UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

DOCTORAL THESIS

Turbulent wake flows: Lie group analysis and conservation laws

Author:
Ashleigh Jane Hutchinson

Supervisors:
Professor D P Mason
Professor F M Mahomed

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

in the

Faculty of Science
School of Computer Science and Applied Mathematics

March 2016

Declaration of Authorship

I, Ashleigh Jane Hutchinson, declare that this thesis titled, 'Turbulent wake flows: Lie group analysis and conservation laws' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

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Date: 02/03/2016

Practically everything that is useful in turbulence theory is a scaling law.

P G Saffman

Abstract

We investigate the two-dimensional turbulent wake and derive the governing equations for the mean velocity components using both the eddy viscosity and the Prandtl mixing length closure models to complete the system of equations. Prandtl's mixing length model is a special case of the eddy viscosity closure model. We consider an eddy viscosity as a function of the distance along the wake, the perpendicular distance from the axis of the wake and the mean velocity gradient perpendicular to the axis of the wake. We calculate the conservation laws for the system of equations using both closure models. Three main types of wakes arise from this study: the classical wake, the wake of a self-propelled body and a new wake is discovered which we call the combination wake. For the classical wake, we first consider the case where the eddy viscosity depends solely on the distance along the wake. We then relax this condition to include the dependence of the eddy viscosity on the perpendicular distance from the axis of the wake. The Lie point symmetry associated with the elementary conserved vector is used to generate the invariant solution. The profiles of the mean velocity show that the role of the eddy viscosity is to increase the effective width of the wake and decrease the magnitude of the maximum mean velocity deficit. An infinite wake boundary is predicted from this model. We then consider the application of Prandtl's mixing length closure model to the classical wake. Previous applications of Prandtl's mixing length model to turbulent wake flows, which neglected the kinematic viscosity of the fluid, have underestimated the width of the boundary layer. In this model, a finite wake boundary is predicted. We propose a revised Prandtl mixing length model by including the kinematic viscosity of the fluid. We show that this model predicts a boundary that lies outside the one predicted by Prandtl. We also prove that the results for the two models converge for very large Reynolds number wake flows. We also investigate the turbulent wake of a self-propelled body. The eddy viscosity closure model is used to complete the system of equations. The Lie point symmetry associated with the conserved vector is derived in order to generate the invariant solution. We consider the cases where the eddy viscosity depends only on the distance along the wake in the form of a power law and when a modified version of Prandtl's hypothesis is satisfied. We examine the effect of neglecting the kinematic viscosity. We then discuss the issues that arise when we consider the eddy viscosity to also depend on the perpendicular distance from the axis of the wake. Mean velocity profiles reveal that the eddy viscosity increases the boundary layer thickness of the wake and decreases the magnitude of the maximum mean velocity. An infinite wake boundary is predicted for this model. Lastly, we revisit the discovery of the combination wake. We show that for an eddy viscosity depending on only the distance along the axis of the wake, a mathematical relationship exists between the classical wake, the wake of a self-propelled body and the combination wake. We explain how the solutions for the combination wake and the wake of a self-propelled body can be generated directly from the solution to the classical wake.

Acknowledgements

I thank my thesis supervisors Professors D P Mason and F M Mahomed.

Professor D P Mason I am immensely grateful for your support and encouragement. You have inspired my passion for research and my quest to leave my mark.

Professor F M Mahomed, it is with much pride that I acknowledge your invaluable supervision. I am in awe of your great achievements and brilliant ideas.

I also thank M J Woolway for his assistance with compiling this thesis.

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This research is dedicated to the following special people:

My father W J Hutchinson, for sharing and understanding my ultimate life goal. I will not give up.

My mother M J Hutchinson, your cups of tea and kind words helped me overcome difficult times. Without you I would not be where I am.

My fiancé P A De Almeida, your unwavering support and encouragement will always be remembered. I am a more positive person because of you.

My sister L S Hutchinson, you have always been so proud of me. I hope I never disappoint you.

My brother L P Hutchinson, I hope I inspire you to make your mark.

My best friend D Newman, for listening to my worries and helping me overcome them. You lifted my spirits when no one else could. I am truly grateful.

I also dedicate this thesis to one of my academic heroes L Prandtl. Your ideas outlived your death and have given me great inspiration. I wish I had known you.

Chapter 1

Introduction to turbulent wake flows

1.1 Introduction to the wake

The problem of the wake is fascinating and of much practical interest. Wakes are formed when a free flowing laminar fluid with a constant speed passes an obstructing body which is aligned with the mainstream flow. Turbulent wakes are formed from mainstream flows characterised by a large Reynolds number. The presence of the obstruction results in turbulent downstream flow. The symmetric wake can be separated into two types: the 'classical' wake and the 'momentumless' wake. The Reynolds averaged equations are used to determine the mean motion for turbulent flows. A closure model is required to obtain a complete system of differential equations. Algebraic closure models implement the eddy viscosity formulation [1]. The mixing length formulation [2] is a special case of the eddy viscosity model. Boussinesq [3] introduced the concept of an eddy viscosity which unlike the dynamic viscosity, is not a property of the fluid. The effect of the turbulence on the mean flow manifests itself as an increase in the apparent viscosity of the fluid. In this thesis we consider the equations describing the flow in a turbulent two-dimensional wake. The system of equations is ultimately completed by using the eddy viscosity closure model. The equations and boundary conditions are presented in terms of the x- and y- mean velocity components and in terms of a stream function.

The two-dimensional steady flow of the laminar wake of an incompressible Newtonian fluid behind a thin symmetric fixed planar body aligned with the mainstream flow, known as a classical wake, was first studied by Goldstein [4]. A study of the momentumless laminar wake behind a thin symmetric self-propelled body was first undertaken by Birkhoff and Zorantello [5]. The two-fluid laminar classical wake and wake of a self-propelled body was later investigated by Herczynski, Weidman and Burde [6]. The partial differential equation (PDE) for the flow in the wake was derived from the Navier-Stokes equation in the boundary layer approximation. This equation was reduced to an ordinary differential equation (ODE) governing the similarity flow.

The turbulent planar wake has been discussed by Tennekes and Lumley [7]. Equations for the mean velocities using the eddy viscosity closure model were formulated. A constant eddy viscosity was chosen. A solution governing the similarity flow was obtained and the results were compared with experimental observations. Existing solutions for the turbulent planar wake are similarity solutions. Similarity solutions can be obtained when the eddy viscosity is a power law of the distance along the axis of the wake and when the kinematic viscosity is neglected. Similarity solutions cannot be obtained for an effective viscosity which is the sum of the kinematic viscosity and the eddy viscosity, and in general when the eddy viscosity depends on the distance perpendicular to the axis of the wake.

In this thesis, the eddy viscosity closure model is used and the kinematic viscosity is not neglected. An eddy viscosity is considered depending on the distance along the axis of the wake; the distance perpendicular to the axis of the wake; and the mean velocity gradient perpendicular to the axis of the wake. The fluid flow in the classical wake and the wake of a self-propelled body is described by the same governing equations. For wakes with an infinite boundary such as those found for eddy viscosities depending on the spacial variables only, the boundary conditions are identical for both the classical wake and the wake of a self-propelled body. The difference between the two problems lies within the conserved quantity. For the classical wake the conserved quantity is the drag force [4] and for the wake of a self-propelled body the drag is zero and the conserved quantity is the second moment of the axial momentum deficit [5]. This work examines the equations for the two-dimensional turbulent classical wake and the wake of a self-propelled body. An eddy viscosity depending on the spacial variables only is first considered for wakes with infinite boundaries. We then include the dependence of the eddy viscosity on the mean velocity gradient perpendicular to the axis of the wake in order to investigate models that predict finite wake boundaries.

Other types of wake flows, such as the two-dimensional laminar classical wake of a shear thinning fluid [8] and the laminar axisymmetric classical wake for power-law fluids [9], have been considered. For shear thickening flows, the boundary of the wake is finite. A finite wake boundary is obtained for the turbulent axisymmetric wake in [10] which makes use of a closure model developed by Prandtl called the mixing length model [2]. Part of this thesis will be devoted to the closure model developed by Prandtl [2]. The apparent stress is in the form of the product of a length squared-called the 'mixing' length-and the square of the mean velocity gradient perpendicular to the axis of the wake. The kinematic viscosity of the fluid is neglected because it is assumed that the turbulent viscosity is much greater than the kinematic viscosity. This model predicts a finite boundary for the wake. Prandtl assumed large Reynolds numbers and by neglecting the kinematic viscosity term, he subsequently obtained a similarity solution. The mixing length was assumed to be proportional to the width of the boundary layer. However, on comparison of the predicted results with experimental observations, it was found that the predicted boundary calculated from the model lay inside the actual boundary near to the obstructing object [2]. In [10] a second approximation to the motion was implemented and the new model predicted a boundary that lies outside of the boundary predicted by Prandtl. It will be shown that Prandtl's model can be modified by including the kinematic viscosity. The boundary predicted by this new model also lies outside the boundary predicted by Prandtl. Inclusion of the kinematic viscosity requires the use of Lie point symmetry methods in order to obtain an analytical solution. Certain results that were previously assumed, such as the proportionality relationship between the mixing length and width of the wake, can now be proven.

All of the wakes mentioned above are symmetric about the axis of the wake. Another type of wake, called the 'wall-wake', was studied in [11] for laminar flows and in [12] for turbulent flows. Wall-wake flows are formed due to the presence of a body that is situated on the boundary wall. The wall occupies the entire lower half of the plane. The fluid sticks to the boundary wall. Here the wake is situated within a boundary layer. This problem is not symmetric about the axis of the wake as the fluid is confined to the upper half of the plane. The governing equations for this problem differ to that of the classical wake and the wake of a self-propelled body. We do not consider this problem further.

Systematic methods have been formulated for solving problems in fluid mechanics using a Lie symmetry approach. A number of the applications of symmetry methods to problems in turbulence can be found in [13–15]. The problem of the wake is encapsulated in the class of problems in fluid mechanics with homogeneous boundary conditions for which a conserved quantity is required for their solution. Another important problem area which requires a conserved quantity to complete the solution is jet flows. Mason [16] for a laminar two-dimensional jet and Ruscic and Mason [17] for a laminar axisymmetric jet, applied symmetry methods and derived the group invariant solution using a linear combination of the Lie point symmetries of the equation governing the flow. The conserved quantity and the boundary conditions for the jet were used to solve for the arbitrary constants in the linear combination of Lie point symmetries. The turbulent two-dimensional jet, whose governing equations were formulated using the eddy viscosity closure model, was investigated by Mason and Hill [18]. Again, a linear combination of Lie point symmetries was used to generate the group invariant solution. It was found that the Lie point symmetries only existed provided that the eddy viscosity satisfied a first order linear PDE. Higher order symmetries have been considered in [19, 20] and applied to jet flows.

The conserved quantity for a jet plays a central role in the method of solution. The same holds true for the wake. Conserved quantities can be difficult to derive. Much progress has been made recently on deriving conserved quantities using conservation laws for the governing PDEs. A systematic approach has been developed to find the conserved quantity for the jet [21]. The conservation laws are first derived and the conserved quantity can then be determined by integrating one of the conservation laws across the jet, chosen to be compatible with the boundary conditions of the problem.

Methods to calculate the conservation laws of a PDE can be found in [22–26]. A summary on the different approaches for calculating conservation laws is provided in [27]. In this thesis we will derive the conservation laws for the PDEs for the two-dimensional turbulent wake using the multiplier method developed by Steudel [28]. In [28], it is not shown how to construct the components of a conserved vector explicitly from a given multiplier. Further information on this approach can be found in [27, 29]. In particular, it is shown in [27, 29] that the multiplier method yields all non-trivial conservation laws of a PDE. An algorithmic approach to the multiplier method is given in [23–25]. The purpose of this part of the study is to verify if the conserved quantities of physical importance that are obtained belong only to the classical wake and the wake of a self-propelled body. Indeed we discover that another conserved quantity with potential physical significance exists. Not only could this conserved quantity be of physical interest but we also show that a simple relationship between the solutions of the three problems exists.

A modification, due to Kara and Mahomed [30], of the Lie symmetry method was introduced for problems with a conserved quantity. They first derived the condition for a Lie point symmetry to be associated with a conserved vector [30, 31]. Instead of using a linear combination of all the Lie point symmetries of the PDE to derive an invariant solution, the Lie point symmetry associated with the conserved vector that generates the conserved quantity was used. This method is more direct. It has been applied to laminar jet flows by Naz and Naeem [32], to a turbulent jet by Mason and Hill [33] and to turbulent flow of a compressible fluid in a tube by Anthonyrajah and Mason [34]. Because a Lie point symmetry associated with a conserved vector of the PDE is used to reduce the PDE to an ODE, the ODE can be integrated at least once by the double reduction theorem of Sjöberg [35]. Further work on symmetries and conservation laws for differential equations can be found in [23, 24, 27, 36, 37].

The momentum partial differential equation for the turbulent wake described by an eddy viscosity depending on the spacial variables only, is a linear parabolic diffusion-type equation. For an eddy viscosity depending on the distance along the axis of the wake only, the momentum equation reduces to a linear heat equation. The conservation laws for general linear parabolic partial differential equations are calculated in [38]. The potential symmetries are also considered. A significant amount of research has been conducted on diffusion equations. Steinberg and Wolf [39] studied the connection between the moments of the diffusive equation and its symmetries. They showed that the zeroth and first moments are conserved for the classical heat equation. In [40] the partial Lagrangian approach was used to calculate the Noether-type operators of the classical heat equation. Work on the classical symmetry groups and the weak symmetry groups of the linear heat equation can be found in the texts [41, 42] and [41, 43–45] respectively. Mansfield [46] studied a nonclassical group analysis of the linear heat equation and nonclassical reductions are investigated in [47].

A detailed study on the non-linear reaction-diffusion equation with variable coefficients can be found in [48]. In [48], a group classification is performed and the local conservation laws are obtained. Lie symmetry methods are also implemented to obtain some exact solutions. A large number of studies on diffusion-convection equations have also been undertaken. The conservation laws are discussed in [49], and in [50] the conservation laws are used in order to construct the corresponding potential systems. In [51], classical symmetry reductions are obtained and optimal systems are found. The invariants for each optimal system are derived. Group classifications are provided for the non-linear diffusion-convection equation in [52] and the non-linear variable-coefficient diffusion-convection equation in [53, 54]. The invariant solutions of the non-linear heat equation with conduction and a source term can be found in [55]. An interesting study by Bluman and Kumei [56] considered the invariance properties of the heat equation with a conduction term. A particular form for the conduction term was used. Oron [57] calculated some of the symmetries for the non-linear heat equation. Although a general formula for conservation laws was derived in [38] for linear parabolic equations, when we include the continuity equation in our analysis we obtain another non-trivial conservation law that cannot be generated by this formula. Direct comparisons of the results obtained in this thesis and the work done previously can be found in [58].

This thesis is outlined as follows:

- Chapter 2: the mathematical models for the turbulent classical wake and the turbulent wake of a self-propelled body are presented. The Reynolds averaged boundary layer equations for the turbulent wake using the eddy viscosity closure model are derived. The eddy viscosity in this case is a function of the spacial variables and the mean velocity gradient perpendicular to the axis of the wake. The general form chosen for the eddy viscosity can be applied to both problems pertaining to an infinite wake boundary and a finite wake boundary. The boundary conditions are derived.

- Chapter 3: a systematic method using a Lie symmetry approach is presented in order to generate the conserved quantities for the turbulent wake equation described by eddy viscosity. An eddy viscosity depending on the spacial variables and the mean velocity gradient perpendicular to the axis of the wake is first considered. The elementary conserved quantity is derived. This problem concerns the application of Prandtl's mixing length model and a revised version of it to the turbulent classical wake. This approach is later shown to generate a finite boundary for the wake. We then consider an eddy viscosity depending on the spacial variables only. The multiplier approach is used in order to derive a basis of conservation laws for the governing equations expressed both in terms of the velocity components and the stream function. The conserved quantities for the classical wake and the wake of a self-propelled body are derived. A third conserved quantity is obtained. This conserved quantity is generated using the same boundary conditions perpendicular to the axis of the wake at $\pm\infty$ as that of the classical wake and the wake of a self-propelled body.

- Chapter 4: in this chapter the Lie point symmetry associated with the elementary conserved vector is determined for the turbulent classical wake with an infinite boundary. We first consider the eddy viscosity to be a function of only the distance along the axis of the wake and solve for the stream function. Mean velocity profiles are plotted for an eddy viscosity in the form of a power law and the results are compared for a range of power laws and with the laminar classical wake. We then take the eddy viscosity to be a function of the distance along the axis of the wake and the perpendicular distance from the axis of the wake. Various forms of the eddy viscosity are analysed and mean velocity profiles are again compared with those obtained for the laminar wake. .

- Chapter 5: the purpose of this chapter is to apply Lie symmetry methods to the turbulent wake of a symmetric self-propelled body. From the conserved vector obtained in Chapter 3, we calculate the Lie point symmetry associated with this conserved vector and derive the invariant solution. We consider an eddy viscosity in the form of a power law of the distance along the axis of the wake and plot the mean velocity profiles. We also examine the negative effects of excluding the kinematic viscosity as opposed to including it. We include a discussion on the anticipated difficulties that arise when we consider the eddy viscosity to be a function of both the distance along the axis of the wake and the perpendicular distance from the axis of the wake.

- Chapter 6: we consider the application of a revised Prandtl mixing length model in which the kinematic viscosity is not neglected and solve for the stream function. Mean velocity profiles are plotted with the purpose of examining the impact of the strength of the turbulence on the mean velocity and the width of the wake. It is shown numerically that the width of the wake in the revised Prandtl model is finite. We derive and discuss the results from implementing Prandtl's mixing length model to the turbulent classical wake. A detailed comparison of the two models is provided. It is proved that the revised Prandtl model predicts a boundary that lies outside the one predicted by Prandtl. We also show that with the revised Prandtl model the mathematical form of the mixing length can be derived and need not be assumed as it was with Prandtl's model.

- Chapter 7: the conservation law obtained in Chapter 3 that does not belong to any known wake problem is discussed. The Lie point symmetry corresponding to the conservation law is derived and the invariant solution is obtained. The solution to this problem is shown to provide the link between the solutions to the classical wake and the wake of a self-propelled body for an eddy viscosity depending on the distance along the axis of the wake only. We deduce the mathematical link between these solutions and discuss the significance of this result.

- Chapter 8: conclusions are presented in this chapter.

A large portion of the research in Chapters 3 and 7 can be found in [58]. Chapters 4, 5 and 6 refer to work done in [59], [60] and [61] respectively.

Chapter 2

Derivation of the governing equations

2.1 Derivation

In this section we consider a two-dimensional turbulent wake of a symmetric body. Cartesian coordinates (x, y) are used. The classical wake, shown in Figures 2.1 and 2.3, represents the flow past a slender symmetric body aligned with a uniform mainstream flow. The obstructing body is stationary. The origin of the coordinate system is defined to be at the trailing edge of the body. The velocity of the mainstream flow is denoted by U and only consists of a component in the x-direction. In Figure 2.1 the wake has an infinite boundary and in Figure 2.3 the wake has a finite boundary. This is as a result of the closure model used to complete the system of equations. We will consider both cases. The wake behind a self-propelled body is shown in Figure 2.2. It differs from the classical wake in that the obstructing body propels itself in the x-direction at a constant speed resulting in zero momentum deficit. The origin of this coordinate system is chosen to be stationary with respect to the moving body at the trailing edge. However, since this coordinate system is moving at a constant speed, the mainstream velocity U here is defined as the relative velocity of the mainstream flow. For high Reynolds number flows, the obstructing object causes instabilities resulting in the presence of turbulence in the wake downstream of the body.

For the symmetric wakes above, namely, the classical wake and the wake of a self-propelled body, the governing equations and boundary conditions are the same. They differ in the conserved quantities that they satisfy. The position of the boundary for the classical wake also differs depending on the closure model used.

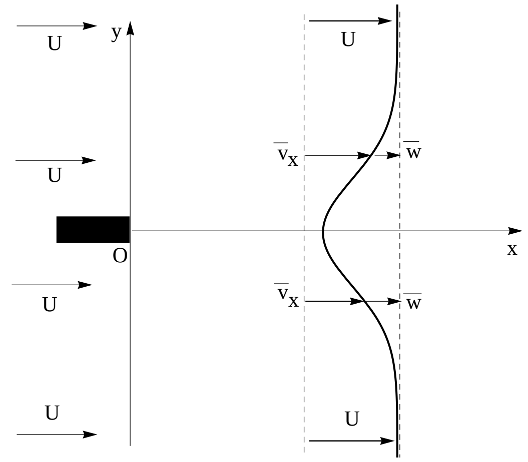

FIGURE 2.1: Two-dimensional classical wake behind a thin symmetric planar body.

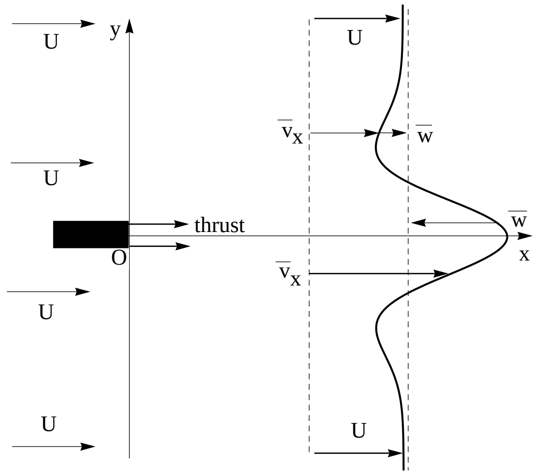

FIGURE 2.2: Two-dimensional wake behind a slender symmetric self-propelled body. The mean velocity deficit is negative in a neighbourhood of the x-axis.

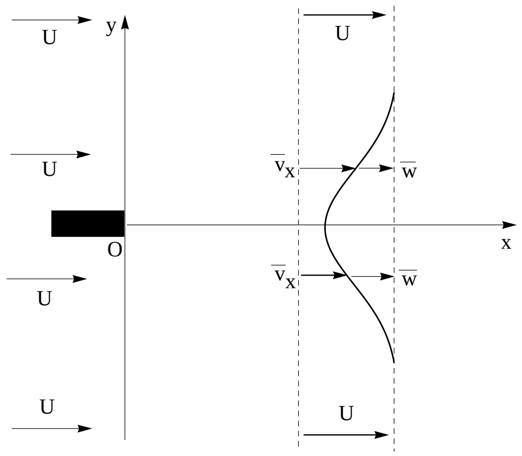

FIGURE 2.3: Finite two-dimensional classical wake behind a thin symmetric planar body aligned with a uniform flow.

One way to model turbulence is to separate the flow into a mean flow component and a fluctuation term which is imposed on the mean flow. The velocity components, v_x and v_y, and the pressure p are written as

$$v_x = \overline{v}_x + v_x', \quad v_y = \overline{v}_y + v_y', \quad p = \overline{p} + p', \tag{2.1}$$

where the mean flow variables $\overline{v}_x(x,y)$, $\overline{v}_y(x,y)$ and $\overline{p}(x,y)$ are defined as time averages [62]:

$$\overline{v}_x(x,y) = \frac{1}{T}\int_t^{t+T} v_x(x,y,t)\,dt, \quad \overline{v}_y(x,y) = \frac{1}{T}\int_t^{t+T} v_y(x,y,t)\,dt,$$

$$\overline{p}(x,y) = \frac{1}{T}\int_t^{t+T} p(x,y,t)\,dt. \tag{2.2}$$

The time interval T is sufficiently large to ensure that these time averages are independent of time. The time averages of the fluctuations are zero. However, the same is not true for squares and products of the fluctuations. We will see that these non-zero terms can be expressed in terms of Reynolds stresses. Their presence affects the mean flow by an apparent increase in the viscosity.

Since the fluid is incompressible,

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial v_x'}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} + \frac{\partial v_y'}{\partial y} = 0. \tag{2.3}$$

Taking the time average of (2.3) gives

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0, \tag{2.4}$$

and therefore from (2.3) and (2.4),

$$\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y} = 0. \tag{2.5}$$

The mean fluid velocity and the fluctuations both satisfy the continuity equation.

The Navier-Stokes equation is written in terms of the mean flow components. By using the continuity equation (2.5) for the fluctuation the x- and y- components of the Reynolds averaged equation may be expressed respectively as

$$\rho\left(\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\overline{p} - \rho\overline{v_x' v_x'} + \mu\frac{\partial \overline{v}_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\rho\overline{v_x' v_y'} + \mu\frac{\partial \overline{v}_x}{\partial y}\right), \tag{2.6}$$

$$\rho\left(\overline{v}_x\frac{\partial\overline{v}_y}{\partial x}+\overline{v}_y\frac{\partial\overline{v}_y}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\rho\overline{v'_xv'_y}+\mu\frac{\partial\overline{v}_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\overline{p}-\rho\overline{v'_yv'_y}+\mu\frac{\partial\overline{v}_y}{\partial y}\right). \tag{2.7}$$

The viscosity of the fluid μ and the density ρ are both constant. Furthermore, by using the continuity equation (2.4) for the mean flow, (2.6) and (2.7) can be written in the form

$$\rho\left(\overline{v}_x\frac{\partial\overline{v}_x}{\partial x}+\overline{v}_y\frac{\partial\overline{v}_x}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\overline{p}-\rho\overline{v'_xv'_x}+2\mu\frac{\partial\overline{v}_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\rho\overline{v'_xv'_y}+\mu\left(\frac{\partial\overline{v}_x}{\partial y}+\frac{\partial\overline{v}_y}{\partial x}\right)\right), \tag{2.8}$$

$$\rho\left(\overline{v}_x\frac{\partial\overline{v}_y}{\partial x}+\overline{v}_y\frac{\partial\overline{v}_y}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\rho\overline{v'_xv'_y}+\mu\left(\frac{\partial\overline{v}_y}{\partial x}+\frac{\partial\overline{v}_x}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(-\overline{p}-\rho\overline{v'_yv'_y}+2\mu\frac{\partial\overline{v}_y}{\partial y}\right). \tag{2.9}$$

Hence,

$$\rho\left(\overline{v}_x\frac{\partial\overline{v}_x}{\partial x}+\overline{v}_y\frac{\partial\overline{v}_x}{\partial y}\right)=\frac{\partial}{\partial x}\overline{\tau}_{xx}+\frac{\partial}{\partial y}\overline{\tau}_{yx}, \tag{2.10}$$

$$\rho\left(\overline{v}_x\frac{\partial\overline{v}_y}{\partial x}+\overline{v}_y\frac{\partial\overline{v}_y}{\partial y}\right)=\frac{\partial}{\partial x}\overline{\tau}_{xy}+\frac{\partial}{\partial y}\overline{\tau}_{yy}, \tag{2.11}$$

where

$$\overline{\tau}_{ik}=-\overline{p}\delta_{ik}+2\mu\overline{D}_{ik}-\rho\overline{v'_iv'_k}, \tag{2.12}$$

and $\overline{D}_{ik}$ is the rate-of-strain tensor for the mean velocity field defined by

$$\overline{D}_{ik}=\frac{1}{2}\left(\frac{\partial\overline{v}_i}{\partial x_k}+\frac{\partial\overline{v}_k}{\partial x_i}\right). \tag{2.13}$$

In order to obtain a closed system of differential equations for the mean velocity, Boussinesq [3] assumed that the Reynolds stress tensor $-\rho\overline{v'_iv'_k}$ is related to the mean rate-of-strain tensor $\overline{D}_{ik}$ in the same way as the stress tensor τ_{ik} is related to the rate-of-strain tensor D_{ik} for the laminar flow of an incompressible Newtonian fluid. He assumed that

$$-\rho\overline{v'_iv'_k}=2\mu_T\overline{D}_{ik}=\mu_T\left(\frac{\partial\overline{v}_i}{\partial x_k}+\frac{\partial\overline{v}_k}{\partial x_i}\right), \tag{2.14}$$

where μ_T is the eddy viscosity. Unlike the dynamic viscosity μ, the eddy viscosity μ_T is not a property of the fluid. It is a property of the flow and can depend on the spacial coordinates, x and y, as well as on the mean velocity components $\overline{v}_x$ and $\overline{v}_y$ and the mean velocity gradients. An eddy viscosity depending on the coordinate x and the velocity gradient $\dfrac{\partial\overline{v}_x}{\partial y}$ can be used to generate the form of the Reynolds stresses given by Prandtl's model and the revised version of it. We will assume that

$\mu_T = \mu_T\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)$ only. Thus (2.12) becomes

$$\overline{\tau}_{ik} = -\overline{p}\delta_{ik} + 2(\mu + \mu_T)\overline{D}_{ik}. \tag{2.15}$$

For a laminar flow the constitutive equation (2.15) reduces to the Navier-Poisson law for an incompressible Newtonian fluid. The effective kinematic viscosity is defined as [64]

$$E = \frac{\mu + \mu_T}{\rho} = \nu + \nu_T. \tag{2.16}$$

Equations (2.10) and (2.11) become

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial x}\left(2E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)\frac{\partial \overline{v}_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)\left(\frac{\partial \overline{v}_x}{\partial y} + \frac{\partial \overline{v}_y}{\partial x}\right)\right), \tag{2.17}$$

$$\overline{v}_x \frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_y}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \frac{\partial}{\partial x}\left(E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)\left(\frac{\partial \overline{v}_x}{\partial y} + \frac{\partial \overline{v}_y}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(2E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)\frac{\partial \overline{v}_y}{\partial y}\right). \tag{2.18}$$

We now impose the boundary layer approximation. Although there is no solid boundary in the wake the boundary layer approximation can be applied because there is a region of sharp change perpendicular to the axis of the wake. We first make equations (2.17) and (2.18) dimensionless. The characteristic speed in the x- direction is U, where U is the speed of the uniform mainstream flow upstream of the body for the classical wake and the relative mainstream flow for the wake of a self-propelled body. The characteristic length in the x-direction is L, where L is an estimate of the length downstream of the body over which the reduction of velocity in the wake is not negligible.

For $E = E(x, y)$ we let $E_0 = \nu + \nu_{T_0}$, where ν_{T_0} is the characteristic turbulent viscosity, be the characteristic effective kinematic viscosity. For $E = E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)$ we still define the characteristic eddy viscosity to be ν_{T_0} but the characteristic effective viscosity E_C is now the sum $\nu_C + \nu_{TC}$, where ν_C and ν_{TC} denote the characteristic kinematic viscosity and characteristic turbulent viscosity respectively. We define E_C in this way in order to compare flows with different viscosities and to examine the effect of turbulence. In particular, when we study Prandtl's mixing length model we set $\nu = 0$ in equation (2.16).

From this we can obtain a characteristic length and speed in the y- direction. From boundary layer theory [62] the characteristic width of the boundary layer is $\delta = L/\sqrt{Re}$, where we will define the Reynolds number for the mean flow for $E = E(x, y)$ and $E = E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)$ by respectively,

$$Re = \frac{UL}{E_0}, \tag{2.19}$$

$$Re = \frac{UL}{E_C}. \tag{2.20}$$

From the conservation of mass equation (2.4), which is not approximated, the characteristic velocity in the y- direction, V, is

$$V = \frac{\delta U}{L} = \frac{U}{\sqrt{Re}}. \tag{2.21}$$

The characteristic pressure is ρU^2. We define for $E = E(x, y)$:

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta} = \frac{\sqrt{Re}}{L} y,$$

$$\overline{v}_x{}^* = \frac{\overline{v}_x}{U}, \quad \overline{v}_y{}^* = \overline{v}_y \frac{\sqrt{Re}}{U}, \quad \overline{p}^* = \frac{p}{\rho U^2}, \quad E^*(x^*, y^*) = \frac{E(x, y)}{E_0}, \tag{2.22}$$

where the Reynolds number Re is given by equation (2.19). For $E = E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)$ we use the same dimensionless variables as in (2.22) but with E^* as the exception:

$$E^*\left(x^*, y^*, \frac{\partial \overline{v}_x^*}{\partial y^*}\right) = \frac{E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}\right)}{E_C}, \tag{2.23}$$

and the Reynolds number Re is given by equation (2.20). Expressed in dimensionless variables, equations (2.17) and (2.18) become

$$\overline{v}_x{}^* \frac{\partial \overline{v}_x^*}{\partial x^*} + \overline{v}_y^* \frac{\partial \overline{v}_x^*}{\partial y^*} = -\frac{\partial \overline{p}^*}{\partial x^*} +$$

$$\frac{2}{Re} \frac{\partial}{\partial x^*}\left(E^*\left(x^*, y^*, \frac{\partial \overline{v}_x^*}{\partial y^*}\right) \frac{\partial \overline{v}_x^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(E^*\left(x^*, y^*, \frac{\partial \overline{v}_x^*}{\partial y^*}\right)\left(\frac{\partial \overline{v}_x^*}{\partial y^*} + \frac{1}{Re} \frac{\partial \overline{v}_y^*}{\partial x^*}\right)\right), \tag{2.24}$$

$$\frac{1}{Re}\overline{v}_x^*\frac{\partial \overline{v}_y^*}{\partial x^*}+\frac{1}{Re}\overline{v}_y^*\frac{\partial \overline{v}_y^*}{\partial y^*}=-\frac{\partial \overline{p}^*}{\partial y^*}+$$

$$\frac{\partial}{\partial x^*}\left(E^*\left(x^*,y^*,\frac{\partial \overline{v}_x^*}{\partial y^*}\right)\left(\frac{1}{Re}\frac{\partial \overline{v}_x^*}{\partial y^*}+\frac{1}{Re^2}\frac{\partial \overline{v}_y^*}{\partial x^*}\right)\right)+\frac{2}{Re}\frac{\partial}{\partial y^*}\left(E^*\left(x^*,y^*,\frac{\partial \overline{v}_x^*}{\partial y^*}\right)\frac{\partial \overline{v}_y^*}{\partial y^*}\right). \quad (2.25)$$

The conservation of mass equation, (2.4), remains unchanged in the new variables. Since $\delta = L/\sqrt{Re}$, a boundary layer exists provided $\sqrt{Re} \gg 1$. Neglecting terms of order $1/\sqrt{Re}$ and smaller, and suppressing the star to help keep the notation simple, equations (2.24) and (2.25) reduce to

$$\overline{v}_x\frac{\partial \overline{v}_x}{\partial x}+\overline{v}_y\frac{\partial \overline{v}_x}{\partial y}=-\frac{\partial \overline{p}}{\partial x}+\frac{\partial}{\partial y}\left(E\left(x,y,\frac{\partial \overline{v}_x}{\partial y}\right)\frac{\partial \overline{v}_x}{\partial y}\right), \quad (2.26)$$

$$\frac{\partial \overline{p}}{\partial y}=0. \quad (2.27)$$

The remaining equation is the conservation of mass equation (2.4).

We now consider the spacial gradient of the mean pressure $\overline{p}$. Since $\overline{p}$ does not depend on y, by (2.27), its value at any position x is determined by the corresponding mainstream conditions of the flow which are satisfied at $y = \pm y_b(x)$, where $y_b(x)$ is the boundary of the wake. If the boundary of the wake is infinite then $y_b(x) = \infty$. The uniform flow in the mainstream is in the x - direction with constant speed U. It is inviscid and satisfies Euler's equation, the x- component of which gives

$$0=-\frac{dp}{dx}. \quad (2.28)$$

Equation (2.26) therefore reduces to

$$\overline{v}_x\frac{\partial \overline{v}_x}{\partial x}+\overline{v}_y\frac{\partial \overline{v}_x}{\partial y}=\frac{\partial}{\partial y}\left(E\left(x,y,\frac{\partial \overline{v}_x}{\partial y}\right)\frac{\partial \overline{v}_x}{\partial y}\right). \quad (2.29)$$

Equations (2.4) and (2.29) are the dimensionless equations for the mean velocity components, $\overline{v}_x$ and $\overline{v}_y$, in a two-dimensional turbulent wake.

