Asymptotically unbiased inference for a panel VAR model with *p* lags

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Borradores de ECONOMÍA

No. 1059
2018

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Abstract

Panel dynamic estimators with fixed effects are biased due to the incidental parameters problem. At this regard, Hahn and Kuersteiner (2002) proposed an estimator to correct this issue. However, they only consider a panel VAR (PVAR) model with one lag. In this paper we extend this bias correction, its asymptotic and small sample properties for a more general case, a PVAR model with p lags. The simulation results indicate that the bias corrected estimator outperforms the OLS panel VAR estimator when sample size in time dimension is small, and when the persistence of the model is low. In these cases, the proposed estimator improves significantly in terms of both, the reduction of bias and mean square error.

Keywords. Panel VAR models; bias correction; restricted OLS.

JEL Classification. C33, C51, C13.

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Inferencia asintótica insesgada para un modelo panel VAR con p rezagos

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Resumen

Los estimadores de los parámetros de un modelo panel dinámico de efectos fijos son sesgados debido al problema de parámetros incidentales. Al respecto, Hahn y Kuersteiner (2002) proponen un estimador para corregir este problema. Sin embargo, ellos consideran únicamente un modelo panel VAR con un sólo un rezago. En este documento analizamos las propiedades asintóticas y de muestra pequeña del estimador corregido por sesgo para un caso más general, un modelo PVAR con p rezagos. Los resultados de las simulaciones indican que el estimador corregido por sesgo tiene un mejor desempeño con respecto al estimador panel VAR MCO cuando la dimensión temporal de la muestra (T) es pequeña, y cuando la persistencia del modelo es baja. En estos casos, el estimador propuesto presenta una disminución significativa en términos de sesgo, y de error cuadrático medio.

Palabras clave. Modelos Panel VAR; corrección de sesgo; MCO restringido.

Códigos JEL. C33, C51, C13.

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1 Introduction

Given the recent availability of data, panel VAR models are very useful since they allow to explore both, the dependence relationships across observations, and the dynamics of the process. Recent applications of these methodologies are found in Love and Zicchino (2006), Lof and Malinen (2014) and Ouyang and Li (2018), among others.

However, OLS estimator of fixed effects panel VAR models suffers from a Hurwicz type of bias. Nickell (1981) presents a detailed analysis of these inconsistencies for the first order autoregressive model, providing an analytical expression for the bias of order $O(T^{-1})$. He also shows that his analytical computations derived from his formulas coincide with the Monte Carlo results of Nerlove (1967) and Maddala (1971).

More recently, Lee et al. (2018) considered the estimation for a dynamic panel of infinite order autoregressive process in the presence of individual effects. They propose a bias corrected fixed effect estimator based on a theoretical asymptotic bias term. They employ double asymptotics under which n and T tend to infinity. However, this paper only analyze the univariate case, they do not consider panel VAR models.

At this regard, Hahn and Kuersteiner (2002) find and correct the asymptotic bias of the OLS estimator when both n and T are large. However, they only consider the case of a panel VAR with one lag.

The objective of this paper is to extend the methodology proposed by Hahn and Kuersteiner (2002) for a panel VAR model allowing up to p lags. We provide the asymptotic bias corrected estimator and found its asymptotic distribution. Finally, we study small sample properties of this estimator using Monte Carlo simulations. To our knowledge this is the fist time that this bias corrections is propose for this model.

This paper is organized as follows. Section 2 contains the theoretical development of our methodology. Section 3 provides Monte Carlo simulations to show the performance of our estimator in a small sample set up. Finally, section 4 concludes. All proofs are relegated to the Appendix.

2 Methodology

In this section we propose an asymptotic bias correction for the OLS estimators of a fixed panel VAR model with p lags. This is an extension of the results of Hahn and Kuersteiner (2002), hereafter HK.

In our framework we considered the following panel VAR model with fixed effects,

$$y_{it} = \alpha_i + \theta_1 y_{it-1} + \cdots + \theta_p y_{it-p} + \epsilon_{it}, \tag{1}$$

where y_{it} is an m-dimensional vector and ϵ_{it} is i.i.d. normal. To calculate the asymptotic bias we use the panel VAR(1) representation of (1) as the first step. Next, we find the asymptotic distribution of the OLS estimator.

After applying the within transformation to (1) we obtain,

$$\ddot{y}_{it} = \theta_1 \ddot{y}_{it-1} + \cdots + \theta_p \ddot{y}_{it-p} + \ddot{\epsilon}_{it} \tag{2}$$

where, $\ddot{y}_{it} = y_{it} - \frac{1}{T}\sum_{t=1}^{T} y_{it}$, and $\ddot{\epsilon}_{it} = \epsilon_{it} - \frac{1}{T}\sum_{t=1}^{T} \epsilon_{it}$.

2.1 VAR(1) representation

Following Lutkepohl (2005) in a panel context, a panel VAR(1) representation of (2) is the following,

$$\ddot{Y}_{it} = \Theta \ddot{Y}_{it-1} + \ddot{U}_{it}. \tag{3}$$

where,

$$\ddot{Y}_{it} := \begin{bmatrix} \ddot{y}_{it} \\ \ddot{y}_{it-1} \\ \vdots \\ \ddot{y}_{it-p+1} \end{bmatrix}, \quad \ddot{U}_{it} := \begin{bmatrix} \ddot{\epsilon}_{it} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \Theta := \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{p-1} & \theta_p \\ I_m & 0 & \cdots & 0 & 0 \\ 0 & I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & I_m & 0 \end{bmatrix}$$

After applying the vec operator to (3) we obtain,

$$\ddot{Y}_{it} = \left(\ddot{Y}'_{it-1} \otimes I_{mp}\right) \operatorname{vec}(\Theta) + \ddot{U}_{it}. \tag{4}$$

In this representation Θ contains identities and block-zero restrictions given by

$$\operatorname{vec}(\Theta) := \zeta = Rb + r, \tag{5}$$

where,

$$R := \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_{mp} \end{bmatrix}_{m^2p^2 \times m^2p}, \quad r := \operatorname{vec} \begin{bmatrix} \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{I}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{I}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m \\ \vdots & \vdots & \ddots & \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{I}_m & \mathbf{0}_m \end{bmatrix}_{m^2p^2 \times 1},$$

$$b := \operatorname{vec} \begin{bmatrix} \theta_{\mathbf{1}} & \cdots & \theta_{\mathbf{p-1}} & \theta_{\mathbf{p}} \end{bmatrix}_{m^2p \times 1} \text{ and } \mathbf{C}_j := \begin{matrix} (1) & & (j) & & (mp) \\ \end{matrix} \begin{bmatrix} \mathbf{0}_m & \cdots & \mathbf{I}_m & \cdots & \mathbf{0}_m \\ \mathbf{0}_m & \cdots & \mathbf{0}_m & \cdots & \mathbf{0}_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0}_m & \cdots & \mathbf{0}_m & \cdots & \mathbf{0}_m \end{bmatrix}_{mp \times m^2p},$$

$j = 1, \ldots, mp.$

2.2 Biased OLS

To estimate b in (3) and (5) we use the methodology proposed in (Lutkepohl, 2005, pp 194-196) for restricted OLS estimators in a panel VAR context.

