# Firewall Argument for Acoustic Black Holes

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## Abstract

We investigate the firewall paradox proposed by AMPS [1] by first explaining the Information Paradox together with Hawking's derivation of the thermal radiation emitted from a evaporating black hole [28]. We then ask if one can apply arguments similar to that of Hawking and AMPS in the regime of fluid mechanics, which was first considered by Unruh [59]. We assume that a black hole, with a geometry conformal to the Schwarzschild metric, can be formed in a fluid. The sonic hole or "dumb" hole, which is characterized by an acoustic event horizon, is the locus of points at which the background fluid is traveling at the local speed of sound. Since sound disturbances are coupled to the background fluid and travel at the speed of sound, the acoustic event horizon affects sound disturbances in a manner analogous to how gravitational black holes affect light [62]. Like a gravitational black hole, which evaporates by emitting Hawking radiation, we check if an acoustic black hole will emit in a similar kind of radiation in the form of phonons. This is done by constructing a massless scalar field describing phonon propagation and treating the acoustic black hole just like a gravitational black hole. We apply the arguments put forth by Hawking and AMPS and see if there is any validity to an "acoustic firewall" as this would require certain physical phenomena emerging from sub-atomic scales.

## Acknowledgments

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Sincerely,

Luca Pontiggia

# Contents

# 1. Introduction

Einstein's general theory of relativity (published in 1916) predicted the existence of objects so massive that nothing - not even light - could escape. These objects became known as black holes. The first black holes were indirectly observed in 1971 by looking at the effect that an unknown object had on the X-ray binary star Cygnus X-1 [42]. Fast forward to current times, a large number of papers have been written on various kinds of black holes: static black holes, rotating black holes, even black holes in higher dimensions. Apart from the extraordinary properties of black holes, the immense gravity surrounding them gives us a theoretical playground to explore how the physics of the very large (general relativity) interacts with the physics of the very small (quantum mechanics). This question led efforts for a consistent theory allowing a unification between these two realms. Quantum gravity is one such theory, although it is not yet complete.

Hawking wrote a paper predicting that black holes are not everlasting objects that continuously absorb, but rather a type of thermodynamic object that actually emits radiation, (of a quantum nature) causing it to eventually evaporate. This radiation is known as Hawking radiation. His argument led to the uncomfortable conclusion that if his prediction is true, then there exists some fundamental inconsistency in either special relativity, or quantum mechanics. This inconsistency is made apparent in Hawking's paradox. Immediately, physicists began either trying to disprove Hawking or search for resolutions to the paradox.

As the preliminaries, chapter 2,3 and 4 expand on many of the concepts required for the reader to have a sufficient grasp of the material covered in chapters 5 and 6. Seeing as there are three pillar theories which are present within the thesis - general relativity, quantum mechanics and quantum field theory- chapter 2 introduces the essential mathematical and or physical technical tools. It is assumed the reader has had undergraduate/honours courses in the above topics and thus chapter 1 is not meant as a substitute to textbooks in the relevant topics. That said, for the purpose of highlighting the importance of topics like particle creation in curved space, certain ideas are quite thoroughly explained. Chapter 2 gives the reader a run through of the properties pertaining to black holes - from their geometric nature to their quantum mechanical nature. Most importantly, chapter 2 gives a detailed analysis of how particles are created around a black hole. Chapter 3 is a review of Hawking's paradox, commonly referred to as the information paradox as Hawking's paradox is a problem in quantum information loss. The information paradox in chapter 4 is the origin for many of the arguments and concepts in chapter 5. We attempt to give a somewhat detailed overview of Hawking's original paper and explain why the information problems generates conflict between quantum mechanics and special relativity.

The proposed resolution to the information paradox is highlighted in chapter 5. The idea of complementarity attempts to settle issues raised from the chapter 4. I use the word 'attempts', as although for a few years black hole complementarity appeared to be the answer, in 2012 a paper known as "AMPS" [1] highlighted an inconsistency in black hole complementarity. This became known as the firewall paradox, which is also introduced in chapter 4. The firewall paradox ultimately states that something strange happens to a free falling observer as they cross the event horizon of a black hole, namely, they encounter a hot dense wall and burn up even before they are able to enter - something which should not happen according to the equivalence principle. The problem with these paradoxes - and their variations - is that at some point they require an understanding of the internal structure of black holes. This is something not easily accessed due to the inability of successfully merging general relativity and quantum mechanics in a single complete theory.

Chapter 6 is the climax of the thesis culminating from the work done in previous chapters. We examine the essential aspects of a black hole and the firewall paradox in a different setting as a way to circumvent any difficulties encountered when using more advanced theories; Such a setting is fluid mechanics. We build a sonic black hole model and examine which features of gravitational black holes translate to acoustic black holes. A lot of the work on this topic was originally done by Unruh [59] and elaborated further by Visser [62]. It is interesting to see that in fact a similar radiation to the Hawking radiation appears, in the form of phonons (sound particles). The idea is to see if one also predicts an 'acoustic firewall'. We then present some techniques to 'look' for a firewall and question whether a firewall makes sense in the context of fluid mechanics as an effective theory.

# Chapter 2

# Mathematical Toolbox and Definitions

Often when one reads through a physics paper with partial knowledge on the subject at hand, it is easy to get lost behind the mathematical and physics jargon. The purpose then of this chapter is to cover all the key concepts of the various theories which are most prevalent in this thesis. Namely, general relativity, quantum mechanics and quantum field theory. The break down is done in such a way that the language we use to explain the later chapters is contained in the first few chapters, with the majority of mathematical details contained in this chapter. Derivations and detailed calculations are only used when necessary with the intention to stress the significance of any results and their application to later chapters. The general relativity section will cover ideas from how one mathematically describes the presence of gravity in a non-Newtonian theory to how one can illustrate the idea of an infinite *spacetime* in a compact picture known as a *Penrose* diagram. The section on quantum mechanics introduces important tools such as *density matrices* and combines some quantum information theory to introduce *entanglement entropy.* Lastly, the section on quantum field theory uses the language of quantum fields to explain how certain properties of these fields differ in the presence of gravity compared with no gravity, and how this difference brings about particle creation in a quantum *vacuum.*

## 2.1 General Relativity

In essence, general relativity can by divided in two areas: spacetime curvature and energy. Consider the Einstein equation:

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{2.1.1}$$

$G^{\mu\nu}$ is known as Einstein's tensor, it tells us something about the curvature of spacetime. $T^{\mu\nu}$ is known as the energy momentum tensor and describes the energy distribution in space. To understand general relativity is to understand this equation and its solutions. For our purpose, it is the left hand side, the curvature of spacetime, which we focus on. The Einstein tensor can be rewritten as

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R, \tag{2.1.2}$$

where $R^{\mu\nu}$ and $R$ are the Ricci tensor and Ricci scalar respectively. Since curvature is very well expressed as a mathematical concept, we use the tools of differential geometry to give a comprehensive analysis of what spacetime curvature is. Differential geometry allows us to define spacetimes as having a *manifold* structure which then allows us to introduce metrics, vector spaces and more importantly tensors.

Once the more rigorous mathematical tools are defined, we can then use them to construct a spacetime interval which encodes the fact that space and time are intrinsically linked. We then are able to discuss what *geodesics* are and how curved geodesics represent the presence of a gravity field. Finally, *Penrose diagrams* will aid in giving a schematic and intuitive picture of what the spacetime really is along with its *causal* properties.

### 2.1.1 Differential Geometry

#### 2.1.1.1 Manifolds

One can imagine spacetime to be expressed as a four dimensional continuum, whereby one requires four numbers to assign coordinates to spacetime. Locally, one can describe such a spacetime with a $\mathbb{R}^{1,3}$ structure, just as one can describe the surface of the earth locally as $\mathbb{R}^2$ even though it is actually a curved sphere. In general, however, one wants to solve for spacetime geometries without making any reference to a specific global spacetime structure. Thus, we need a mathematical tool to describe the spacetime structure in an abstract sense. The discussion of manifolds will be in a general relativity context, and so a spacetime manifold $\mathcal{M}$ is a set of points that define coordinate independent events. The notion of a manifold captures this abstract idea of spacetime having curvature with potentially complicated topologies, yet in certain regions can look locally flat like $\mathbb{R}^n$. The manifold can be subdivided into an *atlas*, i.e a collection of *charts*.

A chart is formally a subset $U$ of $\mathcal{M}$, containing a *one-to-one* map $X^r : U \to \mathbb{R}^n$. We call $X^r$ a coordinate map. An atlas is thus the union of all charts which cover the manifold. Each chart can be mapped to $\mathbb{R}^n$. It is this mapping which gives us coordinate systems like $(x^1, x^2, x^3 ... x^n)$. Each chart comes with their own coordinate mappings as can be seen in figure 2.1.

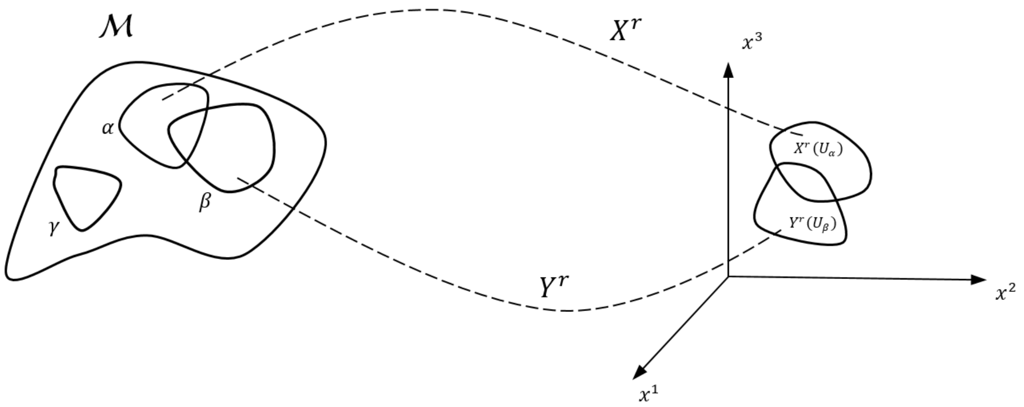


Figure 2.1: Coordinate mapping of various charts on the manifold, $\mathcal{M}$, to $\mathbb{R}^3$ (the time coordinate has been suppressed). $X^r$ and $Y^r$ are the coordinate mappings of charts $\alpha$ and $\beta$ respectively.

In the overlapping regions, there exists a set of transformations which allow one to go from one coordinate space to another. This region thus needs to be well defined. In other words, the transformation between coordinate functions should be infinitely differentiable, $C^\infty$. If the manifold $\mathcal{M}$ is $C^\infty$, then the overlapping regions will automatically be $C^\infty$.

#### 2.1.1.2 Vector Spaces

Vectors are generally introduced as an 'object' with magnitude and direction. This is encapsulated in the idea of an arrow pointing along some direction from one point to another as $x, y, z$ coordinates. The problem with this 'arrow' idea of a vector is that it works only in $\mathbb{R}^n$ or flat space. The more general way of introducing vectors is by defining them on some manifold. Thereafter we will see why we were able to refer to vectors as 'arrows' in $\mathbb{R}^n$. The velocity vector, defined as the tangent to some displacement curve, was another way of describing a vector. This definition turns out to be a useful one, as it extends to a manifold. We are able to define a curve on a manifold as a $C^\infty$ curve which is a map $\gamma(\lambda) : \mathbb{R} \to \mathcal{M}$ such that the image in $\mathbb{R}^n$ is $C^\infty$, where $\lambda$ is the parameterization.

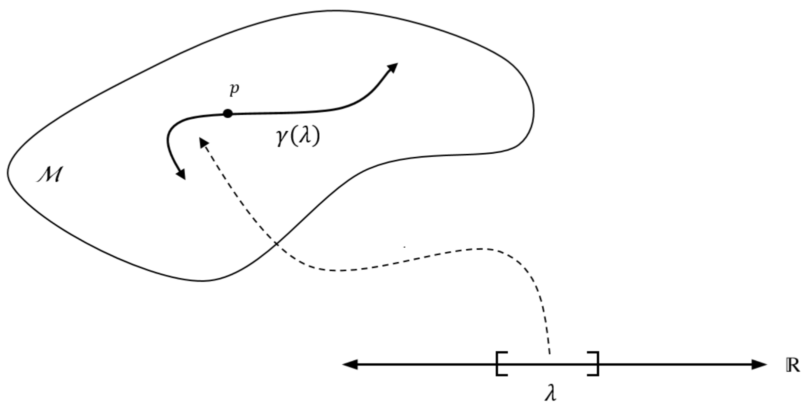


Figure 2.2: The curve defined as a map which takes a point on the real line and maps it to a point $p$ on the manifold.

In other words, a curve on a manifold takes a point in $\mathbb{R}$, given by the value of $\lambda$, and maps it to a point on the manifold which we call $p$. And so, a vector is defined as being tangent to the curve at $p$ on the manifold

$$vf = \frac{df}{d\lambda} \quad ; \; \forall f \in C^{\infty}(\mathcal{M}), \tag{2.1.3}$$

where $v$ is called a *directional derivative* and it maps $f \to \frac{df}{d\lambda}$ at the point $p$ on the manifold. One can figuratively think of $v$ as '$\frac{d}{d\lambda}$'.

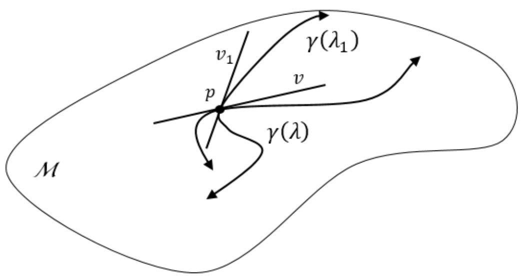


Figure 2.3: Two curves each going through point $p$ will have their own tangent vectors defined by taking the directional derivative of each curve function at $p$.

We are not restricted to having a single curve passing through point $p$ on the manifold. Imagine we introduce a new curve $\gamma_1(\lambda_1)$ as seen in figure 2.2 also passing through point $p$ for some value of $\lambda_1$. We can then find the tangent to this curve at $p$ in similar way as before

$$v_1 f = \frac{df}{d\lambda_1} \quad ; \; \forall f \in C^{\infty}(\mathcal{M}), \tag{2.1.4}$$

The common operations on vectors can be seen by the following

- Change "length" of $v$ by reparameterisation of $\lambda \to a\lambda$, $a \in \mathbb{R}$.

- Change direction by changing curve.

We can continue to add more and more curves which pass through $p$, each one defining a new tangent vector $\{v, v_1, v_2 \dots v_n\}$. The set of all such vectors at $p$ forms a vector space, also called the *tangent space* of $p$ : $V_p(\mathcal{M})$. A different point on the manifold will have its own separate tangent space, the set of all these spaces on the manifold is called the *tangent bundle*, $T(\mathcal{M})$.

A result of being in curved space is that we cannot relate different tangent spaces in a simple manner. In flat space, however, where the manifold is (for illustration purposes) $\mathbb{R}^3$, one can move the different tangent spaces on top of one another. Vectors are drawn as arrows from the origin to some point described by a set of unique coordinates (line $\vec{q}$, $\vec{w}$ and $\vec{w}'$ in figure 2.4). One is actually identifying a tangent plane which is based at the origin of $\mathbb{R}^3$. Imagine we draw some curve elsewhere on the flat manifold and locate a point through it, then define a new tangent vector $\vec{w}$. We then draw another vector at the origin and call it $\vec{w}'$. The idea is that in flat space $\vec{w}$ is the same vector as $\vec{w}'$.

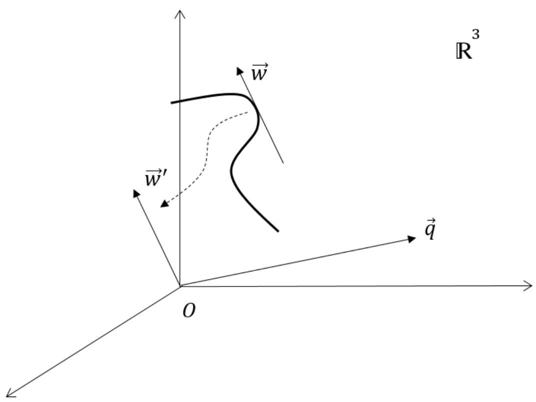


Figure 2.4: In flat space, we are able to simply shift vectors around due to the fact that $\mathcal{M}$ and $T(\mathcal{M})$ are identical.

This is because the tangent bundles of $\mathbb{R}^3$ are merely copies of the single tangent plane based at the origin, that is $\mathcal{M}$ and $T(\mathcal{M})$ are identical. You then have a completely well defined way of transporting vectors all over the plane. We call this *parallel transport.* This is a unique feature to euclidean space as one cannot uniquely define parallel transport in curved space.

Another important quantity is the dual vector or covector. A dual vector is a map from $V_p(\mathcal{M}) \to \mathbb{R}$ i.e.

$$\langle \tilde{u} | \tilde{v} \rangle \in \mathbb{R}. \tag{2.1.5}$$

We denote the space of covectors as $V_p^*(\mathcal{M})$ elements of which are dual vectors to elements in $V_p(\mathcal{M})$. For a given basis $\{e_a\}$ of $T_p(\mathcal{M})$, there exists a dual basis $\{w_a\}$ of $V_p^*(\mathcal{M})$ such that

$$\langle w^a | e_b \rangle = \delta^a{}_b. \tag{2.1.6}$$

From this we can see how the map is represented in components, let

$$\tilde{V} = V^a e_a, \tag{2.1.7}$$

$$\tilde{U} = U_b w^b, \tag{2.1.8}$$

then

$$\begin{aligned}\langle \tilde{U} | \tilde{V} \rangle &= U_b V^a \langle w^a | e_b \rangle \\ &= U_b V^a \delta^a{}_b \\ &= U_a V^a,\end{aligned}$$

which is a sum over components and returns a number. Only once we introduce the metric a little later on, can we think of the covectors and vectors being equivalent. From a mathematical prospective, they are quite different. Vectors are defined as tangents to a curve whereas covectors are a mapping from vectors to real numbers.

An important feature of vectors is the way they transform under a change in coordinates. Objects which transforms like vectors are often considered to be vectors. The transformation law for a vector $A$ is given by

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}. \tag{2.1.9}$$

Covectors instead transform with indices down

$$B'_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} B_{\mu}. \tag{2.1.10}$$

Comparing these equations, the partial derivative for the covariant transformation has the primed coordinate in the denominator, and the vector transformation has the primed coordinate in the numerator. Often covectors are also called *1-form*, *covariant vectors* or dual vectors as we saw above.

#### 2.1.1.3 Tensors

The notion of tensors arises when one considers quantities which have some varying dependence on vector-like quantities, take displacements as an example. The measurement of stress on an object will yield a number depending on the direction you measure. Here the directions are the line of force which is exerted through a point, call it $\vec{l}$ and the the normal vector $\vec{n}$ to the plane on which the point resides. Due to the linear dependence of the force on $\vec{l}$ and $\vec{n}$ one only requires to take measurements in three linearly dependent directions and all other measurement readings will have a linear dependence on those measurements. So a tensor is a *multilinear* map from vectors or dual vectors into numbers. In the case above, the tensor which maps sets of $(\vec{l}, \vec{n})$ to numbers which represent a value of the force is known as the stress tensor of a body at some point. To define the tensor it is important to understand the notion of a dual vectors as it appears in the definition. Let

$V$ and $V^*$ be the vector and dual vector space respectively as defined previously, then a tensor, $T$ of type $(k, l)$ over $V$ is a multilinear map [64]

$$T : \underbrace{V^* \times \ldots \times V^*}_{k} \times \underbrace{V \times \ldots \times V}_{l} \to \mathbb{R}. \tag{2.1.11}$$

Given $k$ dual vectors and $l$ vectors, the tensor $T$ returns a number, and it does so in such a manner that if one fixes all but one of the vectors or covectors, it is a linear map on the remaining variable. Hence a $(1, 0)$ is a vector, note that $V^{**}$ is a double dual vector, which is associated with the vector $V$ and $(0, 1)$ is a dual vector. And so according to our example of the stress tensor, a tensor of type $(1, 1)$ is a multilinear map from $V^* \times V \to \mathbb{R}$.

Without going into too much detail, a more intuitive way of understanding a tensor is by looking at how it transforms. We define the $n$-th rank tensor with components $T^{\mu\nu\ldots}{}_{\alpha\beta\ldots}$ which transform when one changes coordinate systems according to the transformation rules. We have seen in the section on vector spaces how vectors and covectors transform, the general tensor transformation is the following

$$\tilde{T}^{\mu\nu\ldots}{}_{\alpha\ldots} = \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\gamma}} \cdots \frac{\partial x^{\sigma}}{\partial x'^{\alpha}} \cdots T^{\beta\gamma\ldots}{}_{\sigma\ldots}\,. \tag{2.1.12}$$

Just like a vector is a quantity which transforms like a vector with upper indices, and a covector is something which transforms like a covector with lower indices, a tensor is something which transforms like a tensor. These are some examples of how tensors of various rank transform.

| Rank $n$ | Components | Common Name | Symbol | Transformation Law | Example |
|---|---|---|---|---|---|
| 0 | $4^0 = 1$ | Invariant scalar | $\phi$ | $\phi' = \phi$ | Rest mass $m$ |
| 1 | $4^1 = 4$ | Vector | $A^\mu$ | $A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$ | four momentum $p^\mu$ |
| | | covector | $A_\mu$ | $A'_\mu = \frac{\partial x^\mu}{\partial x'^\nu} A_\nu$ | gradient of a scaler $\partial_\mu \phi$ |
| 2 | $4^2 = 16$ | tensor | $T^{\mu\nu}$ | $T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}$ | Energy Stress Tensor $T^{\mu\nu}$ |
| | | | $T^\mu{}_\nu$ | $T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} T^\alpha{}_\beta$ | Kroneker delta $\delta^\mu{}_\nu$ |
| | | | $T_{\mu\nu}$ | $T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}$ | Metric Tensor $g_{\mu\nu}$ |
| 3 | $4^3 = 64$ | tensor | $M^{\mu\nu}{}_\alpha$ | $M'^{\mu\nu}{}_\alpha = \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial x'^\nu}{\partial x^\gamma} \frac{\partial x^\sigma}{\partial x'^\alpha} M^{\beta\gamma}{}_\sigma$ | |
| 4 | $4^4 = 256$ | tensor | $R^\alpha{}_{\beta\mu\nu}$ | $R'^\alpha{}_{\beta\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} R^\gamma{}_{\delta\sigma\lambda}$ | Rieman Tensor $R^\alpha{}_{\beta\mu\nu}$ |

Tensors with rank 2 or greater can have any combination of indices up or down, one can raise or lower indices using a metric (discussed in the next section). Tensors are particularly useful in physics for their invariance under a choice of coordinates, that is, tensors will have exactly the same form in no matter what coordinate system one uses. This allows us to express physics in a coordinate free manner as long as we give a tensorial structure to physical quantities. Laws which are coordinate independent are referred to as *manifestly covariant*. All laws expressed in such forms are automatically consistent with relativity, in that all laws will be true in any *Inertial Reference Frame*. It is for this reason that one strives when possible to express physics using tensors.

#### 2.1.1.4 Metric

Intuitively the metric can be thought of as a quantity which describes distances, that is to say, how squared distances are related to displacements. Think of the distance formula in Euclidean space

$$ds^2 = (x^1 - x^2)^2 + (y^1 - y^2)^2 + (z^1 - z^2)^2. \tag{2.1.13}$$

It relates the displacement between two points, to their square distances. It is more precise to use the notion of infinitesimal displacements and distances. These are characterized by the tangent vector. The metric should thus be a linear map of $V_p \times V_p$ into numbers, i.e,

$$g : V_p \times V_p \to \mathbb{R} \tag{2.1.14}$$

The metric should also be a tensor of rank 2 with the following properties:

1 **Symmetric**

$$\forall v_1.v_2 \in V_p, \quad g(v_1, v_2) = g(v_2, v_1).$$

2 **Non-degenerate**: The only case where

$$g(v_1, v) = 0 \;\; \forall \; v \;\in\; V_p$$

is when $v_1 = 0$.

3 **Bilinear**: let $v_1, v_2, v_3 \in V_p$ be tangent vectors at some point $p$, and $a, b \in \mathbb{R}$ be some real numbers then, the metric $g_p$ at point $p$ on the manifold is bilinear if

$$g_p(av_1 + bv_2, v_3) = ag_p(v_1, v_3) + bg_p(v_2, v_3), \quad \text{and}$$
$$g_p(v_3, av_1 + bv_2) = ag_p(v_3, v_1) + bg_p(v_3, v_2).$$

In essence a metric is an inner product on the tangent space at each point.
Formally the metric $g$ can be expanded in some coordinate basis.

$$g = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{2.1.15}$$

In practice we write

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1.16}$$

where the decomposition can always be found in some orthonormal basis $v_1 \ldots\ldots v_n$ of the tangent space at $p$, i.e,

$$g_p(v_\mu, v_\nu) = \begin{cases} 0 & \text{if } \mu \neq \nu \\ \pm 1 & \text{if } \mu = \nu \end{cases}.$$

The $\pm$ sign is related to the signature. The signature of the metric describes the number of "$+$" and "$-$" signs which occur in the metric, metrics which signatures "$+, +, +, \ldots, +$" are said to be *positive* *definite*, such metrics are also called Riemannian. Metrics with signature "$-, +, +, \ldots, +$" are called Lorentzian metrics[1]. Lorentzian metrics are used in general relativity to describe spacetime, thus, they are also referred to as spacetime metrics.

The metric $g$ can also be viewed as a linear map which takes $V$ into $V^*$, $v \to g(\cdot, v)$. Where the dot refers to the fact that only one argument of the metric has been filled, it is a sort of "place holder". Since $g$ is non-degenerate, this map is a one-to-one and onto, therefore, its inverse $g^{-1}$ exists. This allows us to form a one-to-one correspondence between vectors and covectors (dual vectors). In other words, given some element $v_1 \in V_p$ (a vector belonging to the tangent space at point $p$), we can use $g_p$ to turn $v_1$ into $v_1^* \in V_p^*$. This is what is known as raising or lowering indices

$$A_\mu = g_{\mu\nu} A^\nu ,$$

where $A^\mu$ is some vector and $A_\nu$ is a covector belonging to the dual space of $A^\mu$. The correspondence $v^\mu \longleftrightarrow v_\nu^*$ gives rise to an *isomorphism* between $V$ and $V^*$. Since the *isomorphism* relies on a choice of basis, there is no natural way of identifying $V$ with $V^*$, unless as we have seen above more structure is given on $V$ as a basis or a metric.

### 2.1.2 Spacetime Interval

When we want to assign some structure to the space around us, we introduce coordinates. Coordinates allow us to define points on some manifold using a grid system. In $\mathbb{R}^2$ we can set up a coordinate system with a grid composed of horizontal and vertical parallel lines. Any point within this orthogonal grid structure can be uniquely specified via its horizontal and vertical position relative to another point. These are known as Cartesian coordinates. The choice of coordinates are arbitrary. One usually adopts a set of coordinates based on the underlying symmetry of the topological space. Since we live in a space which looks locally like $\mathbb{R}^{1,3}$, three spatial coordinates and one time, we are able to assign some coordinate system allowing us to uniquely define where events[2] occur relative to us.

A fundamental principle of special relativity is that preferred inertial observers do not exist. The act of assigning some arbitrary set of labels to events is independent of the observer, thus, coordinates are not a property of the geometry of spacetime itself. 'Physics', given as the collective description of laws which govern the Universe, exist independent of our relative position and choice of coordinates. This puts more significance to an observer independent quantity with an absolute meaning which all observers agree on.
This observer independent quantity is known as the *spacetime interval* $I$

$$I = -c^2(\Delta t)^2 + \left[(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2\right], \tag{2.1.17}$$

[1] Different literature sometimes uses the convention $+, -, -, \ldots, -$.

[2] An event is a point in spacetime. It is characterized by a time label $t$, given by a clock located at some position coordinates , $x^1, x^2, x^3$.

where we will also commonly use $x^1, x^2, x^3 \equiv x, y, z$.

$I$ is to spacetime what distance is to Euclidean space. It is the theorem of Pythagoras of spacetime. More technically, we refer to spacetime as having a manifold structure of $\mathbb{R}^4$, with metric $\eta_{\mu\nu}$, whereby

$$\eta_{\mu\nu} = \sum_{a,b=0} \eta_{ab}(dx^a)_\mu (dx^b)_\nu,$$

and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. This metric possesses geodesic curves which are straight lines, hence we use the term *flat space* or Minkowski space. Flat space geometry is unique, in the sense that one can define parallel lines as ‘ lines that will never meet’. Such a definition does not make sense on a curved space.[3] The spacetime metric thus represents the intrinsic geometry of spacetime, it so happens that the geometry on which the spacetime interval $I$ is defined is flat. In general, spacetime can have any kind of geometry, in particular, as we will see a bit later, spacetime in the presence of gravity is curved. As long as the metric is Lorentzian, we can use it to describe distances between spacetime events for which the spacetime interval $I$ is observer independent. Since all inertial observers agree on the value of the spacetime interval, they must all agree on whether $I$ is positive, zero or negative. This enables us to classify the interval into 3 distinct categories

I. **space-like** interval

$$I > 0, \quad c^2\Delta t^2 < \Delta x^2 + \Delta y^2 + \Delta z^2. \tag{2.1.18}$$

II. **light-like** interval

$$I = 0, \quad c^2\Delta t^2 = \Delta x^2 + \Delta y^2 + \Delta z^2. \tag{2.1.19}$$

III. **time-like** interval

$$I < 0, \quad c^2\Delta t^2 > \Delta x^2 + \Delta y^2 + \Delta z^2. \tag{2.1.20}$$

In order to have a clear picture of what these categories signify, we introduce the light cone. Writing $I$ as

$$I = -c^2(t - t_A)^2 + \left[(x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2\right]. \tag{2.1.21}$$

allows us to see that $I$ is the separation interval between some point $(t, \vec{x})$ and the observers coordinates $(t_A, \vec{x}_A)$. Consider the set of events such that they have a 0 time interval $I$, we set the observes coordinates at the origin such that $t_A = 0, x_A = y_A = z_A = 0$. Eqn. (2.1.21) becomes

$$0 = -(ct)^2 + x^2 + y^2 + z^2. \tag{2.1.22}$$

This is the equation of a cone in spacetime.

[3] Think of an atlas. There is no such concept of parallel grid lines. Extended geodesics intersect at least twice.

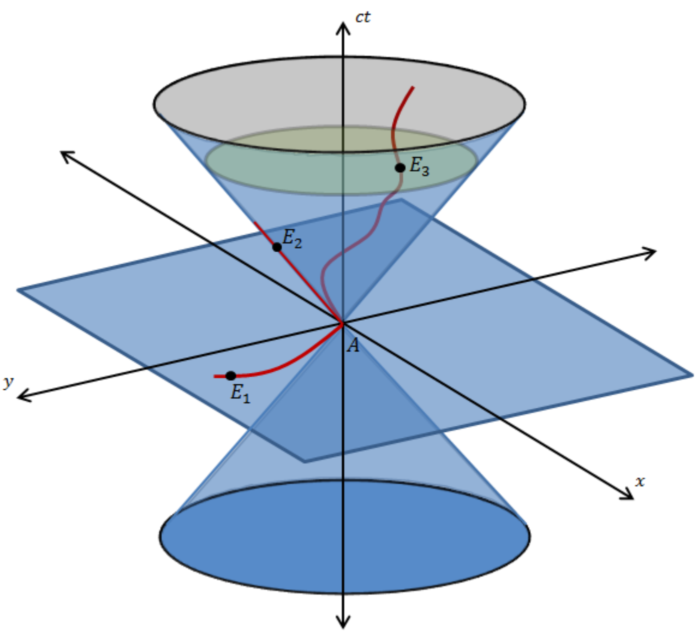


Figure 2.5: The three dimensional representation of a light cone for an observer. $E_1, E_2, E_3$ label different events which occur at various spacetime intervals relative to observer at A. $E_1$ is *space-like*, $E_2$ is *light-like*, $E_3$ is *time-like* .

Physically eqn. (2.1.22) characterizes both the outward propagation of a light pulse emitted at time $t_A = 0$ and position $\vec{x}_A = 0$, and the inward propagation of some light pulse emitted at an earlier time. The lower half of the cone represents all the set of events from which light can be emitted to reach $A$. Similarly, the upper half of the cone represents all sets of events which can be reached by $A$. Points on the surface of the cone have a zero spacetime interval, all such events which lie on the surface are said to be *light-like separated*. And so, *light-like separated* events are these events which are connected by a physical entity travelling at the speed of light. Using the light cone, one can now unambiguously understand the 3 different categories.

**Space-like interval** ($I > 0$)

Events which have a positive spacetime interval are said to be *space-like separated*. These are events that occur in spacetime which cannot affect each other. Since the speed of light is a constant, there is a limit to the reach that information can have. As can be seen two events which are space-like separated occur, there is no reference frame which exists that allows an observer to view the event at the same position in space, as this would require the information of the event to travel at a speed faster than the light. There does however exist a frame for which two space-like separated events are observed to occur at the same time. A possible *space-like* separated event from and observer could be considered event $E_1$ in figure 2.5.

**Light-like interval** ($I = 0$)

As explained above, events with a null spacetime interval are said to be *light-like*. Light-like separated events are thus events that can be linked by a light signal or by any other signal or particle that travels at the speed of light. Such events will lie on the surface of the light cone. Thus, the light cone at $A$ divides spacetime around $A$ into six mutually exclusive regions or sets of events. The light cone itself consists of three of these sets. The first region is given by event $A$(apex of the cone)in figure 2.5. Second is the set of all events that are *light-like* separated from $A$ and which lie in the future of $A$ (these lie on the "top" conical surface represented by the event $E_2$ in figure 2.5, called the future light cone). The third is the set of all events that are light-like separated from $A$ which lie in the past of $A$ (these lie on the "bottom" conical surface in figure (2.5) called the past light cone). The remaining three regions are those separated from each other by the light cone surface. These are the inside of the future light cone, the inside of the past light cone and the remaining events that lie outside the light cone. All events lying inside either the future or past light cones are time-like separated and all events that lie outside are space-like separated.

**Time-like interval** ($I < 0$)

Events which have a negative time interval are called *time-like*, these are all events which are found strictly inside either the future or past light cone regions. The worldline of any massive object, will always be contained inside the time-like regions of the light cone as represented by event $E_3$ in figure 2.5. If the worldline were to cross the boundary of the light cone, that particle would have traveled faster then the speed of light, which is forbidden. Events that place within the light cone can b casually related.

The rules for distinguishing between Lorentzian spacetime interval, apply for all spacetimes, independent of their geometry. The interval is a statement of the causal nature of physics, which is introduced by the fact that there exists a speed limit for all information traveling in the universe.

### 2.1.3 Curvature of Spacetime and Geodesics

The statement that matter curves spacetime is often used to summarize Einstein's theory of General Relativity. The idea that spacetime is curved is a very important one, it is how gravity presents itself, and so to better understand the concept of how spacetime curves, we can unpack the statement into two explanations; the first, is qualitative, it refers to how objects 'move through' spacetime, and is the more intuitive approach. The second, quantitatively gives a way to measure the curvature of spacetime, and is more mathematically inclined.

**How objects move through space**

Classically we think of the trajectory of a *free falling* object as the one followed given a specific set of initial conditions. If two objects have the exact same initial conditions they will follow the exact same trajectory, likewise, if they have differing initial conditions, they will follow different trajectories. It seems that there is no unique trajectory between two sets of points. There is an idea of general relativity called the *Geodesic Hypothesis*[4], this says that a free particle follows a *geodesic* in spacetime.

A geodesic is simply put, the shortest distance between two points. This can be viewed as a form of *Fermat's principle*, which states that "the path taken between two points by a ray of light is the path that can be traversed in the least time". On a flat surface (Euclidean space), for example a sheet of paper, a geodesic is a straight line. On a curved surface (Non-Euclidean), like a sphere, a geodesic would be a great circle. A space's geometric characteristics therefore defines unique geodesic paths in that space. But we just stated that two objects with different initial conditions, which travel through the exact same space, will follow different trajectories and thus there is no unique path. Is there a contradiction? The answer is no. The parabolic path traced by such objects are defined by considering a three dimensional space; the geodesic hypothesis only makes sense in the context of spacetime, not just space. If the path of the projectiles were plotted in 4 dimensional spacetime, three spatial dimensions and one time dimension, the paths would in fact follow similar geodesics. In other words, they do not follow the exact same geodesics with the same end points, but the geodesics have the same radius of curvature determined by the spacetime geometry. On earth the radius of curvature of the geodesic followed by say a rock and bullet is approximately 1 light year.