In dimensionless form the mainstream velocity is unity in the x- direction and zero in the y-direction. We write

$$\overline{v}_x(x,y)=1-\overline{w}(x,y), \quad \overline{v}(x,y)=0+\overline{v}(x,y), \quad (2.30)$$

where $\overline{w}(x,y)$ is the mean velocity deficit in the wake in the x-direction and we have replaced $\overline{v}_y$ with $\overline{v}$ for simplicity. Substituting (2.30) into (2.4) and (2.29) gives

$$-\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0, \tag{2.31}$$

$$\frac{\partial \overline{w}}{\partial x} - \overline{w}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} = \frac{\partial}{\partial y}\left(E\left(x, y, \frac{\partial \overline{w}}{\partial y}\right)\frac{\partial \overline{w}}{\partial y}\right). \tag{2.32}$$

Suppose now that we are sufficiently far downstream that $\overline{w}$ and $\overline{v}$ are small such that products and powers of $\overline{w}$ and $\overline{v}$ can be neglected. Then (2.32) reduces to the diffusion equation

$$\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left(E\left(x, y, \frac{\partial \overline{w}}{\partial y}\right)\frac{\partial \overline{w}}{\partial y}\right). \tag{2.33}$$

We now formulate the problem in terms of a stream function $\psi(x,y)$ defined by

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y}, \quad \overline{v}(x,y) = \frac{\partial \psi}{\partial x}. \tag{2.34}$$

This formulation ensures that the continuity equation is identically satisfied and reduces the number of unknowns and equations from two to one. Equation (2.33) takes the form

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left(E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right)\frac{\partial^2 \psi}{\partial y^2}\right). \tag{2.35}$$

In the next section the boundary conditions and conserved quantities are discussed.

2.1.1 Boundary conditions and conserved quantities

2.1.1.1 Classical wake and wake of a self-propelled body with $E = E(x,y)$

The boundary conditions on $\overline{w}$ and $\overline{v}$ are deduced as follows. As y tends to $\pm\infty$ the mean velocity deficit in the x-direction tends very slowly to zero. We assume that, as well as $\overline{w}$, the first derivative of $\overline{w}$ with respect to y tends to zero:

$$\overline{w}(x,\pm\infty) = 0, \quad \frac{\partial \overline{w}}{\partial y}(x,\pm\infty) = 0, \quad x \geq 0. \tag{2.36}$$

Also, the x- axis is an axis of symmetry of the wake. The mean velocity deficit $\overline{w}(x,y)$ is a maximum with respect to y at each point of the positive x - axis and therefore

$$\frac{\partial \overline{w}}{\partial y}(x,0) = 0, \quad x \geq 0. \tag{2.37}$$

Lastly, along the positive x-axis, the y- component of the mean velocity is zero:

$$\overline{v}(x,0) = 0, \quad x \geq 0. \tag{2.38}$$

In terms of the stream function the boundary conditions (2.36) to (2.38) become, respectively, for $x \geq 0$,

$$\frac{\partial \psi}{\partial y}(x, \pm\infty) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x, \pm\infty) = 0, \tag{2.39}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x,0) = 0, \quad \frac{\partial \psi}{\partial x}(x,0) = 0. \tag{2.40}$$

We will see later that not all the boundary conditions are independent of one another.

For the classical wake, the conserved quantity is obtained by integrating equation (2.35) from $y = -\infty$ to $y = \infty$ at a fixed point x [4]. The conserved quantity is

$$\int_{-\infty}^{\infty} \frac{\partial \psi}{\partial y} dy = D, \tag{2.41}$$

where D is proportional to the drag force. This method works for a constant eddy viscosity and an effective viscosity of the form $E(x, y)$.

In order to derive the conserved quantity for the wake behind a self-propelled body we multiply equation (2.35) by y^2 and then integrate across the wake from $y = -\infty$ to $y = \infty$ at a fixed point x [5]. This method works for a constant eddy viscosity and for an eddy viscosity depending on only the distance along the axis of the wake.

The conserved quantity for the wake of a self-propelled body is given by [5]

$$\int_{-\infty}^{\infty} y^2 \frac{\partial \psi}{\partial y} dy = K, \tag{2.42}$$

and since the drag force is also zero

$$\int_{-\infty}^{\infty} \frac{\partial \psi}{\partial y} dy = 0. \tag{2.43}$$

The constant K is proportional to the second moment of the axial momentum deficit.

2.1.1.2 Classical wake with $E = E\left(x, y, \dfrac{\partial \overline{w}}{\partial y}\right)$

The boundary conditions, in terms of the stream function ψ, are for $x \geq 0$,

$$\frac{\partial \psi}{\partial y}(x, \pm y_b) = 0, \qquad \frac{\partial^2 \psi}{\partial y^2}(x, \pm y_b) = 0, \tag{2.44}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x, 0) = 0, \qquad \frac{\partial \psi}{\partial x}(x, 0) = 0, \tag{2.45}$$

where the boundary $y = \pm y_b(x)$ is unspecified. If the wake extends to infinity in the y-direction then $y_b(x) = \infty$.

The conserved quantity for the wake is the drag force D [4]. Integrating (2.35) across the wake from $y = -y_b(x)$ to $y = +y_b(x)$ at a fixed point x gives

$$\int_{-y_b(x)}^{y_b(x)} \frac{\partial^2 \psi}{\partial x \partial y}(x, y)\, dy - E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right) \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=-y_b(x)}^{y=y_b(x)} = 0, \tag{2.46}$$

which on imposing the second derivative boundary condition in (2.44) and the theorem for differentiation under an integral sign [63] reduces to

$$\frac{d}{dx} \int_{-y_b(x)}^{y_b(x)} \frac{\partial \psi}{\partial y}(x, y)\, dy - y_b'(x) \frac{\partial \psi}{\partial y}(x, y_b(x)) - y_b'(x) \frac{\partial \psi}{\partial y}(x, -y_b(x)) = 0. \tag{2.47}$$

Hence, from the first derivative boundary condition in (2.44)

$$\int_{-y_b(x)}^{y_b(x)} \frac{\partial \psi}{\partial y}(x, y)\, dy = D, \tag{2.48}$$

where D is a dimensionless constant proportional to the drag on the body [4]. Equation (2.48) is used to determine the position of the boundary $y = \pm y_b(x)$. We note that

$$E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right) \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=-y_b(x)}^{y=y_b(x)} = 0,$$

if $\dfrac{\partial^2 \psi}{\partial y^2}$ vanishes or if $E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right)$ vanishes at $y = \pm y_b(x)$. We can thus also consider a model that replaces the second derivative boundary condition in (2.44) with

$$E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right)\bigg|_{y=\pm y_b(x)} = 0. \tag{2.49}$$

We will not investigate this further.

The right-hand side of all the boundary conditions is zero. The boundary conditions can therefore be described as homogeneous boundary conditions. As in other problems described by Prandtl's boundary layer equations no condition is placed on $\overline{v}_y = \dfrac{\partial \psi}{\partial x}$ at the boundaries $y = \pm\infty$ and $y = \pm y_b(x)$.

Chapter 3

Derivation of the conservation laws for the two-dimensional turbulent wake using operator methods

In this chapter we calculate the conservation laws for the turbulent wake. The relevant theory is outlined in Section 3.1. In Section 3.2 we consider the turbulent classical wake with an eddy viscosity depending on the spacial variables and the velocity gradient in the y- direction. This model describes a wake with a finite boundary given by Prandtl's model and a revised Prandtl model which is formulated in Chapter 6. The elementary conserved vector is derived in terms of the stream function and the velocity components. The conserved quantity is also obtained. In Section 3.3 we analyse wakes with infinite boundaries that are generated by eddy viscosities depending on the spacial variables only. We calculate the conserved vectors in terms of the velocity components and the stream function. In Section 3.4 the conserved quantities for the classical wake for $E = E(x, y)$ and the wake of a self-propelled body for $E = E(x)$ are derived. A third conserved quantity for $E = E(x)$ is obtained. This conserved quantity is generated using the same boundary conditions at $y = \pm\infty$ as that of the classical wake and the wake of a self-propelled body. Conclusions for this chapter are given in Section 3.5.

3.1 Conservation law theory

In this section, the theory and techniques required for the remainder of this chapter are provided. In particular, conservation law theory is discussed.

Consider a p-th order system of PDEs of n independent variables and m dependent variables:

$$F_j(x,\psi,\psi_{(1)},...,\psi_{(p)})=0, \quad j=1,\ 2,\ ...,\ m, \tag{3.1}$$

where $x=(x^1,x^2,...,x^n)$ and $\psi=(\psi^1,\psi^2,...,\psi^m)$ denote the independent and dependent variables respectively. The collection of i-th order partial derivatives of the dependent variables is denoted by $\psi_{(i)}$. System (3.1) is assumed to be of maximal rank and is locally solvable.

The total derivative of ψ^j with respect to x^i is written as $\psi^j_i=D_i(\psi^j)$, $\psi^j_{ik}=D_kD_i(\psi^j)$, ..., where the total derivative operator with respect to x^i is given by

$$D_i=\frac{\partial}{\partial x^i}+\psi^j_i\frac{\partial}{\partial\psi^j}+\psi^j_{ik}\frac{\partial}{\partial\psi^j_k}+..., \quad i=1,2,...,n. \tag{3.2}$$

The Euler operator, which annihilates a differential function if and only if the differential function is a total divergence, is defined by

$$E^*_{\psi^j}=\frac{\partial}{\partial\psi^j}+\sum_{s\geq1}(-1)^sD_{i_1}...D_{i_s}\frac{\partial}{\partial\psi^j_{i_1...i_s}}, \quad j=1,2,...,m,\ \ i_1\leq i_2\leq...\leq i_s. \tag{3.3}$$

The symmetry operator is

$$X=\xi^i\frac{\partial}{\partial x^i}+\eta^j\frac{\partial}{\partial\psi^j}+\sum_{s\geq1}\zeta^j_{i_1...i_s}\frac{\partial}{\partial\psi^j_{i_1...i_s}}, \quad i_1\leq i_2\leq...\leq i_s, \tag{3.4}$$

where $\zeta^j_{i_1...i_s}$ are given by

$$\zeta^j_i=D_i(\eta^j)-\psi^j_kD_i(\xi^k), \tag{3.5}$$

$$\zeta^j_{i_1,...i_s}=D_{i_s}(\zeta^j_{i_1...i_{s-1}})-\psi^j_{ki_1...i_{s-1}}D_{i_s}(\xi^k), \quad s>1. \tag{3.6}$$

A set of functions $T^1,\ T^2,\ ...,\ T^n$ consists of the components of a local conservation law if and only if they satisfy

$$D_iT^i=0, \tag{3.7}$$

for all solutions of (3.1).

The characteristic or multiplier approach developed by Steudal [28], can be used to generate conservation laws for systems of PDEs. The multiplier Λ has components $\Lambda^1, \Lambda^2, ..., \Lambda^n$ which can depend on the independent and dependent variables as well as the partial derivatives of at most up to p-th order of the dependent variables. The multiplier Λ for the system of PDES (3.1) has the property that

$$\Lambda^j F_j(x, \psi, \psi_{(1)}, ..., \psi_{(k)}) = D_j T^j, \tag{3.8}$$

for all functions ψ. The determining equations for the multiplier Λ are found by applying the Euler operator to equation (3.8) which gives

$$E^*_{\psi^j}\left[\Lambda^k F_k\right] = 0, \quad j = 1, 2, ..., m. \tag{3.9}$$

The determining equations (3.9) are satisfied for all functions $\psi = (\psi^1, \psi^2, ..., \psi^m)$ and not only for solutions of the system (3.1). Since the partial derivatives of ψ^j are therefore independent, the determining equations (3.9) are solved by separating by powers and products of the partial derivatives of ψ^j. Once the multiplier Λ has been determined we now consider for ψ solutions of the system of PDEs (3.1). The components of the conserved vector T are calculated by performing elementary manipulations on equation (3.8).

The conserved vector $T = (T^1, T^2, ..., T^n)$ is invariant under the action of the generator (3.4) provided [30, 31]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2, ..., n, \tag{3.10}$$

where X is prolongated to as high an order in the derivatives as required. Any generator (3.4) satisfying (3.10) is a Lie point symmetry of the system of PDES (3.1). Once the Lie point symmetry X has been obtained the invariant solution corresponding to this X can be derived.

3.2 Elementary conserved quantity for the turbulent classical wake with $E = E\left(x, y, \dfrac{\partial^2\psi}{\partial y^2}\right)$

In this section we consider the elementary conserved vector for the turbulent classical wake with $E = E\left(x, y, \dfrac{\partial^2\psi}{\partial y^2}\right)$. It is shown in Chapter 6 that this model predicts a finite wake boundary. We first calculate the elementary conserved vector in terms of the stream function. Since, for the elementary conserved vector the multiplier Λ is simply equal to 1, once we have calculated the elementary conserved vector in terms of the stream function we can use the definition of the stream function, namely, $\overline{w} = \dfrac{\partial\psi}{\partial y}$ and $\overline{v} = \dfrac{\partial\psi}{\partial x}$, in order to directly find the elementary conserved vector in terms of the velocity components. In this section we assume that $\dfrac{\partial E}{\partial \psi_{yy}} \neq 0$.

3.2.1 Elementary conserved vector in terms of the stream function

In this section we derive the elementary conserved vector for the partial differential equation (2.35). In [23, 24, 29] it is shown that there is a one-to-one correspondence between non-trivial conserved vectors and non-zero multipliers. The suffix notation ψ_x, ψ_y, ψ_{xy}, ... is used when x, y, ψ and the partial derivatives of ψ are regarded as independent variables. The notation $\dfrac{\partial\psi}{\partial x}$, $\dfrac{\partial\psi}{\partial y}$, $\dfrac{\partial^2\psi}{\partial x\partial y}$, ... is used when ψ and the partial derivatives of ψ are regarded as dependent variables which are functions of the independent variables x and y.

In order to derive the elementary conserved vector we let x, y, ψ and all partial derivatives of ψ be independent variables. We can write equation (2.35) as

$$\psi_{xy} - \frac{\partial E}{\partial y}(x, y, \psi_{yy})\psi_{yy} - \frac{\partial E}{\partial \psi_{yy}}\psi_{yyy}\psi_{yy} - E(x, y, \psi_{yy})\psi_{yyy} = 0. \tag{3.11}$$

A conservation law for the partial differential equation (2.35) satisfies

$$\left(D_1 T^1 + D_2 T^2\right)\big|_{(3.11)} = 0, \tag{3.12}$$

where

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx}\frac{\partial}{\partial \psi_x} + \psi_{yx}\frac{\partial}{\partial \psi_y} + \ldots, \tag{3.13}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \dots. \tag{3.14}$$

The components of the corresponding conserved vector $T = (T^1, T^2)$ are

$$T^1 = T^1(x, y, \psi, \psi_x, \psi_y, \dots), \quad T^2 = T^2(x, y, \psi, \psi_x, \psi_y, \dots). \tag{3.15}$$

The elementary conserved vector for the partial differential equation (2.35) is

$$T^1(\psi_y) = \psi_y, \quad T^2(x, y, \psi_{yy}) = -E(x, y, \psi_{yy})\psi_{yy}, \tag{3.16}$$

which can be verified directly by substituting (3.16) into (3.12).

3.2.2 Elementary conserved vector in terms of the velocity components

Using the definition of the stream function ψ, the conserved vector in (3.16) in terms of the velocity components is

$$T^1(\overline{w}) = \overline{w}, \quad T^2(x, y, \overline{w}_y) = -E(x, y, \overline{w}_y)\overline{w}_y. \tag{3.17}$$

From the continuity equation (2.31) we also obtain another elementary conserved vector, namely,

$$T^1 = -\overline{w}, \quad T^2 = \overline{v}. \tag{3.18}$$

3.2.3 Conserved quantity

On solutions $\psi = \psi(x, y)$ of the governing partial differential equation (2.35), the components T^1 and T^2 of the conserved vector can be regarded as functions of x and y. Therefore, we have for the stream function

$$\frac{\partial}{\partial x} T^1(x, y, \psi(x, y), \psi_x(x, y), \psi_y(x, y), \dots) = D_x T^1, \tag{3.19}$$

$$\frac{\partial}{\partial y} T^2(x, y, \psi(x, y), \psi_x(x, y), \psi_y(x, y), \dots) = D_y T^2, \tag{3.20}$$

and for the velocity components

$$\frac{\partial}{\partial x} T^1(x, y, \overline{w}, \overline{v}, \overline{w}_x, \overline{w}_y, \overline{v}_x, \overline{v}_y, ...) = D_x T^1, \tag{3.21}$$

$$\frac{\partial}{\partial y} T^2(x, y, \overline{w}, \overline{v}, \overline{w}_x, \overline{w}_y, \overline{v}_x, \overline{v}_y, ...) = D_y T^2, \tag{3.22}$$

and so

$$D_x T^1 + D_y T^2 = \frac{\partial}{\partial x} T^1 + \frac{\partial}{\partial y} T^2. \tag{3.23}$$

For a conserved vector the left hand side of equation (3.23) vanishes and the conservation law (3.23) can be written as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{3.24}$$

The elementary conserved quantity for the classical wake with $E = E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right)$ can be derived using both the velocity components and the stream function. In terms of the velocity components, using T^1 and T^2 defined in (3.17), the conserved quantity is found by integrating (3.24) across the wake from $y = -y_b(x)$ to $y = +y_b(x)$ at a fixed point x. Integrating (3.24) across the wake gives

$$\int_{-y_b(x)}^{y_b(x)} \frac{\partial \overline{w}}{\partial x}(x, y) dy - E\left(x, y, \frac{\partial \overline{w}}{\partial y}\right) \frac{\partial \overline{w}}{\partial y}\bigg|_{y=-y_b(x)}^{y=y_b(x)} = 0. \tag{3.25}$$

Since $\overline{w} = \frac{\partial \psi}{\partial y}$ we can use the second derivative boundary condition in (2.44) and the theorem for differentiation under an integral sign [63] to obtain

$$\frac{d}{dx} \int_{-y_b(x)}^{y_b(x)} \overline{w}(x, y) dy - y_b'(x) \overline{w}(x, y_b(x)) - y_b'(x) \overline{w}(x, -y_b(x)) = 0, \tag{3.26}$$

and from the first derivative boundary condition in (2.44)

$$\int_{-y_b(x)}^{y_b(x)} \overline{w}(x, y) dy = D, \tag{3.27}$$

where D is a dimensionless constant proportional to the drag on the body [4]. Equation (3.27) is used to determine the position of the boundary $y = \pm y_b(x)$.

Integrating equation (3.24) across the wake from $y = -y_b(x)$ to $y = +y_b(x)$ at a fixed point x using the conserved vector given in (3.18) results in

$$\int_{-y_b(x)}^{y_b(x)} \left(-\frac{\partial \overline{w}}{\partial x}(x,y) + \frac{\partial \overline{v}}{\partial y}(x,y)\right) dy = 0, \tag{3.28}$$

since the continuity equation is identically satisfied. The conserved vector in (3.18) does not generate a conserved quantity.

In terms of the stream function, using T^1 and T^2 defined in (3.16), and integrating (3.24) across the wake from $y = -y_b(x)$ to $y = +y_b(x)$ at a fixed point x results in

$$\int_{-y_b(x)}^{y_b(x)} \frac{\partial^2 \psi}{\partial x \partial y}(x,y)dy - E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right) \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=-y_b(x)}^{y=y_b(x)} = 0, \tag{3.29}$$

and by using the first and second derivative boundary conditions in (2.44) and the theorem for differentiation under an integral sign [63] we obtain

$$\int_{-y_b(x)}^{y_b(x)} \frac{\partial \psi}{\partial y}(x,y)dy = D, \tag{3.30}$$

which is equivalent to the expression in (3.27).

3.3 Conserved vectors for the turbulent wake with $E = E(x,y)$

In this section, we consider wakes with infinite boundaries. These are described by eddy viscosities depending on the spacial variables only. We are no longer restricted to the elementary conserved vector. We first calculate the conserved vectors in terms of the velocity components and then we consider the stream function. We then proceed to calculate the conserved quantities.

3.3.1 Conserved vectors for the turbulent wake in terms of the velocity components

In terms of the x- and y- mean velocity components $\overline{v}_x$ and $\overline{v}$ the dimensionless governing equations for the two-dimensional turbulent wake are

$$-\frac{\partial \overline{w}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0, \tag{3.31}$$

$$\frac{\partial \overline{w}}{\partial x}=\frac{\partial}{\partial y}\left(E(x,y)\frac{\partial \overline{w}}{\partial y}\right), \tag{3.32}$$

where the dimensionless mean velocity deficit $\overline{w}(x,y)$ is defined from (2.30)

$$\overline{v}_x(x,y)=1-\overline{w}(x,y). \tag{3.33}$$

In this section we apply the multiplier method to equations (3.31) and (3.32). We use the following notation: the suffix notation $\overline{w}_x$, $\overline{w}_y$, $\overline{v}_x$, $\overline{v}_y$, ... is used when x, y, $\overline{w}$, $\overline{v}$ and the partial derivatives of $\overline{w}$ and $\overline{v}$ are regarded as independent variables. The notation $\frac{\partial \overline{w}}{\partial x}, \frac{\partial \overline{v}}{\partial x}, \frac{\partial \overline{w}}{\partial y}, \frac{\partial \overline{v}}{\partial y}, \frac{\partial^2 \overline{w}}{\partial x\partial y}$, ... is used when $\overline{w}$ and $\overline{v}$ and the partial derivatives of $\overline{w}$ and $\overline{v}$ are regarded as dependent variables which are functions of the independent variables x and y.

In order to derive the conserved vectors we let x, y, $\overline{w}$, $\overline{v}$ and all partial derivatives of $\overline{w}$ and $\overline{v}$ be independent variables. We can write (3.31) and (3.32) as

$$-\overline{w}_x+\overline{v}_y=0, \tag{3.34}$$

$$\overline{w}_x-\frac{\partial E}{\partial y}\overline{w}_y-E(x,y)\overline{w}_{yy}=0. \tag{3.35}$$

Given the multiplier $\Lambda=(\Lambda_1,\Lambda_2)$, we have the property that the conserved form of (3.34) and (3.35) is given by

$$\Lambda_1\left(\overline{w}_x-\frac{\partial E}{\partial y}\overline{w}_y-E(x,y)\overline{w}_{yy}\right)+\Lambda_2\left(-\overline{w}_x+\overline{v}_y\right)=D_1T^1+D_2T^2, \tag{3.36}$$

for all functions $\overline{w}(x,y)$ and $\overline{v}(x,y)$ where

$$D_1=D_x=\frac{\partial}{\partial x}+\overline{w}_x\frac{\partial}{\partial \overline{w}}+\overline{v}_x\frac{\partial}{\partial \overline{v}}+\overline{w}_{xx}\frac{\partial}{\partial \overline{w}_x}+\overline{v}_{xx}\frac{\partial}{\partial \overline{v}_x}+\overline{w}_{xy}\frac{\partial}{\partial \overline{w}_y}+\overline{v}_{xy}\frac{\partial}{\partial \overline{v}_y}+..., \tag{3.37}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \overline{w}_y \frac{\partial}{\partial \overline{w}} + \overline{v}_y \frac{\partial}{\partial \overline{v}} + \overline{w}_{yy} \frac{\partial}{\partial \overline{w}_y} + \overline{v}_{yy} \frac{\partial}{\partial \overline{v}_y} + \overline{w}_{yx} \frac{\partial}{\partial \overline{w}_x} + \overline{v}_{yx} \frac{\partial}{\partial \overline{v}_x} + \ldots, \tag{3.38}$$

and T^1 and T^2 are the components of the conserved vector $T = (T^1, T^2)$. The determining equations for the multiplier Λ are given by

$$E^*_{\overline{w}} \left[\Lambda_1 \left(\overline{w}_x - \frac{\partial E}{\partial y}(x,y)\overline{w}_y - E(x,y)\overline{w}_{yy} \right) + \Lambda_2 \left(-\overline{w}_x + \overline{v}_y \right) \right] = 0, \tag{3.39}$$

$$E^*_{\overline{v}} \left[\Lambda_1 \left(\overline{w}_x - \frac{\partial E}{\partial y}(x,y)\overline{w}_y - E(x,y)\overline{w}_{yy} \right) + \Lambda_2 \left(-\overline{w}_x + \overline{v}_y \right) \right] = 0, \tag{3.40}$$

where the operators $E^*_{\overline{w}}$ and $E^*_{\overline{v}}$ are the standard Euler operators that annihilate divergence expressions:

$$E^*_{\overline{w}} = \frac{\partial}{\partial \overline{w}} - D_x \frac{\partial}{\partial \overline{w}_x} - D_y \frac{\partial}{\partial \overline{w}_y} + D_x^2 \frac{\partial}{\partial \overline{w}_{xx}} + D_x D_y \frac{\partial}{\partial \overline{w}_{xy}} + D_y^2 \frac{\partial}{\partial \overline{w}_{yy}} - \ldots, \tag{3.41}$$

$$E^*_{\overline{v}} = \frac{\partial}{\partial \overline{v}} - D_x \frac{\partial}{\partial \overline{v}_x} - D_y \frac{\partial}{\partial \overline{v}_y} + D_x^2 \frac{\partial}{\partial \overline{v}_{xx}} + D_x D_y \frac{\partial}{\partial \overline{v}_{xy}} + D_y^2 \frac{\partial}{\partial \overline{v}_{yy}} - \ldots. \tag{3.42}$$

For components of the multiplier Λ of the form $\Lambda_1 = \Lambda_1(x,y)$ and $\Lambda_2 = \Lambda_2(x,y)$, equations (3.39) and (3.40) become

$$\left[-D_x \frac{\partial}{\partial \overline{w}_x} - D_y \frac{\partial}{\partial \overline{w}_y} + D_y^2 \frac{\partial}{\partial \overline{w}_{yy}} \right] \left[\Lambda_1 \left(\overline{w}_x - \frac{\partial E}{\partial y}(x,y)\overline{w}_y - E(x,y)\overline{w}_{yy} \right) \right] +$$

$$\left[D_x \frac{\partial}{\partial \overline{w}_x} \right] \left[\Lambda_2 \overline{w}_x \right] = 0, \tag{3.43}$$

$$D_y \Lambda_2 = 0, \tag{3.44}$$

which, after simplifying gives

$$\frac{\partial \Lambda_1}{\partial x} - \frac{\partial \Lambda_2}{\partial x} + \frac{\partial}{\partial y} \left(\frac{\partial \Lambda_1}{\partial y} E(x,y) \right) = 0, \tag{3.45}$$

$$\Lambda_2 = \Lambda_2(x). \tag{3.46}$$

It is difficult to derive definite results when $E(x,y)$ depends on y. Two cases will be considered for which definite results can be derived.

3.3.1.1 Case (i): $\Lambda_1 = c_1$, $E = E(x,y)$

Equation (3.45) reduces to

$$\frac{\partial \Lambda_2}{\partial x}(x) = 0, \tag{3.47}$$

and therefore

$$\Lambda_2 = c_2, \tag{3.48}$$

where c_2 is a constant. This result holds for arbitrary $E(x, y)$. The left-hand-side of (3.36) with $\Lambda_1 = c_1$ and $\Lambda_2 = c_2$ can be written in divergence form because c_1 and c_2 are multipliers. Now by elementary manipulations,

$$c_1\left[\overline{w}_x - \frac{\partial E}{\partial y}\overline{w}_y - E(x,y)\overline{w}_{yy}\right] + c_2\left[-\overline{w}_x + \overline{v}_y\right] =$$

$$D_x\left[c_1\overline{w} + c_2(-\overline{w})\right] + D_y\left[c_1(-E(x,y)\overline{w}_y + c_2\overline{v}\right], \tag{3.49}$$

for arbitrary functions $\overline{w}(x, y)$ and $\overline{v}(x, y)$. When $\overline{w}(x, y)$ and $\overline{v}(x, y)$ are solutions of the system of PDEs, (3.31) and (3.32), (3.49) becomes

$$D_x\left[c_1\overline{w} + c_2(-\overline{w})\right] + D_y\left[c_1(-E(x,y)\overline{w}_y + c_2\overline{v}\right] = 0. \tag{3.50}$$

Two conserved vectors are obtained. Let $c_1 = 1$ and $c_2 = 0$. Then

$$T^1 = \overline{w}, \quad T^2 = -E(x,y)\overline{w}_y. \tag{3.51}$$

Let $c_1 = 0$ and $c_2 = 1$. Then

$$T^1 = -\overline{w}, \quad T^2 = \overline{v}. \tag{3.52}$$

The conserved vectors (3.51) and (3.52) are the elementary conserved vectors for the system (3.31) and (3.32).

3.3.1.2 Case (ii): $\Lambda_1 = \Lambda_1(y)$, $E = E(x)$

Equation (3.45) reduces to

$$-\frac{d\Lambda_2}{dx} + E(x)\frac{d^2\Lambda_1}{dy^2} = 0. \tag{3.53}$$

Differentiating equation (3.53) with respect to y gives

$$\frac{d^3\Lambda_1}{dy^3}(y) = 0, \tag{3.54}$$

and therefore

$$\Lambda_1(y) = a_1 y^2 + a_2 y + a_3, \tag{3.55}$$

where a_1, a_2 and a_3 are constants. Equation (3.53) becomes

$$\frac{d\Lambda_2}{dx}(x) = 2a_1 E(x), \tag{3.56}$$

so that

$$\Lambda_2(x) = 2a_1 \int_0^x E(\alpha)d\alpha + a_4, \tag{3.57}$$

where a_4 is a constant. Equation (3.57) holds for arbitrary $E(x)$. We have established that the left-hand-side of (3.36) with $E = E(x)$ and Λ_1 and Λ_2 given by (3.55) and (3.57) can be written in divergence form. By elementary manipulations we find that

$$\begin{aligned}
&\left(a_1 y^2 + a_2 y + a_3\right)\left(\overline{w}_x - E(x)\overline{w}_{yy}\right) + \left(2a_1 \int_0^x E(\alpha)d\alpha + a_4\right)\left(-\overline{w}_x + \overline{v}_y\right) \\
&= D_x \left[a_1 \left(y^2 - 2\int_0^x E(\alpha)d\alpha \right)\overline{w} + a_2\left(y\overline{w}\right) + a_3\left(\overline{w}\right) + a_4\left(-\overline{w}\right)\right] + \\
&D_y \left[a_1 \left(-y^2 E(x)\overline{w}_y + 2yE(x)\overline{w} + 2\int_0^x E(\alpha)d\alpha\overline{v}\right) + \right. \\
&\left. a_2\left(-yE(x)\overline{w}_y + E(x)\overline{w}\right) + a_3\left(-E(x)\overline{w}_y\right) + a_4\left(\overline{v}\right)\right],
\end{aligned} \tag{3.58}$$

for arbitrary functions $\overline{w}(x,y)$ and $\overline{v}(x,y)$. When $\overline{w}(x,y)$ and $\overline{v}(x,y)$ are solutions of the system (3.31) and (3.32) then equation (3.58) reduces to

$$\begin{aligned}
&D_x \left[a_1 \left(y^2 - 2\int_0^x E(\alpha)d\alpha \right)\overline{w} + a_2\left(y\overline{w}\right) + a_3\left(\overline{w}\right) + a_4\left(-\overline{w}\right)\right] + \\
&D_y \left[a_1 \left(-y^2 E(x)\overline{w}_y + 2yE(x)\overline{w} + 2\int_0^x E(\alpha)d\alpha\overline{v}\right) + \right. \\
&\left. a_2\left(-yE(x)\overline{w}_y + E(x)\overline{w}\right) + a_3\left(-E(x)\overline{w}_y\right) + a_4\left(\overline{v}\right)\right] = 0.
\end{aligned} \tag{3.59}$$

Four conserved vectors are obtained by setting all except one of a_1, a_2, a_3 and a_4 to zero in turn. For $a_1 = 1$, $a_2 = a_3 = a_4 = 0$ the conserved vector is

$$T^1 = \left(y^2 - 2\int_0^x E(\alpha)d\alpha\right)\overline{w}, \quad T^2 = -y^2 E(x)\overline{w}_y + 2yE(x)\overline{w} + 2\int_0^x E(\alpha)d\alpha\overline{v}. \tag{3.60}$$

For $a_1 = 0$, $a_2 = 1$, $a_3 = a_4 = 0$ we have

$$T^1 = y\overline{w}, \quad T^2 = -yE(x)\overline{w}_y + E(x)\overline{w}. \tag{3.61}$$

For $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 0$ the conserved vector is

$$T^1 = \overline{w}, \quad T^2 = -E(x)\overline{w}_y. \tag{3.62}$$

Lastly, for $a_1 = a_2 = a_3 = 0$, $a_4 = 1$ we have for the conserved vector

$$T^1 = -\overline{w}, \quad T^2 = \overline{v}. \tag{3.63}$$

The conserved vectors (3.62) and (3.63) are the same as (3.51) with $E = E(x)$ and (3.52).

We have therefore established the following result for a multiplier with components

$$\Lambda_1 = \Lambda_1(y), \quad \Lambda_2 = \Lambda_2(x, y), \tag{3.64}$$

of the system of PDEs, (3.31) and (3.32). If $\dfrac{\partial E}{\partial y} \neq 0$ then there are two conserved vectors given by (3.51) and (3.52) and any conserved vector of the system with a multiplier of the form (3.64) is a linear combination of these two conserved vectors.

If $E = E(x)$ there are two further conserved vectors given by (3.60) and (3.61) and any conserved vector with a multiplier of the form (3.64) is a linear combination of the four conserved vectors given by (3.60) to (3.63).

3.3.2 Conserved vectors for the turbulent wake in terms of the stream function

When x, y, ψ and the partial derivatives of ψ are regarded as independent variables, equation (2.35) with $E = E(x, y)$ is written as

$$\psi_{xy} - \frac{\partial E}{\partial y}(x, y)\psi_{yy} - E(x, y)\psi_{yyy} = 0. \tag{3.65}$$

Consider a multiplier Λ of the form $\Lambda = \Lambda(x, y)$. The conserved form of (3.65) is given by

$$\Lambda\left(\psi_{xy} - \frac{\partial E}{\partial y}(x, y)\psi_{yy} - E(x, y)\psi_{yyy}\right) = D_1 T^1 + D_2 T^2, \tag{3.66}$$

for all functions $\psi(x, y)$ where the total derivative operators D_x and D_y are defined by

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \ldots, \tag{3.67}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \ldots, \tag{3.68}$$

and T^1 and T^2 are the components of the conserved vector $T = (T^1, T^2)$. The determining equation for the multiplier Λ is given by

$$E_\psi^* \left[\Lambda \left(\psi_{xy} - \frac{\partial E}{\partial y}(x,y)\psi_{yy} - E(x,y)\psi_{yyy} \right) \right] = 0, \tag{3.69}$$

where the operator E_ψ^* is the standard Euler operator that annihilates divergence expressions:

$$E_\psi^* = \frac{\partial}{\partial \psi} - D_x \frac{\partial}{\partial \psi_x} - D_y \frac{\partial}{\partial \psi_y} + D_x^2 \frac{\partial}{\partial \psi_{xx}} + D_x D_y \frac{\partial}{\partial \psi_{xy}} + D_y^2 \frac{\partial}{\partial \psi_{yy}} - \ldots. \tag{3.70}$$

The determining equation for Λ becomes

$$D_x D_y(\Lambda) - D_y^2 \left(\frac{\partial E}{\partial y} \Lambda \right) + D_y^3 (\Lambda E) = 0, \tag{3.71}$$

which after simplifying gives

$$\frac{\partial^2 \Lambda}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \left(\frac{\partial \Lambda}{\partial y} E \right) = 0. \tag{3.72}$$

We will again consider the cases where E depends on y and when E is independent of y in order to obtain conclusive results.

3.3.2.1 Case (i): $\Lambda = c_1$, $E = E(x,y)$

For $\Lambda = c_1$ where c_1 is a constant, equation (3.72) is satisfied for arbitrary $E(x,y)$. Since $\Lambda = c_1$ is a multiplier, we can write the left-hand-side of (3.66) in divergence form:

$$c_1 \left[\psi_{xy} - \frac{\partial E}{\partial y}(x,y)\psi_{yy} - E(x,y)\psi_{yyy} \right] = D_x \left[c_1 \psi_y \right] + D_y \left[c_1 (-E(x,y)\psi_{yy} \right], \tag{3.73}$$

for arbitrary $\psi(x,y)$. When $\psi(x,y)$ is a solution of equation (3.65) then (3.73) reduces to

$$D_x \left[c_1 \psi_y \right] + D_y \left[c_1 (-E(x,y)\psi_{yy} \right] = 0. \tag{3.74}$$

By setting $c_1 = 1$ a single conserved vector is obtained:

$$T^1 = \psi_y, \quad T^2 = -E(x,y)\psi_{yy}, \tag{3.75}$$

which is the elementary conserved vector for equation (3.65).