Under the usual minimization problem,

$$\hat{b} = \arg\min_b \sum_{i=1}^{n} \sum_{t=1}^{T} \left[\ddot{Y}_{it}' - b' R' \left(\ddot{Y}_{it-1} \otimes I\right) - r' \left(\ddot{Y}_{it-1} \otimes I\right)\right] \times \left[\ddot{Y}_{it} - \left(\ddot{Y}_{it-1}' \otimes I\right) Rb - \left(\ddot{Y}_{it-1}' \otimes I\right) r\right].$$

we obtain the following OLS estimator of b,

$$\hat{b} = \left[\sum_{i=1}^{n} \sum_{t=1}^{T} R' \left(\ddot{Y}_{it-1} \ddot{Y}_{it-1}' \otimes I_{mp}\right) R\right]^{-1} \left\{ \sum_{i=1}^{n} \sum_{t=1}^{T} \left[R' \left(\ddot{Y}_{it-1} \otimes I_{mp}\right) \ddot{Y}_{it}\right] - \sum_{i=1}^{n} \sum_{t=1}^{T} \left[R' \left(\ddot{Y}_{it-1} \ddot{Y}_{it-1}' \otimes I_{mp}\right) r\right] \right\}. \tag{6}$$

Then, given (5) and (6) the estimator of vec(Θ) is,

$$\hat{\zeta} = R\left[\sum_{i=1}^{n}\sum_{t=1}^{T} R'\left(\ddot{Y}_{it-1}\ddot{Y}'_{it-1} \otimes I_{mp}\right) R\right]^{-1} \left\{ \sum_{i=1}^{n}\sum_{t=1}^{T}\left[R'\left(\ddot{Y}_{it-1} \otimes I_{mp}\right)\ddot{Y}_{it}\right] - \sum_{i=1}^{n}\sum_{t=1}^{T}\left[R'\left(\ddot{Y}_{it-1}\ddot{Y}'_{it-1} \otimes I_{mp}\right) r\right]\right\} + r. \tag{7}$$

Theorem 1 *Let y_{it} be generated by* (1)*, and let all the innovations ϵ_{it} be independent for all i and t. Under the conditions assumed in HK and the representation* (3)*, $\sqrt{nT}vec(\hat{b} - b_0)$ has the following asymptotic distribution,*

$$\sqrt{nT}(\hat{b} - b_0) \to \mathcal{N}\left(-\sqrt{\rho}\ \mathbf{A}(I \otimes I - (\Theta_0 \otimes I))^{-1} vec(\Omega), \mathbf{A}(\Omega \otimes \Upsilon)\mathbf{A}'\right), \tag{8}$$

where,

$$\mathbf{A} = \left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} R'\left(\ddot{Y}_{it-1}\ddot{Y}'_{it-1} \otimes I_{mp}\right) R\right]^{-1} R', \tag{9}$$

$\Omega \equiv E\left[U_{it}U'_{it}\right]$[1]*,* $\rho \equiv \frac{n}{T}$ *and* $\Upsilon \equiv \Omega + \Theta\Omega\Theta' + \Theta^2\Omega\left(\Theta'\right)^2 + \cdots$*.*

Proof. See Appendix A ■

Thus, Theorem 1 shows that the limiting distribution of the estimator of b is not centered at zero.

2.3 Bias estimation

Given the noncentrality term of (8), $-\sqrt{\rho}\ \mathbf{A}(I \otimes I - (\Theta_0 \otimes I))^{-1}\text{vec}(\Omega)$, an asymptotic bias corrected OLS estimator of b is given by,

$$\hat{\hat{b}} = \hat{b} + \tfrac{1}{T}\ \mathbf{A}(I \otimes I - (\hat{\Theta} \otimes I))^{-1}\text{vec}(\hat{\Omega}),$$

After some algebra,

[1] U_{it} is defined in a similar way as $\ddot{U}_{it}$, but without applying the within transformation to the model.

$$\text{vec}\left(\hat{\hat{\Theta}}\right) = R\hat{\hat{b}} + r,$$

with,

$$\begin{aligned}\hat{\hat{b}} = &\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} R'\left(\ddot{Y}_{it-1}\ddot{Y}'_{it-1}\otimes I_{mp}\right)R\right]^{-1} \\ &\times\left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[R'\left(\ddot{Y}_{it-1}\otimes I_{mp}\right)\ddot{Y}_{it}\right]\right. \\ &-\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[R'\left(\ddot{Y}_{it-1}\ddot{Y}'_{it-1}\otimes I_{mp}\right)r\right] \\ &\left.+\frac{1}{T}R'_{m^2p\times m^2p^2}\left[(I_{mp}\otimes I_{mp}-(\hat{\Theta}\otimes I_{mp}))^{-1}\text{vec}(\hat{\Omega})\right]\right\},\end{aligned} \tag{10}$$

where $\hat{\Theta}$ is the panel VAR OLS estimator of Θ, and

$$\text{vec}(\hat{\Omega})_{m^2p^2\times 1} = \left[I_{mp}\otimes I_{mp}-\left(\hat{\Theta}\otimes\hat{\Theta}\right)\right]\text{vec}\left(\hat{\Upsilon}\right),$$

$$\hat{\Upsilon} = \frac{1}{nT}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\ddot{Y}_{it-1}\ddot{Y}'_{it-1}\right)_{mp\times mp}.$$

Next, we show that $\sqrt{nT}\text{vec}\left(\hat{\hat{b}}-b_0\right)$ is centered at zero and find the asymptotic distribution of $\hat{\hat{b}}$.

Theorem 2 *Let y_{it} be generated by* (1)*, and let the innovations ϵ_{it} be independent for all i and t. Under the conditions assumed in HK and the representation* (3)*, $\sqrt{nT}vec\left(\hat{\hat{b}}-b_0\right)$ has the following asymptotic distribution,*

$$\sqrt{nT}\ vec\left(\hat{\hat{b}}-b_0\right)\to\mathcal{N}\Big(0,\mathbf{A}\left(\Omega\otimes\Upsilon\right)\mathbf{A}'\Big) \tag{11}$$

Proof. See Appendix B ■

3 Monte Carlo simulations

In order to evaluate the small sample performance of the proposed estimator we perform a Monte Carlo experiment. We compare both the biased (OLS) and unbiased panel VAR estimators, $\hat{b}$ and $\hat{\hat{b}}$, described in the previous section.

The DGP is as follows,

$$y_{it} = \alpha_i + \theta_1 y_{it-1} + \theta_2 y_{it-2} + \epsilon_{it}, \tag{12}$$

where, $y_{it} \in \mathbb{R}^2$, θ_1 and θ_2 are 2×2 matrices, $\alpha_i \sim \mathcal{N}(0, I_2)$ is independent across i, and $\epsilon_{it} \overset{i.i.d.}{\sim} \mathcal{N}(0, \Omega)$ with $\Omega = \begin{pmatrix} 1 & \omega \\ \omega & 1 \end{pmatrix}$. [2]

We allow for sample sizes $n = \{100, 200\}$ and $T = \{5, 10, 15\}$. We also evaluate different values of the covariance of the innovations, $\omega = \{0, 0.4, 0.8\}$. In addition, we consider five models (Model 1 to Model 5) with different levels of persistence, according to the values of the parameters of θ_1 and θ_2 presented in Table 1. We run 5000 simulations for each specification.

Table 1: Parameters θ_1 and θ_2 of the panel VAR model

Model	θ_1	θ_2	Maximum eigenvalue of $(I - \theta_1 z - \theta_2 z^2)$
Model 1	$\begin{bmatrix} 0.06 & 0.05 \\ -0.06 & 0.08 \end{bmatrix}$	$\begin{bmatrix} 0.03 & -0.02 \\ -0.03 & 0.04 \end{bmatrix}$	$\lambda_{max} = 0.27$
Model 2	$\begin{bmatrix} 0.10 & -0.10 \\ -0.12 & 0.10 \end{bmatrix}$	$\begin{bmatrix} 0.09 & 0.08 \\ 0.10 & 0.05 \end{bmatrix}$	$\lambda_{max} = 0.41$
Model 3	$\begin{bmatrix} 0.30 & -0.20 \\ 0.15 & 0.20 \end{bmatrix}$	$\begin{bmatrix} 0.15 & -0.10 \\ 0.10 & 0.15 \end{bmatrix}$	$\lambda_{max} = 0.59$
Model 4	$\begin{bmatrix} 0.40 & 0.15 \\ -0.30 & 0.30 \end{bmatrix}$	$\begin{bmatrix} 0.25 & 0.10 \\ -0.20 & 0.20 \end{bmatrix}$	$\lambda_{max} = 0.75$
Model 5	$\begin{bmatrix} 0.60 & 0.10 \\ -0.10 & 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.20 & 0.10 \\ -0.10 & 0.20 \end{bmatrix}$	$\lambda_{max} = 0.83$

[2]The initial values of y_{it} are computed as the unconditional mean of the process.

The results of the simulations are shown in Table 2 for Model 1, and in Tables 3 to 6 of Appendix C for Models 2 to 5. Columns 4 to 11 of these tables present the bias of both, the OLS estimator, $\hat{\theta}_j$, and the bias corrected estimator, $\hat{\hat{\theta}}_j$, for $j = 1, 2$. The last four columns report the ratio of the RMSE of $\hat{\hat{\theta}}_j$ and the RMSE of $\hat{\theta}_j$, for $j = 1, 2$. Values smaller than 1 indicate a better performance for the bias corrected estimator.