Another important idea which indirectly highlights that gravity is the curvature of spacetime is the *equivalence principle*; if a freely falling reference frame is an inertial reference frame, then it should be physically equivalent to a freely floating frame, similarly a rest frame on the surface of the earth should be equivalent to a frame in deep space which is accelerating at 9.81 $m \cdot s^{-2}$. One may ask how can you distinguish then between an accelerating frame, and a frame inside a gravitational field, or more specifically, how can you even tell that you are in a real gravitational field? The answer is in the presence of *tidal forces* around a gravitating body. Imagine two rooms, one freely floating in deep space, and the other freely falling in a gravitational field. Spread across the rooms are a collection of balls, all initially at rest. In the freely floating frame, the balls will remain in exactly the same positions as time goes by, but in the freely falling frame, the balls will begin to move towards or away from eachother depending on their positions in the room. This effect is due to the tidal force

[4]One possible motivation for the Geodesic hypothesis comes from the fact that an object experience the same acceleration in a gravitational field irrespective of their mass, this is different, for example to electrostatics where the charge of the particle influences its acceleration, and so it seems that in actual fact spacetime is responsible for how gravity is manifested.

within the gravitational field.

From general relativity's point of view, we interpret tidal forces as converging/diverging geodesics in spacetime. Clearly in flat space (the freely floating frame) geodesics which begin parallel will always remain parallel, hence the balls remain fixed. In a curved spacetime, geodesics which originally are parallel, will eventually diverge/converge hence the balls moving towards/away from each other. And so once we understand the geometric structure of the spacetime near a gravitating object, we can calculate geodesics and thus predict how freely falling objects will move.

**Geodesics**

As mentioned briefly in the above, by determining a geodesic we can understand what the spacetime structure is. The statement that a geodesic is the shortest path for a free particle, can alternatively be stated as: a geodesic is the path of longest proper time for a free particle. Let's explore this concept in a bit more detail. Consider two paths through flat spacetime, path $P_f$ is the path of a particle which is in an interstitial reference frame. We chose the inertial reference frame such that the particle has a zero velocity. Path $P_{Nf}$ is the path of a particle in an a non-inertial reference frame. We ensure that it is a non-inertial frame by requiring that some events $A$ and $B$ are common to both paths, thus if path $P_{Nf}$ is different to path $P_f$ it must be non inertial at some point to meet such a condition

the invariant

$$P_f = ds'^2 = -c^2 dt'^2 + 0 + 0 + 0, \tag{2.1.23}$$

$$P_{Nf} = ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2.1.24}$$

is the same for all observers, thus

$$d\tau = \sqrt{-ds^2}, \tag{2.1.25}$$

and so for $P_{Nf}$

$$d\tau = \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2} \tag{2.1.26}$$

$$= dt\sqrt{1 - v^2}. \tag{2.1.27}$$

Obviously $0 < v < 1$, so $d\tau$ or in that case $\tau$ will be a maximum when $v = 0$ or along the free particles world line, hence out of all possible world lines that connect two events, the world line of a free particle is the one that maximises $\tau$, where

$$\tau = \int \sqrt{-ds^2} = \int_0^1 d\sigma \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}. \tag{2.1.28}$$

The notion of finding a path which maximizes or minimizes some quantity is exactly the same as the action $S$. The action looks for the trajectory which minimizes or finds the extremum of the Lagrangian

$$S = \int_{t_A}^{t_B} L(q_i, \dot{q}_i) dt. \tag{2.1.29}$$

Such a trajectories satisfy the Euler-Lagrange equation

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}. \tag{2.1.30}$$

This trajectory is analogous to the geodesic, that is, the geodesic extremizes the Lagrangian

$$L(x^\alpha, \dot{x}^\alpha) = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x^\nu}}, \tag{2.1.31}$$

by satisfying

$$0 = \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha}. \tag{2.1.32}$$

Equation (2.1.32) is a set of N differential equations, thus in our spacetime this would correspond to 4 differential equations. By applying the chain rule and noting that $\frac{d\tau}{d\sigma} = L$, with some algebra and renaming of labels, one can rewrite (2.1.32) into the geodesic equation

$$0 = \frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\beta}{d\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{2.1.33}$$

This is a far more convenient form of equation (2.1.32) as it directly contains the spacetime metric tensor instead of having it hidden away in the Lagrangian. Solving for such an equation would leave one with coordinates $x^\alpha(\tau)$ for events along the geodesic.

**Describing curvature Mathematically**

Without going into too much detail and mathematical rigor, there is one mathematical object which characterizes curvature, the Riemann Tensor $R^\alpha{}_{\mu\nu\sigma}$. Since spacetime is a manifold in $\mathbb{R}^4$, the Riemann tensor describes the curvature of the spacetime manifold[5]. It makes an important appearance in the equation of geodesic deviation

$$\left(\frac{d^2 n}{d\tau^2}\right)^\alpha = -R^\alpha{}_{\mu\nu\sigma} \tag{2.1.34}$$

The equation of geodesic deviation comes from looking at how two infinitesimally close geodesics move relative to each other. The derivation is explained very clearly in [41], this will just be a summary to highlight the key ideas.

Imagine two free particles traveling along adjacent geodesics described by two functions $x^\alpha(\tau)$ (reference particle) and $\tilde{x}^\alpha(\tau) \equiv x^\alpha(\tau) + n^\alpha(\tau)$ where $\tau$ is the proper time as measured by the reference particle, and $n^\alpha(\tau)$ is an **infinitesimal separation four-vector** which extends between the two particles at some time $\tau$. Clearly the change in $n^\alpha(\tau)$ describes how the geodesics deviate, two initially parallel geodesics which remain parallel implies that the particles have zero relative acceleration, that is, $d^2 n^\alpha/d\tau^2 = 0$. But from (2.1.34), a zero relative particle acceleration will therefore imply a zero Riemann Tensor.

We can therefore say that if a Riemann Tensor is zero everywhere, initially parallel geodesics will

[5] The Riemann Tensor describes the curvature of any manifold not just ones which describe spacetime.

remain parallel, hence we have flat space if not, curvature is present. Thus calculating the Riemann tensor gives us a quantifiable way of distinguishing whether a given spacetime is curved or not, something which cannot be seen directly from the metric. The Riemann Tensor is defined by

$$R^{\alpha}{}_{\mu\nu\sigma} \equiv \partial_{\nu}\Gamma^{\alpha}{}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\nu\gamma}\Gamma^{\gamma}{}_{\mu\sigma} - \Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\gamma}{}_{\mu\nu}, \tag{2.1.35}$$

hence the Riemann tensor can be calculated by computing each of the Christoffel symbols defined as

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\left[\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right] \tag{2.1.36}$$

The absolute derivative or covariant derivative is also a useful object to define. It is the extension of the normal directional derivative. We define in in the covariant derivative of some vector field $A(x^{\sigma})$ as

$$\nabla_{\alpha}A^{\mu} \equiv \frac{\partial A^{\mu}}{\partial x^{\alpha}} + \Gamma^{\mu}{}_{\alpha\nu}A^{\nu}. \tag{2.1.37}$$

The covariant derivative is a second rank tensor. Note, neither the Christoffel symbol nor the partial derivative alone are tensors, it is only the combination of the two, which transform like a second rank tensor.

With this we are able to calculate what $G^{\mu\nu}$ for any given metric by using (2.1.2) repeated here

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{2.1.38}$$

### 2.1.4 Penrose Diagrams

The causal structure of spacetime allows us to identify if different events in spacetime are allowed to communicate with each other. Conceptualizing this may not be too difficult, the problem comes if, rather than looking at specific regions within the spacetime structure, one wants to consider the entire infinite size of spacetime as a single and finite picture. How does one neatly describe in a usable and illustrative manner a spacetime which is infinite in scale?

The answer is by making a conformal transformation of the metric structure describing the spacetime. This compactifies all of spacetime into a single diagram, also known as a compact manifold. We call such diagrams conformal Penrose diagrams, or just Penrose diagrams.

In general, what determines the causal structure of spacetime are the null geodesics represented by $ds^2 = 0$.

The advantage of using conformal transformations can be seen by the following argument:

Say we have some spacetime with an associated metric $g_{\mu\nu}$, we can make a conformal transformation of $g_{\mu\nu}$ to get

$$\tilde{g}_{\mu\nu} = e^{w(x)}g_{\mu\nu}. \tag{2.1.39}$$

The two metrics will describe the same spacetime geometry if and only if their causal structure is the same. This statement is equivalent to saying that null geodesics in one spacetime must also be null geodesics in the other spacetime.

Suppose there is some path in spacetime on which there exists a set of points such that

$$\tilde{g}_{\mu\nu}dx^{\mu}dx^{\nu} = 0. \tag{2.1.40}$$

Going back to the original $g_{\mu\nu}$ spacetime, the same set of points would be described by

$$e^{w(x)}g_{\mu\nu}dx^{\mu}dx^{\nu} = 0. \tag{2.1.41}$$

Since $e^{w(x)}$ is never 0, we require that

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = 0. \tag{2.1.42}$$

This tells us that null geodesics in the transformed spacetime are also null geodesics in the old spacetime. Two spacetimes related by such transformations share the same null geodesics, thus share the same causal structure.

The simplest example of a Penrose diagram is that of Minkowski spacetime,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2,$$

or alternatively using common notation of $u, v$, where, $u = t + r$ and $v = t - r$,

$$ds^2 = -dudv + \left(\frac{u-v}{2}\right)^2 d\Omega^2. \tag{2.1.43}$$

Suppose we make the following coordinate transformation

$$u' = \arctan(u), \tag{2.1.44}$$

$$v' = \arctan(v). \tag{2.1.45}$$

This is already a promising transformation as the range of $u', v'$ is restricted to $u', v' \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We find that

$$\begin{aligned} dv' &= \frac{1}{(1+v^2)}dv & du' &= \frac{1}{(1+u^2)}du \\ &= \frac{1}{1+\tan^2(v')}dv & &= \frac{1}{1+\tan^2(u')}du \\ dv' &= \cos^2(v')dv & du' &= \cos^2(u')du \end{aligned}$$

The metric (2.1.43) becomes

$$ds^2 = -\sec^2(v')\sec^2(u')du'dv' + \left(\frac{\tan u' - \tan v'}{2}\right)^2 d\Omega^2. \tag{2.1.46}$$

For our purposes we always assume constant angular coordinates, $d\Omega = 0$,

$$ds^2 = -\sec^2(v')\sec^2(u')du'dv'$$

We can think of $\sec^2(v')\sec^2(u')$ as being some conformal transformation, which just scales the metric, and so the line elements retains its structure of eq. (2.1.43), less the $d\Omega^2$ term

$$ds^2 = -du'dv'.$$

And so the effect of this conformal transformation has been to shrink infinity to the boundary lines of the Penrose diagram illustrated below.

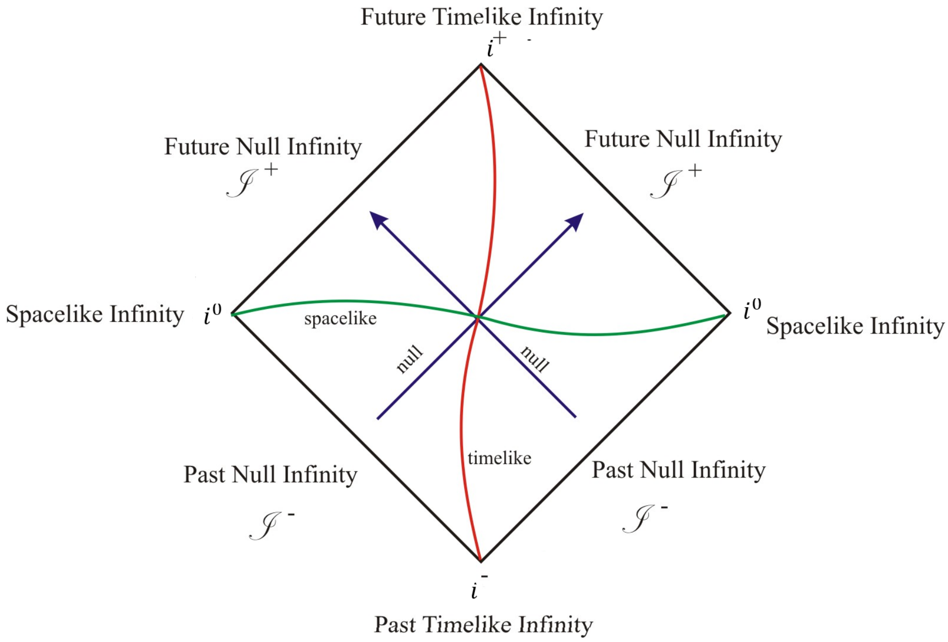


Figure 2.6: The Penrose diagram representing the causal structure of the two-dimensional Minkowski spacetime.

$\mathscr{I}^+$ and $\mathscr{I}^-$ are boundary lines representing future and past null infinity respectively. These are null boundaries as only null rays will begin or end on these boundaries. As mentioned previously, the two spacetime structures share null geodesics thus in both Minkowski diagrams and Penrose diagrams we draw null geodesics at $45^o$ angles. $i^+$ is the future time-like infinity and $i^-$ is the past time-like infinity. All time-like particles will converge on these two points. The points $i_0$ represent space-like infinity. Such diagrams emphasize the fact that only so much information is accessible to any single observer. One can only witness events occurring for as long as they happen inside this *causal diamond*.

What is of more interest to us, are the Penrose diagrams of a black hole. The rough definition of a black hole is something by which light cannot escape, that is, no causal signals can ever reach an observer outside the black hole from behind the event horizon. We use the Penrose diagram to encode this unusual causal structure. We begin with the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

where we now use units of $c = 1$. Consequently $GM$ has the same units of $r$, the radial coordinate. In order to explore the black hole we want to be able to extend over the event horizon. We want to see the event horizon as some regular surface instead of a region in which a coordinate pathology arises. We thus use the Kruskal coordinates.

$$r^* = \int \frac{dr}{1 - \frac{2GM}{r}} = r + 2GM \ln\left[\frac{r}{2GM} - 1\right], \tag{2.1.47}$$

where $r^*$ is a new radial coordinate chosen such that the $dr$ part of the metric will now be simply like a warp factor in $r$ multiplying a flat part of the metric $dt^2$. We define

$$U = -2GMe^{(\bar{u}/2GM)} \quad ; \quad \bar{u} = t - r^* \tag{2.1.48}$$

$$V = 2GMe^{(\bar{v}/2GM)} \quad ; \quad \bar{v} = t + r^* \tag{2.1.49}$$

This then makes the metric regular at $r = 2GM$. These coordinates give the Schwarzschild solution. Which can be see in figure 2.7 below

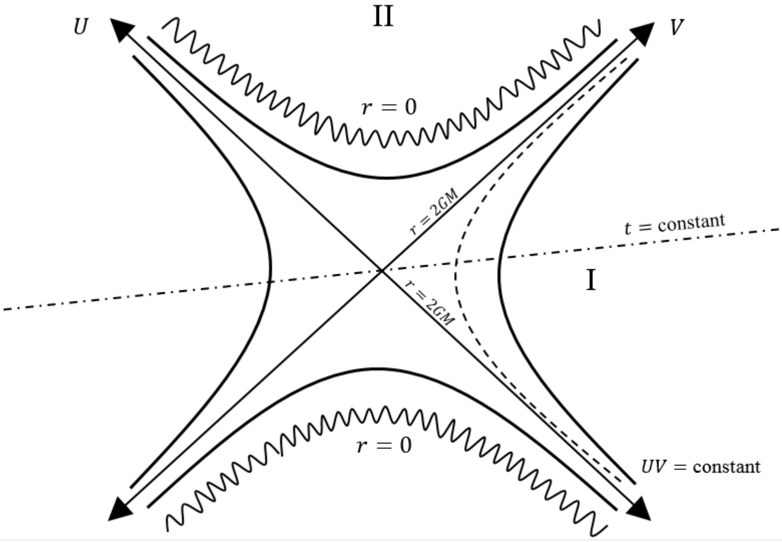


Figure 2.7: The maximally extended Schwarzschild solution. At $r > 2GM$, i.e in region **I** we recover the Schwarzschild solution.

We achieve the Pernose diagram of this particular representation of a black hole my making a very similar transformation as in (2.1.44) and (2.1.45)

$$P = \arctan\left(\frac{V}{2GM}\right), \tag{2.1.50}$$

$$Q = \arctan\left(\frac{U}{2GM}\right). \tag{2.1.51}$$

This will give us the maximally extended Schwarzschild Penrose diagram in figure 2.8. The use of the word 'maximally extended' comes from finding a solution to Einstein's equation that does not present any coordinate singularities. The Schwarzschild solution is not maximally extended to start with, so we redefine the spacetime coordinates so that the space around the coordinate singularity becomes regular. Geodesics within a maximally extended spacetime will physically end on geometric singularities or carry on to asymptotic infinity.

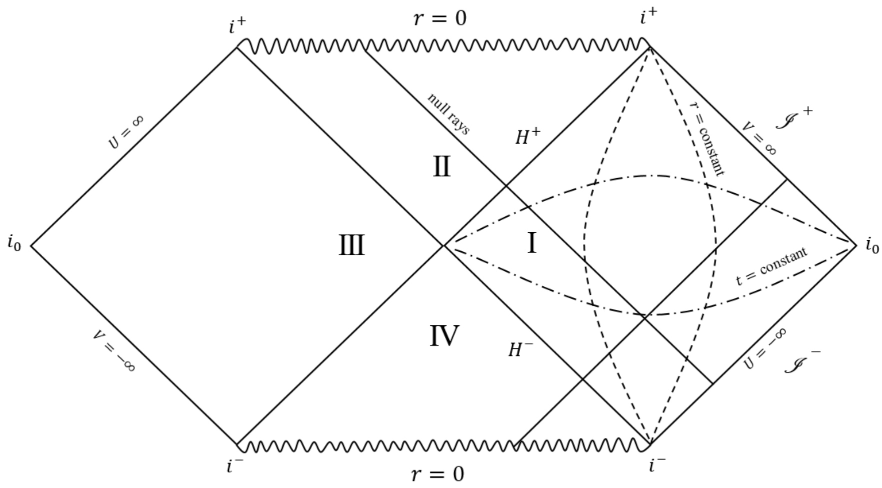


Figure 2.8: The Penrose diagram of a maximally extended Schwarzschild Black Hole, regions **II** and **IV**

are the black hole and white hole regions respectively bounded of by surfaces $H^+, H^-$. Observers who haven not fallen into or have come out of a white hole are resitricted to regions **I** and **III**. $i^+, i^-, i_0$ all have the same definitions as for figure 2.6

If we were an observer and decide to go into the black hole, it is the horizon $H^+$ that we cross, $H^-$ is more of an artificial boundary as that is the null surface representing the boundary of a "white hole" in that someone sitting in region **I** can never influence region **II**. The Penrose diagram that we will use is that of a collapsed star which has formed into a black hole. The Penrose diagram thus will be a hybrid of a flat space Minkowski Penrose diagram in the asymptotic past, and a Schwarzschild black hole in the asymptotic future as represented in figure 2.10

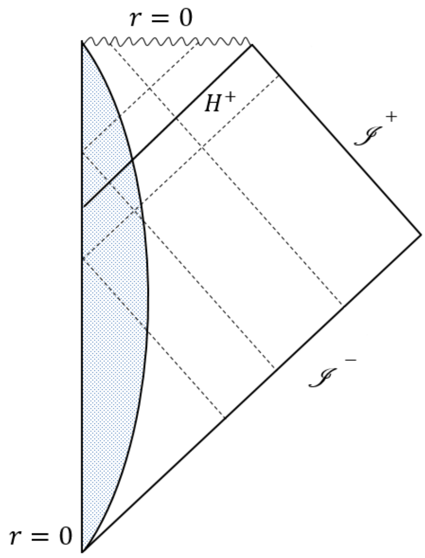


Figure 2.9: Penrose diagram of a collapsed star which has formed into a black hole.

The event horizon of a black hole is not actually something which is locally defined. Imagine when one 'looks' at a black hole in astrophysics, not directly, but via its gravitational influence on the orbits of various stars around it, one sees it as something real and which exists 'now'. Yet $H^+$ is a mathematical structure which is defined from the Penrose diagram, it is a boundary of our vision, a causal boundary. It is defined by looking at the far future and saying that no observer in region **II** of figure 2.8 can ever influence the far future. This is actually a very strong statement.

So what this is distinguishing between, is the formal definition of an event horizon as being something to do with the asymptotic property of spacetime, and a more local definition which has come to be called an apparent event horizon. This is where no matter in what direction one shines a torch, the radially outgoing light rays will always converge to a point onto the singularity. The apparent event horizon and a real event horizon are the same for a Schwarzschild black hole, but can be different for other black hole systems.

## 2.2 Quantum Mechanics

### 2.2.1 Density Matrices

The density matrix which we refer to as $\rho$ is another way of speaking about probabilities, it refers to situations which are more general to just knowing the state vector of the system. How would you describe a system of which you may only have some, to no degree of knowledge of how it was 'prepared'? Incidentally if you have no knowledge then the density matrix is proportional to the unit matrix, that is, it has all equal *eigenvalues*, of $\frac{1}{N}$, where $N$ is the total number of states. This just means complete randomness, all possibilities are equally weighted. The density matrix is the quantum analogue of knowing the probability distribution of classical states,[6] the condition that probabilities must add up to 1 is encoded by looking at the trace of $\rho$

$$\mathrm{Tr}(\rho) = 1, \tag{2.2.1}$$

the trace is the sum of the eigenvalues, $\lambda_i, i = 1 \ldots N > 0$, where $N$ is the dimension of the vector space. The eigenvalues themselves of $\rho$ can be thought of as the probabilities for different states. To ensure that probabilities $(\lambda_i)$ are real, as complex probabilities do not make sense, we require that $\rho$ be Hermitian. Each eigenvalue goes with some eigenstate. In the space of states there is a basis comprised of $N$ mutually orthogonal vectors, where each vector is an eigenvector with an eigenvalue $\lambda_i$. Thus, one can simply think of the $\lambda_i$'s as the probabilities that the system was prepared in the $i^{th}$ eigenvector of $\rho$. If all eigenvalues are equal you know nothing of what state the system is in.

In the limit of maximal knowledge, we have a pure state $|\psi\rangle$ implying it has probability 1 of being in that state, that is, you know in what state the system was created. When the system is described by a pure state, one can still introduce a density matrix defined, as a projection operator which projects onto that pure state $|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi|. \tag{2.2.2}$$

The pure state has one non-zero eigenvalue. A non-pure state is referred to as a *mixed state*, meaning the density matrix simply has more than one non-zero eigenvalue. Whether or not a state is pure, the rule for finding average values of some observable $\bar{M}$ is

$$\bar{M} = \mathrm{Tr}(\rho M) = \sum_i \langle i|\rho M|i\rangle, \tag{2.2.3}$$

where $|i\rangle$ is any basis, not necessarily the one in which $\rho$ is diagonal. Note, the trace of a product of matrices, whether or not the matrices commute, does not matter on the cyclic order of the operators

$$\mathrm{Tr}(XYZ) = \mathrm{Tr}(ZXY), \tag{2.2.4}$$

[6] These are commonly referred to as probability density, density in phase space, or density in coordinate space.

whereas

$$\mathrm{Tr}(XYZ) \neq \mathrm{Tr}(XZY). \tag{2.2.5}$$

The trace is also invariant under a changes of basis - it does not matter in what basis of the vector space one uses to calculate the trace.

These ideas can be seen quite nicely with two examples. The first where we have a strictly pure state $|\psi\rangle$, and the second more general case, where the state is mixed. Let $|\psi\rangle$ be a pure state, then the average value is

$$\begin{aligned} \bar{M} &= \sum_i \langle i|\psi\rangle\langle\psi|M|i\rangle \\ &= \sum_i \langle\psi|M|i\rangle\langle i|\psi\rangle \\ \bar{M} &= \sum_i \langle\psi|M|\psi\rangle, \end{aligned} \tag{2.2.6}$$

where we used the fact the $\sum_i |i\rangle\langle i| = \mathbb{1}$, the unit matrix. And so, not surprisingly, for as long as you are dealing with a pure state the average of $M$ is just the expectation value of $M$. Suppose we now have a mixed state, for the sake of the example we do the calculation in a basis in which $\rho$ is diagonal.

$$\begin{aligned} \bar{M} &= \sum_i \langle i|\rho M|i\rangle \qquad (2.2.7) \\ &= \sum_{i,j} \langle i|\rho|j\rangle\langle j|M|i\rangle. \end{aligned}$$

Since we are working in the basis in which $\rho$ is diagonal, then we know the only matrix elements $\langle i|\rho|j\rangle$ which exists are when $i = j$. These matrix elements are just the eigenvalues $\lambda$ thus

$$\begin{aligned} \bar{M} &= \sum_{i,j} \langle i|\rho|j\rangle\langle j|M|i\rangle, \\ &= \sum_{i,j} \lambda_i\delta_{ij}\langle i|j\rangle\langle j|M|i\rangle, \\ &= \sum_j \lambda_j\langle j|j\rangle\langle j|M|j\rangle, \\ \bar{M} &= \sum_j \lambda_j\langle j|M|j\rangle. \end{aligned} \tag{2.2.8}$$

This reads that the average value $M$ is some expectation value of $M$ in the $j^{th}$ state multiplied or weighed by $\lambda_j$, which is the probability of the $j^{th}$ state. As mentioned before, we can think of the eigenvalues as the probabilities of being in the $j^{th}$ state, but for this case, only in the basis in which $\rho$ is diagonal.

### 2.2.2 Quantum Information

#### 2.2.2.1 Entropy

Classically, if you are given a probability distribution $P_i$, then the entropy $S$ is defined as

$$S = -\sum_i P_i \ln(P_i). \tag{2.2.9}$$

If only one of the $P$'s is non zero, i.e $P_1 = 1$ sum of all probabilities must be 1, then $\ln 1 = 0$ therefore $S = 0$. What this is saying is that if you have complete knowledge, then the entropy is 0. In the case of complete ignorance, each $P_i = \frac{1}{N}$ and then

$$S = -\sum_i P_i \ln(P_i) \tag{2.2.10}$$

$$= -N\frac{1}{N}\ln(\frac{1}{N}) \tag{2.2.11}$$

$$S = \ln N, \tag{2.2.12}$$

for a completely unknown system, the entropy is just the logarithm of the number of states. And so

$$S_{\text{pure}} = 0 \tag{2.2.13}$$

$$S_{\text{random}} = \ln N \tag{2.2.14}$$

Roughly speaking, the entropy is the logarithm of the number of states which are important in the probability distribution, and so it is a measure of the degree of ignorance.

The quantum analogy of entropy using density matrices is

$$S = -\text{Tr}\rho \ln \rho, \tag{2.2.15}$$

conveniently if one works in a basis in which $\rho$ is diagonal one can write the entropy as

$$S = -\sum_i \lambda_i \ln \lambda_i. \tag{2.2.16}$$

Using $N$ as the dimensions of our space as before one gets the same result as for the classical case

$$S_{\text{pure}} = 0, \tag{2.2.17}$$

$$S_{\text{random}} = \ln N. \tag{2.2.18}$$

#### 2.2.2.2 Entanglement

In a classical system, the idea of a pure state would be a probability distribution which has only one point that is non zero, in other words a definite piece of information about which state you're in. If you have complete knowledge about a combined system, you have complete knowledge about either of the subsystems which make up the combined system. In quantum mechanics this is not the case, instead one gets an entanglement of states.

In quantum mechanics entanglement arises when we have more then one system. Suppose we have a system composed of two parts, with a basis labeled by a state or observable '$a'$ for one of the subsystems, and '$b'$ for the other, to give the state $|ab\rangle$. Imagine a case of two spins, in that case we could have $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. The label $a$ could also represent the 2 spin states, and $b$ could label 4 excited states of an atom. Either way altogether the total number of states would be the product of the number of configurations of subsystem $a$ with subsystem $b$. In the above two examples that would be 4 and 8 respectively.

Say that

$$a = 1, \dots N, \tag{2.2.19}$$

$$b = 1, \dots n, \tag{2.2.20}$$

where $n$ and $N$ need not be the same, in fact $a$ and $b$ could be two very different subsystems. The most general state of the combined system that one can write is

$$\sum_{a,b} \Psi(a,b)|a,b\rangle, \tag{2.2.21}$$

with normalization condition of $\sum_{a,b} \Psi(a,b)\Psi^*(a,b) = 1$. Suppose we are doing an experiment which does not involve $b$ but only the $a$'s degrees of freedom. We assign the observable $M$ for the subsystem $a$, this means mathematically that when $M$ acts on a state vector $|ab\rangle$ it may act on the coordinate $a$ but will leave the $b$ coordinate alone. $M$ is therefore only a matrix in the $a$ subsystem which, hence it ignores $b$. Now we can calculate the expectation value of $M$ in the state of the combined system

$$\bar{M} = \sum_{a',a,b',b} \Psi^*(a'b')\langle a'b'|M|ab\rangle\Psi(ab). \tag{2.2.22}$$

As we stated above the matrix $M$ does nothing to $b$ but it is a matrix as for as $a$ is concerned, this means that $b' = b$ as it is left unchanged. The definition that $M$ is an operator that acts on the $a$ subsystem and which is passive to $b$, is simply the statement that the matrix is diagonal in the $b$'s, in that $b'$ and $b$ have to be the same otherwise the matrix vanishes. So $M$ is only a matrix in the $a$ space, thus we can write the expectation value as

$$\bar{M} = \sum_{a',a}\sum_{b} \Psi^*(a'b)M_{a'a}\Psi(ab). \tag{2.2.23}$$

If we ignore for a moment the sum of $a', a$ and focus on the sum over $b$ and group everything that depends on $b$

$$\begin{aligned} \bar{M} &= \sum_{a',a}\sum_{b} M_{a'a}\Psi^*(ab)\Psi(a'b) && (2.2.24)\\ &= \sum_{a',a} M_{a'a}\rho_{aa'}, && (2.2.25) \end{aligned}$$

where the sum over $b$ removes any $b$ dependence. We get a density matrix $\rho_{aa'}$. This expression written in terms of matrix elements is the trace of the product of $M$ times $\rho$

$$\sum_{aa'} M_{a'a}\rho_{aa'} = \mathrm{Tr}(M\rho). \tag{2.2.26}$$

We can see this by using the definition of the trace

$$\mathrm{Tr}(M\rho) = \sum_{a'} \langle a'|M\rho|a'\rangle \tag{2.2.27}$$

$$= \sum_{aa'} \langle a'|M|a\rangle\langle a|\rho|a'\rangle \tag{2.2.28}$$

$$= \sum_{aa'} M_{a'a}\rho_{aa'}. \tag{2.2.29}$$

We find that by ignoring $b$ and asking what are the rules for the $a$ subsystem when it is in a combined system with a unique state vector (which is a pure state for the combined system). You find that the $a$ subsystem is described by a density matrix. When you have a combined composite system, these subsystems may be described by the single state vector whereby they are entangled. But If you look at one alone, it is described by a density matrix ,and in general, that density matrix will be that of a mixed state. There is however a special circumstance in which the density matrix will be a pure state.

The condition on the wavefunction $\Psi(ab)$ when the individual subsystems are in pure states is that the wavefunction needs to factorize into a product of a function of $a$ and a function of $b$,

$$\Psi(ab) = \phi(a)\chi(b) \tag{2.2.30}$$

All this is saying, is that the combined system is composed of a pure state $\phi(a)$ and $\chi(b)$. To see what the density matrix for the $a$ subsystem is, we use the definition above to get

$$\rho_{aa'} = \sum_{a,a',b} \phi(a)\chi(b)\phi^*(a')\chi^*(b) \tag{2.2.31}$$

$$= \sum_{a,a'} \phi(a)\phi^*(a') \sum_{b} \chi(b)\chi^*(b) \tag{2.2.32}$$

$$\rho_{aa'} = \sum_{a,a'} \phi(a)\phi^*(a'), \tag{2.2.33}$$

since we normalized $\sum_b \chi(b)\chi^*(b) = 1$. In this case the expectation value of $M$ would just be

$$\bar{M} = \sum_{a'a} \phi^*(a')M_{aa'}\phi(a) \tag{2.2.34}$$

$$= \langle\phi|M|\phi\rangle. \tag{2.2.35}$$

In other words, the expectation value of $M$ or any observable that involves only $a$, is exactly the same had state vector been just $\phi$. Thus the $a$'s degrees of freedom are completely described by the state $\phi$. So when the wavefuntion is a product of two factors, each subsystem is in itself a pure state, and that pure state is either $\phi(a)$ or $\chi(b)$.

The way of discerning how pure a state vector is, or, how well the wavefunction resembles a product, is by calculating the entropy. If the entropy is small, then you are near a situation of a product state, if the entropy is large - somewhere near the maximum - you are deeply entangled, meaning you are far from a product state. Thus entanglement entropy is a tool we use to measure 'how pure' a state is. The way we measure these entanglement entropies is by making use of the reduced density matrices.

#### 2.2.2.3 Reduced Density Matricies

Previously we saw that entropy gave us a degree of ignorance of the state. The nature of entropy that one uses to discuss entangled states is very different from the thermal entropy. One uses the entanglement entropy in the description of density matrices. Although it was not stated explicitly in the previous section on entanglement, the density matrix we found $\rho_{aa'}$ is a reduced density matrix. To make this more clear, consider the example

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_a|\downarrow\rangle_b - |\downarrow\rangle_a|\uparrow\rangle_b\right). \tag{2.2.36}$$

$|\psi\rangle$ is in a pure state composed of two subsystems $a$ and $b$ just like in the section before, one cannot write $|\psi\rangle$ into a product of the two subsystems and so we have that the subsystems are entangled with each other. As per the rules of constructing a density matrix of a pure state, we can construct the density matrix $\rho_{ab}$ as given in (2.2.2).

$$\begin{aligned}\rho_{ab} &= |\psi\rangle\langle\psi| \\ &= \frac{1}{2}\left(|\uparrow\rangle_a|\downarrow\rangle_b\langle\uparrow|_a\langle\downarrow|_b\right) - \frac{1}{2}\left(\uparrow\rangle_a|\downarrow\rangle_b\langle\downarrow|_a\langle\uparrow|_b\right) - \frac{1}{2}\left(\downarrow\rangle_a|\uparrow\rangle_b\langle\uparrow|_a\langle\downarrow|_b\right) - \frac{1}{2}\left(\downarrow\rangle_a|\uparrow\rangle_b\langle\downarrow|_a\langle\uparrow|_b\right).\end{aligned}$$

In order to describe the $a$ subsystem we have to construct the density matrix of subsystem $a$. We do this by tracing out the $b$ spin subsystem

$$\rho_a = \mathrm{Tr}_b(\rho_{ab}) \tag{2.2.37}$$

$$= {}_b\langle\uparrow|\rho_{ab}|\uparrow\rangle_b + {}_b\langle\downarrow|\rho_{ab}|\downarrow\rangle_b \tag{2.2.38}$$

$$= \frac{1}{2}|\uparrow\rangle_a\langle\uparrow|_a + \frac{1}{2}|\downarrow\rangle_a\langle\downarrow|_a. \tag{2.2.39}$$

From $\rho_a$ we can now deduce that there is a probability of $\frac{1}{2}$ to find $a$ as a spin-up and a probability of $\frac{1}{2}$ to find it as a spin-down. Because this reduced density matrix is completely random, it is maximally entangled with $b$.

So for a general quantum system composed of two subsystems $A$ and $B$, the Hilbert space $\mathcal{H}$ is a tensor product

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.2.40}$$

We can choose $\{|i\rangle\}$ to be an orthonormal basis for $\mathcal{H}_A$ and $\{|j\rangle\}$ for $\mathcal{H}_B$, the general state is then expressed as

$$|\Psi\rangle = \sum_{i,j} c_{ij}|i\rangle \otimes |j\rangle, \tag{2.2.41}$$

then the reduced density matrix for $A$ in the basis $\{|i\rangle\}$ is

$$\langle i|\rho_{AB}|i\rangle = \rho_A(i,i') = \sum_j c_{ij}c^*_{i'j}, \tag{2.2.42}$$

similarly the reduced density matrix for $B$ in the basis $\{|j\rangle\}$ is

$$\langle j|\rho_{AB}|j\rangle = \rho_B(j,j') = \sum_i c_{ij}c^*_{ij'}. \tag{2.2.43}$$

Note that the requirement that our combined state be pure is not a necessary one. One can construct density matricies of subsystems of a combined system which is itself mixed. And so we can formally define the **entanglement entropy** as

$$S_A = -\mathrm{Tr}\left(\rho_A \ln \rho_A\right), \tag{2.2.44}$$

$$S_B = -\mathrm{Tr}\left(\rho_B \ln \rho_B\right). \tag{2.2.45}$$

$$\tag{2.2.46}$$

This entropy as mentioned earlier is of a very different nature, entanglement entropy comes from an inherent indeterminacy in the state of a subsystem because of its quantum mechanical correlations with another subsystem. It should be noted that the second law of thermodynamics only concerns thermal entropy, so the entanglement entropy can increase or decrease with time.