3.3.2.2 Case (ii): $\Lambda = \Lambda(y)$, $E = E(x)$

Equation (3.72) reduces to

$$\frac{\partial^3 \Lambda}{\partial y^3}(y) = 0, \tag{3.76}$$

which therefore gives a multiplier of the form

$$\Lambda(y) = a_1 y^2 + a_2 y + a_3, \tag{3.77}$$

which is satisfied for arbitrary $E(x)$ where a_1, a_2 and a_3 are constants. In divergence form, equation (3.65) is written as

$$(a_1 y^2 + a_2 y + a_3)(\psi_{xy} - E(x)\psi_{yyy}) = D_x\left[(a_1 y^2 + a_2 y + a_3)\psi_y\right] +$$

$$D_y\left[a_1\left(-y^2\psi_{yy} + 2y\psi_y - 2\psi\right)E(x) + a_2(-y\psi_{yy} + \psi_y)E(x) + a_3(-\psi_{yy}E(x))\right], \tag{3.78}$$

for arbitrary $\psi(x,y)$. When $\psi(x,y)$ is a solution to equation (3.65) then equation (3.78) becomes

$$D_x\left[(a_1 y^2 + a_2 y + a_3)\psi_y\right] +$$

$$D_y\left[a_1\left(-y^2\psi_{yy} + 2y\psi_y - 2\psi\right)E(x) + a_2\left(-y\psi_{yy} + \psi_y\right)E(x) + a_3\left(-\psi_{yy}E(x)\right)\right] = 0. \tag{3.79}$$

Three conserved vectors are obtained by setting all except one of a_1, a_2 and a_3 to zero in turn. For $a_1 = 1$, $a_2 = a_3 = 0$ the conserved vector is

$$T^1 = y^2\psi_y, \quad T^2 = \left(-y^2\psi_{yy} + 2y\psi_y - 2\psi\right)E(x). \tag{3.80}$$

For $a_1 = 0$, $a_2 = 1$, $a_3 = 0$ we have

$$T^1 = y\psi_y, \quad T^2 = -yE(x)\psi_{yy} + E(x)\psi_y, \tag{3.81}$$

and finally for $a_1 = a_2 = 0$, $a_3 = 1$ the conserved vector is

$$T^1 = \psi_y, \quad T^2 = -E(x)\psi_{yy}. \tag{3.82}$$

3.4 Conserved quantities for the two-dimensional wake

In this section we calculate the conserved quantities for the classical wake and the wake of a self-propelled body. We also discuss a third conservation law belonging to a wake which we call the combination wake. This wake is important since it has the same boundary conditions at $y = \pm\infty$ as that of the classical wake and the wake of a self-propelled body.

On solutions $\psi = \psi(x, y)$ of the governing partial differential equation (2.35), the components T^1 and T^2 of the conserved vector can be regarded as functions of x and y. Therefore, we have for the stream function

$$\frac{\partial}{\partial x} T^1(x, y, \psi(x, y), \psi_x(x, y), \psi_y(x, y), ...) = D_x T^1, \tag{3.83}$$

$$\frac{\partial}{\partial y} T^2(x, y, \psi(x, y), \psi_x(x, y), \psi_y(x, y), ...) = D_y T^2, \tag{3.84}$$

and for the velocity components

$$\frac{\partial}{\partial x} T^1(x, y, \overline{w}, \overline{v}, \overline{w}_x, \overline{w}_y, \overline{v}_x, \overline{v}_y, ...) = D_x T^1, \tag{3.85}$$

$$\frac{\partial}{\partial y} T^2(x, y, \overline{w}, \overline{v}, \overline{w}_x, \overline{w}_y, \overline{v}_x, \overline{v}_y, ...) = D_y T^2, \tag{3.86}$$

and therefore

$$D_x T^1 + D_y T^2 = \frac{\partial}{\partial x} T^1 + \frac{\partial}{\partial y} T^2. \tag{3.87}$$

For a conserved vector the left hand side of equation (3.87) vanishes and the conservation law (3.87) can be written as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{3.88}$$

A general form for the conserved vector components in terms of the stream function for Case (ii) in Section 3.3.2.2 is given by

$$T^1 = (a_1 y^2 + a_2 y + a_3)\psi_y,$$

$$T^2 = a_1(-y^2\psi_{yy} + 2y\psi_y - 2\psi)E(x) + a_2(-y\psi_{yy} + \psi_y)E(x) + a_3(-\psi_{yy})E(x). \quad (3.89)$$

Equation (3.89) also holds for Case (i) in Section 3.3.2.1 with $a_1 = a_2 = 0$ and $E = E(x, y)$.

Using (3.88) and T^1 and T^2 defined by (3.89), and integrating (3.88) across the wake from $y = -\infty$ to $y = +\infty$ at a fixed point x gives

$$\frac{d}{dx}\int_{-\infty}^{\infty}(a_1 y^2 + a_2 y + a_3)\frac{\partial\psi}{\partial y}(x, y)dy +$$
$$\left[a_1\left(-y^2\frac{\partial^2\psi}{\partial y^2} + 2y\frac{\partial\psi}{\partial y} - 2\psi\right)E(x) + a_2\left(-y\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\psi}{\partial y}\right)E(x) + a_3\left(-\frac{\partial^2\psi}{\partial y^2}\right)E(x)\right]\Bigg|_{y=-\infty}^{y=\infty}$$
$$= 0, \quad (3.90)$$

which on imposing the boundary conditions in (2.39) reduces to

$$\frac{d}{dx}\int_{-\infty}^{\infty}(a_1 y^2 + a_2 y + a_3)\frac{\partial\psi}{\partial y}(x, y)dy - \left[2a_1\psi(x, y)E(x)\right]\Big|_{y=-\infty}^{y=\infty} = 0. \quad (3.91)$$

The conserved vector in terms of the velocity components can be written for Case (ii) in Section 3.3.1.2 as the linear combination:

$$T^1 = a_1\left(y^2 - 2\int_0^x E(\alpha)d\alpha\right)\overline{w} + a_2\left(y\overline{w}\right) + a_3(\overline{w}) + a_4(-\overline{w}),$$

$$T^2 = a_1(-y^2E(x)\overline{w}_y + 2yE(x)\overline{w} + 2\int_0^x E(\alpha)d\alpha\overline{v}) +$$
$$a_2(-y\overline{w}_y + \overline{w})E(x) + a_3(-E(x)\overline{w}_y) + a_4(\overline{v}). \quad (3.92)$$

Equation (3.92) also holds for Case (i) in Section 3.3.1.1 with $a_1 = a_2 = 0$ and $E = E(x, y)$.

Integrating (3.88) across the wake from $y = -\infty$ to $y = +\infty$ at a fixed point x with (T^1, T^2) given by (3.92) results in

$$\frac{d}{dx}\int_{-\infty}^{\infty}\left[a_1\left(y^2 - 2\int_0^x E(\alpha)d\alpha\right)\overline{w} + a_2(y\overline{w}) + a_3(\overline{w}) + a_4(-\overline{w})\right]dy$$

$$+\left[a_1\left(-y^2E(x)\overline{w}_y+2yE(x)\overline{w}+2\int_0^x E(\alpha)d\alpha\overline{v}\right)\right.$$

$$\left.\left.+a_2(-y\overline{w}_y+\overline{w})E(x)+a_3(-E(x)\overline{w}_y)+a_4(\overline{v})\right]\right|_{y=-\infty}^{y=\infty}=0, \tag{3.93}$$

and using the boundary conditions in (2.36) we obtain

$$\frac{d}{dx}\int_{-\infty}^{\infty}\left[a_1\left(y^2-2\int_0^x E(\alpha)d\alpha\right)\overline{w}+a_2(y\overline{w})+a_3(\overline{w})+a_4(-\overline{w})\right]dy+$$

$$\left.\left[2a_1\int_0^x E(\alpha)d\alpha\overline{v}+a_4(\overline{v})\right]\right|_{y=-\infty}^{y=\infty}=0. \tag{3.94}$$

We now consider the conserved quantities for the classical wake and the wake of a self-propelled body. They can be derived in terms of both the stream function and the velocity components.

3.4.1 Classical wake

In terms of the stream function we consider equation (3.91). For the classical wake $a_1 = a_2 = 0$ and $a_3 = 1$. Thus we have

$$\frac{d}{dx}\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}(x,y)dy=0, \tag{3.95}$$

and hence

$$\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}(x,y)dy=D, \tag{3.96}$$

where D is a dimensionless constant proportional to the drag on the body [4, 6].

In terms of the velocity components, we have that $a_1 = a_2 = a_4 = 0$ and $a_3 = 1$. Equation (3.94) reduces to

$$\frac{d}{dx}\int_{-\infty}^{\infty}\overline{w}(x,y)dy=0. \tag{3.97}$$

Therefore,

$$\int_{-\infty}^{\infty}\overline{w}(x,y)dy=D, \tag{3.98}$$

which is equivalent to the expression obtained in (3.96).

3.4.2 Wake of a self-propelled body

For the wake of a self-propelled body we set $a_2 = a_3 = 0$ and $a_1 = 1$ in equation (3.91):

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2 \frac{\partial\psi}{\partial y}(x,y)\,dy - 2E(x)\left[\psi(x,y)\right]\Big|_{y=-\infty}^{y=\infty} = 0. \tag{3.99}$$

But since the drag D on a self-propelled body is zero,

$$D = \int_{-\infty}^{\infty} \frac{\partial\psi}{\partial y}(x,y)\,dy = \left[\psi(x,y)\right]\Big|_{y=-\infty}^{y=\infty} = 0, \tag{3.100}$$

and therefore gives the conserved quantity

$$\int_{-\infty}^{\infty} y^2 \frac{\partial\psi}{\partial y}(x,y)\,dy = K, \tag{3.101}$$

where the constant K is proportional to the second moment of the axial momentum deficit [5].

In equation (3.94) we set $a_2 = a_3 = a_4 = 0$ and $a_1 = 1$. We thus obtain

$$\frac{d}{dx}\int_{-\infty}^{\infty}\left(y^2 - 2\int_0^x E(\alpha)\,d\alpha\right)\overline{w}\,dy + 2\int_0^x E(\alpha)\,d\alpha\left[\overline{v}(x,y)\right]\Big|_{y=-\infty}^{y=\infty} = 0. \tag{3.102}$$

We can rewrite equation (3.102) as

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2\overline{w}(x,y)\,dy - 2\frac{d}{dx}\left(G(x)\int_{-\infty}^{\infty}\overline{w}\,dy\right) + 2G(x)\left[\overline{v}(x,y)\right]\Big|_{y=-\infty}^{y=\infty} = 0, \tag{3.103}$$

where

$$G(x) = \int_0^x E(\alpha)\,d\alpha. \tag{3.104}$$

Thus

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2\overline{w}(x,y)\,dy - 2\frac{dG}{dx}\int_{-\infty}^{\infty}\overline{w}\,dy - 2G(x)\int_{-\infty}^{\infty}\frac{\partial\overline{w}}{\partial x}\,dy + 2G(x)\int_{-\infty}^{\infty}\frac{\partial\overline{v}}{\partial y}\,dy = 0. \tag{3.105}$$

Since the drag on a self-propelled body is zero,

$$D = \int_{-\infty}^{\infty}\overline{w}(x,y)\,dy = 0, \tag{3.106}$$

and therefore equation (3.105) reduces to

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2\overline{w}(x,y)\,dy + 2G(x)\int_{-\infty}^{\infty}\left(-\frac{\partial\overline{w}}{\partial x} + \frac{\partial\overline{v}}{\partial y}\right)dy = 0. \tag{3.107}$$

But from the continuity equation

$$-\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0, \tag{3.108}$$

and therefore

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2 \overline{w}(x,y)\,dy = 0. \tag{3.109}$$

Hence

$$\int_{-\infty}^{\infty} y^2 \overline{w}(x,y)\,dy = K, \tag{3.110}$$

which is equivalent to (3.101).

We see from (3.101) and (3.110) that when the effective viscosity is of the form $E = E(x)$, the conserved quantity does not depend explicitly on the effective viscosity.

3.4.3 Combination wake

In terms of the stream function the combination wake arises from the case $a_1 = a_3 = 0$ and $a_2 = 1$ in equation (3.91). Equation (3.91) becomes

$$\frac{d}{dx}\int_{-\infty}^{\infty} y\frac{\partial \psi}{\partial y}(x,y)\,dy = 0, \tag{3.111}$$

giving the conserved quantity

$$\int_{-\infty}^{\infty} y\frac{\partial \psi}{\partial y}(x,y)\,dy = S. \tag{3.112}$$

We do not know the physical interpretation of the constant S.

In terms of the velocity components, using T^1 and T^2 defined in (3.92) and setting $a_1 = a_3 = a_4 = 0$ and $a_2 = 1$, equation (3.94) becomes

$$\frac{d}{dx}\int_{-\infty}^{\infty} y\overline{w}(x,y)\,dy = 0, \tag{3.113}$$

and therefore

$$\int_{-\infty}^{\infty} y\overline{w}(x,y)\,dy = S, \tag{3.114}$$

which is equivalent to expression (3.112) found for the stream function.

3.4.4 Remaining case

In terms of the velocity components there is one remaining case, $a_1 = a_2 = a_3 = 0$, $a_4 = 1$. It does not occur for the stream function formulation. Equation (3.94) becomes

$$-\frac{d}{dx}\int_{-\infty}^{\infty} \overline{w}(x,y)\,dy + \left[(\overline{v}(x,y))\right]\Big|_{y=-\infty}^{y=\infty} = 0, \tag{3.115}$$

which can be written as

$$\int_{-\infty}^{\infty}\left(-\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{v}}{\partial y}\right) dy = 0. \tag{3.116}$$

From the continuity equation, (3.108), equation (3.116) is identically satisfied and does not lead to a conserved quantity.

3.5 Conclusions

In this chapter the conserved quantities were derived for the turbulent wake. The conserved vectors were first obtained which were expressed in terms of the velocity components and the stream function by using the multiplier method. We first considered the case where $E = E\left(x, y, \dfrac{\partial^2\psi}{\partial y^2}\right)$ which models wakes with finite boundaries. For the governing equations in terms of the velocity components, two elementary conserved vectors were found. One of the elementary conserved vectors was used to generate the conserved quantity for the classical wake. The other did not generate a conserved quantity. In terms of the stream function, one conserved vector was found. This conserved vector was equivalent to the elementary conserved vector in terms of the velocity components used to generate the conserved quantity for the classical wake. We then considered an effective viscosity of the form $E = E(x, y)$ which predicts an infinite wake boundary. It is difficult to obtain conclusive results when the eddy viscosity depends on the variable y. We therefore considered two cases. For arbitrary $E(x, y)$, finding the conserved vectors from the governing equations in terms of the velocity components lead to two elementary conserved vectors as a result of assuming that the multipliers were constants. One was the elementary conserved vector which generated the conserved quantity for the classical wake and the other did not generate a conserved quantity. In terms of the velocity components, for $E = E(x)$, four conserved vectors were obtained. The first belonged to the wake of a self-propelled body and generated its conserved quantity as given in [5]. The second appeared to possibly correspond to a new type of wake which we called the combination wake. The third conserved vector was the conserved vector for the classical wake and the remaining conserved vector was an elementary conserved vector which did not generate a conserved quantity. In terms of the stream function we again considered two cases. For a constant multiplier and arbitrary $E(x, y)$ one conserved vector was obtained which was the elementary conserved vector. For $E = E(x)$ three conserved vectors were obtained. Two belonged to the classical wake and the wake of a self-propelled body and the third belonged to the combination wake.

To conclude, the conserved quantities for the classical wake and the wake of a self-propelled body could be obtained from the conserved vectors found in terms of the velocity components and the stream function. Calculating the conserved vectors lead to the discovery of a new wake which we called the combination wake.

Chapter 4

Solutions for the turbulent classical wake using Lie symmetry methods

In this chapter we consider the turbulent classical wake as shown in Figure 2.1. We examine the case for an eddy viscosity depending on the spacial variables x and y. An outline of this chapter is as follows. In Section 4.1, the Lie point symmetry associated with the elementary conserved vector is determined. In Section 4.2, we consider the eddy viscosity to be a function of only the distance along the axis of the wake and solve for the stream function for the turbulent classical wake. Mean velocity profiles are plotted for an eddy viscosity in the form of a power law and the results are compared for a range of power laws and with the laminar classical wake. In Section 4.3 we consider the eddy viscosity to be a function of the distance along the wake and the perpendicular distance from the axis of the wake. Various forms of the eddy viscosity are investigated and mean velocity profiles are again compared with those obtained for the laminar wake. Finally, conclusions are presented in Section 4.4.

4.1 Elementary conserved vector and associated Lie point symmetry

Recall that the governing equation for the turbulent classical wake in terms of the stream function is given by

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2 \psi}{\partial y^2}\right), \tag{4.1}$$

subject to

$$\frac{\partial \psi}{\partial y}(x, \pm\infty) = 0, \tag{4.2}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x, \pm\infty) = 0, \tag{4.3}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x, 0) = 0, \tag{4.4}$$

$$\frac{\partial \psi}{\partial x}(x, 0) = 0. \tag{4.5}$$

In terms of the stream function the conserved quantity for the turbulent classical wake is

$$\int_{-\infty}^{\infty} \frac{\partial \psi}{\partial y} dy = D. \tag{4.6}$$

The elementary conserved vector for the turbulent classical wake with $E = E(x, y)$ was calculated in Chapter 3. It is given by

$$T^1(\psi_y) = \psi_y, \quad T^2(x, y, \psi_{yy}) = -E(x, y)\psi_{yy}. \tag{4.7}$$

The conserved vector $T = (T^1, T^2)$ is invariant under the action of the Lie point symmetry

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}, \tag{4.8}$$

of the PDE (4.1) provided [30, 31]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2. \tag{4.9}$$

Equation (4.9) consists of two components, namely,

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{4.10}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0, \tag{4.11}$$

where X is prolongated to as high an order in the derivatives as required. Since the conserved vector (4.7) depends on ψ_y and ψ_{yy} we require the second prolongation of X in the form

$$X^{[2]} = X + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{22} \frac{\partial}{\partial \psi_{yy}}, \tag{4.12}$$

where [65]

$$\zeta_2 = D_2(\eta) - \psi_k D_2(\xi^k), \tag{4.13}$$

$$\zeta_{22} = D_2(\zeta_2) - \psi_{2k} D_2(\xi^k). \tag{4.14}$$

Consider first the invariance condition (4.10). Now T^1 is given by (4.7) and

$$X(T^1) = \zeta_2 = \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^1}{\partial y}\psi_x - \frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y - \frac{\partial \xi^2}{\partial y}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_y^2. \tag{4.15}$$

Condition (4.10) becomes

$$\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \frac{\partial \xi^1}{\partial y}\psi_x - \frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y + E(x,y)\frac{\partial \xi^1}{\partial y}\psi_{yy} + E(x,y)\frac{\partial \xi^1}{\partial \psi}\psi_y\psi_{yy} = 0, \tag{4.16}$$

which does not depend on $\xi^2(x, y, \psi)$. Equating coefficients of derivatives of ψ to zero we obtain

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x, y, \psi), \quad \eta = \eta(x). \tag{4.17}$$

Consider next the second invariance condition (4.11). Using (4.17), ζ_{22} simplifies to

$$\zeta_{22} = -\frac{\partial^2 \xi^2}{\partial y^2}\psi_y - 2\frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 - \frac{\partial^2 \xi^2}{\partial \psi^2}\psi_y^3 - 2\frac{\partial \xi^2}{\partial y}\psi_{yy} - 3\frac{\partial \xi^2}{\partial \psi}\psi_y\psi_{yy}, \tag{4.18}$$

and (4.11) becomes

$$E(x,y)\left(\frac{\partial^2 \xi^2}{\partial y^2}\psi_y + 2\frac{\partial^2 \xi^2}{\partial y \partial \psi}\psi_y^2 + \frac{\partial^2 \xi^2}{\partial \psi^2}\psi_y^3 + \left(2\frac{\partial \xi^2}{\partial y} - \frac{d\xi^1}{dx}\right)\psi_{yy} + 3\frac{\partial \xi^2}{\partial \psi}\psi_y\psi_{yy}\right)$$
$$- \left(\xi^1 \frac{\partial E}{\partial x} + \xi^2 \frac{\partial E}{\partial y}\right)\psi_{yy} - \frac{\partial \xi^2}{\partial x}\psi_y - \frac{\partial \xi^2}{\partial \psi}\psi_x\psi_y = 0. \tag{4.19}$$

Now $E(x, y) \neq 0$. We separate (4.19) according to powers and products of the partial derivatives of ψ. The coefficient of $\psi_y\psi_{yy}$ gives

$$\frac{\partial \xi^2}{\partial \psi} = 0, \tag{4.20}$$

and therefore $\xi^2 = \xi^2(x, y)$. Condition (4.19) simplifies and the coefficients of ψ_y and ψ_{yy} give respectively

$$E(x,y)\frac{\partial^2 \xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0, \tag{4.21}$$

$$\xi^1(x)\frac{\partial E}{\partial x} + \xi^2(x,y)\frac{\partial E}{\partial y} = \left(2\frac{\partial \xi^2}{\partial y} - \frac{d\xi^1}{dx}\right)E(x,y). \tag{4.22}$$

The Lie point symmetry associated with the conserved vector (4.7) is

$$X = \xi^1(x)\frac{\partial}{\partial x} + \xi^2(x, y)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}, \tag{4.23}$$

provided that the effective viscosity $E(x, y)$ satisfies equations (4.21) and (4.22).

The derivation of the associated Lie point symmetry is simpler than that of the Lie point symmetries of the partial differential equation. Only the second prolongation of X was required while the derivation of the Lie point symmetries of the partial differential equation (4.1) would require the third prolongation of X which contains many terms. The associated Lie point symmetry has to satisfy two invariance conditions which are easier to solve than the one large invariance condition for the Lie point symmetries of the partial differential equation. The results from the first condition (4.10) greatly simplified the derivation and solution of the second invariance condition (4.11).

In the sections that follow we will investigate the invariant solution when $E = E(x)$ and $E = E(x, y)$.

4.2 Eddy viscosity a function only of x

We initially consider the eddy viscosity to be a function of x only. We then consider an eddy viscosity which is a power law in x and then the approximation in which the kinematic viscosity is neglected so that the effective viscosity is a power law in x.

4.2.1 General results for eddy viscosity a function of x only

When the eddy viscosity depends on x only equation (4.22) becomes

$$\frac{d\xi^1}{dx} + \frac{E'(x)}{E(x)}\xi^1(x) = 2\frac{\partial \xi^2}{\partial y}(x, y). \tag{4.24}$$

Differentiating (4.24) with respect to y we have

$$\frac{\partial^2 \xi^2}{\partial y^2}(x, y) = 0. \tag{4.25}$$

Equation (4.21) then reveals that

$$\frac{\partial \xi^2}{\partial x}(x, y) = 0, \tag{4.26}$$

and therefore we can conclude that

$$\xi^2(y) = a_1 y + a_2, \tag{4.27}$$

where a_1 and a_2 are constants. Substituting (4.27) into (4.24) we obtain

$$\frac{d\xi^1}{dx} + \frac{E'(x)}{E(x)}\xi^1(x) = 2a_1, \tag{4.28}$$

which is a linear first order ODE with integrating factor $E(x)$. The solution is

$$\xi^1(x) = \frac{1}{E(x)}\left[a_3 + 2a_1 \int_0^x E(x')dx'\right], \tag{4.29}$$

where

$$a_3 = \xi^1(0)E(0). \tag{4.30}$$

The Lie point symmetry of (4.23) becomes

$$X = \frac{1}{E(x)}\left[a_3 + 2a_1 \int_0^x E(x')dx'\right]\frac{\partial}{\partial x} + (a_1 y + a_2)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}. \tag{4.31}$$

The constant a_1 plays an important role in the Lie point symmetry (4.31). In the analysis that follows, we will assume that $a_1 \neq 0$ so that the invariant solution depends on the effective viscosity. Let

$$X^* = \frac{X}{a_1}, \quad a_3^* = \frac{a_3}{a_1}, \quad a_2^* = \frac{a_2}{a_1}, \quad \eta^*(x) = \frac{\eta(x)}{a_1}. \tag{4.32}$$

Suppressing the star to help keep the notation simple the Lie point symmetry (4.31) becomes

$$X = \frac{1}{E(x)}\left[a_3 + 2\int_0^x E(x')dx'\right]\frac{\partial}{\partial x} + (y + a_2)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}. \tag{4.33}$$

Now $\psi = \Psi(x, y)$ is an invariant solution of the PDE (4.1) with $E = E(x)$ generated by the Lie point symmetry associated with the elementary conserved vector provided

$$X(\psi - \Psi(x, y))\big|_{\psi=\Psi} = 0, \tag{4.34}$$

that is, provided

$$\frac{1}{E(x)}\left(a_3 + 2\int_0^x E(x')dx'\right)\frac{\partial\Psi}{\partial x} + \left(y + a_2\right)\frac{\partial\Psi}{\partial y} = \eta(x). \tag{4.35}$$

The differential equations of the characteristic curves of (4.35) are

$$\frac{E(x)dx}{a_3 + 2\int_0^x E(x')dx'} = \frac{dy}{y + a_2} = \frac{d\Psi}{\eta(x)}. \tag{4.36}$$

The first pair of terms give

$$\frac{y + a_2}{(a_3 + 2\int_0^x E(x')dx')^{1/2}} = b_1, \tag{4.37}$$

where b_1 is a constant. The first and last terms in (4.36) give

$$\Psi - G(x) = b_2, \tag{4.38}$$

where b_2 is a constant and

$$G(x) = \int^x \frac{\eta(x')E(x')}{a_3 + 2\int_0^{x'} E(\alpha)d\alpha}dx'. \tag{4.39}$$

The general solution of the first order linear PDE (4.35) is

$$b_2 = F(b_1), \tag{4.40}$$

where F is an arbitrary function. Since $\Psi = \psi$ we have

$$\psi = F(\xi) + G(x), \tag{4.41}$$

where

$$\xi = \frac{y + a_2}{(a_3 + 2\int_0^x E(x')dx')^{1/2}}. \tag{4.42}$$

We now express the conserved quantity (4.6) in terms of the similarity variables. Using (4.41) and (4.42) at a fixed point x on the axis of the wake, (4.6) becomes

$$\int_{-\infty}^{\infty} \frac{dF}{d\xi}d\xi = D. \tag{4.43}$$

Equation (4.43) is independent of x unconditionally and does not give a relation between the constants in the associated Lie point symmetry (4.33). When a linear combination of the Lie point symmetries is used a relation between the constants in the linear combination is derived for the conserved quantity to be independent of x when expressed in the similarity variables.

We now determine the constants a_2, a_3 and the function $\eta(x)$ in the Lie point symmetry (4.33). Consider first a_2. The boundary condition (4.4) imposes the following restriction:

$$F''(\xi|_{y=0}) = 0, \tag{4.44}$$

and by differentiating (4.44) with respect to x we obtain

$$F'''(\xi|_{y=0})\frac{a_2}{(a_3 + 2\int_0^x E(x')dx)^{3/2}}E(x) = 0. \tag{4.45}$$

However,

$$\frac{\partial^2 \overline{w}}{\partial y^2}(x,0) = F'''(\xi|_{y=0})\frac{1}{(a_3 + 2\int_0^x E(x')dx')^{3/2}} < 0, \tag{4.46}$$

as $y = 0$ is a local maximum for $\overline{w}(x, y)$. Thus

$$F'''(\xi|_{y=0}) \neq 0, \tag{4.47}$$

and from (4.45) this forces us to set $a_2 = 0$. Hence $\xi = 0$ when $y = 0$. Consider next $\eta(x)$. Now

$$\frac{\partial \psi}{\partial x} = -\xi\frac{dF}{d\xi}\frac{E(x)}{(a_3 + 2\int_0^x E(x')dx')} + \frac{dG}{dx}. \tag{4.48}$$

But

$$\overline{w}(x, y) = \frac{\partial \psi}{\partial y} = \frac{dF}{d\xi}\frac{1}{(a_3 + 2\int_0^x E(x')dx')}, \tag{4.49}$$

and since $\overline{w}(x,0)$ is finite it follows that $F'(0)$ is finite. Hence from (4.48) the boundary condition (4.5) gives that $G'(x) = 0$. Therefore, from (4.39) we have the following condition:

$$\frac{E(x)\eta(x)}{a_3 + 2\int_0^x E(x')dx'} = 0, \tag{4.50}$$

and since $E(x)$ is non-zero, we must have that $\eta(x) = 0$ giving that $G(x) = 0$. Finally consider a_3. The order of magnitude of the effective half-width of the wake, $W(x)/2$, at position x is given by the value of y when $\xi = O(1)$. Thus

$$\frac{W(x)}{2} = O\left(\left[a_3 + 2\int_0^x E(x')dx'\right]^{1/2}\right). \tag{4.51}$$

If we assume that the effective width of the wake is valid for $x \to 0$, then, as $x \to 0$, we expect the effective width of the wake to also tend to zero. From this we obtain $a_3 = 0$.

Thus we have

$$\psi(x, y) = F(\xi), \tag{4.52}$$

where

$$\xi = \frac{y}{(2\int_0^x E(x')dx')^{1/2}}, \tag{4.53}$$

and the Lie point symmetry simplifies to

$$X = \frac{2\int_0^x E(x')dx'}{E(x)}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.54}$$

The constants a_2, a_3 and $\eta(x)$ in the Lie point symmetry were all obtained from physical considerations and cannot be specified arbitrarily.

Substituting (4.52) and (4.53) into the PDE (4.1) yields the ODE

$$\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} = 0. \tag{4.55}$$

The boundary conditions (4.2), (4.3) and (4.4) become

$$\frac{dF}{d\xi}(\pm\infty) = 0, \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0,$$

$$\frac{d^2F}{d\xi^2}(0) = 0. \tag{4.56}$$

The boundary condition (4.5) is identically satisfied. The solution of (4.55) is also subject to the conserved quantity (4.43) where the constant D is given. Once the problem has been solved for $F'(\xi)$, $\overline{w}(x, y)$ and $\overline{v}_y(x, y)$ are obtained from

$$\overline{w}(x, y) = \frac{1}{(2\int_0^x E(x')dx')^{1/2}}\frac{dF}{d\xi}, \tag{4.57}$$

$$\overline{v}_y(x, y) = -\frac{E(x)}{2\int_0^x E(x')dx'}\xi\frac{dF}{d\xi}. \tag{4.58}$$

The solution for $F(\xi)$ need only be obtained if it is required to calculate $\psi(x, y)$. Otherwise it is sufficient to determine $F'(\xi)$.

We can rearrange the ODE (4.55) as follows:

$$\frac{d^3F}{d\xi^3} + \frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0, \tag{4.59}$$

which when integrated with respect to ξ yields

$$\frac{d^2F}{d\xi^2} + \xi\frac{dF}{d\xi} = c_1. \tag{4.60}$$

Using the boundary condition (4.56) at $\xi = 0$ and noting from (4.57) that $F'(0)$ is finite since $\overline{w}(x,0)$ is finite we obtain $c_1 = 0$. Therefore, we now need to solve

$$\frac{d^2F}{d\xi^2} + \xi\frac{dF}{d\xi} = 0, \tag{4.61}$$

which is a first order ODE for $F'(\xi)$. From equation (4.61) it is noted that the boundary conditions (4.2) and (4.3) are not independent of one another. The solution is

$$\frac{dF}{d\xi} = c_2 \exp\left(-\frac{\xi^2}{2}\right), \tag{4.62}$$

where the constant c_2 is determined in terms of the drag force on the body by the conserved quantity (4.43):

$$c_2 = \frac{D}{\sqrt{2\pi}}, \tag{4.63}$$

and thus

$$\frac{dF}{d\xi} = \frac{D}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right). \tag{4.64}$$

Integrating (4.64) with respect to ξ from 0 to ξ gives

$$F(\xi) = \frac{D}{\sqrt{2\pi}} \int_0^{\xi} \exp\left(-\frac{\xi^{*2}}{2}\right) d\xi^* + F(0). \tag{4.65}$$

Since $\overline{w}(x, y)$ and $\overline{v}_y(x, y)$, defined by (2.34), do not depend on an additive constant in $\psi(x, y)$ we choose $F(0) = 0$. Equation (4.65) with $F(0) = 0$ may be written in the form

$$F(\xi) = \frac{D}{2}\text{erf}\left(\frac{\xi}{\sqrt{2}}\right), \tag{4.66}$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du. \tag{4.67}$$

Equation (4.66) determines the stream function by (4.52). The solution for $F(\xi)$ has the same form as that for a classical laminar wake [4, 6].

The mean velocity deficit in the wake in the x-direction is

$$\overline{w}(x,y) = \frac{D}{2\sqrt{\pi}}\exp\left(-\frac{\xi^2}{2}\right)\frac{1}{(\int_0^x E(x')dx')^{\frac{1}{2}}}. \tag{4.68}$$

The x- and y- components of the mean velocity are given by respectively,

$$\overline{v}_x(x,y) = 1 - \overline{w}(x,y) = 1 - \frac{D}{2\sqrt{\pi}}\exp\left(-\frac{\xi^2}{2}\right)\frac{1}{(\int_0^x E(x')dx')^{\frac{1}{2}}}, \tag{4.69}$$

$$\overline{v}_y(x,y) = -\frac{D}{2\sqrt{2\pi}}\frac{E(x)}{\int_0^x E(x')dx'}\xi\exp\left(-\frac{\xi^2}{2}\right), \tag{4.70}$$

where ξ is given by (4.53).

Prandtl's hypothesis [66] states that the eddy viscosity is constant across a boundary layer and is proportional to the product of the maximum mean velocity and the width of the layer. The eddy viscosity considered in this section is independent of y and is therefore constant across the wake. The wake tends to infinity in the $\pm\, y$-directions but because the mean velocity deficit (4.68) falls off exponentially with y an effective width can be defined as twice the value of y for which the argument of the exponential is -1:

$$W(x) = 4\sqrt{\int_0^x E(x')dx'}. \tag{4.71}$$

Also from (4.68) the maximum value of the mean velocity deficit is

$$\overline{w}(x,0) = \frac{D}{2\sqrt{\pi}}\frac{1}{(\int_0^x E(x')dx')^{\frac{1}{2}}}, \tag{4.72}$$

and therefore

$$\overline{w}(x,0)W(x) = \frac{2D}{\sqrt{\pi}}, \tag{4.73}$$

which is a consequence of the conserved quantity (4.6). If we can replace the maximum mean velocity by the maximum mean velocity deficit in Prandtl's hypothesis then for a classical wake it is satisfied if the eddy viscosity is constant independent of x and y. We will not impose Prandtl's hypothesis but we will include its predictions in the comparison of results.

4.2.2 Eddy viscosity a power law in x

Consider an eddy viscosity which is a power law in x. Then the dimensionless effective viscosity has the form

$$E(x) = \frac{\nu}{\nu + \nu_{T_0}} + \frac{\nu_{T_0}}{\nu + \nu_{T_0}} x^{\beta}, \quad \beta > -1. \tag{4.74}$$

By increasing ν_{T_0} continuously from zero the transition from a laminar wake to a turbulent wake can be investigated. For well developed turbulence in the wake the ratio ν_{T_0}/ν can equal 1000 or larger. By using (4.74),

$$\int_0^x E(x')dx' = \frac{x}{\left(1 + \dfrac{\nu_{T_0}}{\nu}\right)} \left(1 + \frac{\nu_{T_0}}{\nu(1+\beta)} x^{\beta}\right), \tag{4.75}$$

and the Lie point symmetry which generates the invariant solution for the turbulent wake is from (4.54)

$$X = 2 \left(\frac{1 + \dfrac{\nu_{T_0}}{\nu(1+\beta)} x^{\beta}}{1 + \dfrac{\nu_{T_0}}{\nu} x^{\beta}} \right) x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \tag{4.76}$$

The Lie point symmetry (4.76) is not a scaling symmetry. The invariant solution for $\overline{w}(x, y)$ and $\overline{v}_y(x, y)$ therefore cannot be derived using elementary scaling methods to obtain similarity solutions.

The mean velocity deficit (4.68) is

$$\overline{w}_T(x, y) = \frac{D}{2\sqrt{\pi x}} \left(\frac{1 + \dfrac{\nu_{T_0}}{\nu}}{1 + \dfrac{\nu_{T_0}}{\nu(1+\beta)} x^{\beta}} \right)^{\frac{1}{2}} \exp\left[-\frac{y^2}{4x} \left(\frac{1 + \dfrac{\nu_{T_0}}{\nu}}{1 + \dfrac{\nu_{T_0}}{\nu(1+\beta)} x^{\beta}} \right) \right]. \tag{4.77}$$

The maximum value for the mean velocity deficit occurs on the axis of the wake, $y = 0$, and is

$$\overline{w}_T(x, 0) = \frac{D}{2\sqrt{\pi x}} \left(\frac{1 + \dfrac{\nu_{T_0}}{\nu}}{1 + \dfrac{\nu_{T_0}}{\nu(1+\beta)} x^{\beta}} \right)^{\frac{1}{2}}. \tag{4.78}$$

An effective width $W_T(x)$ of the turbulent wake can be defined as twice the value of y for which the argument in the exponential is -1:

$$W_T(x) = 4\sqrt{x}\left(\frac{1 + \dfrac{\nu_{T_0}}{\nu(1+\beta)}x^{\beta}}{1 + \dfrac{\nu_{T_0}}{\nu}}\right)^{\frac{1}{2}}. \tag{4.79}$$

We observe that

$$\overline{w}_T(x,0)W_T(x) = \frac{2D}{\sqrt{\pi}}, \tag{4.80}$$

which is a special case of (4.73) and is a consequence of the conserved quantity (4.6).