The simulation exercises suggest that both estimators still suffer from some bias issues. In general, the bias decreases as the sample sizes n and T increase, as expected. Moreover, the bias corrected estimator shows an improvement in terms of both, reduction of the bias and reduction of the RMSE.

The bias corrected estimator shows smaller bias and RMSE compared to the OLS estimator, specially for small values of T ($\{5, 10\}$), low covariance of the innovations ($\omega = \{0, 0.4\}$) and low persistence ($\lambda_{max} < 0.6$). In particular, the bias is reduced in 7 out of 8 cases for $T = 5$, $\omega = \{0, 0.4\}$ and $\lambda_{max} = 0.27$ as shown in Table 2.[3] These findings hold for every $n = \{100, 200\}$.

The results presented in Tables 3 to 6 of Appendix C also show that the bias and the RMSE of the bias corrected estimator deteriorate as the persistence of the model, λ_{max}, increases. These results are consistent with the findings of Hahn and Kuersteiner (2002).

[3]The 8 cases refer to the number of elements of θ_1 and θ_2.

Table 2: Small sample performance of the panel VAR estimators

n	T	ω	Bias of $\hat{\theta}_1$ and $\hat{\theta}_2$				Bias of $\hat{\hat{\theta}}_1$ and $\hat{\hat{\theta}}_2$				RMSE ratio			
			$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\hat{\theta}}_1$		$\hat{\hat{\theta}}_2$		$\hat{\hat{\theta}}_1/\hat{\theta}_1$		$\hat{\hat{\theta}}_2/\hat{\theta}_2$	
100	5	0	-0.36	0.01	-0.36	0.04	-0.18	0.02	-0.32	0.03	0.53	1.13	0.89	0.94
			0.02	-0.37	-0.01	-0.37	0.01	-0.19	0.00	-0.33	1.07	0.53	1.01	0.89
100	10	0	-0.13	0.01	-0.13	0.02	-0.03	0.01	-0.12	0.01	0.37	1.12	0.95	0.95
			0.00	-0.13	-0.01	-0.13	-0.01	-0.03	0.00	-0.13	1.11	0.36	0.98	0.96
100	15	0	-0.08	0.00	-0.08	0.01	-0.01	0.01	-0.08	0.01	0.38	1.09	0.98	0.96
			0.00	-0.08	-0.01	-0.08	0.00	-0.01	0.00	-0.08	1.08	0.38	0.98	1.00
100	5	0.4	-0.37	0.01	-0.37	0.04	-0.19	0.02	-0.33	0.03	0.54	1.14	0.89	0.95
			0.02	-0.37	-0.02	-0.35	0.01	-0.18	-0.01	-0.31	1.08	0.52	1.00	0.89
100	10	0.4	-0.13	0.00	-0.14	0.02	-0.03	0.01	-0.13	0.01	0.39	1.12	0.95	0.95
			0.01	-0.13	-0.01	-0.13	0.00	-0.03	0.00	-0.12	1.10	0.38	0.98	0.96
100	15	0.4	-0.08	0.00	-0.09	0.01	-0.01	0.00	-0.08	0.01	0.42	1.09	0.98	0.96
			0.00	-0.08	-0.01	-0.08	0.00	-0.01	0.00	-0.08	1.08	0.40	0.98	1.00
100	5	0.8	-0.38	0.02	-0.42	0.08	-0.21	0.03	-0.37	0.07	0.61	1.16	0.89	0.94
			0.01	-0.35	-0.06	-0.30	-0.01	-0.16	-0.05	-0.26	1.12	0.52	0.97	0.90
100	10	0.8	-0.14	0.01	-0.16	0.04	-0.04	0.01	-0.15	0.03	0.52	1.13	0.95	0.95
			0.00	-0.13	-0.03	-0.10	-0.01	-0.02	-0.02	-0.10	1.12	0.50	0.96	0.98
100	15	0.8	-0.08	0.00	-0.10	0.02	-0.02	0.01	-0.10	0.02	0.56	1.09	0.98	0.96
			0.00	-0.08	-0.02	-0.06	-0.01	0.00	-0.02	-0.06	1.09	0.56	0.96	1.01
200	5	0	-0.36	0.01	-0.36	0.04	-0.18	0.01	-0.32	0.03	0.51	1.15	0.89	0.89
			0.02	-0.37	-0.01	-0.37	0.01	-0.19	0.00	-0.33	1.03	0.52	0.99	0.89
200	10	0	-0.13	0.01	-0.13	0.02	-0.03	0.01	-0.12	0.01	0.31	1.14	0.95	0.91
			0.00	-0.14	-0.01	-0.13	0.00	-0.03	0.00	-0.13	1.10	0.30	0.96	0.96
200	15	0	-0.08	0.00	-0.08	0.01	-0.01	0.01	-0.08	0.01	0.29	1.11	0.98	0.92
			0.00	-0.08	-0.01	-0.08	0.00	-0.01	0.00	-0.08	1.08	0.29	0.96	1.00
200	5	0.4	-0.37	0.01	-0.37	0.04	-0.19	0.02	-0.33	0.03	0.53	1.16	0.89	0.90
			0.02	-0.37	-0.02	-0.35	0.01	-0.18	-0.01	-0.31	1.05	0.51	0.98	0.89
200	10	0.4	-0.13	0.00	-0.14	0.02	-0.03	0.01	-0.13	0.01	0.32	1.14	0.95	0.91
			0.00	-0.13	-0.01	-0.13	-0.01	-0.03	0.00	-0.12	1.11	0.31	0.95	0.96
200	15	0.4	-0.08	0.00	-0.08	0.01	-0.01	0.01	-0.08	0.01	0.31	1.10	0.98	0.93
			0.00	-0.08	-0.01	-0.08	0.00	-0.01	0.00	-0.08	1.09	0.30	0.96	1.00
200	5	0.8	-0.38	0.02	-0.42	0.08	-0.21	0.03	-0.37	0.07	0.58	1.18	0.89	0.90
			0.00	-0.35	-0.06	-0.30	-0.02	-0.16	-0.05	-0.26	1.14	0.49	0.93	0.90
200	10	0.8	-0.14	0.01	-0.16	0.04	-0.04	0.01	-0.15	0.03	0.42	1.14	0.94	0.92
			0.00	-0.13	-0.03	-0.10	-0.01	-0.02	-0.02	-0.10	1.13	0.38	0.92	0.97
200	15	0.8	-0.08	0.00	-0.10	0.02	-0.01	0.01	-0.10	0.02	0.44	1.09	0.98	0.94
			0.00	-0.08	-0.02	-0.06	-0.01	0.00	-0.01	-0.06	1.09	0.42	0.94	1.01

Simulations are based on 5000 replications for model 1 presented in Table 1, $\theta_1 = \begin{bmatrix} 0.06 & 0.05 \\ -0.06 & 0.08 \end{bmatrix}$ and $\theta_2 = \begin{bmatrix} 0.03 & -0.02 \\ -0.03 & 0.04 \end{bmatrix}$, the maximum eigenvalue of the roots of the autoregressive polynomial is $\lambda_{max} = 0.27$. The third column represents the covariance of the innovations, ω. Columns 4-7 indicate the bias of the OLS PVAR estimator, $\hat{\theta}_j$; similarly, columns 8-11 indicate the bias of the bias corrected PVAR estimator, $\hat{\hat{\theta}}_j$, for $j = 1, 2$. The last 4 columns represent the ratio of the RMSE of $\hat{\hat{\theta}}_j$ and the RMSE of $\hat{\theta}_j$, for $j = 1, 2$. Values smaller than 1 indicate a better performance for the bias corrected estimator.

4 Final remarks

In this paper we extend the results of Hahn and Kuersteiner (2002) for deriving an analytical expression for the bias of the fixed effects estimator panel VAR model. Instead of considering one lag, we analyze a model with p lags. We provide the asymptotic distribution of the bias corrected estimator when both n and T are large. We also study its small sample properties using Monte Carlo simulations.

The simulations indicate that our bias corrected estimator outperforms the OLS panel VAR estimator when T is small ($T = 5, 10$), and when the persistence of the model is low. In these cases, the proposed estimator improves significantly in terms of both, the reduction of bias and mean square error.

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Appendix A Proof of Theorem 1

As in HK, Theorem 1 is obtained from Lemmas 5 and 6 given below. Lemmas 1 to 4 are used to prove Lemma 5.