The entanglement entropy of a subsystem is zero only if the state $|\Psi\rangle$ of the total system is an uncorrelated product state. Denote the dimension of $\mathcal{H}_B$ by $|B|$ and that of $\mathcal{H}_A$ by $|A|$. If $|A| > |B|$, then the maximum value of $S_B$ is

$$S_B = \ln|B|, \tag{2.2.47}$$

which corresponds to a completely random state for $B$.

Entanglement entropy satisfies two important inequalities [43]. The first is called subadditivity and is given by

$$|S_A - S_B| \leq S_{AB} \leq S_A + S_B, \tag{2.2.48}$$

where $S_{AB}$ is the entanglement entropy of the combined system $A, B$. The second is called the law of strong subadditivity which involves three subsystems $A$, $B$ and $C$ and states

$$S_{ABC} + S_B \leq S_{AB} + S_{BC}, \tag{2.2.49}$$

with $S_{ABC}$ is the entanglement entropy of the combined tripartite system $A, B, C$. The strong sub-additivity inequality can be seen quite nicely by the following picture proof of of the three systems as seen in [66]

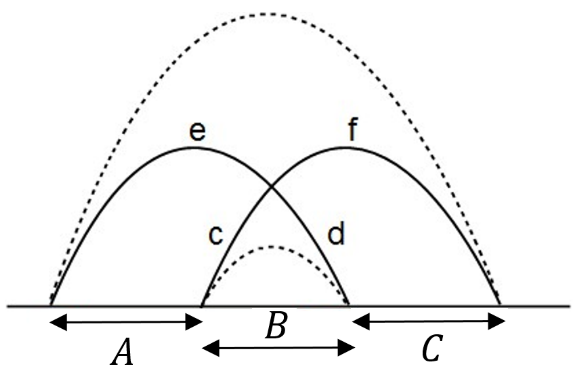


Figure 2.10: The horizontal line at the base represents a quantum field theory, whose space is subdivided into the regions $A, B, C$ The solid lines are the surfaces $ed$ and $cf$ used to calculate the entropies of the regions $AB$ and $BC$ respectively. By reconnecting the lines, one obtains surfaces $cd$ and $ef$, which must have more area than the minimal surfaces anchored to $B$ and $ABC$.

#### 2.2.2.4 Maximal Entanglement

**Monogamy of entanglement**

In quantum mechanics entanglement is monogamous. Monogamy is one of the most fundamental properties of entanglement. The monogamy is expressed as follow: "If two qubits $a$ and $a$ are maximally quantumly correlated they cannot be correlated at all with a third qubit $c$" [12]. In general, there is a trade-off between the amount of entanglement between qubits $a$ and $b$ and the same qubit $a$ with qubit $c$. This is mathematically expressed by the Coffman-Kundu-Wootters (CKW) monogamy inequality given in [12]

$$C^2_{ab} + C^{ac} \leq C^2_{a(bc)} \tag{2.2.50}$$

where $C_{ab}, C_{ac}$ are the concurrences (the simultaneous set of occurrences) between $a$ and $b$ respectively and between $a$ and $c$, while $C_{a(bc)}$ is the concurrence between subsystems $a$ and $bc$. As an example we see the monogamy of entanglement by considering a triple spin $-\frac{1}{2}$ particles $A, B, C$. If particle $A$ is maximally entangled with $B$, they are in a singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, then particle $A$ cannot be entangled with particle $C$. If this were true then $AB$ would too be entangled with $C$ and therfore should be described by a mixed-density matrix, but the singlet state $AB$ is pure and so $A$ cannot be entangled with $C$. This is why one refers to entanglement as being monogamous.

## 2.3 Quantum Field Theory

### 2.3.1 The Scalar Field in Flat Space

We refer to the scalar field $\phi(\vec{x},t)$ an operator valued field. The general form of $\phi(\vec{x},t)$ is built using the harmonic oscillator expansion.

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[F(\vec{k})\, e^{ix^\mu k_\mu}\, \hat{a}_{\vec{k}} \;+\; F^*(\vec{k})\, e^{-ix^\mu k_\mu}\, \hat{a}^\dagger_{\vec{k}}\right], \tag{2.3.1}$$

where $\vec{k}$ $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ are defined to be the annihilation and creation operators respectively. There are a set of such operators for every oscillator of momentum $\vec{k}$. With a few assumption for our field one can show that $F(\vec{k})$ is the same function evaluated at any momentum. This means that $F(\vec{k})$ has to be a constant as it does not actually depend on the momentum $k$. Since $F(\vec{k})$ must be a constant, we can choose it to be 1 as it will be easier to calculate amplitudes as we will not have to worry about normalizing the wavefunction. The final form of our scalar field is

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[e^{ix^\mu k_\mu} \hat{a}_{\vec{k}} \;+\; e^{-ix^\mu k_\mu}\, \hat{a}^\dagger_{\vec{k}}\right]. \tag{2.3.2}$$

According to various texts, the Lorentz invariant factor may be different, and instead of the integral over momentum, one may have a sum over momentum in the case when one is dealing with a discrete set of momenta.

$$\phi(\vec{x},t) = L^{-3/2} \sum_{\vec{k}} \frac{1}{2\omega_{\vec{k}}} \left(e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} \;+\; e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}}\, \hat{a}^\dagger_{\vec{k}}\right). \tag{2.3.3}$$

So far the scalar field has been motivated without the presence of any interactions and gravity one important property of the scalar field is the vacuum. The reason for its importance, is that the definition of a vacuum is actually observer dependent in curved space, however due to the fact that observers are Poincaré invariant in flat space implies that the vacuum is well defined for every observer as we shall see in the next section

#### 2.3.1.1 The Vacuum

The vacuum state is a state that contains no particles: $|0\rangle$. Although the vacuum state has no particles present, a quantum vacuum, is by no means an empty space, void of anything inside. In fact, there are quantum fluctuations of energy exist in the vacuum. The fluctuations of energy take the form of particles and anti-particles which come into existence and annihilate each other almost immediately.

In fact, the quantum vacuum for electrodynamics was the first vacuum in the field to be developed by Feynman, Tomonaga and Schwinger. Such vacuum interaction can be illustrated by the use of Feynman diagrams. When the quantum field can be described with some perturbative expansion the vacuum state behaves a lot like the ground state in a quantum mechanical oscillator, in such a case the vacuum expectation value vanishes.

We characterize the vacuum by:

$$\hat{a}_{\vec{k}} \left|0\right\rangle = 0, \text{for all momentum } \vec{k}. \tag{2.3.4}$$

Similarly:

$$P_\mu \left|0\right\rangle = 0, \tag{2.3.5}$$
$$e^{iaP} \left|0\right\rangle = 1 \cdot \left|0\right\rangle = \left|0\right\rangle, \tag{2.3.6}$$
$$e^{i\theta L} \left|0\right\rangle = \left|0\right\rangle, \tag{2.3.7}$$

$$\phi(\vec{x}, t) \left|0\right\rangle = \left|1\right\rangle, \tag{2.3.8}$$

that is we create a one particle state. Note, the notion of measuring the position of a particle at time $t$ and position $\vec{x}$, is not the same as creating a particles at time $t$ and position $\vec{x}$. We still cannot measure the position. This is due to the fact that position and momentum of particle are conjugated to each other. In other words, position and momentum are related by Fourier transform. Hence, if a particle is created with great precision in its momentum it will be difficult to measure its position. To see this clearly consider the following figures

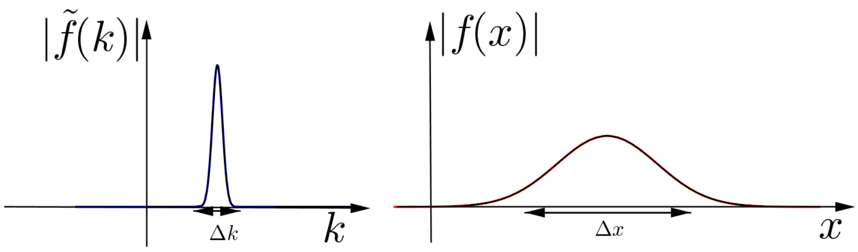


Figure 2.11: The narrow curve represent a probability of finding a particle in momentum space. The wide curve represents the probability of finding the same particle in position space.

Next we try:

$$\phi(\vec{x}_2, t_2)\phi(\vec{x}_1, t_1) \left|0\right\rangle = \phi(\vec{x}_2, t_2) \left|1\right\rangle = \left|0\right\rangle + \left|2\right\rangle \tag{2.3.9}$$

The above creates either a 2 particle state or a no particle state. The way we pick out the no particle state is by taking the overlap with the vacuum state.

$$\left\langle 0\right| \phi(\vec{x}_2, t_2)\phi(\vec{x}_1, t_1) \left|0\right\rangle. \tag{2.3.10}$$

This is the amplitude that a particle created at position $\vec{x}_1$ and time $t_1$ propagates to a position $\vec{x}_2$

and time $t_2$.

We mentioned above that the vacuum is not a void with zero energy, but when one does a further analysis of what the energy is one oddly gets an infinite energy density. The infinity comes from the divergence of the vacuum energy and looks to lowest order like

$$\sum_k \frac{1}{2}\omega = \left(\frac{L^2}{4\pi}\right)^{(n-1)/2} \frac{1}{\Gamma((n-1)/2)} \int_0^\infty \sqrt{k^2+m^2}k^{n-2}dk, \tag{2.3.11}$$

which diverges like $k^n$ for large $k$. This divergence implies some infinite energy density. This comes from the fact that the *zero point energy* has no upper bound, any arbitrarily large value of $k$ is possible. In flat space, the way such a problem is circumvented is by *renormalizing* or crudely rescaling the zero point energy by an equally large amount as to match what the observable are in an experimental setup. Doing so allows one to not deal with infinite vacuum energy divergences.

Something which needs emphasizing is the fact that all the above is defined in flat space. Complications follow when one extends the definition of a vacuum state in flat space to that of a vacuum state for a quantum field in the presence of gravity, or more precisely, curved space.

### 2.3.2 The Scalar Field in Curved Space

Quantum fields in curved space have an interesting phenomena attached to them whereby particle production occurs in the presence of a time independent gravitational field. The curved space quantum field theory is an extension of the flat space or Minkowski spacetime quantum field theory. The most important feature of this extension - what gives rise to particle production - is that the vacuum state does not generalize in a simple manner. Instead concepts like 'vacuum states' and 'particles' become vague as different observers will have different definitions.

#### 2.3.2.1 Particle Definition

Physically we define particles as entities that display detectable behaviour, so naturally the physical definition of a vacuum would be when no particles are detected. It turns out that this is not a good definition. Reason being, that particle number actually depends on the nature of the detectors motion. For example, a free falling particle detector will not always measure the same particle number as a non-inertial accelerating detector. The notion of disagreeable measurement of particles at different positions in space is not confined only to non-inertial detectors, but it is true also in curved space. Minkowski space is special in that all inertial observers agree on the vacuum definition (2.3.4). This is because the vacuum as defined in (2.3.4) is invariant under the Poincaré group, and so are all the set of inertial observers in Minkowski space.

A common practice when approaching problems dealing with particles in curved space is to define two regions which look asymptotically like Minkowski space, call them past infinity and future infinity.

This allows one to define particles measured at early and late times in a way that is acceptable to all inertial observers. Typically the remote future and past regions are referred to as the *in* and *out* regions. This concept will be further expanded in chapter 4.

#### 2.3.2.2 Loss of Symmetries

The treatment of particles and vacuum states is not something that is easily generalize. The clarity of definition for particles only holds true for flat space. the main reason for this, is that flat space possesses symmetries which are not shared to a general spacetime. For a classical field theory, it is relatively simple to generalize from flat to curved spacetime. That is because there is a clean separation between the field equations and its solutions. The field equations can be easily generalized to curved spacetime in an entirely local and covariant manner. In quantum field theory, 'states' are the analogs of 'solutions in classical field theory. However, properties of these states are deeply embedded in the usual formulations of quantum field theory. An important example of this is the Poincaré invariance of the vacuum state in Minkowski spacetime.

The Poincaré symmetries are symmetries which belong to the Poincaré group which is the group of *isometries* of Minkowski spacetime, this includes a total of 10 symmetries

- Time translation
- Translation through any of the three spatial directions
- Rotations about any of the three spatial axis
- Boosts along any any of the three spatial axis

The most important consequence of not having Poincaré symmetry is related to the loss of time translations, to see this consider a solution to the Klein Gordon equation

$$\psi(t,\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{2\omega_{\vec{k}}} \left( e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + e^{i\omega_k t - i\vec{k}\cdot\vec{x}} \hat{a}^{\dagger}_{\vec{k}} \right) \tag{2.3.12}$$

States belonging to the free Klein-Gordon field are given the same interpretation as we have seen thus far:vacuum states denoted by $|0\rangle$. Single particle states are of the form $\alpha^{\dagger}|0\rangle$. At the core of the vacuum and particle definition is the ability to decompose the field into its positive and negative frequency parts as can be seen in (2.3.12). The time translation invariance in Minkowski spacetime is vital to this decomposition. In curved spacetime, which does not share all the Poincaré symmetries, there is no natural notion of positive and negative frequency solutions. Consequently, no natural notion of 'vacuum states' or 'particles'.

#### 2.3.2.3 Particle Creation

From Scalar field quantum field theory, we saw that the mode expansion of the field takes the form

$$\phi = \sum_{\omega} \left( \hat{a}_{\omega} f_{\omega} + \hat{a}^{\dagger}_{\omega} f^{*}_{\omega} \right), \tag{2.3.13}$$

where we expressed (2.3.3) succinctly by replacing the exponential functions with $f_\omega, f^*_\omega$. As long as spacetime is locally flat, we can define the unique vacuum state by

$$\hat{a}_\omega|0\rangle = 0 \tag{2.3.14}$$

As mentioned previously, in curved space a problem appears, this has to do with the use of the words "locally flat" and "unique". In flat space quantum field theory, these two words go hand in hand when describing vacuum states for a particular field. It is the Poincaré symmetry on flat space that enables us to define a unique[7] vacuum state that all inertial observers will agree on. In curved space, as mentioned above, Poincaré symmetry is broken, so with it the ability to define an observer independent vacuum state. A more concise way of saying this is that, in curved space there is no canonical definition of particles. This can be shown explicitly by considering two observers separated by a region of curvature.

An observer $a$ at position $\vec{x}_a$ using time coordinate $t$, will have a field expansion as seen in eqn. (2.3.13), with particle states being created according to the creation operator $\hat{a}^\dagger_\omega$; for example, a 1-particle state would be:

$$|\psi\rangle = \hat{a}^\dagger_\omega|0\rangle_a, \tag{2.3.15}$$

where we add the notation $|0\rangle_a$ to denote the vacuum state as define by observer $a$. Since in curved space there is no unique definition of time, an observer $b$ at position $\vec{x}_b$ using a different time coordinate $\tilde{t}$, will have a mode expansion for same field

$$\phi = \sum_\omega \left(\hat{b}_\omega h_\omega + \hat{b}^\dagger_\omega h^*_\omega\right), \tag{2.3.16}$$

with

$$\hat{b}_\omega|0\rangle_b = 0. \tag{2.3.17}$$

For future purposes, one can think of such a situation existing around a black hole, observer $a$ is situated at some position just before the event horizon is created in the region we will refer to as the "in-region", observer $b$ is located some distance away after black hole is formed in a region called the "out-region". It is in this region, between the in and out regions, whereby spacetime curves. The functions $f_\omega, f^*_\omega$ within expansion (2.3.13) and $h_\omega, h^*_\omega$ within expansion (2.3.16) can be given the following orthogonal properties defined in their asymptotically flat spacetime regions

$$\begin{aligned}
(f_\omega, f_{\omega'}) &= (h_\omega, h_{\omega'}) = \delta_{\omega\omega'} \\
(f^*_\omega, f^*_{\omega'}) &= (h^*_\omega, h^*_{\omega'}) = -\delta_{\omega\omega'} \\
(f_\omega, f^*_{\omega'}) &= (h_\omega, h^*_{\omega'}) = 0
\end{aligned}$$

where $(f, h)$ is an inner product defined as

$$(f, h) \equiv -i \int d\Sigma^\mu (f\partial_\mu h^* - h^*\partial\mu f)$$

[7] There is no unique vacuum in flat space, only a unique vacuum for all inertial observers

It is important to note that although these functions are defined by their asymptotic properties, they are solutions of the same wave equation of the field $\phi$ everywhere in spacetime. The implication of this is that the mode expansion for the two different observers are equal,

$$\phi = \sum_{\omega} \left(\hat{a}_\omega f_\omega + \hat{a}^\dagger_\omega f^*_\omega\right) = \sum_{\omega} \left(\hat{b}_\omega h_\omega + \hat{b}^\dagger_\omega h^*_\omega\right), \tag{2.3.18}$$

taking the inner product on either side with $(\phi, f_{\omega'})$

$$\begin{aligned}
\sum_{\omega} \left(\hat{a}_\omega (f_\omega, f_{\omega'}) + \hat{a}^\dagger_\omega (f_\omega, f^*_{\omega'})\right) &= \sum_{\omega} \left(\hat{b}_\omega (h_\omega, f_{\omega'}) + \hat{b}^\dagger_\omega (h^*_\omega, f_{\omega'})\right)\\
\sum_{\omega} \left(\hat{a}_\omega \delta_{\omega\omega'} + \hat{a}^\dagger_\omega (0)\right) &= \sum_{\omega} \left(\hat{b}_\omega (h_\omega, f_{\omega'}) + \hat{b}^\dagger_\omega (h^*_\omega, f_{\omega'})\right)\\
\hat{a}_{\omega'} &= \sum_{\omega} \left(\hat{b}_\omega (h_\omega, f_{\omega'}) + \hat{b}^\dagger_\omega (h^*_\omega, f_{\omega'})\right)\\
\hat{a}_{\omega'} &= \sum_{\omega} \left(\alpha_{\omega\omega'} \hat{b}_\omega + \beta_{\omega\omega'} \hat{b}^\dagger_\omega\right)
\end{aligned} \tag{2.3.19}$$

Similarly taking the inner product $(\phi, h_{\omega'})$

$$\begin{aligned}
\sum_{\omega} \left(\hat{b}_\omega (h_\omega, h'_\omega) + \hat{b}^\dagger_\omega (h_\omega, h^*_\omega)\right) &= \sum_{\omega} \left(\hat{a}_\omega (f_\omega, h_{\omega'}) + \hat{a}^\dagger_\omega (f^*_\omega, h_{\omega'})\right)\\
\sum_{\omega} \left(\hat{b}_\omega (\delta_{\omega\omega'}) + \hat{b}^\dagger_\omega (0)\right) &= \sum_{\omega} \left(\hat{a}_\omega (f_\omega, h_{\omega'}) + \hat{a}^\dagger_\omega (f^*_\omega, h_{\omega'})\right)\\
\hat{b}_{\omega'} &= \sum_{\omega} \left(\hat{a}_\omega (h_\omega, f_\omega) + \hat{a}^\dagger_\omega (h_\omega, f^*_{\omega'})\right) \quad ;\text{relabelling } \omega' \longleftrightarrow \omega\\
\hat{b}_{\omega} &= \sum_{\omega'} \left(\hat{a}_\omega (h_\omega, f_{\omega'}) + \hat{a}^\dagger_\omega (h_\omega, f^*_{\omega'})\right)\\
\hat{b}_{\omega} &= \sum_{\omega'} \left(\alpha_{\omega\omega'} \hat{a}_{\omega'} + \beta_{\omega\omega'} \hat{a}^\dagger_{\omega'}\right)
\end{aligned} \tag{2.3.20}$$

Equations (2.3.19) and (2.3.20) are known as Bogoliubov transformations and $\alpha_{\omega\omega'}$, $\beta_{\omega\omega'}$ are called Bogoliubov coefficients. As mention earlier $\hat{a}_\omega, \hat{a}^\dagger_\omega$ are the annihilation and creation operators for observer $a$ which we said is situated in the in-region, thus has vacuum state defined by

$$\hat{a}_\omega |0\rangle_{\text{in}} = 0 \ , \ \forall \omega,$$

where $|0\rangle_{\text{in}}$ describes the situation where no particles are present initially. Similarly $\hat{b}_\omega, \hat{b}^\dagger_\omega$ belong to observer $b$ situated in the out-region with a vacuum state defined by

$$\hat{b}_\omega |0\rangle_{\text{out}} = 0 \ , \ \forall \omega,$$

where $|0\rangle_{\text{out}}$ describes the situation where no particles are present at late times. And so observer $a$'s vacuum state satisfies

$$0 = \hat{a}_\omega |0\rangle_{\text{in}} = \sum_{\omega} \left(\alpha_{\omega\omega'} \hat{b}_\omega + \beta_{\omega\omega'} \hat{b}^\dagger_\omega\right) \tag{2.3.21}$$

Following the analysis given by Mathur [38], we look to solve this equation. We suppose we have just a single mode such that, the sum disappears, the equation reduces to

$$(\hat{b} + \gamma \hat{b}^\dagger)|0\rangle_{\text{in}} = 0,$$

with solution of the form

$$|0\rangle_{\text{in}} = Ce^{\mu\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_{\text{out}},$$

where is a number, $\mu$ is a number to be determined and $C$ is a normalization constant. Expanding the exponential

$$e^{\mu\hat{b}^\dagger\hat{b}^\dagger} = \sum_n \frac{\mu^n}{n!}(\hat{b}^\dagger\hat{b}^\dagger)^n$$

we have that

$$\begin{aligned}
\hat{b}(\hat{b}^\dagger\hat{b}^\dagger) &= (\hat{b}\hat{b}^\dagger)\hat{b}^\dagger \\
&= (1+\hat{b}^\dagger\hat{b})\hat{b}^\dagger \\
&= \hat{b}^\dagger + \hat{b}^\dagger(\hat{b}\hat{b}^\dagger) \\
&= \hat{b}^\dagger + \hat{b}^\dagger(1+\hat{b}^\dagger\hat{b}) \\
&= 2\hat{b}^\dagger + (\hat{b}^\dagger\hat{b}^\dagger)\hat{b}
\end{aligned}$$

where we make use of the commutation relation $[\hat{b},\hat{b}^\dagger] = 1$, one will find that

$$\hat{b}(\hat{b}^\dagger\hat{b}^\dagger)^2 = 4\hat{b}^\dagger(\hat{b}^\dagger\hat{b}^\dagger) + (\hat{b}^\dagger\hat{b}^\dagger)^2\hat{b}$$

which then generalizes to

$$\hat{b}(\hat{b}^\dagger\hat{b}^\dagger)^n = 2n\hat{b}^\dagger(\hat{b}^\dagger\hat{b}^\dagger)^{n-1} + (\hat{b}^\dagger\hat{b}^\dagger)^n\hat{b}$$

This relation allows us to do the following

$$\begin{aligned}
\hat{b}e^{\mu\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_{\text{out}} &= \hat{b}\left(\sum_n \frac{\mu^n}{n!}(\hat{b}^\dagger\hat{b}^\dagger)^n\right)|0\rangle_{\text{out}} \\
&= \left(\sum_n \frac{\mu^n}{n!}\hat{b}(\hat{b}^\dagger\hat{b}^\dagger)^n\right)|0\rangle_{\text{out}} \\
&= \left(\sum_n \frac{\mu^n}{n!}(2n\hat{b}^\dagger(\hat{b}^\dagger\hat{b}^\dagger)^{n-1} + (\hat{b}^\dagger\hat{b}^\dagger)^n\hat{b})\right)|0\rangle_{\text{out}} \\
&= \left(\sum_n \frac{\mu^n}{n!}(2n\hat{b}^\dagger(\hat{b}^\dagger\hat{b}^\dagger)^{n-1})\right)|0\rangle_{\text{out}} + \left(\sum_n \frac{\mu^n}{n!}\hat{b}^\dagger\hat{b}^\dagger)^n\hat{b}\right)|0\rangle_{\text{out}} \\
&= 2\mu\hat{b}^\dagger\left(\sum_n \frac{\mu^{n-1}}{(n-1)!}(\hat{b}^\dagger\hat{b}^\dagger)^{n-1}\right)|0\rangle_{\text{out}} + \left(e^{\mu\hat{b}^\dagger\hat{b}^\dagger}\right)\hat{b}|0\rangle_{\text{out}} \\
&= 2\mu\hat{b}^\dagger e^{\mu\hat{b}^\dagger\hat{b}^\dagger}
\end{aligned}$$

putting $\mu = -\frac{\gamma}{2}$, we get that

$$|0\rangle_{\text{in}} = Ce^{-\frac{\gamma}{2}\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_{\text{out}} \tag{2.3.22}$$

So the vacuum state

$$|0\rangle_{\text{in}} = C|0\rangle_{\text{out}} + C_1\hat{b}^\dagger\hat{b}^\dagger|0\rangle_{\text{out}} + C_2\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger|0\rangle_{\text{out}} \tag{2.3.23}$$

has the form that it is composed partly of an out vacuum state, and the rest 2 type $b$ particles, 4 type $b$ particles and so on.

By summing over all possible modes, $\gamma$ will now become a matrix to get the solution

$$|0\rangle_{\text{in}} = Ce^{-\frac{1}{2}\sum_{m.n}\hat{b}^{\dagger}_{m}\gamma_{mn}\hat{b}^{\dagger}_{n}}|0\rangle_{\text{out}}. \tag{2.3.24}$$

Going back to the original picture of two observers $a$ and $b$; this calculation illustrates that when we have two observers who are in a region of curvature, they see particles appearing in each others supposed vacuum state. Observer $a$ will have a different mode expansion to observer $b$, consequently they will have differing definitions of their vacuum states. According to observer $b$'s point of view, the vacuum state for observer $a$, $|0\rangle_a$ is actually seen to be a state populated by particles defined as

$$|0\rangle_a = C|0\rangle_b + C_1\hat{b}^{\dagger}\hat{b}^{\dagger}|0\rangle_b + C_2\hat{b}^{\dagger}\hat{b}^{\dagger}\hat{b}^{\dagger}\hat{b}^{\dagger}|0\rangle_b, \tag{2.3.25}$$

where $\hat{b}^{\dagger}|0\rangle_b = |1\rangle_b$. Observer $a$ however will not see his vacuum state populated with any particles as, to him, $|0\rangle_a$ is a the state with no particles.

What we make clear is that the vacuum defined for one observer may be filled with particles according to some other observers definition, as seen above and explicitly shown in eqn (2.3.23).

Although curved space proposes a difficulty in defining particles, one can take measurements at asymptotic infinity where the spacetime metric looks approximately like $\eta_{\mu\nu}$, the flat spacetime metric. In flat space the vacuum is well defined, and so the *in* and *out* regions are both in some asymptomatic flat space region, separated by a region in which curvature exists.

# Chapter 3

# Black Holes

The original idea of black holes - although at that point they weren't called black holes - does not come from Einstein's theory of general relativity but rather from Laplace. As described in [40], in 1796 Laplace was doing calculations on escape velocity. He realized the existence of a scenario where the ratio of the mass and size of a planet would have an associated escape velocity equal to or greater that of the speed of light[1]. And so the idea that there exists an object whereby its gravitational pull was so strong, that not even light can escape, was born.

Later Schwarzschild theorized the existence of black holes by finding a particular solution to Einstein's equations. One of the distinguishing features of a black hole is the *event horizon*. This is located at the Schwarzschild radius and is 'point of no return'. The first observation came studying binary star systems. In these systems, two stars orbit each other moving in generally predictable ways because of the gravitational attraction between the stars. A single star was observed moving as if there were a massive object nearby, but with no other star in evidence, this 'invisible' companion was a black hole. Associated with such black holes is the gas from the visible star which spiral at very high speed around the black hole. This action creates enormous heat and X-ray radiation, which was also detected through observations.

Black holes were initially thought to be inert, static and everlasting massive bodies that exists in the universe. They were studied for their immense gravitational properties. Around black holes the effects of gravity - such as time dilation - become highly noticeable. Clocks close to a black hole, as seen from an external observer, appear to run slower and slower. Even more interesting is when

[1]Laplace calculated that the earth would have to be 1.8 millimeters in diameter with its same mass to have an escape velocity equal to the speed of light [35]

one attempts to describe what happens at the event horizon. According to the metric that was first used to describe spacetime around a black hole (the Schwarzschild metric) radial coordinates and time coordinate switch roles, light gets infinitely red shifted and certain metric components become divergent. As the study of black holes progressed, physicists found that black holes weren't actually what they first thought them to be.

In 1974 Hawking showed that black holes can be described as a thermodynamic-like object, which radiates away its energy and has an associated entropy. Since the radiation is emitted due to particle creation at the event horizon, which is of a quantum nature, black holes appear to be a playground whereby various realms of physics interact in a manner than cannot be seen elsewhere. Essentially black holes can be viewed as both these classical objects significantly bending spacetime, or as a quantum object that emits radiation and exhibits thermodynamic behaviour.

## 3.1 Geometric Properties

### 3.1.1 Schwarzschild Metric

When solving Einstein's equations one looks for a metric solution describing a specific geometry. An exact solution of Einstein equation was found in 1916 by Schwarzschild for a spherically symmetric mass in empty space. The metric components for Schwarzschild's spacetime were found using a set of coordinates we now call *Schwarzschild coordinates.* In units where $c = 1$, the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{r_s}{r}}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{3.1.1}$$

describes a spherically symmetric time independent spacetime geometry, where $r_s$ is the Schwarzschild radius. It is the simplest non-trivial solution with a zero cosmological constant ($\Lambda = 0$) and provides a spring board for exploring different geometries. As $r \to \infty$ the metric tends towards that of the flat spacetime metric, this seems like a reasonable result, as far from the gravitational source, one should not experience any gravitational effects. There are four interesting features of the Schwarzschild metric: the radial coordinate $r$, the Schwarzschild radius $r_s$, the time coordinate $t$ and the phenomenon of gravitational redshift.

**Radial coordinate,** $r$

The only way to understand the meaning of the radial coordinate, $r$, appearing in the metric is by looking at the metric equation. The radial coordinate represents the radial separation between two points. In flat space $\Delta s = r_A - r_B$, however, in the Schwarzschild metric, $dr$ does not describe the radial separation between such points. Consider a case where $r >> r_s$ and $dt = d\theta = d\phi = 0$, the metric becomes

$$ds^2 = \frac{dr^2}{\left(1 - \frac{r}{r_s}\right)} \qquad \Rightarrow \qquad ds = \frac{dr}{\sqrt{1 - \frac{r}{r_s}}}. \tag{3.1.2}$$

Two points with radial coordinates $r_A$ and $r_B$ will thus have a separation of

$$\Delta s = \int ds = \int_{r_A}^{r_B} dr \left(1 - \frac{r}{r_s}\right)^{-\frac{1}{2}} \tag{3.1.3}$$

$$\approx \int_{r_A}^{r_B} dr \left(1 + \frac{r_s}{2r}\right) \tag{3.1.4}$$

$$= r_B - r_A + \frac{r_s}{2} \ln\left(\frac{r_B}{r_A}\right), \tag{3.1.5}$$

where we used the binomial expansion since $r >> r_s$. And so the $r$ coordinate is actually a circumferential radial coordinate and not a purely radial coordinate. This is a consequence of the fact that spacetime is curved, and not flat.

**Schwarzschild radius,** $r_s$

Consider the geodesic equation for the for an object with coordinates $x^\gamma$

$$\frac{d^2 x^\gamma}{d\tau^2} = -g^{\gamma\alpha}\left(\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\right) u^\mu u^\nu, \tag{3.1.6}$$

for an object at rest the spatial components of the four velocity $u^\mu$ are zero. The time component $u^t \equiv \frac{dt}{d\tau} \neq 1$ as the coordinate time $t$ and proper time $\tau$ as measured by the objects clock are not the same even if the object is at rest. Making use of the invariant quantity $u^\mu g_{\mu\nu} u_\nu = -1$ to find $u^t$, and the fact that the object is at rest, the geodesic equation, of only the radial coordinate, evaluates to

$$\frac{d^2 r}{d\tau^2} = -g^{rr}\left(0 - \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\right)\left(u^t\right)^2 \tag{3.1.7}$$

$$= -\frac{1}{2}\left(1 - \frac{r_s}{r}\right)\frac{\partial}{\partial r}\left(1 - \frac{r_s}{r}\right)\left(1 - \frac{r_s}{r}\right)^{-1} \tag{3.1.8}$$

$$\frac{d^2 r}{d\tau^2} = -\frac{r_s}{2r^2}. \tag{3.1.9}$$

Comparing this to Newton's law of gravity:

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2}, \tag{3.1.10}$$

we see that the predicted Schwarzschild acceleration will agree with the Newtonian result at large radii ($r >> r_s$) if and only if $r_s = 2GM$.

The source of the gravitational field in Schwarzschild description of spacetime is that of a black hole with mass $M$. By studying the features of this metric one is actually studying the features of a black hole. One feature of particular interest is the event horizon located at the Schwarzschild radius. It is at the event horizon where spacetime behaves rather non-intuitively. For future use we will write the Schwarzschild metric in the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{r}}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{3.1.11}$$

where $c = 1$ and $r_S = 2GM$

#### Time coordinate

Imagine having a clock at rest with a fixed $r$ coordinate ($dr = d\theta = d\phi = 0$). Using the metric equation, the invariant proper time $d\tau^2 = -ds^2$ gives

$$\Delta\tau = \int \sqrt{\left(1 - \frac{2GM}{r}\right) dt^2 + 0 + 0 + 0} \tag{3.1.12}$$

$$= \sqrt{1 - \frac{2GM}{r}} \int dt$$

$$\Delta\tau = \sqrt{1 - \frac{2GM}{r}} \Delta t. \tag{3.1.13}$$

In a region of flat spacetime ($r = \infty$) we see that $\Delta\tau = \Delta t$. So the clock of some fixed observer runs at the same rate as a clock for a moving observer. The '$t$-clock' is always at rest at infinity, and so as an object decreases its $r$ coordinate the '$\tau$-clock' begins to run at a slower rate as seen by the $t$-clock, that is, time intervals are made longer. Relating this to black holes, as observers move closer towards the event horizon of a black hole, their decreasing $r$-coordinate results in their clock to tick at a slower rate according to someone at a fixed point in space very far away. Note, the feature of clocks ticking at different rates is not a property of black holes specifically, but a property of curved spacetime. An important result of this time discrepancy is gravitational redshift.

#### Gravitational Redshift

Consider two crests of light emitted from a source at $r_E$, with wavelength $\lambda_E$. One can measure the proper time between two successive crests emission events which we call $\Delta\tau$. In units $c = 1$ time is measured in meters, thus is equal to the wavelength at $r_E$. The $t$-coordinate difference will be $\Delta t$. If we let the wave travel through a region of spacetime with curvature, the $\Delta\tau$ at point of reception $r_R$ will be the same as at point of emission. Since $\Delta t$ must be the same, by using equation (3.1.13)

$$\lambda_E = \Delta\tau_E = \sqrt{1 - \frac{2GM}{r_E}} \Delta t \qquad\qquad \lambda_R = \Delta\tau_R = \sqrt{1 - \frac{2GM}{r_R}} \Delta t$$

$$\Rightarrow \frac{\lambda_R}{\lambda_E} = \frac{\sqrt{1 - \frac{2GM}{r_R}}}{\sqrt{1 - \frac{2GM}{r_E}}}. \tag{3.1.14}$$

This is the redshift formula. When $r_E < r_R$, (3.1.14) is greater than one. Thus the light becomes more and more redshifted as $r_R$ moves further outward away from the point of emission. At $r_E = 2GM$, one gets seemingly infinite redshift. This is one of a few result which will be addressed in the following section.

### 3.1.2 Event Horizon

At a radial coordinate of $r = 2GM$ of a black hole, one finds the event horizon. The event horizon is the dividing region between the interior and the exterior regions of a black hole. Any light signal emitted in the exterior region can propagate to infinity, however, any light signal emitted in the interior is confined to this region. This division is thus a causal division as illustrated in the Penrose diagram 2.7. According to the equivalence principle this division does not imply anything special will happen at the event horizon. Up until now, we haven't motivated why the event horizon is the point at which light cannot escape. We observe what features arise from the Schwarzschild metric at $r = 2GM$. This will goive us insight into what the event horizon is, and why once someone crosses the event horizon their inevitable future is towards the singularity.
According to the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{r}}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

we get the following.

