance of such skills in mathematical problem solving. Moreover, in improving educational practices that focus on the development of students' metacognitive skilfulness, researchers and educational practitioners alike should not exclusively aim to train students in becoming 'metacognitive', but also on developing the quality of students' metacognitive behaviour.

Appendices

Appendix A: Examples of Metacognitive Decision Points and Indicators

This appendix gives examples of all metacognitive decision points and indicators of the four categories of metacognitive skills, as discussed in Chapter 5, and as seen from Tables 5.6 and 5.7 and Figure 5.2.

Similar to the discussions on observations 1 to 4 of Chapter 6 containing parts of the transcribed protocols, the following conventions are also used: completed turns of each speaker have been numbered sequentially; the symbols [ ] are used for non-verbal actions; … indicates either pause, interruption in the speech, or a jump from one turn to another which do not follow sequentially from each other. The main stream degree student is Dean, while Will is the extended degree student (pseudonyms are used throughout). The initials D, W and R respectively refer to Dean, Will and the researcher.

Mathematical concepts and terminology that appear in this appendix are discussed in Chapter 6.

Orientation: Local Assessment of Knowledge or Knowledge Building (O-LAK)

The Orientation category only consist of one metacognitive decision point; 'local assessment of knowledge or knowledge building' (O-LAK). This decision point on its own consists of five metacognitive indicators. These are discussed below with examples to illustrate how the appropriate indicators were applied in the transcribing of protocols.

1. O-APK: Activating Prior Knowledge

The student considers and/or uses domain specific knowledge relevant to the problem. This may also include the student considering and/or remembering problems that he had done in the past.

<u>Example</u>

Observation 2, question 3: Considering the expression $\cos(n\pi)$ contained within the given series $\sum \frac{1}{n}\cos(n\pi)$, Will tried to relate it to the identities and properties of the cos-function and its graph.

474. W: Is there no cos law that says, I don't know, there has to be one or something? Because oh, it alternates between one and negative one.

…

475. W: The cos, the cos function. Graph? Function? I don't know…

<u>Example</u>

Observation 2, question 1: Dean orientated himself around the given question, realising that he had done it before.

49. D: I remember doing this question…

50. R: Where did you remember doing it?

51. D: Yesterday. [*laughs*]

52. R: In a class?

53. D: No, it's part of the exercises from the textbook.

2. O-BMM: Building a Mental Model

The student tries to represent the problem in his own words in order to make sense of the problem. With this the student engages with the question in an attempt to reformulate and/or simplify the problem; or tries to get an appropriate perspective of the problem and reformulate the problem into that perspective. The student may also paraphrase what is asked for. In essence, this metacognitive decision point presupposes that the student has some form of 'mental picture' of what the problem entails, in trying to make better sense of the problem.

<u>Example</u>

Observation 4, question 3: The students were given the general definition of the inverse sine-hyperbolic function $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. Dean asked if the left-hand side of the equation is equivalent to the right-hand side with respect to this definition.

595. D: [*points to the given definition*] What does that mean? That this is equivalent to that?

<u>Example</u>

Observation 4, question 3: In response to Dean's question in line 595 above, Will noted that the function $g(x) = \sinh^{-1} x$ is equivalent to $\ln(x + \sqrt{x^2 + 1})$, and that they should try and determine a relation between g and the definition.

596. W: Um, this means we can use this [*points to the given definition*] as our g of x instead of the sinh.

…

599. W: And then find the relationship with that.

3. O-IMP: Identifying and Repeating Important Information (to be remembered)

Here the student either writes down and/or identifies important key facts of the problem. It includes the student selecting relevant information in solving the problem.

<u>Example</u>

Observation 2, question 2: Dean pointed to the factorial $(2n+1)!$ of the given series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}$, since the factorial played a prominent role in solving the problem.

285. D: How do I solve factorial?

<u>Example</u>

Observation 3, question 1: The students were given the following question:

(1.) Determine a power series centred around $a = 0$ for $f(x) = \sqrt[4]{1+x^3}$

in which Will noted the importance of using the fact that the series is centred around the point of $a = 0$.

110. W: …So they've told you it's centred around a is equal to zero, and you know that that's only a Maclaurin and a binomial…

4. O-RRO: Rereading at Orientation

The student either rereads the question to understand the question better or realises that he did not read the question properly.

<u>Example</u>

Observation 4, question 1: Will quickly skimmed through all the given questions of the task, before returning to the first question in realising that he did not read it.

12. W:	Ah, now at least I know it's a power series they're talking about. I hadn't read the question. [*starts reading question 1*]

5. O-UES: Using an External Source

The student turns to either his personal notes or textbook in considering domain specific knowledge relevant to the problem in order to get a better and/or appropriate perspective of the problem.

<u>Example</u>

Observation 3, question 1: The students turned to their textbook, in which Dean noted they were working with a power series centred around $a = 0$. In consulting the textbook, Dean mentioned that they could disregard the use of the method of Taylor Series.

95. D:	Where's that? I missed it. [*turns to his textbook*]
96. R:	May I ask, why are you turning to your textbooks now?
97. D:	I just want to see this here sir, because… [*pages through his textbook*]

98. R: Mmm?

99. D: Because it says determine a power series centred around a equal to zero. So then this [*reads the question off the task 3 sheet*] when it says a is equal to zero, you can already eliminate the Taylor series [*pointing to his textbook*]

Planning: Proposed Idea (P-PI) and New Idea (P-NI)

The category Planning consists of two different, but related metacognitive decision points: the Proposed Idea (strategy, plan) coded as P-PI, and New Idea (strategy, plan) coded as P-NI. P-NI applies to instances where the student discarded the original plan and implemented a new strategy on how to solve the problem. P-PI consists of three indicators, while P-NI consists of two indicators. This is discussed below.

1. P-FAP: Formulating Action Plan (falls under P-PI)

The student considers steps and/or strategies in order to solve the problem. The student may describe an approach to be used in solving the problem. Subgoaling and estimating the answer to the problem also forms part of P-FAP. There are cases in which the student does not always outline the steps of the strategy that need to be followed, but merely notes what strategy needs to be implemented. This indicator also includes instances where the student does set up a sequence of steps (a step-by-step action plan). What is of importance is that the student does not work on a trial-by-error basis. Such trial-by-error is rather considered to form part of the Orientation category. The main point of P-FAP is that the student considers a plan of action that needs to be followed *before solving the problem*.

Example

Observation 2, question 1: Dean considered the strategy of applying the alternating series test to the given question. He mentioned the steps (the two properties of the alternating series test[87]) that need to be done before actually solving the problem.

63. D: So but what does the…

…

65. D: …the properties of the alternating series, they said that b_{n+1}

…

67 .D: …must be greater than b_n so…

…

69. D: And the limit of b_n must be, equal zero.

…

71. D: Those are the two alternating series properties.

2. P-CDWS: Considering Different Ways of the Solving the Problem (when falling under P-PI)

Here the student considers a number of different strategies on how to solve the problem *before solving the problem*.

Note that P-CDWS as sub-metacognitive decision point also occurs *after* a proposed strategy had been applied and discarded. Hence this code also falls under P-NI which is discussed later.

[87] The two properties of the alternating series test were discussed in Chapter 6, Observation 2.

Example

Observation 3, question 1: Dean considered different strategies on how to solve the problem. He mentioned that they could use a Maclaurin and binomial series expansion, but a Taylor series expansion can be discarded.

99. D: Because it says determine a power series centred around a equal to zero. So then this [*reads the question off the Task 3 sheet*] when it says a is equal to zero, you can already eliminate the Taylor series.

100. W: Mmm.

101. D: And you can look at the Maclaurin and the binomial series, binomial series, and then I'm thinking like this root four you can write it as…

102. W: Brackets.

3. P-OT: Organising Thought by Questioning Oneself (falls under P-PI)

In questioning himself, the student considers what steps a strategy consists of.

Example

Observation 2, question 1: The students considered the strategy of the alternating series test in solving the problem. Will questioned himself on one of the properties of the alternating series.

72. W: Um, the limit of b_n must be greater than zero?

P-NI: New Idea

This metacognitive decision point occurs where the student discards the previous (or original) proposed plan. It consists of two indicators: P-DCS and P-CDWS.

For the case of P-NI, P-CDWS is used in a different manner to that of P-PI above: P-CDWS now applies to instances of metacognitive behaviour[88] *after* a proposed strategy had been applied and discarded. This is illustrated in the example below.

1. P-CDWS: Considering Different Ways of the Solving the Problem (falls under P-NI)

Here the student considers a number of different strategies on how to solve the problem after discarding a previous strategy.

<u>Example</u>

Observation 3, question 3: With this question the students had to determine a power series centred around $a = 0$ for $f(x) = \sin^2 x$. Moreover, they were also given the hint that $\sin^2 x = (1 - \cos 2x)/2$. Furthermore, one of the main objectives of this question was that the students had to use the hint.

Dean considered using the method of determining a Maclaurin series or a binomial series for the given function f after discarding their previous strategy (that of implementing a geometric power series).

533. D: a centred around zero, so we can use…

534. D: …the Maclaurin and the binomial series. I'm just thinking how the derivative of $\cos 2x$ works. $\cos 2x$ will always be, if you derive this,

[88] As noted before, the terms metacognitive skills, metacognitive behaviour and metacognitive activities are used interchangeably.

this will fall away. [*considers using a Maclaurin Series or a binomial Series*]

535. R: What do you want to differentiate?

536. D: [*points to* $\cos 2x$] This guy here. To like use the, to find the f^n, the nth (derivative), you know, so I'm just trying… [*considers in using the strategy of determining a Maclaurin Series*]

538. R: Oh, so are you going to apply a method similar like here? [*points to the previous question 2, in which the students used the method of Taylor series in which they had to determine the nth derivative, but now applying it to the case of a Maclaurin series*]

537. D: Like this one. [*points to the previous question*] I'm just thinking it will be easier to use that or to just use a binomial series. [*considers using a Maclaurin Series or a Binomial Series*]

2. P-DCS: Decision to Change Strategy (on basis of former interim outcomes)

The student considers an alternative (new) strategy to apply after discarding a previous (or original) strategy. As noted before, this metacognitive decision point occurs where the student discards the previous (or original) proposed plan and hence is subsumed under P-NI.

<u>Example</u>

Observation 3, question 3: The students initially used the method of the Binomial series in solving the question. In realising that this strategy was not helpful, Will suggested that they should consider a method in one of their examples as discussed in their lectures; that of a geometric power series.

548. R: You guys, you both said you're going to use the Binomial series.

549. W: We could use the other one. There's the one we first learned… [*proposing a different plan/strategy on how to solve the problem*]

550. R: Other what?

…

555. R: Tell me, what other one?

556. W: …there was a way that the lecturer first taught it.

557. R: Please tell me that.

…

558. W: [*looks in his book*] Um, I'll try and find it.

...

561. W: We use these [*shows R the textbook*] Like, uh, $f(x)$ is equal to one all over one minus x. And we could like represent that differently, obviously. [*points to an example in his textbook of a geometric power series, which is different to the initial strategy they implemented; that of a Binomial series expansion*]

Execution: Local Assessment of Accuracy of Procedure (E-LAPA) and Local Assessment of Usefulness/Reasonableness of Procedure (E-LAPR)

The Execution category consists of the two metacognitive decision points: local assessments in which the student either checks/monitors the accuracy of his working (E-LAPA); or monitors/evaluates the usefulness or reasonableness of his working (E-LAPR). These two metacognitive decision points consist of a number of different indicators as discussed below.

1. E-EAP: Executing the Action Plan

The student monitors the implementation of the proposed strategy (action plan) in keeping track of the progress being made, in verifying that the results obtained do provide an answer to the solution statement. Alternatively, when the student is keeping track of the progress being made, he also considers if his work adheres to the steps of the proposed plan.

Furthermore, E-EAP can either fall under E-LAPA or E-LAPR, depending on the situation in which the metacognitive behaviour occurs.

In the case of E-LAPA, E-EAP involves the student monitoring his working in checking if it agrees accurately (correctly) to the steps of the implemented plan (strategy).

For the case of E-LAPR, we have that E-EAP is exemplified when the student monitors his working in checking if the implemented plan is useful, that is, the reasonableness (practicality) and/or appropriateness of the implemented plan.

Example

Observation 2, question 1: After deciding to use the proposed plan of the alternating series test, Dean mentioned what is the first step of the test that needs to be implemented. This is an example of an instance of E-LAPA.

90. D: So sir, I'm just proving my first property [*writes while he speaks*] and then b_n plus one will be equals to… b_n of n plus one over n plus one.

Example

Observation 2, question 2: Dean was proving that the second property of the alternating series test holds true. In proving this for this particular problem, Dean questioned if it was possible to evaluate the limit of a sequence containing a factorial. This is an example of an instance of E-LAPR.