First, it is important to note that the vec operator of the panel VAR - OLS estimator of b, given in (6), can be expressed in terms of the innovations as follows,

$$(\hat{b} - b_0) = \left[R' \left(\sum_{i=1}^{n} \sum_{t=1}^{T} \ddot{Y}_{it-1} \ddot{Y}'_{it-1} \otimes I_{mp} \right) R \right]^{-1} R' \times \sum_{i=1}^{n} \sum_{t=1}^{T} \left(\ddot{Y}_{it-1} \otimes I_{mp} \right) \ddot{U}_{it}.$$

Lemma 1 *Let Y_{it} be generated by the representation given in* (3) *without applying the within transformation, and also let,*

$$S^*_{nt} = \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(U^*_{it-1} \otimes I_{mp} \right) \left(U_{it} - \bar{U}_i \right) \tag{13}$$

Then, under conditions 1, 2 *and* 3 *of HK:*

$$\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(\left(Y_{it-1} - \bar{Y}_{i-} \right) \otimes I_{mp} \right) \left(U_{it} - \bar{U}_i \right) = S^*_{nT} + o_p(1) \tag{14}$$

where $\bar{Y}_{i-} = \frac{1}{T-1} \sum_{t=1}^{T-1} Y_{it}$ *and* $\bar{U}_i = \frac{1}{T} \sum_{t=1}^{T} U_{it}$.

Proof.

Similar to the reparametrization (3) of equation (2), model (1) can be represented as a panel VAR(1) as,

$$Y_{it} = \tilde{\alpha}_i + \Theta Y_{it-1} + U_{it} \tag{15}$$

with $\tilde{\alpha}_i = [\alpha'_i, 0', \ldots, 0']'$.

Expression (15) can be written in the following way,

$$Y_{it} = \Theta_0^t Y_{i0} + (I - \Theta_0)^{-1}(I - \Theta_0^t)\tilde{\alpha}_i + \Theta_0^{t-1} U_{i1} + \Theta_0^{t-2} U_{i2} + \cdots + U_{it} \tag{16}$$

In the stationary case, when $\lim_{w\to\infty}\Theta_0^w = 0$, the stationary approximation of Y_{it} is given by:

$$U_{it}^* \equiv \sum_{j=0}^{\infty} \Theta_0^j U_{it-j},\ t \geq 1, \tag{17}$$

$$Y_{it}^* \equiv (I_{mp} - \Theta_0)^{-1}\tilde{\alpha}_i + U_{it}^*,\ t \geq 0 \tag{18}$$

Using (18), $\sum_{t=1}^{T}\left(Y_{it-1}^* \otimes I\right)\left(U_{it} - \bar{U}_i\right)$ can be expressed as,

$$\sum_{t=1}^{T}\left(Y_{it-1}^* \otimes I\right)\left(U_{it} - \bar{U}_i\right) = \sum_{t=1}^{T}\left[\left((I \otimes \Theta_0^{-1})\tilde{\alpha}_i\right) \otimes I\right]\left(U_{it} - \bar{U}_i\right) + \sum_{t=1}^{T}\left(U_{it-1}^* \otimes I\right)\left(U_{it} - \bar{U}_i\right) \tag{19}$$

Given that the first term of the right hand side of (19) is zero and (13),

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(Y_{it-1}^* \otimes I\right)\left(U_{it} - \bar{U}_i\right) = S_{nT}^* \tag{20}$$

Based on (20) and the fact that $\sum_{t=1}^{T}\left(\bar{Y}_{i-} \otimes I\right)\left(U_{it} - \bar{U}_i\right) = 0$, to prove the lemma we need to show,

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(Y_{it-1} \otimes I)\left(U_{it} - \bar{U}_i\right) = \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(Y_{it-1}^* \otimes I\right)\left(U_{it} - \bar{U}_i\right) + o_p(1)$$

From (16), (17) and (18) we are able to show that,

$$Y_{it} = Y_{it}^* + \Theta_0^t(Y_{i0} - U_{i0}^*) - (I - \Theta_0)^{-1}\Theta_0^t\tilde{\alpha}_i, \tag{21}$$

and after some algebra,

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(Y_{it-1} \otimes I)\left(U_{it} - \bar{U}_i\right) - \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(Y_{it-1}^* \otimes I)\left(U_{it} - \bar{U}_i\right)$$

$$= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t Y_{i0} \otimes I)\left(U_{it} - \bar{U}_i\right) \tag{22}$$

$$- \frac{1}{\sqrt{nT}}\left((I - \Theta_0)^{-1} \otimes I\right)\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t\tilde{\alpha}_i \otimes I)\left(U_{it} - \bar{U}_i\right) \tag{23}$$

$$- \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t U_{i0}^* \otimes I)\left(U_{it} - \bar{U}_i\right), \tag{24}$$

we need to show that the terms (22), (23) and (24) are $o_p(1)$. We start by analyzing (22) and (23).

To prove that (22) is $o_p(1)$, we first calculate the variance of the inner summation of this expression,

$$\text{Var}\left(\sum_{t=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\left(U_{it}-\bar{U}_i\right)\right)=\sum_{t,s=1}^{T}E\left[(\Theta_0^t Y_{i0}\otimes I)\left(U_{it}-\bar{U}_i\right)\left(U_{is}-\bar{U}_i\right)'(Y_{i0}'\Theta_0'^s\otimes I)\right] \tag{25}$$

After some algebra,

$$\begin{aligned}\text{Var}\left(\sum_{t=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\left(U_{it}-\bar{U}_i\right)\right)=&\sum_{t=1}^{T}\left((\Theta_0^t Y_{i0}\otimes I)\Omega(Y_{i0}'\Theta_0'^t\otimes I)\right)\\&-\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\Omega(Y_{i0}'\Theta_0'^s\otimes I)\end{aligned} \tag{26}$$

Thus, the vec operator of the variance of (22) is,

$$\text{vec}\left[\text{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\left(U_{it}-\bar{U}_i\right)\right)\right]=\frac{1}{nT}\text{vec}\left\{\sum_{i=1}^{n}\text{Var}\left[\sum_{t=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\left(U_{it}-\bar{U}_i\right)\right]\right\}, \tag{27}$$

replacing (26) in (27),

$$\begin{aligned}\text{vec}\left[\text{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\left(U_{it}-\bar{U}_i\right)\right)\right]=&\frac{1}{nT}\text{vec}\left\{\sum_{i=1}^{n}\left[\sum_{t=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\Omega(Y_{i0}'\Theta_0'^t\otimes I)\right.\right.\\&\left.\left.-\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}(\Theta_0^t Y_{i0}\otimes I)\Omega(Y_{i0}'\Theta_0'^s\otimes I)\right]\right\}.\end{aligned} \tag{28}$$

Using Kronecker product properties we can rewrite the first term of the right hand side of equation (28),

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{vec}\Big[(\Theta_0^tY_{i0}\otimes I)\Omega(Y_{i0}^{'}\Theta_0^{'t}\otimes I)\Big] = \frac{1}{nT}\sum_{t=1}^{T}\left[(\Theta_0\otimes I)\otimes(\Theta_0\otimes I)\right]^t \text{vec}\sum_{i=1}^{n}\Big[(Y_{i0}\otimes I)\Omega(Y_{i0}^{'}\otimes I)\Big]$$

Then,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{vec}\Big[(\Theta_0^tY_{i0}\otimes I)\Omega(Y_{i0}^{'}\Theta_0^{'t}\otimes I)\Big] = \frac{1}{nT}\Bigg[\left[\mathbf{I}-((\Theta_0\otimes I)\otimes(\Theta_0\otimes I))\right]^{-1}\Bigg(\left[(\Theta_0\otimes I)\otimes(\Theta_0\otimes I)\right] - \left[(\Theta_0\otimes I)\otimes(\Theta_0\otimes I)\right]^{T+1}\Bigg)\Bigg]\text{vec}\sum_{i=1}^{n}\Big[(Y_{i0}\otimes I)\Omega(Y_{i0}^{'}\otimes I)\Big] \tag{29}$$