In order to determine the effect of the turbulence on the wake we compare with the laminar wake. For the laminar wake

$$E = \frac{\nu}{\nu + \nu_{T_0}}, \tag{4.81}$$

where E has been scaled by $\nu + \nu_{T_0}$ as in (4.74). Then

$$\int_0^x E(x')dx' = \frac{x}{1 + \dfrac{\nu_{T_0}}{\nu}}, \tag{4.82}$$

and the Lie point symmetry (4.54) for the laminar wake is the scaling symmetry

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.83}$$

The turbulence alters only $\xi^1(x)$ in (4.76). From (4.68) the velocity deficit in the x-direction is

$$w_L(x,y) = \frac{D}{2\sqrt{\pi x}}\left(1 + \frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}} \exp\left[-\frac{y^2}{4x}\left(1 + \frac{\nu_{T_0}}{\nu}\right)\right]. \tag{4.84}$$

The maximum velocity deficit for the laminar wake is

$$w_L(x,0) = \frac{D}{2\sqrt{\pi x}}\left(1 + \frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}}, \tag{4.85}$$

while the effective width of the laminar wake is

$$W_L(x) = \frac{4\sqrt{x}}{\left(1 + \dfrac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}}}. \tag{4.86}$$

We again have the product

$$w_L(x,0)W_L(x) = \frac{2D}{\sqrt{\pi}}. \tag{4.87}$$

From (4.85) and (4.78) the ratio of the maximum velocity deficit in the laminar and turbulent wakes is

$$\frac{w_L(x,0)}{\overline{w}_T(x,0)} = \left[1 + \frac{\nu_{T_0}}{\nu(1+\beta)}x^\beta\right]^{\frac{1}{2}}. \tag{4.88}$$

The maximum velocity deficit in the laminar wake is greater than that in the turbulent wake and the ratio increases as ν_{T_0} increases and the wake becomes more turbulent. From (4.79) and (4.86) the ratio of the effective width of the turbulent wake to the effective width of the laminar wake is

$$\frac{W_T(x)}{W_L(x)} = \left[1 + \frac{\nu_{T_0}}{\nu(1+\beta)}x^\beta\right]^{\frac{1}{2}}. \tag{4.89}$$

The effective width of the turbulent wake is greater than that of the laminar wake because of the increase in diffusion of vorticity due to the eddy viscosity.

In Figure 4.1 the dependence of the effective width and maximum mean velocity deficit on the exponent β is investigated. The x- component of the velocity in a laminar wake v_x^L, where

$$v_x^L = 1 - w_L(x,y) = 1 - \frac{D}{2\sqrt{\pi x}}\left(1 + \frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}} \exp\left[-\frac{y^2}{4x}(1 + \frac{\nu_{T_0}}{\nu})\right], \tag{4.90}$$

is compared with the x- component of the mean velocity $\overline{v}_x$ in a turbulent wake

$$\overline{v}_x = 1 - \overline{w}(x,y) = 1 - \frac{D}{2\sqrt{\pi x}}\left(\frac{1 + \frac{\nu_{T_0}}{\nu}}{1 + \frac{\nu_{T_0}}{\nu(1+\beta)}x^\beta}\right)^{\frac{1}{2}} \exp\left[-\frac{y^2}{4x}\left(\frac{1 + \frac{\nu_{T_0}}{\nu}}{1 + \frac{\nu_{T_0}}{\nu(1+\beta)}x^\beta}\right)\right], \tag{4.91}$$

at $x = 1/2$ for a range of values of β. Since the velocity deficit is scaled by the magnitude of the uniform mainstream flow, to prevent the formation of a region of back-flow at the axis of the wake, the maximum velocity deficit should not exceed unity. Since the maximum velocity deficit in a laminar wake is greater than that in a turbulent wake, it is sufficient to consider a laminar wake. The maximum velocity deficit in a laminar wake is less than unity provided

$$\frac{D}{2\sqrt{\pi x}}\left(1 + \frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}} \le 1, \tag{4.92}$$

that is, provided

$$x \geq x_{min} = \frac{D^2}{4\pi}\left(1 + \frac{\nu_{T_0}}{\nu}\right). \tag{4.93}$$

In Figure 4.1, $D = 0.1$ and $\nu_{T_0}/\nu = 500$ so that $x_{min} = 0.399$. When $x = 1/2$, the eddy viscosity decreases as the exponent β increases, the diffusion of vorticity therefore decreases as β increases and the width of the wake decreases and the maximum velocity deficit increases (to satisfy (4.80)) as β increases. The curve $\beta = 0$ corresponds to Prandtl's hypothesis. From (4.78) the maximum mean velocity deficit for $\beta > 0$ is greater than that of $\beta = 0$ provided

$$x < \left(1 + \beta\right)^{1/\beta}, \tag{4.94}$$

which is satisfied for $x \leq 1$, in agreement with Figure 4.1. For $x = 2$, for instance, the maximum mean velocity deficit for $\beta = 1$ is the same as that for $\beta = 0$ but for $\beta = 2$, 3 and 4 it is less than that for $\beta = 0$ and therefore from Figure 4.1 it will be small. For $x > 1$ the order of the curves is reversed with greatest velocity deficit for $\beta = 1$ and least for $\beta = 4$.

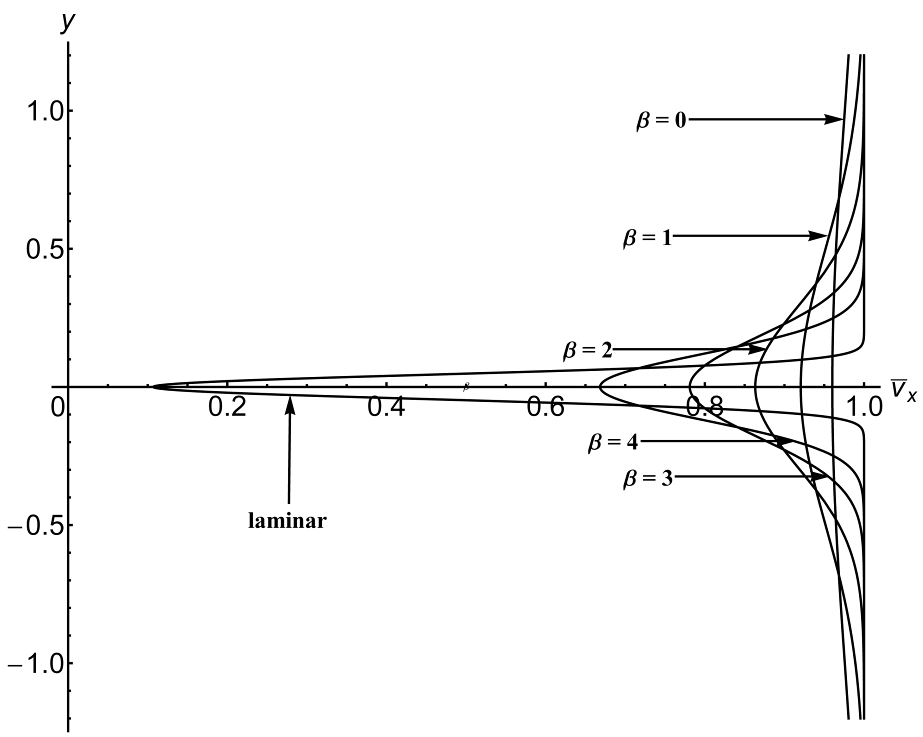

FIGURE 4.1: Velocity in a laminar wake $v_x^L(x, y)$ and the mean velocity in a turbulent wake $\overline{v}_x(x, y)$ plotted against y at $x = 1/2$ with $D = 0.1$ and $\nu_{T_0}/\nu = 500$ for $\beta = 0$, 1, 2, 3 and 4.

Consider next the dependence of the mean velocity deficit on the strength of the turbulence. Since the characteristic effective viscosity in (4.74), $\nu_{T_0} + \nu$, depends on ν_{T_0} it is replaced by $\nu + \nu_{T_C}$ where ν_{T_C} is suitably chosen. It is readily verified that

$$\overline{w}_T(x, y) = \frac{D}{2\sqrt{\pi x}} \left(\frac{1 + \frac{\nu_{T_c}}{\nu}}{1 + \frac{\nu_{T_0}}{\nu(1+\beta)} x^\beta} \right)^{\frac{1}{2}} \exp\left[-\frac{y^2}{4x} \left(\frac{1 + \frac{\nu_{T_c}}{\nu}}{1 + \frac{\nu_{T_0}}{\nu(1+\beta)} x^\beta} \right) \right], \tag{4.95}$$

$$w_L(x, y) = \frac{D}{2\sqrt{\pi x}} \left(1 + \frac{\nu_{T_c}}{\nu}\right)^{\frac{1}{2}} \exp\left[-\frac{y^2}{4x} \left(1 + \frac{\nu_{T_c}}{\nu}\right) \right], \tag{4.96}$$

and that (4.93) is replaced by

$$x \geq x_{min} = \frac{D^2}{4\pi} \left(1 + \frac{\nu_{T_c}}{\nu}\right). \tag{4.97}$$

In Figure 4.2 the x- component of the mean velocity for a laminar wake and the x-component of the mean velocity for a turbulent wake are plotted against y for a range of values of ν_{T_0}/ν and $\beta = 2$. As ν_{T_0}/ν increases the eddy viscosity increases and therefore the diffusion of vorticity in the wake increases. The width of the wake therefore increases and, in order to satisfy (4.80), the maximum velocity deficit decreases.

4.2.3 Effective viscosity a power law in x

Consider now well developed turbulence in which

$$\frac{\nu_{T_0}}{\nu} \gg 1. \tag{4.98}$$

The scaled effective viscosity (4.74) approximates to

$$E(x) = x^\beta, \quad \beta > -1. \tag{4.99}$$

The effective viscosity can therefore be approximated by a power law in x.

When (4.99) is satisfied the Lie point symmetry (4.76) reduces to the scaling symmetry

$$X = \frac{2}{1+\beta} x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \tag{4.100}$$

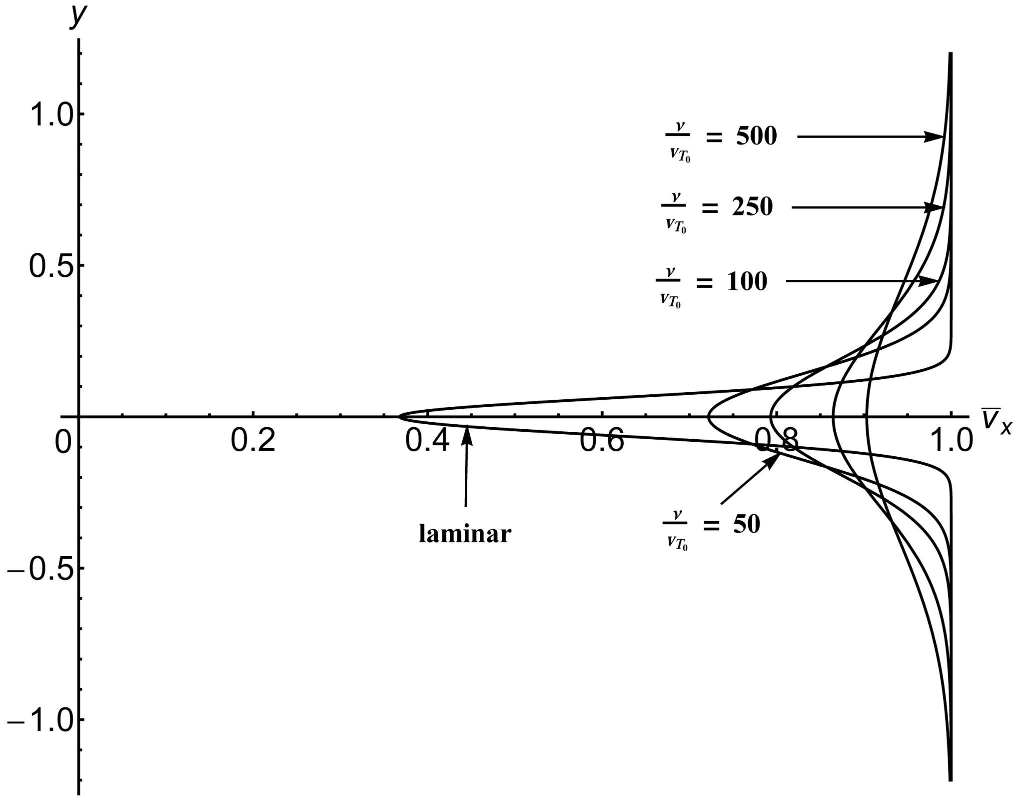

FIGURE 4.2: Velocity in a laminar wake $v_x^L(x, y)$ and the mean velocity in a turbulent wake $\overline{v}_x(x, y)$ plotted against y at $x = 1/2$ with $D = 0.1$, $\nu_{T_c}/\nu = 250$, $\beta = 2$ and for $\nu_{T_0}/\nu = 50,\ 100,\ 250$ and 500. There is no backflow provided $x \geq x_{min} = 0.197$.

The mean velocity deficit (4.77) simplifies to

$$\overline{w}_T(x, y) = \frac{D}{2}\left(\frac{1+\beta}{\pi x^{1+\beta}}\right)^{\frac{1}{2}} \exp\left[-\frac{(1+\beta)}{4}\frac{y^2}{x^{1+\beta}}\right], \tag{4.101}$$

and the velocity deficit in the laminar wake becomes

$$w_L(x, y) = \frac{D}{2\sqrt{\pi x}}\left(\frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}} \exp\left[-\frac{\nu_{T_0}}{4\nu}\frac{y^2}{x}\right]. \tag{4.102}$$

The ratios (4.88) and (4.89) simplify to

$$\frac{w_L(x,0)}{\overline{w}_T(x,0)} = \frac{W_T(x)}{W_L(x)} = \left[\frac{\nu_{T_0}}{\nu(1+\beta)}x^{\beta}\right]^{\frac{1}{2}}, \tag{4.103}$$

and condition (4.93) on the model for no backflow along the axis of the wake becomes

$$x \geq x_{min} = \frac{D^2}{4\pi}\frac{\nu_{T_0}}{\nu}. \tag{4.104}$$

4.3 Eddy viscosity a function of x and y

A part of Prandtl's hypothesis is that the eddy viscosity is constant across a boundary layer [66]. In this section we will allow the eddy viscosity to depend on y as well as x to investigate how the y dependence affects the wake.

We return to equations (4.21) and (4.22). When $E = E(x, y)$ we were not able to find the general solution for $\xi^2(x, y)$. When $E = E(x)$ the general solution for $\xi^2(x, y)$ is a linear function of y with constant coefficients given by (4.27). When $E = E(x, y)$ we looked for a special solution for $\xi^2(x, y)$ as a linear function of y of the form

$$\xi^2(x, y) = a_1(x)y + a_2(x), \tag{4.105}$$

where $a_1(x)$ and $a_2(x)$ are arbitrary functions of x. Substituting (4.105) into equation (4.21) and separating by powers of y gives that a_1 and a_2 are constants and (4.27) is re-derived. We consider $a_1 \neq 0$ and take $a_1 = 1$ which is equivalent to making the transformation (4.32). This gives

$$X = A(x)\frac{\partial}{\partial x} + \left(y + a_2\right)\frac{\partial}{\partial y} + \eta(x)\frac{\partial}{\partial \psi}, \tag{4.106}$$

and (4.22) becomes

$$A(x)\frac{\partial E}{\partial x} + (y + a_2)\frac{\partial E}{\partial y} = \left(2 - \frac{dA}{dx}\right)E(x, y), \tag{4.107}$$

where we let $\xi^1(x) = A(x)$.

Now $\psi = \Psi(x, y)$ is an invariant solution of the PDE (4.1) if (4.34) is satisfied, that is, if

$$A(x)\frac{\partial \Psi}{\partial x} + (y + a_2)\frac{\partial \Psi}{\partial y} = \eta(x). \tag{4.108}$$

The differential equations of the characteristic curves of (4.108) are

$$\frac{dx}{A(x)} = \frac{dy}{y + a_2} = \frac{d\Psi}{\eta(x)}. \tag{4.109}$$

The first pair of terms give

$$\frac{y + a_2}{\exp(B(x))} = d_1, \tag{4.110}$$

where d_1 is a constant and

$$B(x) = \int^x \frac{dx}{A(x)}. \tag{4.111}$$

The first and last terms in (4.109) give

$$\Psi - G(x) = d_2, \tag{4.112}$$

where d_2 is a constant and

$$G(x) = \int^x \frac{\eta(x)}{A(x)} dx. \tag{4.113}$$

The general solution of (4.108) is

$$d_2 = F(d_1), \tag{4.114}$$

where F is an arbitrary function. Since $\Psi = \psi$ we obtain

$$\psi(x, y) = F(\xi) + G(x), \tag{4.115}$$

where

$$\xi = \frac{y + a_2}{\exp(B(x))}. \tag{4.116}$$

From the boundary conditions (4.4) and (4.5) we can again deduce that $a_2 = 0$ and $\eta(x) = 0$. Thus

$$\psi(x, y) = F(\xi), \quad \xi = \frac{y}{\exp(B(x))}, \tag{4.117}$$

and the Lie point symmetry simplifies to

$$X = A(x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.118}$$

The effective viscosity $E(x, y)$ satisfies equation (4.107) with $a_2 = 0$. The differential equations of the characteristic curves of (4.107) are

$$\frac{dx}{A(x)} = \frac{dy}{y} = \frac{dE}{\left(2 - \frac{dA}{dx}\right)E}. \tag{4.119}$$

The first pair of terms give (4.110) with $a_2 = 0$. The first and last terms give

$$\frac{EA(x)}{\exp(2B(x))} = d_3, \tag{4.120}$$

where d_3 is a constant. The general solution of (4.107) is

$$d_3 = S(d_1), \tag{4.121}$$

where S is an arbitrary function. Thus

$$E(x,y) = \frac{\exp(2B(x))}{A(x)} S(\xi), \tag{4.122}$$

where ξ is given by (4.117).

We introduce the notation

$$H(x) = \exp(B(x)), \tag{4.123}$$

and express the problem in terms of $H(x)$. The stream function and effective viscosity are

$$\psi(x,y) = F(\xi), \tag{4.124}$$

$$E(x,y) = H(x)H'(x)S(\xi), \tag{4.125}$$

where

$$\xi = \frac{y}{H(x)}. \tag{4.126}$$

The Lie point symmetry is

$$X = \frac{H(x)}{H'(x)} \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \tag{4.127}$$

We define the effective width of the wake, $W(x)$, to be twice the value of y when $\xi = 1$:

$$W(x) = 2H(x). \tag{4.128}$$

Since the effective width of the wake vanishes at $x = 0$,

$$H(0) = 0. \tag{4.129}$$

The PDE (4.1) reduces to the ODE

$$\frac{d}{d\xi}\left[S(\xi)\frac{d^2F}{d\xi^2}\right] + \frac{d}{d\xi}\left(\xi \frac{dF}{d\xi}\right) = 0, \tag{4.130}$$

where $F(\xi)$ is subject to the boundary conditions

$$\frac{d^2F}{d\xi^2}(0) = 0, \quad \frac{dF}{d\xi}(\pm\infty) = 0, \tag{4.131}$$

and to the conserved quantity

$$\int_{-\infty}^{\infty} \frac{dF}{d\xi} d\xi = D. \tag{4.132}$$

The boundary condition (4.5) is identically satisfied. Once the problem has been solved for $F'(\xi)$, $\overline{w}(x, y)$ and $\overline{v}_y(x, y)$ are obtained from

$$\overline{w}(x, y) = \frac{1}{H(x)} \frac{dF}{d\xi}, \tag{4.133}$$

$$\overline{v}_y(x, y) = -\frac{H'(x)}{H(x)} \xi \frac{dF}{d\xi}. \tag{4.134}$$

Consider now the solution of the ODE (4.130). Since the wake is symmetric about the x-axis, $S(\xi)$ is an even function of ξ. Also $S(\xi) > 0$ for $-\infty \le \xi \le \infty$, $S(0)$ is finite and $S(\pm\infty)$ is a positive constant since $E(x, y)$ is finite and non-zero as $y \to \pm\infty$. Integrating (4.130) with respect to ξ, imposing the boundary condition (4.131) at $\xi = 0$ and noting that $F'(0)$ is finite because $\overline{w}(x, 0)$ is finite, we obtain

$$\frac{d^2F}{d\xi^2} + \frac{\xi}{S(\xi)} \frac{dF}{d\xi} = 0. \tag{4.135}$$

Equation (4.135) is a first order ODE for $F'(\xi)$. The solution is

$$\frac{dF}{d\xi} = b \exp\left[-\int_0^{\xi} \frac{\gamma}{S(\gamma)} d\gamma\right], \tag{4.136}$$

where b is a constant. Since $S(\gamma)$ is an even function of γ and $S(\pm\infty)$ is a positive constant the integral in (4.136) diverges to $+\infty$ as $\xi \to \pm\infty$ and the boundary condition (4.131) at $\xi = \pm\infty$ is satisfied. By using the conserved quantity (4.132) it can be verified that

$$b = \frac{D}{2\int_0^{\infty} \exp\left[-\int_0^{\xi} \frac{\gamma}{S(\gamma)} d\gamma\right] d\xi}. \tag{4.137}$$

Consider an effective viscosity of the form

$$E(x, y) = \frac{\nu + \nu_{T_0} K(\xi)}{\nu + \nu_{T_0}}, \tag{4.138}$$

where $K(\xi)$ has still to be specified. Then from (4.125),

$$\frac{\nu + \nu_{T_0} K(\xi)}{(\nu + \nu_{T_0}) S(\xi)} = H(x) H'(x). \tag{4.139}$$

It follows by the technique of separation of variables or by differentiating (4.139) with respect to ξ that each side of (4.139) is a constant, λ. We take $\lambda = 1$. Thus

$$S(\xi) = \frac{1 + \frac{\nu_{T_0}}{\nu} K(\xi)}{1 + \frac{\nu_{T_0}}{\nu}}, \quad H(x)H'(x) = 1, \tag{4.140}$$

and since $H(0) = 0$,

$$H(x) = \sqrt{2x}, \tag{4.141}$$

which gives

$$\xi = \frac{y}{\sqrt{2x}}. \tag{4.142}$$

The velocity deficit (4.133) can be expressed as

$$\overline{w}(x, y) = \frac{D \exp\left[-\left(1 + \frac{\nu_{T_0}}{\nu}\right) I(\xi)\right]}{2\sqrt{2x} \int_0^\infty \exp\left[-\left(1 + \frac{\nu_{T_0}}{\nu}\right) I(\gamma)\right] d\gamma}, \tag{4.143}$$

where

$$I(\xi) = \int_0^\xi \frac{\gamma}{1 + \frac{\nu_{T_0}}{\nu} K(\gamma)} d\gamma. \tag{4.144}$$

The x-component of the mean velocity $\overline{v}_x$ is given by

$$\overline{v}_x(x, y) = 1 - \overline{w}(x, y) = 1 - \frac{D \exp\left[-\left(1 + \frac{\nu_{T_0}}{\nu}\right) I(\xi)\right]}{2\sqrt{2x} \int_0^\infty \exp\left[-\left(1 + \frac{\nu_{T_0}}{\nu}\right) I(\gamma)\right] d\gamma}. \tag{4.145}$$

From (4.127) and (4.141) the Lie point symmetry is the scaling symmetry

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.146}$$

For $K(\xi)$ we choose

$$K_1(\xi) = 0, \;\; K_2(\xi) = \exp(-\xi^2), \;\; K_3(\xi) = \frac{1}{1 + \xi^2}, \;\; K_4(\xi) = 1. \tag{4.147}$$

Then

$$I_1(\xi) = \frac{1}{2}\xi^2, \quad I_4(\xi) = \frac{\xi^2}{2\left(1 + \frac{\nu_{T_0}}{\nu}\right)}, \tag{4.148}$$

where I_n is defined in terms of $K_n(\xi)$. It can be verified that (4.143) reduces to (4.84) for a laminar wake for $K_1(\xi) = 0$. Also $K_4(\xi) = 1$ describes a turbulent wake with constant eddy viscosity in agreement with Prandtl's hypothesis. We have

$$\overline{w}_4(x, y) = \frac{D}{2\sqrt{\pi x}} \exp\left(-\frac{1}{2}\xi^2\right). \tag{4.149}$$

Now

$$I_1(\xi) > I_2(\xi) > I_3(\xi) > I_4(\xi), \tag{4.150}$$

and therefore the maximum velocity deficit satisfies

$$\left(1 + \frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}} \frac{D}{2\sqrt{\pi x}} = \overline{w}_1(x,0) > \overline{w}_2(x,0) > \overline{w}_3(x,0) > \overline{w}_4(x,0) = \frac{D}{2\sqrt{\pi x}}. \tag{4.151}$$

There will be no backflow in the wakes if (4.93) is satisfied. It follows from the conserved quantity (4.132) that the effective widths of the wakes satisfy

$$\frac{4\sqrt{x}}{\left(1 + \frac{\nu_{T_0}}{\nu}\right)^{\frac{1}{2}}} = W_1(x) < W_2(x) < W_3(x) < W_4(x) = 4\sqrt{x}. \tag{4.152}$$

In Figure 4.3 the mean velocity (4.145) for the laminar wake and the three turbulent wakes described by (4.147) are plotted against ξ. The graphs confirm the inequalities (4.151) for the maximum velocity deficit and (4.152) for the effective width. The y-dependence of the eddy viscosity can only enter through the similarity variable ξ. The effect of the y-dependence of the eddy viscosity is to decrease the effective width of the wake and increase the mean velocity deficit. The effect is greater when the eddy viscosity decreases more rapidly with y.

4.4 Conclusions

The boundary layer approximation was imposed on the Reynolds averaged equations which were then formulated in terms of a stream function for the velocity deficit. Unlike the boundary layer formulation for jet flow problems a further approximation was made that the squares and products of the mean velocity deficit could be neglected.

The conserved vector for the classical wake was the elementary conserved vector which could be obtained directly by inspection from the PDE for the velocity deficit.

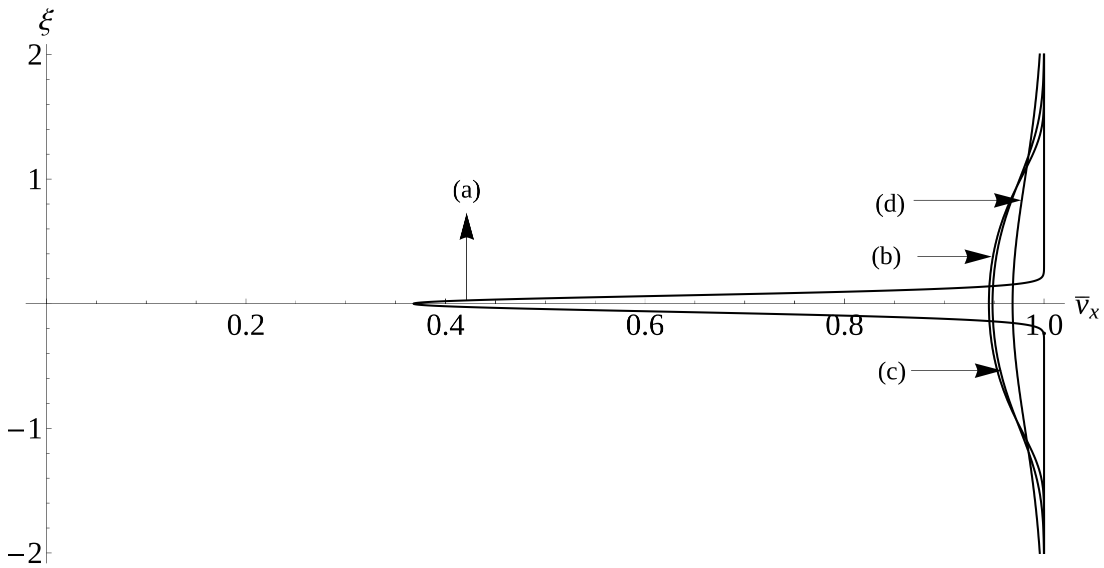

FIGURE 4.3: Mean velocity $\overline{v}_x(x, y)$ given by (4.145) plotted against $\xi = y/\sqrt{2x}$ at $x = \frac{1}{2}$ for $D = 0.1$, $\nu_{T_0}/\nu = 250$ and (a) laminar wake with $K(\xi) = 0$, (b) turbulent wake with $K(\xi) = \exp[-\xi^2]$, (c) turbulent wake with $K(\xi) = (1+\xi^2)^{-1}$ and (d) turbulent wake with $K(\xi) = 1$.

The two invariance conditions that had to be satisfied by the Lie point symmetry associated with the elementary conserved vector depended on only the first and second prolongations of the symmetry and were easier to work with than one large invariance condition for the full group of Lie point symmetries of the PDE which depends on prolongations up to third order.

When the effective viscosity is a function of x only, that is $E = E(x)$, the general solution for the Lie point symmetry could be derived. The ordinary differential equation and boundary conditions were the same as for the laminar wake but the similarity variable was different. When the eddy viscosity is a power law in x and the kinematic viscosity is not neglected, the Lie point symmetry is not a scaling symmetry. The solution generated by this symmetry is important as it allowed the transition from a laminar wake to a turbulent wake to be investigated by gradually increasing the strength of the turbulence. The solution cannot be derived by using a scaling transformation to find the similarity variable. A Lie point symmetry analysis was required. When the kinematic viscosity was neglected the Lie point symmetry reduced to a scaling symmetry.

When $E = E(x, y)$, that is, when the eddy viscosity is not constant across the wake, the general solution for the associated Lie point symmetry could not be found. A solution was derived for the special case in which $\xi^2(x, y)$ is a linear function of y

which was shown to have constant coefficients as for the laminar wake. An analytical solution for the effective viscosity and the mean velocity deficit was derived. This compares with two-dimensional turbulent jet flow for which the general solution for the Lie point symmetry could be derived [18].

The effect of the eddy viscosity was to increase the diffusion of vorticity across the wake and therefore increase the effective width of the wake. Because of the conserved quantity for the classical wake, (4.6), an increase in the effective width causes a decrease in the maximum velocity deficit for the wake. The product of the effective width of the wake with the maximum velocity deficit is approximately equal to the dimensionless drag at all positions x on the axis of the wake. This result may be useful in estimating the effective width of a classical turbulent wake because the maximum mean velocity deficit is readily obtained from the solution for the mean velocity deficit. As the turbulent wake evolves from the laminar wake and the strength of the turbulence increases, the effective width of the wake increases and its maximum velocity deficit decreases at each point on its axis.

Prandtl's hypothesis on the eddy viscosity has been applied successfully to free boundary layer flows. It states that the eddy viscosity is constant across the layer and is proportional to the product of the maximum mean velocity and the width of the layer. If the maximum mean velocity can be replaced by the maximum mean velocity deficit then Prandtl's hypothesis applied to the classical wake would require the eddy viscosity to be constant. This is a consequence of the conserved quantity for the classical wake. The mean velocity deficit can depend on y only through the similarity variable. We found that, compared with a constant eddy viscosity, an eddy viscosity that depended on y decreased the effective width and increased the maximum mean velocity deficit of the wake. The stronger the dependence on y the greater the effect.

Chapter 5

Lie symmetry methods applied to the turbulent wake of a symmetric self-propelled body

In this chapter we consider the turbulent wake of a self-propelled body as shown in Figure 2.2. We first examine the case for an eddy viscosity depending on the spacial variable x only. An outline of this chapter is as follows. In Section 5.1 the Lie point symmetry associated with the conserved vector that was obtained in Chapter 3 in terms of the stream function is found. The invariant solution is derived in Section 5.2. An eddy viscosity in the form of a power law of the distance along the axis of the wake is considered and plots of the mean velocity profiles are provided. We also examine the negative effects of excluding the kinematic viscosity as opposed to including it. Section 5.3 contains a discussion on the anticipated difficulties that arise when we consider the eddy viscosity to be a function of both the distance along the axis of the wake and the perpendicular distance from the axis of the wake. Conclusions are given in Section 5.4.

5.1 Conserved vector and associated Lie point symmetry

In terms of the stream function $\psi(x,y)$ the governing equation for $E = E(x,y)$ is given by

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial^2\psi}{\partial y^2}\right). \tag{5.1}$$

The boundary conditions, for $x \geq 0$, are

$$\frac{\partial\psi}{\partial y}(x,\pm\infty) = 0, \quad \frac{\partial^2\psi}{\partial y^2}(x,\pm\infty) = 0, \tag{5.2}$$

$$\frac{\partial\psi}{\partial x}(x,0) = 0, \quad \frac{\partial^2\psi}{\partial y^2}(x,0) = 0. \tag{5.3}$$

The conserved quantity is obtained for the wake behind a self-propelled body by multiplying equation (5.1) by y^2 and then integrating across the wake from $y = -\infty$ to $y = \infty$ [5]. This method works for a constant eddy viscosity and for an eddy viscosity depending on only the distance along the axis of the wake. We will see that we cannot solve the equation when the eddy viscosity also depends on the distance perpendicular to the axis of the wake. In this chapter we neglect the dependence of the eddy viscosity on the distance perpendicular to the axis of the wake but we will discuss it briefly in Section 5.3.

According to [5] the conserved quantity for the wake of a self-propelled body is given by

$$\int_{-\infty}^{\infty} y^2\frac{\partial\psi}{\partial y}dy = K, \tag{5.4}$$

and since the drag force is also zero

$$\int_{-\infty}^{\infty} \frac{\partial\psi}{\partial y}dy = 0. \tag{5.5}$$

The conserved vector with $E = E(x)$ was calculated in Chapter 3. It is given by

$$T^1 = y^2\psi_y, \tag{5.6}$$

$$T^2 = -E(x)y^2\psi_{yy} + 2E(x)y\psi_y - 2E(x)\psi, \tag{5.7}$$

which generates the conserved quantity given in equation (5.4). When $E = E(x)$ the conserved quantity for a wake of a self-propelled body has the same form as for a laminar wake of a self-propelled body.