378. D: But now I can't take the limit. Ah!

379. R: You can. Says who?

380. D: Oh, the factorial and now I'm thinking.

381. R: You're thinking what?

382. D: How else to do it? Because…

383. W: It's a factorial.

384. D: If …

385. W: There's so much that …

386. D: I don't, I don't know … this function.

2. E-ECS: Evaluating the Current Situation

The indicators E-EAP and E-ECS are closely related but differ in description. E-ECS applies to the local monitoring of *particular* instances of the problem solving process, while E-EAP applies to the *global* monitoring in adhering to the proposed plan (strategy). Verbal and non-verbal behaviours concerned with the monitoring and checking of activities had to be related to the proposed plan in order for such behaviour to be coded E-EAP.

Instances in which E-ECS applies are characterised by either (i) the student taking control of learning process / problem solving process; (ii) checking his work; or (iii) drawing away from the problem to see what has been done and/or where the solution is leading to. Comments and questions such as "what are you doing?", "what am I doing?", "this is not getting us anywhere", "I think that it is the answer", "I have used all the given conditions, now I will start…", and "wait, we forgot to use…" are instances that are coded as E-ECS.

Furthermore, E-ECS can either fall under E-LAPA or E-LAPR, depending on the situation in which the metacognitive behaviour occurs.

In the case of E-LAPA, E-ECS is exemplified by the student checking/monitoring the accurateness of his working (for example a particular step or part of the implemented proposed plan).

In the case of E-LAPR, E-ECS entails the student checking the usefulness/reasonableness of his work.

<u>Example</u>

Observation 2, question 1: In applying the alternating series test, Dean was trying to prove that the sequence $f(n) = b_n = \frac{\ln n}{n}$ is decreasing. In order to show this he considered determining the derivative of $f(x) = \frac{\ln x}{x}$ in showing that the first derivative is negative ($f'(x) < 0$) and whether he should use the quotient rule to determine $f'(x)$. This instance is an example of E-LAPA.

140. D: So if it's like… Say f of x equals $\ln x$ over x and then if you derive that you get f of x equals, what's this, the quotient rule?

<u>Example</u>

Observation 2, question 2: After trying to evaluate the limit of a sequence containing a factorial, Dean realised that it was possible (practical/reasonable/appropriate) to evaluate such a limit. This instance is an example of E-LAPR.

412. D: I was like just a realisation or an epiphany that I can apply the limit because it's n. You can apply the limits of sequences because it's n and the integers.

Example

Observation 3, question 2: The students were determining a power series centred around $x = 1$ for the function $f(x) = \ln x$. In determining a Taylor series expansion for this function, the students had to determine the nth derivative, $f^n(x)$. Dean calculated the nth derivative of f in the point $a = 1$ to be $f^{(n)}(1) = (-1)^{n+1}(n-1)!$ but still had difficulty with how the value $f(1) = 0$ relates to the nth derivative. He mentioned that the above expression for the nth derivative only holds true for first derivative $f'(1)$ and the successive derivatives, but not for $f(1) = 0$. With this instance we have that Dean was considering the usefulness (appropriateness) of the procedure of determining the nth derivative. This instance is an example of E-LAPR.

295. D: [*looking at what he has written*] This doesn't work. It's not going to… This only works from there [*points to the first derivative of* f]. Oh, but that zero doesn't make sense… [*point to* $f(1) = 0$]

3. E-NUL: Note-taking, Underlining, Circling, Highlighting, Writing out of Work in an Orderly Manner

Here the student either makes notes; underline; circle or highlight important words or mathematical expressions; or writes out his work in an orderly manner to structure the work. Because of the way it is defined, this indicator falls exclusively under E-LAPA.

Example

Observation 4, question 3: The students had to determine a power series centred around $a = 0$ for $g(x) = \sinh^{-1} x$, given the general definition of this function, namely $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. In applying the method of determining a Taylor series expansion for the function g, the students had to obtain the nth derivative of g. After considering what Dean obtained for $g'(x)$, Will started writing out and simplifying the expression of $g'(x)$ in order to structure his work.

685. W: I'm just going to write it neatly now.

686. R: Ok, you're going simplify?

687. W: Yes.

688. R: Ok.

689. W: So I can read it myself.

Example

Observation 2, question 2: After proving that the first and second property of the alternating series test were true, Dean circles the two properties of the alternating series test in his work.

401. D: Let's see. [*writes as he speaks*] As n approaches this big number b_n approaches zero. So therefore… this can be seen as a limit. I'm saying this can be seen as a limit because [*points to what he has written as he speaks*] n is approaching infinity and both of them approach zero so this can be seen as that second property where the limit of b_n must approach infinity, must equal zero, sorry. So this is the first property. Second property was realised over the first one there, second one there. [*circles the two properties in his work*]

4. E-EDKT: Error Detection and Correction and Keeping Track

The student primarily focuses either on the accuracy/precision of his calculations; keeping track if he made errors (in some cases also correcting them); checking his calculations and answers by recalculating; or monitoring his behaviour in avoiding negligent mistakes. Because of the way it is defined, this indicator falls exclusively under E-LAPA, similar to that of E-NUL above.

<u>Example</u>

Observation 3, question 2: The students are determining a power series for the function $f(x) = \ln x$, in which they need to determine the nth derivative of f. Will noted that he made a mistake in calculating the first few derivatives of f.

237. D: [*points to W's work*]

238. D: What! How the hell…

239. W: Oh, ok….

…

241. W: [*laughs*]

242. R: What's going on here? Please, please verbalise for me what's going on.

243. W: I made a mistake with my derivatives.

244. R: What?

245. W: I wasn't actually thinking. Because it's minus 2. Well, when you get here it becomes minus 2.

246. R: You said you weren't thinking. What does that mean if you were not thinking?

247. W: I wasn't paying attention to what I was doing. I just…

Example

Observation 2, question 1: In showing that the sequence $\{b_n\}$ is decreasing, Dean intended to write $b_{n+1} < b_n$, but he accidently wrote $b_{n+1} > b_n$. Dean realised his mistake and corrected the mistake.

160. **D**: Whoops, wrong one. Wrong one. b_n is greater than b_{n+1}.

5. Claiming Progress in Understanding (E-CLU)

The student mentions/acknowledges that he understands either the problem solving process; how an applied strategy is helpful in solving the problem; or a fellow peer's explanation/working. This indicator can form part either of E-LAPA for example in the case of claiming understanding in terms of the accuracy of any working during the problem solving process; or E-CLU can fall under E-LAPR when the student claims understanding in the usefulness/reasonablness of a certain procedure. The beelow instance is an example of E-LAPA.

Example

Observation 2, question 2: After Dean discussed the definition of the factorial function, as well as that the sequence $\left\{\frac{1}{(2n+1)!}\right\}$ is decreasing, Will acknowledges that he understands Dean's argument.

299. **W**: No, I get what he's saying. It's decreasing because it's one over the factorial and when you're doing factorials you go from the largest to the smallest in the whole… You go from the largest to one, basically. So it's going to be one over one, ok, that's one.

6. E-CMC: Checking Memory Capacity

The student recalls information on how to solve the problem or mentions that he has difficulty in remembering certain information. This indicator can either fall under E-LAPA or E-LAPR depending on the situation.

Example

Observation 1, question 1: The students are determining the convergence/divergence of the given series $\sum \frac{n^2}{n^3+1}$. With this they consider the corresponding sequence $\{a_n\} = \left\{\frac{n^2}{n^3+1}\right\}$ of the series as well as the limit of $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^3+1}$, with $f(n) = a_n$. This is an example of E-LAPA.

21. D: And if the limit of f of x exists and then the series will converge here.

…

23. D: If I remember correctly. [*writes while he speaks*] So limit x approaches infinity f of x equals to limits…

Example

Observation 1, question 2: As the students are determining the convergence/divergence of the given series, Dean mentions he has difficulty in recalling his 'rules' (work). This instance is an example of E-LAPA.

146. D: Sir I'm forgetting now. Sir you forget all your rules…

…

148. D: I forget all my rules.

7. E-FSA: Finding Similarities, Analogies

The student considers a part of a problem or the problem as a whole, in making comparisons between this and other work they have done in the past. Similar to previous indicators, this indicator can either fall under E-LAPA or E-LAPR depending on the situation.

<u>Example</u>

Observation 2, question 3: Dean compared the problem at hand with the exercise questions (on the alternating series test) from their textbook. This instance is an example of E-LAPR.

536. D: Well there's also a question like this in our exercises and it looks hard for us, but then you see this again and then…

…

537. D: The alternating series and you're like 'Yay' and they become easy…

<u>Example</u>

Observation 4, question 2: Dean made a comment in saying that the second question is easier compared to the first question of the given task. This instance is an example of E-LAPR.

418. D: [*writes while he speaks*] … find the first derivative… At least this is easier than the first question.

Example

Observation 4, question 2: Will was comparing the procedure of repeatedly differentiating $\sinh x$ in asking if it was similar to repeated differentiation of $\sin x$. This instance is an example of E-LAPA.

433. R: You're thinking? You're looking at Dean's work and you're thinking something Will.

434. W: I'm thinking isn't it like the normal one where the derivative of sine and cos…

435. D: Yes, but there's no minus sign.

436. W: Is there no minus?

…

437. D: It just goes cosh… It just goes sinh, cosh, sinh, cosh.

438. W: Really?

439. D: Yes

440. W: Oh, ok. I thought there was a minus.

8. E-GSO: Giving Suggestions to Others

Students give suggestions to each other on either how to solve the problem; errors they have made in their calculations or definitions; or any misconceptions they may have.

Similar to previous indicators, this indicator can either fall under E-LAPA or E-LAPR depending on the situation. Examples of these are given below.

Example

Observation 3, question 2: Dean corrected Will's work in mentioning that some of the factors in an expression will cancel out. This instance is an example of E-LAPA.

370. W: [*laughs, then writes as he speaks*] n multiplied by n minus one multiplied by n minus two factorial.

371. D: [*points to paper*] But then aren't you going to cancel this one?

372. W: Then you won't cancel this, you'll just cancel this and you'll…

373. D: But you can't cancel this one with the top on it because this is n minus one factorial. [*addresses Will and points to Will's work*]

374. W: Oh! Ok. Ok. Yes, I'm sorry. I thought it was just n minus one.

375. D: Oh no, it's a factorial.

Verification: Local Assessment of the Accuracy of a Result (V-LARA), and Local Assessment of the Reasonableness/Usefulness of a Result (V-LARR)

The Verification category consists of the two metacognitive decision points: local assessments in which the student either checks or verifies the accuracy of his working and/or calculations, or if the steps of the proposed plan had been followed after the problem had been solved (V-LARA); or local assessments in which the student reflects on the solution process whether it does makes sense (V-LARR), which may involve the student making conclusions or commenting on his own personal habits, or considering the reasonableness/usefulness of a process and/or strategy.

1. V-VG: Student Reviews his Work in General

The student considers the solution of the problem overall in evaluating the outcomes (solution) whether it reflects the adequate problem understanding, analysis, planning and/or implementation. The student also checks if the solution satisfies the conditions of the problem. Also, the student may check if the solution process makes sense after the problem had been solved. This indicator can either fall under V-LARA or V-LARR depending on the situation.

Example

Observation 1, question 3: After solving all three questions of the task, Dean realised and verified that the nth term test for divergence was not a helpful strategy to apply to the three questions. This example is an instance of V-LARR.

390. D: I was just thinking of the test for divergence shows that if the limit's not equal to zero or if it doesn't exist then the whole series is divergent, so… But none of our limits were not equal to zero… so we can't use the divergence theory there.

Example

Observation 4, question 3: After obtaining their solution, Dean forgot to use x^2 instead of x in his series expansion. This instance is an example of V-LARA.

894. D: No sir, I forgot this squared followed by that x there, so…

2. V-CON: Concluding

With concluding, the student relates the answer to the question. The student recapitulates on what has been done while drawing conclusions from and/or referring to the problem statement. The student may also relate his conclusions to the subject matter of the question. This indicator can either fall under V-LARA or V-LARR depending on the situation.

Example

Observation 2, question 3: After completing question 3 and noting that the properties of the applied strategy (the alternating series test), Dean concludes that the given series is convergent. This is an example of V-LARA.

556. D: So since this, both properties are met again, I can write down the conclusion. Ah, let's do that.

Example

Observation 1, question 2: After solving the problem, Dean concludes that the series seems to be convergent according to his 'logic' in comparison to how they were taught during lectures ('mathematically'). In this sense, Dean is depending on the reasonableness of his evaluation in concluding that the series is convergent. Thus, this example is an instance of V-LARR.

211. D: Well sir I don't know, I'm forgetting my let's prove it mathematically, but logically if I look at it like that as n increases the terms of each successive term does get smaller.

…

213. D: So I say I think it's convergent.

3. V-REF: Reflection on the Learning Process

After solving the problem, the student reflects on what he has learnt during the problem solving process. This indicator falls exclusively under V-LARR.

Example

Observation 3, question 2: In reflecting on the solution of the problem after solving it, Will mentioned that he had difficulty in obtaining nth derivative of $f(x) = \ln x$, from the first few derivatives of f, namely $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}$, ... ('the list of derivatives' below).