**1.** A clock at rest at $r = 2GM$ registers no time:

$$\Delta\tau = \sqrt{1 - \frac{2GM}{r}}\Delta t \tag{3.1.15}$$

$$\Delta\tau = 0. \tag{3.1.16}$$

**2.** Light is infinitely redshifted

$$\frac{\lambda_R}{\lambda_E} = \frac{\sqrt{1 - \frac{2GM}{r_R}}}{0}, \tag{3.1.17}$$

moreover the redshift formula breaks down at $r < 2GM$

**3.** Given the equations of motions for massive and massless particles,

$$\frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right)\sqrt{1 - \frac{1}{e^2}\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)} \;\; ; \; m > 0 \tag{3.1.18}$$

$$\frac{d\phi}{dt} = \frac{\ell}{er^2}\left(1 - \frac{2GM}{r}\right) \;\; ; \; m > 0 \tag{3.1.19}$$

$$\frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right)\sqrt{1 - \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}} \;\; ; \; m = 0 \tag{3.1.20}$$

$$\frac{d\phi}{dt} = \frac{b}{r^2}\left(1 - \frac{2GM}{r}\right) \;\; ; \; m = 0 \tag{3.1.21}$$

for a distant observer such particles -even light- seem to 'freeze at $r = 2GM$

**4** The metric component $g_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1}$ goes to infinity implying $\frac{ds}{dr}$ diverges.

These singularities all involving infinities come from either

1) A geometric singularity.

2) A coordinate singularity.

#### Geometric singularity

This is more of a physical issue embedded in the geometry of spacetime. The assumption that spacetime is not so curved that you can always take a small enough patch which looks flat, breaks down at the geometric singularity, in fact, the center of a black hole possesses such a singularity. An example of a geometric structure whereby one cannot make the curvature look smooth is at the point of a cone.

#### Coordinate singularity

This implies the coordinate system we chose to describe the spacetime is flawed, but the geometric spacetime structure itself is fine. For example consider a sphere, the latitude longitude coordinate system has a $g_{\phi\phi}$ metric component which goes to zero at the poles. This results in a coordinate singularity, but this is only because the $\phi$ coordinate is not well defined at the poles. There is no unique longitudinal line to describe the coordinate at the poles, as all longitudinal lines cross through the poles. The fact that $g_{tt}$ goes to zero in the Schwarzschild metric resembles the same problem.

The problem with $g_{rr} \to \infty$ can be addressed by noting that the actual physical distance between two radial coordinates $R$ and $r = 2GM$ is actually finite.

$$\Delta s = R\sqrt{1 - \frac{2GM}{R}} + 2GM \tanh^{-1} \sqrt{1 - \frac{2GM}{R}}, \tag{3.1.22}$$

so for example by simple substitution the physical distance between $r = 3GM$ and $r = 2GM$ is $3.05GM$, which is definitely not infinite. One can also show that the proper time for a particle falling from rest at $r = R$ to $r = 0$ is

$$\Delta\tau = \frac{\pi R^{3/2}}{\sqrt{8GM}} \tag{3.1.23}$$

the fact that we do not observer problems that would arise as a result of a geometric divergences at $r = 2GM$ implies we are dealing with a coordinate pathology.

#### An infalling observer

Imagine yourself as an infalling observer approaching a black hole, there would be a few noticeable points. If the black hole were in complete empty space, you would not actually be able to physically see a black hole. However, were there a backdrop of stars, you would notice the distortion of light rays around the black hole (gravitational lensing). This is due to the large spacetime curvature, bending light rays. Up until the event horizon, with enough thrust, you will always be able to escape the black hole. However at a distance of around $r = 4GM$, any circular orbit becomes very unstable meaning that were you in a rocket, the rocket could be propelled out of orbit into the black hole. Your proper time would continue as normal. However a distant external observer will notice the relative slowing down of the your clock. Falling further in, at approximately $r = 3GM$ you reach the *photon sphere*. This is the point at which photons actually orbit the black hole. An interesting result of this is that a photon leaving your head will orbit the black hole and reach your eyes, allowing you to see the back of your own head. Once you reach the event horizon at $r = 2GM$, there is no escaping. As you cross the horizon, the universe will appear brighter and brighter. Upon crossing it, the gravitational effects may become physically noticeable. In a typical sized black hole, the tidal forces are weak enough that you would continue to fall before being torn apart. Eventually the tidal forces will be so great that you would experience what is often referred to as'spaghetiffication', the stretching of any body due to extreme tidal fores. This would be the last thing and you, as an infalling observer, would notice as as you fall into a classically non-rotating black hole.

We can mathematically present the reasoning why once the event horizon crossed, it is physically impossible to get back out.

**Inside the event horizon**

Consider for now the flat spacetime metric $\eta_{\mu\nu}$ with $c = 1$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

A property of Lorentzian metrics is that the he spatial components of the metric are all positive and the time coordinate is negative. Thus for any diagonal[2] spacetime metric $g_{qq}$, all the components $g_{qq} > 0$ signify space components and similarly the component with $g_{qq} < 0$ signifies a time component. The important physical distinction between space and time coordinates is: one can move freely in space, but one can only move forward in time.

When $r < 2GM$ the $g_{tt} = -\left(1 - \frac{2GM}{r}\right)$ and $g_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1}$ components change signs, that is, $g_{tt} > 0$ and $g_{rr} < 0$. This signifies a change in what used to be the radial coordinate and the time coordinate,leaving $r$ to become a time coordinate and $t$ to become a radial coordinate. The most important fact about this change is that a particle cannot remain at a fixed radial coordinate as much as a particle cannot remain at a fixed time coordinate in flat space. The $r$ coordinate still retains its geometric structure and meaning within the Schwarzschild metric, the change in sign just means that in order for time to flow it cannot remain at any fixed $r$, but rather must move to go forward in time. This tells us that the inevitable future of any particle which crosses the $r = 2GM$ 'membrane' is to fall towards the singularity. No particle can escape the black hole, just as no particle can travel back in time.

Although the Schwarzschild coordinates give us a good intuitive idea of a black hole and of the

[2]Off diagonal metrics do not follow the simple rule. One often has cross terms like $dtdx$ which can have any sign

event horizon, due to the coordinate singularity and the fact that coordinate definitions change, they do not aid to calculations when crossing the event horizon. In search of ways to describe the geometry both inside (but still far from $r = 0$) and outside the event horizon, there are a different set of coordinates one can use.

### 3.1.3 New coordinates

The idea of introducing a new set of coordinates is to remove the shortfalls of the Schwarzschild coordinate, namely the coordinate singularity at $r = 2GM$. We will first explore the Eddington-Finkelstein coordinates as we make use of them later on. Thereafter we briefly introduce the Kruskal-Szekeres coordinates as they are widely used, especially when studying more complicated black hole systems.

**Eddington-Finkelstein coordinates**

We can first define a *tortoise coordinate*

$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|. \tag{3.1.24}$$

We next introduce null geodesic coordinates

$$v = t + r^*, \tag{3.1.25}$$

$$u = t - r^*, \tag{3.1.26}$$

and so infalling radial null geodesics are characterized by $v =$ constant, while the outgoing ones satisfy $u =$ constant. Lets see what the metric will look like using these coordinates. Letting

$$t = v - r - 2GM \ln\left|\frac{r}{2GM} - 1\right|, \tag{3.1.27}$$

we have that

$$dt = dv - dr - 2GM\left[\frac{\frac{dr}{2GM}}{\frac{r}{2GM} - 1}\right], \tag{3.1.28}$$

$$dt = dv - dr\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{3.1.29}$$

The Schwarzschild metric can then be written as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\left[dv^2 - 2dvdr\left(1 - \frac{2GM}{r}\right)^{-1} + \left(1 - \frac{2GM}{r}\right)^{-2} dr^2\right] + r^2 d\Omega^2$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{3.1.30}$$

If one looks at the behaviour of light rays, we get $ds^2 = 0$. For some fixed angles $d\phi = d\theta = 0$,

$$0 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dvdr, \tag{3.1.31}$$

giving two solutions:

$$\frac{dv}{dr} = \begin{cases} 0 & \text{infalling,} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{outgoing.} \end{cases}$$

At $r = 2GM$, $dvdr = 0$. This corresponds to the light being 'trapped' at $r = 2GM$ surface. This surface is acting as a point of no return, once a light ray dips below it, it can never come back. Furthermore when $r \leq 2GM$, $\frac{r}{2GM} < 1$, thus as $v$ increases $r$ must decrease, again indicating that once the event horizon is crossed, the inevitable fate of any particle or light ray is to continue falling towards $r = 0$. The implication is that any signal at $r < 2GM$ can never influence a point at $r > 2GM$. This is not to say that an observer falling through the event horizon will experience anything unusual, to the observer there is nothing irregular at the even horizon. Just as the coordinates suggest, the geometry at the event horizon is as regular as anywhere else.

Note though, that we could have also chosen the coordinate $u$ instead of $v$. $v$ represented future directed paths, but we can just as easily choose past-directed paths. This would yield a metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - 2dudr + r^2 d\Omega^2 \tag{3.1.32}$$

So we have an extended spacetime in two different directions, one to the future and one to the past. One can further improve these coordinates such that we have one manifold extending over both pieces of spacetime (future and past), these are the Kruskal-Szekeres coordinates.

**Kruskal-Szekeres coordinates**

If we define the coordinates

$$u^2 - v^2 = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \qquad \text{and} \qquad t = 2GM \ln\left|\frac{u+v}{u-v}\right| \tag{3.1.33}$$

where $v$ is the timelike coordinate The Kruskal-Szekeres metric looks as follows[3]

$$ds^2 = \frac{32(GM)^3}{r} e^{-r/2GM}(dv^2 - du^2) + r^2 d\Omega^2 \tag{3.1.34}$$

it is a diagonal metric, which is far nicer than the awkward and sometimes misleading non-diagonal metrics. The most noticeable properties of this metric are the following

- There is still a pathology at $r = 0$, since the pathology is a geometric one.

- The $v$ coordinate is always timelike as $g_{vv} < 0$ and $u$ is always a spacelike coordinate as $g_{uu} > 0$.

- There is no coordinate pathology at $r = 2GM$ as no metric component is infinite.

[3]For a more detailed derivation of the KS coordinate system, one can look at the General relativity notes of Sean M. Carroll [11].

- Particles with mass follow paths such that $dv > |du|$.

- Radial light worldlines follow paths such that $du = \pm dv$.

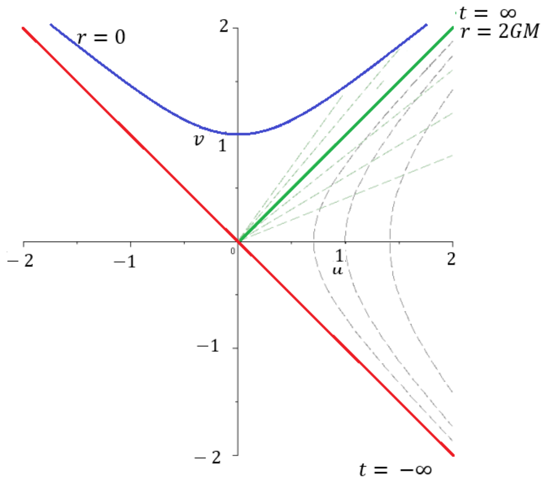


Figure 3.1: The blue solid line is the $r = 0$ hyperbola. The grey dashed curves represent hyperbolas where $r_0 > 2GM$. Lines of constant $t_0$ are represented by the green dashed lines meeting at the origin. The red diagonal line with slope $-1$ represents $t = -\infty$ and the heavy green line with slope $+1$ is $t = +\infty$. The event horizon is represented by the This green line and the upper half of the red line. The region between the $r = 2GM$ lines and the $r = 0$ hyperbola represents the interior region of the black hole. All world lines move upwards in the diagram, and must end when they reach the $r = 0$ hyperbola.

## 3.2 Thermodynamic Properties

No matter what choice of coordinates one chooses, at $r = 0$ there is an infinite mass energy concentration creating a singularity in the geometry of spacetime. This singularity breaks down the fundamental assumptions of general relativity. One might think that any quantum mechanical features of the black hole close to the singularity are irrelevant to any external observer[4]. In the 1970's however, Hawking showed that black holes in fact do have quantum-mechanical aspects which can be seen by observers at infinity. He discovered that the area of a black hole's event horizon must always increase[41]. If we compare this to the second law of thermodynamics which states that in an isolated system the entropy must always increase, there appears to be a similarity. This and other similarities became the four laws of black hole mechanics, which are analogues of the four laws of classical thermodynamics.

### 3.2.1 Zeroth Law of Black Hole Mechanics

Classically the zeroth law states that the temperature $T$ is constant for a system in thermal equilibrium. In terms of a black hole, the law is stated as the surface gravity $\kappa$ of a stationary black hole is constant over its event horizon[44]. When studying surface gravity one needs to make the distinction between stationary geometries and static ones, which is a subclass class stationary geometries.
**Stationary:** These geometries admits an asymptotically time-like killing vector. It has a time-independent metric while containing $dtdx^i$ cross terms in the coordinate expression of the metric.
**Static:** Static geometries in the context of the metric admits a time-like killing vector, which is irrotational. The metric will also be time dependent but instead will have no $dtdx^i$ cross terms in the coordinate expression of the metric.
More technically the surface gravity $\kappa$ of a stationary black hole can be defined as the magnitude of the gradient of the norm of the *horizon generating Killing field* [5] [32]

$$\kappa^2 :\equiv -\left(\nabla^\alpha|\chi|\right)\left(\nabla_\alpha|\chi|\right) \tag{3.2.1}$$

### 3.2.2 First Law of Black Hole Mechanics

The first law states that energy is conserved.

$$\delta E = T\delta S - P\delta V. \tag{3.2.2}$$

Since heat is a form of energy, this means an object that is heating up must be getting energy from somewhere. Likewise, if an object is cooling down, the energy it loses must be gained by something

[4] An external observer is unable to observe anything lying inside the event horizon.

[5] Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances on the object. If $L$ is the Lie derivative, a Killing vectorfield $X$ satisfies $L_X g = 0$

else. For black holes, the first law expresses the conservation of energy by relating the change in mass $M$ to the change in area $A$, angular momentum $J$ and charge $Q$. This is expressed as

$$\delta M = \frac{1}{8\pi G}\kappa\delta A + \Omega\delta J + \Phi\delta Q, \tag{3.2.3}$$

where $\Omega$ is the angular velocity and $\Phi$ is the electrostatic potential. The entropy of the black hole is then related to the surface area. The surface gravity then plays the role of temperature by seeing that classically the term $TdS$ is in the black hole case $\frac{1}{8\pi G}\kappa\delta A$. Which gives that

$$T = \frac{\kappa}{2\pi}, \tag{3.2.4}$$

$$S = \frac{A}{4G}. \tag{3.2.5}$$

### 3.2.3 Second Law of Black Hole Mechanics

The second law is expressed by Hawking's area theorem. It was shown by Hawking that the area of a black hole's event horizon can never decreases. Basically the area theorem says that: "if the *Cosmic censorship* holds, that is, there are no *naked singularities*, then the cross sectional area of a future event horizon cannot be decreasing anywhere" [32]. The second law of Black hole looks analogous to the statement of the second law of thermodynamics. This states that entropy can never decrease. However, it has not been discussed yet, but in the next section 3.3 we see that black holes radiate, and in Chapter 4 we see that this radiation causes the black hole to evaporate. If the black hole evaporates, surely the event horizon decreases and thus violating the proposed second law of black hole mechanics? The answer is yes, thus we have the generalized second law

#### 3.2.3.1 Generalized Second Law

In the second law of Black hole mechanics we stated that the area of the event horizon never decreases, but can only increase. When considering the black hole alone, there appears to be a violation of the black hole entropy. Then the question can be asked: is the area of the black hole really a quantity which resembles entropy?. Once again the answer is yes, based on the following argument.

Indeed when viewing the black hole as a quantum system, the ordinary second law and area theorem seem to be violated as in the evaporation of the black hole one has $\delta A < 0$. If we consider the radiation emitted then $\delta S$ of the radiation is increasing. By considering the black hole and radiation as one complete system, one obtains the generalized entropy, $S'$. This is was proposed by Bekenstein [6] as

$$S' = S_{\text{matter}} + \frac{1}{4}k\frac{Ac^3}{G\hbar}, \tag{3.2.6}$$

we have reintroduced units of $c$ and $\hbar$. Since a decrease in $S$ is compensated by an increase in $A$, and similarly a decrease in $A$ is compensated by an increase in $S$ then the generalized second law takes the form

$$\delta S' = \delta S_{\text{matter}} + \delta S_{\text{BH}} \geq 0, \tag{3.2.7}$$

or more concisely

$$\delta S' \geq 0. \tag{3.2.8}$$

The Generalized entropy has the simple interpretation that it can be viewed as nothing but the ordinary thermodynamic entropy for a black hole. This implies that the actual entropy of a black hole is

$$S_{BH} = \frac{Ac^3}{4G\hbar}, \tag{3.2.9}$$

where the subscript refers to Bekenstein Hawking. This relationship often hints at the deep underlying connection between gravitation, quantum theories and statistical physics.

There is also a **Third Law of black hole mechanics** which states that the surface gravity of the horizon cannot be reduced to zero in any finite number of steps. This is like the thermodynamic case where the temperature cannot go to absolute zero. The four laws of black hole mechanics strongly resemble the four laws of thermodynamics allowing us to draw parallels, and begin investigating the structure of black holes.

## 3.3 Quantum Mechanical Properties

Particle creation was discussed in section 2.3.2 as a feature of quantum fields in curved space, the mechanism however was not discussed. For the purpose of the thesis we will discuss the mechanism in which particles are created in the vicinity of a black holes, within the framework of quantum field theory.

### 3.3.1 Particle Creation

#### 3.3.1.1 Intuitive Picture

Consider a toy model where modes of a quantum field behave like harmonic oscillators. When we are in an excited state of some oscillator, $|n\rangle$, then we have $n$ particles of a particular mode. Amplitudes of the these modes can be described by the Lagrangian

$$L = \frac{1}{2}\dot{a}^2 - \frac{1}{2}\omega^2 a^2, \tag{3.3.1}$$

where $\omega$ represents the frequency of the various modes that the system can have and $a$ is the amplitude of each mode. The vacuum state would be $|0\rangle_\omega$, which corresponds to a vacuum wavefunction which is represented below:

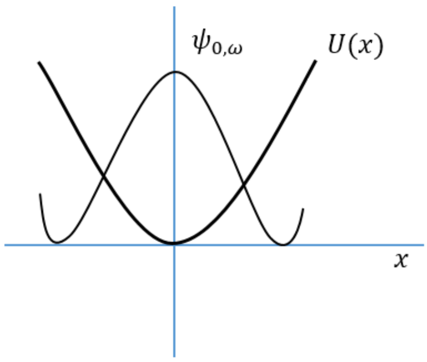


Figure 3.2: Vacuum state of some mode $\omega$ for a given potential.

Suppose we change the potential resulting in a new set of modes with frequency $\omega'$. The Lagrangian describing the new mode amplitudes becomes

$$L = \frac{1}{2}\dot{a}^2 - \frac{1}{2}\omega'^2 a^2. \tag{3.3.2}$$

Under the change in potential the new vacuum state $|0\rangle_{\omega'}$ looks like

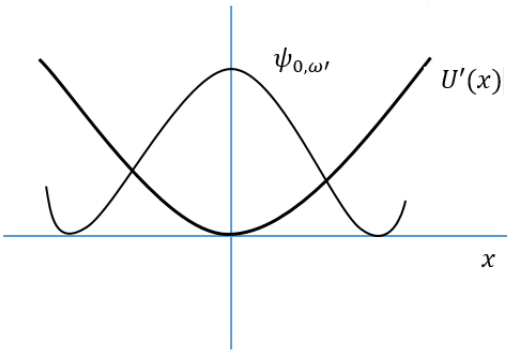


Figure 3.3: Vacuum state of some mode $\omega'$ for a given potential.

If we slowly change the potential such that the vacuum wavefunction changes adiabatically, the wavefunction will always describe the vacuum state no matter what we change the potential to. On the other hand, were we to change the potential rapidly, the wavefunction describing the $\omega$ vacuum, will not have time to evolve. As a result when the new potential is reached the wavefunction is still in the old state corresponding to an excited state for the $\omega'$ mode.

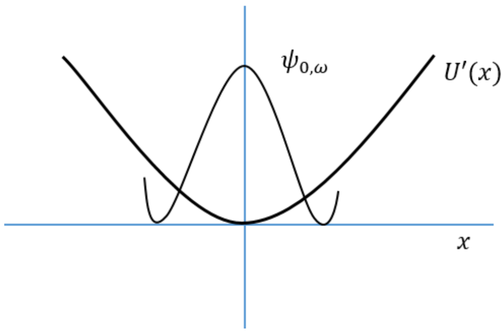


Figure 3.4: Vacuum state of $\omega$ modes is now an excited state for the for a $\omega'$ modes after a rapid change in the potential.

The wavefunctions is not a vacuum wavefunction for $\omega$, but one can expand it in terms of the excited wavefunction $|n\rangle_{\omega'}$ describing the $n$ excitation levels of the harmonic oscillator for frequency $\omega'$

$$|0\rangle_{\omega} = c_0|1\rangle_{\omega'} + c_1|1\rangle_{\omega'} + c_2|2\rangle_{\omega'} + \dots \tag{3.3.3}$$

Thus under slow changes of the harmonic oscillator, modes remain unchanged, whereas fast changes result in these modes to become populated with particles.

The terms ‘fast’ change and ‘slow’ change are relative to the natural frequency ( $\omega^{-1}$) of the modes. Thus a fast change in potential implies a change which occurs on a timescale much smaller than $\Delta T \sim \omega^{-1}$. A slow change refers to a timescale much larger than $\Delta T$. [38]

#### 3.3.1.2 Mechanism of Particle Creation

Mathur presents a thorough and pedagogical analysis of how particles are produced around a black hole in both [38] and [37]. What follows in this next section is a review of [38], focusing on his ideas surrounding particle creation. In the context of a curved spacetime, the varying potential we saw before is replaced by a varying metric. For black holes, the time dependence of the metric describing the surrounding spacetime will be central to the creation of particles. We start off by using the Schwarzschild metric, however, a slight contradiction appears. The metric is time independent.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{r}}\right) dr^2 + r^2 d\Omega^2, \tag{3.3.4}$$

Time independent metrics cannot give rise to particle production. The oversight here lies in the fact that the Schwarzschild metric does not cover the entire spacetime around a black hole, it only covers the exterior region. Using a spacetime metric that covers all regions of the black hole, will result in a time dependent metric. We can immediately see this by considering three radial geodesics, illustrated in figure 3.5 below.

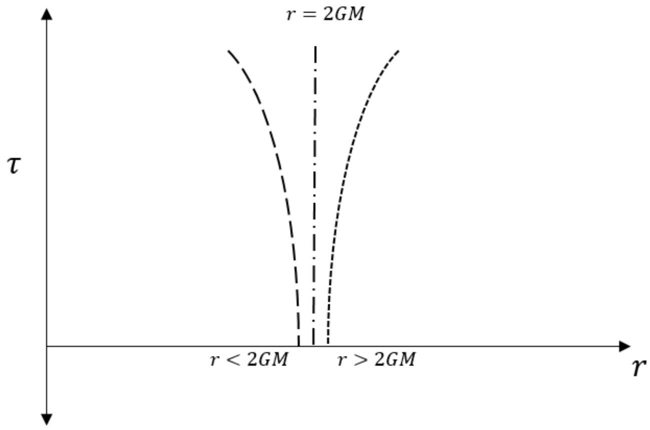


Figure 3.5: Figure adapted my Mathur [38]. The evolution of three geodesics, one inside the the event horizon $r < 2GM$, one along the horizon at $r = 2M$, and the last geodesic just outside the horizon $r > 2GM$.

Photons following geodesics which 'straddle' the horizon at $r = 2GM$, remain on the horizon. Since $ds^2 = 0$ in the metric, $dr = 0$ and so these photons appear to orbit the black hole. The second geodesic is slightly outside the horizon. As photons move radially outward along this geodesic, they will move further and further away from the black hole as time progresses. For the last case, photons following geodesics which are slightly inside the horizon can never escape. Furthermore they cannot remain at a fixed radius, instead will have to continue to move towards the center of the black hole as we saw in section 3.1.2 and 3.1.3. These interior and exterior geodesics 'peel' away from the horizon, and so, close to the horizon geodesics outside fall off to infinity, geodesics inside fall into $r = 0$.

The stretching of spacetime is indicative of a time dependent metric across the horizon. By analyzing how wavemodes evolve along a set of constant *space-like slices*, we will see how particle production arises. The prescription for describing these space-like slices is adopted from Mathur in [38]. By the term 'space-like slice', we actually mean a surface of constant $t$ call it $S$. All points on this surface are space-like separated. We want the surface to extend across the event horizon, whereby events will be space-like separated everywhere. The surface is constructed out of three separate surfaces. The first surface is one which extends from the event horizon out to infinity, we call it $S_{\text{out}}$. From chapter 3, we saw that as one crosses the event horizon of a Schwarzschild black hole, the time and radial coordinates of the Schwarzschild metric switch place. Inside the event horizon a space-like slice actually becomes a surface of constant $r$, this surface will be called $S_{\text{in}}$. We then smoothly connect these slices with a surface which extends across the horizon called $S_{\text{con}}$. This surfaces is chosen in such a way that the transition from $S_{\text{out}}$ to $S_{\text{in}}$ is everywhere a smooth space-like surface.

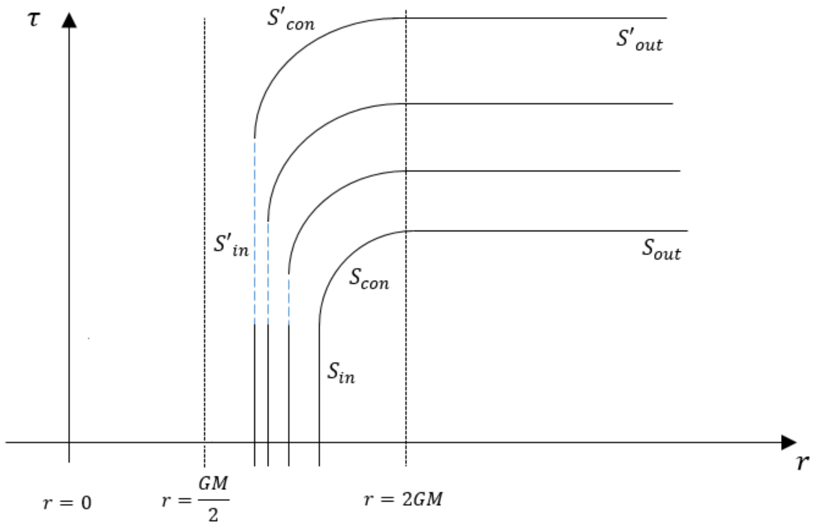


Figure 3.6: The evolution of a space-like slice composed of three space-like surfaces $S_{\text{in}}$, $S_{\text{con}}$, $S_{\text{out}}$. The slices evolves in such a way that it asymptotically reaches $\frac{GM}{2}$, this allows the value of $r$ to stay away from the singularity.

The intrinsic geometry of $S_{\text{out}}$ and $S'_{\text{in}}$ as they evolve to $S'_{\text{out}}$ and $S'_{\text{con}}$, remain somewhat constant. Instead if we consider the evolution of $S_{\text{in}}$, it moves little in the $r$ coordinate, but has to stretch a lot in order to remain connected to the entire space-like slice. The extension in $S_{\text{in}}$ is represented by the dotted line in figure 3.6.

**Evolution of wavemodes along space-like slices**

Imagine now we have a vacuum mode starting out at past null infinity. These ingoing wavemodes then scatter off at $r = 0$ and become outgoing modes, note, at this point there is no singularity. What we are interested, is in the behaviour of the outgoing modes as they evolve. By combining the ideas of how geodesics peel off from the horizon and the stretching of the spacetime slices, we get a good picture of what happens to the outgoing modes. The vacuum mode which has an initial constant phase is placed on some initial slice. The important point is in observing the manner in which the wavelength of the mode - which is drawn as a wavepacket - changes at various points along the space-like-slice surface. This is shown in figure 3.7

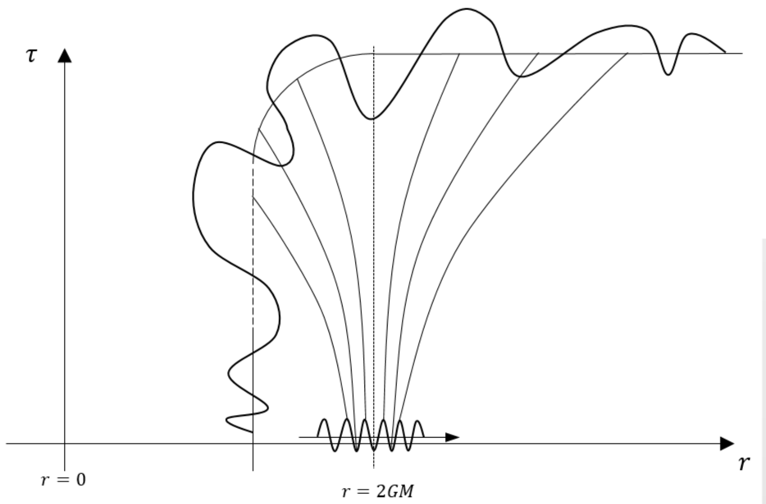


Figure 3.7: The distorted evolution of wavelengths corresponding to a vacuum mode on an initial space-like slice situated across the horizon.

Points along the wavemode evolve differently depending on their position on the slice. The parts of the initial wavemode which evolved on the surface $S_{\text{in}}$ distorted the most compared to the other points along $S_{\text{con}}$ and $S_{\text{out}}$. This is due to the large amount of stretching that takes place along $S_{\text{in}}$ for the later slice $S'_{\text{in}}$ to remain connected everywhere. Thus, the wavelength on the late space-like slice is no longer constant, but distorted. It is the differing evolution of geodesics on either side of the event horizon, consequently, the distortion of the wavemode which creates particles.

**Evolution of wavemodes**

Since the spacetime geometry around the the event horizon looks locally flat, we can use the null coordinates $y^+, y^-$ to locally describe spacetime along the initial slices which are in the vicinity of the event horizon. From the field expansion in (2.3.3)

$$\phi(y^-) = \sum_k \left( \hat{a}_{\vec{k}}^{iky^-} + \hat{a}_{\vec{k}}^\dagger e^{-iky^-} \right), \tag{3.3.5}$$

which is presented in a slightly different form for our discussion. Modes in our null coordinates can be expressed as the positive frequency modes $e^{iky^-}$, which multiply the annihilation operator, and negative frequency modes $e^{-iky^-}$ which multiply the creation operators. We choose the initial vacuum mode in such a way that it is only composed of positive frequency modes. Using these null coordinates, we examine how coordinates on the initial slice evolve to null coordinates on the late slice. The coordinates on the late slice are given by

$$X^+ = t + r \qquad\qquad X^- = t - r. \tag{3.3.6}$$

We need to consider the evolution in two regions: outside the event horizon $y^- < 0$ and inside the event horizon $y^- > 0$, where we let $y^- = 0$ represent the horizon itself. Modes on the late slice will be of the form $e^{iKX^-}$ thus we look for relations that look like

$$X^- = X^-(y^-). \tag{3.3.7}$$

From [38] it turns out that the coordinates have the relations:
For $y^- < 0$ and $|y^-|$ very small,

$$X^- = -(GM) \ln \left( -\frac{y^-}{GM} \right), \tag{3.3.8}$$

where the factor $GM$ was included for correct dimensions. For inside the event horizon, $y^- > 0$, the null coordinates are not particularly well defined. We can however take advantage of the way in which we sliced the geometry, to then use coordinates $Y^+, Y^-$ defined on $S_{\text{in}}$ similar to the null coordinates $X^+, X^-$ defined on $S_{\text{out}}$. Using the $Y^-$, we get the relation

$$Y^- = -(GM) \ln \left( \frac{y^-}{GM} \right), \tag{3.3.9}$$

giving us a similar distortion for modes just inside the horizon.

The important point here is that the initial vacuum mode straddles both sides of the horizon. Due to the very different manner in which geodesics behave on either side of the horizon, modes will stretch in a logarithmic manner as they evolve on $S_{\text{in}}$ and $S_{\text{out}}$ to $S'_{\text{in}}$ and $S'_{\text{out}}$ . Due to this distortion, a single initial mode straddling the event horizon on the early slice becomes a combination of modes on the later separate slices. The splitting of modes on $S'_{\text{in}}$ and $S'_{\text{out}}$ produces particles that are in a mixed state.
Naively, one may think that it is the logarithmic distortion of the modes which are responsible for the creation of particles. Not exactly, remember that in the simple picture of a changing potential, the potential had to change ‘quickly’. Thus the length scale on which changes occur is important. In our case what is changing is the wavelength of the oscillations of the modes close to the event horizon.

More precisely, the change of one oscillation relative to the next one. Consider a Fourier mode $e^{iky^-}$ on the initial slice with wavelength much smaller than $GM$

$$\lambda = \frac{2\pi}{k} = \epsilon GM; \quad \epsilon << 1. \tag{3.3.10}$$

The range of spacetime we consider in which the oscillation occur is $-GM < y^- < 0$, note we are outside the event horizon. Within such a range the number of oscillations are of the order $\frac{1}{\epsilon}$. Now consider how this oscillation will evolve along the late slice within the range $-\alpha < y^- < -\alpha + \epsilon$. It will have a wavelength

$$\lambda_1 = |\delta X^-| = \left| \left( \frac{dX^-}{dy^-} \right) \delta y^- \right| \tag{3.3.11}$$

$$= -(GM) \left( -\frac{1}{|y^-|} \right) \delta y^- \tag{3.3.12}$$

$$= GM \frac{(-\alpha) - (-\alpha + \epsilon)}{|-\alpha|} \tag{3.3.13}$$

$$= \frac{GM}{\alpha} \epsilon. \tag{3.3.14}$$

The subsequent oscillation on the initial slice will evolve within a range $-\alpha - \epsilon < y^- < -\alpha$ and so will have wavelength

$$\lambda_2 = GM \frac{(-\alpha - \epsilon) - (-\alpha)}{|-\alpha + \epsilon|} \tag{3.3.15}$$

$$= \frac{GM}{\alpha - \epsilon} \epsilon \tag{3.3.16}$$

If we compare the two wavelengths

$$\frac{\lambda_2}{\lambda_1} \approx \frac{\alpha}{\alpha - \epsilon} \approx 1. \tag{3.3.17}$$

The meaning of this is that adjacent oscillations, although they deform, they actually deform almost uniformly. However, under a uniform change no particle production can occur. This is analogous to the toy model where a uniform and slow change of the potential did not cause particle production. More precisely, the creation of particles is given by the mixing of positive and negative frequencies. In the case above we do not get a significant amount of mixing, that is, imagine the initial mode evolved to $e^{ik\mu y^-}$, where $\mu$ is some positive constant. To check for particles, one would compute $(f, g)$ with $f = e^{iky^-}$ and $g = e^{ik\mu y^-}$

$$(f, g) = -i \int dy^- e^{iky^-} e^{ik\mu y^-} \to 0 \tag{3.3.18}$$

since the signs of the Fourier modes are the same, only integrals with $e^{iky^-}$and $e^{-iky^-}$ will be non-zero. So the conclusion is the same as in the toy model, there was not enough stretching over the length scale of oscillations of the wavemode to allow the productions of particles.

This mixing of frequencies happens within a range very close to the event horizon where $|y^-| \sim \epsilon$. In such a range one cannot break up a number of oscillations into wavepackets which each deform uniformly, but instead one has to consider the evolution of the wavemode for a few oscillations. The highly non uniform deformations along the late slice will occur on $S_{\text{in}}$ and $S_{\text{out}}$ right near the event horizon. Thus it is in this region where the particle pair productions take place.

# Chapter 4

# Information Paradox

We saw that the evolution of space-like slices led to particles being created at the horizon. These particles are created in pairs, the member of the pair which escapes off to infinity is known as Hawking radiation. With the conjecture of the Hawking radiation, followed the idea that black holes should then evaporate. The black hole evaporation occurs via the emitted radiation removing energy from the black hole thus decreasing its mass. Upon doing calculations, it soon became apparent to Hawking that there is a serious issue in believing that a black hole evaporates; the information is lost.

Quantum mechanics requires unitary evolution when describing any quantum process. If one considers the evolution of a state which is initially pure, the final state by unitary evolution must also be pure. Similarly a mixed state has to evolve to a mixed state. The particle pairs which come into existence are in a pure state. Together they make up the pure system, but individually, they are in a mixed state. What we observe as the Hawking radiation are the particle pairs which flew off to infinity. If the black hole evaporates, all that we are left with is the Hawking radiation which remains in a mixed state. The problem is, they no longer have their partner pairs to be mixed with. So the initial pure system has evolved into a mixed system, violating the requirement for unitarity. If instead we want unitarity to hold, then we have lost information as the mixed Hawking radiation has lost its entanglement with infalling partner pairs. This is essentially the information paradox.