Proceeding with the second term of the right hand side of equation (28) in a similar manner, we get that,

$$\begin{aligned}\frac{1}{nT}\text{vec}\Bigg\{\sum_{i=1}^{n}\Bigg[\sum_{t=1}^{T}(\Theta_0^tY_{i0}\otimes I)\Omega(Y_{i0}^{'}\Theta_0^{'t}\otimes I) - \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}(\Theta_0^tY_{i0}\otimes I)\Omega(Y_{i0}^{'}\Theta_0^{'s}\otimes I)\Bigg]\Bigg\} = \\ \frac{1}{nT}\Bigg[\left[\mathbf{I}-((\Theta_0\otimes I)\otimes(\Theta_0\otimes I))\right]^{-1}\Bigg(\left[(\Theta_0\otimes I)\otimes(\Theta_0\otimes I)\right] \\ - \left[(\Theta_0\otimes I)\otimes(\Theta_0\otimes I)\right]^{T+1}\Bigg)\Bigg]\text{vec}\sum_{i=1}^{n}\Big[(Y_{i0}\otimes I)\Omega(Y_{i0}^{'}\otimes I)\Big] \\ -\frac{1}{nT^2}\Bigg[\Big(\left[\mathbf{I}-(\Theta_0\otimes I)\right]^{-1}\left[(\Theta_0\otimes I)-(\Theta_0\otimes I)^{T+1}\right]\Big)\otimes \\ \Big(\left[\mathbf{I}-(\Theta_0\otimes I)\right]^{-1}\left[(\Theta_0\otimes I)-(\Theta_0\otimes I)^{T+1}\right]\Big)\Bigg]\text{vec}\sum_{i=1}^{n}\Big[(Y_{i0}\otimes I)\Omega(Y_{i0}^{'}\otimes I)\Big] \\ = o(1).\end{aligned} \tag{30}$$

Therefore, (22) is $o_p(1)$ since it converges in L^2. Following a similar procedure we get that (23) is $o_p(1)$. Then, to conclude the proof of Lemma 1 it only remains to establish that (24) is $o_p(1)$ as follows.

Given the independence between U_{it} and U_{i0}^*, the expected value of (24) is zero. And the variance of the vec operator of the inner summation of (24) without taking in consideration $\bar{U}_i$ is,

$$\begin{aligned}
\text{vec}\left[\text{Var}\left(\sum_{t=1}^{T}(\Theta_0^t U_{i0}^* \otimes I)U_{it}\right)\right] &= \sum_{t_1,t_2=1}^{T}\left[(\Theta_0^{t_1}\otimes I)\otimes(\Theta_0^{t_2}\otimes I)\right] \\
&\quad \times \text{vec}\left(E[(U_{i0}^*\otimes I)U_{it_1}U_{it_2}^{'}(U_{i0}^{*'}\otimes I)]\right) \\
&= \sum_{t_1,t_2=1}^{T}\left[(\Theta_0^{t_1}\otimes I)\otimes(\Theta_0^{t_2}\otimes I)\right] \qquad (31)\\
&\quad \times \text{vec}\left(E[U_{it_1}U_{it_2}^{'}]\otimes E[U_{i0}^{*}U_{i0}^{*'}] + \mathcal{K}_0(t_1,t_2)\right),
\end{aligned}$$

where $\mathcal{K}_0(t_1,t_2) = E[(U_{i0}^*\otimes I)U_{it_1}U_{it_2}^{'}(U_{i0}^{*'}\otimes I)] - E[U_{it_1}U_{it_2}^{'}]\otimes E[U_{i0}^{*}U_{i0}^{*'}]$ contains elements of the form $\text{cum}_{j_1,\ldots,j_4}(U_{i0}^*, U_{i0}^*, U_{it_1}, U_{it_2})$; given condition 2 of HK this term is finite.

Analyzing the first expression of the right hand side of (31), we obtain that it converges to a constant as follows,

$$\begin{aligned}
&\sum_{t_1,t_2=1}^{T}\left[(\Theta_0^{t_1}\otimes I)\otimes(\Theta_0^{t_2}\otimes I)\right]\times\text{vec}\left(E[U_{it_1}U_{it_2}^{'}]\otimes E[U_{i0}^{*}U_{i0}^{*'}]\right) \\
&\quad = \sum_{t=1}^{T}\left[(\Theta_0^{t}\otimes I)\otimes(\Theta_0^{t}\otimes I)\right]\text{vec}\left[\Omega\otimes E(U_{i0}^{*}U_{i0}^{*'})\right] \\
&\quad = \Bigg[\left[I\otimes I\otimes I\otimes I - ((\Theta_0\otimes I)\otimes(\Theta_0\otimes I))\right]^{-1}\left[((\Theta_0\otimes I)\otimes(\Theta_0\otimes I))\right. \\
&\qquad \left. - ((\Theta_0\otimes I)\otimes(\Theta_0\otimes I))^{T+1}\right]\Bigg]\text{vec}\left[\Omega\otimes E[U_{i0}^{*}U_{i0}^{*'}]\right].
\end{aligned}$$

On the other hand, the second term of the right hand side of (31) is bounded by,

$$\begin{aligned}
&\sum_{t_1,t_2=1}^{T}\left|\left|((\Theta_0^{t_1}\otimes I)\otimes(\Theta_0^{t_2}\otimes I))\right|\right|\,||\text{vec}(\mathcal{K}_0(t_1,t_2))|| \\
&\qquad \leq \sup_{t_1,t_2}||\text{vec}(\mathcal{K}_0(t_1,t_2))||\sum_{t_1,t_2=1}^{T}\left|\left|((\Theta_0^{t_1}\otimes I)\otimes(\Theta_0^{t_2}\otimes I))\right|\right| < \infty.
\end{aligned}$$

The last two results imply that $\text{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t U_{i0}^*\otimes I)U_{it}\right) = o(1)$.

To prove that (24) is $o_p(1)$, in addition to the previous result, we need to show that the second term of (24), $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(\Theta_0^t U_{i0}^* \otimes I)\bar{U}_i$, is $o_p(1)$,

$$
\begin{aligned}
&\left\|\text{vec}\left[\text{Var}\left(\sum_{t=1}^{T}(\Theta_0^t U_{i0}^* \otimes I)\bar{U}_i\right)\right]\right\| \\
&= \left\| T^{-2}\sum_{t_1,\ldots,t_4=1}^{T}((\Theta_0^{t_2}\otimes I)\otimes(\Theta_0^{t_1}\otimes I))\text{vec}(E[(U_{i0}^*\otimes I)U_{it_3}U_{it_4}^{'}(U_{i0}^{*'}\otimes I)])\right\| \\
&\leq \left\|\left[(I-(\Theta_0\otimes I))((\Theta_0\otimes I)-(\Theta_0\otimes I)^{T+1})\right]\otimes\left[(I-(\Theta_0\otimes I))((\Theta_0\otimes I)-(\Theta_0\otimes I)^{T+1})\right]\right\| \\
&\quad \left\|T^{-1}\text{vec}(\Omega\otimes E[U_{i0}^*U_{i0}^{*'}])+T^{-2}\text{vec}(\mathcal{K}_0(t_3,t_4))\right\| \\
&\leq \left\|\left[(I-(\Theta_0\otimes I))((\Theta_0\otimes I)-(\Theta_0\otimes I)^{T+1})\right]\otimes\left[(I-(\Theta_0\otimes I))((\Theta_0\otimes I)-(\Theta_0\otimes I)^{T+1})\right]\right\| \\
&\quad \left\|T^{-1}\text{vec}(\Omega\otimes E[U_{i0}^*U_{i0}^{*'}]\right\| + T^{-2}\text{vec}\left\|\mathcal{K}_0(t_3,t_4)\right\| \\
&= O(T^{-1})
\end{aligned}
$$

The previous two results imply that (24) is $o_p(1)$. Since (22), (23) and (24) are $o_p(1)$,

$$
\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\left(Y_{it-1}-\bar{Y}_{i-}\right)\otimes I_{mp}\right]\left(U_{it}-\bar{U}_i\right) = S_{nT}^* + o_p(1)
$$

■

Lemma 2 *Let* Y_{it} *be generated by the representation given in* (15)*. Under conditions 1,2 and 3 of HK,*

$$
\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(U_{it-1}^*\otimes I)\bar{U}_i = \sqrt{\rho}\left[I\otimes I-(\Theta_0\otimes I)\right]^{-1} vec(\Omega) + o_p(1) \qquad (32)
$$