395. W: Mmm. Um, I'm actually thinking more [*points to what Dean has written*] because this is where I struggle, I can't make this link from all of these [*points to the list of derivatives*] …I can't get to that [*points to the nth derivative*].

4. V-CPH: Commenting on Personal Habits

Here we have that the student comments on either his own personal habits as problem solver; his beliefs of mathematics; or the difficulty of the task. This indicator falls exclusively under V-LARR (similar to V-REF).

Example

Observation 1, question 2: Dean mentioned his confusion on the terminology of sequences and series, and that he forgets the correct application of the tests of convergence/divergence of series.

219. D: It's just about I'm confused. Like I forget… I need to go and do more Maths, ok, like after this I'll go and study it. [*laughs*]

<u>Example</u>

Observation 2, question 1: After solving the problem, Dean commented on the mathematical procedures and knowledge from first year they had to remember and must be able to use in their current second year mathematics.

257. D: Even though we don't remember it as much as we did last year, because there we were tested on it, it's just somewhere there, but at least we can call upon the what we have to know the, the detail, but we can call upon our best knowledge of that stuff.

Appendix B: Example of Observation Transcript

Dean & Will Observation 1

Speaker and Utterance		Comment/Interpretation	MC CODE
1	Ruan (R): Ok, so just say, 'I'm Dean'		
2	Dean (D): I'm Dean		
3	Ruan: Ok, cool.		
4	Will (W): I'm Will		
5	R: Ok, so it's Will… We've got Will here and we've got Dean there. Shot guys, there's the first problem, you can start on it. Ok, and remember to verbalise your thoughts the whole time and speak to each other the whole time.		
6	W: Ok	Students starting on the first problem: $$\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$$	
7	D: Ah, y		
8	W: Ok, looking		
9	D: ?		
10	W: Ja… looking at it… I think we should actually divide by the highest power	Although it is not explicit to the reader the students are considering the sequence $a_n = \frac{n^2}{n^3+1}$ Although it is not clear to the reader here already, what the students want to do is evaluate the following limit $$\lim_{n \to \infty} \frac{n^2}{n^3+1}$$ by dividing by the highest power in above fraction. What Will is proposing is to divide the whole fraction by n^3	**P-FAP*** **P-PI***
11	W: The highest power… We could have the… <u>Remember when you could make the sequence as function?</u>		**P-FAP** **P-PI (underlined words**

		correspond to the code)
12 W: Yes *[nods]*		
13 D: So sum to infinity equals 2n squared over n 3 plus 1.	Dean writing down the question.	
14 R: Ok, tell us what you're gonna do. Are you gonna apply a specific test or something? What's… What's going on in your mind? Verbalise everything what is in your mind, guys and speak to…		
15 D: A ? It's a sequence but we want to make it a function because this sequence is defined everywhere where the domain of n is.	Dean explains here that he is changing the sequence to a function, by converting all the n's to x's	
16 W: *[nods]*		
17 D: So from this we can make it the function of f of x equals x squared over n x cubed plus 1.		
18 W: Right.		
19 D: Ok, then let's write it down. *[writes while he speaks]* So f of x equals to x squared over x cubed plus 1. Ok, then. So ? x is a kind of, f of x is a continuous function	Here the students are already implementing the proposed plan	
20 W: *[nods]*		
21 D: And if the limit of f of x exists and then the series will converge there.		**E-LAPR** (lines 21 and 23 are linked and hence the same code)
22 W: Mmm, right, it will		
23 D: <u>If I remember correctly.</u> *[writes while he speaks]* So limit x approaches infinity, f of x equals to limits…	Note that the underlined statement links back to line 21. Students are determining the following limit in order to show that the given series is convergent: $$\lim_{x\to\infty}\frac{x^2}{x^3+1}$$	**E-CMC** (lines 21 and 23 are linked and hence the same code)
24 R: Dean, can you tell us what methods are you		

	applying here? What test or…		
25	D: I forgot the proper names Sir, but	This is extremely NB as Dean throughout the observation has difficulty remembering terminology and definitions	
26	R: What do you want to show?		
27	01:53		
28	D: If the, the limits of a function exists or the limits equals to zero exists and then the series will also converge because if the function of that series converges, the limits will converge with the… the series will converge with this.	This line is NB as we shall see later in the observation in line 322.	This line is an extension, just a further explanation, clarification of what Dean is doing in lines 21 and 23.
29	R: Ok. What do you say about that, Will?		
30	W: I say that's accurate because first we have to prove by doing this whole substituting x in that n is an integer and there's already a test that's being done to show that you actually can do this.		
31	R: Ok		
32	W: So I think we're on the right track.		**E-LAPR*-RE** BUT this is at a **SOCIAL LEVEL (is at social level because of me intervening in line 29)** (this line is linked to line 30 above, Will is making a local assessment of where they are at the moment with the procedure of solving the problem) BUT this is at a **SOCIAL LEVEL (is at social level because of me intervening in line 29)**

33	R: Cool. Just go forward, super.	What happens in the following lines is that the students are evaluating the limit $$\lim_{x\to\infty}\frac{x^2}{x^3+1}$$ by dividing the whole fraction throughout by x^3	
34	D: *[speaks quietly while he writes]* 1 over x, 1 plus x cubed, ok, equals to limit. No, I lie, this is zero over 1 plus others, it's 1.	Dean made a mistake in his writing and corrects himself (cf. the underlined words).	**E-EDKT** **E-LAPA**
35	W: 1 over x. All over 1 plus 1 over x cubed.	Will are saying these words concurrently with Dean's words in line 34. Will is suggesting to Dean what the steps to follow. It seems Will is on the same track as Dean.	**E-GSO*** **E-LAPA***
36	D: So now there you have a function with the limits so you can ? with the highest power of x as the denominator	Dean has completed in evaluating the limit of the sequence and now tells Ruan what he (Dean) has done while referring to his work	
37	R: Ok, cool.		
38	D: And then you end up with the limit is x ? to infinity. 1 over x over 1 plus 1 over x cubed.	Dean has completed in evaluating the limit of the sequence and now tells Ruan what he (Dean) has done while referring to his work	Purely cognitive here
39	R: Ok, cool		
40	D: And if you take the limit as x approaches infinity you end up with zero	Dean has completed in evaluating the limit of the sequence and now tells Ruan what he (Dean) has done while referring to his work	Purely cognitive here
41	03:25		
42	R: Ok, so you get your answer as zero. Super.		
43	D: Yes		
44	R: Ok.		
45	W: *[still writing]* Infinity, infinity cubed	Will still be busy in evaluating the limit.	
46	R: Ok, cool. Now what?		
47	D: I saw f since this limit exists n approaches zero then the function also converge, the series converges.		
48	R: The series?		

49	D: Oh, the sequence, sorry. *[laughs]*		
50	R: Ok, so the sequence converges, but you've got a series there.		
51	D: No, I said I'm not supposed to say it's a sequence, <u>I keep getting mixed up between those two terms.</u>		Dean confused about the terminology. It is **RE** because of my question on line 48 which is linked to line 47 in which he switches terminology around. **V-LARR-RE SOCIAL LEVEL (is at social level because of me intervening in line 50)**
52	R: No problem. No, cool.		
53	D: So the series will converge because the function has a limit over that thing.	Dean talks slowly here and pauses between words. From his facial expression it *seems* that the is thinking and reflecting on and regulating his words, while he is talking to Ruan. *May it be* because of the conversation in line 48 – 51? I think this is the case – I can only infer here.	**V-LARR-RE SOCIAL LEVEL (is at social level because of me intervening in line 50)**
54	W: Yes		
55	R: You agree, Will?		
56	W: Yes, I do.		
57	R: Ok, so let me get this straight. You guys showed that that functions limit is zero so the sequence also goes to zero.	While I am talking here Dean is silently talking to himself and reflecting on what he said in line 53	
58	W: *[nods]*		
59	R: So the series is also convergent. Is that what you're saying to me?		
60	D: I forget		Links to line 51 in which Dean said *'I keep getting mixed up between those two terms'*
61	W: No, it's…		

62	D: I keep getting confused between the sequence and series.		Links to line 51 in which Dean said *'I keep getting mixed up between those two terms'*
63	R: Ok, let's just zoom in. What do we have here? Is this a series or a sequence.		
64	*[video zooms in onto the question paper]*		
65	D: That's a series.		
66	R: Ok, you want to show the series either convergent or divergent? Ok, so what did you guys just do now? Tell me, what did you just do now?		
67	*[D and V chuckle]*		
68	04:43		
69	D: We found that if, if it's a series then by the limit, by the diverging test, divergence test says that if it's not zero or it's not, if it doesn't approach a specific number then the sequence is convergent, <u>the series is con… divergent, sorry.</u>	While Dean is talking here, there are a lot of pauses between his words and one can see from his facial expression that he is thinking about what he is talking and reflecting on his words. It seems there is a monitoring from his side on his own words. Considering the underlined words it seems Dean is not sure what the theory behind the work entails – also refer to line 71 below.	
70	R: Ok		
71	D: And, but that converse doesn't work. So if it is zero it doesn't mean that the series is a convergent scenario to make sure that the series converges also. <u>But I forgot how to do that.</u>		Does Dean lack resources? **V-LARR-RE – because of the underlined words SOCIAL LEVEL (is at social level because of me intervening in line 66)**
72	R: Ok, so the sequence goes to zero.	Interrogation of students by me	
73	W: Yes. The series…	Will reacting to me	
74	R: Ok, so the sequence goes to zero, what does it tell you about the series?	Interrogation of students by me	

75	W: It tells you that the series is divergent. Oh, no,	Will reacting to me	
76	D: These will be converging because you keep on adding smaller and smaller terms and then the series will go to a specific number, it will converge to a number.		
77	W: Right		
78	R: Ok, so what do you deduce?		
79	D: That this series is convergent.		
80	R: You both agree on that?		
81	05:43		
82	W: No, isn't it not convergent?	Will reflecting on what is done and said so far, but because of Ruan's interrogation – in line 80 above.	
83	D: Why?		
84	W: Because it's equal to zero when a_n is equal to zero, is not equal to zero then it's convergent.	This line follows from line 82 – NB! Will pauses a lot here between his words, from his facial expression it *seems* that the is reflecting on his own words. But it also *seems* that Will is not sure what he is talking about.	**V-VG-RE*** This line links to line 82. The same code applies to both these lines. **V-LARR*-RE SOCIAL LEVEL (is at social level because of me intervening in line 80)**
85	D: That's… For the divergence test it says that if the limit of a, of a series or the function of a series is not zero or it doesn't exist then it diverges, but if the series…	Dean is reacting to Will's words in line 84. Dean almost like giving a justification why the series is convergent according to him. Note that we are already in the category **VERIFICATION** as the students are making comments after the problem's solutions has been obtained.	**V-VG** **V-LARR**
86	W: Oh, then it does, ok. Right.	It *seems* Will is following what Dean is saying	
87	R: Ok, what's your conclusion now?		
88	D: That this series is convergent, <u>I hope. *[chuckles]*</u>	Again here we see Dean is uncertain about theory behind the work.	
89	W: Mmm	It *seems* that Will agrees, but from his facial expression it also *seems* that Will is uncertain	

		about what Dean has mentioned in above lines 85, 88.	
90	06:16		
91	R: Ok, so Dean you say it's convergent?		
92	D: Yes, that's what I think.		
93	R: Will? Are you still suspicious?		
94	W: *[chuckles]* No, I'm just thinking about that, um…	**Cf. lines 94 and 96 as one comment.**	Will reflecting here on what has been done and said so far **V-VG-RE*** **V-LARR*-RE** **(at social level because of my intervening in line 93)**
95	R: Ok, what are you thinking? Tell us. Please.		
96	W: I think it is conver… I think it is convergent and ja… I just… No, I just had the reasoning wrong behind it. *[chuckles]*		**Code follows still from line 94 above. Will reflecting here on what has been done and said so far.**
97	R: Ok, cool, so you guys are fine with that problem now?		
98	D: I think so. *[chuckles]*		
99	R: You think so. Ok. Do you want to proceed to the next problem?		
100	D +W: Ja		
101	R: Ok, super. Ok, just put those papers aside.		
102	R: Shall we go to the next problem? Also the next one there. Also the next one. there. There we go. And also that series there is it convergent or divergent?		
103	*[D + V chat together quietly]*		
104	R: And remember to verbalise everything what's going through your head.		
105	07:12	Students starting on the second problem now:	