It is important to understand how Hawking came to this conclusion. To do so, we first go through his derivation of the emitted radiation which is seen to emit at a thermal temperature of

$$T = \frac{\hbar c^3}{8\pi G M k_B} \approx \frac{1.227 \times 10^{23}\text{kg}}{M} K. \tag{4.0.1}$$

A typical sized black hole of a couple of solar masses ($M_{\odot} = 1.98892 \times 10^{30}$kg) would have a temperature of $\approx 60nK$ to $\approx 60pK$. The black hole is therefore seen to act as a perfect black body.

Thereafter we resume the discussion of chapter 3, section 3.3.1 and give a more through explanation as how the evaporation of black hole gives us the information problem. We then show that if we enforce unitary evolution then the equivalence principle comes under question. The information paradox is one that puts fundamental theories in disagreement with one another, such paradoxes are vital to the evolution of physics.

## 4.1 Hawking Radiation

We begin with a detailed overview of Hawking's derivation [28] of the thermal spectrum for a black hole, by first calculating what the quantum field around a Schwarzschild black hole looks like. Thereafter calculating the spectrum of emitted particles. We will not make any specific reference to information loss in this section, as it is mainly devoted to deriving the spectrum of emitted particles.

Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{r}}\right) dr^2 + r^2 d\Omega^2. \tag{4.1.1}$$

Such a geometry is created in spacetime when a star collapses to form a black hole. Although the Schwarzschild coordinates are nice for getting an intuitive understanding of what a black hole is, the presence of the coordinate singularity at $r = 2GM$ is an inconvenient problem for any questions aimed at explaining what actually happens as one crosses the event horizon, making any subsequent analysis and progress somewhat intractable. To circumvent this problem, we change to the Eddington-Finkelstein coordinates as given in section 3.1.3, with the coordinate transformation repeated here

$$t = v - r - 2GM \ln\left|\frac{r}{2GM} - 1\right|$$

The Eddington-Finkelstein coordinates allow us to introduce the advanced time $v = t + r^*$ and the retarded time $u = t - r^*$, where

$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|. \tag{4.1.2}$$

$v$ is the null geodesic which straddles the event horizon. Photons traveling along $v$ would never escape, nor fall into the black hole. The main feature of these new coordinates is made evident by considering the Penrose diagram of a collapsing star which has just reached its Schwarzschild radius

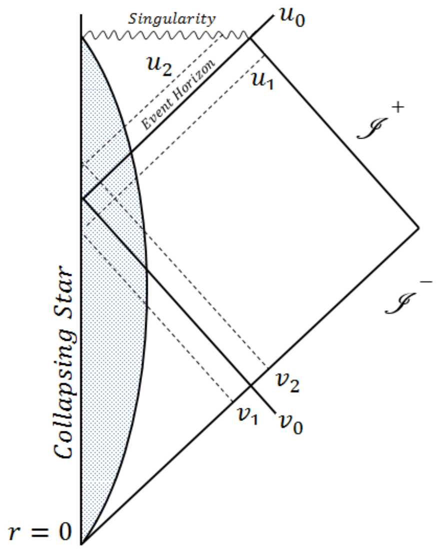


Figure 4.1: Null trajectories $v_0, v_1, v_2$ originating on the surface $\mathscr{I}^-$, representing past null infinity, whereby $v_1, v_2$ have the property that, $v_1 < v_0$ and $v_2 > v_0$. $v_0$ is the last any null ray can escape to infinity.

Using the definitions of $v, u$ we can then ask questions of what happens to the phases of various modes of a quantum field as they follow trajectories $v_1$ and $v_2$. This was done in a general sense in section 4 with the discussion of constant space-like slices. In figure 4.1 $v_1$ represents the last time that any light-like geodesic will escape the collapsing star, such geodesics end on the surface $\mathscr{I}^+$ representing future null infinity. $v_2$ represents the time immediately after the horizon is formed. Light-like geodesics along $v_2$ will inevitably fall into the now formed black hole, such geodesics end at the singularity. This distinct separation between null trajectories which fall into or escape the black hole creates a setting whereby particle production occurs as described in section 3.3.1. The created particles which radiate out from the black hole is the Hawking radiation.

As mentioned previously, such a black hole is formed at the collapse of a star (figure 4.1). We imagine that the black hole was formed at some time in the past by gravitational collapse. This is reasonable as we are primarily interested in particle emission at late times (long after the collapse occurs). The geometry of the black hole formed by the collapsing star is described using the Schwarzschild metric (3.1.1) repeated here

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{r}}\right) dr^2 + r^2 d\Omega^2. \tag{4.1.3}$$

The Schwarzschild metric is the spherically symmetric vacuum solution to Einstein's equations, implying $T^{\mu\nu} = 0$ and $R = 0$. We want to construct a quantum field around the black hole, so, in the absence of matter, we choose to describe spacetime with the massless scalar field $\phi$ whose equations of motion are given by the massless Klein-Gordon equation

$$-g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0 \quad \text{or} \quad \Box\phi = 0, \tag{4.1.4}$$

where $\nabla_\mu$ is the covariant derivative. Motivated by the spherical symmetry of the black hole, the field $\phi$ can be assumed to have a spherically symmetric solution of the form given in [28]

$$\phi = r^{-1}R_{\omega\ell}(r)Y_{\ell m}(\theta,\varphi)e^{-i\omega t}, \tag{4.1.5}$$

where $Y_{\ell m}$ are the spherical harmonics and $R_{\omega\ell}$ is the radial function. Using the above field $\phi$, the equations of motion lead to the Regge-Wheeler equations of motion [51]

$$-\frac{d^2R_{\omega\ell}(r)}{dr^{*2}} + \left[V(r) - \omega^2\right]R_{\omega\ell}(r) = 0, \tag{4.1.6}$$

where $V(r)$ is the gravitational potential

$$V(r) = \left(1 - \frac{2GM}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2GM}{r^3}\right). \tag{4.1.7}$$

The Regge-Wheeler equation of motion is rather difficult to solve exactly, but since we are not interested at the regions close to the collapsing body (measurements are done at asymptotic infinity), we restrict ourselves to solutions where $r \to \pm\infty$. Under such conditions the gravitational potential $V(r \to \infty) = 0$ and so the differential equation (4.1.6) reduces to

$$\frac{d^2R_{\omega\ell}(r)}{dr^{*2}} + \omega^2R_{\omega\ell}(r) = 0, \tag{4.1.8}$$

which has a general solution proportional to

$$R_{\omega\ell}(r) \propto e^{\pm i\omega r^*}. \tag{4.1.9}$$

Using this, our choice for the field (4.1.5) becomes

$$\phi \sim r^{-1}e^{\pm i\omega r^*}e^{-i\omega t}Y_{\ell m}(\theta,\varphi) = 0. \tag{4.1.10}$$

For $e^{i\omega r^*}$

$$\phi \sim \frac{e^{-i\omega(t-r^*)}}{r}Y_{\ell m}(\theta,\varphi), \tag{4.1.11}$$

$$\phi \sim \frac{e^{-i\omega u}}{r}Y_{\ell m}(\theta,\varphi), \tag{4.1.12}$$

$$\phi \sim h(r,\omega). \tag{4.1.13}$$

Similarly for $e^{-i\omega r^*}$

$$\phi \sim \frac{e^{-i\omega(t+r^*)}}{r}Y_{\ell m}(\theta,\varphi), \tag{4.1.14}$$

$$\phi \sim \frac{e^{-i\omega v}}{r}Y_{\ell m}(\theta,\varphi), \tag{4.1.15}$$

$$\phi \sim f(r,\omega). \tag{4.1.16}$$

For simplicity we have suppressed the quantum numbers $\theta, \varphi$, and in future we write $f(r,\omega) \equiv f_\omega$ and $h(r,\omega) \equiv h_\omega$. Our goal is to calculate what are the Bogoliubov transformations between the set of ingoing and outgoing modes $f(r,\omega) \equiv f_\omega$ and $h(r,\omega) \equiv h_\omega$ respectively. The coefficients of Bogoliubov transformations will then tell us what is the spectrum of the emitted particles. We first however need to impose boundary conditions on the modes and then do an analysis of these modes within the various regions to obtain relationships between the modes.

The function $f_\omega$ represents the set of basis functions of the ingoing modes of $\phi$ which are chosen to be pure positive ($f_\omega \sim e^{-i\omega v}$). They represent solutions to the field $\phi$ at early times ($t \to -\infty$) and which originate on the Cauchy surface $\mathscr{I}^-$. One commonly refers to the function $f_\omega$ as the set of basis functions in which the modes can be expanded. The field in this in-region can be expressed as a mode expansion of the form (2.3.13)

$$\phi_{\text{in}} = \sum_\omega \left( f_\omega \hat{a}_\omega + f_\omega^* \hat{a}_\omega^\dagger \right), \tag{4.1.17}$$

with the vacuum defined as

$$\hat{a}_\omega |0\rangle_{\text{in}} = 0 \quad \forall \omega, \tag{4.1.18}$$

and commutation relations

$$[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega') \tag{4.1.19}$$

$$[a_\omega, a_{\omega'}] = [a_\omega^\dagger, a_{\omega'}^\dagger] = 0. \tag{4.1.20}$$

The out-region instead has modes which both fall into the black hole and those which reach some observer at infinity. Thus we need to distinguish between such modes. An observer in the out region collecting information on the Cauchy surface $\mathscr{I}^+$, cannot access information carried by modes which fall into the black hole, as this would require a violation of causality.The field $\phi_{\text{out}}$ of the entire out region is then composed of the modes which fall into the black hole, and modes which reach $\mathscr{I}^+$ expressed as

$$\phi_{\text{out}} = \sum_\omega \left( h_\omega \hat{b}_\omega + h_\omega^* \hat{b}_\omega^\dagger + q_\omega \hat{c}_\omega + q_\omega^* \hat{c}_\omega^\dagger \right). \tag{4.1.21}$$

The decomposition of the solutions of $\phi$ in the various regions can be seen in figure 4.2. Solutions $h_\omega$ describe outgoing modes which reach the observer at $\mathscr{I}^+$ and are also chosen to be pure positive ( $h_\omega \sim e^{-i\omega u}$) and are purely outgoing. The modes in the out region are defined by the creation and annihilation operators $\hat{b}_\omega, \hat{b}_\omega^\dagger$. The vacuum state is defined as

$$\hat{b}_\omega |0\rangle_{\text{out}} = 0 \quad \forall \omega, \tag{4.1.22}$$

with commutation relations

$$[b_\omega, b_{\omega'}^\dagger] = \delta(\omega - \omega'), \tag{4.1.23}$$

$$[b_\omega, b_{\omega'}] = [b_\omega^\dagger, {b^\dagger}_{\omega'}] = 0. \tag{4.1.24}$$

The modes $q_\omega$ fall into the black hole ending at the singularity. They are solutions to the wave equation that are not outgoing, so contain no Cauchy data on $\mathscr{I}^+$. It is unclear how to define $q_\omega$ and whether one can impose the positive frequency condition [18], and if so, with respect to what. Nevertheless, we are still able to continue forward, as although we are unable to specify what $q_\omega$ is, it lands up not playing a role in the computation of the spectrum of emitted particles [18, 28].

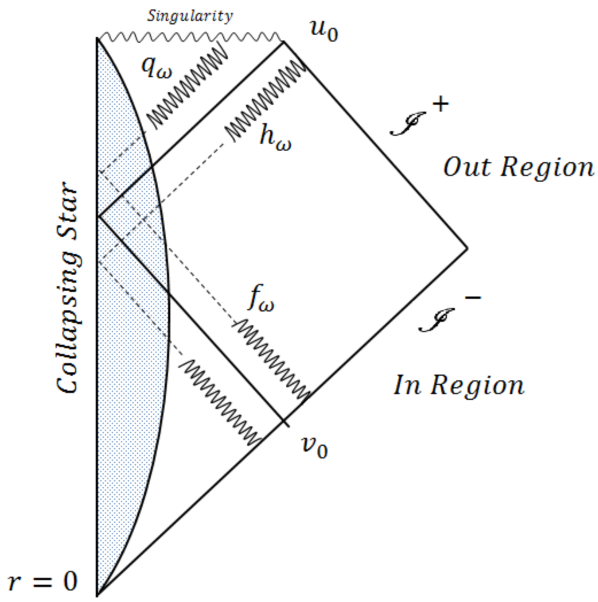


Figure 4.2: The composition of the field $\phi$ described by solutions given in the in-region ($f_\omega$) and solutions in the out-region ($h_\omega$, $q_\omega$).

The field $\phi$ at any point in time can be expressed as a linear combination of a complete set of solutions to the Klein-Gordon equation. This is made evident when expressing $\phi$ as solutions to the in and out regions represented by eqn. (4.1.17) and (4.1.21) respectively. More precisely $\phi$ can either be expressed by the complete set of solutions in the basis of $f_\omega$, taken as incoming modes, or by the complete set of solutions in the basis of $h_\omega$ and $q_\omega$, taken as outgoing modes. Reversing time, the outgoing modes $h_\omega$ and $q_\omega$, ending on $\mathscr{I}^+$ and at the singularity respectively, now look like they originate from either $\mathscr{I}^+$ or the singularity and developed into $f_\omega$. Essentially the outgoing modes can be expressed in terms of the ingoing modes. We represent this mathematically as

$$h_\omega = \sum_{\omega'} \alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}, \tag{4.1.25}$$

$$q_\omega = \sum_{\omega'} \chi_{\omega\omega'} f_{\omega'} + \eta_{\omega\omega'} f^*_{\omega'}, \tag{4.1.26}$$

giving the relation between oscillators

$$\hat{b}_\omega = \sum_{\omega'} \alpha^*_{\omega\omega'} \hat{a}_{\omega'} - \beta^*_{\omega\omega'} \hat{a}^\dagger_{\omega'}, \tag{4.1.27}$$

$$\hat{c}_\omega = \sum_{\omega'} \chi^*_{\omega\omega'} \hat{a}_{\omega'} - \eta^*_{\omega\omega'} \hat{a}^\dagger_{\omega'}, \tag{4.1.28}$$

where $\alpha_{\omega\omega'}, \beta_{\omega\omega'}$ and $\chi_{\omega\omega'}, \eta_{\omega\omega'}$ are the Bogoliubov coefficients we initially set out to find. This result is directly analogous to how we first introduced the Bogoliubov coefficients eqn. (2.3.19) by looking at what the vacuum state for some observer $b$ looks like relative to another observer $a$ in a region of spacetime which is curved.

### 4.1.1 The Collapsing star

The discussion thus far has concentrated on how the modes within the various regions are related to eachother by imposing certain boundary conditions. Now that we have found the set of Bogoliubov transformations, all that is left is to find an analytic expression of the modes $h_\omega$ and $f_\omega$ and then compute the coeffcients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$. This is done according to Hawking [28] by first modeling the collapsing star.

According to [18], the collapsing star can be modeled as a single thin shell. This allows us to come up with a way to describe $h_\omega$. We then compute the flux of particles emitted by the thin shell by calculating the large phase shift that $f_\omega$ undergoes. This phase shift occurs when ingoing modes, $f_\omega$, pass through the shell just before the horizon is formed. From the phase shift we obtain the Bogoliubov coefficients $\beta_{\omega\omega'}, \beta^*_{\omega\omega'}$. Since ingoing modes had very high frequencies[1] as they pass through the collapsing body, Hawking described their propagation using *geometric optics*. This leads to a $v =$ constant ingoing ray and a $u =$ constant outgoing ray. By the geometric optics approximation one can perform a ray tracing analysis leading to ingoing rays with the following form $u = g(v)$ or, inversely $v = g^{-1}(u) \equiv G(u)$, that is, they are functions of each other. The asymptotic forms for the in and outgoing modes are given as in [18]

$$f_{\omega\ell m} \sim \frac{Y_{\ell m}(\theta\phi)}{\sqrt{4\pi\omega}r} \times \begin{cases} e^{-i\omega v} & , \quad \text{on } \mathscr{I}^- \\ e^{-i\omega G(u)} & , \quad \text{on } \mathscr{I}^+ \end{cases}$$

and

$$h_{\omega\ell m} \sim \frac{Y_{\ell m}(\theta\phi)}{\sqrt{4\pi\omega}r} \times \begin{cases} e^{-i\omega u} & , \quad \text{on } \mathscr{I}^+ \\ e^{-i\omega g(v)} & , \quad \text{on } \mathscr{I}^- \end{cases}$$

the ray tracing then leads to Hawking's result [28]

$$u = g(v) = -4GM \ln\left(\frac{v_0 - v}{C}\right) \tag{4.1.29}$$

$$v = G(u) = v_0 - Ce^{-u/4M} \tag{4.1.30}$$

where $C$ is some constant and $M$ is the mass of the black hole. We now prove this result for the case a thin spherical shell according to [18].

[1]Modes dominating the surface $\mathscr{I}^+$ were all largely redshifted as they left the spherical shell, thus had to have a very large frequency as they pass through.

The collapsing shell consists of an interior region and an exterior region. The interior region can be described by the metric

$$ds^2_{\text{in}} = -dT^2 + dR^2 + r^2 d\omega^2, \tag{4.1.31}$$

with null coordinates $V = T + r$ and $U = T - r$, just as with $v$ and $u$. Outside the shell we have the Schwarzschild metric

$$ds^2_{\text{out}} = -\left(1 - \frac{2GM}{R}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{R}}\right) dR^2 + R^2 d\Omega^2. \tag{4.1.32}$$

where $R \equiv R(t)$ as the shell radius is collapsing. We require that the intrinsic geometry of the shell be continuous across the shell boundary and thus $ds^2_{\text{in}} = ds^2_{\text{out}}$ giving the condition

$$\begin{aligned} -dT^2 + dR^2 &= -\left(1 - \frac{2GM}{R}\right) dt^2 + \left(\frac{1}{1 - \frac{2GM}{R}}\right) dR^2 \\ -1 + \left(\frac{dR}{dT}\right)^2 &= -\left(1 - \frac{2GM}{R}\right)\left(\frac{dt}{dT}\right)^2 + \left(1 - \frac{2GM}{R}\right)^{-1}\left(\frac{dR}{dT}\right)^2. \end{aligned} \tag{4.1.33}$$

To find how $v$ and $u$ are related we are left with having to find three relations: 1) $v$ and $V$, 2) $V$ and $U$, 3) $U$ and $u$. Such relations are represented schematically in the figure 4.3 below

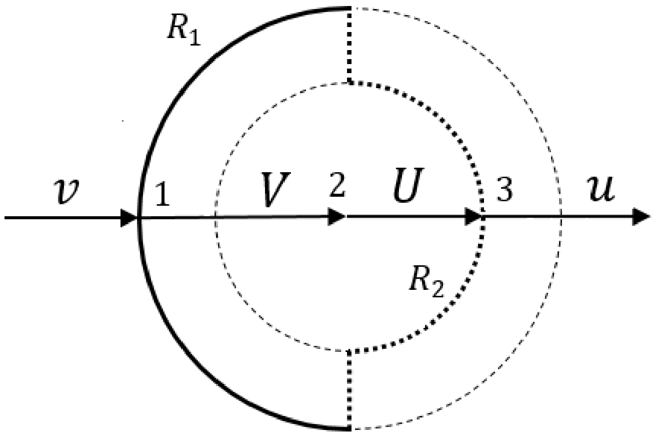


Figure 4.3: An ingoing ray $u$ enters the shell with radius $R_1$ represented by the solid line, it passes through the interior of the sphere as a null ray $V$. The ray null ray $U$ then exits the now slightly collapsed shell with smaller radius $R_2$ represented at the dark dotted line, to then become an outgoing ray $u$.

1) The ingoing ray $v$ enters the shell which has radius $R_1 > 2GM$. If we look at the terms $\frac{dR}{dT}$ and $\left(1 - \frac{2GM}{R}\right)^{-1}$ in (4.1.33), we see that they are finite and approximately constant. Similarly $\frac{dt}{dT}$ is also approximately constant, and so $t \propto T$. Lastly since $r^*$ as given in (4.1.2) close to $R_1$ is a linear function in $r$, the relation between the ingoing ray on the exterior, $v$, and on the interior, $V$, have a simple linear relation

$$V(v) = av + b, \tag{4.1.34}$$

where $a$ and $b$ constants.

2) At the center $r = 0$ where $U = T - r$ and $V = T + r$ we simply get that

$$U(V) = V \tag{4.1.35}$$

3) The shell now has collapsed to a radius $R_2$ close to $2GM$, thus let $T_0$ be the time when $R_2 = 2GM$, and for some $T$ less than, but close to $T_0$, we can express

$$R(T) \approx 2GM + A(T_0 - T), \tag{4.1.36}$$

where $A$ is some constant. Pluging this result in (4.1.33) we get

$$\begin{aligned}\left(\frac{dt}{dT}\right)^2 &\approx \left(\frac{R(T) - 2GM}{2GM}\right)^{-2} \left(\frac{dR}{dT}\right)^2 \\ &\approx \left[\frac{(2GM + A(T_0 - T)) - 2GM}{2GM}\right]^{-2} (-A)^2 \\ &= \left(\frac{2GM}{T - T_0}\right)^2\end{aligned}$$

Integrating this and looking at $T \to T_0$ one gets

$$t \sim -2GM \ln\left(\frac{T_0 - T}{B}\right) \tag{4.1.37}$$

similarly

$$r^* \sim 2GM \ln\left(\frac{R - 2GM}{2GM}\right) \sim 2GM \ln\left(\frac{A(T_0 - T)}{2GM}\right) \tag{4.1.38}$$

and so

$$\begin{aligned}u = t - r^* &\sim - - 2GM \ln\left(\frac{T_0 - T}{B}\right) - 2GM \ln\left(\frac{A(T_0 - T)}{2GM}\right) \\ &\sim -2GM \ln\left(\frac{A(T_0 - T)^2}{2BGM}\right) \\ &\sim -4GM \ln\left(\frac{T_0 - T}{B'}\right); \quad B' = \sqrt{\frac{A}{2GBM}}\end{aligned}$$

in this limit we also have on the inside of the shell that

$$U = T - R \sim T - (2GM + A(T_0 - T)) \tag{4.1.39}$$

$$\sim T(1 + A) - 2GM - AT_0 \tag{4.1.40}$$

Finally joining these last two results with (4.1.34) and (4.1.35) and with some algebra, one obtains (4.1.29). This result was derived by only considering a star made of a single thin shell. Hawking gave a much more general argument by dividing the collapsing star into a series of collapsing shells [18]. As the null ray enters and exits each shell, each null coordinate is a linear function of the preceding one, until we come to the exit from the last shell. At this point, the retarded time $u$ in the exterior spacetime is a logarithmic function of the previous coordinate, and hence also a logarithmic function of $v$, which leads to the same result as (4.1.29).

Using the expression for $h_\omega$ and the results above, we have that

$$h_\omega \sim \begin{cases} e^{-i4GM\omega \ln\left[\frac{v_0 - v}{C}\right]} & , \quad v < v_0 \\ 0 & , \quad v > v_0 \end{cases}$$

The modes for $v > v_0$ belong to $q_\omega$ and these fall into the black hole, thus have no information on $\mathscr{I}^+$.

### 4.1.2 Spectrum of Emitted Particles

We are now in a position to compute the Bogoliubov coefficients. Doing so, will give us an important identity below

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi GM\omega}|\beta_{\omega\omega'}|^2, \tag{4.1.41}$$

which we will prove. We then compute the particle number for a distant observer located at infinity by finding the expectation value $\langle 0|b_\omega b_\omega^\dagger|0\rangle$. Since the particle number will contain the Bogoliubov coefficients, we use the identity to then express everything in terms of $\beta_{\omega\omega'}$ which we shall now compute.

We begin by taking a Fourier transform of

$$h_\omega = \int_0^\infty d\omega' \left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}\right) \tag{4.1.42}$$

where $h_\omega$ is expressed as an integral since we are looking at an infinite amount of modes. More specifically, we consider the space over which we integrate to be a sphere with very large radius. This leads to

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv \sqrt{\frac{\omega'}{\omega}} e^{i\omega' v_0} e^{-i\omega g(v)}, \tag{4.1.43}$$

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0} e^{-i\omega g(v)}. \tag{4.1.44}$$

By making a change of variables to $s = v_0 - v$ for $\alpha_{\omega\omega'}$ and $s = -s$ for $\beta_{\omega\omega'}$ and using that $h_\omega = e^{-i4GM\omega \ln\left[\frac{v_0 - v}{C}\right]}$, we get

$$\alpha_{\omega\omega'} = \frac{1}{2\pi} \int_0^\infty ds \sqrt{\frac{\omega'}{\omega}} e^{i\omega' v_0} e^{-i\omega' s} e^{i4GM\omega \ln[s/C]}, \tag{4.1.45}$$

$$\beta_{\omega\omega'} = \frac{1}{2\pi} \int_{-\infty}^0 ds \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0} e^{-i\omega' s} e^{i4GM\omega \ln[-s/C]}. \tag{4.1.46}$$

Now we are left with the task of computing the integrals. In equation (4.1.45) we do the integral by choosing an appropriate contour. The contour along the real axis from 0 to $\infty$ can be closed by a quarter circle at infinity and by the contour along the imaginary axis from $-i\infty$ to 0. The integral from 0 to 1 along the real $s$-axis equals the integral from 0 to $-i\infty$ along the imaginary $s$-axis. The integral goes to infinity since there are no poles present in the enclosed quadrant of the complex plane.

The idea is the same for (4.1.46), whereas now the contour from $-\infty$ to 0 is joined by a quarter circle from $-i\infty$ to 0 giving a closed contour. By the same reasoning the integral from $-\infty$ to 0 is the same as $-i\infty$ to 0. This allows us to put $s = is'$ to give

$$\alpha_{\omega\omega'} = -\frac{1}{2\pi}\int_{-\infty}^{0} ds\,\sqrt{\frac{\omega'}{\omega}}e^{i\omega' v_0}e^{\omega' s'}e^{i4GM\omega\ln\left[is'/C\right]}, \tag{4.1.47}$$

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\int_{-\infty}^{0} ds\,\sqrt{\frac{\omega'}{\omega}}e^{-i\omega' v_0}e^{\omega' s'}e^{i4GM\omega\ln\left[-is'/C\right]}. \tag{4.1.48}$$

Taking the branch cut in the complex plane along the negative real axis, one gets gets a single-valued natural logarithm function that comes from the multiple-valued complex logarithm in the above equations. So for $s' < 0$

$$\ln\left[\frac{is'}{C}\right] = \ln\left[\frac{-i|s'|}{C}\right] = -i\frac{\pi}{2} + \ln\left[\frac{|s'|}{C}\right] \tag{4.1.49}$$

and

$$\ln\left[-\frac{is'}{C}\right] = \ln\left[\frac{i|s'|}{C}\right] = i\frac{\pi}{2} + \ln\left[\frac{|s'|}{C}\right] \tag{4.1.50}$$

So (4.1.47) and (4.1.48) become

$$\alpha_{\omega\omega'} = -i\frac{1}{2\pi}e^{i\omega' v_0}e^{2\pi\omega MG}\int_{-\infty}^{0} ds\,\sqrt{\frac{\omega'}{\omega}}e^{\omega' s'}e^{i4GM\omega\ln\left[|s'|/C\right]}, \tag{4.1.51}$$

$$\beta_{\omega\omega'} = i\frac{1}{2\pi}e^{-i\omega' v_0}e^{-2\pi\omega MG}\int_{-\infty}^{0} ds\,\sqrt{\frac{\omega'}{\omega}}e^{\omega' s'}e^{i4GM\omega\ln\left[|s'|/C\right]}. \tag{4.1.52}$$

This leads to an important result for the part of the wavepacket that was propagated backwards in time through the collapsing sphere before the event horizon was formed.

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi GM\omega}|\beta_{\omega\omega'}|^2 \tag{4.1.53}$$

Let us recall a few chosen properties of these basis modes from chapter 2.

$$(f_\omega, f_{\omega'}) = (h_\omega, h_{\omega'}) = \delta_{\omega\omega'} \tag{4.1.54}$$

$$(f^*_\omega, f^*_{\omega'}) = (h^*_\omega, h^*_{\omega'}) = -\delta_{\omega\omega'} \tag{4.1.55}$$

$$(f_\omega, f^*_{\omega'}) = (h_\omega, h^*_{\omega'}) = 0 \tag{4.1.56}$$

If we rewind the evolution of the $h_\omega$ modes from $\mathscr{I}^+$ to $\mathscr{I}^-$, we see there is a distinction that needs to be made between two modes, $h^{(1)}_\omega$ and $h^{(2)}_\omega$. It turns out that we are only interested in the $h^{(2)}_\omega$ component. This is the part of the wavemode which travels through the collapsing body. Let $h^{(1)}_\omega$ denote the components of the part of the wave packet that, when propagated backwards in time, is scattered from outside the collapsing star then travels back, reaching $\mathscr{I}^-$ with the same frequency $\omega$ as from when it originated on $\mathscr{I}^+$. The matching in frequency is because $h^{(1)}_\omega$'s blueshift when approaching the mass and its redshift when going away from the mass cancel each other exactly.

Since $h_\omega^{(1)}$ and $h_\omega^{(2)}$ travel in disjoint regions($v_1 > v_0$ and $v_2 < v_0$ respectively) they are orthogonal [10].

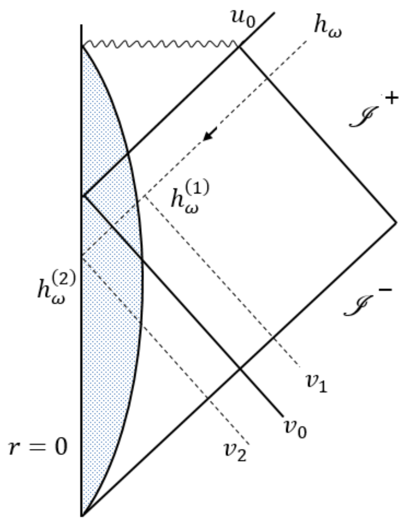


Figure 4.4: Modes $h_\omega$ ending on $\mathscr{I}^+$ propagated backwards can be expressed as a linear combination of modes $h_\omega^{(1)}$ and $h_\omega^{(2)}$.

We encode this by writing $h_\omega = h_\omega^{(1)} + h_\omega^{(2)}$, thus resulting in

$$(h_\omega, h_{\omega'}) = (h_\omega^{(1)}, h_{\omega'}^{(1)}) + (h_\omega^{(2)}, h_{\omega'}^{(2)}) \tag{4.1.57}$$

figure 4.4 gives a representation of the composition of $h_\omega$. From (4.1.25) and the normalised properties of the modes we get that

$$(h_\omega^{(1)}, h_{\omega'}^{(1)}) = \Gamma(\omega)\delta_{\omega\omega'}, \tag{4.1.58}$$
$$(h_\omega^{(2)}, h_{\omega'}^{(2)}) = (1 - \Gamma(\omega))\,\delta_{\omega\omega'}. \tag{4.1.59}$$

Where $\Gamma(\omega)$ represents the fraction of the packet of frequency $\omega$ that would propagate backwards from $\mathscr{I}^+$ through the collapsing body back to $\mathscr{I}^-$. Using the fact that

$$(h_\omega, h_{\omega'}) = \int d\omega' \left(\alpha_{\omega\omega'}\alpha^*_{\omega\omega'} - \beta_{\omega\omega'}\beta_{\omega\omega'}\right), \tag{4.1.60}$$

then

$$\begin{aligned}
\Gamma(\omega)\delta(\omega - \omega') &= \int d\tilde{\omega} \left(\alpha_{\omega\tilde{\omega}}\alpha^*_{\omega'\tilde{\omega}} - \beta_{\omega\tilde{\omega}}\beta_{\omega'\tilde{\omega}}\right) && (4.1.61)\\
&= \int d\tilde{\omega} \left(|\alpha_{\omega\tilde{\omega}}|^2 - \beta_{\omega\tilde{\omega}}|^2\right) && (4.1.62)\\
&= \int d\tilde{\omega} \left(e^{8\pi GM\omega}|\beta_{\omega\tilde{\omega}}|^2 - \beta_{\omega\tilde{\omega}}|^2\right) && (4.1.63)\\
&= \left(e^{8\pi GM\omega} - 1\right) \int d\tilde{\omega} |\beta_{\omega\tilde{\omega}}|^2, && (4.1.64)
\end{aligned}$$

where we used the relation given by (4.1.53) to eliminate $\alpha_{\omega\tilde{\omega}}$. Lastly we can replace the delta function with

$$\delta(\omega-\omega') = \lim_{T\to\infty}\frac{1}{2\pi}\int_{T/2}^{T/2} dte^{i(\omega-\omega')t}, \tag{4.1.65}$$

to finally get

$$\lim_{T\to\infty}\frac{1}{2\pi}\Gamma(\omega) = \left(e^{8\pi GM\omega}-1\right)\int d\tilde{\omega}|\beta_{\omega\tilde{\omega}}|^2. \tag{4.1.66}$$

To find the spectrum of emitted particles we calculate the expectation value of the number operator $\hat{n}_\omega$. Since the information about the created particles should be contained in $b_\omega$, the number operator is just $\hat{n}_\omega = b_\omega b_\omega^\dagger$ and so

$$\langle 0|b_\omega b_\omega^\dagger|0\rangle = \int d\omega'|\beta_{\omega\omega'}|^2. \tag{4.1.67}$$

Combining this with (4.1.66) we get

$$\langle 0|b_\omega b_\omega^\dagger|0\rangle = \lim_{T\to\infty}\frac{1}{2\pi}\Gamma(\omega)\frac{1}{e^{8\pi GM\omega}-1}. \tag{4.1.68}$$

At late times, this tell us what is the number of created particles per unit angular frequency and per unit time that passes through a surface $r = R$, with $R >> r_s$

$$\frac{\Gamma(\omega)}{2\pi}\frac{1}{e^{8\pi GM\omega}-1}. \tag{4.1.69}$$

As stated above, the quantity $\Gamma(\omega)$ is the fraction of a purely outgoing wave packet that when propagated from $\mathscr{I}^+$ backward in time would enter the collapsing body just before the black hole had formed. At late times this fraction is the same as the fraction of the wave packet that would enter the black hole. So $\Gamma(\omega)$ is interpreted as the probability that a purely incoming wave packet starting from $\mathscr{I}^-$ at late times will be 'absorbed' by the black hole. And so, by noting that

$$\frac{1}{e^{2\pi\omega/k}-1} \tag{4.1.70}$$

is the Bose-Einstein distribution with temperature $T = \frac{k}{2\pi}$ one gets that from (4.1.69) the statement that Hawking made about black holes: *A Schwarzschild black hole emits and absorbs radiation exactly like a gray body of absorptivity* $\Gamma(\omega)$ *and temperature* $T$ *given by*

$$kT = \frac{1}{8\pi MG} \tag{4.1.71}$$

$$= \frac{\kappa}{2\pi} \tag{4.1.72}$$

where $k$ is the Boltzmann constant and $\kappa = \frac{1}{4\pi MG}$ is the surface gravity.

## 4.2 Problems with Information in the Hawking Radiation

One can think of the information paradox as a problem about reconstructing an initial state. A classical black hole has no hair, implying that it does not possess any degrees of freedom to store the information about the collapsed matter such that it is available to an outside observer. This means that if you have an initial pure state $|\psi\rangle$ which then collapses to form a black hole, you are insensitive to all the properties of that state except for the mass $M$, angular momentum $J$ and charge $Q$. In other words, all black holes described by a collapsed state with $M$, $J$ and $Q$, irrespective of any other properties of the initial state $|\psi\rangle$, are effectively the same black hole. Now if the black hole evaporates thermally via quantum process and we collect all the Hawking radiation, the question is, how can we construct the initial state $|\psi\rangle$ from this radiation? The radiation is sensitive to the physics near the horizon, but since black holes have no hair, the radiation does not take along with it the initial properties of $|\psi\rangle$, only the mass $M$, angular momentum $J$ and charge $Q$. This is a problem as the black hole evaporation is allowing different physical states to devolve into the same state. In quantum mechanics all information about a physical system at one point in time should determine its state at any other time. But according the what happens classically when a black hole evaporates, the evolution of the state is non-unitary as the information of the state is lost through the evaporation of the black hole.

One does have the option to assume that no information is lost, but then one has to give up on unitarity, something which is fundamental to quantum mechanics. So the information paradox lies in looking at what state the Hawking pairs are in, and by what process does the black hole evaporate.