Proof.
Consider the expected value of the following expression,

$$
\begin{aligned}
E\Big[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(U^*_{it-1}\otimes I)\bar{U}_i\Big] &= \frac{n}{\sqrt{nT}}\frac{1}{T}\sum_{t_1,t_2=1}^{T}E[(U^*_{it_1-1}\otimes I)U_{it_2}] \\
&= \sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t_1,t_2=1}^{T}E\left[\sum_{j=1}^{\infty}\text{vec}\left(U_{it_2}U^{'}_{it_1-j}\Theta_0^{'j-1}\right)\right] \\
&= \sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t_1,t_2=1}^{T}\left[\sum_{j=1}^{\infty}\left(\Theta_0^{j-1}\otimes I\right)\text{vec}\left[E\left(U_{it_2}U^{'}_{it_1-j}\right)\right]\right] \\
&= \sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t=1}^{T-1}\sum_{j=0}^{t-1}(\Theta_0^j\otimes I)\text{vec}(\Omega) \\
&= \sqrt{\frac{n}{T}}\frac{1}{T}\left[\sum_{t=1}^{T}\sum_{j=0}^{t-1}(\Theta_0^j\otimes I)\text{vec}(\Omega)-\sum_{j=0}^{T-1}(\Theta_0^j\otimes I)vec(\Omega)\right] \qquad (33)
\end{aligned}
$$

Using the Cesàro averages, we get that,

$$
\frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}(\Theta_0^j\otimes I)=\left[I\otimes I-(\Theta_0\otimes I)\right]^{-1}+o(1) \qquad (34)
$$

Given (33), (34) and the fact that $\frac{1}{T}\sum_{j=0}^{T-1}(\Theta_0^j\otimes I)vec(\Omega)=o(1)$, then,

$$
E\Big[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(U^*_{it}\otimes I)\bar{U}_i\Big]=\sqrt{\frac{n}{T}}\left[I\otimes I-(\Theta_0\otimes I)\right]^{-1}\text{vec}(\Omega)+o(1) \qquad (35)
$$

On the other hand, in order to prove that the inner summation of the left hand side of (32) converges in probability to its expected value, we show that the variance of this term is $o(1)$ as follows,

$$\text{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(U^{*}_{it-1}\otimes I\right)\bar{U}_i\right)$$

$$= E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\left(\left(U^{*}_{it_1-1}\otimes I\right)U_{it_2}-E\left[\left(U^{*}_{it_1-1}\otimes I\right)U_{it_2}\right]\right)\left(\left(U^{*}_{it_3-1}\otimes I\right)U_{it_4}-E\left[\left(U^{*}_{it_3-1}\otimes I\right)U_{it_4}\right]\right)'\right]$$

$$= E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(\Theta_0^{j_1}U_{it_1-j_1-1}\otimes I\right)U_{it_2}U'_{it_4}\left(U'_{it_3-j_2-1}\Theta_0^{'j_2}\otimes I\right)\right] \tag{36}$$

$$- E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(\Theta_0^{j_1}U_{it_1-1}\otimes I\right)U_{it_2}E\left[U'_{it_4}\left(\Theta_0^{j_2}U_{it_3-j_2-1}\otimes I\right)'\right]\right] \tag{37}$$

$$- E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}E\left[\left(\Theta_0^{j_1}U_{it_1-j_1-1}\otimes I\right)U_{it_2}\right]U'_{it_4}\left(\Theta_0^{j_2}U_{it_3-j_2-1}\otimes I\right)'\right] \tag{38}$$

$$- \frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}E\left[\left(\Theta_0^{j_1}U_{it_1-j_1-1}\otimes I\right)U_{it_2}\right]E\left[U'_{it_4}\left(\Theta_0^{j_2}U_{it_3-j_2-1}\otimes I\right)'\right]. \tag{39}$$

We will start by proving that (37) is $o(1)$. Following a similar procedure to the one used in (35), we have that,

$$E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(\Theta_0^{j_1}U_{it_1-1}\otimes I\right)U_{it_2}E\left[U'_{it_4}\left(\Theta_0^{j_2}U_{it_3-j_2-1}\otimes I\right)'\right]\right]$$

$$= \frac{1}{T}\left[I\otimes I-\left(\Theta_0\otimes I\right)\right]^{-1}\text{vec}(\Omega)\text{vec}(\Omega)'\left[I\otimes I-\left(\Theta_0\otimes I\right)\right]'^{-1}+o(1). \tag{40}$$

In a similar manner, it can be shown that (38) and (39) are $o(1)$. Taking a closer look at (36) we have that,

$$E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(\Theta_0^{j_1}U_{it_1-j_1-1}\otimes I\right)U_{it_2}U'_{it_4}\left(U'_{it_3-j_2-1}\Theta_0^{'j_2}\otimes I\right)\right]$$

$$= E\left[\frac{1}{T^3}\sum_{t_1,\cdots,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(\Theta_0^{j_1}\otimes I\right)\left(U_{it_1-j_1-1}\otimes I\right)U_{it_2}U'_{it_4}\left(U'_{it_3-j_2-1}\otimes I\right)\left(\Theta_0^{'j_2}\otimes I\right)\right], \tag{41}$$

Given the cumulant summability assumption (41) is $o(1)$. This result in addition to the previous ones indicate that,

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(U^{*}_{it-1}\otimes I)\bar{U}_i=\frac{1}{\sqrt{nT}}E\left[\sum_{i=1}^{n}\sum_{t=1}^{T}(U^{*}_{it-1}\otimes I)\bar{U}_i\right]+o_p(1).$$

Finally, the preceding result and (35) imply that (32) holds.

■

Lemma 3 *Assume that U_t is a sequence of i.i.d random vectors with $E[U_t]=0$ for all t. Then $cum_{j_1,...,j_k}(U_{t_1},...,U_{t_k})=0$ unless $t_1=t_2=\cdots=t_k$. For notational purposes, $cum(j_1,...,j_k)\equiv cum_{j_1,...,j_k}(U_t,...,U_t)$.*

Lemma 4 *Assume that conditions 1, 2 and 3 of HK hold. Then,*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(U^{*}_{it-1}\otimes I)U_{it}\to\mathcal{N}(0,\Omega\otimes\Upsilon+\mathcal{K})\tag{42}$$

where $\mathcal{K}=\sum_{t=-\infty}^{\infty}\mathcal{K}(t,0)$ y $\mathcal{K}(t_1,t_2)\equiv E[(U^{*}_{it-1}\otimes I)U_{it_1}U'_{it_2}(U^{*\prime}_{it_2-1}\otimes I)]-E[U_{it_1}U'_{it_2}]\otimes E[U^{*}_{i0}U^{*\prime}_{i0}]$.

If, additionally, all innovations U_{it} are independent for all i,t,

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(U^{*}_{it-1}\otimes I)U_{it}\to\mathcal{N}(0,\Omega\otimes\Upsilon)\tag{43}$$

Proof.
It is necessary to check the generalized Lindeberg-Feller conditions for joint asymptotic normality as in Theorem 2 of Phillips and Moon (1999). A sufficient condition for this result to hold is that for all $\ell\in\mathbb{R}^{m^2}$ such that $\ell\ell'=1$, $E\big[\big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\ell'(U^{*}_{it-1}\otimes I)U_{it}\big)^4\big]<\infty$ uniformly over i and T. Let $z_{it}=\ell'(U^{*}_{it-1}\otimes I)U_{it}$, with $E[z_{it}]=0$, then,

$$\begin{aligned}\frac{1}{T^2}\sum_{t_1,...,t_4=1}^{T}E[z_{it_1}z_{it_2}z_{it_3}z_{it_4}]=&\frac{1}{T^2}\sum_{t_1,...,t_4=1}^{T}[\mathrm{Cov}(z_{it_1},z_{it_2})\mathrm{Cov}(z_{it_3},z_{it_4})\\&\quad+\mathrm{Cov}(z_{it_1},z_{it_3})\mathrm{Cov}(z_{it_2},z_{it_4})]\\&+\frac{1}{T^2}\sum_{t_1,...,t_4=1}^{T}[\mathrm{Cov}(z_{it_1},z_{it_4})\mathrm{Cov}(z_{it_2},z_{it_3})\\&\quad+\mathrm{cum}(z_{it_1},z_{it_2},z_{it_3},z_{it_4})],\end{aligned}$$

where,

$$\begin{aligned}
\mathrm{Cov}(z_{it}, z_{is}) &= \ell' E[(U^*_{it-1} \otimes I) U_{it} U'_{is} (U'^*_{is-1} \otimes I)]\ell \\
&= \ell' \mathrm{vec}(E[U^*_{it-1} U'_{is}]) \mathrm{vec}(E[U^*_{is-1} U'_{it}])' \ell \\
&\quad + \ell' E[U_{it} U'_{is}] \otimes E[U^*_{it-1} U^{*\prime}_{is-1}]\ell \\
&\quad + \sum_{j_1,\ldots,j_4=0}^{m-1} \ell_{j_3 m + j_1 + 1} \ell_{j_4 m + j_2 + 1} \mathrm{cum}_{j_1,\ldots,j_4}(U^*_{it-1}, U_{it}, U^*_{is-1}, U_{is}) \\
&= 0 + \ell' (\Omega \otimes E[U^*_{it-1} U^{*\prime}_{is-1}])\ell \cdot 1\{t = s\} \\
&\quad + \sum_{j_1,\ldots,j_4=0}^{m-1} \ell_{j_3 m + j_1 + 1} \ell_{j_4 m + j_2 + 1} \mathrm{cum}_{j_1,\ldots,j_4}(U^*_{it-1}, U_{it}, U^*_{is-1}, U_{is}).
\end{aligned}$$