		$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$$	
106	D: Ah, let's see. So but if we can… *Because if we can… because we can always like start at the divergence test because it's the easiest to* apply. And then from that… So if this… <u>so if you can apply that limit again like we did on top there and then if that limit is not zero or approaches it or does not exist then this will be divergent. So let's try that. Oh (?) it's so (?)</u>	Dean referring to what he has done in the first problem of this task, he is making a comparison almost – he is referring to the strategy he applied in the previous problem. Dean is relating the current question to the previous problem done.	**P-FAP** Underlined words are: **P-PI** The words in italic are: **O-LAK**
107	R: Will, is something going on in your head?		
108	W: *[laughs]* Yes, I'm…		
109	R: Ok. But if it's going on in your head could you please verbalise it for us?		
110	W: Right.		
111	R: Cool		
112	W: I'm thinking this will eventually end up as that 1 all over n to the power r	Will is mentioning here what he is thinking of: considering that the series that the given series will eventually become $1/n^r$ He writes down $1/n^r$	**O-LAK*** and **P-PI*** Lines 112 and 116 needs to be considered as one instance, i.e. they need to be considered together.
113	D: ?		
114	W: I don't really		
115	D: geometric series	Dean considering what Will is referring to and mentioning that Will is referring to the geometric series	
116	W: Right. I think it's eventually going to lead to that. And then we know that that tends to zero I think.	Will is mentioning here that he is thinking of, considering the strategy of the geometric series – this is shown by the word 'Right 'in order to solve the problem. He mentions that $1/n^r$ is eventually going to zero.	Codes here still follow and apply here from line 112 above.
117	D: Wasn't that a geometric series what…? A geometric series is like a a… *[goes to write on the*	Dean reacting to Will's suggestion of using the geometric series. Dean writes down	**E-EAP** **E-LAPR**

	question paper] Whoops, can I write on this?	$ar^{n-1}$to which he then refers to as his geometric series.	
118	R: Yes, you can write on there. You can do whatever you want to.		
119	D: It's a times r to the n minus 1. a times r to the n minus 1, that's a geometric series.	Dean reacting to Will's suggestion of using the geometric series. **Dean is executing Will's plan in order to show Will that Will's plan is not helping** as will be seen in the lines below…	The code from line 117 applies here as well. Lines 117, 119 and 121 and 123 to be seen as one instance together.
120	W: Mmm		
121	D: Then only if r, this r, *[points to the question paper]* is between zero and 1	Dean reacting to Will's suggestion of using the geometric series. Dean discusses the properties of the geometric series while referring to r in ar^{n-1}	The code from line 117 applies here as well. Lines 117, 119 and 121 and 123 to be seen as one instance together.
122	W: Is greater than zero		
123	D: Ja, between minus 1 and 1 and then this will converge in a geometric series.	Dean reacting to Will's suggestion of using the geometric series. Dean is stating the properties of the geometric series	The code from line 117 applies here as well. Lines 117, 119 and 121 and 123 to be seen as one instance together.
124	W: Oh. Oof *[shakes his head]*	*Seems* like Will is realising that he is wrong in considering the Geometric series, because of what Dean has explained in above lines BUT I AM NOT SURE OF THIS!	
125	D + W: *[laugh]*		
126	D: Oh, do we have Maths today? Oh, we do.		
127	W: We do at the end.		
128	D: Snap		
129	*[D and W laugh]*		
130	D: Oh, let's see. Never mind(?).		
131	R: Whatever's going on in your head, verbalise it. Utter it please.		
132	D: Ok, so now with this, the same question, we're going to apply the divergence test to it to see if that holds true for the series. So the limit here is of as x approaches infinity let's	**Dean again refers to the previous problem done, strategy done in the previous problem. Dean has discarded Will's proposed plan of the geometric series as seen above, in showing to Will that**	**P-PI**

	make this f of x equals to the limit(?) as x approaches infinity. e to the power of 1 over x over x squared.	**the geometric series is not a viable solution in solving the problem.** Moreover, Dean now starts to execute his proposed plan of applying the same strategy as was done in the first problem, i.e. determining the limit $\lim_{n\to\infty}\frac{e^{\frac{1}{n}}}{n^2}$ and converting all the n's to x's. That is, he is considering the following limit $$\lim_{n\to\infty}\frac{e^{\frac{1}{n}}}{n^2}$$ By changing all the n's to x's	
133	W: Mmm.		
134	D: To the power. We can't divide it by the highest power?		**E-ECS** **E-LAPA**
135	W: No you can't so…	I can only guess that Will is on the same track as Dean since in the previous problem the students divided by the highest power and here it *seems* Will is suggesting to Dean that they cannot divide by the highest power – not enough info for me to code this instance as a metacognitive.	**E-LAPA***
136	09:25		
137	D: Well let's… the fourth	Both students now evaluating the limit $$\lim_{n\to\infty}\frac{e^{1/n}}{n^2}$$	
138	W: It's find the fourth(?)		
139	D: Because when this goes to, if the x 1 of, e to the power 1 over x goes to infinity	Lines 139 – 143: it *seems* both students are on the same track and monitoring each other's thoughts, since they are looking at each other and nodding their heads. Both have substituted infinity into the above limit.	
140	W: Right		
141	D: if the e fall here it's 1		
142	W: It's 1, ja		
143	W + D: Over infinity.		

144	D: Can you apply it like that?		Code from line 134 also applies here. The words/turns of Dean from line 134 till here are linked and tied together.
145	W: Ja, I think so. <u>I'm not actually sure. *[laughs]*</u>	Will's comment shows to me that he is not sure, understands the content of the work well enough – lacking resources, content knowledge?	Code from line 135 also applies here. The words/turns of Will from line 135 till here are linked and tied together
146	D: <u>So I'm forgetting now. So (sir?) you forget all your rules in this situation</u>		Cf. line 148 as well **E-CMC** **E-LAPR**
147	R: Sorry, just say…		
148	D: <u>I forget all my rules in these situations.</u>		Cf. line 146 as well. These lines are linked together. **E-LAPR**
149	R: What situations?		
150	W: *[laughs]*		
151	D: When you're being video recorded.		
152	D +W: *[laugh]*		
153	D: But I…		
154	R: I'm sorry.		
155	D: I forget, can… it's possible to do that because if x was infinity in both of them, this top part would go to one because one over x equals to zero and x power zero is one.	Dean reflecting if what he has done is correct, he is reflecting on if he solved $\lim_{n\to\infty}\frac{e^{1/n}}{n^2}$ correctly	The codes from lines 146 and 148 also applies here.
156	W: And you can only use L'Hospital's Rule if both of them go to infinity.	Will mentions that they can only use L'Hospital's Rule if both the numerator and denominator of $\lim_{n\to\infty}\frac{e^{1/n}}{n^2}$ goes to infinity	**E-LAPA***
157	D: If both of them go to infinity.		
158	W: Only one is going to infinity.		Code of line 156 above also applied here
159	D: And the other one ? Ah		

160	W: Ja, the numerator is going to 1.	Will notes here that the strategy of L'Hospital's Rule fails	Code of line 156 above also applied here
161	D: Ah *[laughs]* That's not that, I don't believe that.	Dean expresses unhappy about that they cannot use L'Hospital	
162	W: I think it is(?). No, I think it's zero. *[laughs]*	Here the students are looking at each other, asking each other to confirm what is going on and if they are correct	
163	D: *[looks at V]* 1 over, 1 over infinity is zero.	Here the students are looking at each other, asking each other to confirm what is going on and if they are correct	
164	W: Mmm. Ja, 1 over infinity's zero and then again we still get 1 over infinity.	Here the students are looking at each other, asking each other to confirm what is going on and if they are correct. They note that $\lim_{n\to\infty} \frac{e^{1/n}}{n^2}$ becomes $1/\infty$	
165	D: I'm just… Ja, I'm thinking if… I'm looking at the series		
166	W: Ja		
167	D: As n… As n gets larger and larger this top term becomes smaller and smaller	Dean is referring back and considering the given series, considering what can be done in order to solve the problem	Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together. **E-LAPR** Dean is referring here (implicitly) in which he is taking the limit as n tends to infinity b by considering how the terms $\frac{e^{\frac{1}{n}}}{n^2}$ become smaller and smaller
168	W: Right *[nods]*		
169	D: Because 1 over e to the power of 1 over an increasingly small number will make this one smaller.	Dean is referring back and considering the given series, considering what can be done in order to solve the problem Here Dean is referring to numerator of the fraction $\frac{e^{\frac{1}{n}}}{n^2}$	Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together.
170	W: Right.		

171	D: And this one will get bigger and bigger which will make the whole series smaller. So…	Dean is referring back and considering the given series, considering what can be done in order to solve the problem. Here Dean is referring to the denominator of the fraction $\frac{e^{\frac{1}{n}}}{n^2}$	Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together.
172	W: *[nods]* So it keeps going closer and closer to ?		
173	D: Keep on getting smaller and smaller so they		Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together.
174	W: Right.		
175	D: Ok, so that's that.		
176	R: Ok, so what's your conclusion?		
177	D: Sir, so like we came across that if you just look at this sequence in its form like that, as n gets bigger and bigger e keeps approaching the smaller number because e it's, e raise the power of the smaller number 1 over 2. e to the third, e to the quarter, e to the fifth and it keeps on getting smaller and smaller until eventually it was 1	Dean now explains to me what they have done in lines 167 – 171, that is, what happens with $\lim_{n \to \infty} \frac{e^{1/n}}{n^2}$ but Dean is pointing to $\sum \frac{e^{1/n}}{n^2}$	Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together.
178	R: Mmm?		
179	D: This one will keep going from its value 2 comma is it 4? Ok, 2 comma something, something *[speaks while the writes]* It will keep going down until it reaches 1.	Dean now explains to me what they have done in lines 167 – 171, that is, what happens with $\lim_{n \to \infty} \frac{e^{1/n}}{n^2}$ but Dean is pointing to $\sum \frac{e^{1/n}}{n^2}$ And here he refers to the numerator of the fraction and to the value of the irrational number $e = 2.71828\ldots$	Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together.
180	R: Mmm?		

181	D: And this one will keep on getting bigger until it reaches infinity.	Dean now explains to me what they have done in lines 167 – 171, that is, what happens with $$\lim_{n \to \infty} \frac{e^{1/n}}{n^2}$$ but Dean is pointing to $$\sum \frac{e^{1/n}}{n^2}$$ Here Dean is referring to the denominator of the fraction	Lines 167, 169, 171, 173, 177, 179, 181 all need to be consider together.
182	R: Ok, cool.		
183	D: And then they'll keep that series will keep on getting, will keep on adding smaller and smaller terms to it		
184 **185**	W + D: So it has to reach a specific number, W: ja.		
186	11:58		
187	R: Ok. So Will wrote something down here. Can you share that with, with, with, um, I'm sure Dean's on the same right track.		Will at this stage wrote $$\lim_{x \to \infty} \frac{e^{1/x}}{x^2} = \frac{e^{1/\infty}}{\infty^2}$$ After substituting in the infinity symbol in x's place, he scratched out the limit symbol - **E-EDKT*** **E-LAPA***
188	W: Ja		
189	R: You're both on the same thing.		
190	W: Um, well it's basically the same thing, it's just that I substituted them in, I went for it.		Lines 190 – 196, is Will's reaction to my question in line 187. Will explains what he did is basically the same as Dean in evaluating the limit; he just did rote substitution when he evaluated the limit by means of

		substituting ∞ into the place of x , i.e. he just wrote precisely $\lim_{x \to \infty} \frac{e^{1/x}}{x^2} = \frac{e^{1/\infty}}{\infty^2}$
191 R: Cool.		
192 W: And then…		
193 R: Cool		
194 W: We saw that infinity remains infinity. One over infinity's zero		
195 R: You've got the limit now zero of that sequence, what now?		
196 W: (?) zero.		
197 R: What now? You've got the limit zero, what now? Please tell me.		
198 D: Uh, so maybe that ? that you said last time that the		
199 W: limit		
200 D: like we said last time… That the limit is approaching zero which means that this, as this function increases, the numbers keep getting smaller and smaller and smaller until they go to zero.		Lines 200, 201, 204: **V-CON-RE*** **V-LARR-RE** (but this is at a **social level**, because of my interrogation in line 197 above)
201 R: Mmm?		
202 D: So if you keep on adding smaller numbers in series you will approach a specific number eventually.		
203 W: *[nods]*		Lines 203, 207 – not clear if Will is really understanding
204 D: So we can say the series is convergent.		
205 12:54		
206 R: Ok, so you say the series is convergent? Do you agree Will?		
207 W: Ja.		
208 R: Ok, just write that down for me on the paper as well guys please.		