We now calculate the Lie point symmetry associated with the conserved vector with components (5.6) and (5.7). The conserved vector $T = (T^1, T^2)$ is invariant under the action of the Lie point symmetry

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}, \tag{5.8}$$

provided [30, 31]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2, \tag{5.9}$$

where X is prolongated to as high an order in the derivatives as required. Equation (5.9) consists of two components, namely,

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{5.10}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.11}$$

From (5.6) the highest derivative of ψ in T^1 is ψ_y and from (5.7) the highest derivative of ψ in T^2 is ψ_{yy}. Thus we require the second prolongation of X. The second prolongation of X, denoted by $X^{[2]}$, is given as

$$X^{[2]} = X + \zeta_2\frac{\partial}{\partial \psi_y} + \zeta_{22}\frac{\partial}{\partial \psi_{yy}}, \tag{5.12}$$

where [65]

$$\zeta_2 = D_2(\eta) - \psi_k D_2(\xi^k), \tag{5.13}$$

$$\zeta_{22} = D_2(\zeta_2) - \psi_{2k} D_2(\xi^k). \tag{5.14}$$

Consider the first invariance condition (5.10). The expansion of ζ_2 is

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \psi_x\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right) - \psi_y\left(\frac{\partial \xi^2}{\partial y} + \frac{\partial \xi^2}{\partial \psi}\psi_y\right). \tag{5.15}$$

Equation (5.10) yields

$$2y\xi^2\psi_y + y^2\frac{\partial \eta}{\partial y} + y^2\frac{\partial \eta}{\partial \psi}\psi_y - y^2\psi_x\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right)$$

$$+\left(E(x)y^2\psi_{yy}-2E(x)y\psi_y+2E(x)\psi\right)\left(\frac{\partial\xi^1}{\partial y}+\frac{\partial\xi^1}{\partial\psi}\psi_y\right)=0. \tag{5.16}$$

Equating the coefficients of the derivatives $\psi_x\psi_y$ to zero gives

$$\xi^1=B(x), \tag{5.17}$$

and (5.16) simplifies to

$$2y\xi^2\psi_y+y^2\frac{\partial\eta}{\partial y}+y^2\frac{\partial\eta}{\partial\psi}\psi_y=0. \tag{5.18}$$

Separating equation (5.18) by ψ_y we obtain

$$\eta=\eta(x,\psi), \tag{5.19}$$

$$2\xi^2+y\frac{\partial\eta}{\partial\psi}=0. \tag{5.20}$$

The first invariance condition therefore gives

$$\xi^1=B(x),\quad \xi^2=-\frac{1}{2}y\frac{\partial\eta}{\partial\psi}(x,\psi),\quad \eta=\eta(x,\psi). \tag{5.21}$$

Consider next the second invariance condition (5.11). Equation (5.15) for ζ_2 simplifies to

$$\zeta_2=\frac{\partial\eta}{\partial\psi}\psi_y-\psi_y\left(\frac{\partial\xi^2}{\partial y}+\frac{\partial\xi^2}{\partial\psi}\psi_y\right). \tag{5.22}$$

Since ξ^1 does not depend on y and ψ, equation (5.14) for ζ_{22} becomes

$$\zeta_{22}=D_2(\zeta_2)-\psi_{22}D_2(\xi^2). \tag{5.23}$$

There is only one term in the second invariance condition (5.11) which depends on ψ_x. It comes from $T^1\dfrac{\partial\xi^2}{\partial\psi}\psi_x$. Separating (5.11) by the derivative product $\psi_x\psi_y$ yields

$$\frac{\partial\xi^2}{\partial\psi}=0. \tag{5.24}$$

Differentiating equation (5.20) with respect to y and using (5.19), we obtain

$$\frac{\partial\eta}{\partial\psi}=-2\frac{\partial\xi^2}{\partial y}, \tag{5.25}$$

and therefore

$$\zeta_2 = -3\psi_y \frac{\partial \xi^2}{\partial y}. \tag{5.26}$$

From (5.25)

$$\frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{5.27}$$

and thus

$$\xi^2(x, y) = a(x)y + b(x). \tag{5.28}$$

Substituting (5.28) back into (5.20), separating by y and integrating the coefficient of y with respect to ψ results in

$$\eta(x,\psi) = -2a(x)\psi + c(x), \quad b(x) = 0. \tag{5.29}$$

From (5.26) and (5.28) we arrive at

$$\zeta_2 = -3a(x)\psi_y, \tag{5.30}$$

and equation (5.14) reduces to

$$\zeta_{22} = -4a(x)\psi_{yy}. \tag{5.31}$$

The second invariance condition (5.11) becomes

$$\begin{gathered} 2E(x)y^2\psi_{yy}a(x) - 4yE(x)a(x)\psi_y + \\ 4a(x)\psi E(x) - 2c(x)E(x) + B(x)E'(x)(-y^2\psi_{yy} + 2y\psi_y - 2\psi) + \\ (-E(x)y^2\psi_{yy} + 2yE(x)\psi_y - 2\psi E(x))B'(x) - y^2\psi_y a'(x)y = 0. \end{gathered} \tag{5.32}$$

Setting the coefficient of ψ_{yy} to zero gives

$$2E(x)a(x) - B(x)E'(x) - E(x)B'(x) = 0. \tag{5.33}$$

Separating (5.32) by derivatives of ψ_y we obtain

$$-4a(x)yE + 2yB(x)E'(x) + 2yB'(x)E - y^3a'(x) = 0. \tag{5.34}$$

Separating (5.34) by powers of y we deduce that

$$a(x) = a_1, \tag{5.35}$$

where a_1 is a constant and equation (5.33) again. The remaining terms in (5.32) give

$$4a(x)E\psi - 2c(x)E - 2B(x)E'(x)\psi - 2EB'(x)\psi = 0. \tag{5.36}$$

Separating by powers of ψ we obtain

$$c(x) = 0, \tag{5.37}$$

and (5.33) again. Thus we have found that

$$\xi^1 = B(x), \ \ \xi^2 = a_1 y, \ \ \eta = -2a_1\psi, \tag{5.38}$$

subject to (5.33). Condition (5.33) can be written as the first order ODE for $B(x)$,

$$\frac{dB}{dx} + \frac{1}{E(x)}\frac{dE}{dx}B = 2a_1. \tag{5.39}$$

This is the same as the condition in a classical wake with $E = E(x)$ for a Lie point symmetry to exist associated with the elementary conserved vector of the PDE (5.1). The solution of (5.39) is

$$B(x) = \frac{1}{E(x)}\left[a_2 + 2a_1\int_0^x E(\alpha)\,d\alpha\right], \tag{5.40}$$

where $a_2 = E(0)B(0)$ is a constant. The Lie point symmetry X associated with the conserved vector with components (5.6) and (5.7) is

$$X = \frac{1}{E(x)}\left[a_2 + 2a_1\int_0^x E(\alpha)\,d\alpha\right]\frac{\partial}{\partial x} + a_1 y\frac{\partial}{\partial y} - 2a_1\psi\frac{\partial}{\partial \psi}. \tag{5.41}$$

We consider the general case in which $a_1 \neq 0$. Without loss of generality we let $a_1 = 1$ and denoting a_2 by a we obtain

$$X = \frac{1}{E(x)}\left[a + 2\int_0^x E(\alpha)\,d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}. \tag{5.42}$$

Unlike the classical wake the associated Lie point symmetry is determined once the effective viscosity has been specified.

5.2 Invariant solution

In this section we find the invariant solution generated by the Lie point symmetry (5.42) associated with the conserved vector of the PDE (5.1) with components (5.6) and (5.7).

Now, $\psi = \Psi(x, y)$ is an invariant solution of the PDE (5.1) with $E = E(x)$ generated by the Lie point symmetry (5.42) provided

$$X(\psi - \Psi(x, y))\big|_{\psi=\Psi} = 0, \tag{5.43}$$

that is, provided $\Psi(x, y)$ satisfies the first order PDE

$$\frac{1}{E(x)}\left(a + 2\int_0^x E(\alpha)d\alpha\right)\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y} = -2\Psi. \tag{5.44}$$

The differential equations of the characteristic curves of (5.44) are

$$\frac{E(x)dx}{a + 2\int_0^x E(\alpha)d\alpha} = \frac{dy}{y} = -\frac{d\Psi}{2\Psi}. \tag{5.45}$$

Solving the first pair of terms gives

$$\frac{y}{(a + 2\int_0^x E(\alpha)d\alpha)^{1/2}} = c_1, \tag{5.46}$$

where c_1 is a constant. The first and last terms in (5.45) give

$$\Psi\left(a + 2\int_0^x E(\alpha)d\alpha\right) = c_2, \tag{5.47}$$

where c_2 is a constant. The general solution of the first order linear PDE (5.44) is

$$c_2 = F(c_1), \tag{5.48}$$

where F is an arbitrary function and since $\Psi = \psi$ we obtain

$$\psi(x, y) = \frac{F(\xi)}{a + 2\int_0^x E(\alpha)d\alpha}, \tag{5.49}$$

where

$$\xi = \frac{y}{(a + 2\int_0^x E(\alpha)d\alpha)^{1/2}}. \tag{5.50}$$

Substituting (5.49) and (5.50) into the PDE (5.1) reduces the PDE to an ODE for $F(\xi)$:

$$\frac{d^3F}{d\xi^3}+\xi\frac{d^2F}{d\xi^2}+3\frac{dF}{d\xi}=0. \tag{5.51}$$

The mean velocity deficit $\overline{w}$ and the mean velocity components $\overline{v}_x$ and $\overline{v}_y$ are given by

$$\overline{w}(x,y)=\frac{\partial\psi}{\partial y}=\frac{1}{\left(a+2\int_0^x E(\alpha)d\alpha\right)^{3/2}}\frac{dF}{d\xi}, \tag{5.52}$$

$$\overline{v}_x(x,y)=1-\overline{w}(x,y)=1-\frac{1}{\left(a+2\int_0^x E(\alpha)d\alpha\right)^{3/2}}\frac{dF}{d\xi}, \tag{5.53}$$

$$\overline{v}_y(x,y)=\frac{\partial\psi}{\partial x}=-\frac{E(x)}{\left(a+2\int_0^x E(\alpha)d\alpha\right)^{2}}\left(2F+\xi\frac{dF}{d\xi}\right). \tag{5.54}$$

Since the mean velocity deficit is finite it follows that $F'(0)$ is finite. In terms of the function $F(\xi)$, the boundary conditions (5.2) and (5.3) become

$$\frac{dF}{d\xi}(\pm\infty)=0,\quad \frac{d^2F}{d\xi^2}(\pm\infty)=0, \tag{5.55}$$

$$F(0)=0,\quad \frac{d^2F}{d\xi^2}(0)=0. \tag{5.56}$$

Consider now the solution of the ODE (5.51) subject to the boundary conditions (5.55) and (5.56). We let

$$W(\xi)=\frac{dF}{d\xi}. \tag{5.57}$$

Multiplying (5.51) by ξ^2-1 and grouping terms we obtain

$$\frac{d}{d\xi}\left(\xi^2\frac{dW}{d\xi}\right)-\frac{d^2W}{d\xi^2}-3\frac{d}{d\xi}(\xi W)+\frac{d}{d\xi}\left(\xi^3W\right)=0, \tag{5.58}$$

and by integrating once with respect to ξ and using the second derivative boundary condition in (5.56) at $\xi=0$ we obtain the first order linear homogeneous ODE

$$\frac{dW}{d\xi}+\frac{\xi(3-\xi^2)}{1-\xi^2}W=0. \tag{5.59}$$

The solution to equation (5.59) is

$$W(\xi)=m(1-\xi^2)\exp[-\xi^2/2], \tag{5.60}$$

where m is a constant. By integrating the first term in (5.60) by parts and imposing the boundary condition $F(0) = 0$ it is readily shown that

$$F(\xi) = m\xi \exp[-\xi^2/2]. \tag{5.61}$$

The constant m cannot be obtained from the boundary conditions (5.55) and (5.56) which are identically satisfied. The constant m is determined by the conserved quantity (5.4). Expressed in terms of $W(\xi)$, the conserved quantity (5.4) becomes

$$\int_{-\infty}^{\infty} \xi^2 W(\xi) = K. \tag{5.62}$$

Using the properties of the Gamma function we find that

$$m = -\frac{K}{2\sqrt{2\pi}}, \tag{5.63}$$

and therefore

$$W(\xi) = -\frac{K}{2\sqrt{2\pi}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.64}$$

and

$$F(\xi) = -\frac{K}{2\sqrt{2\pi}}\xi \exp[-\xi^2/2]. \tag{5.65}$$

From (5.52)-(5.54) the mean velocity deficit and the mean velocity components in the x- and y- directions are

$$\overline{w}(x,y) = -\frac{K}{2\sqrt{2\pi}} \frac{1}{\left(a + 2\int_0^x E(\alpha)\,d\alpha\right)^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.66}$$

$$\overline{v}_x(x,y) = 1 - \overline{w}(x,y) = 1 + \frac{K}{2\sqrt{2\pi}} \frac{1}{\left(a + 2\int_0^x E(\alpha)\,d\alpha\right)^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.67}$$

$$\overline{v}_y(x,y) = \frac{K}{2\sqrt{2\pi}} \frac{E(x)}{\left(a + 2\int_0^x E(\alpha)\,d\alpha\right)^{2}}\xi(3-\xi^2)\exp[-\xi^2/2]. \tag{5.68}$$

Consider now the constant a. We would expect the effective width of the wake to vanish as $x \to 0$. The mean velocity deficit $\overline{w}$ tends exponentially to zero as $y \to \pm\infty$. We define the effective half-width of the wake, $H(x)$, to be the value of y for which the argument of the exponential in (5.66) is -1. Then $\xi = \pm\sqrt{2}$ and

$$H(x) = \sqrt{2}\left(a + 2\int_0^x E(\alpha)\,d\alpha\right)^{1/2}. \tag{5.69}$$

We assume that $\int_0^x E(\alpha)d\alpha \to 0$ as $x \to 0$. Then $H(x) \to 0$ as $x \to 0$ provided $a = 0$. Thus

$$H(x) = 2\left(\int_0^x E(\alpha)d\alpha\right)^{1/2}, \tag{5.70}$$

and the similarity variable ξ becomes

$$\xi(x,y) = \frac{y}{\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}}. \tag{5.71}$$

Equations (5.66)-(5.68) reduce to

$$\overline{w}(x,y) = -\frac{K}{2\sqrt{2\pi}}\frac{1}{\left(2\int_0^x E(\alpha)d\alpha\right)^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.72}$$

$$\overline{v}_x(x,y) = 1-\overline{w}(x,y) = 1+\frac{K}{2\sqrt{2\pi}}\frac{1}{\left(2\int_0^x E(\alpha)d\alpha\right)^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.73}$$

$$\overline{v}_y(x,y) = \frac{K}{2\sqrt{2\pi}}\frac{E(x)}{\left(2\int_0^x E(\alpha)d\alpha\right)^{2}}\xi(3-\xi^2)\exp[-\xi^2/2]. \tag{5.74}$$

Finally we establish some general properties valid for all effective viscosities $E(x)$. The mean velocity deficit vanishes at $\xi = \pm 1$, that is, at

$$y = \pm y_1 = \pm\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2} = \frac{H(x)}{\sqrt{2}}. \tag{5.75}$$

Unlike the classical wake the mean velocity deficit on the x-axis is negative. The turning points of the mean velocity deficit are at $\xi = 0$ and $\xi = \pm\sqrt{3}$ and occur at

$$y = 0, \quad y = \pm y_2 = \pm\sqrt{3}\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2} = \sqrt{\frac{3}{2}}H(x). \tag{5.76}$$

The extremum values are

$$\overline{w}(x,0) = -\frac{K}{2\sqrt{2\pi}}\frac{1}{\left(2\int_0^x E(\alpha)d\alpha\right)^{3/2}}, \tag{5.77}$$

$$\overline{w}(x,\pm y_2) = \frac{K}{\sqrt{2\pi}}\frac{1}{\left(2\int_0^x E(\alpha)d\alpha\right)^{3/2}}\exp[-3/2]. \tag{5.78}$$

The ratio

$$\frac{\overline{w}(x,\pm y_2)}{\overline{w}(x,0)} = -2\exp[-3/2], \tag{5.79}$$

is satisfied for all effective viscosities $E = E(x)$.

As well as being turning points of $\overline{w}$, the mean velocity component $\overline{v}_y$ vanishes at $\xi = \pm\sqrt{3}$. The velocity component $\overline{v}_y$ therefore vanishes on the curves

$$y = \pm\sqrt{3}\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}. \tag{5.80}$$

For $y_2 < y < \infty$, $\overline{v}_y < 0$ while for $0 < y < y_2$, $\overline{v}_y > 0$ and similarly for the lower half of the wake.

The estimate (5.70) of the effective half-width $H(x)$ excludes the outer extrema of the wake and the curve of zero $\overline{v}_y$. A useful alternative estimate of the effective half-width could be

$$H(x) = y_2 = \sqrt{3}\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}. \tag{5.81}$$

Since $a = 0$, the Lie point symmetry which generates the invariant solution when $E = E(x)$ is

$$X = \frac{1}{E(x)}\left[2\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}. \tag{5.82}$$

The turbulence occurs only in the coefficient of $\dfrac{\partial}{\partial x}$.

Finally we derive the condition on the eddy viscosity for Prandtl's hypothesis to be satisfied [66]. For a turbulent boundary layer Prandtl's hypothesis states that the eddy viscosity is constant across the boundary layer and is proportional to the product of the maximum mean velocity and the width of the layer. For a turbulent wake it was proposed that Prandtl's hypothesis applies with the mean velocity replaced by the mean velocity deficit. Since $\nu \ll \nu_T$ we will apply Prandtl's hypothesis to the effective viscosity $E(x)$. Then if Prandtl's hypothesis is satisfied

$$E(x) \propto \overline{w}(x,0)H(x), \tag{5.83}$$

and by using equation (5.77) for $\overline{w}(x,0)$ and (5.70) for $H(x)$ we find that Prandtl's hypothesis is satisfied for the wake behind a self-propelled body provided

$$E(x)\int_0^x E(\alpha)d\alpha = \lambda, \tag{5.84}$$

where λ is independent of x. For the turbulent classical wake the corresponding condition is that $E(x)$ is a constant independent of x. Condition (5.84) will be used in Section 5.2.1 to determine the exponent in the power law for $E(x)$.

5.2.1 Eddy viscosity as a power law in x

Consider a dimensionless effective viscosity of the form

$$E(x) = \frac{\nu}{\nu+\nu_{T_0}} + \frac{\nu_{T_0}}{\nu+\nu_{T_0}} x^{\beta}, \quad \beta > -1. \tag{5.85}$$

Then

$$\int_0^x E(\alpha)\,d\alpha = \frac{x}{\nu+\nu_{T_0}}\left(\nu+\nu_{T_0}\frac{x^{\beta}}{(1+\beta)}\right), \tag{5.86}$$

and the Lie point symmetry X from (5.82) becomes

$$X = \left[\frac{2x}{\nu+\nu_{T_0}x^{\beta}}\left(\nu+\nu_{T_0}\frac{x^{\beta}}{(1+\beta)}\right)\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}, \tag{5.87}$$

which is not a scaling symmetry hence justifying the need to use the Lie point symmetry approach instead of simply searching for a similarity solution. The effective half-width of the wake is given by

$$H(x) = 2\left[\frac{x}{1+\frac{\nu_{T_0}}{\nu}}\left(1+\frac{\nu_{T_0}}{\nu}\frac{x^{\beta}}{(1+\beta)}\right)\right]^{1/2}, \tag{5.88}$$

and the similarity variable ξ becomes

$$\xi(x,y) = \left(1+\frac{\nu_{T_0}}{\nu}\right)^{1/2}\frac{y}{\left[2x\left(1+\frac{\nu_{T_0}}{\nu}\frac{x^{\beta}}{(1+\beta)}\right)\right]^{1/2}}. \tag{5.89}$$

Equations (5.72)-(5.74) become

$$\overline{w}(x,y) = -\frac{K}{8\sqrt{\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)^{3/2}\frac{1}{\left[x\left(1+\frac{\nu_{T_0}}{\nu}\frac{x^{\beta}}{(1+\beta)}\right)\right]^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.90}$$

$$\overline{v}_x(x,y) = 1-\overline{w}(x,y) = 1+\frac{K}{8\sqrt{\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)^{3/2}\frac{1}{\left[x\left(1+\frac{\nu_{T_0}}{\nu}\frac{x^{\beta}}{(1+\beta)}\right)\right]^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.91}$$

$$\overline{v}_y(x,y) = \frac{K}{8\sqrt{2\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)\frac{\left(1+\frac{\nu_{T_0}}{\nu}x^{\beta}\right)}{\left[x\left(1+\frac{\nu_{T_0}}{\nu}\frac{x^{\beta}}{(1+\beta)}\right)\right]^2}\xi(3-\xi^2)\exp[-\xi^2/2]. \tag{5.92}$$

From (5.75) the mean velocity deficit vanishes at

$$y = \pm y_1 = \pm \left[\frac{2x}{1 + \frac{\nu_{T_0}}{\nu}} \left(1 + \frac{\nu_{T_0}}{\nu} \frac{x^\beta}{(1+\beta)} \right) \right]^{1/2}. \tag{5.93}$$

From (5.76) and (5.77) the turning points of the mean velocity deficit are at $\xi = 0$ and $\xi = \pm\sqrt{3}$ and occur at

$$y = 0, \quad y = \pm y_2 = \pm\sqrt{3} \left[\frac{2x}{1 + \frac{\nu_{T_0}}{\nu}} \left(1 + \frac{\nu_{T_0}}{\nu} \frac{x^\beta}{(1+\beta)} \right) \right]^{1/2}. \tag{5.94}$$

The extremum values are

$$\overline{w}(x,0) = -\frac{K}{8\sqrt{\pi}} \left(1 + \frac{\nu_{T_0}}{\nu} \right)^{3/2} \frac{1}{\left[x \left(1 + \frac{\nu_{T_0}}{\nu} \frac{x^\beta}{(1+\beta)} \right) \right]^{3/2}}, \tag{5.95}$$

$$\overline{w}(x, \pm y_2) = \frac{K}{4\sqrt{\pi}} \left(1 + \frac{\nu_{T_0}}{\nu} \right)^{3/2} \frac{1}{\left[x \left(1 + \frac{\nu_{T_0}}{\nu} \frac{x^\beta}{(1+\beta)} \right) \right]^{3/2}} \exp[-3/2]. \tag{5.96}$$

From (5.80) the velocity component $\overline{v}_y$ vanishes on the curves

$$y = \pm\sqrt{3} \left[\frac{2x}{1 + \frac{\nu_{T_0}}{\nu}} \left(1 + \frac{\nu_{T_0}}{\nu} \frac{x^\beta}{(1+\beta)} \right) \right]^{1/2}. \tag{5.97}$$

We compare the turbulent wake of a self-propelled body with the laminar wake of a self-propelled body. When performing the comparison suffices T and L will be used where necessary to denote the turbulent and laminar flow quantities. For the laminar wake of a self-propelled body we denote the half-width by $H_L(x)$, the velocity deficit by w_L and the x- and y- components of the velocities by v_x and v_y respectively. The dimensionless kinematic viscosity of the laminar wake is

$$E_L(x) = \frac{\nu}{\nu + \nu_{T_0}}. \tag{5.98}$$

The Lie point symmetry X_L, from (5.82), becomes

$$X_L = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}. \tag{5.99}$$

The effective half-width of the wake is given by

$$H_L(x) = 2\left[\frac{x}{1+\dfrac{\nu_{T_0}}{\nu}}\right]^{1/2}, \tag{5.100}$$

and the similarity variable ξ becomes

$$\xi_L(x,y) = \left(1+\frac{\nu_{T_0}}{\nu}\right)^{1/2}\frac{y}{\sqrt{2x}}. \tag{5.101}$$

Equations (5.72)-(5.74) reduce to

$$w(x,y) = -\frac{K}{8\sqrt{\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)^{3/2}\frac{1}{x^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.102}$$

$$v_x(x,y) = 1-w(x,y) = 1+\frac{K}{8\sqrt{\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)^{3/2}\frac{1}{x^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \tag{5.103}$$

$$v_y(x,y) = \frac{K}{8\sqrt{2\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)\frac{1}{x^2}\xi(3-\xi^2)\exp[-\xi^2/2]. \tag{5.104}$$

From (5.75) the velocity deficit vanishes at

$$y = \pm y_1^L = \pm\left[\frac{2x}{1+\dfrac{\nu_{T_0}}{\nu}}\right]^{1/2}, \tag{5.105}$$

and the turning points occur at

$$y=0, \quad y = \pm y_2^L = \pm\sqrt{3}\left[\frac{2x}{1+\dfrac{\nu_{T_0}}{\nu}}\right]^{1/2}. \tag{5.106}$$

The extremum values are

$$w(x,0) = -\frac{K}{8\sqrt{\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)^{3/2}\frac{1}{x^{3/2}}, \tag{5.107}$$

$$w(x,\pm y_2^L) = \frac{K}{4\sqrt{\pi}}\left(1+\frac{\nu_{T_0}}{\nu}\right)^{3/2}\frac{1}{x^{3/2}}\exp[-3/2]. \tag{5.108}$$

From (5.80) the velocity component $\overline{v}_y$ vanishes on the curves

$$y = \pm\sqrt{3}\left[\frac{2x}{1+\dfrac{\nu_{T_0}}{\nu}}\right]^{1/2}. \tag{5.109}$$

The ratio of the half-widths of the turbulent to the laminar wake of a self-propelled body is given by

$$\frac{H_T(x)}{H_L(x)} = \left[1+\frac{\nu_{T_0}}{\nu}\frac{x^\beta}{(1+\beta)}\right]^{1/2}, \tag{5.110}$$

which is the same expression found in Chapter 4 for the classical wake. An increase in the strength of the turbulence, that is, an increase in the ratio ν_{T_0}/ν results in an increase in the effective width of the wake. The eddy viscosity increases the diffusion of the velocity deficit from the axis of the wake.

The ratios of the extremum values of the mean velocity deficit of the turbulent wake of a self-propelled body to that of the laminar wake of a self-propelled body are

$$\frac{\overline{w}(x,0)}{w_L(x,0)} = \frac{\overline{w}(x,\pm y_2)}{w_L(x,\pm y_2^L)} = \frac{1}{\left[1+\dfrac{\nu_{T_0}}{\nu}\dfrac{x^\beta}{(1+\beta)}\right]^{3/2}}, \tag{5.111}$$

which means that the magnitude of the mean velocity deficit in the x-direction decreases as the strength of the turbulence increases. We observe that

$$y_1 = \frac{H(x)}{\sqrt{2}}, \quad y_2 = H(x)\sqrt{\frac{3}{2}}, \tag{5.112}$$

and so the actual width of the wake does not increase uniformly as the strength of the turbulence increases; the extremum point y_2 increases slightly faster than y_1.

The first part of Prandtl's hypothesis, that the eddy viscosity is constant across the wake, is satisfied because $E(x)$ does not depend on y. The second part of Prandtl's hypothesis is satisfied provided (5.83) holds which with $E(x)$ given by (5.84) becomes

$$x^{1+2\beta} + (2+\beta)\frac{\nu}{\nu_{T_0}}x^{1+\beta} + (1+\beta)\left(\frac{\nu}{\nu_{T_0}}\right)^2 x = (1+\beta)\left(1+\frac{\nu}{\nu_{T_0}}\right)^2\lambda. \tag{5.113}$$

For $\nu \ll \nu_{T_0}$, equation (5.113) reduces to

$$x^{1+2\beta} = (1+\beta)\lambda, \tag{5.114}$$

which is satisfied provided $\beta = -1/2$. The turbulence decreases in the wake behind a self-propelled body as we move further downstream. This compares for a classical wake for which $\beta = 0$ and the eddy viscosity remains constant for Prandtl's hypothesis to be satisfied. We will not make the Prandtl hypothesis but we will include $\beta = -1/2$ in the analysis of the results.

We first consider the effect of β. Since the magnitudes of the mean velocity deficits differ greatly for different β values, we separate the problem into three cases. In all the cases we let $\nu_{T_0}/\nu = 500$ and $x = 0.25$. For $\beta = -1/2$, we consider $K = 1$ and in order to compare the effect of negative β we also consider $\beta = -1/3,\ -1/4$. For $\beta = 0,\ 1$, we let $K = 0.1$. For $\beta = 2,\ 3,\ 4$, we let $K = 0.0001$ and compare these particular values of β with the laminar wake of a self-propelled body. Plots of the velocity profiles are presented in Figures 5.1, 5.2 and 5.3.

We see that for $x < 1$, as the exponent β increases, the maximum mean velocity of the wake increases and the width of the wake decreases. The reverse situation is true for $x > 1$. We omit the plots for $x > 1$ because the mean velocities are very small and difficult to distinguish. From (5.111) we see that for small β, that is, $-1 < \beta \leq 1$, the maximum mean velocity in the turbulent wake is very small compared with the laminar wake and is thus negligible.

Next we consider the effect of the strength of the turbulence on the x-component of the mean velocity. We replace the characteristic effective viscosity by $\nu + \nu_{T_C}$ where ν_{T_C} is suitably chosen. This approach was also done in Chapter 4 for the classical wake. We have that

$$\overline{v}_x(x,y) = 1 + \frac{K}{8\sqrt{\pi}}\left(1+\frac{\nu_{T_C}}{\nu}\right)^{3/2} \frac{1}{\left[x\left(1+\dfrac{\nu_{T_0}}{\nu}\dfrac{x^\beta}{(1+\beta)}\right)\right]^{3/2}}(1-\xi^2)\exp[-\xi^2/2], \quad (5.115)$$

where

$$\xi = \left(1+\frac{\nu_{T_C}}{\nu}\right)^{1/2} \frac{y}{\left[2x\left(1+\dfrac{\nu_{T_0}}{\nu}\dfrac{x^\beta}{(1+\beta)}\right)\right]^{1/2}}. \quad (5.116)$$

In Figure 5.4 the x- component of the mean velocity for the turbulent wake is plotted against y for a range of values of ν_{T_0}/ν with $\beta = 4$. As ν_{T_0}/ν increases the width of the wake increases and the maximum velocity deficit decreases.

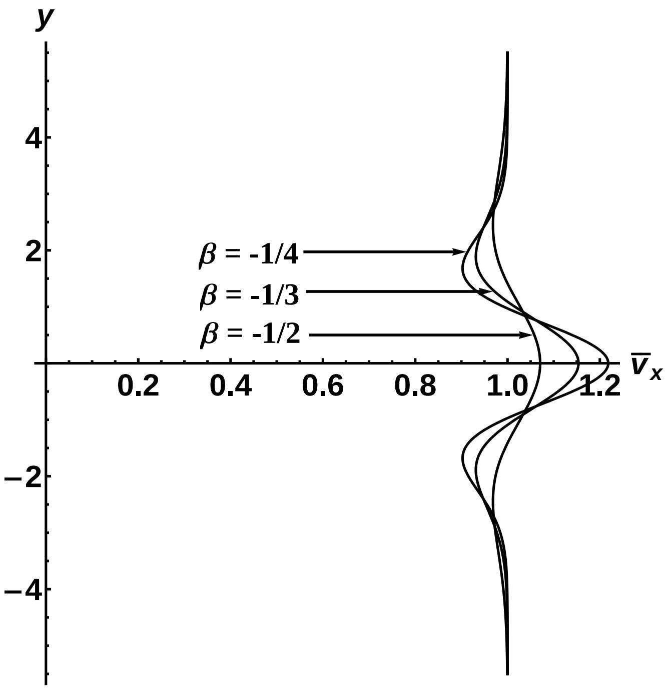

FIGURE 5.1: Plots of the velocity profiles for the turbulent wake $\overline{v}_x(x, y)$ against y at $x = 0.25$ with $K = 1$ and $\nu_{T_0}/\nu = 500$ for $\beta = -1/2,\ -1/3,\ -1/4$.

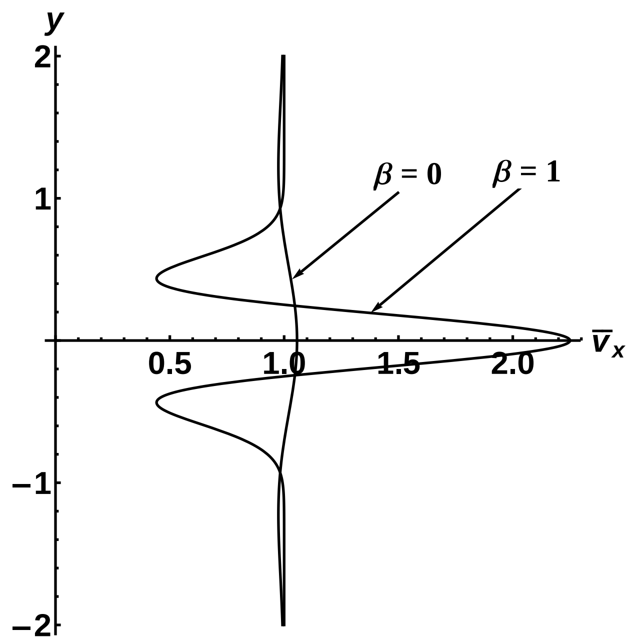

FIGURE 5.2: Plots of the velocity profiles for the turbulent wake $\overline{v}_x(x, y)$ against y at $x = 0.25$ with $K = 0.1$ and $\nu_{T_0}/\nu = 500$ for $\beta = 0,\ 1$.

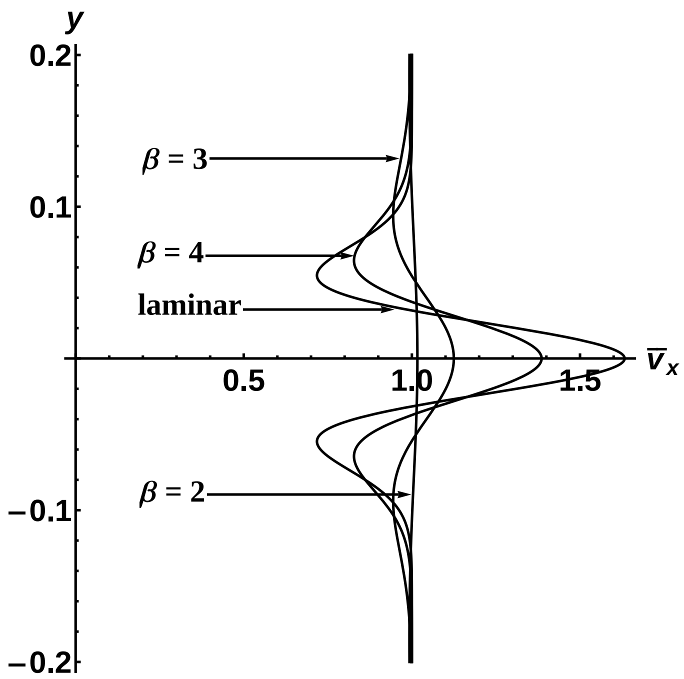

FIGURE 5.3: Plots of the velocity profiles for the laminar wake $v_x(x, y)$ and the turbulent wake $\overline{v}_x(x, y)$ against y at $x = 0.25$ with $K = 0.0001$ and $\nu_{T_0}/\nu = 500$ for $\beta = 2, 3, 4$.

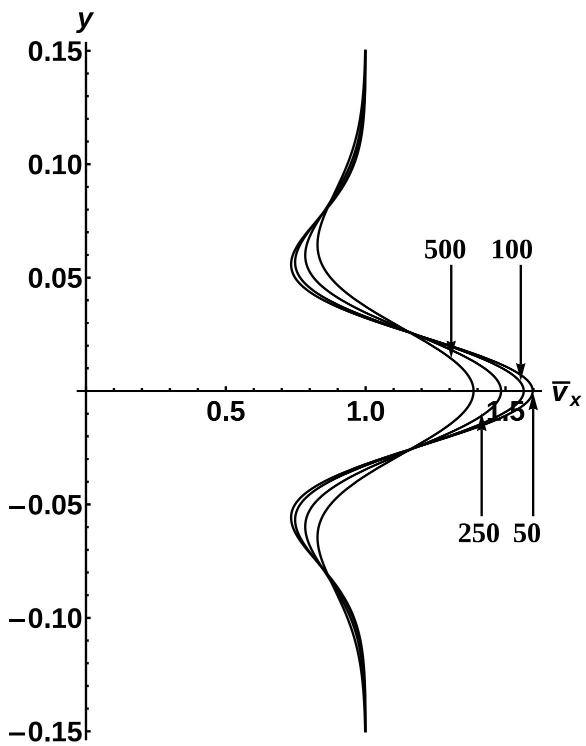

FIGURE 5.4: Mean velocity in a turbulent wake $\overline{v}_x(x, y)$ plotted against y at $x = 0.25$ with $K = 0.0001$, $\nu_{T_c}/\nu = 500$, $\beta = 4$ and for $\nu_{T_0}/\nu = 50, 100, 250$ and 500.

5.2.2 Effective viscosity as a power of x

The ratio ν_{T_0}/ν can be 1000 or larger. We consider the approximation in which

$$\frac{\nu_{T_0}}{\nu} \gg 1. \tag{5.117}$$

The eddy viscosity (5.85) reduces to

$$E(x) = x^{\alpha}, \quad \alpha > -1, \tag{5.118}$$

which is a power law in x. This approximation in which the effective viscosity is considered as a power law in x was investigated for the classical wake in Chapter 4.

The Lie point symmetry is given by

$$X = \frac{2}{(1+\alpha)} x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - 2\psi \frac{\partial}{\partial \psi}, \tag{5.119}$$

which is a scaling symmetry.