Using these results it follows that,

$$\begin{aligned}
&\frac{1}{T^2} \sum_{t_1,\ldots,t_4=1}^{T} E[z_{it_1} z_{it_2} z_{it_3} z_{it_4}] \\
&= 3(\ell' (\Omega \otimes E[U^*_{it-1} U^{*\prime}_{it-1}])\ell)^2 + 6(\ell' (\Omega \otimes E[U^*_{it-1} U^{*\prime}_{it-1}])\ell) \\
&\quad \times \left(\frac{1}{T} \sum_{t,s=1}^{T} \sum_{j_1,\ldots,j_4=0}^{m-1} \ell_{j_3 m + j_1 + 1} \ell_{j_4 m + j_2 + 1} \mathrm{cum}_{j_1,\ldots,j_4}(U^*_{it-1}, U_{it}, U^*_{is-1}, U_{is}) \right) \\
&\quad + 3 \left(\frac{1}{T} \sum_{t,s=1}^{T} \sum_{j_1,\ldots,j_4=0}^{m-1} \ell_{j_3 m + j_1 + 1} \ell_{j_4 m + j_2 + 1} \mathrm{cum}_{j_1,\ldots,j_4}(U^*_{it-1}, U_{it}, U^*_{is-1}, U_{is}) \right)^2 \\
&\quad + \frac{1}{T^2} \sum_{t_1,\ldots,t_4=1}^{T} \sum_{j_1,\ldots,j_4=1}^{m^2} \left(\prod_{k=1}^{4} \ell_{j_k} \right) \mathrm{cum}_{j_1,\ldots,j_4}(z_{it_1}, z_{it_2}, z_{it_3}, z_{it_4}).
\end{aligned}$$

Where the terms involving the higher order cumulants are,

$$\begin{aligned}
\frac{1}{T} \sum_{t,s=1}^{T} \mathrm{cum}_{j_1,\ldots,j_4}(U^*_{it-1}, U_{it}, U^*_{is-1}, U_{is}) &= O(1) \\
\sum_{t_1,\ldots,t_4=1}^{T} \mathrm{cum}_{j_1,\ldots,j_4}(z_{it_1}, z_{it_2}, z_{it_3}, z_{it_4}) &= O(T),
\end{aligned}$$

and they are also independent from $(i, j_1, ..., j_4)$. This proves that,

$$T^{-2} \sum_{t_1,...,t_4=1}^{T} E[z_{it_1} z_{it_2} z_{it_3} z_{it_4}] < \infty,$$

uniformly over i and T. Finally,

$$\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(U^*_{it-1} \otimes I\right) U_{it}\right)^2\right] \\
&= \frac{1}{T} \sum_{t,s=1}^{T} E\left[(U^*_{it-1} \otimes I) U_{it} U^{'}_{is} (U^{*'}_{is-1} \otimes I)\right] \\
&= \frac{1}{T} \sum_{t,s=1}^{T} \text{vec}\left(E[U^*_{it-1} U^{'}_{is}]\right) \text{vec}\left(E[U^*_{is-1} U^{'}_{it}]\right) \\
&\quad + \frac{1}{T} \sum_{t,s=1}^{T} E\left[U_{it} U^{'}_{is}\right] \otimes E\left[U^*_{it-1} U^{*'}_{is-1}\right] + \frac{1}{T} \sum_{t,s=1}^{T} \mathcal{K}(t,s) \\
&= \Omega \otimes \Upsilon + \mathcal{K} + o(1),
\end{aligned}$$

where $\mathcal{K} = \sum_{t_1=-\infty}^{\infty} \mathcal{K}(t_1, 0)$. Note that $\text{vec}\left(E\left[U^*_{it-1} U^{'}_{is}\right]\right) \text{vec}\left(E\left[U^*_{is-1} U^{'}_{it}\right]\right) = 0$ for all t and s; and that,

$$\frac{1}{T} \sum_{t,s=1}^{T} E\left[U_{it} U^{'}_{is}\right] \otimes E\left[U^*_{it-1} U^{*'}_{is-1}\right] = \frac{1}{T} \sum_{t=1}^{T} E\left[U_{it} U^{'}_{it}\right] \otimes E\left[U^*_{it-1} U^{*'}_{it-1}\right] = \Omega \otimes \Upsilon,$$

due to strict stationarity. The second part of the Theorem 2 of Phillips and Moon is derived from Lemma 3, which implies that $\mathcal{K}(t_1, t_2) = 0$ for all t_1 and t_2.
■

Lemma 5

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left((Y_{it-1} - \bar{Y}_{i-}) \otimes I\right) (U_{it} - \bar{U}_i) \\
& \qquad \to \mathcal{N}\left(-\sqrt{\rho}(I \otimes I - (\Theta_0 \otimes I))^{-1} vec(\Omega), \Omega \otimes \Upsilon + \mathcal{K}\right)^{4} \qquad (44)
\end{aligned}$$

[4] As before, $\rho = \frac{n}{T}$.

Proof. This result follows from Lemmas 1, 2 and 4 ■

Lemma 6 *Let* Y_{it} *be generated by the representation given in* (15)*. Under conditions 1, 2 and 3 of HK:*

$$\frac{1}{nT}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}(Y_{it-1}-\bar{Y}_{i-})(Y_{it-1}-\bar{Y}_{i-})^{'}\right)=\Upsilon+o_p(1),$$

where, $\Upsilon=\sum_{j=0}^{\infty}\Theta_0^j\Omega\Theta_0^j$.

Proof. First, let us prove that,

$$\frac{1}{nT}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}(Y_{it-1}-\bar{Y}_{i-})(Y_{it-1}-\bar{Y}_{i-})^{'}\right)=E\left[\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)^{'}\right] + o_p(1) \qquad (45)$$

Using (21), the first term of the right hand side of the previous equation can be expressed as,

$$E\left[\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)\left(Y_{it-1}^{*}-E(Y_{t-1}^{*})\right)^{'}\right]=E\left[\left(Y_{it-1}-E(Y_{it-1})\right)\left(Y_{it-1}-E(Y_{it-1})\right)^{'}\right], \quad (46)$$

given that $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(Y_{it-1}-\bar{Y}_{i-})(Y_{it-1}-\bar{Y}_{i-})^{'}=E\left[\left(Y_{it-1}-E(Y_{it-1})\right)\left(Y_{it-1}-E(Y_{it-1})\right)^{'}\right]+o_p(1)$. and (46),(45) holds true. Then, it only remains to prove that,

$$E\left[\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)^{'}\right]=\Upsilon.$$

Using (17) and (18), we get that,

$$\begin{aligned}E\left[\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)\left(Y_{it-1}^{*}-E(Y_{it-1}^{*})\right)^{'}\right]&=E\left[U_{it-1}^{*}U_{it-1}^{*'}\right]\\&=\sum_{j=0}^{\infty}\Theta_0^j\Omega\Theta_0^j=\Upsilon\end{aligned}$$