209	D: Must I write down the whole sentence that I just ?		
210	R: No, just say it's convergent. Super. Ok, so you both agree the series is convergent, and you're finished with the problem now. Or do you still want to go on with the problem? Do you both feel satisfied that this is the answer?		
211	D: Well I, *[drops his ruler]* Oops. *[picks up the ruler]* Well Sir I don't know, <u>I'm forgetting my let's prove it mathematically, but logically</u> <u>if I look</u> at it like that as n increases the terms of each successive term does get smaller.		Lines 211, 213: **V-CON-RE** Moreover, the underlined words are **V-CPH-RE** **V-LARR-RE** (but this is at a **social level,** because of my interrogation in line 210 above)
212	R: Ok, cool		
213	D: So I say I think it's convergent.		Links to line 211 above.
214	W: Ja, because it does have a limit. The limit does exist.		**V-LARR***
215	R: Ok. Cool. Sweet. Shall we go to the third problem?		
216	W: Whoo *[chuckles]*		
217	D: Ja, ok		
218	R: Are you sure? Or do you want to go on with this problem? Are you ok with this problem with the second one?		
219	D: <u>It's just about I'm confused. Like I forget… I need to go and do more Maths</u>, ok, like after this I'll go and study it. *[laughs]*		Cf. line 222 as well, specifically the underlined words **V-CPH-RE** **Lines 219, 222, 228 to be considered together and the same code applies throughout then.** **V-LARR-RE**

			(at **social level,** because of my question in line 218)
220	R: I just want to remind you when you're with me there's no right or wrong at this stage.		
221	14:08		
222	D: *[laughs]* No, because, ah but as I said before, so logically when I look at it it seems like the terms are getting smaller and smaller and it smaller so it will eventually reach one number. But I just can't prove it mathematically – not yet.		Cf. line 219 as well, specifically the underline words code follows from that above line. Lines 219, 222, 228 to be considered together and the same code applies throughout then.
223	R: Do you have…		
224	D: I get…		
225	R: Do you mean you're having difficulty to write it down mathematically? Is that what?		
226	D: I think like the… The… Last week's lectures were on this stuff. I wasn't actually paying attention, but what our… But when I sit down and look at it I'll remember it.		
227	R: Ok		
228	D: But now it's like I'm thinking about what I remember from class, I forgot how to like apply these things properly.		Lines 219, 222, 228 to be considered together and the same code applies throughout then. Considering the underlined words, Dean is possibly referring in how to do maths in a proper mathematical reasoning

			manner/process/sequence of steps
229	R: Ok. How do you feel about that, Will? Do you have similar experiences?		
230 **231**	W: I do but then my problem is with the deduction. I can get the mathematical proof, it's just like I can do the steps but then the reasoning behind it I think I kind of need more, a *deeper understanding*. Because it's very easy to carry out and just follow the procedure without knowing what you're actually doing. So I…		Lines 230, 231 and 233: **V-LARR*-RE** (at **social level**, because of my question in line 229)
232	R: What do you mean by 'deeper understanding'? More practise, or what?		
233	W: More practise. More practise and with the practise actually making sure you know why when it's a specific series you have to use a specific formula.	Interesting that he mentions justifying what strategy to use (see underlined words)	Code follows from line 230 and still applies here as well.
234	R: Ok. Cool. Thanks gentlemen, shall we go to…		
235	D: Sir(?)		
236	R: Yes Dean?		
237	D: Sir, I'd like… As they say that Maths is like the practise subject, so like last week was the first term and we had so much other things to do so I forgot to do Maths. *[laughs]*	Dean exhibits here mathematical beliefs and metacognitive knowledge	**V-LARR** This code of applies right throughout to line 237, 243, 245.
238	R: No problem. Guys, you know what, I appreciate your honesty.		
239	D: Ja		
240	R: You can say whatever you want to		
241	D: And then		
242	15:53		

243	D: Um, so practise on… Like I just need practise with this. Now I know because there's not many formulas to apply to sequence and series, I know it's the monotonic theorem and the test for divergence.	Lines 243 – 248 mentions he needs more practice on sequences and series	**V-CPH-RE** The code of line 237 applies right throughout to line 237, 243, 245.
244	R: You just feel as mechanical engineers sometimes it's a bit rough?		
245	D: Just more practise and no more of playing soccer. *[laughs]*		The code of line 237 applies right throughout to line 237, 243, 245.
246	W: *[laughs]*		
247	D: Then I'll be ready for this, for this stuff.		
248	R: You know what, I'm not here to judge you		
249	W: Studying.		
250	R: I just want to see how your mind works and I appreciate your time. Cool. Shall we go to number 3 now?		
251	W + D: Yes		
252	R: Super. Thanks. Cool.		
253	W: And I think we'll have to go back to number one, we neglected this 2. Doesn't that change anything?	Will refers back to question one of this task. He is bothered by the n=2 in $\sum_{n=2}^{\infty}\frac{n^2}{n^3+1}$	**V-LARA***
254	D: The n equals 2?		
255	W: Mmm		
256	16:35		
257	D: You started ? the first term(?).		
258	R: Ok, we'll get to that now. Let's start with number, with question 3, we'll get back to that	Ruan dismisses the question	
259	D: Ok		
260	R: because we're recording about 16 minutes already. So it's no problem, cool.		
261	W + D: *[start looking at the next question]*		

262	R: Remember to verbalise what's going on in your mind, please.		
263	W: I'm thinking it's just like the first one.	Lines 263 – 270, both students contributing – different to what was seen at the beginning of the observation? Both students here are considering what strategies to be used in order to solve the problem. Interesting that Will now refers to the strategy used in the first problem.	**P-FAP*** **P-PI***
264	D: I apply that right (?)		
265	W: Dividing by the highest part.		Code of line 263 still applies here.
266	D: I just looked at it and (?) both of these don't (?) go to infinity, so you can't do that.	Not sure what Dean is saying here or what strategy he is trying to employ – it seems he is hinting to L'Hopital's Rule?	
267	W: Mmm		
268	D: If you just like, take it like that.		
269	W: Ja, that's true.		
270	D: Ok, so let's just use that first one again. Ah, what is it? Let f of x equals that. *[writes]* equals x. Over x to the 4 plus 1		
271	W: *[is also working out this problem]* Right. And then…		
272	D: Ah!		
273	R: Remember to verbalise what's going on in your head.		
274	D: We let this function have the same form as the series. *[speaks very quietly while he writes]* Now we're going to try and find the limit to that function over there. So the limit is x approaches infinity, f of x equals to the limit as x approaches infinity again. x over x to the 4 plus 1. So now we can divide it by the higher power. I must start here. So limit x approaches infinity while x cubed over…	Dean is executing the proposed plan. I am not seeing any form of monitoring here. Dean is evaluating the following limit by using the same strategy as in the first problem of this task: $\lim_{x \to \infty} \frac{x}{x^4 + 1}$	.

275	18:05		
276	R: Will, are you thinking?		
277	W: Yes, I am, I am.		
278	R: *[laughs]* Ok, cool.		
279	D: x to the (?) 4. See my personal (?) can reach zero, right?		Video very bad and inaudible – cannot gather what is going on here completely in these lines
280	W: Ja, again it's gonna reach zero(?)		Video very bad and inaudible – cannot gather what is going on here completely in these lines
281	***End of video 1***		Video very bad and inaudible – cannot gather what is going on here completely in these lines
			Video very bad and inaudible – cannot gather what is going on here completely in these lines
282	**Video 2**		Video very bad and inaudible – cannot gather what is going on here completely in these lines
283	D +W: *[laugh]*	Since both students laugh in line 283, it *seems* that there is a problem here, because again as in the previous problems they get the answer of the limit as zero. Laughing indicates the *possibility* that they feel they are on the wrong track and not following the correct procedure? This is indeed the case as we will see in a while.	Video very bad and inaudible – cannot gather what is going on here completely in these lines
284	D: Here's the second one. Right, next time I'll be ready.		
285	W: *[looks at D and laughs]*		

286	R: Next time what, sorry?		
287	D: I'll be ready, Sir.		
288	R: Guys, it doesn't matter.		
289	D: Oh, but		
290	R: You're doing perfect at this stage.		
291	D: ?		
292	R: You're showing your way of thinking and that's all we want, we appreciate that, cool.		
293	W: I get zero	Lines 293 – 295, it *seems* they are on the same track and following each other's' progress.	
294	D: Zero over 1 equals to zero. Have a limit of zero.		
295	W: Ja		
296	D: Oh no		
297	D +W: *[laugh]*		
298	D: Pass the textbook man, let's look at the theory.	Lines 296-298: Dean feels uncomfortable about the situation of getting an answer of zero again. Hence he asks for the textbook	**V-LARR**
299	W: Ja. I mean can we? *[laughs]*		
300	D: Are we allowed to, Sir? *[laughs]*		
301	R: Yes		
302	W: Oh can we	(Will also getting out his textbook)	**V-LARR*** (Will also getting out his textbook)
303	D +W: *[get out their textbooks]*		
304	D: Oh man…		
305	W: That changes everything!		
306	R: I'm sorry.		
307	W: Yes, no, it's fine.		
308	D: Ah, I hate this textbook, it's so heavy.		
309	W: Yes		
310	D: So why can't I just		
311	W: Make a lighter textbook		
312	D: Or like small pages or something because I saw ? it's just like in a different colour.		

313	W: Ok, ?		
314	D: Ah, where were we? There. There.		
315	W: This is far more ?		
316	D: ? might be too because the series		
317	W: No		
318	D: More, more, more. What am I doing there? *[pages back in his textbook]* ? series, I don't do evaluating, ?		
319	W: Oh, stop. Slow down.		
320	R: Sorry?		
321	01:31		
322	D: Like there that's what I… You see, this is what I was talking about, like this theorem 6. If a series oh, that a_n is convergent then the limit is equals to zero.	Dean is referring and pointing to the textbook. He is referring to what he has mentioned in the beginning of this observation, in line 28.	
323	R: Oh, ok		
324	D: But now it says that the, that the inverse is not true, if the limit equals zero you cannot conclude that a_n is convergent. **Ah!**	Dean is referring and point to the textbook – he is referring to the inverse statement of theorem 6 of the textbook.	
325	**R: Why are you making 'Ah!'?**		
326	**D: That's my, my, my voice in my head when I get something wrong(?). *[laughs]***	Dean unhappy, realises a mistake – see lines 328 – 330.	Code holds true for the following lines and these lines are linked: Lines 326, 330, 332. **V-LARR**
327	R: Oh, ok, it's… so that's, you did something wrong?		
328	D: No, of course, when I just… When you do something and you've found that you've done it wrong, like ah…		
329	R: That's no problem, I just wanted to know did you feel you did wrong now?		
330	D: No, it's like a… I'm thinking because like maybe my logic is flawed in that sense because now I'm	Dean realises that they have applied Theorem 6 incorrectly, as we see in the lines below..	**V-REF** Code from line 326 holds the

	applying the wrong theorem because we've been assuming, as it says here, the converse		lines that are linked: Lines 326, 330, 332.
331	R: I understand, don't worry.		
332	D: (?) if the limit is equal to zero and it's not there.		Links to above line 330 as well the code.
333	R: I'm with you. So you're saying to me with all 3 problems so far you did it incorrectly?		
334	D: Let's try that again.		
335	W: I don't think so.	Reaction to my question in line 333	**V-LARR*-RE** Lines 335, 339 linked and same code. **SOCIAL LEVEL** (because of my question in line 333)
336	D: This one is, um…		
337	R: You don't think so?		
338	02:36		
339	W: No, reading theorem 6, we've proved that the limit an is equal to zero, so the series is convergent. If we had found that the limit is not equal to zero…	Reaction to my question in line 333	Lines 335, 339 linked and same code
340	D: What about there, the converse?	Dean responds to Will's words in above line 339, it *seems* Dean is following what Will is saying and correcting him – this is the case as we shall see in a moment.	
341	W: … we would have not proved		
342	D: *[reads from V's textbook]* You see, the converse of theorem 6 is not true. In general if limits of the series is zero it has to be proved then we cannot conclude that the sequence, the series is convergent. That's what I was talking about. Wow. *[chuckles]*		**V-REF** **V-LARR**
343	W: So here…		
344	R: Why are you wowing?		