The mean velocity deficit is

$$\overline{w}_p(x, y) = -\frac{K}{8\sqrt{\pi}} \frac{(1+\alpha)^{3/2}}{x^{\frac{3}{2}(1+\alpha)}} (1-\xi^2) \exp[-\xi^2/2], \tag{5.120}$$

and the effective width is

$$H_p(x) = 2\left[\frac{x^{1+\alpha}}{1+\alpha}\right]^{1/2}. \tag{5.121}$$

By comparing the effective widths and the maximum mean velocities of the cases where the eddy viscosity is a power law in x and where the effective viscosity is a power law in x we can examine the effect of neglecting the kinematic viscosity term $\nu/(\nu+\nu_{T_0})$. We have that for $\alpha = \beta$,

$$\frac{H(x)}{H_p(x)} = \left[\frac{\beta+1}{(1+\frac{\nu_{T_0}}{\nu})x^{\beta}} + \frac{\frac{\nu_{T_0}}{\nu}}{1+\frac{\nu_{T_0}}{\nu}}\right]^{1/2}, \tag{5.122}$$

and $H_p(x) = H(x)$ if

$$x^{\beta} = 1+\beta. \tag{5.123}$$

The condition (5.123) is always true for $\beta = 0$. Since the mean velocity deficit is negligibly small for $x \geq 1$, we only consider $0 < x < 1$. We have that for $\beta > 0$

$$1 + \beta > x^{\beta}, \tag{5.124}$$

and for $-1 < \beta < 0$

$$1 + \beta < x^{\beta}. \tag{5.125}$$

We therefore obtain the following results:

$$H(x) = H_p(x), \quad \beta = 0, \tag{5.126}$$

$$H(x) > H_p(x), \quad \beta > 0, \tag{5.127}$$

$$H_p(x) > H(x), \quad -1 < \beta < 0. \tag{5.128}$$

We see that for very large ν_{T_0}/ν,

$$\frac{H(x)}{H_p(x)} \approx \left[\frac{\beta + 1}{\left(\frac{\nu_{T_0}}{\nu}\right) x^{\beta}} + 1 \right]^{1/2} \approx 1, \tag{5.129}$$

showing that the effect of neglecting the kinematic viscosity term is negligible when considering very large values of the ratio ν_{T_0}/ν.

In general, it appears that the consequences of neglecting the kinematic viscosity term in the effective viscosity manifests itself as an underestimation of the effective width of the wake for $\beta > 0$ and an overestimation of the width of the wake for $-1 < \beta < 0$.

5.3 Eddy viscosity a function of x and y

Recall from Chapter 3 that the components of the conserved vector (T^1, T^2) for the turbulent wake are given by

$$T^1 = \Lambda(y)\psi_y, \tag{5.130}$$

$$T^2 = -E(x,y)\Lambda(y)\psi_{yy} + E(x,y)\frac{\partial \Lambda}{\partial y}\psi_y - \frac{\partial}{\partial y}\left(E(x,y)\frac{\partial \Lambda}{\partial y}\right)\psi, \tag{5.131}$$

provided the equation

$$\frac{\partial^2}{\partial y^2}\left(E\frac{\partial \Lambda}{\partial y}\right) = 0, \tag{5.132}$$

is satisfied.

In order to generate the conserved quantity of Birkhoff and Zorantello [5],

$$\int_{-\infty}^{\infty} y^2 \frac{\partial \psi}{\partial y}(x,y)\,dy = 0, \tag{5.133}$$

we must have that $\Lambda(y) = y^2$. Equation (5.132) becomes

$$\frac{\partial^2}{\partial y^2}\left(yE\right) = 0, \tag{5.134}$$

which gives that

$$E(x,y) = \frac{a(x)}{y} + b(x). \tag{5.135}$$

The form of the eddy viscosity given by (5.135) is not finite at $y = 0$ unless $a(x) = 0$ which reduces to $E(x,y) = E(x)$. This result may indicate that the eddy viscosity cannot depend on the variable y.

5.4 Conclusions

The PDE which describes the turbulent wake behind a thin two-dimensional body expressed in terms of a stream function was investigated. The difference between the classical wake and the wake of a self-propelled boundary lies within the conserved quantity. The conserved vector for the wake behind a self-propelled body was derived for an eddy viscosity depending on both the distance along the wake and the distance perpendicular to the axis of the wake. In order to calculate the Lie point symmetry associated with the conserved vector we first considered an eddy viscosity depending on only the distance along the wake, $E = E(x)$. The invariant solution was then obtained.

An eddy viscosity as a power law in x was compared with the laminar wake behind a self-propelled body. The power was denoted by β. The velocity profiles were plotted for three separate cases due to the large variations in the mean velocity deficit for different values of β. A modified version of Prandtl's hypothesis that stated that the eddy viscosity is proportional to the product of the maximum mean velocity deficit and the effective width of the wake was implemented which derived the value $\beta = -1/2$. This compares with a classical wake for which the modified Prandtl's hypothesis gives $\beta = 0$. This would imply that the turbulence in the wake behind a self-propelled body decreases with distance downstream of the body while for a classical wake it remains approximately constant. For an eddy viscosity as a power law in x, the Lie point symmetry is not a scaling symmetry when the kinematic viscosity is not neglected. The velocity profiles showed that as the exponent β increases, the maximum mean velocity of the wake increases and the effective width decreases. For $-1 < \beta \leq 1$ the maximum mean velocities were small compared with the laminar wake and were thus not compared.

The effect of increasing the strength of the turbulence was also investigated. This approach enabled us to study the transition for a laminar to a turbulent wake. As the strength of the turbulence increases, the maximum mean velocity decreases and the effective width increases.

We also considered the effect of neglecting the kinematic viscosity term in the effective viscosity. We found that the effective width was underestimated for $\beta > 0$ and overestimated for $-1 < \beta < 0$.

When $E = E(x, y)$, that is, when the eddy viscosity is not constant across the wake, the only solution that could be obtained for $E(x, y)$ that also generated the conserved quantity of Birkhoff and Zorantello was not finite at $y = 0$.

Chapter 6

Revised Prandtl mixing length model applied to the two-dimensional turbulent classical wake

In this chapter we develop a revised Prandtl mixing length model by including the kinematic viscosity of the fluid. We compare this new model to Prandtl's mixing length model. This chapter is outlined as follows. In Section 6.1, the Lie point symmetry associated with the elementary conserved vector of the partial differential equation is derived. In Section 6.2, we consider the application of the revised Prandtl mixing length model and solve for the stream function. Mean velocity profiles are plotted with the purpose of examining the impact of the strength of the turbulence on the mean velocity and width of the wake. In Section 6.3 we derive and discuss the results from implementing Prandtl's mixing length model to the turbulent wake. In Section 6.4, a detailed comparison of the two models is provided. It is shown that the revised Prandtl model predicts a boundary that lies outside the one predicted by Prandtl's model. Finally, conclusions are presented in Section 6.4.

6.1 Elementary conserved vector and associated Lie point symmetry

In terms of the stream function $\psi(x, y)$, the velocity components $\overline{v}_x$ and $\overline{v}_y$ are given by

$$\overline{v}_x(x, y) = 1 - \overline{w}(x, y) = 1 - \frac{\partial \psi}{\partial y}, \quad \overline{v}_y(x, y) = \frac{\partial \psi}{\partial x}, \tag{6.1}$$

which ensures that the continuity equation is identically satisfied. The stream function ψ satisfies the partial differential equation

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left(E\left(x, y, \frac{\partial^2 \psi}{\partial y^2}\right)\frac{\partial^2 \psi}{\partial y^2}\right). \tag{6.2}$$

The boundary conditions are for $x \geq 0$,

$$\frac{\partial \psi}{\partial y}(x, \pm y_b) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x, \pm y_b) = 0, \tag{6.3}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x, 0) = 0, \quad \frac{\partial \psi}{\partial x}(x, 0) = 0, \tag{6.4}$$

where the boundary $y = \pm y_b(x)$ is unspecified. If the wake extends to infinity in the y-direction then $y_b(x) = \infty$.

The conserved quantity for the wake is the drag force D [4]. In terms of the stream function it is

$$\int_{-y_b(x)}^{y_b(x)} \frac{\partial \psi}{\partial y}(x, y)\,dy = D, \tag{6.5}$$

where $y_b(x) = \infty$ if the wake extends to infinity in the y-direction. The elementary conserved vector (T^1, T^2) which generates the conserved quantity given in equation (6.5) was calculated in Chapter 3. It is given by

$$T^1(\psi_y) = \psi_y, \quad T^2(x, y, \psi_{yy}) = -E(x, y, \psi_{yy})\psi_{yy}. \tag{6.6}$$

In order to calculate the Lie point symmetry X where

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}, \tag{6.7}$$

that is associated with the elementary conserved vector (6.6), we require the second prolongation of X which is of the form

$$X^{[2]} = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi} + \zeta_2\frac{\partial}{\partial \psi_y} + \zeta_{22}\frac{\partial}{\partial \psi_{yy}}, \tag{6.8}$$

where [65]

$$\zeta_2 = D_2(\eta) - \psi_k D_2(\xi^k), \tag{6.9}$$

$$\zeta_{22} = D_2(\zeta_2) - \psi_{2k} D_2(\xi^k). \tag{6.10}$$

The conserved vector $T = (T^1, T^2)$ is invariant under the action of the Lie point symmetry X if [30, 31]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1,2, \tag{6.11}$$

which, when decomposed into two components, gives

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{6.12}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{6.13}$$

The conserved vector components T^1 and T^2 are given by (6.6). Equation (6.12) becomes

$$\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \psi_x\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right) + E(x,y,\psi_{yy})\psi_{yy}\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right) = 0. \tag{6.14}$$

By separating equation (6.14) by ψ_x, $\psi_x\psi_y$ and ψ_y we obtain

$$\xi^1 = A(x), \quad \xi^2 = \xi^2(x,y,\psi), \quad \eta = B(x). \tag{6.15}$$

In the second invariance condition (6.13), there is only one term depending on $\psi_x\psi_y$. By setting the coefficient of $\psi_x\psi_y$ to zero we obtain

$$\frac{\partial \xi^2}{\partial \psi} = 0. \tag{6.16}$$

Using (6.16), ζ_{22} reduces to

$$\zeta_{22} = -2\psi_{yy}\frac{\partial \xi^2}{\partial y} - \psi_y\frac{\partial^2 \xi^2}{\partial y^2}. \tag{6.17}$$

The invariance condition (6.13) becomes

$$\left(E(x,y,\psi_{yy})+\psi_{yy}\frac{\partial E}{\partial \psi_{yy}}\right)\left(\frac{\partial^2\xi^2}{\partial y^2}\psi_y+2\psi_{yy}\frac{\partial \xi^2}{\partial y}\right)$$

$$-\left(\xi^1\frac{\partial E}{\partial x}+\xi^2\frac{\partial E}{\partial y}+E(x,y,\psi_{yy})\frac{dA}{dx}\right)\psi_{yy}-\psi_y\xi^2_x=0. \tag{6.18}$$

Now $E(x,y,\psi_{yy})\neq 0$. We separate (6.18) according to powers of ψ_y and we thus obtain

$$\frac{\partial^2\xi^2}{\partial y^2}\psi_{yy}\frac{\partial E}{\partial \psi_{yy}}+\frac{\partial^2\xi^2}{\partial y^2}E=\frac{\partial \xi^2}{\partial x}, \tag{6.19}$$

$$A(x)\frac{\partial E}{\partial x}+\xi^2(x,y)\frac{\partial E}{\partial y}-2\frac{\partial \xi^2}{\partial y}\psi_{yy}\frac{\partial E}{\partial \psi_{yy}}=\left(2\frac{\partial \xi^2}{\partial y}-\frac{dA}{dx}\right)E. \tag{6.20}$$

The Lie point symmetry associated with the conserved vector (6.6) is

$$X=A(x)\frac{\partial}{\partial x}+\xi^2(x,y)\frac{\partial}{\partial y}+B(x)\frac{\partial}{\partial \psi}, \tag{6.21}$$

provided that the effective viscosity $E(x,y,\psi_{yy})$ satisfies the pair of first order partial differential equations (6.19) and (6.20).

Now $E=E(x,y)$ has been considered in Chapter 4 for a classical wake with an infinite boundary. For a finite wake boundary the effective viscosity must be equal to the mainstream kinematic viscosity on the boundary. For this reason, we cannot consider an effective viscosity depending on x only as the condition for the effective viscosity reducing to the kinematic viscosity on a finite boundary will not be satisfied. In other words for a finite boundary we must have

$$\frac{\partial E}{\partial \psi_{yy}}\neq 0,$$

and/or

$$\frac{\partial E}{\partial y}\neq 0.$$

Since we are considering Prandtl's model, we will suppose that

$$E=E(x,\psi_{yy}). \tag{6.22}$$

For fully developed turbulent flow, the kinematic viscosity is often neglected and thus the eddy viscosity is approximately zero at the boundary. We will not neglect the kinematic viscosity in the revised Prandtl model.

6.2 Revised Prandtl mixing length model

When implementing Prandtl's mixing length model [2], the turbulent viscosity ν_T is written as

$$\nu_T = l^2(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|, \tag{6.23}$$

where $l(x)$ is called the mixing length. In terms of the dimensionless variables given in (2.22) and (2.23), equation (6.23) becomes

$$\nu_T^* = \frac{1}{\nu_C+\nu_{TC}} l_0^2 \frac{U}{\delta}\, l^{*2}(x^*)\left|\frac{\partial \overline{v}^*_{x^*}}{\partial y^*}\right| = \frac{1}{\nu_C+\nu_{TC}} \nu_{T_0}\, l^{*2}(x^*)\left|\frac{\partial \overline{v}^*_{x^*}}{\partial y^*}\right|, \tag{6.24}$$

where l_0 is the characteristic mixing length. Thus the characteristic turbulent viscosity is

$$\nu_{T_0} = l_0^2\frac{U}{\delta} = \left(\frac{l_0}{L}\right)^2 \mathrm{Re}^{3/2} E_c, \tag{6.25}$$

where Re is defined by (2.20). Omitting the * notation for convenience, in terms of the stream function defined in (6.1), the dimensionless effective viscosity E is written as

$$E(x,\psi_{yy}) = \frac{\nu_{T_0}}{\nu_C+\nu_{TC}} l^2(x)\left|\psi_{yy}\right|. \tag{6.26}$$

The kinematic viscosity is neglected in approximation (6.26). At the boundary defined by $y = \pm y_b(x)$ the eddy viscosity is zero. Approximation (6.26) is suitable for fully developed turbulent flows in regions where the turbulent viscosity is much greater than the kinematic viscosity. Since the eddy viscosity is zero at the boundary $y = \pm y_b(x)$, this model breaks down because it assumes that the kinematic viscosity ν of the fluid can be approximated as zero everywhere. In our model, at the boundary between the turbulent wake and the laminar flow the effective viscosity is equal to the kinematic viscosity where $\nu \neq 0$. We thus impose

$$E(x,\psi_{yy}) = \frac{\nu}{\nu_C+\nu_{TC}} + \frac{\nu_{T_0}}{\nu_C+\nu_{TC}} l^2(x)\left|\psi_{yy}\right|. \tag{6.27}$$

Equation (6.27) reduces to equation (6.26) when $\nu = 0$. From the second derivative boundary condition in (6.3) the effective viscosity defined in (6.27) is equal to the kinematic viscosity at the boundary.

The classical wake is symmetric about the x-axis. We consider the upper half of the wake, $0 \le y < y_b(x)$, in the analysis that follows. The mean velocity deficit $\overline{w}(x, y)$ is given by

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y}. \tag{6.28}$$

From the second derivative boundary condition in (6.4) the velocity deficit is a positive maximum along the x-axis and from the first derivative boundary condition in (6.3) it is zero at $y = y_b(x)$. Thus the velocity deficit is a decreasing function of y and therefore if we consider the upper half of the wake,

$$\frac{\partial \overline{w}}{\partial y}(x,y) = \frac{\partial^2 \psi}{\partial y^2} \leq 0, \;\; 0 \leq y \leq y_b(x). \tag{6.29}$$

In the upper half of the wake we have that $|\psi_{yy}| = -\psi_{yy}$ and therefore

$$E(x,\psi_{yy}) = \frac{\nu}{\nu_C + \nu_{TC}} - \frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l^2(x)\psi_{yy}. \tag{6.30}$$

Substituting (6.30) into (6.19) we obtain

$$\frac{\partial^2 \xi^2}{\partial y^2}\left(\frac{\nu}{\nu_C + \nu_{TC}} - 2\frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l^2(x)\psi_{yy}\right) - \frac{\partial \xi^2}{\partial x} = 0. \tag{6.31}$$

We separate (6.31) by powers of ψ_{yy}:

$$\psi_{yy}: \;\; l^2(x)\frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{6.32}$$

$$\text{remainder}: \;\; \frac{\nu}{\nu_C + \nu_{TC}}\frac{\partial^2 \xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0. \tag{6.33}$$

Since $l(x) \neq 0$ it follows that

$$\frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{6.34}$$

and therefore from (6.33), that $\xi^2 = \xi^2(y)$. Hence from (6.34)

$$\xi^2(x,y) = ay + b, \tag{6.35}$$

where a and b are constants. We consider the general case in which $a \neq 0$ and by dividing the Lie point symmetry (6.21) by a we can set $a = 1$ in (6.35). Substituting (6.35) and (6.30) into equation (6.20) gives

$$2\frac{\nu_{T_0}}{\nu_C + \nu_{TC}}\left(-A(x)l(x)\frac{dl}{dx} + l^2(x)\right)\psi_{yy} = \left(2 - \frac{dA}{dx}\right)\left(\frac{\nu}{\nu_C + \nu_{TC}} - \frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l^2(x)\psi_{yy}\right). \tag{6.36}$$

Separating by ψ_{yy} we have since $\nu \neq 0$,

$$\frac{dA}{dx} = 2, \tag{6.37}$$

and using (6.37)

$$-A(x)\frac{dl}{dx} + l(x) = 0. \tag{6.38}$$

Solving for $A(x)$ and $l(x)$ we obtain

$$A(x) = 2x + A_0, \tag{6.39}$$

$$l(x) = l_1\sqrt{2x + A_0}, \tag{6.40}$$

where A_0 and l_1 are constants.

The Lie point symmetry (6.21) becomes

$$X = (2x + A_0)\frac{\partial}{\partial x} + (y + b)\frac{\partial}{\partial y} + B(x)\frac{\partial}{\partial \psi}. \tag{6.41}$$

Now, $\psi = \Psi(x, y)$ is an invariant solution of the partial differential equation (6.2) (with effective viscosity E given by (6.30)) generated by the Lie point symmetry (6.41) provided

$$X(\psi - \Psi(x, y))\big|_{\psi=\Psi} = 0, \tag{6.42}$$

that is, provided

$$(2x + A_0)\frac{\partial \Psi}{\partial x} + (y + b)\frac{\partial \Psi}{\partial y} = B(x). \tag{6.43}$$

The differential equations of the characteristic curves of (6.43) are

$$\frac{dx}{2x + A_0} = \frac{dy}{y + b} = \frac{d\Psi}{B(x)}. \tag{6.44}$$

The constants of integration are

$$\frac{y + b}{\sqrt{2x + A_0}} = a_1, \quad \Psi - G(x) = a_2, \tag{6.45}$$

where

$$G(x) = \int_0^x \frac{B(\alpha)}{2\alpha + A_0}\, d\alpha, \tag{6.46}$$

and hence since $\Psi = \psi$ the general invariant solution is

$$\psi(x, y) = F(\xi) + G(x), \tag{6.47}$$

where F is an arbitrary function and

$$\xi(x,y) = \frac{y+b}{\sqrt{2x+A_0}}. \tag{6.48}$$

We now determine the constant b and the arbitrary function $B(x)$. A similar argument was used in Chapter 4. The second derivative boundary condition in (6.4) is

$$\frac{d^2F}{d\xi^2}(\xi|_{y=0}) = 0. \tag{6.49}$$

Differentiating (6.49) with respect to x gives

$$\frac{d^3F}{d\xi^3}(\xi|_{y=0})\frac{b}{(2x+A_0)^{3/2}} = 0. \tag{6.50}$$

But since $\overline{w}(x,y)$ has a local maximum at $y=0$ we have

$$\frac{\partial^2\overline{w}}{\partial y^2}(x,0) = \frac{d^3F}{d\xi^3}(\xi|_{y=0})\frac{1}{(2x+A_0)^{3/2}} < 0, \tag{6.51}$$

giving that $F'''(\xi|_{y=0}) \neq 0$. From (6.50) we must have $b=0$.

Consider next $B(x)$. Now

$$\frac{\partial\psi}{\partial x} = \frac{1}{2x+A_0}\left[B(x) - \xi\frac{dF}{d\xi}\right]. \tag{6.52}$$

But

$$\overline{w}(x,y) = \frac{\partial\psi}{\partial y} = \frac{dF}{d\xi}\frac{1}{(2x+A_0)^{1/2}}, \tag{6.53}$$

and since $\overline{w}(x,0)$ is finite it must follow that $F'(0)$ is finite. From (6.52) the first derivative boundary condition in (6.4) therefore gives $B(x)=0$. Thus we have

$$\psi(x,y) = F(\xi), \tag{6.54}$$

where

$$\xi(x,y) = \frac{y}{\sqrt{2x+A_0}}. \tag{6.55}$$

Consider now the conserved quantity (6.5). Since the wake is symmetric about the x-axis, (6.5) can be written as

$$\int_0^{y_b(x)} \frac{\partial\psi}{\partial y}(x,y)\,dy = \frac{D}{2}. \tag{6.56}$$

Expressed in terms of the similarity variables, (6.56) becomes

$$\int_0^{\frac{y_b(x)}{\sqrt{2x+A_0}}} \frac{dF}{d\xi} d\xi = \frac{D}{2}. \tag{6.57}$$

Since D is a constant the upper limit in the integral (6.57) must be a constant which we denote by ξ_b:

$$\frac{y_b(x)}{\sqrt{2x+A_0}} = \xi_b. \tag{6.58}$$

The half-width of the wake at position x on the axis of the wake is therefore

$$y_b(x) = \xi_b\sqrt{2x+A_0}. \tag{6.59}$$

We consider an obstructing object that is slender and its thickness is essentially negligible. In addition, we also assume that $s \ll L$ where s is the length of the object along the $x-$ axis in order to avoid the possible development of turbulent boundary layers on the obstructing body upstream of $x = 0$. We may use the approximation that at $x = 0$ the boundary $y_b(x)$ is zero. For a general symmetric object with $s \ll L$, as $x \to 0$, $y_b(0) = \xi_b\sqrt{A_0}$ and so we can relate the constant A_0 to the thickness of the object. For the remainder of this work we will assume the obstructing object is slender and its length is small in comparison to the characteristic length scale L so that we may set $A_0 = 0$. Thus

$$y_b(x) = \xi_b\sqrt{2x}. \tag{6.60}$$

The Lie point symmetry X reduces to

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{6.61}$$

which is a scaling symmetry. The Lie point symmetry associated with the elementary conserved vector is not in general a scaling symmetry for obstructing objects of a finite thickness and a length comparable to the characteristic length L. The similarity variable ξ becomes

$$\xi(x,y) = \frac{y}{\sqrt{2x}}. \tag{6.62}$$

The mixing length (6.40) reduces to

$$l(x) = l_1\sqrt{2x}. \tag{6.63}$$

The mixing length (6.63) is therefore proportional to the half-width of the wake (6.60).

Prandtl assumed that the mixing length $l(x)$ is approximately proportional to the width of the boundary layer [2]. We have shown that by taking $\nu \neq 0$ that this result can be proved for the classical wake and does not need to be assumed. Equation (6.40) which determines $l(x)$ follows directly from the assumption $\nu \neq 0$ in the separation of (6.36). Hence, although ν is small compared with the eddy viscosity it cannot be neglected everywhere. It must be taken as non-zero adjacent to an interface between laminar and turbulent flows where the kinematic viscosity and the eddy viscosity are the same order of magnitude.

The partial differential equation (6.2) with effective viscosity given by (6.30), when expressed in terms of the similarity variables (6.54) and (6.62) reduces to

$$\frac{d}{d\xi}\left[\left(\frac{\nu}{\nu_C+\nu_{TC}}-\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right]+\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right)=0, \tag{6.64}$$

and the boundary conditions (6.3) and (6.4) become, for the upper half of the wake,

$$\frac{dF}{d\xi}(+\xi_b)=0, \qquad \frac{d^2F}{d\xi^2}(+\xi_b)=0, \tag{6.65}$$

$$\frac{d^2F}{d\xi^2}(0)=0. \tag{6.66}$$

By integrating (6.64) once and using the boundary condition (6.66) we obtain

$$\left(\frac{\nu}{\nu_C+\nu_{TC}}-\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}+\xi\frac{dF}{d\xi}=0. \tag{6.67}$$

The conserved quantity (6.57) is

$$\int_0^{\xi_b}\frac{dF}{d\xi}d\xi=D/2, \tag{6.68}$$

where the constant ξ_b has still to be determined.

In order to calculate the velocity components, since the wake is symmetric about the x-axis, we need only consider the upper half of the wake. In (6.67) we let $W(\xi)=F'(\xi)$ and obtain

$$\left(\frac{\nu}{\nu_C+\nu_{TC}}-\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}l_1^2\frac{dW}{d\xi}\right)\frac{dW}{d\xi}+\xi W=0, \tag{6.69}$$

subject to

$$W(\xi_b)=0, \qquad \frac{dW}{d\xi}(\xi_b)=0, \tag{6.70}$$

and

$$\int_0^{\xi_b} W(\xi) d\xi = D/2. \tag{6.71}$$

The boundary condition (6.66) has already been used to determine the constant of integration in (6.67). We will see later that only one of the two boundary conditions in (6.70) is independent. In terms of W the velocity components are given by

$$\overline{v}_x(x, y) = 1 - \frac{1}{\sqrt{2x}} W(\xi), \tag{6.72}$$

$$\overline{v}_y(x, y) = -\frac{\xi}{2x} W(\xi). \tag{6.73}$$

In order to remove the parameters from (6.69) we substitute the transformation

$$W = W^* A, \quad \xi = \xi^* B, \tag{6.74}$$

where A and B are constants, into (6.69) which becomes

$$\left(\frac{\nu}{\nu_C + \nu_{TC}} \frac{1}{B^2} - \frac{\nu_{T_0}}{\nu_C + \nu_{TC}} \frac{A}{B^3} l_1^2 \frac{dW^*}{d\xi^*} \right) \frac{dW^*}{d\xi^*} + \xi^* W^* = 0. \tag{6.75}$$

By letting

$$A = \frac{1}{l_1^2} \left[\frac{\nu}{\nu_C + \nu_{TC}} \right]^{1/2} \frac{\nu}{\nu_{T_0}}, \quad B = \left[\frac{\nu}{\nu_C + \nu_{TC}} \right]^{1/2}, \tag{6.76}$$

we obtain

$$\left(1 - \frac{dW^*}{d\xi^*} \right) \frac{dW^*}{d\xi^*} + \xi^* W^* = 0, \tag{6.77}$$

where

$$\xi^*(x, y) = \frac{y}{\sqrt{2x}} \left[\frac{\nu_C + \nu_{TC}}{\nu} \right]^{1/2}. \tag{6.78}$$

We keep the star notation for clarity. By solving the quadratic equation (6.77) for $W^{*\prime}(\xi^*)$ we obtain

$$\frac{dW^*}{d\xi^*} = \frac{1 \pm (1 + 4\xi^* W^*)^{1/2}}{2}. \tag{6.79}$$

We take the negative root because from (6.29),

$$\frac{dW^*}{d\xi^*} = \frac{d^2 F^*}{d\xi^{*2}} < 0. \tag{6.80}$$

Thus

$$\frac{dW^*}{d\xi^*} = \frac{1 - (1 + 4\xi^* W^*)^{1/2}}{2}. \tag{6.81}$$

We see from (6.81) that when the boundary condition, $W^*(\xi_b^*) = 0$, is satisfied the boundary condition $W^{*\prime}(\xi_b^*) = 0$ is automatically satisfied. Equation (6.70) therefore consists of only one independent boundary condition. Physically this implies that when $\overline{v}_x(x, y_b) = 0$ then $\frac{\partial \overline{v}_x}{\partial y}(x, y_b) = 0$ which confirms that the eddy viscosity vanishes on the boundary. We therefore need to solve the first order ordinary differential equation (6.81) subject to the boundary condition

$$W^*(\xi_b^*) = 0, \tag{6.82}$$

and the conserved quantity

$$\frac{1}{l_1^2}\left[\frac{\nu}{\nu_C + \nu_{TC}}\right]\frac{\nu}{\nu_{T_0}}\int_0^{\xi_b^*} W^*(\xi^*)\,d\xi^* = D/2. \tag{6.83}$$

The x- and y- velocity components (6.1) are given by

$$\overline{v}_x(x, y) = 1 - \frac{1}{l_1^2}\left[\frac{\nu}{\nu_C + \nu_{TC}}\right]^{1/2}\frac{\nu}{\nu_{T_0}}\frac{1}{\sqrt{2x}}W^*(\xi^*), \tag{6.84}$$

$$\overline{v}_y(x, y) = -\frac{1}{l_1^2}\left[\frac{\nu}{\nu_C + \nu_{TC}}\right]\frac{\nu}{\nu_{T_0}}\frac{\xi^*}{2x}W^*(\xi^*). \tag{6.85}$$

If we consider $W^*(\xi^*)$ on the entire domain from 0 to $+\infty$ we see that W^* is a piecewise function defined by equation (6.81) for $0 < \xi^* < \xi_b^*$ and for $\xi^* \geq \xi_b^*$, $W^*(\xi^*) = 0$. We now solve the differential equation (6.81) subject to (6.82) and (6.83). The numerical solver, NDSolve, in Mathematica, is used which implements a Runge-Kutta method for this problem. The value 1 is used as an initial guess for the boundary ξ_b. In order to ensure that we do not obtain the trivial solution when NDSolve is implemented to solve the ordinary differential equation, (6.81), we modify (6.82) to

$$W^*(\xi_b^*) = \epsilon, \tag{6.86}$$

where we let $\epsilon = 1 \times 10^{-6}$. The conserved quantity is evaluated using the solution obtained from NDSolve and the boundary value ξ_b is updated until the condition

$$\left|\frac{1}{l_1^2}\left[\frac{\nu}{\nu_C + \nu_{TC}}\right]\frac{\nu}{\nu_{T_0}}\int_0^{\xi_b^*} W^*(\xi^*)\,d\xi^* - D/2\right| < 0.0001, \tag{6.87}$$

is satisfied.

Because $\nu \neq 0$, we set the characteristic kinematic viscosity $\nu_C = \nu$. In the analysis that follows we let $D = 0.1$ and consider the cases for which $\nu_{T_0}/\nu = 50$, 150, 250, 500, 1000. In order to analyse the dependence of the mean velocity deficit on the strength of the turbulence we let $\nu_{TC}/\nu = 250$. As the constant l_1 can only be determined from experimental observations, we set it equal to 1 for simplicity. We consider a fixed point $x = 1/2$. In Table 6.1 values are given, for the boundary ξ_b, of the integral

$$I = \int_0^{\xi_b} W(\xi)d\xi, \tag{6.88}$$

whose value must be close to $D/2$ from (6.87), and the maximum mean velocity deficit $\overline{w}(1/2,0)$ which occurs at $y = 0$. In Figure 6.1, the mean velocity in the x-direction $\overline{v}_x(x,y)$ is plotted against ξ at a fixed point $x = 1/2$. This is equivalent to plotting $\overline{v}_x(1/2, By^*)$ against By^*.

The computational results show that the constant factor ξ_b in equation (6.60) for the boundary is finite which gives a numerical proof that the wake is bounded in the y-direction.

Ratio ν_{T_0}/ν	ξ_b	I	$\overline{w}(1/2,0)$
50	0.7476	0.0500	0.1634
100	0.8632	0.0500	0.1381
250	1.0555	0.0500	0.1104
500	1.2365	0.0500	0.0930
1000	1.4544	0.0500	0.0784

TABLE 6.1: Values for the boundary of the wake ξ_b, the conserved quantity I and the maximum velocity deficit $\overline{w}(1/2,0)$ for different values of the turbulence ratio ν_{T_0}/ν.

From Table 6.1 and Figure 6.1 we see that as the ratio ν_{T_0}/ν increases the boundary value ξ_b increases and the maximum mean velocity deficit decreases. The increase in ν_{T_0}/ν means an increase in the eddy viscosity which causes an increase in the diffusion of the mean flow. The increase in diffusion perpendicular to the axis of the wake causes the width of the wake to increase and the maximum velocity deficit to decrease.

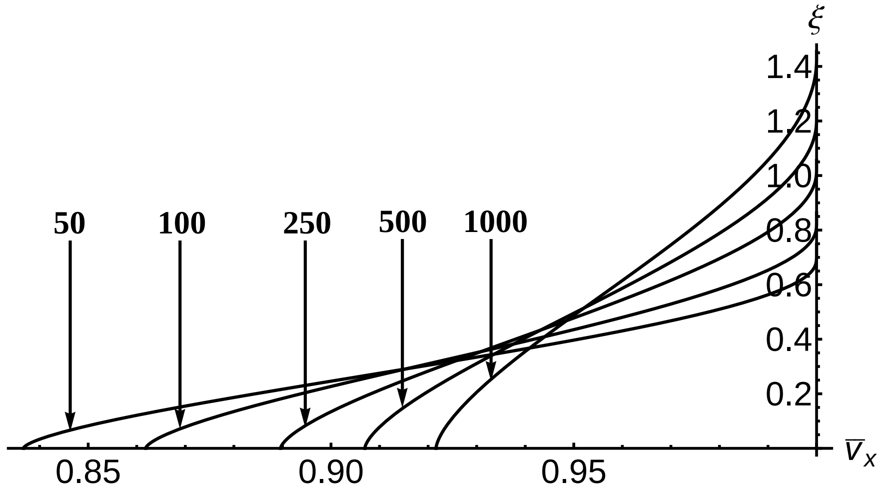

FIGURE 6.1: Velocity profiles for a two-dimensional classical wake with $D = 0.1$, $l_1 = 1$, $x = 1/2$ and $\nu_{TC}/\nu = 250$ for $\nu_{T_0}/\nu = 50,\ 100,\ 250,\ 500,\ 1000$.

6.3 Prandtl's mixing length model

Even if the kinematic viscosity, ν, is very small, setting $\nu = 0$ can lead to significant errors in fluid mechanics, for example, when calculating the drag on a body. In this section we will investigate putting $\nu = 0$ with Prandtl's mixing length model for eddy viscosity.

When $\nu = 0$ the dimensionless effective viscosity (6.30) reduces to the dimensionless eddy viscosity

$$E(x,\psi_{yy}) = -\frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l^2(x)\psi_{yy}, \quad 0 \le y \le y_b(x). \tag{6.89}$$

We keep $\nu_C + \nu_{TC}$ as the characteristic effective viscosity for comparison with the results for $\nu \neq 0$. The stream function satisfies the partial differential equation (6.2) with $E(x,\psi_{yy})$ given by (6.89):

$$\frac{\partial^2\psi}{\partial x\partial y} = -\frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l^2(x)\frac{\partial}{\partial y}\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right], \quad 0 \le y \le y_b(x). \tag{6.90}$$

The elementary conserved vector is

$$T^1 = \psi_y, \quad T^2 = \frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l^2(x)\left(\frac{\partial^2\psi}{\partial y^2}\right)^2. \tag{6.91}$$

The first condition, (6.12), for a Lie point symmetry to be associated with the conserved vector (6.91) gives again the Lie point symmetry (6.21) provided $E(x,\psi_{yy})$ satisfies (6.19) and (6.20). Substituting (6.89) into (6.19) gives

$$2\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}l^2(x)\frac{\partial^2\xi^2}{\partial y^2}\psi_{yy}+\frac{\partial\xi^2}{\partial x}=0, \tag{6.92}$$

which by separating by ψ_{yy} gives again

$$\xi^2=ay+b, \tag{6.93}$$

where a and b are constants. Without loss of generality we again take $a=1$. Substituting (6.89) for $E(x,\psi_{yy})$ and (6.93) for $\xi^2(y)$ into (6.20) gives

$$2\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}\left(-A(x)l(x)\frac{dl}{dx}+l^2(x)\right)\psi_{yy}=-\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}\left(2-\frac{dA}{dx}\right)l^2(x)\psi_{yy}, \tag{6.94}$$

which is similar to (6.36) but with the important difference that the term proportional to ν on the right hand side of (6.36) is absent from (6.94). Unlike (6.36), equation (6.94) does not give two equations, one for $A(x)$ and one for $l(x)$, when separated by ψ_{yy}. It yields only one equation which may be written as

$$\frac{dA}{dx}+\frac{2}{l(x)}\frac{dl}{dx}A(x)=4. \tag{6.95}$$

The mixing length is not determined by the invariance condition but remains arbitrary. This compares with $\nu\neq 0$ for which the mixing length is determined and given by (6.40). From (6.95)

$$A(x)=\frac{1}{l^2(x)}\left(4\int_0^x l^2(\alpha)d\alpha+c\right), \tag{6.96}$$

where c is a constant. The Lie point symmetry (6.21) becomes

$$X=\frac{1}{l^2(x)}\left(4\int_0^x l^2(\alpha)d\alpha+c\right)\frac{\partial}{\partial x}+(y+b)\frac{\partial}{\partial y}+B(x)\frac{\partial}{\partial\psi}. \tag{6.97}$$

Using an argument similar to that in Section 6.2 we again obtain $b=0$, $B(x)=0$ and $c=0$. The Lie point symmetry (6.97) reduces to

$$X=\frac{1}{l^2(x)}\left(4\int_0^x l^2(\alpha)d\alpha\right)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \tag{6.98}$$

which, in general, is not a scaling symmetry unless $l(x)$ is in the form of a power law in x.