### 4.2.1 Pure to Mixed

Reintroducing the discussion from section 3.3.1, a vacuum state described by a wavemode which extends on both sides of the event horizon propagates in time, will deform (their oscillations stretch out) drastically as its evolves through spacetime. This extreme distortion of the wavemode gives rise to the creation of particles. Under such distortions the vacuum state changes to a state of the form

$$e^{-\sum_{ij}\gamma\hat{b}_k^\dagger\hat{c}_k^\dagger}|0\rangle \tag{4.2.1}$$

Where $\hat{b}_k^\dagger$ and $\hat{c}_k^\dagger$ are defined as being the creation operators for the wavemode on the time slice $S_{\text{out}}$ and $S_{\text{in}}$ respectively (refer to section 3.3.1). Since the time-slice has all possible wavelengths, some modes of shorter wavelengths will extend further before significant deformation can take place. We can imagine for every wave mode, which eventually gets distorted non-linearly, there is a quanta created $b_1, b_2, b_3..., b_n$ on the outside of the horizon, and $c_1, c_2, c_3..., c_n$ on the inside of the horizon. We can describe the state of every pair production in the following way [38, 22]

$$|\psi_1\rangle = Ce^{\gamma\hat{b}_1^\dagger\hat{c}_1^\dagger}|0\rangle_{b_1}|0\rangle_{c_1}. \tag{4.2.2}$$

Similarly for the second wavemode

$$|\psi_2\rangle = Ce^{\gamma\hat{b}_2^\dagger\hat{c}_2^\dagger}|0\rangle_{b_2}|0\rangle_{c_2}. \tag{4.2.3}$$

The pairs $b_k$ and $c_k$ for different $k$ lie in different regions thus do not overlap. and so, the overall state on the late time slice is the direct product of the states $|\psi\rangle_k$

$$|\psi\rangle = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3 \otimes \ldots \otimes |\psi\rangle_n. \tag{4.2.4}$$

The key idea in this entire discussion -which is at the heart of problem - is the fact that these states $|\psi\rangle_n$ are entangled states, that is, the particle pairs $b_1$, $c_1$ are maximally entangled with each other [22]. We can write some state $|\psi\rangle_1$ as:

$$|\psi\rangle_1 = C\left(|0\rangle_{b_1} \otimes |0\rangle_{c_1} + \gamma \hat{b}_1^\dagger |0\rangle_{b_1} \otimes \hat{c}_1^\dagger |0\rangle_{c_1} + \frac{\gamma^2}{2} \hat{b}_1^\dagger \hat{b}_1^\dagger |0\rangle_{b_1} \otimes \hat{c}_1^\dagger \hat{c}_1^\dagger |0\rangle_{c_1} + \ldots\right) \tag{4.2.5}$$

$$= C\left(|0\rangle_{b_1} \otimes |0\rangle_{c_1} + \gamma |1\rangle_{b_1} \otimes \gamma |1\rangle_{c_1} + \gamma^2 |2\rangle_{b_1} \otimes \gamma^2 |2\rangle_{c_1} + \ldots\right) \tag{4.2.6}$$

The notation $|n\rangle_{b_1} \otimes |n\rangle_{c_1}$ tells us that $n$ quanta of type $b_1$ are entangled with $n$ quanta of type $c_1$ in the state $|\psi\rangle_1$

Evolution of pure states in quantum mechanics must be unitary. This is a fundamental concept in quantum mechanics. This is where the paradox comes to light. If we consider the pure state $|\psi\rangle_1$ it is composed of the entangled pairs $b_1$ outside the horizon and $c_1$ inside the horizon. To restrict ourselves only to the Hawking radiation seen as quanta $b_1$ and refer to it as pure state is not allowed. If we want to study quanta $b_1$ whilst ignoring quanta $c_1$ we have to find the reduced density matrix for quanta $b_1$. Likewise to consider all the various quanta of the Hawking radiation $b_1, b_2 \ldots b_k$ we need to find the reduced density matrices for each quanta $b_k$ . It is important to observe that the density matrix $\rho_i$ describing some quanta $b_i$ is not a pure quantum state, rather a statistical construct, that is, one cannot see the usual quantum principles of linear superposition and phase interference by looking at $\rho_i$. It is only when looking at the entire state composed of the mixed states[2] $b_i$ and $c_i$ that we get a quantum mechanical description of our state $|\psi\rangle_i$. There is nothing inherently wrong with looking at one of the mixed states, as long as we remember that it is the entangled states of $b_i, c_i$ that form the entangled wave function. The problem arises when the black hole evaporates. Once a black hole evaporates, all we are left with are quanta of type $b_i$. In a naive manner, it seems as if the Hawking radiation described by $b_i$ is now a pure state. This cannot be. A mixed state $b_i$ cannot become a pure state. Conversely, a pure state cannot evolve into a mixed state. We cannot talk about the Hawking radiation as being a mixed state, as there are no more quanta $c_i$ for the remaining $b_i$ to be entangled with.

Quantum Mechanics seems to break down under the process of black hole evaporation and this

[2]mixed state or thermal states described by $b_i$ are mixed with states of type $c_i$ as they are entangled with each other

is what led Hawking to postulate that quantum mechanics by itself is not a consistent theory in the presence of gravity.

What if one demands that unitary evolution of these states does exist, and ignores the fact that for information not to be lost one must give up unitarity?. What one finds, as shown in the next section, is that special relativity now seems to come under fire, the equivalence principle is violated.

### 4.2.2 Assuming Unitarity

The problem with Hawking radiation can be stated neatly in the following information loss question: *Does a pure state on $\mathscr{I}^-$ evolve into a mixed state on $\mathscr{I}^+$ ?*. There are two possible answers:

**Yes**: We know that the evolution of a pure state to a mixed state is not allowed, by the rules of quantum mechanics where time development is a unitary process. If indeed we have such a situation occurring you lose the information of the pure state which you started off with as you end up in a thermal state, hence where the idea of information loss lies.

**No**: We have a pure state on $\mathscr{I}^-$ which evolves to a pure state on $\mathscr{I}^+$, thus retaining the rule of quantum mechanics for unitary evolution.

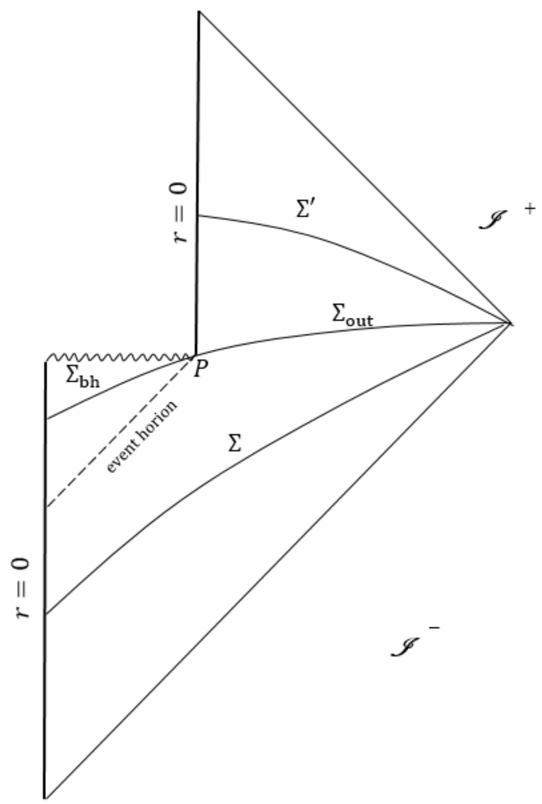


Figure 4.5: The spacetime representation of a black after it has evaporated. One can evolve a spacelike surface which we call a nice slice and evolve it through spacetime. P is the point where the horizon and singularity meet.

For argument's sake, one wishes to enforce that the process by which a black hole evaporates is indeed unitary and thus pick the second response. This has a very important consequence associated with it; namely that we will have to give up the *equivalence principle* as we shall see. The argument below follows from [54] and [49]

Consider figure 4.5. We define an initial pure state $|\Psi(\Sigma)\rangle$ on the Cauchy surface $\Sigma$, this surface is evolved until the singularity is reached, such a surface is denoted $\Sigma_P$. At $P$, $\Sigma_P$ is divided into $\Sigma_{\text{bh}}$ which lies inside the event horizon and $\Sigma_{\text{out}}$ which lies outside the event horizon. The allows us to write the Hilbert space of states on $\Sigma_P$ as a tensor product of Hilbert states on $\Sigma_{\text{bh}}$ and $\Sigma_{\text{out}}$

$$\mathcal{H}_P = \mathcal{H}_{\text{bh}} \otimes \mathcal{H}_{\text{out}}. \tag{4.2.7}$$

By assuming that the evaporation process is unitary, there then exists some unitary scattering matrix $\hat{S}$ such that

$$|\Psi(\Sigma')\rangle = \hat{S}|\Psi(\Sigma)\rangle, \tag{4.2.8}$$

where $|\Psi(\Sigma')\rangle$ is a pure state on the Cauchy surface $\Sigma'$ long after the black hole has evaporated. Note that $|\Psi(\Sigma')\rangle$ had to evolve according to some linear and local evolution from a state $|\Phi(\Sigma_{\text{out}})\rangle$, thus $|\Phi(\Sigma_{\text{out}})\rangle$,which also had to be pure, must have some linear dependence on $|\Psi(\Sigma)\rangle$. This further implying that we can write $|\Psi(\Sigma_P)\rangle$ as a tensor product of states

$$|\Psi(\Sigma_P)\rangle = |\chi(\Sigma_{\text{bh}})\rangle \otimes |\Phi(\Sigma_{\text{out}})\rangle, \tag{4.2.9}$$

where $|\chi(\Sigma_{\text{bh}})\rangle \in \mathcal{H}_{\text{bh}}$ and $|\Phi(\Sigma_{\text{out}})\rangle \in \mathcal{H}_{\text{out}}$. The tensor product itself comes from a local and linear evolution of $|\Psi(\Sigma')\rangle$, but $|\chi(\Sigma_{\text{bh}})\rangle$ cannot depend linearly on $|\Psi(\Sigma)\rangle$ as by our assumption above $|\Phi(\Sigma_{\text{out}})\rangle$ already is solely dependent on it, this brings us to the conclusion that $|\chi(\Sigma_{\text{bh}})\rangle$ has to be independent of the initial state found on the initial slice $\Sigma$.

Another way to look at the situation is the following [58]. Construct a Cauchy surface far from the singularity in a region with low curvature and we choose it such it that crosses most of the outgoing Hawking radiation and also crosses the collapsing body well inside the horizon. Some pure state on this Cauchy surface is thus described by a tensor product

$$|\Psi_{(1)}\rangle = |\phi_{(1)}\rangle_{\text{in}} \otimes |\phi_{(1)}\rangle_{\text{out}} \tag{4.2.10}$$

and another one could be described by

$$|\Psi_{(2)}\rangle = |\phi_{(2)}\rangle_{\text{in}} \otimes |\phi_{(2)}\rangle_{\text{out}}. \tag{4.2.11}$$

Given these two states and the linearity of time evolution in quantum mechanics we could consider a superposition of states

$$\frac{|\Psi_{(1)}\rangle + |\Psi_{(2)}\rangle}{\sqrt{2}} = \frac{|\phi_{(1)}\rangle_{\text{in}} \otimes |\phi_{(1)}\rangle_{\text{out}} + |\phi_{(2)}\rangle_{\text{in}} \otimes |\phi_{(2)}\rangle_{\text{out}}}{\sqrt{2}}. \tag{4.2.12}$$

The only way to make the superposition of states pure is if $|\phi_1\rangle_{\text{in}} = |\phi_2\rangle_{\text{in}}$, then

$$\frac{|\Psi_{(1)}\rangle + |\Psi_{(2)}\rangle}{\sqrt{2}} = \frac{|\phi_{(1)}\rangle_{\text{in}}}{\sqrt{2}} \left(|\phi_{(1)}\rangle_{\text{out}} + |\phi_{(2)}\rangle_{\text{out}}\right) \tag{4.2.13}$$

What this is telling us, is that if the radiation is pure than the body inside the black hole must be unique, hence if information really propagates out encoded in the Hawking radiation, then there must be a mechanism that strips away all information about the collapsing body as the body falls through the horizon long before it reaches the singularity. This is in conflict with the equivalence principle, since for a free falling observer the horizon should not be a special place.

If we are to believe that quantum mechanics must hold true on all these nice slices to allow for the Hawking radiation to remain pure, unitarity as an outside observer is thus in conflict with the equivalence principle. Ideally one wants to keep both unitarity and the equivalence principle, this lead to the idea of *Black Hole Complementarity* as discussed in the next chapter.

# Chapter 5

# Firewall Paradox

The firewall paradox or firewall problem is in a sense a resurrection of the old information paradox albeit in a different form. In short, the information paradox says that if the black hole evaporates via Hawking radiation, all information stored in the black hole disappears. But this violates quantum mechanics. Perhaps the information came back out within the Hawking radiation? The problem is that the information in the black hole can't get out. So the only way it can be in the Hawking radiation (naively) is if what is inside is copied. Having two copies of the information, one inside and one outside, also violates quantum mechanics (the idea of cloning is explained in the no xeroxing principle in section 5.1.1).

The modifications to the current way of thinking came about in the form of *complementarity*. Complementarity suggests that in a way, the information is both inside and outside a black hole, but without allowing any single observer to see both copies. Two different observers, one who remains outside the black hole and one falling into the black hole, will each see the information without ever being able to communicate to each other what they see. To each single observer, no laws of physics appears to be broken, thus no paradox.

The idea of complementarity requires *holography*, an idea developed by 't Hooft [56] and further by Susskind [55]. The holographic principle says that the description of the volume of some space is encoded in its boundary. In the case of the black hole, holography says that the black hole can be replaced by a two dimensional horizon described by two dimensional equations without the presence of gravity. This idea was further backed up to some extent in 1997 when Maldacena conjectured that, under the right circumstances, string theory (a candidate theory of a quantum generalization of general relativity) is equivalent to a quantum field without gravity. This relationship is known as the

"AdS/CFT" correspondence. AdS/CFT showed small black holes can form and evaporate in string theory via a process that can be described by the corresponding quantum field theory (although not very explicitly). This process described was in agreement with all other known processes in any quantum theory and thus preserved information.

One issue with complementarity lies in its lack of equations to accurately describe the black hole evaporation. When Almheiri, Marolf, Polchinski and Sully attempted to find such equations, they instead saw a subtle self-contradiction in complementarity. The argument involves thought experiments whereby one considers *old black holes* and takes into account the entanglement of the radiation at various stages of the lifetime of a black hole. They then showed that given three bits (one belonging to the *early radiation*, *late radiation* and a partner pair of the *late radiation*) of information, they all required to be *maximally entangled* with one another which violates the monogamy of quantum entanglement. Once again, to keep quantum theory, one has to make a drastic change to general relativity: give up the equivalence principle and put a firewall at the horizon which 'burns' everything up before it enters the black hole.

We explain the ideas like the no *xeroxing principle* which led to the development of complementarity by Susskind and his younger co-workers. The postulates as presented for complementarity in [1] are given with a brief motivation. We then show how the authors of the AMPS paper devised a thought experiment highlighting a contradiction between the postulates, leading to a firewall.

## 5.1 Black Hole Complementarity

As shown in the previous section, the Hawking radiation either results in a loss of information or, if one believes that the usual laws of quantum mechanics must hold on nice slices, then unitarity is in conflict with the equivalence principle. So the people who first began to think about complementarity liked both principles, they wanted to keep unitary evolution as seen from an outside observer and they wanted the equivalence principle to hold as seen by a freely falling observer. The ideas of complementarity can be formed by first looking at the no xeroxing principle. This states that a black hole acts as a quantum xeroxing machine which copies information.

### 5.1.1 No Xeroxing Principle

We consider the evolution of a quantum state in the presence of a black hole illustrated below:

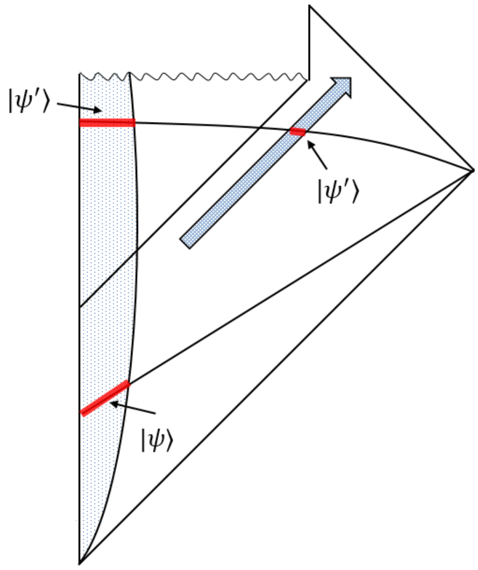


Figure 5.1: A black hole forms by the collapse of an object which starts off in the pure state $|\psi\rangle$. After the black hole has evaporated, the Hawking cloud (the arrow) is in the quantum state $|\psi\rangle$. Note, we adopt the Heisenberg picture so $|\psi\rangle$ evolves into $|\psi\rangle$. The lines crossing the diagram are two time slices.

At some early time when the star is about to collapse, it is in a pure state $|\psi\rangle$. At a later time $|\psi\rangle$ evolves into $|\psi'\rangle$ (we adopt the Heisenberg picture). The arrow represents the Hawking cloud moving out towards infinity. By unitarity, the Hawking cloud is also $|\psi'\rangle$. At a space-like related distance inside the black hole, there still exists the star which continues to collapse and by the equivalence principle, the collapsing star remains in the state $|\psi\rangle$. So we see there are now two states $|\psi'\rangle$, one in the Hawking cloud and one in the collapsing start inside the black hole. Since it didn't matter what $|\psi'\rangle$ was, we get that the black hole behaved like a quantum xeroxing machine:

$$|\psi\rangle \to |\psi\rangle \otimes |\psi\rangle. \tag{5.1.1}$$

This violates the linearity of quantum mechanics. So no observer is allowed to see both copies. In order to see both copies, one has to find a past light cone in figure 5.1 that contains both copies. Susskind and Thorlacius [54] tried to find the best possible strategy to see both copies. An early observer called Alice, falls in the star and carries a bit along with her. As soon as she crosses the horizon, she tries to send this bit over to an observer who is sitting just above the horizon, who we call Bob. If Bob remains above the horizon, he will obviously never be able to see Alice's bit. Bob wants to stay outside long enough to see the Hawking radiation copy of that bit, and then immediately jumps into the black hole to receive the bit that Alice sent. He then wants to verify that both copies of the bit are indeed the same, and see if quantum xeroxing has taken place.

They showed that this does not work. This is due to Page's result which states that you have to wait half the evaporation time of the black hole. This is quite general to entangled systems, as in a maximally entangled system, if you look at a subsystem which is less than half of the original system, it will look like it is in a thermal density matrix. You get no information, even if the whole system is in a pure state. You have to look at a little bit more than half of the subsystem to get your first bit of information out. Assuming you have enough control over the measurement and you know the $S$ matrix of the black hole evaporation, you can arrange for that first bit to be that of Alice's. So Bob has to wait roughly for half of the evaporation of the black hole to measure the Hawking radiation version of Alice's bit, then jump in and try to see the other copy. He fails to do so by a lot [54, 8]. Bob pretty soon hits the singularity no matter what he does, and he has to receive Alice's bit before that happens. Due to the enormous exponential redshift between Alice's in fall time and Bob's in fall time near the horizon. Alice has to send her signal with an exponentially large frequency, that is a quantum bit, whose mass is exponentially larger than the black hole - this is obviously impossible. It is interesting that one misses by a large amount to see both copies.

In 2007, Hayden and Preskill [29] came up with a second strategy where one barely fails to see both copies. Basically, they describe that you have an old black hole and Bob has control over the early Hawking radiation. The radiation is maximally entangled with the black hole. Alice then drops the bit in, and the black hole returns the bit immediately via the Hawking radiation. The time the black hole takes to send out the bit is limited only by the time scale for *scrambling* which, via speculative arguments, shows it to be $O(R \log R)$. This time, Bob only just fails to see both copies

So, at first it seems that we have to choose between two principles: either unitarity or the equivalence principle. What these thought experiments show is that there is another hidden assumption which we could give up instead - *omniscience* - the idea that there has to be a consistent description of the entire global spacetime. If one insists on omniscience, then yes, there are two copies made from one. But if we ask only what any one observer can see, well any one observer does not see quantum xeroxing. And so the idea of complementariy is formed where we sacrifice omniscience in order to preserve unitarity and the equivalence principle.

### 5.1.2 Postulates of Complementarity

The postulates that was originally put forth by [54] go as follows:

**Postulate 1** *From the viewpoint of a distant observer, black hole evaporation is unitary*

What this means is the mechanism of black hole formation and evaporation can be described using standard quantum theory. In particular, there exists a unitary $S$ matrix which describes the evolution from infalling matter to outgoing Hawking radiation. That is we assume that the we have a situation whereby we have a pure state evolving to a pure state.

**Postulate 2** *Outside the horizon, physics can be described, to good approximation, by a set of semi classical field equations which are those of a low energy effective field theory.*

In a semi classical sense, the horizon is a region of low curvature. Thus one can expand the field in terms of creation and annihilation operators and one can apply the normal rules of general relativity.

**Postulate 3** *To a distant observer, a black hole appears as a quantum system whose dimension of states is the exponential of the Bekenstein entropy $S_{BH}(M)$.*

**Postulate 4** *An infalling observer sees 'no drama' at the event horizon.*

Although this was not one of the postulates in the original paper of complementarity, they do state that, with certainty, an infalling observer experiences nothing unusual as per the equivalence principle [1, 54]. In classical general relativity, an infalling observer, apart from visual differences, will experience what another observer experiences floating in empty space with no gravity present.

The claim of complementarity is that there is no experiment that you can perform which will land you in a contradiction with anyone of the postulates. In other words, their point was that we give up the fact that you can do quantum mechanics on nice slices extending across the event horizon, but you can do quantum mechanics sitting outside. The thinking behind this was that it is very difficult to communicate between the inside and outside regions of a black hole. The obvious catch is that although you cannot send signals from the inside to the outside, one can send signals from the outside to the inside.

A motivation on why we have to give up the existence of one nice slice describing both the inside and the outside regions of a black hole is because, if you believe that information comes out, then the state on the outside already encodes for the state on the inside. In other words, no matter what it is you throw into the black hole, you will know about it from the Hawking radiation in the outside region. This is why one doesn't want to be doing usual quantum mechanics on this extended nice slice as you would then, in a sense, be keeping double the information. Instead, you could just be doing quantum mechanics on the outside region or quantum mechanics on the inside region (although it would be a little less precise due to the presence of the singularity). You just can't do both together as this would allow for a cloning of information which is not allowed by rules of quantum mechanics via simple linearity arguments. Essentially, the complementarity point of view is that one can allow this cloning to actually happen as long as the inside region cannot talk to the outside region, and so any one observer won't see quantum xeroxing.

Complementarity thus says that we have to give up the idea that there is an omniscient observer who has a consistent description of the entire global spacetime. It relies on the fact that a fundamental description of nature need only describe experiments that are consistent with causality. In other words, if you consider a causal diamond - construct a past light cone of that end point of some worldline and the future light cone of the starting point - this denotes a region which is in some sense maximal. You cannot probe any region that is larger than the causal diamond. You do, however, require consistent a theory for every such causal diamond, but you don't necessarily need to have a theory for overlapping or multiple causal diamonds, as this is asking for consistency for experiments that no one can actually perform. So, if you have a contradiction between different causal diamonds, say one observer says the information is inside the black hole and another observer says it is in the Hawking radiation, it doesn't actually matter, as they cannot argue with each other. As long as each observer has their own consistent theory, then there is no problem.

We see how complementarity now resolves the issue raised by the Hawking radiation as stated in the previous section. We are able to assume that there is indeed a unitary evolution operator that describes the evaporation of the black hole. We give up on this idea of globally consistent physics over a larger area than any single observers causal diamond. This then means both the outside observer and the infalling observer will have consistent physics in their causal regions. Black hole complementarity was thought to have been the final resolution to these paradoxes and many related ones, until a subtle thought experiment by the team known as 'AMPS' raised an issue which contradicts the postulates put forth by black hole complementarity. This led to the firewall paradox, or the firewall problem which we shall investigate in the next section.

## 5.2 AMPS Argument

The AMPS argument is based on a *Gedankenexperimen*, or thought experiment. This experiment reveals the conflict between the postulates of complementarity. Before we can get into the thought experiment and discuss what the conflict is, it is important to clarify certain concepts and notation that will be used as they are rather subtle. The main ideas is that of an old black hole.

### 5.2.1 Old Black Holes

The idea that nothing can ever escape has been shown to be incorrect. Information can come out of a black hole, be it in a very jumbled state, but it still comes out. As the information comes out, the black hole evaporates slowly, thus decreasing in entropy. According to the generalized second law of black hole mechanics, as the entropy of the black hole decreases, the entropy of the external radiation must increase. This suggests that the information contained in the radiation must increase. One then might ask, how long does one have to wait to retrieve the information of something which entered the black hole? The answer to the question depends on when did that 'something' enter the black hole. Page showed that when a black hole evaporates half of its original mass, or alternatively, when a black hole has lost half of its original entropy, information begins to 'leak' out. We present the Page model which gives this description of the Page time - the point of half entropy - and how this implies that after the Page time, one does not have to wait too long to get the information of that 'something' which was thrown in. Black holes which have reached the Page time and or surpassed it are referred to as *old black holes*.

#### 5.2.1.1 Page Model

The Page time is the point whereby a black hole has emitted half of its initial entropy. This has important consequences in dealing with information within a black hole. The Page model was created with the idea that unitarity is conserved.

Imagine we have a black hole which is emitting radiation, we can then divide our world into that of a black hole and the emitted Hawking radiation.

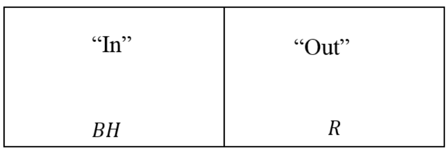


Figure 5.2: The division of our world which is composed of two subsystems that of the black hole, $BH$, and that of the emitted radiation, $R$. The size of these subsystems can vary.

If we want unitarity to hold, then there is some pure state $|\psi_{\mathrm{BH,R}}\rangle$ ( 'BH' for 'Black Hole' and

'R' for 'Radiation') which describes this system. There will also be a unitary operator, $\hat{U}$, such that it evolves $|\psi_{\mathrm{BH,R}}\rangle$ to a very complicated state with a basis of our choice

$$\hat{U}|\Psi_{BH,R}\rangle = \sum_{m,n} A_{mn}|BH\rangle_m|R\rangle_n. \tag{5.2.1}$$

We can construct a density matrix $\rho$ describing this state. In some basis, $\rho$ will have a single entry since it is a pure state as discussed in section 2.2.1. There also exists some complicated unitary transformation $\hat{\mathcal{U}}\rho\mathcal{U}^\dagger$ that diagonalizes $\rho$.

The idea that Page had was to treat this matrix $\hat{\mathcal{U}}$ statistically, as we don't know what is the evolution that propagates the wavefunction since it is very complicated. We therefore treat $\hat{\mathcal{U}}$ as a random unitary matrix[1] and discuss its statistical properties. We then can ask what the properties of the density matrices are in the complicated basis, enabling us to learn important things about the wavefunction like whether or not the radiation *purifies*.

The simplest way to characterize the division of the space is by making use of the relative sizes of the Hilbert spaces. When the black hole evaporates, its entropy decreases. We use this as a measure of the size of its Hilbert space ($\mathcal{H}_{BH}$). Similarly, as this happens, the Hilbert space of the radiation increases ($\mathcal{H}_R$).

As an outside observer, the idea is that we only have access to the information of the radiation and so by the rules of quantum mechanics, we have to find the reduced density matrix $\rho_R$ and study its statistical properties. Let

$$\rho = |\Psi_{BH,R}\rangle\langle\Psi_{BH,R}|, \tag{5.2.2}$$

$$= \sum_{i,j,A,B} \rho_{(A,i,B,j)}|A,i\rangle\langle B,j|. \tag{5.2.3}$$

The size of the Hilbert space $\mathcal{H}_{BH}$ and $\mathcal{H}_R$ is given by $n$ and $m$ respectively, where

$$i,j = 1\dots n\ ; \quad n = e^{S_{BH}} = |\mathcal{H}_{BH}|,$$
$$A,B = 1\dots m\ ; \quad m = e^{S_R} = |\mathcal{H}_R|.$$

We consider $m,n >> 1$, they do not each have to be of similar size. In this case we have

$$\rho_{(A,i,B,j)} = V_{A,i}V^*_{B,j} \tag{5.2.4}$$

$\langle A,i|\Psi_{BH,R}\rangle$ will be the coefficients in the basis $m,n$ in the wavefunction. We also have the normalization $\sum V_{A,i}V^*_{A,i} = 1 = \sum |V_{A,i}|^2$. The next step is to find the reduced density matrix describing the radiation, which is done as prescribed in section 2.2.2.3.

$$\rho_R = \mathrm{Tr}_{BH}\hat{\rho} = \sum_i V_{A,i}V^*_{B,j}. \tag{5.2.5}$$

[1]A random matrix is a matrix where each entry is some random variable which generally has a measure attached to it. The unitary part adds the constraint that the random matrices need to also be unitary

The following ideas were originally conceived by Lubkin [36]. We treat $V_{A,i}$ statistically. The $V_{a,i}$'s are, in this description, random vectors of unit size; in other words, random vectors on a sphere. Lubkin took the simplest probability measure on the sphere to be uniform with respect to the *Haar measure*. Taking the average over a sphere, one normalizes $\int dv = 1$, this gives

$$\int dv V_{A,i} V_{B,j}^{*} = \frac{1}{\mathcal{N}} \delta_{AB} \delta_{ij}, \tag{5.2.6}$$

where $\mathcal{N}$ is some normalization factor. Essentially, this is telling you that the average over specific eigenvalues squared is non-vanishing and the averaging of different matrix elements is zero as they are randomly distributed. We can use this to normalize the density matrix in which we would like that $\mathrm{Tr}\rho_R = 1$. This means

$$\int dv V_{A,i} V_{B,j}^{*} = 1 \Rightarrow \frac{1}{\mathcal{N}} \underbrace{\sum_A \delta_{AA}}_{m} \overbrace{\sum_j \delta_{jj}}^{n} = 1. \tag{5.2.7}$$

Thus, we have that the normalization constant

$$\mathcal{N} = mn. \tag{5.2.8}$$

We now can calculate the purity of $\rho_R$ to find what Page's results were. Purity in quantum mechanics is simply the measure of how pure a quantum state is. We define it by taking the trace squared of the density matrix describing the state. Thus, in our case we calculate

$$\begin{aligned}
\mathrm{Tr}\rho_R^2 &\simeq \int dv \sum_{i,j,A,B} V_{A,i} V_{i,B}^{*} V_{B,j} V_{j,A}^{*} && (5.2.9)\\
&\simeq \frac{1}{(mn)^2} \sum_{i,j,A,B} \delta_{ij}\delta_{ji}\delta_{AA}\delta_{BB} + \delta_{ih}\delta_{ji}\delta_{AB}\delta_{BA} && (5.2.10)\\
&= \frac{1}{(mn)^2}(m^2 n + n^2 m) && (5.2.11)\\
\mathrm{Tr}\rho_R^2 &= \frac{m+n}{mn}. && (5.2.12)
\end{aligned}$$

Thus, the purity $P_R$ can be written in terms of the size of $n, m$

$$P_R^2 = \frac{e^{S_R} + e^{S_{BH}}}{e^{S_R} e^{S_{BH}}}. \tag{5.2.13}$$

This is essentially Page's result. Initially $e^{S_{BH}} >> e^{S_R}$ which gives the purity to be

$$P_R \sim e^{-S_R} \tag{5.2.14}$$

The purity starts at one, telling us that the system begins in a pure state and as $S_R$ gets bigger the purity becomes a very small number. Then after a long time when $e^{S_{BH}} << e^{S_R}$

$$P_R \sim e^{-S_{BH}}, \tag{5.2.15}$$

but this is decreasing as $S_{BH}$ is getting smaller. Formally, when $S_{BH} = 0$ the purity returns to one. This is telling us that, eventually, the radiation is pure.

Using $N_R$ as the average number of emitted particles, we know that $S_R \sim N_R$. Since the entropy of the black hole gets smaller as it evaporates and the entropy of the emitted radiation gets larger, we have $S_{BH} \sim S_{BH}(0) - N_R$. We then define the Page time $N_{\text{Page}}$[2] as the mid point of evaporation

$$S_{BH}(N_{\text{Page}}) = S_R(N_{\text{Page}}). \tag{5.2.16}$$

We can compute the Rényi Entropy $H_2(R) \sim N_R$ which means that the state of the radiation at the page time is roughly thermal. Even though this model is a model of unitary evolution and because we are only looking at part of the system, we have the impression that what we measure contains no information. Drawing a graph to represent the above information gives us the Page curve

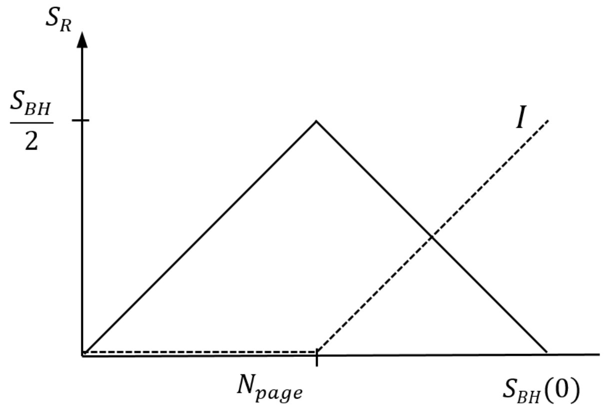


Figure 5.3: The page curve which shows that when the black hole has evaporated half its initial entropy, one reaches the page time. The dotted line represents the information inside the black hole, after the page time, it begins to come out in the form of the emitted Hawking pairs.

Note that the transition is not singular but smooth, it is just exponentially fast so it happens fast. More importantly, the Page time signifies the first time when information begins to exit the black hole, then it goes up to $S_{BH}(0)$ which tells us that we recover all the information that was there. This is also reinforced by the fact that the purity is $P_R = 1$ as shown previously.

The information comes out at a unit rate which was calculated and verified by a thought experiment by Hayden and Preskill [29] who said the following: "at the point that you have an **old black hole**, the information comes out fast with every Hawking photon that is radiated. What we then do is throw in a system of $k$ bits into the black hole and see how fast we can recover the information. It turns out that after $k$ photons that were emitted, you can recover that information. After the page time, black holes behave completely different to what you expect". Old black holes are referred to

[2]One could also define the Page time in terms of an actual time rather than number of particles. Each particle is evaporating on average with Hawking energy equal to the Hawking temperature, so one can convert the particle number to some time.

in [29] as quantum mirrors. Obviously, this assumes that the observer throwing in the $k$ bits has unlimited control over the Hawking radiation. If the observer throws $k$ into a young black hole, they first have to wait for the page time to have been reached, at which point the information will come out almost immediately.

#### 5.2.1.2 Evaporation Time of a Black Hole

It is interesting to see roughly how long it takes for a black hole to evaporate and what the Page time is. We can find this time scale by considering a black hole as a classically thermodynamic object which radiates all its energy via Hawking radiation.

The temperature of a black hole is given by Hawking as

$$T = \frac{\hbar}{8\pi k_B GM}, \tag{5.2.17}$$

and the rate at which an object radiates energy is

$$\frac{dE}{dt} = A\sigma T^4. \tag{5.2.18}$$

$A$ is the surface area of the black hole which has a radius $r = 2GM$ and $\sigma$ is the Stefan Boltzmann constant. In relativistic units, the mass of a black hole $M$ is the energy of the black hole. So we have that

$$\frac{dE}{dt} = -\frac{dM}{dt} \tag{5.2.19}$$

so

$$\frac{dM}{dt} = -A\sigma T^4 \tag{5.2.20}$$

$$= -16\pi^2(2GM)^2\left(\frac{\hbar}{8\pi k_B GM}\right)^4 \tag{5.2.21}$$

$$\int M^2 dM = -\int dt \frac{\sigma\hbar^4}{256\pi^3 k_B^4 G^2}\frac{1}{M^2} \tag{5.2.22}$$

$$\frac{1}{3}M^3 = \frac{\sigma\hbar^4}{256\pi^3 k_B^4 G^2}\frac{1}{M^2}\Delta t_{ev} \tag{5.2.23}$$

$$\Delta t_{ev} = \frac{256\pi^3 k_B^4 G^2}{3\sigma\hbar^4}M^3 \tag{5.2.24}$$

$$\Delta t_{ev} \sim M^3. \tag{5.2.25}$$

So it takes of the order of the mass cubed of the black hole for it to evaporate. Similarly the page time happens at

$$t_{\text{Page}} \sim \frac{1}{2}M^3 \tag{5.2.26}$$

### 5.2.2 Contradiction with Black Hole Complementarity

With the idea of old black holes explained, we can discuss what the contradiction is to complementarity, argued by AMPS [1]. This was done using a simple thought experiment. Consider the figure below

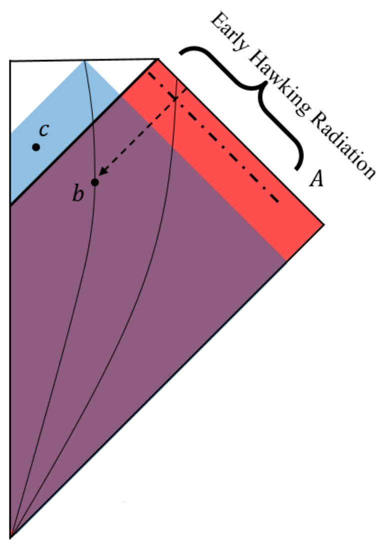


Figure 5.4: The spacetime representing the various causal diamonds of Alice - the infalling observer who passes through point where a $b$ quanta is found. This represents a quanta from the late Hawking radiation. Qunata $c$ is a partner pair of the quanta $b$.