345	D: No, because I… What can I say?		
346	03:15		
347	R: Will, do you agree with him?	During this time (lines 347 - 353) while I am talking to Will, Dean is paging through the textbook.	
348	W: *[after having looked at the textbook]* Now I get it, ja.		Lines 348 and 351 are linked and same code applies
349	R: What do you get?		
350	D: ?		
351	W: Ja, now I get that we were actually proving that it is equal to zero, but that doesn't tell us that it is convergent.		**V-LARR*** Lines 348 and 351 are linked and same code applies
352	R: What is convergent? And who goes to zero?		
353	W: Some of those series.	At this stage Will also blows out his breath and it *seems* he is not sure what Theorem 6 entails. BUT I AM NOT SURE! CANNOT INFER THIS HERE.	
354	D: Yes, I forgot something. I was thinking… Because I keep on forgetting like the, with the, to prove that a series is convergent you must have the limit of the sums is equals to zero.	Dean makes this statement after paging and reading through the textbook. Dean is referring the limit of partial sums – see lines 357, 358 below.	Lines 354, 356, 357, 358 are all linked and in terms of the concept of 'NEW IDEA'. **P-NI**
355	R: Ok		
356	D: <u>But I forgot how to find the sums</u>. *[laughs]* <u>So that's… Let's go look for the sums, where are the sums? Ah.</u>		Lines 354, 356, 357, 358 are all linked and in terms of the concept of 'NEW IDEA'. The underlined words here are using the textbook to get knowledge **O-LAK**. Hence the code **O-UES**.
357	R: What sums? What are you referring to?		
358	D: The limits of partial sums when you, if you… If that limit of partial sums equals to zero it's because all the partial sums add up to a sum which is the sum of		Lines 354, 356, 357, 358 are all linked and in terms of the concept of 'NEW IDEA'.

	the series. And if the limit of the partial sums approaches zero that means that each time you add in the smaller and smaller partial sum to the total sum and that will be a convergent number so uh…		
359	R: Ok		
360	D: We have to find the partial sum. So look for the partial sum.	Dean mentions a strategy they could have used is by using the partial sums.	Lines 354, 356, 357, 358 are all linked and in terms of the concept of ‘NEW IDEA’.
361	W: *[W gets out another book]*		
362	R: Will, and you?		
363	W: No, I’m just trying to get my book to actually *[laughs]*		Lines 363 and 365 same concept and code: **O-LAK*** **O-UES***
364	D: See what it does		
365	W: *[looks through his notebook]* See what the notes actually said.		Lines 363 and 365 same concept and code.
366	R: Are you guys stuck?		
367	04:36		
368	D: Here we are.		
369	W: Ah, mmm *[nods]*		
370	D: No, we’re just… Because I forget like we… As I said, like we’re applying the wrong theorem in the wrong place, we’re using the converse which isn’t true, according to that, like all the time. *So I’m just trying to remember how we’ve done it before. I’m thinking back to class, like finding the partial sums, finding the limits to that, and if that’s convergent, in the limit of that is convergent but we don’t want to know if the limit is zero, we want to study the sequence, if the series is convergent(?)*	This turn/move of Dean is merely just a repition of what he has said. Nothing new, hence no MC code again; as this is not a new instance.	The words in italic links to the code in line 356 of ‘local assessment of knowledge building’. The other words are again just a repetition of what we saw earlier in line 342 above.
371	R: Ok. We’ve been recording now more than 20		

	minutes, do you still want to go on or are you tired now?		
372	D: No, this is nothing, we have 2 hours of Maths, so that's that's where you ?		
373	W: *[laughs]*		
374	05:18		
375	R: Do you still want to go on, gentlemen?		
376	D + W: Yes		
377	R: Ok, no, cool. Super. Thanks for your time, we appreciate it.		
378	W: *[reads the from his notebook?]* It says convergence of a series is defined by the series of <u>partial sums and so on. You are actually right..</u>		Here Will is reading from his personal notes **E-LAPR***
379	D: That was true		
380	W: Ja		
381	D: We wrote those notes in class		
382	W: In the lecture.		
383	D: That's a geometric series that was example, this one, ja, sorry.	Dean points to an example of a geometric series done in Will's notes	
384	W: And our first one wasn't geometric was it? *[pages through his notes]*	Will refers to the first problem done in this task	**V-LARA***
385	D: It was geometric. ? geometric. *[looks in the textbook]* Now that …		
386	*[try and find the answers, D by looking in the textbook and V by looking at his notes]*		
387	06:18		
388	D: No, we can't do that one.		
389	R: Which one? Tell us?		
390	D: I was just thinking of the test for divergence shows that if the limit's not equal to zero or if it doesn't exist then the whole series is divergent, so… But none of our limits were not equals to zero, didn't exist, so we can't use the divergence theory there.	Dean is referring here to Theorem 6 – he calls it the 'divergence theory' as seen below.	**V-VG** **V-LARR** Lines 390 and 396 are linked.
391	W: No		

392	D: That's… So that kicks that one out the window.		
393	R: Sorry, that?		
394	D: That puts that theorem out of the window ?		
395	R: The window. Ok, cool.		
396	D: So you can't even look at that theory because none of our limits equals like none of the limits met the, the criteria for us to apply this theorem.		Code from line 390 applies here as well.
397	R: Will?		
398	07:00		
399	W: No, he's right. Um, well my notes are saying *[smiles]* If the limit as n to infinity of a series a_n is not equal to zero or the limit doesn't exist then the sum of an is divergent. And that's the divergence test. And we can't actually do the divergence test because as you say	Will refers to his personal notes and with 'because as you say' refers and addresses Dean and to what Dean said – shows Will has been following and understanding what Dean has mentioned.	**V-LARR*** Lines 399, 401, 403 are linked
400	D: Because the other…		During this times student are also paging through the textbook and personal notes. **Lines 399, 401, 403 are linked**
401	W: Yes, right	Lines 399, 401, 403 are linked	
402	D: They ? to zero.		
403	W: They're all equal to zero.	Lines 399, 401, 403 are linked	
404	R: Ok cool. What now?		
405	D: So why don't we just try for number 3, why don't we just like I remember in class the lecturer told us if we were not sure you can write down the first few terms of this thing and see what happens.	Dean is referring to a new strategy in writing out the first few terms of the series in order to determine if the series is convergent/divergent.	**P-NI**
406	W: And see what they actually give it, ja.		
407	D: Ja		
408	R: The first 2 terms of what?		
409	*[D and W speak at the same time]*		

410	D: The first 5. So we start at 1 and work our way up to 5 and see if the numbers are getting smaller.		
411	R: The terms of the series or the terms of the partial sums?		
412	D: No, the terms of the series. It says first, the first, no, what is it…		
413	R: Ok, I think		
414	D + W: *[laugh]*	Observation stopped as R had to go to the next observation	

<u>End of video 2</u>

<u>End of Observation 1</u>

Appendix C: Student Tasks

This appendix lists each question of the four respective observations. The reader can use this appendix as accompanying material to gain a better understanding of the discussions of the observations, as outlined in Chapter 6. Furthermore, Appendix C also outlines the correct and 'ideal' solution to each of the questions, followed by the students' solution and/or approach to the questions. Students' transcribed solutions (verbatim) are included for some of the questions.

Task 1, Observation 1

Determine if the following series are convergent/divergent:

1.

$$\sum_{n=2}^{\infty} \frac{n^2}{n^3 + 1}$$

2.

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$$

3.

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

Correct solutions versus students' solutions:

<u>Question 1 of Observation 1</u>

The convergence/divergence of the given series $\sum_{n=2}^{\infty}\frac{n^2}{n^3+1}$ can be determined by a number of different tests, such as the Comparison Test, Limit Comparison Test or the Integral Test. The shortest solution is obtained by applying the Limit Comparison Test, which reads as follows:

Limit Comparison Test

Suppose that Σa_n *and* Σb_n *are series with positive terms. If*

$$\lim_{n\to\infty}\frac{a_n}{b_n}=c$$

where c *is finite and* $c>0$*, then either both series converge or both diverge.*

What is of importance in the Limit Comparison Test is that in order to determine convergence/divergence of a given series Σa_n, one needs to compare it to well-known series, say Σb_n, of which we know its convergence/divergence. Most of the time we use either a geometric series or a p-series.

For the given series $\Sigma a_n=\sum_{n=2}^{\infty}\frac{n^2}{n^3+1}$, we will be using the p-series $\Sigma\frac{1}{n}$ as our comparison series. A p-series is any series of the form $\Sigma\frac{1}{n^p}$, where the series converges for values of $p>1$ and diverges for values $p\leq 1$. Thus for our question, we have that our comparison series $\Sigma b_n=\Sigma\frac{1}{n}$ is divergent.

Furthermore, we have that

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{\frac{n^2}{n^3+1}}{\frac{1}{n}}\right)=\lim_{n\to\infty}\left(\frac{n^3}{n^3+1}\right)=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n^3}}\right)=1$$

Since Σb_n is divergent and above limit is finite and greater than 0, it follows that the given series Σa_n is also divergent by the Limit Comparison Test.

Both students used the nth term test for divergence[89]. In determining $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+1}$ to be 0, they concluded that the given series is convergent. Unfortunately their solution was incorrect and also contains an incorrect application of the nth term test for divergence (this was discussed in Section 6.2.1).

Question 2 of Observation 1

Similar to question 1, the convergence/divergence of the given series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ can be determined by a number of different tests, such as as the Comparison Test, Limit Comparison Test or the Integral Test. The best convergence/divergence test to use in this question is the Integral Test, which reads as follows:

The Integral Test

Suppose f is a continuous, positive, decreasing function on the interval $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(i) *If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*

(ii) *If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

The correct solution to question 2 is set out below:

[89] The nth term test for divergence: If the limit of a sequence $\{a_n\}$ is not zero or does not exist, then the corresponding series $\sum a_n$ is divergent. The test is inconclusive if the limit of the sequence is zero. The series can then either be convergent or divergent, and then another test needs to be used to determine the convergence/divergence of the series.

Let $\Sigma\, a_n = \Sigma \frac{e^{1/n}}{n^2}$, with $a_n = \frac{e^{1/n}}{n^2} = f(n)$, and where $f(x) = \frac{e^{1/x}}{x^2}$ is a continuous, positive, decreasing function on $[1, \infty)$. Determining the improper integral $\int_1^{\infty} f(x)\, dx$, we have that

$$\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\, dx = \int_1^{\infty} e^{1/x} \cdot x^{-2}\, dx = \lim_{t\to\infty} \left[-e^{\frac{1}{x}}\right]_1^t = \lim_{t\to\infty} \left[-e^{\frac{1}{t}}\right] + e = e - 1 < \infty$$

Since the improper integral converges, the corresponding series $\Sigma\, a_n = \Sigma \frac{e^{1/n}}{n^2}$ also converges by the Integral Test.

Again both students applied the nth term test for divergence incorrectly by showing that $\lim_{n\to\infty} a_n = 0$ and hence concluding that the series converges. Although their conclusion that the given series converges is correct, their strategy was faulty. Section 6.2.2 gives a detailed account on how the students approached and solved this question.

<u>Question 3 of Observation 1</u>

Similar to questions 1 and 2, the convergence/divergence of the given series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ can also be determined by either using the Comparison Test, Limit Comparison Test or the Integral Test. The best convergence/divergence test to use in this question is the Limit Comparison Test (similar to the case of question 1). The solution is as follows:

Let $\sum_{n=1}^{\infty} \frac{n}{n^4+1} = \Sigma a_n$ and consider the convergent p-series $\Sigma b_n = \Sigma \frac{1}{n^3}$ as our comparison series. Determining the limit $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)$ we have that

$$\lim_{n\to\infty} \left(\frac{\frac{n}{n^4+1}}{\frac{1}{n^3}}\right) = \lim_{n\to\infty} \left(\frac{n^4}{n^4+1}\right) = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n^4}}\right) = 1$$

Since Σb_n is convergent and above limit is finite and greater than 0, it follows that the given series Σa_n is also convergent by the Limit Comparison Test.

Similar to questions 1 and 2, both students applied the nth term test for divergence incorrectly by showing that $\lim_{n\to\infty} a_n = 0$ and hence concluding that the series converges. In realising their faulty application of the nth term test for divergence, Dean tried a different strategy in determining the convergence/divergence of the series (this is discussed in Section 6.2.3). Unfortunately the researcher had to stop the observation at this point since the next observation with another pair of students was about to start and neither students could determine the convergence/divergence of the given series by using Dean's new proposed strategy.

Task 2, Observation 2

Determine if the following series are convergent/divergent:

1.

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$$

2.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{n}\cos(n\pi)$$

Correct solutions versus students' solutions:

As discussed in Section 6.3, all three series were alternating series in which the students only had to apply one test to determine the convergence/divergence of the given series: the Alternating Series Test.

The Alternating Series Test consiss of two properties that need to be satisfied in order for an alternating series to be convergent. The test reads as follows:

Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ ...$$

with $b_n > 0$, satisfies the following two conditions

(i) $b_{n+1} \leq b_n$ *for all* $n \geq k$ *for some positive integer* k*, and*

(ii) $\lim\limits_{n\to\infty} b_n = 0$

then the series is convergent.

Condition (i) means that the sequence $\{b_n\}$ needs to be decreasing. We say further that a sequence is monotone if it is either increasing or decreasing (to be monotone it cannot be both increasing and decreasing). For example, if we have $b_{n+1} \geq b_n$ for all $n \geq k$, for some k a positive integer, then we say that the sequence $\{b_n\}$ is monotone increasing (or just increasing). For property (ii), the sequence $\{b_n\}$ must only converge to the value 0.

<u>Question 1 of Observation 2</u>

Note that the given alternating series $\sum_{n=2}^{\infty}\frac{(-1)^n \ln n}{n}$ is of the form $\sum_{n=2}^{\infty}(-1)^n b_n$, with $b_n = \frac{\ln n}{n}$.