The kinematic viscosity, ν, no matter how small, is never zero. We have seen that when determining the mixing length, ν cannot be neglected. The result obtained for $\nu \neq 0$, (6.40), does not depend on ν. It can be used for the mixing length when $\nu = 0$. After the mixing length has been found the kinematic viscosity can be set to zero to obtain an approximate solution.

When $\nu \neq 0$ and the obstacle is slender we found in Section 6.2 that

$$l(x) = l_1\sqrt{2x}. \tag{6.99}$$

We will use (6.99) for the mixing length when $\nu = 0$. The Lie point symmetry (6.98) becomes

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{6.100}$$

The invariant solution generated by (6.100) is

$$\psi(x, y) = G(\xi), \tag{6.101}$$

where

$$\xi(x, y) = \frac{y}{\sqrt{2x}}, \tag{6.102}$$

which is the same as for $\nu \neq 0$.

The partial differential equation (6.90) becomes

$$\frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l_1^2 \frac{d}{d\xi}\left[\left(\frac{d^2G}{d\xi^2}\right)^2\right] = \frac{d}{d\xi}\left(\xi\frac{dG}{d\xi}\right), \tag{6.103}$$

subject to the boundary conditions

$$\frac{dG}{d\xi}(+\xi_b) = 0, \quad \frac{d^2G}{d\xi^2}(+\xi_b) = 0, \tag{6.104}$$

$$\frac{d^2G}{d\xi^2}(0) = 0. \tag{6.105}$$

Integrating once and using the second derivative boundary condition at $\xi = 0$ in (6.105) we obtain

$$\frac{\nu_{T_0}}{\nu_C + \nu_{TC}} l_1^2 \left(\frac{dK}{d\xi}\right)^2 = \xi K(\xi), \tag{6.106}$$

where

$$K(\xi) = \frac{dG}{d\xi}. \tag{6.107}$$

Thus

$$\frac{dK}{d\xi} = -\left(\frac{\nu_C + \nu_{TC}}{\nu_{T_0}}\right)^{1/2} \frac{1}{l_1} (\xi K(\xi))^{1/2}, \tag{6.108}$$

where the $-$ sign is taken because in the upper half of the wake $G''(\xi) < 0$ by (6.29).

We use the characteristic viscosity $E_C = \nu_C + \nu_{TC}$, which is the same for all turbulent wakes, in order to investigate the effect of the strength of the turbulence on the wake. We make the transformation

$$K = K^* A, \quad \xi = \xi^* B, \tag{6.109}$$

where A and B are given by equation (6.76). The ordinary differential equation (6.103) becomes

$$\frac{dK^*}{d\xi^*} = -\left(\xi^* K^*(\xi^*)\right)^{1/2}, \tag{6.110}$$

where

$$\xi^* = \left[\frac{\nu_C + \nu_{TC}}{\nu_C}\right]^{1/2} \frac{y}{\sqrt{2x}}. \tag{6.111}$$

The boundary condition is

$$K^*(\xi_b^*) = 0, \tag{6.112}$$

and the solution must satisfy the conserved quantity

$$\frac{1}{l_1^2}\left[\frac{\nu_C}{\nu_C + \nu_{TC}}\right] \frac{\nu_C}{\nu_{T_0}} \int_0^{\xi_b^*} K^*(\alpha)\, d\alpha = D/2. \tag{6.113}$$

The velocity components $\overline{v}_x(x, y)$ and $\overline{v}_y(x, y)$ are given by (6.84) and (6.85) with $W^*(\xi^*)$ replaced by $K^*(\xi^*)$

$$\overline{v}_x(x, y) = 1 - \frac{1}{l_1^2}\left[\frac{\nu_C}{\nu_C + \nu_{TC}}\right]^{1/2} \frac{\nu}{\nu_{T_0}} \frac{1}{\sqrt{2x}} K^*(\xi^*), \tag{6.114}$$

$$\overline{v}_y(x, y) = -\frac{1}{l_1^2}\left[\frac{\nu_C}{\nu_C + \nu_{TC}}\right] \frac{\nu}{\nu_{T_0}} \frac{\xi^*}{2x} K^*(\xi^*). \tag{6.115}$$

Unlike the differential equation (6.81) for $\nu \neq 0$, the differential equation (6.110) for $\nu = 0$ can be solved analytically. A separation of variables can be performed in (6.110) and its solution subject to the boundary condition (6.112) is

$$K^*(\xi^*) = \frac{1}{9}\left(\xi_b^{*3} - 2\xi_b^{*3/2} \xi^{*3/2} + \xi^{*3}\right), \quad 0 \leq \xi^* \leq \xi_b^*. \tag{6.116}$$

For $K^*(\xi^*)$ to be finite (and therefore for $\overline{v}_x(x,y)$ to be finite from (6.114)), ξ_b^* must be finite. Thus the boundary of the wake, $y = y_b(x)$, is finite. Substituting (6.116) into the conserved quantity (6.113) gives

$$\xi_b^* = \sqrt{l_1}\left[\frac{\nu_C+\nu_{TC}}{\nu_C}\right]^{1/4}\left[\frac{\nu_{T_0}}{\nu_C}\right]^{1/4}(10D)^{1/4}, \tag{6.117}$$

and therefore using (6.62) and (6.109),

$$\xi_b = \frac{y_b(x)}{\sqrt{2x}} = \sqrt{l_1}\left[\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}\right]^{1/4}(10D)^{1/4}. \tag{6.118}$$

Hence, the upper half of the wake is $0 \le y \le y_b(x)$ where

$$y_b(x) = \sqrt{l_1}\left[\frac{\nu_{T_0}}{\nu_C+\nu_{TC}}\right]^{1/4}(10D)^{1/4}\sqrt{2x}. \tag{6.119}$$

The x- component of the mean velocity on the axis of the wake where the velocity deficit is a maximum is

$$\overline{v}_x(x,0) = 1-\overline{w}(x,0) = 1-\frac{(10D)^{1/4}}{9\sqrt{l_1}}\left[\frac{\nu_C+\nu_{TC}}{\nu_{T_0}}\right]^{1/4}\frac{1}{\sqrt{2x}}. \tag{6.120}$$

We have used the mixing length derived for $\nu \neq 0$ and the approximate partial differential equation for $\nu = 0$ to derive an approximate analytical result, (6.119), for the half-width of the wake. This supports the conclusion from the numerical solution for $\nu \neq 0$ that the classical wake with Prandtl's mixing length model for eddy viscosity is bounded in the y-direction. We see that as the strength of the turbulence ν_{T_0}/ν_C increases the half-width of the wake $y_b(x)$ increases and the maximum mean velocity deficit decreases. These effects are due to the increase in the diffusion of the mean flow due to an increase in the eddy viscosity.

6.4 Model comparison

In this section we compare the results obtained from Prandtl's mixing length model with $\nu = 0$ and the revised Prandtl mixing length model with $\nu \neq 0$.

In Figures 6.2 and 6.3, the mean velocity in the x-direction deduced from Prandtl's model with $\nu = 0$ is plotted against ξ at $x = 1/2$ and compared with the results obtained from the revised Prandtl model with $\nu \neq 0$. We use the parameter values

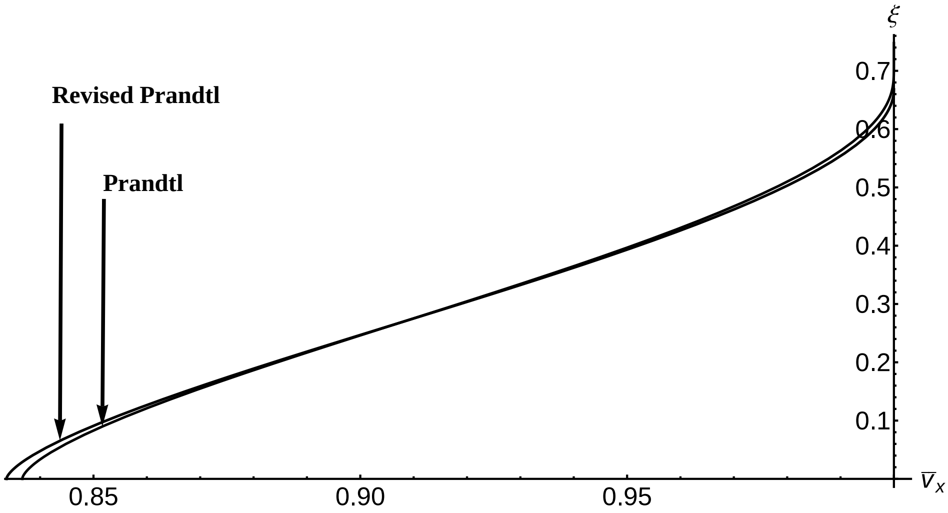

FIGURE 6.2: Velocity profiles for a two-dimensional classical wake with $D = 0.1$, $l_1 = 1$, $x = 1/2$ and $\nu_{TC}/\nu_C = 250$ for $\nu_{T_0}/\nu_C = 50$.

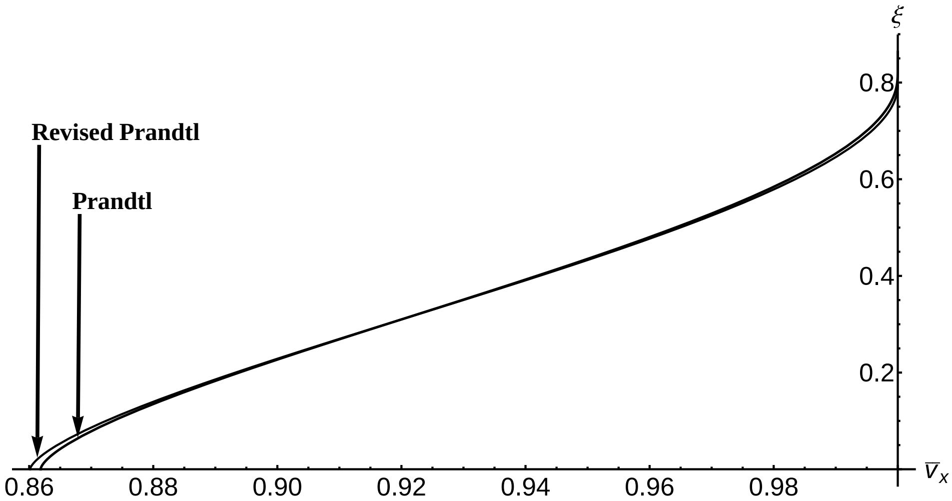

FIGURE 6.3: Velocity profiles for a two-dimensional classical wake with $D = 0.1$, $l_1 = 1$, $x = 1/2$ and $\nu_{TC}/\nu = 250$ for $\nu_{T_0}/\nu = 100$.

$D = 0.1$, $l_1 = 1$ and $\nu_{TC}/\nu_C = 250$ for $\nu_{T_0}/\nu_C = 50,\ 100$. We choose the values 50 and 100 for the turbulence ratio ν_{T_0}/ν_C because at lower Reynolds number flows the differences between the two models are most noticeable. Tables 6.2 and 6.3 summarize the important features.

In Figures 6.2 and 6.3 the greatest difference in the velocity profiles for the two models occurs at $y = 0$ because on the axis $\overline{v}_x(x, y)$ has a local maximum and $\dfrac{\partial \overline{v}_x}{\partial y} = 0$. The eddy viscosity therefore vanishes and the difference between the effective viscosity of the two models is greatest.

From Table 6.2 we conclude that the revised Prandtl model with $\nu \neq 0$ predicts a boundary value ξ_b that lies outside of the one predicted by Prandtl's model with

Ratio ν_{T_0}/ν	Revised Prandtl: ξ_b	Prandtl: ξ_b	Percentage increase
50	0.7476	0.6681	11.90
100	0.8632	0.7945	8.65
250	1.0555	0.9990	5.66
500	1.2365	1.1880	4.08
1000	1.4544	1.4128	2.94

TABLE 6.2: Values for the boundary of the wake ξ_b computed from the revised Prandtl model and compared with the results obtained from Prandtl's model for different values of the turbulence ratio ν_{T_0}/ν_C. The percentage increase is shown.

Ratio ν_{T_0}/ν	Revised Prandtl: $\overline{w}(1/2,0)$	Prandtl: $\overline{w}(1/2,0)$	Percentage decrease
50	0.1634	0.1663	1.74
100	0.1381	0.1399	1.29
250	0.1104	0.1112	0.72
500	0.0930	0.0935	0.53
1000	0.0784	0.0786	0.25

TABLE 6.3: Values for the maximum mean velocity deficit $\overline{w}(1/2,0)$ computed from the revised Prandtl model and compared with the results obtained from Prandtl's model for different values of the turbulence ratio ν_{T_0}/ν_C. The percentage decrease is shown.

$\nu = 0$. A significant increase in the value of the boundary ξ_b occurs for smaller ratios of ν_{T_0}/ν_C. As the ratio ν_{T_0}/ν_C increases the percentage difference between the boundary values predicted by the two models decreases. Prandtl's model was specifically used for very large Reynolds number flows and is shown to be more accurate for these values. However, for smaller Reynolds number flows, an obvious underestimation of the boundary value is observed and thus Prandtl's model should be replaced by its revised version presented in this chapter. In Table 6.3 it is shown that the differences in the mean velocity deficits of the two models is not as significant as the differences between the boundary values. As the turbulence ratio ν_{T_0}/ν_C increases, the differences between the boundary values and mean velocity deficits obtained from the two models decreases.

The two models, Prandtl's mixing length model with $\nu = 0$ and the revised Prandtl mixing length model with $\nu \neq 0$, contribute to the analysis of the wake in different but complimenting ways.

When $\nu \neq 0$ a definite expression for the mixing length was derived from the Lie symmetry analysis. The result for the mixing length was independent of ν and therefore independent of the physical properties of the fluid. However, the ordinary differential equation for the stream function could not be solved analytically. It was not possible to show analytically that the boundary of the wake is finite. By assuming that the boundary is finite, the equation of the boundary, except for a proportionality constant, was obtained from the conserved quantity. The solution of the ordinary differential equation for the stream function and the proportionality constant were obtained numerically. The boundary of the wake lies outside the boundary obtained with $\nu = 0$.

When $\nu = 0$, it was not possible to determine the mixing length. We choose for the mixing length the expression for $\nu \neq 0$ because the kinematic viscosity, although it may be small, is never zero in a real fluid. This agrees with Prandtl's assumption that the mixing length is proportional to the half-width of the wake. The ordinary differential equation for the stream function was solved analytically. It was also proved analytically that the boundary of the wake is finite and the proportionality constant in the equation of the wake boundary was obtained analytically. The boundary of the wake is underestimated. Prandtl's mixing length model is obtained from the revised model presented in Section 6.2 in the limiting case where the kinematic viscosity is set to zero. It is for this reason that Prandtl's model underestimates the width of the wake.

6.5 Conclusions

Even although the kinematic viscosity, ν, is small compared with the eddy viscosity it played an essential part in the modelling process of the turbulent wake. The kinematic viscosity cannot be neglected in the Lie symmetry analysis of the partial differential equation for the stream function because when $\nu = 0$ Prandtl's mixing length cannot be determined. The ordinary differential equation obtained for $\nu \neq 0$ could not be solved analytically, but once the mixing length had been found we could set $\nu = 0$ to obtain an approximate analytical solution. This analytical solution was useful because it showed that the width of the wake is finite which was a guide for the numerical solution when $\nu \neq 0$.

Prandtl [2] neglected the kinematic viscosity compared with the eddy viscosity and therefore had to make an assumption about the mixing length. He assumed that the mixing length was proportional to the half-width of the wake which agrees with the mixing length obtained from the Lie symmetry analysis with $\nu \neq 0$.

The finite boundary of the turbulent classical wake was a significant prediction of Prandtl's mixing length model of the wake. However, by neglecting the kinematic viscosity the half-width of the wake was underestimated. The boundary of the wake with $\nu \neq 0$ lies outside the one predicted by Prandtl. This improvement was achieved without going to a second approximation as was done by Swain [10].

Chapter 7

Mathematical relationship between the different types of two-dimensional turbulent wakes

In this chapter we consider the classical wake, the combination wake and the wake of a self-propelled body. We present the solution to each problem in terms of the stream function and derive the relationship between them. In Chapters 4 and 5 the Lie point symmetry associated with the conserved vector was used to derive the invariant solution for the classical wake with $E = E(x, y)$ and the wake of a self-propelled body with $E = E(x)$. In the sections that follow, since E must be independent of y in order to generate the conserved quantities for the wake of a self-propelled body and the combination wake, we only consider $E = E(x)$ for the classical wake in order to compare the solutions obtained for the three problems. We will use the stream function formulation.

This chapter is outlined as follows: in Section 7.1 the invariant solution corresponding to the conserved vector for the combination wake is derived. In Section 7.2 the results for the classical wake, the combination wake and the wake of a self-propelled body are summarised. In Section 7.3, the mathematical relationship between the three different wakes is investigated. Conclusions for this chapter are given in Section 7.4.

7.1 Combination wake

The combination wake provides the link between the solutions of the classical wake and the wake of a self-propelled body with $E = E(x)$. The same techniques and reasoning provided in the chapters on the classical wake and the wake of a self-propelled body are used in order to obtain the arbitrary constants in the Lie point symmetry for the combination wake. For example, we assume that the effective width of the combination wake also tends to zero as x approaches zero. Since the solution to this problem has not been derived, we include the calculations and a plot of the typical mean velocity profile in this section.

Recall that the conserved vector $T = (T^1, T^2)$ is invariant under the action of the Lie point symmetry

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}, \tag{7.1}$$

provided [30, 31]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2, \tag{7.2}$$

where X is prolongated to as high an order in the derivatives as required. Equation (7.2) consists of two components, namely,

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{7.3}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{7.4}$$

The components of the conserved vector $T = (T^1, T^2)$ were calculated in Chapter 3. In terms of the stream function they are given by (3.81),

$$T^1 = y\psi_y, \quad T^2 = -yE(x)\psi_{yy} + E(x)\psi_y. \tag{7.5}$$

The second prolongation of X, denoted by $X^{[2]}$, is given as

$$X^{[2]} = X + \zeta_2\frac{\partial}{\partial \psi_y} + \zeta_{22}\frac{\partial}{\partial \psi_{yy}}, \tag{7.6}$$

where [65]

$$\zeta_2 = D_2(\eta) - \psi_k D_2(\xi^k), \tag{7.7}$$

$$\zeta_{22} = D_2(\zeta_2) - \psi_{2k} D_2(\xi^k). \tag{7.8}$$

From (7.7) the expansion of ζ_2 is

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial \psi}\psi_y - \psi_x\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right) - \psi_y\left(\frac{\partial \xi^2}{\partial y} + \frac{\partial \xi^2}{\partial \psi}\psi_y\right). \tag{7.9}$$

The first invariance condition (7.3) yields

$$\xi^2\psi_y + y\frac{\partial \eta}{\partial y} + y\frac{\partial \eta}{\partial \psi}\psi_y - y\psi_x\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right)$$
$$+\left(E(x)y\psi_{yy} - E(x)\psi_y\right)\left(\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^1}{\partial \psi}\psi_y\right) = 0. \tag{7.10}$$

The coefficients of $\psi_y\psi_{yy}$ and ψ_y are set to zero giving

$$\xi^1 = B(x). \tag{7.11}$$

Equation (7.10) becomes

$$\xi^2\psi_y + y\frac{\partial \eta}{\partial y} + y\frac{\partial \eta}{\partial \psi}\psi_y = 0. \tag{7.12}$$

Separating by the partial derivative ψ_y in equation (7.12) yields

$$\psi_y: \quad \xi^2 + y\frac{\partial \eta}{\partial \psi} = 0, \tag{7.13}$$

$$\text{remainder}: \quad \frac{\partial \eta}{\partial y} = 0. \tag{7.14}$$

Therefore, from the first invariance condition, we have

$$\xi^1 = B(x), \quad \xi^2 = -y\frac{\partial \eta}{\partial \psi}(x,\psi), \quad \eta = \eta(x,\psi). \tag{7.15}$$

Since there is only one term in the second invariance condition (7.4) which contains ψ_x, we can set its coefficient to zero giving

$$\frac{\partial \xi^2}{\partial \psi} = 0. \tag{7.16}$$

From equations (7.13) and (7.16) we obtain

$$\xi^2(x,y) = ya(x), \quad \eta(x,\psi) = -a(x)\psi + b(x). \tag{7.17}$$

The expressions (7.7) and (7.8) for ζ_2 and ζ_{22} reduce to

$$\zeta_2 = -2a(x)\psi_y, \quad \zeta_{22} = -3a(x)\psi_{yy}. \tag{7.18}$$

The second invariance condition (7.4) becomes

$$2yE(x)a(x)\psi_{yy} - 2E(x)a(x)\psi_y - yB(x)\frac{dE}{dx}\psi_{yy} + B(x)\frac{dE}{dx}\psi_y +$$
$$\frac{dB}{dx}E(x)(-y\psi_{yy} + \psi_y) - y^2\frac{da}{dx}\psi_y = 0. \tag{7.19}$$

Setting the coefficients of ψ_{yy} and ψ_y to zero gives, respectively,

$$\psi_{yy}: \quad -2a(x)E(x) + \frac{dB}{dx}E(x) + B(x)\frac{dE}{dx} = 0, \tag{7.20}$$

$$\psi_y: \quad -2a(x)E(x) + \frac{dB}{dx}E(x) + B(x)\frac{dE}{dx} - y^2\frac{da}{dx} = 0. \tag{7.21}$$

From equations (7.20) and (7.21) we obtain

$$a(x) = a_1, \tag{7.22}$$

$$\frac{dB}{dx} + \frac{1}{E(x)}\frac{dE}{dx}B(x) = 2a_1, \tag{7.23}$$

where a_1 is a constant.

We therefore have

$$\xi^1 = B(x), \;\; \xi^2 = a_1 y, \;\; \eta = -a_1\psi + b(x), \tag{7.24}$$

subject to (7.23) and $b(x)$ is an arbitrary function. Equation (7.23) is the same as equations (4.28) and (5.39) obtained for the classical wake and the wake of a self-propelled body with $E = E(x)$. The solution of (7.23) is

$$B(x) = \frac{1}{E(x)}\left[a_2 + 2a_1\int_0^x E(\alpha)d\alpha\right], \tag{7.25}$$

where $a_2 = E(0)B(0)$ is a constant. The Lie point symmetry X associated with the conserved vector is

$$X = \frac{1}{E(x)}\left[a_2 + 2a_1\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + a_1 y\frac{\partial}{\partial y} + (-a_1\psi + b(x))\frac{\partial}{\partial \psi}. \tag{7.26}$$

For the general case with $a_1 \neq 0$, without loss of generality we let $a_1 = 1$. The Lie point symmetry X reduces to

$$X = \frac{1}{E(x)}\left[a_2 + 2\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (-\psi + b(x))\frac{\partial}{\partial \psi}. \tag{7.27}$$

We now calculate the invariant solution generated by the Lie point symmetry (7.27). For $E = E(x)$, $\psi = \Psi(x, y)$ is an invariant solution of the PDE (2.35) generated by the Lie point symmetry (7.27) provided

$$X(\psi - \Psi(x, y))\big|_{\psi=\Psi} = 0, \tag{7.28}$$

that is, provided $\Psi(x, y)$ satisfies the first order PDE

$$\frac{1}{E(x)}\left(a_2 + 2\int_0^x E(\alpha)d\alpha\right)\frac{\partial \Psi}{\partial x} + y\frac{\partial \Psi}{\partial y} = -\Psi + b(x). \tag{7.29}$$

The differential equations of the characteristic curves of (7.29) are given by

$$\frac{E(x)dx}{a_2 + 2\int_0^x E(\alpha)d\alpha} = \frac{dy}{y} = \frac{d\Psi}{b(x) - \Psi}. \tag{7.30}$$

Solving the first pair of terms and first and last terms gives, respectively,

$$\frac{y}{(a_2 + 2\int_0^x E(\alpha)d\alpha)^{1/2}} = c_1, \tag{7.31}$$

$$\Psi\left(a_2 + 2\int_0^x E(\alpha)d\alpha\right)^{1/2} - G(x) = c_2, \tag{7.32}$$

where c_1 and c_2 are constants and

$$G(x) = \int_0^x \frac{E(\alpha)b(\alpha)}{\left[a_2 + 2\int_0^\alpha E(\beta)d\beta\right]^{1/2}}d\alpha. \tag{7.33}$$

The general solution of (7.29) is

$$c_2 = F(c_1), \tag{7.34}$$

where F is an arbitrary function. Since $\Psi = \psi$ we have that the solution is given by

$$\psi(x, y) = \frac{F(\xi) + G(x)}{\left(a_2 + 2\int_0^x E(\alpha)d\alpha\right)^{1/2}}, \tag{7.35}$$

where the similarity variable is defined by

$$\xi = \frac{y}{\left(a_2 + 2\int_0^x E(\alpha)d\alpha\right)^{1/2}}. \tag{7.36}$$

We use the following reasoning to determine $b(x)$. Integrating the boundary condition

$$\overline{v}_y(x, 0) = \frac{\partial \psi}{\partial x}(x, 0) = 0, \tag{7.37}$$

with respect to x gives

$$\psi(x, 0) = k_1, \tag{7.38}$$

where k_1 is a constant independent of x. We specify the arbitrary additive constant k_1 in the stream function by choosing

$$\psi(x, 0) = 0. \tag{7.39}$$

Then from (7.35),

$$F(0) + G(x) = 0. \tag{7.40}$$

Hence,

$$\frac{dG}{dx} = 0, \tag{7.41}$$

and therefore from (7.33)

$$\frac{E(x)b(x)}{\left(a_2 + 2\int_0^x E(\alpha)d\alpha\right)^{1/2}} = 0. \tag{7.42}$$

Because $E(x) \neq 0$, we have $b(x) = 0$ and since $G(x) = 0$ it follows from (7.40) that

$$F(0) = 0. \tag{7.43}$$

The Lie point symmetry (7.27) becomes

$$X = \frac{1}{E(x)}\left[a_2 + 2\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi}, \tag{7.44}$$

and $\psi(x,y)$ reduces to

$$\psi(x,y) = \frac{F(\xi)}{\left(a_2 + 2\int_0^x E(\alpha)\,d\alpha\right)^{1/2}}, \tag{7.45}$$

where ξ is given by equation (7.36).

Substituting (7.45) and (7.36) into the PDE (2.35) results in the ODE

$$\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + 2\frac{dF}{d\xi} = 0. \tag{7.46}$$

Also,

$$w(x,y) = \frac{\partial\psi}{\partial y} = \frac{1}{\left[a_2 + 2\int_0^x E(\alpha)\,d\alpha\right]}\frac{dF}{d\xi}, \tag{7.47}$$

$$\frac{\partial w}{\partial y}(x,y) = \frac{\partial^2\psi}{\partial y^2} = \frac{1}{\left[a_2 + 2\int_0^x E(\alpha)\,d\alpha\right]^{3/2}}\frac{d^2F}{d\xi^2}, \tag{7.48}$$

$$v(x,y) = \frac{\partial\psi}{\partial x} = -\frac{E(x)}{\left[a_2 + 2\int_0^x E(\alpha)\,d\alpha\right]^{3/2}}\left[F(\xi) + \xi\frac{dF}{d\xi}\right]. \tag{7.49}$$

The boundary conditions are

$$w(x,\pm\infty) = 0: \quad \frac{dF}{d\xi}(\pm\infty) = 0, \tag{7.50}$$

$$\frac{\partial w}{\partial y}(x,\pm\infty) = 0: \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0, \tag{7.51}$$

$$v(x,0) = 0: \quad F(0) = 0, \tag{7.52}$$

where we have used the reasoning that the mean velocity deficit is finite giving that $F'(0)$ must be finite. We have seen that the boundary condition $F(0) = 0$ can also be obtained from the definition of the stream function. We have not included the boundary condition $F''(0) = 0$ because this condition applies to wakes that are symmetric about the $x-$ axis and this wake is asymmetric about the $x-$ axis. We will see later that a property of this solution is $F'(0) = 0$.

In order to solve ODE (7.46) we multiply through by ξ and group terms as follows:

$$\frac{d}{d\xi}\left(\xi\frac{d^2F}{d\xi^2} + \xi^2\frac{dF}{d\xi}\right) - \frac{d^2F}{d\xi^2} = 0. \tag{7.53}$$

Integrating with respect to ξ and using the boundary conditions in (7.50) and (7.51) we obtain

$$\xi\frac{d^2F}{d\xi^2} + \xi^2\frac{dF}{d\xi} - \frac{dF}{d\xi} = 0. \tag{7.54}$$

When imposing the boundary conditions in (7.50) and (7.51) we assumed the stronger conditions

$$\xi \frac{d^2F}{d\xi^2} \to 0 \text{ and } \xi^2 \frac{dF}{d\xi} \to 0 \text{ as } \xi \to \pm\infty. \tag{7.55}$$

Equation (7.54) is a first order ODE in F'. The solution is

$$\frac{dF}{d\xi} = m\xi \exp\left[-\frac{\xi^2}{2}\right], \tag{7.56}$$

where the constant m is determined from the conserved quantity (3.112) which in terms of the new variables F and ξ is given by

$$m \int_{-\infty}^{\infty} \xi^2 \exp\left[-\frac{\xi^2}{2}\right] d\xi = S. \tag{7.57}$$

Solving for m using elementary properties of the gamma function, we obtain

$$m = \frac{S}{\sqrt{2\pi}}, \tag{7.58}$$

giving

$$\frac{dF}{d\xi} = \frac{S}{\sqrt{2\pi}} \xi \exp\left[-\frac{\xi^2}{2}\right]. \tag{7.59}$$

The solution for $F(\xi)$ is

$$F(\xi) = \frac{S}{\sqrt{2\pi}} \left(1 - \exp\left[-\frac{\xi^2}{2}\right]\right), \tag{7.60}$$

where the arbitrary constant of integration is calculated from the boundary condition $F(0) = 0$. We see from (7.59) that the condition $F'(0) = 0$ is satisfied. In order to obtain (7.54) we could have used the boundary condition $F'(0) = 0$ to calculate a value of zero for the constant of integration on the right-hand-side of the equation. However, we were not initially aware that the condition $F'(0) = 0$ might be physically plausible.

We now determine the constant a_2. The effective half-width of the wake, $H(x)$, is defined to be the value of y for which the argument of the exponential in (7.59) is -1. The same definition was used for the classical wake and wake of a self-propelled body. We have

$$H(x) = \sqrt{2}\left(a_2 + 2\int_0^x E(\alpha)\, d\alpha\right)^{1/2}. \tag{7.61}$$

We use the reasoning that the effective half-width of the wake tends to zero as $x \to 0$. We assume that $\int_0^x E(\alpha)\, d\alpha \to 0$ as $x \to 0$. Then $H(x) \to 0$ as $x \to 0$ provided $a_2 = 0$.

Thus

$$H(x) = 2\left(\int_0^x E(\alpha)\,d\alpha\right)^{1/2}, \tag{7.62}$$

and the similarity variable ξ becomes

$$\xi(x,y) = \frac{y}{\left(2\int_0^x E(\alpha)\,d\alpha\right)^{1/2}}, \tag{7.63}$$

which is the same expression as found for the classical wake and wake of a self-propelled body.

The Lie point symmetry associated with the conserved vector is

$$X = \frac{1}{E(x)}\left[2\int_0^x E(\alpha)\,d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi}, \tag{7.64}$$

and the solution for $\psi(x,y)$ is

$$\psi(x,y) = \frac{S}{2\sqrt{\pi}\left(\int_0^x E(\alpha)\,d\alpha\right)^{1/2}}\left[1-\exp\left(-\frac{\xi^2}{2}\right)\right]. \tag{7.65}$$

In Figure 7.1, the typical mean velocity profile of the combination wake is shown. The velocity deficit is positive for $y > 0$ and negative for $y < 0$. The velocity deficit is zero at $y = 0$.

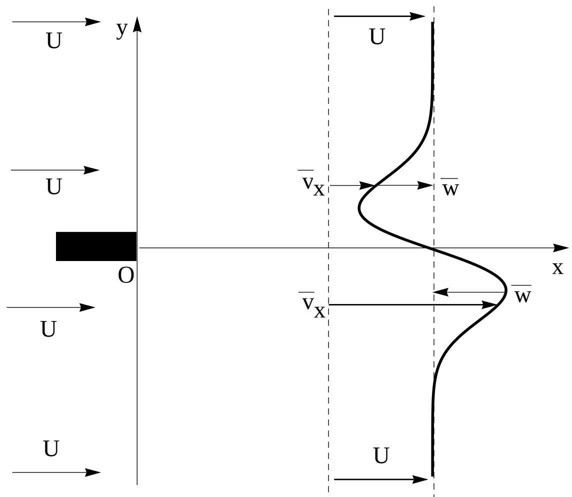

FIGURE 7.1: Two-dimensional combination wake behind a slender symmetric body.

7.2 Comparison of the solutions

In this section we briefly review each type of wake. We use alternative notation to that of Chapters 4 and 5 in order to distinguish between each solution. We will define the solutions to the ODEs obtained from the first reduction as follows: for the classical wake we denote the solution by F, for the combination wake the solution is denoted by G and the wake of a self-propelled body is given by P.

7.2.1 Classical wake

The Lie point symmetry X associated with the elementary conserved vector for the turbulent classical wake with $E = E(x)$ is given by

$$X = \frac{1}{E(x)}\left[2\int_0^x E(\alpha)d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{7.66}$$

The stream function ψ is

$$\psi(x, y) = F(\xi), \tag{7.67}$$

where

$$\xi = \frac{y}{\left(2\int_0^x E(\alpha)d\alpha\right)^{1/2}}, \tag{7.68}$$

and the function $F(\xi)$ satisfies the ODE

$$\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} = 0, \tag{7.69}$$

subject to the boundary conditions

$$\frac{dF}{d\xi}(\pm\infty) = 0, \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0, \tag{7.70}$$

$$F(0) = 0, \quad \frac{d^2F}{d\xi^2}(0) = 0, \tag{7.71}$$

and the conserved quantity

$$\int_{-\infty}^{\infty}\frac{dF}{d\xi}d\xi = D. \tag{7.72}$$

When equation (7.69) is integrated with respect to ξ we obtain

$$\frac{d^2F}{d\xi^2} + \xi\frac{dF}{d\xi} = c_1, \tag{7.73}$$

where c_1 is an arbitrary constant of integration. In order to obtain the value $c_1 = 0$ we can use the second derivative boundary condition at $\xi = 0$ in (7.71) or the conditions at $\xi = \pm\infty$ in (7.70). The solution, therefore, to ODE (7.69) is completely specified by the conditions

$$\frac{dF}{d\xi}(\pm\infty) = 0, \quad \frac{d^2F}{d\xi^2}(\pm\infty) = 0, \tag{7.74}$$

$$F(0) = 0, \tag{7.75}$$

and the conserved quantity (7.72). It was discussed earlier in Chapter 4 that the boundary conditions are not independent of one another. The solution to (7.69) subject to the boundary conditions (7.74), (7.75) and the conserved quantity (7.72) is

$$F(\xi) = \frac{D}{\sqrt{2\pi}} \int_0^{\xi} \exp\left[-\frac{\xi^{*2}}{2}\right] d\xi^*. \tag{7.76}$$

7.2.2 Combination wake

The Lie point symmetry X_C associated with the conserved vector for the turbulent combination wake with $E = E(x)$ is given by

$$X_C = \frac{1}{E(x)}\left[2\int_0^x E(\alpha)\,d\alpha\right]\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi}. \tag{7.77}$$

The stream function ψ satisfies

$$\psi(x,y) = \frac{G(\xi)}{\left(2\int_0^x E(\alpha)\,d\alpha\right)^{1/2}}, \tag{7.78}$$

where ξ is given by equation (7.68) and the function $G(\xi)$ is the solution of the ODE

$$\frac{d^3G}{d\xi^3} + \xi\frac{d^2G}{d\xi^2} + 2\frac{dG}{d\xi} = 0, \tag{7.79}$$

subject to the boundary conditions

$$\frac{dG}{d\xi}(\pm\infty) = 0, \quad \frac{d^2G}{d\xi^2}(\pm\infty) = 0, \tag{7.80}$$

$$G(0) = 0, \quad \frac{dG}{d\xi}(0) = 0, \tag{7.81}$$

and the conserved quantity

$$\int_{-\infty}^{\infty} \xi\frac{dG}{d\xi}\,d\xi = S. \tag{7.82}$$

Integrating (7.79) once with respect to ξ gives

$$\xi \frac{d^2G}{d\xi^2} + \xi^2 \frac{dG}{d\xi} - \frac{dG}{d\xi} = c_1, \tag{7.83}$$

where c_1 is an arbitrary constant of integration. In order to obtain $c_1 = 0$ we can use the first derivative boundary condition at $\xi = 0$ in (7.81) or the conditions at $\xi = \pm\infty$ in (7.80). The solution to ODE (7.79) is completely specified by the conditions

$$\frac{dG}{d\xi}(\pm\infty) = 0, \quad \frac{d^2G}{d\xi^2}(\pm\infty) = 0, \tag{7.84}$$

$$G(0) = 0, \tag{7.85}$$

and the conserved quantity (7.82).