Alice is an observer identified by the thin black worldline which passes through a point where a $b$ quanta is located, her causal diamond is illustrated by the shaded blue region. Bob, an observer who remains outside, has a casual region shaded in red. Their overlapping regions is shaded in purple, and the black dotted lines denote the early Hawking radiation.

One waits for the black hole to evaporate half of its mass, or more technically waits for it to reach its Page time. In doing so, all the early Hawking radiation is maximally entangled with what remains of the black hole, or equivalently is entangled with the remaining radiation that still needs to be emitted from the black - the late Hawking radiation.

Now consider one of the quanta in the late Hawking radiation. If the early radiation is maximally entangled with the late radiation, then there exists some quanta in the early radiation $b_E$ which must be entangled with a quanta[3] $b$. We will want to focus on this quanta $b$. We arrange such that Alice crosses the horizon when $b$ is still close to the horizon (approximately a distance smaller than the Schwarzschild radius from the horizon) such that one can apply flat space intuition on the scale of the system involving the quanta $b$. Alice then asks a question about one of the late quanta that hasn't evaporated by the time she decided to jump in, but which is on its way out to Bob as she crosses the horizon. Is this quanta entangled with any other quanta?

From the first time that the black hole evaporates, a small subspace of Hawking radiation bits will fall back into the black hole. As the black hole evaporates, the number of these early radiation states

[3]The quanta can be thought of as some wave packet of size $O(r_s)$.

increase. If we assume Bob waits long enough such that the black hole has evaporated 2/3 of its mass, we can describe the state of the black hole as [1]:

$$|\psi\rangle = \sum_i |\psi_E\rangle \otimes |\psi_L\rangle. \tag{5.2.27}$$

Since the black hole has been evaporating for a long time, the Hilbert space containing the states which describe the black hole are now primarily composed of $|\psi_E\rangle$ (this is the state composed of all the infalling bits of quanta $c_i$). Hence, the subspace of early radiation is now much larger than that of the late radiation. This allows us to approximate the density matrix describing the late radiation as unity[1]. By unitarity, as particle pair creation takes place, the outgoing quanta $b$ is entangled with some earlier quanta $b_E$ emitted from the black hole a while back - the late radiation is entangled with the early radiation. We therefore have that $b_E$ and $b$ are maximally entangled with each other. But as Alice falls passed the event horizon, by the second and fourth postulate, she finds an answer to her question regarding the bit $b$ of radiation, Alice observes quanta $b$ is maximally entangled with the partner quanta $c$ inside the black hole [1].

There is a problem, between which modes of the various quanta are really entangled. From the law of strong subadditivity briefly introduced in section 2.2.2.3, two systems which are maximally entangled with each other cannot also be maximally entangled with another system (this is another form of monogamy of entanglement), as we shall see.

**Strong subadditivity**

Consider three systems as seen in [1]

$A \Rightarrow$Early radiation (before page time)

$B \Rightarrow$Late radiation (after page time)

$C \Rightarrow$ingoing partners of B

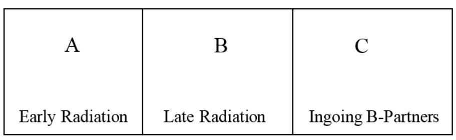


Figure 5.5: The partitioning of an entire black hole system of 3 subsystems.

The strong subadditivity as briefly shown in 2.2.2.3 says that

$$S_{ABC} + S_B \leq S_{AB} + S_{BC},$$

or equivalently by assuming a purifier $D$ of the $ABC$ system

$$S_A + S_C \leq S_{AB} + S_{BC}, \tag{5.2.28}$$

if you consider the entropy of the early and late radiation which is purifying after the page time, that is, its entropy is decreasing, we have that $S_{AB} < S_A$. The entropy $S_{BC}$ is approximately zero as it is a pure state. So we write

$$S_A - S_{AB} + S_C \leq 0. \tag{5.2.29}$$

This is clearly a contradiction as $S_C$ is by itself not in a pure state thus $S_C \neq 0$. What the strong subaddativity is saying in terms of inequalities is exactly the same argument about how the outgoing radiation quanta have to be both entangled with the ingoing quanta, to satisfy equivalence principal, and must also be entangled with the early radiation, to satisfy unitarity.

It seems that one must give up either:

1. **Unitarity**: Which stated that $b$ is maximally entangled with early Hawking radiation.
2. **QFT**: Evolution of the mode $b$ from a created pair at the event horizon.
3. **Equivalence Principle**: Entanglement of $b$ with $c$.

Option two makes the least sense, as everything is on scales greater then the Plank scale. AMPS argued that option three is the most conservative choice [1]. This is still a radical conclusion. The lack of short distance quanta $b$ with quanta $c$ inside the horizon implies a divergent stress tensor at the stretched horizon - a firewall. What AMPS argues is that Alice will not actually cross the event horizon in a peaceful manner, but rather, she will hit a firewall, burn up and never enter the black hole.

### 5.2.3 The Firewall

One may wonder why the pairs of the late radiation and the ingoing modes[4] have to be entangled at all What if $b$ and $c$ weren't entangle? What would go wrong? We can understand the correlation of modes at the horizon by the following picture. Imagine two people, one waving his hands and the other sitting up and sitting down. You then tell a third person to correlate his behaviour with either of the two other people. The third person can choose to either wave their hands or sit up and down but he can't follow them both. If you believe that black holes are unitary in their evolution, then $b$ and $b_E$ are correlated. This implies $b$ and $c$ cannot be correlated, so why does this lead to a paradox? $b$ and $c$ do not have to, by all certainty, be correlated but if you believe in unitarity then, with full certainty, $b$ and $b_E$ are definitely correlated. This is the only way one can get the entropy to go down

[4]Depending on the context, one interchanges between the terms bits/quanta/modes. They however all maintain the same meaning.

to zero. So the question arises, is there still a problem if $b$ and $c$ are not correlated? This seems to go against why there was any paradox in the first place.

Consider a free system. According to the firewall, there should be an energy divergence at the horizon, so the energy for a free field looks like

$$H = \int d^3x \left[ \frac{\pi^2}{2} - \frac{\nabla\phi \cdot \nabla\phi}{2} + m^2\phi^2 \right]. \tag{5.2.30}$$

For now, imagine we have a wavefuntion which looks like

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle \right), \tag{5.2.31}$$

then study a few correlation functions. We can think about the following operator $O_z^{(1)}$ which acts only on particle 1

$$(O_z^{(1)} \otimes 1)|\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle \right), \tag{5.2.32}$$

taking the overlap with $\psi$

$$\langle\psi|(O_z^{(1)} \otimes 1)|\psi\rangle = \frac{1}{2}(1 - 0 + 0 - 1) = 0. \tag{5.2.33}$$

Similarly,

$$\langle\psi|\mathbb{1}) \otimes (O_z^{(2)}|\psi\rangle = 0. \tag{5.2.34}$$

If instead, we do the following

$$\langle\psi|(O_z^{(1)} \otimes O_z^{(2)})|\psi\rangle = \frac{1}{2}(1 - 0 - 0 + 1) = 1, \tag{5.2.35}$$

now, we can write the above expectation value as

$$\langle\psi|(O_z^{(1)} \otimes \mathbb{1} \cdot \mathbb{1} \otimes O_z^{(2)})|\psi\rangle \neq 1. \tag{5.2.36}$$

The expectation value of each operator individually is zero, but when put together one gets a non-zero answer. What we see happening is that $O_z^{(1)}$ and $O_z^{(2)}$ are correlated. When one was positive, the other one was positive. When one was negative, the other one was also negative. Before in (5.2.33) and (5.2.34) when we summed all the positives and negatives and we got zero. Now, because both operators are both positive and negative at the same spots. when we sum things up it is not zero. When these kinds of situations arise, in which the one observable expectation value is zero and the second observable is also zero, but the product of expectation values is not zero, then we know that the observables are correlated [15].

Based on this, we look at the following correlation function:

$$\langle 0|\phi(0,x)\phi(0,y)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)}. \tag{5.2.37}$$

This is telling us that the quantum vacuum for the scalar field is highly entangled. When we let $x \to y$, the expression goes to infinity so it is infinitely correlated, which makes sense as we can see by looking at the energy expression. If one wants to calculate $\nabla\phi$, we have

$$\nabla\phi = \frac{\phi(x+\epsilon) - \phi(x)}{|\epsilon|}. \tag{5.2.38}$$

As we let $\epsilon$ tend to zero, it blows up unless the value of $\phi(x)$ becomes exactly the value of $\phi(x+\epsilon)$. So at very short distances, we require the fields to behave in exactly the same way. This tells you that you need infinite correlation.

This is the situation with modes at the horizon. We have the horizon with modes $b$ on the inside and modes $c$ on the outside. These two modes are not behaving in the same way as $b$ is already entangled with the early radiation due to unitarity, thus by the arguments above, the energy blows to infinity at the horizon - the firewall.

# Chapter 6

# Acoustic Black Holes

Due to the extreme nature of black holes and the uncertainties surrounding their interior, it becomes a problematic task to tackle the firewall problem head on. An alternative approach, is to study black hole analogues in a regime which is well understood. One such regime is fluid mechanics. In general, classical mechanics is a rigorous subject with strong foundations in mathematics, particularly differential geometry and topology. Almost every problem is well understood, thus to pose the question if one can study a black hole analogue using fluid mechanics, is a well worth problem to pursue.

The modeling of a black hole has been discussed in a paper by Visser [62] who elaborated on Unruh's idea, that it is indeed possible to theoretically set up a conformal analogue to a black hole - called *dumb holes* or acoustic black holes. By studying these analogues, we are able to see what physics is intrinsic to the black hole object and what is a consequence of the limitations of the theory used. The remarkable result with dumb holes, is that the phenomena of Hawking radiation is also predicted, where in the acoustic analogue, the radiation is composed of emitted phonons (acoustic disturbance particles).

Unruh proposes a way to study high frequency behaviour of the Hawking modes at the creation of the black hole. This is of particular interest as if one traces back the emitted thermal radiation to the black hole, these modes have such frequencies, that the corresponding energy of the modes would represent something more massive than the entire Universe [59]. This is obviously absurd. The purpose of using fluids is that although the theory is complete, when a fluid is moving at sonic velocities the propagation of sound waves can be described in the same way as the propagation of scalar waves[60]. In fact we study what the ingoing modes and outgoing modes of the radiation would look like if we construct a phonon scalar field around a dumb hole.

The ultimate purpose of looking for a black hole analogue which emits Hawking evaporation is to try and perform experiments where one can observe and measure something like the evaporation of a black hole or the Hawking radiation. In doing so, find clues that will lead to progress in the firewall paradox for the gravitational case. Jacobson also discusses black hole analogues not just in fluids but in condensed matter physics[33][30]. Both him and Koike propose a way of setting up a black hole on a thin film of $^3He - A$ [30], although in such a model Hawking radiation is not observed. Gerace and Carusotto [23] theoretically study "flowing super fluid of exciton-polaritons in a one-dimensional semiconductor microcavity". Via computational methods, they find quantitative predictions for observable quantities and point out clear and accessible signatures of analogue Hawking radiation.

For our purposes if Black hole evaporation happens in the same way for dumb holes, then one can ask if the paradoxes like the information loss and the firewall problem arise. If so, what does this mean? Can we learn anything new of the gravitational black holes. Do these paradoxes exist due to inconsistencies in our theory, or do they exist in which ever theoretical framework we use to set up these black holes?

## 6.1 Building a Model

The first step in the analysis is to lay down the ground work introducing certain ideas in fluids. The goal is to find a suitable description for a fluid enabling us to define how acoustic disturbances on some background flow couple to a background metric. By making the acoustic metric resembles the Schwarzschild metric, one can introduce the dumb. We the look at what are the shared properties between black holes.

### 6.1.1 Obtaining a Schwarzschild-Analogue Metric

Before we can set up a metric for which acoustic disturbances couple to, we require a few conditions for the fluid:

- Barotropic (a function of pressure only). This allows us to introduce the enthalpy $h$ and define $\nabla h = \frac{1}{\rho}\nabla p$.
- Inviscid, i.e. no viscosity.
- Irrotational, $\vec{\nabla} \times \vec{v} = 0$, where $\vec{v}$ is the velocity vector field of the flow. This allows us to introduce a velocity potential $\phi$, s.t $\vec{v} = -\nabla\phi$.

In fluid mechanics, homogeneous fluids are described by two equations; the continuity equation and Euler's equation:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{6.1.1}$$

$$p\frac{d\vec{v}}{dt} = \rho\left[\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}\vec{v})\right] = \vec{F}. \tag{6.1.2}$$

Using the assumptions of our fluid Euler's equation (6.1.2) reduces to:

$$-\partial_t \phi + h + \frac{1}{2}(\nabla\phi)^2 + \psi + \Phi = 0, \tag{6.1.3}$$

where $\Phi$ is an external driving force and $\psi$ is the gravitational potential.

For some assumed background with known properties, sound is just small fluctuations in the medium. Hence if we have some background density, pressure and velocity potential $(\rho_0, p_0, \phi_0)$, then fluctuations in such properties will represent these acoustic disturbances. We represent fluctuations as follows:

$$\rho = \rho_0 + \delta\rho_1 + \mathcal{O}(\delta^2)$$
$$p = p_0 + \delta p_1 + \mathcal{O}(\delta^2)$$
$$\phi = \phi_0 + \delta\phi_1 + \mathcal{O}(\delta^2)$$

We then linearize (6.1.1) and (6.1.2) around this assumed background. It is important to note that the exact motion can be described in terms of the variables $\rho, p, \phi$. We are only interested in the disturbances of the average bulk motion of the fluid $(\delta\rho_1, \delta p_1, \delta\phi_1)$. Linearizing the equations as described in [62], we obtain the wave equation:

$$-\partial_t\left(\frac{\partial\rho}{\partial p}\rho_0\left(\partial_t\phi_1+\vec{v}_0\right)\cdot\nabla\phi_1\right)+\vec{\nabla}\cdot\left(\rho_0\nabla\phi_1-\frac{\partial\rho}{\partial p}\rho_0\left(\partial_t\phi_1+\vec{v}_0\right)\cdot\nabla\phi_1\right)\right)=0. \tag{6.1.4}$$

This equation describes the propagation of the linearised scalar potential. Once $\phi_1$ is determined, we can solve for $\rho_1$ and $p_1$, Thus (6.1.4) completely determines the propagation of acoustic disturbances. Using the fact that the speed of sound is defined locally as $c^{-2}=\frac{\partial\rho}{\partial p}$, (6.1.4) simplifies to:

$$0=\left[\partial_t^2\phi_1+\partial_t\left(\vec{v}_0\cdot\nabla\phi_1\right)+\left(\vec{\nabla}\cdot\vec{v}_0\right)\partial_t\phi_1+\left(\vec{\nabla}\cdot\vec{v}_0\right)\left(\vec{v}_0\cdot\nabla\phi_1\right)\right]-c^2\nabla^2\phi_1. \tag{6.1.5}$$

As discussed by Unruh [59], for a slightly different version of the wave equation, $\phi_1$ can be considered as a massless scalar field. Quoting the acoustic metric from Visser [62] we have that:

$$ds^2=\frac{\rho_0}{c}\left[-c^2dt^2+\left(dx^i-v_0^idt\right)\delta_{ij}\left(dx^i-v_0^jdt\right)\right] \tag{6.1.6}$$

Our goal is to make a transformation allowing us to set up the metric in a form where it describes the geometry of a black hole. In other words, we want a metric which looks like the Schwarzschild metric for a gravitational black hole in empty space. Since the Schwarzschild metric admits spherical symmetry, we change to spherical coordinates for our acoustic metric, and make the assumption that our fluid velocity depends only on the radial component.

$$d_x=dr \qquad v_0\hat{r}=v_0 \tag{6.1.7}$$

$$d_y=rd\theta \qquad v_0\hat{\theta}=0 \tag{6.1.8}$$

$$d_z=r\sin\theta d\phi \qquad v_0\hat{\phi}=0 \tag{6.1.9}$$

The line element (6.1.6) becomes:

$$\begin{aligned}
ds^2&=\frac{\rho_0}{c}\left[-c^2dt^2+\delta_{ij}dx^idx^j-\delta_{ij}v_0^idx^idt-\delta_{ij}dx^iv_0^j+\delta_{ij}v_0^iv_0^jdt^2\right]\\
&=\frac{\rho_0}{c}\left[-c^2dt^2+dr^2+r^2d\theta^2\sin^2\theta+r^2d\phi^2-v_0drdt-v_0dtdr+v_0^2dt^2\right]\\
ds^2&=\frac{\rho_0}{c}\left[-(c^2-v_0^2)dt^2-2v_0drdt+dr^2+r^2\left(d\theta^2\sin^2\theta+d\phi^2\right)\right].
\end{aligned} \tag{6.1.10}$$

As it is in this form, one cannot clearly see the link to the Schwarzschild metric. To do this we introduce the Gullstrand–Painlevé metric:

$$ds^2=-\left(c^2-\frac{2GM}{r}\right)dt_{\mathrm{GP}}^2\pm2\sqrt{\frac{2GM}{r}}\,drdt_{\mathrm{GP}}+dr^2+r^2\left(d\theta^2\sin^2\theta+d\phi^2\right). \tag{6.1.11}$$

Comparing (6.1.10) with (6.1.11), it looks like we could can set $v_0 = \sqrt{\frac{2GM}{r}}$. The only problem is this violates the continuity equation, that is, $\vec{\nabla} \cdot \left(\rho\sqrt{\frac{2GM}{r}}\right) \neq 0$. Thus, following the discussion of [62] the closest we are able to get is metric which is conformal to (6.1.11), $\frac{p_0}{c} \propto r^{-\frac{3}{2}}$. However, if one stays in a region close to the event horizon, the conformal factor can simply be taken to be a constant [62]. The metric thus looks like:

$$ds^2 = \frac{\rho_0}{c}\left[-\left(c^2 - \frac{2GM}{r}\right)dt_{\mathrm{GP}}^2 \pm 2\sqrt{\frac{2GM}{r}}\,drdt_{\mathrm{GP}} + dr^2 + r^2\left(d\theta^2\sin^2\theta + d\phi^2\right)\right] \tag{6.1.12}$$

The last step to visualizing the metric in Schwarzschild coordinates, is by setting $c = 1$ and making the following transformation:

$$dt_{\mathrm{GP}} = dt - \frac{\sqrt{\frac{2GM}{r}}}{1 - \frac{2GM}{r}} \tag{6.1.13}$$

Substituing this into (6.1.12) and letting $a = \frac{2GM}{r}$:

$$\begin{aligned} ds^2 &= \rho_0\left[-(1-a)\left(dt + \frac{\sqrt{a}}{1-a}\right)^2 + 2\sqrt{a}dr\left(dt \pm \frac{\sqrt{a}}{1-a}dr\right) + dr^2 + r^2\left(d\theta^2\sin^2\theta + d\phi^2\right)\right] \\ &= \rho_0\left[-(1-a)dt^2 + (1-a)^{-1}dr^2 + r^2 d\Omega^2\right] \end{aligned}$$

We choosing the positive sign in the metric as we chose the coordinate patch which covers the future horizon and acoustic black hole singularity. Thus finally we have:

$$ds^2 = \rho_0\left[-\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2\right] \tag{6.1.14}$$

or, introducing the speed of sound back, and $\sqrt{\frac{2GM}{r}} = v_0$

$$ds^2 = \frac{\rho_0}{c}\left[-\left(c^2 - v_0^2\right)dt^2 + \left(\frac{c^2}{c^2 - v_0^2}\right)dr^2 + r^2 d\Omega^2\right], \tag{6.1.15}$$

as seen in [17].

We have successfully managed to write our acoustic metric in a way which is conformal to the Schwarzschild metric. We now look to define an event horizon and surface gravity. Which becomes important when looking at the Hawking radiation of phonons.

### 6.1.2 Properties of a Dumb Hole

The metric (6.1.14) is conformal to the Gullstrand–Painlevé coordinates form of the Schwarzschild metric. By only considering the static acoustic black hole, we are able to identify an event horizon and a way to define surface gravity, allowing us to check for various black hole phenomena. In the case of a static geometry one finds that the surface gravity in terms of the normal derivative is given according to [62]:

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v^2)}{\partial n} = c\frac{\partial(c - v)}{n} \tag{6.1.16}$$

If we instead now consider a stationary case ,the question then is: how similar is the dumb hole to a stationary black hole? If one looks at the arguments put forth by Visser in [62], the notion of surface gravity may be difficult to formulate. One of the reasons for this is that there is no justification of a zeroth law of black hole mechanics in the acoustic case. The zeroth law in general relativity is proven by making use of Einstein's equation and imposing certain energy condition. Whereas in the acoustic case, the governing equations are the Navier-Stokes equations which are not analogous to Einstein's equation. One therefore cannot apply the same argument for an acoustic black hole, thus there is no reason to suspect a zeroth law. Another issue is that non-axisymmetric black holes will lose their energy then relax to a axisymmetric configuration. In our fluid model the presence of external driving forces deny the need for acoustic black holes to dynamically relax to axisymmetry [62]. Furthermore there is no reason to expect an acoustic event horizon to in general be a Killing horizon.

What does remain from gravitational black holes are the notions of an apparent horizon and an acoustic event horizon(the boundary of all phonon null geodesics that do not go to infinity). It so happens that these meanings coincide for the Schwarzschild metric. The apparent horizon is defined as a two surface which acts as the boundary between an inner and outer trapped surface. Here the normal component of the fluid velocity is everywhere equal to the local speed of sound. From the definition of the apparent horizon, we decompose the velocity close to the horizon into a normal and tangential component [62]:

$$\vec{v} = \vec{v}_\perp + \vec{v}_\parallel \qquad\qquad ; \vec{v}_\parallel = v\hat{n}. \tag{6.1.17}$$

This allows one to introduce the following field

$$L^\mu = (1; v^i_{0\parallel}). \tag{6.1.18}$$

By calculating $(L \cdot \nabla)L$ we see that:

$$L^\alpha \nabla_\alpha L^\mu = \frac{1}{2c}\frac{\partial(c^2 - v_\perp^2)}{\partial n}(1; v^i_{0\parallel}). \tag{6.1.19}$$

Comparing this result with the standard definition of surface gravity: $L^\alpha \nabla_\alpha L^\mu = \frac{g_H}{c} L^\mu$

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v_\perp^2)}{\partial n} = c\frac{\partial(c - v_\perp)}{\partial n}. \tag{6.1.20}$$

This result was seen in [62] and agrees with different calculations given in [59]

## 6.2 Analogue of Hawking Radiation

The existence of Hawking radiation can be established by the same arguments given in the thesis thus far. Unruh [59] states that the propagation of sound waves in a hypersonic fluid flow are exactly the same as the propagation of scalar waves. From this we set out to describe what the phonon scalar field will look like for a dumb hole. Any resemblance to the results given by Hawking [28] further justify the idea that dumb holes radiate via phonon emission. In fact we do not expect to see any real deviation from the gravitational case, the purpose of the calculation is more of a check. What we find is that the expression of the scalar field around the acoustic black hole is the same. Calculations made in [61] agree with these results, although for a more specific case.

### 6.2.1 Phonon Scalar Field

Consider for now the normal Schwarzschild metric.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{6.2.1}$$

This is the empty space spherically symmetric solution to Einstein's equations. We do not need to worry about how the black hole is formed, as we only consider the exterior metric after the collapse. Any vacuum states define in Minkowski spacetime, no longer exist. That is, vacuum states of some observer are not Poincaré invariant in curved space. In a quantum field theory of curved spaces, to find the rate of particle production around a black hole we calculate the Bogoliubov transformations between the in and out modes. This is the same procedure Hawking used in calculating the spectrum of the emitted radiation, which was derived in chapter 4. Similarly, we could calculate what would be the spectrum of emitted particles for an acoustic black hole. The exact expression can be quoted by both [59] and [17] who calculated them using different approaches. In my approach, I would like to follow closely Hawking's derivation and apply as much as possible, the same analysis for the acoustic case.

We imagine that we are able to set up a dumb hole in $3 + 1$ dimensions, where the flow of the fluid is purely radial and exceeds the speed of sound at some point. At such a point we have an acoustic horizon. In the acoustic setting we set the background flow $v_0 = \sqrt{\frac{2GM}{r}}$, where $M$ represents the strength of the 'source' of the acoustic black hole. In empty space the Klein Gordon equation of a massless scalar field becomes:

$$\partial_\mu \partial^\mu \phi = 0$$
$$\Box \phi = 0.$$

In the acoustic scenario, the equations of motion are not determined by the Klein Gordon Equation, but rather the equation of continuity and Euler's equation. As we saw before, the wave equation

(6.1.4), repeated here

$$0=\left[\partial_t^2\phi_1+\partial_t\left(\vec{v}_0\cdot\nabla\phi_1\right)+\left(\vec{\nabla}\cdot\vec{v}_0\right)\partial_t\phi_1+\left(\vec{\nabla}\cdot\vec{v}_0\right)\left(\vec{v}_0\cdot\nabla\phi_1\right)\right]-c^2\nabla^2\phi_1 \tag{6.2.2}$$

describes the what we now call scalar field $\phi_1$ originally describing the acoustic fluctuations in the background field $\phi_0$. By finding a solution for $\phi_1$ we can describe $\phi_1$ using an orthonormal mode expansion in terms of creation and annihilation operators, just like the mode expansion of a scalar field in curved space.

We change the metric to one that is continuous across the event horizon by using Eddington-Finkelstein coordinates.
Let

$$\begin{aligned}\tau=v-r^*; \qquad r^*&=r+2GM\ln\left|\frac{r}{2GM}-1\right|\\ dt=dv-dr^*; \qquad dr^*&=dr\left(1-\frac{2GM}{r}\right)^{-1}\end{aligned}$$

Thus, the metric (6.1.14) becomes:

$$\begin{aligned}ds^2&=\rho_0\left[-\left(1-\frac{2GM}{r}\right)(dv-dr^*)^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\Omega^2\right], && (6.2.3)\\ &=\rho_0\left[-\left(1-\frac{2GM}{r}\right)dv^2+2dvdr+r^2d\Omega^2\right]. && (6.2.4)\\ & && (6.2.5)\end{aligned}$$

Setting $\sqrt{\frac{2GM}{r}}=v_0$ and reintroducing factors of $c$,

$$ds^2=\frac{\rho_0}{c}\left[-\left(c^2-v_0^2\right)dv^2+2dvdr+r^2d\Omega^2\right]. \tag{6.2.6}$$

This result is reproduced by Fang and Zhou [17].

**Solution to the wave equation**

In the gravitational case we solve (4.1.6) to find what the scalar field looks like. Whereas, in the acoustic case we need to solve the wave equation to find a solution for $\phi_1$.
Since we are dealing with a dumb hole which is spherically symmetric. We convert the wave equation into spherical coordinates:

$$\vec{v}_0=v_r\hat{r}+v_\theta\hat{\theta}+v_\varphi\hat{\varphi} \tag{6.2.7}$$

$$\nabla\phi_1=\frac{\partial\phi_1}{\partial r}\hat{r}+\frac{\partial\phi_1}{\partial\theta}\hat{\theta}+\frac{\partial\phi_1}{\partial\varphi}\hat{\varphi} \tag{6.2.8}$$

$$\nabla\cdot\vec{v}_0=\frac{1}{r^2}\frac{\partial\left(r^2v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} \tag{6.2.9}$$

We already required that $\vec{v}_0=\sqrt{\frac{2GM}{r}}$. This means that there is only a radial component for $v_0$ and so $\vec{v}_0=v_r\hat{r}$, where $v_r=\sqrt{\frac{2GM}{r}}$. According to the continuity equation, making such substitution adds the constraint that the background density must become a function of $r$ such that $\rho_0 \propto r^{-\frac{2}{3}}$. This is not an issue as in the wave equation we have eliminate $\rho_0$.

The wave equation now becomes

$$0 = \partial_t^2\phi_1 + \partial_t\left(v_r\frac{\partial\phi_1}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial\left(r^2v_r\right)}{\partial r}\right)\partial_t\phi_1 + \left(\frac{1}{r^2}\frac{\partial\left(r^2v_r\right)}{\partial r}\right)\left(v_r\frac{\partial\phi_1}{\partial r}\right) - c^2\left(\frac{1}{r^2}\frac{\partial(r^2\frac{\partial\phi_1}{\partial r})}{\partial r}\right) \tag{6.2.10}$$

$$0 = \partial_t^2\phi_1 + v_r\partial_t\partial_r\phi_1 + \left(\frac{2v_r}{r} + \partial_r v_r\right)\partial_t\phi_1 + \left(\frac{2v_r^2}{r} + v_r\partial_r v_r - \frac{2}{r}\right)\partial_r\phi_1 - \partial_r^2\phi_1. \tag{6.2.11}$$

Substituting $v_r = \left(\frac{2GM}{r}\right)^{\frac{1}{2}}$, and noting that $\partial_r v_r = -\frac{1}{2}\frac{2GM}{r^2}\left(\frac{2GM}{r}\right)^{-\frac{1}{2}}$, we have:

$$0 = \partial_t^2\phi_1 + \left(\frac{2GM}{r}\right)^{\frac{1}{2}}\partial_t\partial_r\phi_1 + \frac{3}{2r}\left(\frac{2GM}{r}\right)^{\frac{1}{2}}\partial_t\phi_1 + \left(\frac{3GM}{r^2} - \frac{2}{r}\right)\partial_r\phi_1 - \partial_r^2\phi_1. \tag{6.2.12}$$

This is a second order linear differential equation in both $t$ and $r$. We convert this into a time independent differential equation by making the following ansatz: $\phi_1(r,t) = \phi_\omega(r)e^{-i\omega t}$. This will give us an identity which we use to the solve for $\phi_\omega(r)$.

Substituting the ansatz, and noting that $\partial_t^2\phi_1(t,r) = \omega^2\phi_\omega(r)e^{i\omega t}$, and $\partial_t\phi_1(r,t) = -i\omega\phi_\omega(r)e^{i\omega t}$ we get

$$0 = \left[\omega^2\phi_\omega(r) - \left(\frac{2GM}{r}\right)^{\frac{1}{2}} i\omega\partial_r\phi_\omega(r) - \frac{3}{2r}\left(\frac{2GM}{r}\right)^{\frac{1}{2}} i\omega\phi_\omega(r) + \left(\frac{3GM}{r^2} - \frac{2}{r}\right)\partial_r\phi_\omega(r) - \partial_r^2\phi_\omega(r)\right]e^{i\omega t}. \tag{6.2.13}$$

Since $e^{-i\omega t}$ is never equal to zero, we require that the terms inside the bracket equal to zero. Using $\frac{\partial}{\partial r} = \frac{\partial r^*}{\partial r}\frac{\partial}{\partial r^*} = \left(1 - \frac{2GM}{r}\right)^{-1}\frac{\partial}{\partial r^*}$ we are able to express the differential equation in terms of $r^*$. This gives us

$$0 = \omega^2\phi_\omega(r) - \left(\frac{2GM}{r}\right)^{\frac{1}{2}} i\omega\left(1 - \frac{2GM}{r}\right)^{-1}\partial_{r^*}\phi_\omega(r) - \frac{3}{2r}\left(\frac{2GM}{r}\right)^{\frac{1}{2}} i\omega\phi_\omega(r) - \tag{6.2.14}$$

$$\left(\frac{3GM}{r^2} - \frac{2}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1}\partial_{r^*}\phi_\omega(r) + \left(1 - \frac{2GM}{r}\right)^{-2}\partial_{r^*}^2\phi_\omega(r)$$

$$0 = \frac{d^2}{dr^{*2}}\phi_\omega(r) - \left(1 - \frac{2GM}{r}\right)\left[\frac{3GM}{r^2} - \frac{2}{r} + \left(\frac{2GM}{r}\right)^{\frac{1}{2}} i\omega\right]\frac{d}{dr^*}\phi_\omega(r) + \left[\omega^2\left(1 - \frac{2GM}{r}\right) - \frac{3}{2r}\left(\frac{2GM}{r}\right)^{\frac{1}{2}} i\omega\right]$$

$$0 = \frac{d^2}{dr^{*2}}\phi_\omega(r) - P(r)\frac{d}{dr^*}\phi_\omega(r) + \left[\omega^2Q(r) + S(r)\right]\phi_\omega(r) = 0. \tag{6.2.15}$$

By the same arguments used in solving the (4.1.6), we only consider the asymptotic solution $r = \infty$ as this is where an observer would be making the measurements.

In the limit that $r \to \infty$ ; $P(r) = S(r) = 0$ and $Q(r) = 1$, leaving us with the following differential equation:

$$0 = \frac{d^2}{dr^{*2}}\phi_\omega(r) + \omega^2\phi_\omega(r) = 0 \tag{6.2.16}$$

Solutions to (6.2.16) have the form

$$\phi_\omega(r) = Ce^{\pm i\omega r^*} \tag{6.2.17}$$

Thus combining with our ansatz, we get that

$$\phi_1(r,t) = C_1 e^{-i\omega(t+r^*)} \tag{6.2.18}$$

$$\phi_1(r,t) = C_2 e^{-i\omega(t-r^*)} \tag{6.2.19}$$

This is a nice result as comparing this to Hawking's result we defined

$$\phi \sim \frac{e^{-i\omega(t+r^*)}}{r} Y_{\ell m}(\theta,\varphi) \sim A_1 e^{-i\omega(t+r^*)} \sim h(r,\omega) \tag{6.2.20}$$

$$\phi \sim \frac{e^{-i\omega(t-r^*)}}{r} Y_{\ell m}(\theta,\varphi) \sim A_2 e^{-i\omega(t-r^*)} \sim f(r,\omega) \tag{6.2.21}$$

Continuing with similar notation, we define

$$\phi_1(r,t) = C_1 e^{-i\omega(t+r^*)} = h'(r,\omega) \tag{6.2.22}$$

$$\phi_1(r,t) = C_1 e^{-i\omega(t+r^*)} = f'(r,\omega). \tag{6.2.23}$$

There is no angular dependence due to our choice in the flow. By defining $h'(r,\omega)$ as outgoing modes from the dumb hole and $f'(r,\omega)$ as the ingoing modes of the dumb hole. We could then compute the Bogoliubov transformation between the ingoing and outgoing modes. Doing so would be an exercise in repetition as we have the same expression of the basis modes, and all assumption made by Hawking can be continued to the dumb hole. By such arguments we indeed believe that the dumb hole will have a spectrum of emitted particles which is approximately thermal.