When proving that the first property of the Alternating Series Test holds true, we note that for $\lim\limits_{n\to\infty} b_n$ we have

$$\lim_{n\to\infty}\frac{\ln n}{n} = \frac{\infty}{\infty}$$

Since ∞/∞ is an indeterminate form, we need to apply L'Hospital's Rule (this rule was briefly mentioned in Section 6.3.1). This rule is used in evaluating limits which have to do with indeterminate forms $0/0$ or ∞/∞ and states the following:

L'Hospital's Rule

If f and g are continuous and differentiable functions on an open interval I that contains a and we have that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

or

$$lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0},$$

then

$$lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

In order to apply L'Hospital's Rule to question 1 of observation 2, we need to consider the continuous function $f(x) = \frac{\ln x}{x}$, for $x \geq 2$ where $f(n) = b_n$.

Since $\lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$, L'Hospital's Rules gives

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)} \lim_{x \to \infty} \frac{1}{x} = 0$$

and thus $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\ln n}{n} = 0$ as well.

When proving the second property of the Alternating Series Test, we need to show that the sequence $\{b_n\} = \left\{\frac{\ln n}{n}\right\}$ is decreasing. In order to do this, we consider the function $f(x) = \frac{\ln x}{x}$, with $f(n) = b_n$ and prove that f is decreasing for x sufficiently large. In order to do this, we need to show that $f'(x) < 0$ for x sufficiently large[90].

By using the Quotient Rule for differentiation we have

$$f'(x) = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

where $f'(x) < 0$ when $1 - \ln x < 0,$ that is, when $x > e$.

Since $f'(x) < 0$ for all $x > e$, it follows that $f(x) = \frac{\ln x}{x}$ is decreasing, and hence the corresponding sequence $\{b_n\} = \left\{\frac{\ln n}{n}\right\}$ is also decreasing.

Since both properties of the Alternating Series Test are satisfied it follows that the given series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ is convergent.

Both Dean and Will were able to show the above properties in their work. Although their solutions were similar to the above outlined solution, they still had difficulty in reasoning how to show the above properties while solving the question. Their reasoning and approach on how to solve the question is discussed in Section 6.3.1.

Question 2 of Observation 2

Note that the given alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$ is of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$, where $b_n = \frac{1}{(2n+1)!}$.

Clearly $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, which proves the first property of the Alternating Series Test. Furthermore, for the sequence $\{b_n\} = \left\{\frac{1}{(2n+1)!}\right\}$, we have that

$$\frac{1}{(2(n+1)+1)!} < \frac{1}{(2n+1)!}$$

[90] A function f is said to be decreasing if $f'(x) < 0$ for all x in that interval. Also, a sequence $\{a_n\}$, with $a_n = f(n)$, will also be decreasing for all $x \geq k$ for $k \in \mathbb{Z}^+$ if $f'(x) < 0$ for all $x \geq k$.

that is, $b_{n+1} < b_n$, and thus $\{b_n\}$ is decreasing, proving the second property of the Alternating Series Test.

Since both properties of the Alternating Series Test are satisfied, it follows that the given series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$ is convergent.

Similar to question, both students were able to show the above properties in their work. Their solutions were similar to the above outlined solution, yet not that complete as the desired result. The students' reasoning and approach on how to solve the question is discussed in Section 6.3.2.

<u>Question 3 of Observation 2</u>

As discussed in Section 6.3.3, the given series of this question was not given in an explicit form of an alternating series $\sum(-1)^n b_n$. The crux of the question is that one should realise that for n even $\cos(n\pi) = 1$ and for n odd $\cos(n\pi) = -1$, hence simplifying the given series as:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

It is then easy to show that the above series is convergent by the Alternating Series Test in considering the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$. Clearly $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$, which proves the first property of the Alternating Series Test. Further, we have that

$$\frac{1}{n+1} < \frac{1}{n}$$

for all $n \geq 1$, that is, $b_{n+1} < b_n$, proving that the sequence $\{b_n\}$ decreasing.

Similar to questions 1 and 2, both students were able to show the above properties in their work. Their solutions were similar to the above outlined solution. Students' reasoning and approach on how to solve the question is discussed in Section 6.3.3.

Task 3, Observation 3

1. Determine a power series centred around $a = 0$ for $f(x) = \sqrt[4]{1+x^3}$.

2. Determine a power series centred around $a = 1$ for $f(x) = \ln x$.

3. Determine a power series centred around $a = 0$ for $f(x) = \sin^2 x$.
(Hint: given $\sin^2 x = (1 - \cos 2x)/2$)

Correct solutions versus students' solutions:

Question 1 of Observation 3

Since the power series is centred around $a = 0$ and the function $f(x) = (1 + x^3)^{1/4}$ is of the form $(1 + x)^k$, the power series of f can be written as binomial power series which is of the form

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k \cdot (k-1)}{2!} x^2 + \frac{k \cdot (k-1) \cdot (k-2)}{3!} x^3 + \cdots$$

where the nth term of the series is

$$\frac{k \cdot (k-1) \cdot (k-2) \cdots \big(k - (n-1)\big)}{n!} x^n$$

and $\binom{k}{n}$ is referred to as the binomial coefficient with

$$\binom{k}{n} = \frac{k!}{n!\,(k-n)!}$$

Using the above definition of the binomial series, with $k = \frac{1}{4}$ and replacing x by x^3, we have

$$(1+x^3)^{1/4} = 1 + \frac{1}{4}x^3 + \frac{\frac{1}{4} \cdot \left(\frac{1}{4} - 1\right)}{2!}(x^3)^2 + \frac{\frac{1}{4} \cdot \left(\frac{1}{4} - 1\right) \cdot \left(\frac{1}{4} - 2\right)}{3!}(x^3)^3 + \cdots$$

$$= 1 + \frac{1}{4}x^3 - \frac{1 \cdot 3}{4^2 \cdot 2!}(x^3)^2 + \frac{1 \cdot 3 \cdot 7}{4^3 \cdot 3!}(x^3)^3 - \frac{1 \cdot 3 \cdot 7 \cdot 11}{4^4 \cdot 4!}(x^3)^4 + \cdots$$

and the desired result is

$$f(x) = (1+x^3)^{1/4} = 1 + \frac{1}{4}x^3 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (3+4n)}{4^{n+2}\,(n+2)!} x^{3n+6}$$

Neither of the students obtained the above result. Both students only focused on the 'compact' form of the binomial series given by $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. By substituting a $1/4$ in the place of k and replacing x with x^3, they obtained the following answer:

$$(1+x^3)^{1/4} = \sum_{n=0}^{\infty} \binom{1/4}{n} x^{3n}.$$

As noted in Section 6.4.1, both students did realise that they could not have a non-integer value in the binomial coefficient $\binom{k}{n}$. Still, both students were happy with their solution and proceeded to question 2.

Question 2 of Observation 3

Since the power series is centred around $a = 1$, the function $f(x) = \ln x$ can be expanded as a Taylor series. As outlined in Section 6.4.2, a Taylor series for a function $y = f(x)$ centred around $x = a$, is a power series of the form:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where the coefficients of the series are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

First we determine a few derivatives of f in the point $x = 1$, in order to obtain an expression for c_n:

$$f(x) = \ln x \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{1 \cdot 2}{x^3} \qquad f'''(1) = 1 \cdot 2$$

$$f^{(4)}(x) = \frac{-1 \cdot 2 \cdot 3}{x^4} \qquad f^{(4)}(1) = -1 \cdot 2 \cdot 3$$

$$f^{(5)}(x) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{x^5} \qquad f^{(5)}(1) = 1 \cdot 2 \cdot 3 \cdot 4$$

$$f^{(6)}(x) = \frac{-1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{x^6} \qquad f^{(5)}(1) = -1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$$

From the above it follows that for $n \geq 1$

$$c_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$$

where $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$.

Thus the Taylor series for $f(x) = \ln x$, around $x = 1$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Only Dean obtained the above desired result, but also with some difficulty and in making some mistakes (as discussed in Section 6.4.2). Both Dean and Will made the mistake in initially evaluating the derivatives of f in the point $x = 0$, instead of $x = 1$. Dean was first to realise this mistake and corrected his work, while Will did not. Only when I intervened and encouraged the students to work together, Dean pointed out Will's mistake in evaluating the derivatives in the point $x = 1$. Moreover, Dean pointed out to Will that he differentiated incorrectly in which Will corrected his work.

As noted in Section 6.4.2, Dean was able to obtain an expression for $f^{(n)}(1)$, namely $f^{(n)}(1) = (-1)^{n+1}(n-1)!$. Dean mostly had difficulty with how the value

$f(1) = \ln 1 = 0$ was related to his expression of $f^{(n)}(1)$. After some time he realised that $f(1) = 0$ did not affect the power series representation and that it could be ignored. Will did not proceed in determining an expression for $f^{(n)}(1)$ and for most of the time quietly observed Dean working – it was only Dean who obtained a complete solution to this question. As a final answer Dean obtained $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ and was able to realise that the starting index of the series was not $n = 0$, but $n = 1$ and corrected his error in obtaining the desired result.

Question 3 of Observation 3

Since the power series is centred around $a = 0$, we need to determine a Maclaurin series for $f(x) = \sin^2 x$. A Maclaurin series by definition, is a Taylor series centred around the point $x = 0$, which is of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

The easiest and quickest solution to the question is to use the standard Maclaurin series representation of $\cos x$ and some algebra in conjunction with the given identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Using the definition of a Maclaurin series is not advisable, since repeated differentiation of $\sin^2 x$ will result in more complicated derivatives, making it difficult to determine a general expression for $c_n = \frac{f^{(n)}(0)}{n!}$.

Using the given given identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, we first determine a Maclaurin series for $\cos 2x$. Knowing that the standard Maclaurin series for $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

we then have that

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

and thus

$$1 - \cos 2x = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 - 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

Consequently

$$\sin^2 x = \frac{1}{2}\,(1 - \cos 2x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$$

giving the desired result.

Neither of the students considered the above approach and obtained the desired result. As seen from Section 6.4.3, both students were mostly orientating themselves about the question and considered different options on how to solve the problem. None of their proposed strategies was helpful in solving the problem. Since both students had to leave for a lecture, they were not able to solve the problem and also did not write down any solutions.

Task 4, Observation 4

1. Determine the first four non-zero terms of the power series, centred around $a = 0$ for $f(x) = \arcsin x$.

2. Determine a power series centred around $a = 0$ for $f(x) = \sinh x$.

3. Determine a power series, centred around $a = 0$ for $g(x) = \sinh^{-1} x$ if given

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

4. Determine if the following series is absolutely convergent, conditionally convergent, or divergent:

4.1

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

4.2

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$$

4.3

$$\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$$

Correct solutions versus students' solutions:

Question 1 of Observation 4

Since the power series is centred around $a = 0$, one can try to determine the Maclaurin series for $f(x) = \arcsin x$. This is quite difficult since repeated differentiation would result in f having more complex derivatives. It is easier to first consider the derivative of $f(x) = \arcsin x$ in relating it to a function of which the power series is easier to obtain. This is possible by considering that

$$\arcsin x = \int f'(x)\, dx = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

Since $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$ and the series is centred at $= 0$, one can determine a binomial series[91] expansion for f'.

[91] A binomial power series is a power series, centred around the point $a = 0$ and of the form:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k \cdot (k-1)}{2!} x^2 + \frac{k \cdot (k-1) \cdot (k-2)}{3!} x^3 + \cdots$$

where the nth term of the series is $\frac{k \cdot (k-1) \cdot (k-2) \cdots (k-(n-1))}{n!} x^n$.

Since we only need to determine the first four terms of the power series of f, we only need to determine the first four terms of the power series of f'. With $k = -\frac{1}{2}$ and replacing x with $-x^2$, it follows that the first four terms of f', according to the binomial series expansion are as follows:

$$f'(x) = (1 + (-x^2))^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \cdots$$
$$= 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}x^4 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6 + \cdots$$

Taking into account that

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

it follows from the above that

$$\arcsin x = \int \left(1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}x^4 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6 + \cdots\right)dx$$
$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2^2\cdot 2!}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\cdot\frac{x^7}{7} + \cdots$$

which gives the desired result.

As noted in Section 6.5.1, the students were able to obtain a similar (but not the exact correct) answer to the above. Furthermore, it was Dean who guided the problem solving process that resulted in both students having a similar solution structure (although not the same answers and results). Both students focussed on using the 'compact' form of the binomial series, namely

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

instead of the expanded form as illustrated in correct solution. Below are the transcribed solutions (verbatim) of Dean and Will as recorded during the

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Furthermore, $\binom{k}{n}$ is referred to as the binomial coefficient with $\binom{k}{n} = \frac{k!}{n!(k-n)!}$

observation (note the mathematical errors in the students' work, as well as imprecise mathematical notation).