The solution to (7.79) subject to the boundary conditions (7.84), (7.85) and the conserved quantity (7.82) is

$$G(\xi) = \frac{S}{\sqrt{2\pi}} \left(1 - \exp\left[-\frac{\xi^2}{2}\right]\right). \tag{7.86}$$

7.2.3 Wake of a self-propelled body

For the wake of a self-propelled body the Lie point symmetry X_P associated with the conserved vector for $E = E(x)$ is

$$X_P = \frac{1}{E(x)} \left[2 \int_0^x E(\alpha)\, d\alpha\right] \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - 2\psi \frac{\partial}{\partial \psi}. \tag{7.87}$$

The invariant solution is

$$\psi(x, y) = \frac{P(\xi)}{2 \int_0^x E(\alpha)\, d\alpha}, \tag{7.88}$$

where ξ is given by (7.68) and $P(\xi)$ must satisfy the ODE

$$\frac{d^3P}{d\xi^3} + \xi \frac{d^2P}{d\xi^2} + 3\frac{dP}{d\xi} = 0, \tag{7.89}$$

subject to the boundary conditions

$$\frac{dP}{d\xi}(\pm\infty) = 0, \quad \frac{d^2P}{d\xi^2}(\pm\infty) = 0, \tag{7.90}$$

$$P(0) = 0, \quad \frac{d^2P}{d\xi^2}(0) = 0, \tag{7.91}$$

and the conserved quantity

$$\int_{-\infty}^{\infty} \xi^2 \frac{dP}{d\xi} d\xi = K. \tag{7.92}$$

Integrating (7.89) once with respect to ξ gives by (5.58)

$$\left(1-\xi^2\right)\frac{d^2P}{d\xi^2} + \xi\left(3-\xi^2\right)\frac{dP}{d\xi} = c_1, \tag{7.93}$$

where c_1 is an arbitrary constant of integration. In order to obtain the value $c_1 = 0$ we can use the second boundary condition at $\xi = 0$ in (7.91) or the conditions at $\xi = \pm\infty$ in (7.90). The solution is derived completely using the boundary conditions

$$\frac{dP}{d\xi}(\pm\infty) = 0, \quad \frac{d^2P}{d\xi^2}(\pm\infty) = 0, \tag{7.94}$$

$$P(0) = 0, \tag{7.95}$$

and the conserved quantity (7.92). Equation (7.93) is independent of $P(\xi)$ which is required to be able to use the boundary conditions at $\xi = \pm\infty$.

The solution to (7.89) subject to (7.94), (7.95) and (7.92) is

$$P(\xi) = -\frac{K}{2\sqrt{2\pi}}\xi \exp\left[-\frac{\xi^2}{2}\right]. \tag{7.96}$$

From the discussion above, the boundary condition $\psi(x,0) = 0$ is needed to obtain the solution for the stream function $\psi(x,y)$. It is not required to obtain the velocity deficit. For each wake, the condition $\overline{v}_y(x,0) = 0$ was used. The condition that the velocity deficit is an extremum on the axis of the wake was used for the classical wake to obtain one of the arbitrary constants in the Lie point symmetry. We did not know whether any condition at $y = 0$ on the velocity deficit was satisfied for the combination wake. However, once the solution for the combination wake is obtained, it can be easily verified that the x-component of the velocity deficit is zero on the axis of the wake. The same reasoning can be applied to the wake of a self-propelled body. Once the solution to the problem of the wake behind a self-propelled body has been derived, it is easily shown that the velocity deficit is a maximum on the axis of the wake.

The classical wake appears to be the base problem to which the other two are linked.

We note that the solution for the combination wake is linearly proportional to the first derivative with respect to y of the classical wake solution. The solution for the wake of a self-propelled body is proportional to the second derivative with respect to y of the solution for the classical wake. We explore this relationship in detail in the next section.

7.3 Mathematical relationship between the solutions

In this section we show that the solutions for the combination wake and the wake of a self-propelled body can be generated directly from the solution for the classical wake with $E = E(x)$. Once the three conservation laws have been found, it is then assumed that two other wakes, besides the classical wake, exist.

Recall that the PDE and boundary conditions for the infinite classical wake with $E = E(x)$ are

$$\frac{\partial^2 \psi}{\partial x \partial y} = E(x)\frac{\partial^3 \psi}{\partial y^3}, \tag{7.97}$$

subject to

$$\frac{\partial \psi}{\partial y}(x, \pm\infty) = 0, \tag{7.98}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x, \pm\infty) = 0, \tag{7.99}$$

$$\frac{\partial^2 \psi}{\partial y^2}(x, 0) = 0, \tag{7.100}$$

$$\frac{\partial \psi}{\partial x}(x, 0) = 0. \tag{7.101}$$

In terms of the stream function the conserved quantity for the turbulent classical wake is

$$\int_{-\infty}^{\infty} \frac{\partial \psi}{\partial y}(x, y)\, dy = D. \tag{7.102}$$

The stream function is

$$\psi(x, y) = F(\xi) = \frac{D}{\sqrt{2\pi}} \int_0^{\xi} \exp\left[-\frac{\xi^{*2}}{2}\right] d\xi^*, \tag{7.103}$$

where

$$\xi = \frac{y}{\left(2\int_0^x E(\alpha)\, d\alpha\right)^{1/2}}, \tag{7.104}$$

and we imposed the additional condition

$$\psi(x,0) = F(0) = 0. \tag{7.105}$$

Equation (7.97) admits two other conserved quantities. We denote the solutions that satisfy these conserved quantities by $\rho(x, y)$ and $\phi(x, y)$. We then have that ρ and ϕ satisfy equation (7.97) and the boundary conditions (7.98), (7.99) and (7.101). At this stage we do not know if ρ and ϕ satisfy (7.100). In addition, we have

$$\int_{-\infty}^{\infty} y\frac{\partial\rho}{\partial y}dy = S, \tag{7.106}$$

$$\int_{-\infty}^{\infty} y^2\frac{\partial\phi}{\partial y}dy = K. \tag{7.107}$$

For each wake problem we will assume that the similarity variable ξ is the same. The arbitrary constant in ξ was calculated to be zero in Chapters 4 and 5 for the classical wake and the wake of a self-propelled body by assuming that the effective width of the wake tends to zero as $x \to 0$. In Section 7.1 we assumed that the effective width of the combination wake tends to zero as $x \to 0$. By using the same ξ, we are imposing the condition that for each wake the effective width tends to zero as $x \to 0$.

The effective viscosity E must be independent of y in order to calculate the conserved quantities for the combination wake and the wake of a self-propelled body. Since $E = E(x)$ and (7.97) and the boundary conditions are linear, if ψ is a solution then the $n-th$ partial derivatives of ψ with respect to y, namely,

$$\psi_n(x, y) = \frac{\partial^n\psi}{\partial y^n}, \quad n \geq 1, \tag{7.108}$$

are also solutions. It is precisely this fact that allows us to determine the solutions for the combination wake and the wake of a self-propelled body from the solution for the classical wake.

In terms of the similarity variable ξ and the function $F(\xi)$ given in (7.103), the solutions ψ_n are

$$\psi_n(x, y) = \frac{F^{(n)}(\xi)}{\left(2\int_0^x E(\alpha)d\alpha\right)^{n/2}}. \tag{7.109}$$

For $n = 0$ equation (7.109) reduces to the solution for the classical wake.

The conserved quantities must be invariant in terms of the new variables $F(\xi)$ and ξ. The conserved quantities are related to the first derivative of the stream function with respect to y which, for ψ_n, is

$$\frac{\partial \psi_n}{\partial y} = \frac{F^{(n+1)}(\xi)}{\left(2\int_0^x E(\alpha)d\alpha\right)^{(n+1)/2}}. \tag{7.110}$$

The conserved quantities can be expressed as

$$\int_{-\infty}^{\infty} y^\alpha \frac{\partial \psi_n}{\partial y} dy = \int_{-\infty}^{\infty} y^\alpha \frac{F^{(n+1)}(\xi)}{\left(2\int_0^x E(\alpha)d\alpha\right)^{n/2}} d\xi, \tag{7.111}$$

where $\alpha = 0,\ 1,\ 2.$ The function y^α is the multiplier for the problem of interest. We have shown that α can take the values $0,\ 1,\ 2$. In order to express equation (7.111) in terms of the variable ξ only, we must have

$$\frac{y^\alpha}{\left(2\int_0^x E(\alpha)d\alpha\right)^{n/2}} = \xi^m, \quad m \geq 0. \tag{7.112}$$

We want to find the values of n that generate the required conserved quantities. For $\alpha = 0$, $n = 0 = m$ which corresponds to the problem of the classical wake. For $\alpha = 1$, $n = 1$ giving $m = 1$. The conserved quantity becomes

$$\int_{-\infty}^{\infty} y \frac{\partial \psi_1}{\partial y} dy = \int_{-\infty}^{\infty} \xi F''(\xi) d\xi = S^*, \tag{7.113}$$

where S^* is not necessarily equal to S, but is is proportional to S. For $\alpha = 2$, we must have that $n = 2$ giving $m = 2$. The conserved quantity becomes

$$\int_{-\infty}^{\infty} y^2 \frac{\partial \psi_2}{\partial y} dy = \int_{-\infty}^{\infty} \xi^2 F'''(\xi) d\xi = K^*, \tag{7.114}$$

where K^* is proportional to K. Using $F(\xi)$ defined in (7.103) we have

$$S^* = -D, \tag{7.115}$$

$$K^* = 2D. \tag{7.116}$$

If we consider a constant multiple of the functions ψ_1 and ψ_2, such as $\alpha_1\psi_1$ and $\beta_1\psi_2$, we can set

$$\alpha_1 = -\frac{S}{D}, \tag{7.117}$$

$$\beta_1 = \frac{K}{2D}, \tag{7.118}$$

which gives the required conserved quantities

$$-\frac{S}{D}\int_{-\infty}^{\infty} y\frac{\partial\psi_1}{\partial y}dy = S, \tag{7.119}$$

$$\frac{K}{2D}\int_{-\infty}^{\infty} y^2\frac{\partial\psi_2}{\partial y}dy = K. \tag{7.120}$$

The function $F(\xi)$ defined in (7.103) has some very interesting properties:

$$F^{(n)}(\pm\infty) = 0, \quad n \geq 1, \tag{7.121}$$

$$F^{(n)}(0) = 0, \quad \text{n even}, \tag{7.122}$$

$$F^{(n)}(0) = \frac{1}{\sqrt{2\pi}}(-1)^{Floor[n/2]}(n-2)(n-4)(n-6)...1, \quad \text{n odd}, \tag{7.123}$$

where the function Floor is defined as follows:

$$\text{Floor}\left[\frac{n}{2}\right] = \frac{n}{2}, \;\; \text{n even}, \tag{7.124}$$

$$\text{Floor}\left[\frac{n}{2}\right] = \frac{n-1}{2}, \;\; \text{n odd}. \tag{7.125}$$

From (7.121) we have

$$\frac{\partial^2\psi_n}{\partial y^2}(x,\pm\infty) = 0, \quad \frac{\partial\psi_n}{\partial y}(x,\pm\infty) = 0, \tag{7.126}$$

and therefore the boundary conditions (7.98) and (7.99) are satisfied for all ψ_n. Now,

$$\frac{\partial\psi_n}{\partial x}(x,y) = -\frac{E(x)}{\left(2\int_0^x E(\alpha)d\alpha\right)^{n/2+1}}\left(\xi F^{(n+1)}(\xi) + nF^{(n)}(\xi)\right), \tag{7.127}$$

which from (7.121) gives

$$\frac{\partial\psi_n}{\partial x}(x,\pm\infty) = 0. \tag{7.128}$$

From (7.122) we also have that

$$\frac{\partial^2\psi_n}{\partial y^2}(x,0) = 0, \quad \text{n even}, \tag{7.129}$$

and

$$\frac{\partial\psi_n}{\partial x}(x,0) = 0, \quad \text{n even}. \tag{7.130}$$

Thus, for each function ψ_n, we have that ψ_n satisfies the PDE (7.97) and the boundary conditions (7.98) and (7.99). For n even, the conditions in (7.100) and (7.101) are also satisfied. Because the PDE in (7.97) is linear and the boundary conditions are homogeneous, any constant multiple of ψ_n will also satisfy the PDE and boundary conditions (7.98) and (7.99) and if n is even the conditions (7.100) and (7.101) will also be satisfied.

Consider the function $\alpha_1\psi_1$ where α_1 is defined in (7.117). It satisfies the partial differential equation (7.97) and the boundary conditions (7.98), (7.99) as well as the conserved quantity (7.119). However, it does not satisfy (7.100) and (7.101) because n is odd in (7.127). We therefore consider the stream function

$$\rho(x,y) = -\frac{S}{D}\psi_1(x,y) + \alpha_2(x) = -\frac{S}{D}\frac{\partial\psi}{\partial y} + \alpha_2(x), \tag{7.131}$$

and choose $\alpha_2(x)$ so that $\rho(x,y)$ satisfies (7.101). The addition of $\alpha_2(x)$ to $\psi_1(x,y)$ does not alter the properties of $\psi_1(x,y)$ because $\alpha_2(x)$ is a function of x only. The stream function $\rho(x,y)$ satisfies (7.101) provided $\rho(x,0)$ is a constant independent of x. The line $y=0$ is a streamline and a stream function is constant along a streamline. By using (7.103) for $\psi(x,y)$, (7.131) becomes

$$\rho(x,y) = -\frac{S}{\sqrt{2\pi}}\frac{1}{(2\int_0^x E(\alpha)\,d\alpha)^{1/2}}\exp[-\xi^2/2] + \alpha_2(x). \tag{7.132}$$

We choose $\rho(x,0)=0$. Thus

$$\alpha_2(x) = \frac{S}{\sqrt{2\pi}}\frac{1}{(2\int_0^x E(\alpha)\,d\alpha)^{1/2}}, \tag{7.133}$$

and therefore,

$$\rho(x,y) = \frac{S}{\sqrt{2\pi}}\frac{1}{(2\int_0^x E(\alpha)\,d\alpha)^{1/2}}\left(1-\exp\left[-\frac{\xi^2}{2}\right]\right), \tag{7.134}$$

which corresponds to the solution given by (7.65) for the combination wake.

We now consider the function $\beta_1\psi_2$ where β_1 is defined in (7.118). It also satisfies the partial differential equation (7.97) and the boundary conditions (7.98), (7.99), (7.100) and (7.101) as well as the conserved quantity in (7.120). Unlike ψ_1, it satisfies (7.100) and (7.101) because n is even in (7.127). Thus $\psi_2(x,0)$ is constant along the streamline $y=0$ and takes the value $\psi_2(x,0)=0$. We let

$$\phi(x,y) = \frac{K}{2D}\psi_2 = \frac{K}{2D}\frac{\partial^2\psi}{\partial y^2}. \tag{7.135}$$

Using (7.103) for $\psi(x,y)$, we obtain

$$\phi(x,y) = \frac{K}{2D}\frac{F''(\xi)}{(2\int_0^x E(\alpha)d\alpha)}, \tag{7.136}$$

and is equivalent to

$$\phi(x,y) = -\frac{K}{2\sqrt{2\pi}}\frac{1}{\left(2\int_0^x E(\alpha)d\alpha\right)}\xi \exp\left[-\frac{\xi^2}{2}\right], \tag{7.137}$$

which is the same as the solution found in Chapter 5 for the turbulent wake of a self-propelled body with $E = E(x)$.

We have shown that it is possible to generate the solution for the combination wake and the wake of a self-propelled body from the solution to the classical wake with $E = E(x)$.

When obtaining the constants of integration in the solution of the ordinary differential equations for $F(\xi)$, $G(\xi)$ and $P(\xi)$ and also when deriving the conserved quantities, stronger boundary conditions than (7.98)and (7.99) are required. These stronger boundary conditions are of the form

$$\xi^m \left.\frac{dF}{d\xi}\right|_{\xi=\pm\infty} = 0, \quad \xi^n \left.\frac{d^2F}{d\xi^2}\right|_{\xi=\pm\infty} = 0, \tag{7.138}$$

where m and n are positive integers. Since the solutions derived for $F(\xi)$, $G(\xi)$ and $P(\xi)$ all tend to zero exponentially like $\exp\left[-\xi^2/2\right]$ as $\xi \to \pm\infty$ it is readily verified that the stronger boundary conditions are indeed satisfied by the solutions derived.

7.4 Conclusions

Three conservation laws for the partial differential equation for the two-dimensional turbulent wake equation with $E = E(x)$ were obtained. In order to generate two of the conservation laws, the eddy viscosity had to be independent of the variable y. Two of the conservation laws belong to the classical wake and the wake of a self-propelled body. The other law appears to be a new wake problem, which we called the combination wake. Lie symmetry methods were applied to the combination wake in order to obtain the invariant solution. It is not clear as to the physical significance of the combination wake. This wake is asymmetric about the x- axis and has a zero mean velocity deficit on the axis of the wake. Although the physical importance of this problem has yet to be established, the mathematical significance was immediately clear. The combination wake was found to be the link between the classical wake and the wake of a self-propelled body. If we differentiate the solution for the classical wake with respect to y, we discover that the combination wake solution is the sum of a term proportional to this derivative and a function of x. Furthermore, the derivative with respect to y for the solution to the combination wake is proportional to the solution for the wake of a self-propelled body.

This discovery initiated the task of determining whether the solutions for the combination wake and the wake of a self-propelled body can be directly obtained from the solution for the classical wake. We had to assume that the similarity variable is the same for each problem which meant that the effective width of each wake had to tend to zero as x approached zero. When $E = E(x)$ all y derivatives of the stream function are also solutions to the equation. By enforcing the condition that the conserved quantities must remain invariant under the change of variables that was used for the classical wake in order to reduce the PDE to an ODE, it was found that the first two y derivatives of the stream function satisfied the other two conserved quantities. It was subsequently shown that the solutions for the combination wake and the wake of a self-propelled body could indeed be generated directly from the classical wake solution without needing to calculate the Lie point symmetry associated with the conserved vectors.

Further work will be conducted in this area. A systematic method needs to be developed in order to explain in detail the mathematical connection between the solutions to problems with homogeneous boundary conditions that require a conservation law to complete their solution. Possible links between the solutions for jet flow problems will be investigated.

Chapter 8

Conclusions

The governing equations for the two-dimensional turbulent wake in terms of the mean velocity components were derived by applying the boundary layer approximation to the Reynolds averaged equations. The system of equations was completed by using the eddy viscosity and Prandtl mixing length closure models. The equations were expressed in terms of the y-component of the mean velocity, $\overline{v}_y$, and the mean velocity deficit in the x-direction, $\overline{w}$. In addition, it was assumed that we were sufficiently far downstream of the obstruction allowing for products and powers of $\overline{v}_y$ and $\overline{w}$ to be neglected. A stream function was introduced which reduced the system of equations from two to one equation. We consider an eddy viscosity E which depends on the distance along the wake, x, the perpendicular distance from the axis of the wake, y, and the mean velocity deficit gradient, $\dfrac{\partial \overline{w}}{\partial y}$. We first studied an eddy viscosity as a function of the spacial variables x and y only. This particular form of the eddy viscosity predicts an infinite wake boundary. For $E = E(x, y)$, the resulting diffusion equation, which was written in terms of a stream function, applied to both the turbulent classical wake and the turbulent wake of a self-propelled body. The turbulent classical wake and the turbulent wake of a self-propelled body were shown to have identical boundary conditions at $y = 0$ and at $y = \pm\infty$. The boundary conditions are homogeneous and thus a conserved quantity is required. The two problems differ in the conserved quantity that they satisfy. An eddy viscosity $E = E\left(x, \dfrac{\partial \overline{w}}{\partial y}\right)$ can be used to generate the form of the Reynolds stresses required for Prandtl's model and the revised version of it. For $E = E\left(x, \dfrac{\partial \overline{w}}{\partial y}\right)$, the boundary conditions at $y = 0$ do not differ from the case where $E = E(x, y)$. However, since a finite wake boundary is predicted for this model, the mainstream matching conditions were imposed at the boundary of the wake at $y = \pm y_b(x)$ and not at $y = \pm\infty$. In order to obtain the conserved quantity for the wake of a self-propelled body, we had to assume that $E = E(x)$ only. Therefore, models that predict finite wake boundaries can only be applied to the classical wake problem.

In Chapter 3 we calculated the conservation laws of the turbulent wake equation in terms of the velocity components and the stream function. The elementary conserved vector which was obtained directly from the governing equations generated the conserved quantity, namely, the drag force, for the classical wake with $E = E\left(x, y, \frac{\partial \overline{w}}{\partial y}\right)$. In order to generate the conserved quantity for the turbulent wake of a self-propelled body as derived by Birkhoff and Zorantello [5], we had to neglect the y-dependence of the eddy viscosity and simply let $E = E(x)$. The multiplier method was used to calculate the conserved vectors for the two-dimensional turbulent wake equation with $E = E(x)$. The governing equations expressed in terms of the velocity components and the stream function were considered. Three physically significant conservation laws were found. Two of the conservation laws belonged to the classical wake and the wake of a self-propelled body. The third one, did not pertain to any known conservation law. As a result, another type of wake, which we called the combination wake, was discovered.

In Chapters 4 and 5, the turbulent classical wake with $E = E(x, y)$ and the turbulent wake of a self-propelled body with $E = E(x)$ were investigated further. Lie symmetry methods were used in order to generate the invariant solution associated with the conserved vector for each problem. Two invariance conditions had to be satisfied by each Lie point symmetry associated with a conserved vector. These conditions depended on only the first and second prolongations of the symmetry. This method is easier than working with one large invariance condition for the full group of Lie point symmetries of the partial differential equation which depends on prolongations up to third order.

Previous studies of these problems considered similarity solutions that can be obtained by neglecting the kinematic viscosity. Since we do not neglect the kinematic viscosity, Lie symmetry methods were required for this study in order to produce analytical solutions that were not similarity solutions. A modified version of Prandtl's hypothesis was also considered for both wake problems. The modified version stated that the eddy viscosity is constant across the boundary layer and proportional to the product of the maximum mean velocity deficit and the width of the layer and was applied to both types of wakes. Mean velocity profiles were plotted for each type of wake. It was concluded that the role of the eddy viscosity was to increase the diffusion of vorticity across the wake hence increasing the effective width of the wake.

In Chapter 6 an eddy viscosity of the form $E = E\left(x, \dfrac{\partial \overline{w}}{\partial y}\right)$ was considered which produced the correct form of the Reynolds stresses as predicted by Prandtl's mixing length model. This model predicts a finite wake boundary. A revised Prandtl mixing length model was developed. Unlike with Prandtl's model, the kinematic viscosity was included in the revised model. The elementary conserved vector was used to generate the invariant solution for the problem. In Prandtl's mixing length theory there is only one length scale, namely, the mixing length. We considered a mixing length $l(x)$ as an arbitrary function of x. We then determined the forms that $l(x)$ must satisfy for an invariant solution to exist. The Lie point symmetry with the kinematic viscosity included is a scaling symmetry. If the kinematic viscosity is neglected, Lie symmetry methods showed that other analytical solutions that are not simply similarity solutions can also be obtained. Inclusion of the kinematic viscosity lead to better predictions of the behaviour of the wake.

In Chapter 7, we revisited the combination wake problem. Lie symmetry methods were used to determine the invariant solution. With $E = E(x)$ the governing equation reduced to a linear PDE when expressed in terms of the stream function. It was due to this equation now being linear that we could determine the solutions for the combination wake and the wake of a self-propelled body directly from the solution for the classical wake. Once we assumed that the similarity variable is the same for each wake problem, it was not difficult to produce a simple method of obtaining the solutions for the combination wake and the wake of a self-propelled body from the classical wake solution.

This research showed that Lie symmetry methods can be used as a means to further our understanding of the problem of the two-dimensional turbulent wake described by eddy viscosity. Not only were these methods used to produce the invariant solutions, but they also lead to the discovery of the combination wake. The relationship between mathematical modelling and symmetry methods is of great importance. Physical arguments such as defining an effective width of a wake, were used to determine the arbitrary constants in the Lie point symmetry. This proved that when using Lie symmetry methods one should never lose sight of the physics of the problem. From a modelling perspective, the importance of including the kinematic viscosity could not be overstated. The kinematic viscosity determined the equation of the boundary of the wake. Even although the kinematic viscosity is much less than the eddy viscosity it could not be neglected. If it is neglected an additional assumption is required to obtain the equation of the wake boundary [2, 10, 62]. Once the wake boundary has been found the kinematic viscosity can be neglected although it does lead to a small difference in the predictions. In the literature, the classical wake and the wake of a self-propelled body are treated separately. The application of the multiplier method to derive the conservation laws allowed the classical wake, the combination wake and the wake of a self-propelled body to be analysed together and the three conservation laws to be obtained in one calculation. This demonstrated the relationship between the wakes and provided unification of the theory of two-dimensional wakes. Lie symmetry methods were required in order to calculate an analytical solution when the kinematic viscosity was included. The mathematical relationship between the wake solutions indicates that we have not yet fully unravelled and understood the link between Lie symmetry methods and the physics of a model.

Bibliography

[1] Cebeci T. Analysis of turbulent flows, Elsevier, London, 2004.

[2] Prandtl L, in: Proceedings of the Second International Congress for Applied Mechanics, Zurich 1926, p 62.

[3] Boussinesq J. Théorie de l'écoulement tourbillant. Mém. prés. Acad. Sci. 23 (1877) 46.

[4] Goldstein S. On the two-dimensional steady flow of a viscous fluid behind a solid body. Proc. R. Soc. Lond. Math. Phys. Eng. Sci. 142 (1933) 545-561.

[5] Birkhoff G, Zorantello E H. Jets, Wakes and Cavities. Academic Press, New York, 1957.

[6] Herczynski A, Weidman P D, Burde G I. Two-fluid jets and wakes. Phys. Fluids 16 (2004) 1037-1048.

[7] Tennekes H, Lumley J L. A First Course in Turbulence, MIT Press, Cambridge, Massachusetts and London 1971, pp. 113-124.

[8] Rotem Z. A note on boundary-layer solutions for pseudoplastic fluids. Chem. Engng. Sci. 21 (1966) 618-620.

[9] Weidman P D, Van Atta C W. The laminar axisymmetric wake for power-law fluids. Acta Mech. 146 (2001) 239-245.

[10] Swain L M. On the turbulent wake behind a body of revolution. Proc. R. Soc. Lond. Ser. A 125 (1929) 647-659.

[11] Hunt J C R. A theory for the laminar wake of a two-dimensional body in a boundary layer. J. Fluid Mech. 49 (1971) 159-178.

[12] Hunt J C R, Jackson P S. Wakes behind two-dimensional surface obstacles in turbulent boundary layers. J. Fluid Mech. 64 (1974) 529-563.

[13] Unal G. Application of equivalence transformations to inertial subrange of turbulence. Lie Groups Appl. 1 (1994) 232-240.

[14] Ibragimov N H, Unal G. Lie groups in turbulence. Lie Groups Appl. 1 (1994) 98-103.

[15] Oberlack M. A unified approach for symmetries in plane parallel turbulent shear flows. J. Fluid Mech. 42 (2001) 299-328.

[16] Mason D P. Group invariant solution and conservation law for a free laminar two-dimensional jet, J. Nonlinear Math. Phys. 9 (Supplement 2) (2002) 92-101.

[17] Ruscic I, Mason D P. Group invariant solution and conservation law for a steady laminar axisymmetric free jet. Quaest. Math. 27 (2004) 171-183.

[18] Mason D P, Hill D L. Group invariant solution for a two-dimensional turbulent free jet described by eddy viscosity. J. Nonlinear Math. Phys. 15 (Supplement 1) (2008) 146-160.

[19] Vinogradov A M. A spectral sequence associated with a nonlinear differential equation and algebra-geometric foundations of Lagrangian field theory with constraints. Soviet Math. Dokl. 19 (1978) 144-148.

[20] Vinogradov A M. Symmetries and conservation laws of partial differential equations: basic notions and results. Acta Appl. Math. 15 (1989) 3-21.

[21] Naz R, Mason D P, Mahomed F M. Conservation laws and conserved quantities for the laminar two-dimensional and radial jets. Nonlinear Anal.: Real World Appl. 10 (2009) 2641-2651.

[22] Naz R, Mahomed F M, Mason D P. Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. Appl. Math. Comput. 205 (2008) 212-230.

[23] Anco S C, Bluman G W. Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. Euro. J. Appl. Math. 13 (2002) 545-566.

[24] Anco S C, Bluman G W. Direct construction method for conservation laws of partial differential equations Part II: General treatment. Euro. J. Appl. Math. 13 (2002) 567-585.

[25] Anco S C, Bluman G. Direct Construction of Conservation Laws from Field Equations. Phys. Rev. Lett. 78 (1997), 2869-2873.

[26] Wolf T. A comparison of four approaches to the calculation of conservation laws. Euro. J. Appl. Math. 13 (2002), 129-152.

[27] Bluman G W, Cheviakov A F, Anco S C. Applications of Symmetry Methods to Partial Differential Equations. Springer- Verlag New York, Applied Mathematical Sciences 168, 2010.

[28] Steudel H. Über die zuordnung zwischen invarianzeigenschaften und erhaltungssätzen. Zeitschrift für Naturforschung 17 (1962) 129-132.

[29] Olver P J. Applications of Lie Groups to Differential Equations. Springer- Verlag New York, Graduate Texts in Mathematics 107 (1986).

[30] Kara A H, Mahomed F M. Relationship between symmetries and conservation laws. Int. J. Theor. Phys. 39 (2000) 23-40.

[31] Kara A H, Mahomed F M. A basis of conservation laws for partial differential equations. J. Nonlinear Math. Phys. 9 (Supplement 2) (2002) 60-72.

[32] Naz R, Naeem I. Group invariant solutions for two-dimensional free, wall and liquid jets having finite fluid velocity at orifice, in: World Academy of Science, Engineering and Technology, vol 43, 2008, pp 275-281.

[33] Mason D P, Hill D L. Invariant solution for an axisymmetric turbulent free jet using a conserved vector. Commun. Nonlinear Sci. Numer. Simulat. 18 (2013) 1607-1622.

[34] Anthonyrajah M, Mason D P. Conservation laws and invariant solution in the Fanno model for turbulent compressible flow. Math. Comput. Appl. 15 (2010) 529-542.

[35] Sjöberg A. Double reduction of PDEs from the association of symmetries with conservation laws with applications. Appl. Math. Comput. 184 (2007) 608-616.

[36] Bluman G W, Anco S C. Symmetry and Integration Methods for Differential Equations. Springer- Verlag New York, Applied Mathematical Sciences 154, 2002.

[37] Krasilshchik I S, Vinogradov A M. Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, American Mathematical Society, Translations of Mathematical Monographs, Book 182, 1999.

[38] Popovych R O, Kunzinger M, Ivanova N M. Conservation laws and potential symmetries of linear parabolic equations. Acta. Appl. Math. 100 (2008) 113-185.

[39] Steinberg S, Wolf K B. Symmetry, conserved quantities and moments in diffusive equations. J. Math. Anal. Appl. 80 (1981) 36-45.

[40] Kara A H, Mahomed F M. Noether-type symmetries and conservation laws via partial Lagrangians. Nonlinear Dynamics 45 (2006) 367-383.

[41] Bluman G W, Cole J D. Similarity Methods for Differential Equations. Applied Mathematical Sciences, Vol 13, Springer, Berlin, 1974.

[42] Bluman G W, Cole J D. The general similarity solution of the heat equation. J. Math. Mech. 18 (1969) 1025-1042.

[43] Olver P J, Rosenau P. The construction of special solutions to partial differential equations. Phys. Lett. A 114 (1986) 107-112.

[44] Olver P J, Rosenau P. Group-Invariant Solutions of Differential Equations. SIAM J. Appl. Math. 47 (1987) 263-278.

[45] Pucci E, Saccomandi G. On the weak symmetry groups of partial differential equations. J. Math. Anal. Appl. 163 (1992) 588-598.

[46] Mansfield E L. The nonclassical group analysis of the heat equation. J. Math. Anal. Appl. 231 (1999) 526-542.

[47] Arrigo D J, Hickling F. On the determining equations for the nonclassical reductions of the heat and Burger's equation. J. Math. Anal. Appl. 270 (2002) 582-589.

[48] Vaneeva O O, Johnpillai A G, Popovych R O, Sophocleous C. Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities. J. Math. Anal. Appl. 330 (2007) 1363-1386.

[49] Popovych R O, Ivanova N M. Hierarchy of conservation laws of diffusion-convection equations. J. Math. Phys. 46, 043502 (2005) 1-22.

[50] Ivanova N M. Conservation laws and potential systems of diffusion-convection equations. Proc. Institute Math. NAS Ukraine. 50 (2004) Part 1, 149-152.

[51] Edwards M P. Classical symmetry reductions of nonlinear diffusion-convection equations. Phys. Lett. A 190 (1994) 149-154.

[52] Yung C M, Verburg K, Baveye P. Group classification and symmetry reductions of the non-linear diffusion-convection equation $u_t = (D(u)u_x)_x - K'(u)u_x$. Int. J. Non-Linear Mech. 29 (1994) 273-278.

[53] Popovych R O, Ivanova N M. New results on group classification of nonlinear diffusion-convection equations. J. Phys. A 37 (2004) 7547-7565.

[54] Ivanova N M, Sophocleous C. On the group classification of variable-coefficient nonlinear diffusion-convection equations. J. Comput. Appl. Math. 197 (2006) 322-344.

[55] Dorodnitsyn V A. On invariant solutions of the equation of non-linear heat conduction with a source. U.S.S.R. Comput. Maths. Math. Phys. 22 (1982) 115-122.

[56] Bluman G, Kumei S. On the remarkable nonlinear diffusion equation $(\partial/\partial x)[a(u+b)^{-2}(\partial u/\partial x)] - (\partial u/\partial t) = 0$. J. Math. Phys. 21 (1980) 1019-1023.

[57] Oron A. Some symmetries of the nonlinear heat and wave equations. Phys. Lett. A 118 (1986) 172-176.

[58] Hutchinson A J. A unified theory for turbulent wake flows described by eddy viscosity. Int. J. Non-Linear Mech. 81 (2016) 40-54.

[59] Hutchinson A J, Mason D P, Mahomed F M. Solutions for the turbulent classical wake using Lie symmetry methods. Commun. Nonlinear Sci. Numer. Simulat. 2015; 23: 51-70.

[60] Hutchinson A J, Mason D P. Lie symmetry methods applied to the turbulent wake of a symmetric self-propelled body. Appl. Math. Modelling. In Press.

[61] Hutchinson A J, Mason D P. Revised Prandtl mixing length model applied to the two-dimensional turbulent classical wake. Int. J. Non-Linear Mech. 77 (2015) 162-171.

[62] Schlichting H. Boundary-Layer Theory, Sixth Edition, M^{c}Graw-Hill, New York, 1968 (Chapters 8, 18 and 19).

[63] Gillespie R P. Integration, Oliver and Boyd, Edinburgh (1959) 113-116.

[64] Pope S B. Turbulent Flows, Cambridge University Press, Cambridge, 2000, pp 92-95.

[65] Ibragimov N H, Anderson R L. One-parameter transformation groups, in: Ibragimov N H (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Vol 1: Symmetries, Exact Solutions and Conservation Laws. CRC Press, Boca Raton 1994, pp 7-14.

[66] Glauert M B. The wall jet. J. Fluid. Mech. 1 (1956) 625-643.