### 6.2.2 Black Hole Evaporation and Information Loss

The treatment of the dumb hole has led us to a point, in the semi-qualitative analysis, which is in direct correspondence with the results from section 4.1. The outgoing and ingoing modes of the phonon field have the same form as the scalar field around a black hole. This result is verified in a more explicit manner in [61] but in a slightly different context. [61] obtains the exact wave solution near the acoustic horizon in terms of the confluent Huen functions. Their analysis also results in the conclusion that there exists the emission of Hawking radiation with a thermal spectrum which is in agreement with [59, 62]. Further they find the resulting phononic Hawking radiation of scalar particles is given by

$$|N_\omega|^2 = \frac{1}{e^{\frac{2\pi}{\kappa_0}\omega} - 1} = \frac{1}{e^{\frac{\hbar\omega}{k_B T_0}} - 1}. \tag{6.2.24}$$

This too has a thermal character analogous to the black hole spectrum Hawking found in [28] and which we showed in 4.1 by equation (4.1.69). By this result they show that, both the rotating and canonical black holes, behave not just as thermal bodies but as actual black bodies. In the case of acoustic terminology, "dumb bodies"

After the confirmation of phononic Hawking radiation in a dumb hole, the next question which follows is: are these phonon pairs created at the horizon correlated or not? The reason for this is that it is the correlations between modes which result in the information paradox. As explained in section 4.2, if the pairs are correlated and the black hole evaporated, the remaining radiation has nothing to be correlated with thus there seems to be a loss of information. We stated that one may treat the dumb hole as the same as a black hole case, this would lead to the conclusion that the phonon pairs are indeed correlated, as they are created in a quantum mechanical manner around an event horizon, just as with a black hole. Visser [63] explains however, perhaps the fact that pairs are correlated or not is not as important as distinguishing whether one is dealing with an *event horizon* or an *apparent horizon*

For the gravitational black hole, the information loss problem comes from the fact that correlations between modes are forever 'hidden' behind the event horizon. These modes only disappear once the black hole completely evaporates. In the acoustic case, even though correlations may very well exist, the acoustic horizon is defined as the point where the normal component of the fluid velocity reaches the local speed of sound[62]. This horizon is not required to be a true event horizon, in fact one can make the horizon appear and disappear freely by changing the velocity of the fluid flow. The 'fickleness' of the acoustic horizon means that any correlations that were temporarily hidden are eventually revealed, before the dumb hole actually evaporates.

**Old dumb hole**

As a qualitative check, we can see what order of magnitude is for the evaporation time of an acoustic black hole, consequently what the page time is. From [59] we have that the temperature of the thermal radiation emitted is

$$T = \frac{\hbar}{2\pi k_B} g_H = \frac{\hbar}{2\pi k_B} \frac{dv}{dr}\bigg|_{r=2GM}, \tag{6.2.25}$$

if we use the fact that $v = \sqrt{\frac{2GM}{r}}$ we have that for an acoustic black hole

$$T = \frac{\hbar}{8\pi k_B GM}, \tag{6.2.26}$$

which is the same result as for the gravitational case. Doing the same calculations, the evaporation time for an acoustic black hole is also

$$t_{ev}(\text{Dumb hole}) \sim M^3. \tag{6.2.27}$$

It seems most likely that any acoustic horizon will disappear before the evaporation of any dumb hole takes place. Such black hole systems actually behave like a 'leaky furnace' in that the radiation is approximately that of a Plankian spectrum rather than a thermal spectrum, a subtle yet important distinction[63]. A thermal spectrum (a spectrum whereby modes are uncorrelated) is strongly dependent on the existence of a true event horizon, such that any possible correlation are forever hidden. Systems that have apparent horizons or trapped horizons which display a Plankian spectrum of radiation cannot be thermal, as they do not hide the correlations forever.

Ultimately Visser explains that the presence of apparent horizons does not signify the existence of any information problem. This presents some doubt of whether or not any information problem even exists for dumb holes. Instead there is a lot of reason to believe that normal unitary evolution and standard physics behaves as expected.

## 6.3 An acoustic Firewall

Based on the ideas above, the presence of a firewall does not seem likely. This stems from the fact that there is no information problem. Lets do a recap of how the information problem lead to the firewall and then see how absence of any information problem implies an absence of a firewall. Recall the information paradox led to the conclusion that either unitarity or the equivalence principle is violated as discussed in section 4.2. Black hole complementarity came to the rescue and stated that physics need only be consistent on local causal diamonds, thus no experimental observation can lead to any contradictions. AMPS then argued that, if one assumes unitary evolution, then one is dealing with radiation which is in a thermal state. This radiation then becomes entangled with any radiation which exits after the page time. However, modes which exit after the page time, will themselves be entangled with their partner modes which fall back into the black hole.

In the acoustic analogue where instead of event horizons one has apparent horizon, what breaks down is the fact that the early radiation is not entangled with the late radiation, and if there is no entanglement with early and late radiation then there is no issue of multiple entanglement and thus no firewall[5].

### 6.3.1 Looking for a Firewall

For our purposes, by all the arguments put forth by Hawking and AMPS, we keep the notion that a firewall should exist. According to AMPS [1] for the firewall to exist one needs understand what the physics is at the smallest of scales. As the name suggests, there is something very dramatic happening for an observer at the horizon, namely they burn up. This is coming from the presence of high energy Planck scale modes at the horizon. In the gravity context, we do not have a detailed description of what the microscopic physics is. Since we do not have a complete theory of quantum gravity, other than to say there is a firewall, we do no know any further detail of what the firewall structure looks like. In the context of the black hole analogues, the solution to the effective field theory of hydrodynamics is the dumb hole. Effective field theories, by design, have within them cut-off scales. Such scales may be at the IR(infra red) limit or the UV (ultra violet) limit, in any case the effective field theory is only valid within the cut-off regimes. Hydrodynamics has a cut-off at the inter molecular spacing of the fluid, this is because the Navier-Stokes equation, which is the governing equation, is only valid under the continuum approximation. This states that you can always model the fluid as a continuous body.

Just like at a black hole event horizon, at the sonic horizon there is a huge accumulation of energy. What this is saying is that the effective field theory is breaking down, by consequence, you need new physics to replace the effective field theory of hydrodynamics. One is looking at energy scales, which are sensitive to distances shorter than the inter molecular spacing of the molecules in the fluid. In this regime, hydrodynamics is not a good approximation. The effective theory at this new spacing is the kinetic theory of gasses, which is well understood. And so, if there is indeed a firewall, we are seeing new evidence of microscopic physics, and therefore should find a very strong signal for this. Within the fluid regime, when there is a large concentration of energy in a tiny region of the fluid, we have the initial condition for fully developed turbulence. Thus turbulence is a signal for the break down of the effective field theory, and it is what we first looked for within our model.

#### 6.3.1.1 Turbulent Behaviour at the Horizon

In our model the flow has only radial dependence and far away from the center of the dumbhole, the flow is laminar. If the flow were to become very turbulent at the horizon, one may attribute this to the presence of a firewall. To understand why this is so it is important to go into some detail into what is turbulence, and how one characterizes it.

#### What is Turbulence

There are two types of flows. Laminar flow is when the fluid flows in parallel layers, with no interruption between the layers. Such flows generally occur at low fluid velocities or technically at low *Reynold's number*. The Reynolds number, $Re$, is a dimensionless parameter important in discerning whether certain flow conditions will lead to laminar or turbulent flow. Essentially, laminar flow is ordered flow with no cross currents perpendicular to the direction of the flow. Turbulent flow is chaotic and highly non-linear in its motion, yet it is still described by the Navier-Stokes equation

$$\partial_t v = -(v \cdot \nabla v + \nabla p) + v(\nabla^2 \vec{v}) + f. \tag{6.3.1}$$

One way we compute turbulent flow is according to RANS(Reynolds Averages Navier-Stokes). We do this to find the decomposition of the background flow into some time independent laminar flow and fluctuations of the laminar part, which not need be small [20]:

$$v(\vec{x}) = \bar{v}(\vec{x}) + v'(\vec{x}). \tag{6.3.2}$$

One then would substitute $v(\vec{x})$ into the Navier-Stokes to obtain two new sets of equations: one for the mean flow($\bar{v}(\vec{x})$), and one for the turbulent flow ($v'(\vec{x})$). According to [48] and [57] the characteristic features of turbulent flow are

1 **Three-Dimensional**. The turbulence always presents itself in three dimensional flow. If we perform a time averaging of the equations, the flow can be regarded as two-dimensional assuming the background geometry is two dimensional

2 **Continuum**. All process in the flow must be larger than the inter molecular distances as to adhere to the continuum approximation.

3 **Large Reynolds Numbers**. Turbulent flows occurs for very large values of the Reynolds number. The specific Reynolds number is determined by the system. Two such example can be seen: turbulent flow in pipes occurs at $Re_D \simeq 2300$, and in boundary layers the transition occurs at $Re_x \simeq 500000$.

4 **Dissipation**. This is the manner in which kinetic energy is transferred from small eddies into thermal energy. Small eddies gain their energy from the larger eddies who in turn receive energy from even larger eddies. Conversely large eddies transfer their energy into smaller eddies who then themselves break up into smaller eddies. The process of transferring energy from large eddies to the smallest eddies is known as the *cascade process*.

5 **Irregularity**. This expresses the chaotic nature of turbulent flow. The flow composed of many different scales which we call eddy sizes. A *turbulent eddy* is region in space for a period in time whereby the flow is rotational and is subsequently dissipated via the cascade process. Turbulence has a characteristic velocity and length scales. These are simply called the velocity and length scale. The largest eddies have sizes of the same order as the flow geometry. Viscous forces eventually dissipate the smallest eddies into thermal energy. resulting in a an increase of temperature[1].

6 **Diffusivity**. Turbulence results in an increase of diffusivity, this increase the exchange of momentum in places like boundary layers. The increase of diffusivity also increases friction, subsequently heat transfer for internal flows.

A flow which begins laminar and turns turbulent is indicative that there is some process depositing energy into the system. The more energy deposited the higher the turbulent behaviour. Reverting back slightly, in our model it is the firewall which is depositing all the energy into the fluid, thus we suspect that the flow should become turbulent as one approached the horizon where the firewall should be present. It is important to understand at what scales turbulent behaviour is manifest and at what scales energy dissipation occurs.

#### Turbulence Scale

The largest length scales $\ell_0$ and velocity scale $v_0$ in turbulent flow in general are in the order of the background flow geometry. The kinetic energy of the mean low is removed via the large scales $\ell_0$ and $v_0$. The time scale of the kinetic energy loss is comparable to the large scales, that is

$$\frac{\partial \bar{v}}{\partial x} \sim t_0^{-1} \sim \frac{v_0}{\ell_0} \tag{6.3.3}$$

[1]It is in fact this process of dissipation of the eddies into temperature increases that interests us, as any process which happen on the micro scales of the theory can give us insight in the existence of a firewall or not

Via the cascade process (figure 6.1), energy is transferred from the largest scale to the smallest scale. This is because at the smallest scales of the system, viscous effects become non-negligible and the kinetic energy is transferred into thermal energy. This dissipation which we denote with $\epsilon$ is the energy per unit time per unit mass, and has dimensions $[m^2/s^3]$. The smallest scales at which dissipation occurs are called the Kolmogorov scales, whose velocity scale is denoted by $v_\eta$, length scale by $\ell_\eta$ and time scale by $\tau_\eta$. There are important ratios with specific power laws which relate the energy dissipation with the Kolmogorov scales.Assuming the Kolmogorov scales are only determined by viscosity, $\nu$, and dissipation, $\varepsilon$, one gets

$$v_\eta = (\nu v)^{1/4}; \qquad l_\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}; \qquad \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \tag{6.3.4}$$

Jut like the kinetic energy, the viscosity is too destroyed by viscous forces, so it makes sense in assuming that viscosity is a factor in determining these scales. The greater the dissipation, the more energy there is to be transformed from kinetic energy to thermal energy, thus the larger the velocity gradients must be.

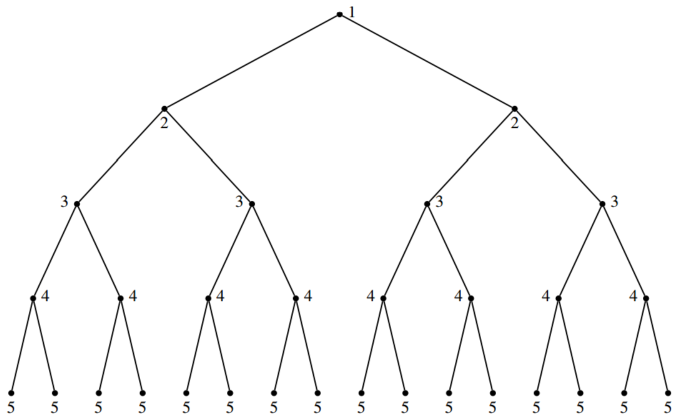


Figure 6.1: Figure according to [14] showing a family tree of turbulent eddies showing five generations. The large original eddy (point 1) is 1st generation [9].

### Energy Spectrum

An important law that describes fully developed turbulence is a statement that the energy spectrum $E(\kappa)$ follows a $\kappa^{-5/3}$ within a suitable range. The larger the Reynolds number, the wider this range. To see this, consider the following diagram

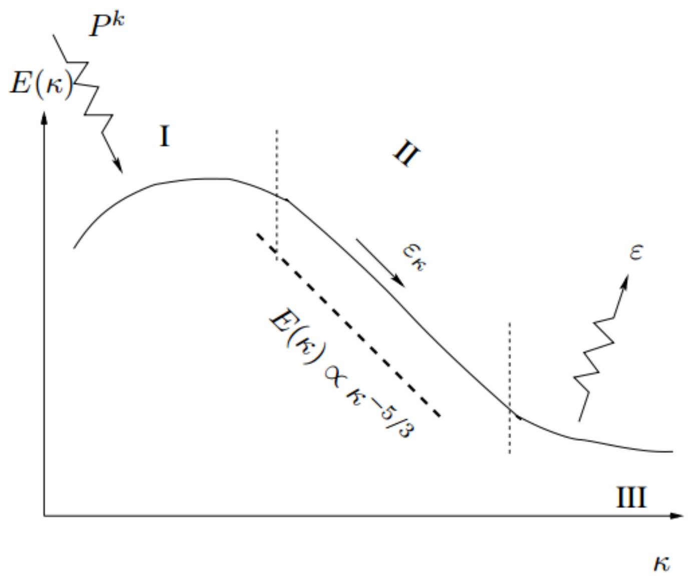


Figure 6.2: Figure according to [14] of the kinteic energy,k, for turbulent flow. Region **I** is the range for the large, energy containing eddies.

The turbulent scales extend from where the largest eddies are found, to the where dissipation occurs at the smallest of scales. It is therefore convenient to study the kinetic energy of the eddy size by going into wavenumber space:

$$E(\kappa)d\kappa. \tag{6.3.5}$$

This expresses the energy contribution per eddy to the turbulent kinetic energy. The contributions come from scales with wavenumber $\kappa$ to $\kappa + d\kappa$. By integrating over the whole wavemumber space we can obtain the total turbulent kinetic energy. $E$ is the kinetic energy per unit wavenumber of eddy size $\kappa \propto \ell_\kappa^{-1}$. The kinetic energy is the sum of the kinetic energy of the three fluctuating velocity [14] components, that is

$$k = \frac{1}{2}\left(\overline{v_1'^2} + \overline{v_2'^2 v_3'^2}\right) = \frac{1}{2}\overline{v_i' v_i'} \tag{6.3.6}$$

We see the spectrum of $E$ in figure 6.2, where regions **I,II,III** have the following meanings.

**I** This region contains most of the large eddies with velocity and length scales $v_0$ and $\ell_0$ respectively. The eddies carry most of the energy in this region. The energy transfer occurs between the large eddies and the mean flow via what is called the *production term* $P^k$. The energy extracted is transferred to th slightly smaller scales.

**II** In this region we find inertial subrange, here were require fully developed turbulence (large Reynolds number). This is the region where the cascade process occurs. The eddies in this region are independent of both the large eddies in region **I**, and the smallest energy dissipating eddies in region **III**. [14] argues that the eddies in the inertial subrange can be characterized by the "*spectral transfer* of energy per unit time", $(\varepsilon)$, and the eddy sizes, $1/\kappa$. So via dimensional analysis one gets that

$$\begin{aligned} E \quad &= \quad \kappa^a \qquad \varepsilon^b \\ [m^3/s^2] &= [1/m] \quad [m^2/s^3] \end{aligned} \tag{6.3.7}$$

Giving an equations for $[m]$:

$$3 = -3a + 2b, \tag{6.3.8}$$

and an equation for $[t]$:

$$-2 = -3b. \tag{6.3.9}$$

This gives us that $b = 2/3$ and $a = -5/3$. Inserting these into (6.3.7), we obtain

$$E(\kappa) = C_K \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}}, \tag{6.3.10}$$

where $C_K$ is the Kolmogorov constant. This is where the Kolmogorov spectrum law or the $-5/3$ law appears. It states: "if the flow is fully turbulent, the energy spectra should exhibit $-5/3$ - decay in the inertial region" [14]

**III** This is the dissipation range. It is in this range where the energy dissipation from the eddies occur. $\varepsilon$ in the transport equation, governs the energy transfer from turbulent kinetic energy to thermal energy, creating an increase in temperature. The Kolmogorov scales describe all eddies scales in the dissipation range.

The $-5/3$ law is sometimes more commonly referred to as the the 2/3's law. According to the definition in [20], the **two thirds law** states that [20] : "In a turbulent flow at very high Reynolds number, the mean square velocity increment $\langle(\delta v(\ell))^2\rangle$ between two points separated by a distance $\ell$ behaves approximately as the two-thirds power of the distance". If the energy spectrum power law is given by

$$E(\kappa) \propto \kappa^{-n} \tag{6.3.11}$$

where we saw that $n = 5/3$. Then the velocity field has homogeneous increments and the second order spatial structure function is also a power law

$$\langle |v(r') - v(r)|^2 \rangle \propto |r' - r|^{n-1}, \tag{6.3.12}$$

putting $n = 5/3$ gives us

$$\langle |v(r') - v(r)|^2 \rangle \propto |r' - r|^{2/3}. \tag{6.3.13}$$

So we see the two laws are equivalent.

### Looking for Turbulence around an acoustic black hole

The useful feature of fully developed turbulence is the two thirds power law. This due to the presence of velocity correlators, which are relatively simple to calculate (assuming one has the expression for the turbulent velocity term). Essentially for our given model, what we would like to do is calculate what the velocity correlators are close to the event horizon and look for a two thirds power law. This would indicate the presence of fully developed turbulence, however it would not guarantee it. Of course if there is no two thirds power law, then there is no turbulence. The reason we look for turbulent behaviour, is to somehow justify the presence of a firewall and look for a break down of the effective field theory. The presence of Hawking radiation around an acoustic black hole suggests the presence of a firewall by the same arguments of chapters 4 and 5. Although there may be some speculation as to the validity of a firewall by arguments made earlier in the section, for our purposes we assume that there exists a firewall. Since the energy scale at which the firewall manifests itself is very small compared to the characteristic length scale of the theory, we need some description in the fluids theory which too has energy dissipation occurring at very small scales of the theory. The energy dissipation via smaller and smaller eddies to the molecular scale is the kind of energy dissipation mechanism that could indicate a firewall. In other words, highly turbulent behaviour close to the acoustic event horizon indicates the presence of many densely packed eddies dissipating the energy into a very small region in the fluid -signaling a firewall.

### Velocity correlators

Velocity correlators can be expressed by calculating the Reynolds Stress Tensor

$$B_{ik} = \langle (v_{2i} - v_{1i})(v_{2k} - v_{1k}) \rangle \tag{6.3.14}$$

where $\vec{v}_2$ and $\vec{v}_1$ are the fluid velocities at two neighboring points, and the angle brackets denote a time averaging. The time average can also be defined as a spatial average by the Taylor Hypothesis

$$\langle v_{1i} v_{2k} \rangle = \frac{1}{V_\Omega} \int_\Omega v_{1i}(\vec{x}) v_{2k}(\vec{x}) d^3x \tag{6.3.15}$$

We define the radius vector which points from 1 to 2 $\vec{r} = \vec{r}_2 - \vec{r}_1$. The velocity variation over small distances is due to small eddies. In an idealized system we can consider a flow where there exists *isotropy* and*homogeneity*. From these assumption one can find various relations between correlation functions. Since the local turbulence is isotropic, the tensor $B_{ik}$ cannot depend on any direction in space. The only vector that can appear, is the radius vector $\vec{r}$. The general form of such a symmetrical tensor of rank two as shown in [34] is

$$B_{ik} = A(r)\delta_{ik} + B(r) n_i n_k, \tag{6.3.16}$$

where $\vec{n}$ is a unit vector in the direction of $\vec{r}$. By choosing the coordinate axis such that one of the axis is in the direction of $\vec{n}$. The velocity component along the axis is $v_r$, and the velocity perpendicular to $\vec{n}$ is $v_t$. $B_r r$ is the mean square relative velocity of two fluid particles along the radial line joining them. $B_{tt}$ is the mean square transverse velocity of one particle relative to another. This means $n_r = 1$ and $n_t = 0$, thus we express the tensor $B_{ik}$ as

$$B_{ik} = B_{tt}(r)(\delta_{ik} - n_i n_k) + B_{rr}(r) n_i n_k. \tag{6.3.17}$$

Using the definition of the Reynolds stress tensor we get

$$B_{ik} = \langle v_{1i} v_{1k} \rangle + \langle v_{2i} v_{2k} \rangle - \langle v_{1i} v_{2k} \rangle - \langle v_{2i} v_{1k} \rangle. \tag{6.3.18}$$

From here one would substitute in the expressions of $\vec{v}$ of the turbulent flow. If indeed the flow were turbulent, one would find the relation between the velocity correlation functions $B_{rr}$ and $B_{tt}$ to be proportional to $r^{2/3}$ as shown in [34]. This would be the two thirds power law, indicating fully developed turbulence.

There is an important issue which needs to be stressed. The velocity $v_r$ and $v_t$ are the components of the **turbulent flow** not the background flow. They are the decomposition of the $v'(\rho)$ fluctuations of the background flow. And so one cannot naively just use the required expression of the background flow $v(\rho) = \sqrt{\frac{2GM}{\rho}}$ found in section 6.1.1 and decompose this into $v_r$ and $v_t$. Note, we used $\rho$ instead of $r$ as to not confuse the notation that $\rho$ is a radial coordinate for the acoustic black hole, and $r$ is the radial distance between two points in the fluid. The turbulent flow is achieved by performing a Reynolds averaging of our $v(\rho)$ by finding

$$v(\rho) = \bar{v}(\rho) + v'(\rho), \tag{6.3.19}$$

but this is by no means a trivial exercise, and one cannot just find what the mean part of our function is and the turbulent part is. Furthermore we already described the fluctuations in our background flow to characterize acoustic disturbances as shown in section 6.1.1 which lead to a description acoustic disturbances. Thus we need to find a way to describe the turbulent flow in a way that does not interfere with our definition for the acoustic fluctuations. This would require a much more detailed analysis of our model to find a statistical description of the turbulent flow near the event horizon. There is even a chance the the model will not yield any statistical description of turbulent flow as it is too simple. Such investigations would require future work and a more in depth analysis of the modeling of turbulence.

#### 6.3.1.2 Divergence at the Horizon

Just as in section 5.2.3, we saw that an energy divergence at the horizon has the correct firewall signature, perhaps the best way forward is by looking for a similar kind of energy divergence in the fluid. A possible starting point would be looking at the Bondi accretion problem in astrophysics. The

Bondi problem asks what is the spherical accretion rate of a supersonic gas by a gravitating point $M$.

It is not the accretion rate of the fluid that we are interested in, but rather the manner in which one models the Bondi problem. The Bondi problems states that as the gas accretes onto the mass, the surface begins to grow. This gas gets compressed to look like a solid 'wall' of gas, allowing no more gas to pass through. The problem them studies what happens to this fluid(gas) as it continues to accrete. We could model the firewall in a similar manner, we construct a spherical $3+1$ dimensional dumb hole as an object with mass $M$. At the horizon there exists an impenetrable barrier which we assume to be the firewall. One could put this assumption in, as the firewall is meant to be something which allows no information to pass through. In shock wave analysis, this resembles the case when supersonic fluid strikes a solid barrier resulting in the sudden halt of the fluid as it collides with the surface of the barrier. When a flow sharply changes from supersonic to subsonic in a short period of time, this sends out a shock wave. The transition from supersonic to subsonic occurs not smoothly, but in one sharp jump. This shock transition layer occupies approximately a few mean free paths for elastic gas collision. It is this shock wave boundary which has the properties of a firewall

Thus we first look at the energy stress tensor of the fluid which can be initially seen as a sort of 'dust', that is far from the black hole where the pressure is approximately zero. The dust energy momentum tensor can be written as

$$T^{\alpha\beta} = \rho \begin{pmatrix} c^2 & cu_x & cu_y & cu_z \\ cu_x & u_x u_x & u_x u_y & u_x u_z \\ cu_y & u_y u_x & u_y u_y & u_y u_z \\ cu_z & u_z u_x & u_z u_y & u_z u_z \end{pmatrix}$$

or more succinctly

$$T^{\alpha\beta} = \rho_0 u^\alpha u^\beta. \tag{6.3.20}$$

Taking the divergence of the energy momentum tensor:

$$\partial_\beta T^{\alpha\beta} = 0, \tag{6.3.21}$$

leads to two very important equations. In the case when the free index $\alpha = 0$, we have

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0, \tag{6.3.22}$$

which is equation of continuity of a fluid as seen in (6.1.1):

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0. \tag{6.3.23}$$

Similarly if we consider the case when the free index $\alpha = 1, 2, 3$ we get

$$\partial_\beta T^{1\beta} = 0, \partial_\beta T^{2\beta} = 0, \partial_\beta T^{3\beta} = 0. \tag{6.3.24}$$

When combined give these three equations

$$\frac{\partial(\rho\vec{u})}{\partial t} + \frac{\partial(\rho u_x \vec{u})}{\partial x} + \frac{\partial(\rho u_y \vec{u})}{\partial y} + \frac{\partial(\rho u_z \vec{u})}{\partial z} = 0, \tag{6.3.25}$$

then using the equation of continuity (6.1.1) this becomes

$$\rho \left[\frac{\partial(\vec{u})}{dt} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = 0, \tag{6.3.26}$$

which is the Navier-Stokes Equation. This is the equivalent of Newton's 2nd Law in classical mechanics. From this we see that the dust-energy momentum tensor leads to the governing equations of fluids, namely the continuity equation and the Navier-Stokes equation.

Since close to the horizon, one can not neglect pressure anymore, we can model the fluid as a perfect fluid,that is, it is characterized by two scalar quantities:
the proper density of the fluid

$$\rho_0 = \rho_0(x), \qquad \sim \frac{\text{mass}}{\text{volume}}, \tag{6.3.27}$$

pressure of the fluid

$$p = p_0(x), \qquad \sim \frac{\text{energy}}{\text{volume}}, \tag{6.3.28}$$

and one vector quantity: 4-velocity of the flow

$$u^\alpha = \frac{dx^\alpha}{d\tau}. \tag{6.3.29}$$

And so, the perfect fluid tensor requires the following form

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right) u^\alpha u^\beta - pg^{ab}. \tag{6.3.30}$$

It is interesting to note that when the conservation equations,

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{6.3.31}$$

is applied to 6.3.30, one also obtains two equations: a type of equation of continuity equation and a geodesic 'look-a-like' equation of motion, however, with an added forcing function arising from the derivative of the pressure. In general relativity, to have a complete model of some matter-energy distribution, we need an equation of state for the matter-energy system. This describes the interaction of matter with pressure and temperature. In general, this can be written as:

$$p = p(\rho_0, T). \tag{6.3.32}$$

From here onward one would need to find an appropriate equation of state for our model. Ideally one wouldn't necessarily start off with the fact that the fluid is striking a 'solid' object, instead one would study the energy momentum tensor at the horizon, and based on the specific construction of situation at the horizon, do some energy considerations to see if there are indeed energy spikes. One could then say that this may be due to the shockaves of the fluid encountering the firewall creating the high energy divergences.

# 7. Conclusion

The goal of the thesis is two fold: to present ideas surrounding black hole physics, in particular the firewall problem by 'AMPS' [1] and discuss how the firewall problem could be extended to black hole analogues in fluid mechanics. To do this however, many introductory concepts and tools were presented in the preceding chapters. Chapters 1 and 2 explained all the basic notations and ideas associated with general relativity, quantum mechanics and quantum field theory. General relativity is the framework in which we study spacetime geometries, we place particular emphasis on curved spacetime geometries which are synonymous with the presence of gravity. Einstein's equation

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{7.0.1}$$

says that objects with mass will curve spacetime, in turn spacetime will dictate the motion of all objects within it. One object that exhibits very interesting features is a black hole. This is due to quantum effects that occur around the black hole. To study such effects we require quantum field theory and quantum information theory. Quantum field theory allows us to define a mode expansion for a scalar field around the back hole

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[ e^{ix^\mu k_\mu} a_{\vec{k}} \; + \; e^{-ix^\mu k_\mu} \, a^\dagger_{\vec{k}} \right]. \tag{7.0.2}$$

With the scalar field comes the vacuum state, defined to be

$$\alpha_{\vec{k}} \left|0\right\rangle_a = 0, \text{for all momentum } \vec{k}. \tag{7.0.3}$$

The crucial aspect in curved space is that unlike in flat space, the vacuum state is not Poincaré invariant. This lack of symmetry prevents invariance under time translations, consequently different observers ($a$ and $b$) will have different notions of time leading to a different definition of their vacuum state

$$\alpha_{\vec{k}} \left|0\right\rangle_b = 0, \text{for all momentum } \vec{k}. \tag{7.0.4}$$

Since two observers will define their vacuum states differently, in general a vacuum for one observer may be populated with particles according to another observer's definition of what the vacuum state is. Due to the time dependent metric of a black hole, described using Eddington-Finklstein coordinates, we show that particle production occurs at the event horizon. The mechanism in which these particles are produced is analogous to how particles are produces under a rapid changing of the potential in some quantum system. The spacetime is 'sliced' along space-like surfaces, as described by

Mathur [38]. Vacuum wavemodes that evolve on these surfaces along the horizon distort significantly very near the horizon. The distortion of vacuum wavemodes give rise to the production of particle pairs which are entangled with each other. One partner pair falls into the black hole, while the other partner goes to infinity.

The fact that particles appear to radiate from the black hole, and that the particles pairs are entangled with each other, create a problem. Hawking [28] showed that black holes radiate with a spectrum that is approximately thermal. The black hole then appears to radiate its energy with a temperature

$$T = \frac{\hbar c^3}{8\pi G M k_B}. \tag{7.0.5}$$

For example, a black hole the size of the moon would be 2.7 K, for a solar mass black hole the temperature would be even lower at around $6 \times 10^{-8}$ K. The work of Bekenstein and Hawking showed that black holes can be viewed as real thermodynamic objects that exhibit laws which are analogues to the four laws of thermodynamics. The most noticeable resemblance to the second law of thermodynamics is the generalized second law of black hole mechanics

$$\delta S' = \delta S_{\text{matter}} + \delta S_{\text{BH}} \geq 0. \tag{7.0.6}$$

This resemblance allows one to identify normal thermodynamic entropy to the generalized black hole entropy as

$$S_{BH} = \frac{Ac^3}{4G\hbar}. \tag{7.0.7}$$

The significance of the generalized second law is that Black holes are allowed to evaporate, in fact a simple calculation shows that the time for a black hole to evaporate is of the order $\sim M^3$. As remarkable a conclusion as this is, it comes with an underlying paradox: if the black hole evaporates, what happens to everything that falls into the black hole? According to Hawking the information comes out in the quanta of radiation emitted from the radiation. This poses a even deeper problem: if particle pairs produced at the horizon are entangled, and one pair escapes to the infinity as Hawking radiation while the other falls into the black hole, then what state is the Hawking radiation in once the black hole evaporates? It can't be pure, as the Hawking pairs were originally in a mixed state due to their entanglement. The Hawking pairs also cannot still be in a mixed state, since due to the evaporated black hole, there are no more partner pairs inside the black hole to be entangled with. This is the information paradox as most commonly presented in terms of the loss of information. Chapter 3 was devoted to reproducing Hawking's derivation of the thermal spectrum of black hole, and the analysis of what the information paradox really is.

Once black hole complementarity was introduced, a lot of issues surrounding the information problem were resolved. Complementarity states that in order to preserve unitary evolution of black hole evaporation and the equivalence principal for observers near the horizon, we must give up on omniscience.

The ability that we are able to define a single consistent theory over all of spacetime seemed too great a statement. Instead, as long as any single theory is consistent within every causal boundary, then there does not exist any experiment which could find any contradictions. Black hole complementarity was condensed into four postulates [54] which, if correct, would always hold true no matter what experiment one may performs. 'AMPS' however suggested just that; a thought experiment which puts the postulates of complementarity in disagreement with each other. If according to the postulates the black hole evaporation is indeed unitary, then once the black hole has sufficiently surpassed its Page time, the new bits of Hawking quanta emitted from the black hole(late radiation) are maximally entangled with the bits already evaporated (early radiation). Due to the monogamy of entanglement, the late radiation cannot be entangled with any other system. AMPS showed that instead, the late radiation is indeed entangled with another system. As the late hawking quanta are created, again by the postulates, they should be maximally entangled with their partner pairs falling into the black hole. The disagreement between entanglement of states of the late radiation put the postulates in contradiction with one another. The 'conservative' conclusion by AMPS was to introduce a large energy divergence at the horizon, which was called a 'firewall'. As some observer approaches the event horizon of an old black hole, instead of experiencing nothing special as per the equivalence principal, they will encounter the firewall and burn up, never to enter the black hole, thus sacrificing the equivalence principal. The dynamical properties of the firewall are not well understood, nor is the requirement for a firewall the only conclusive one.

Perhaps the question one asks about the firewall are too difficult to answer with current knowledge of the inner structure of a black hole. Since a firewall signifies the presence of extremely short scale physics of the theory, choosing a different framework to study black holes may turn out advantageous. Fluid mechanics and solid state physics are such regimes. We focus on the fluid mechanics model of black holes primarily developed by Unruh [59, 60] and further expanded by Visser [62, 3, 4, 63]. The solid states black holes analogues are discussed by Jacobson [30, 33]. Acoustic black hole analogues are called 'dumb holes'. The metric for a dumb hole was found to be

$$ds^2 = \frac{\rho_0}{c}\left[-\left(c^2 - v_0^2\right)dt^2 + \left(\frac{c^2}{c^2 - v_0^2}\right)dr^2 + r^2 d\Omega^2\right], \tag{7.0.8}$$

which is conformal to the Schwarzschild metric of a black hole. The acoustic event horizon of a dumb hole is a null surface for phonon geodesics, and so it behaves in the same way as the event horizon of a black hole. By applying Hawking's arguments for a black hole to a dumb hole we see that we can apply a phonon scalar field around the dumb hole. We find that the ingoing and outgoing modes of the field resemble that of the Schwarzschild black hole that Hawking used. From this we discuss how, by applying the same analysis to a dumb hole, we will find that there will be a spectrum of emitted phonons from the dumb hole. The radiative behaviour of dumb holes was again originally discussed by Unruh [59] with subsequent calculation done by [17, 23] showing analogue Hawking radiation in acoustic systems. With the existence of Hawking-like radiation, we argue towards the manifestation of an acoustic version of all issues pertaining to information loss and the presence of a firewall. The idea is to study what the firewall would look like in a fluid. Our first approach was to look for turbulence in the fluid model, as this would signal the breakdown of the effective field theory of hydrodynamics. A large energy deposit in the fluid by the firewall will create a setting for turbulent behaviour near the event horizon. We give a basic introduction to what turbulence is and what are the signatures for turbulence in a fluid. The main idea we have is to calculate velocity correlators near the horizon and look for a particular two-thirds power law between velocity correlator components. Unfortunately this method is more difficult than originally intended, mainly due to the requirement that one needs to perform a Reynolds averaging over the background flow. This then separates the background flow into two components: a time independent laminar flow and fluctuations of the laminar flow. Turbulent flow is described by these fluctuations, however we already reserved the fluctuations to describe acoustic disturbances. One would need a statistical description for the turbulent component to then calculate the velocity correlators. What one can also think of doing is to look for some energy divergence at the event horizon of the energy momentum tensor for the dumb hole. If the firewall were true then the energy divergence could be viewed as a shock wave boundary where the pressure and temperature sharply increase by several orders of magnitude. We suggest how to setup the analysis but, to advance further, a more precise description of the state parameters are needed in order to gather more information.

In terms of the acoustic analogue, future work lies in building a more precise model by introducing what the state parameters of the background flow may look like. One will lose some generality of the model, which is something that is not so easy to let go of. Unfortunately, although fluid mechanics is a well studied subject, unless one has a precise description of the system, many of the analysis with regard to turbulence become statistical in nature. It may even be worthwhile to borrow some ideas from astrophysics where the accretion of matter onto stars is studied. For a simple model the flow equation of the gas accreting onto the star looks similar to the flow of our model. By investigating further one may find tools valuable to the scope of the thesis. One could also deviate and study analogues in solid state physics, but the reason why fluid mechanics is favorable is that it is governed by the Navier-Stokes equation. This equation is only valid for as long as we can use the continuum approximation of the fluid. Since the firewall suggests a break down of the effective field theory at the shortest of distances hydrodynamics, any suggestion that an acoustic firewall may exist implies that the physics responsible for the firewall cannot be described using fluid mechanics, as it exists at scales smaller than the cut off limit of the continuum approximation (atomic scales). From this we conclude that one needs an underlying theory which acts at scales smaller than what we are currently able to work with in quantum field theory. Such a conclusion reinforces the need of a fully developed quantum theory of gravity.

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