Dean' Solution:

$$\int \frac{d}{dx}\sin^{-1}(x) = \int \frac{1}{\sqrt{1-x^2}}$$

$$= \int (1-x^2)^{-1/2} = \int \left(1+(-x^2)\right)^{-1/2} = \int \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$= \int \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n} (-1)^n x^{2n+1}}{2n+1}$$

As discussed in Section 6.5.1, Dean reminded himself that he only had to determine the first four terms of the power series. Moreover, Dean realised that he was unable to determine the binomial coefficient of each term, since he could not have negative factorials. Turning to his textbook for help, Dean considered the expanded form of the binomial series in determining the first four terms of the series. His solution (verbatim) is transcribed below:

$$n = 0, \quad T_1 = x \qquad n = 1, \quad T_2 = \frac{-\frac{1}{2}x^3}{3}$$

$$n = 2, \quad T_3 = \frac{k(k-1)}{2!}\,x^2 = \frac{\frac{3}{4}x^5}{2! \times 5} = \frac{\frac{3}{4}x^5}{5 \cdot 2!} = \frac{3x^5}{20 \cdot 2!}$$

$$n = 3, \quad T_4 = \frac{k\,(k-1)(k-2)}{3!}x^3 = \frac{-\frac{15}{8}x^7}{3! \cdot 7} = \frac{-15x^7}{64 \cdot 3!}$$

Will's Solution:

Will's solution was more 'fragmented' compared to that of Dean's. As noted earlier, it was Dean who guided the problem solving process, which resulted in Will only writing a partial solution to his answer and leaving out certain steps and values in his calculations. This can be seen from his transcribed (verbatim) solution below (note the mathematical errors as well as imprecise mathematical notation in Will's work):

$$\frac{d}{dx}f(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arcsin x = \int \frac{1}{\sqrt{1-x^2}} = \int (1(-x^2)^{-1/2} = \int \sum_{n=\infty}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$= \int \sum_{n=\infty}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n} = \sum_{n=\infty}^{\infty} \frac{\binom{-\frac{1}{2}}{n} (-1)^n x^{2n+1}}{2n+1}$$

As discussed in Section 6.5.1, Dean more than once had to assist and explain to Will how to use the expanded form of the binomial series and in determining the first four terms of the power series of $f(x) = \arcsin x$. Moreover, it was only Dean who obtained the first four terms. Will was still behind in completing the question and copied Dean's solution, but only partially (although Dean's solution was not entirely correct). The remainder of Will's solution is transcribed below:

$$f(x) = \sin^{-1} x = \sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n} (-1)^n x^{2n+1}}{2n+1}$$

$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)!}{\left(-\frac{1}{2}-n\right)!\,n!} = 1 + kx + \frac{k\cdot(k-1)}{2!}x^2 + \frac{k\cdot(k-1)\cdot(k-2)}{3!}x^3$$

$$n(0) = x \quad n(1) = \frac{-\left(\frac{1}{2}\right)x^3}{3} = -\frac{1}{6}x^3$$

$$n(2) = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!\,(2(2)+1)}x^5 = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!\,5}x^5 = \frac{\left(\frac{3}{4}\right)}{2!\,(5)}x^5 = \frac{\left(\frac{3}{4}\right)}{10}x^5 = \frac{3}{40}x^5$$

$$n(3) = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!\,(2(3)+1)}x^7 = \left(-\frac{1}{2}\right)($$

<u>Question 2 of Observation 4</u>

Since we need to determine a power series for $f(x) = \sinh x$ around $a = 0$, we can use the definition of a Maclaurin series to obtain the desired result. This is set out below:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Determining the first six derivatives of f we have

$$\begin{aligned}
f(x) &= \sinh x & f(0) &= 0 \\
f'(x) &= \cosh x & f'(0) &= 1 \\
f''(x) &= \sinh x & f''(0) &= 0 \\
f'''(x) &= \cosh x & f'''(0) &= 1 \\
f^{(4)}(x) &= \sinh x & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cosh x & f^{(5)}(0) &= 1 \\
f^{(6)}(x) &= \sinh x & f^{(5)}(0) &= 0
\end{aligned}$$

Thus it follows that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 0 + \frac{1}{1!}x + 0 + \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \cdots$$

and the desired result

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Below are the transcribed solutions (verbatim) of Dean and Will respectively. A detailed account on how the students collaborated in solving the question is given in Section 6.5.2. The reader would note Will's incomplete solution, compared to that of Dean's solution.

Dean's Solution:

$$f(x) = \sinh x \qquad a = 0$$

$$f(x) = \sinh x \qquad f(0) = 0$$

$$f'(x) = \cosh x \qquad f'(0) = 1$$

$$f''(x) = \sinh x \qquad f''(0) = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}$$

Will's Solution:

$$a = 0 \qquad f(x) = \sinh x \qquad f(0) =$$

$$f'(x) = \cosh x$$

$$f''(x) = +\sinh x$$

$$f'''(x) = +\cosh x$$

$$f''''(x) = \sinh x$$

<u>Question 3 of Observation 4</u>

Since the power series is centred around $a = 0$, one can determine the Maclaurin series for $g(x) = \sinh^{-1} x$ by repeated differentiation of the given definition,

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$. Unfortunately this is not the most elegant approach since the derivatives of g would become more and more complex. The ideal method of solving the problem would be to use the following relationship from first year calculus:

$$\sinh^{-1} x = \int \frac{1}{\sqrt{1+x^2}}\, dx$$

that is, $g'(x) = \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$.

Using the above, one follows a similar strategy as in question 1 in solving question 3. The solution to the question is set out below, in which we use the definition of a binomial series and the above relationship.

We first determine a binomial series for $g'(x) = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2}$. Considering the expanded form of the binomial series

$$(1+x)^k = = 1 + kx + \frac{k\cdot(k-1)}{2!}x^2 + \frac{k\cdot(k-1)\cdot(k-2)}{3!}x^3 + \cdots$$

with $k = -\frac{1}{2}$ and replacing x by x^2, we then have

$$\begin{aligned}
(1+x^2)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{3!}(x^2)^3 + \cdots \\
&= 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}(x^2)^2 - \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}(x^2)^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 2!}(x^2)^4 - \cdots \\
&= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n\, n!}x^{2n}
\end{aligned}$$

Hence the desired result

$$\begin{aligned}
\sinh^{-1} x = \int (1+x^2)^{-1/2}\, dx &= \int\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n\, n!}x^{2n}\right) dx \\
&= x + \sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n\,(2n+1)n!}x^{2n+1} + C
\end{aligned}$$

where C is an integration constant.

From the discussion in Section 6.5.3, it was noted that the students battled with deciding what strategy to use. Initially both of them tried to determine a Maclaurin series for $g(x) = \sinh^{-1} x$ by repeated differentiation of the given definition, $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$. Since the derivatives of g would become more and more complex, I intervened by interrogating the students to evaluate the usefulness of their strategy they were implementing. This led me in giving them a hint of what the derivative of $\sinh^{-1} x$ is, which enabled them to solve the problem. The students proceeded in following a similar approach to that of the outlined solution above. Yet again, they only focussed on using the 'compact' form of the binomial series and did not obtain the desired power series for g, as given in the correct solution. Below are the transcribed solutions (verbatim) of Dean and Will's work respectively, illustrating how they initially tried in determining the derivatives of g.

Dean's Solution:

$$g(x) = \ln\left(x + \sqrt{x^2+1}\right) = \ln\left(x + x\sqrt{1 + 1/x^2}\right) = \ln\left(x(1 + \sqrt{1 + 1/x^2}\right)$$

$$\frac{d}{dx} g(x) = \frac{1}{x + \sqrt{x^2+1}} \times \left(1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right)$$

$$= \frac{1}{x + \sqrt{x^2+1}} \times \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right) = \frac{1}{x + \sqrt{x^2+1}} \times \left(\frac{\sqrt{x^2+1} + 2x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1} + 2x}{x\sqrt{x^2+1} \cdot (x^2+1)}$$

After giving the students a hint of what the derivative of $\sinh^{-1} x$ is, Dean proceeded in writing the following:

$$\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

$$f(x) = \sinh^{-1}(x) = \int \frac{1}{\sqrt{1+x^2}}\,dx = \int (1+x^2)^{-1/2}\,dx$$

$$= \int \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n}\,dx = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1}$$

Will's Solution:

$$a = 0 \qquad g(x) = \sinh^{-1} x \qquad \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$= \ln\left(x + x\sqrt{1+1/x^2}\right)$$

$$g(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$g'(x) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} + \frac{x}{x+\sqrt{x^2+1}+\sqrt{x^2+1}\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}} + \frac{x}{x}$$

$$\int (1+x^2)^{-1/2}$$

$$\int \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n} = \sum_{n=0}^{\infty} \int \binom{-\frac{1}{2}}{n} x^{2n}\,dx = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1}$$

Question 4 of Observation 4

This question is similar to the questions of observations 1 and 2, but slightly more advanced because one needs to consider both conditional convergence and absolute convergence for the series. In order to solve the question, we first need to familiarise ourselves about what is meant by these two types of convergence. The definitions are as follows:

> *Absolute convergent: A series Σa_n is absolute convergent if the series of absolute values converges, that is, the series $\Sigma|a_n|$ is convergent.*

Conditionally convergent: A series Σa_n is conditionally convergent if the series of absolute values diverges, that is, the series $\Sigma|a_n|$ is divergent, but Σa_n is convergent.

Note that if a series is not absolutely convergent, it is either conditionally convergent or divergent. When solving these type of questions, it is always best to first test for absolute convergence. If the series is not absolutely convergent, one then proceeds in testing if the series is conditionally convergent or divergent. What is also of importance is that if a series is absolutely convergent, it is also convergent, in other words, absolute convergence implies convergence.

Below are the correct solutions to the three given questions:

Question 4.1 of Observation 4

Let

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \sum_{n=2}^{\infty} a_n$$

We first consider the series of absolute values:

$$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

In order to determine the convergence of $\Sigma|a_n|$, we use the Comparison Test which reads as follows:

Suppose that Σa_n and Σb_n are series with positive terms.

(i) If Σb_n converges and $a_n \leq b_n$ for n sufficiently large, then Σa_n is convergent.

(ii) If Σb_n diverges and $a_n \geq b_n$ for n sufficiently large, then Σa_n is divergent.

In order to solve the question we let the series Σb_n be the divergent p-series $\Sigma \frac{1}{n}$ (p-series were discussed in Task 1, Observation 1 of this Appendix).

Furthermore, we also have that $n > \ln n$ for n sufficiently large, or equivalently $\frac{1}{n} < \frac{1}{\ln n}$.

Thus, since $b_n < |a_n|$ it follows by the Comparison Test that $\Sigma|a_n|$ is divergent, and hence the series Σa_n is not absolutely convergent.

We now proceed in determining if the series is conditionally convergent or divergent. Note that the given series $\Sigma a_n = \Sigma \frac{(-1)^n}{\ln n}$ is an alternating series. By the Alternating Series Test (which is discussed in Section 6.3 and Task 2, Observation 2 of this Appendix) it is easy to show that the given series is convergent by letting the sequence $\{b_n\} = \left\{\frac{1}{\ln n}\right\}$.

In conclusion, since $\Sigma|a_n|$ is divergent, but Σa_n is convergent, it follows that Σa_n is conditionally convergent.

<u>Question 4.2 of Observation 4</u>

Let

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} a_n$$

Note that the given series is actually an alternating series, since for n odd we have $\cos n\pi = -1$, and n even $\cos n\pi = 1$, and thus

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

We first consider if the above series is absolutely convergent, that is, we consider the convergence/divergence of the series $\Sigma|a_n|$. Note that $\Sigma|a_n| = \Sigma\frac{1}{n}$ is a divergent p-series, and hence $\Sigma|a_n|$ is divergent.

Considering if the given series is conditionally convergent, we note that $\Sigma a_n = \Sigma \frac{(-1)^n}{n}$ is an alternating series. It is easy to show by the Alternating Series Test that the given series is convergent, by letting the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$,.

Thus, since $\Sigma|a_n|$ is divergent, but Σa_n is convergent, it follows that Σa_n is conditionally convergent.

<u>Question 4.3 of Observation 4</u>

Let

$$\sum_{n=0}^{\infty}(-1)^n e^{-n^2} = \sum_{n=0}^{\infty} a_n$$

The best approach in solving the question is by using the Ratio Test, which states the following:

Ratio Test

Consider a series Σa_n.

(i) *If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series Σa_n is absolutely convergent (and therefore convergent).*

(ii) *If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series Σa_n is divergent.*

(iii) *If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive: that is, no conclusion can be drawn about the convergence or divergence of Σa_n.*

Applying the Ratio Test to the given series $\sum_{n=0}^{\infty}(-1)^n e^{-n^2} = \sum_{n=0}^{\infty} a_n$, it follows that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{-(n+1)^2}}{e^{-n^2}}\right| = \lim_{n\to\infty} e^{-(2n+1)} = 0 < 1$$

and hence the given series is absolutely convergent.

As discussed in Section 6.5.4, the students only finished question 4.1. Furthermore, Dean was the only one that wrote out a solution to this question, which is transcribed (verbatim) below. Note that Dean's writing of the mathematics is imprecise and that he only wrote down what he deemed was necessary in explaining the solution to Will.

Dean's Solution:

$$\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{\ln n} \;\; \substack{\leq \\ \geq} \;\; \frac{1}{n}$$

A complete discussion on how the students approached the problem is given in Section 6.5.4.

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