■

Appendix B Proof of Theorem 2

Using (10) we have,

$$\sqrt{nT}\left(\hat{\hat{b}}-b_0\right)=\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}R'\left(\ddot{Y}_{it-1}\ddot{Y}_{it-1}'\otimes I_{mp}\right)R\right]^{-1}R'$$
$$\times\left\{\sqrt{\frac{n}{T}}\left[(I_{mp}\otimes I_{mp}-(\hat{\Theta}\otimes I_{mp}))^{-1}\mathrm{vec}(\hat{\Omega})\right]\right. \tag{47}$$
$$\left.+\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\ddot{Y}_{it-1}\otimes I_{mp}\right)\ddot{U}_{it}\right\}.$$

Recalling that,

$$\Upsilon=\sum_{j=0}^{\infty}\Theta_0^j\Omega\left(\Theta_0'\right)^j,$$

thus,

$$\mathrm{vec}\left(\Upsilon\right)=\sum_{j=0}^{\infty}\left(\Theta_0\otimes\Theta_0\right)^j\mathrm{vec}\left(\Omega\right)$$
$$=\left(I-\left(\Theta_0\otimes\Theta_0\right)\right)^{-1}\mathrm{vec}\left(\Omega\right),$$

given that $\hat{b}$ is consistent and (5), we have,

$$\sqrt{\tfrac{n}{T}}\left[(I_{mp}\otimes I_{mp}-(\hat{\Theta}\otimes I_{mp}))^{-1}\mathrm{vec}(\hat{\Omega})\right]=\sqrt{\tfrac{n}{T}}\left[(I_{mp}\otimes I_{mp}-(\Theta_0\otimes I_{mp}))^{-1}\mathrm{vec}(\Omega)\right]+o_p(1). \tag{48}$$

Combining (48) with Lemma 5, when all innovations U_{it} are i.i.d., we get,

$$\tfrac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\ddot{Y}_{it-1}\otimes I_{mp}\right)\ddot{U}_{it}+\sqrt{\tfrac{n}{T}}\left[(I_{mp}\otimes I_{mp}-(\hat{\Theta}\otimes I_{mp}))^{-1}\mathrm{vec}(\hat{\Omega})\right]$$
$$\xrightarrow{d}\mathcal{N}\left(\mathbf{0},\Omega\otimes\Upsilon\right).$$

Therefore,

$$\sqrt{nT}\left(\hat{\hat{b}} - b_0\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \mathbf{A}\left(\Omega \otimes \Upsilon\right)\mathbf{A}'\right). \tag{49}$$

where,

$$\mathbf{A} = \left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} R'\left(\ddot{Y}_{it-1}\ddot{Y}_{it-1}' \otimes I_{mp}\right) R\right]^{-1} R'.$$

Appendix C Monte Carlo

Table 3: Small sample performance of the panel VAR estimators

n	T	ω	Bias of $\hat{\theta}_1$ and $\hat{\theta}_2$				Bias of $\hat{\hat{\theta}}_1$ and $\hat{\hat{\theta}}_2$				RMSE ratio			
			$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\hat{\theta}}_1$		$\hat{\hat{\theta}}_2$		$\hat{\hat{\theta}}_1/\hat{\theta}_1$		$\hat{\hat{\theta}}_2/\hat{\theta}_2$	
100	5	0	-0.40	-0.01	-0.39	-0.02	-0.22	-0.01	-0.35	0.00	0.55	1.10	0.89	0.95
			-0.01	-0.38	-0.03	-0.37	-0.01	-0.20	-0.01	-0.33	1.12	0.54	0.90	0.90
100	10	0	-0.15	-0.01	-0.15	-0.01	-0.03	-0.02	-0.14	0.00	0.36	1.13	0.97	0.95
			-0.01	-0.14	-0.01	-0.14	-0.01	-0.03	0.00	-0.13	1.13	0.36	0.93	0.97
100	15	0	-0.09	-0.01	-0.09	-0.01	-0.01	-0.01	-0.09	0.00	0.36	1.10	1.01	0.97
			-0.01	-0.08	-0.01	-0.08	-0.01	-0.01	0.00	-0.08	1.10	0.37	0.96	1.01
100	5	0.4	-0.40	-0.01	-0.40	-0.02	-0.22	-0.01	-0.35	0.00	0.56	1.11	0.89	0.97
			-0.02	-0.38	-0.04	-0.37	-0.02	-0.20	-0.01	-0.33	1.11	0.54	0.90	0.90
100	10	0.4	-0.14	-0.01	-0.15	-0.01	-0.03	-0.01	-0.14	0.00	0.38	1.12	0.97	0.97
			-0.01	-0.14	-0.02	-0.14	-0.02	-0.03	0.00	-0.13	1.13	0.37	0.92	0.97
100	15	0.4	-0.09	-0.01	-0.09	-0.01	-0.01	-0.01	-0.09	0.00	0.39	1.09	1.01	0.98
			-0.01	-0.08	-0.01	-0.08	-0.01	-0.01	0.00	-0.08	1.10	0.40	0.94	1.01
100	5	0.8	-0.41	0.00	-0.41	-0.01	-0.23	0.01	-0.36	0.01	0.60	1.12	0.90	1.02
			-0.03	-0.37	-0.05	-0.36	-0.03	-0.19	-0.02	-0.32	1.11	0.56	0.93	0.90
100	10	0.8	-0.15	0.00	-0.16	0.00	-0.04	-0.01	-0.15	0.01	0.49	1.11	0.97	1.01
			-0.02	-0.13	-0.03	-0.13	-0.02	-0.02	-0.01	-0.12	1.13	0.50	0.92	0.97
100	15	0.8	-0.09	0.00	-0.09	0.00	-0.01	-0.01	-0.09	0.01	0.52	1.08	1.00	1.01
			-0.01	-0.08	-0.02	-0.08	-0.01	-0.01	-0.01	-0.08	1.09	0.55	0.95	1.01
200	5	0	-0.40	-0.01	-0.39	-0.02	-0.22	-0.01	-0.35	0.00	0.54	1.10	0.89	0.90
			-0.01	-0.38	-0.03	-0.37	-0.01	-0.20	-0.01	-0.33	1.11	0.53	0.81	0.89
200	10	0	-0.15	-0.01	-0.15	-0.01	-0.03	-0.02	-0.14	0.00	0.30	1.14	0.97	0.91
			-0.01	-0.14	-0.01	-0.14	-0.01	-0.03	0.00	-0.13	1.14	0.29	0.88	0.97
200	15	0	-0.09	-0.01	-0.09	-0.01	-0.01	-0.01	-0.09	0.00	0.28	1.12	1.01	0.93
			-0.01	-0.08	-0.01	-0.08	-0.01	-0.01	0.00	-0.08	1.11	0.28	0.92	1.01
200	5	0.4	-0.40	-0.01	-0.40	-0.02	-0.22	-0.01	-0.35	0.00	0.55	1.10	0.89	0.94
			-0.02	-0.38	-0.04	-0.37	-0.01	-0.20	-0.01	-0.33	1.10	0.53	0.82	0.89
200	10	0.4	-0.15	-0.01	-0.15	-0.01	-0.03	-0.01	-0.14	0.00	0.32	1.13	0.97	0.95
			-0.01	-0.14	-0.02	-0.14	-0.02	-0.03	0.00	-0.13	1.14	0.31	0.86	0.97
200	15	0.4	-0.09	-0.01	-0.09	-0.01	-0.01	-0.01	-0.09	0.00	0.29	1.11	1.01	0.97
			-0.01	-0.08	-0.01	-0.08	-0.01	-0.01	0.00	-0.08	1.11	0.29	0.91	1.01
200	5	0.8	-0.41	0.00	-0.41	-0.01	-0.23	0.01	-0.36	0.01	0.58	1.12	0.89	1.02
			-0.03	-0.37	-0.05	-0.36	-0.03	-0.18	-0.02	-0.32	1.11	0.53	0.86	0.90
200	10	0.8	-0.15	0.00	-0.16	0.00	-0.04	-0.01	-0.15	0.01	0.40	1.11	0.96	1.02
			-0.02	-0.13	-0.03	-0.13	-0.02	-0.02	-0.01	-0.12	1.14	0.38	0.88	0.97
200	15	0.8	-0.09	0.00	-0.09	0.00	-0.01	0.00	-0.09	0.01	0.41	1.08	1.00	1.02
			-0.01	-0.08	-0.02	-0.08	-0.02	0.00	-0.01	-0.08	1.10	0.42	0.91	1.01

Simulations are based on 5000 replications for model 2 presented in Table 1, $\theta_1 = \begin{bmatrix} 0.10 & -0.10 \\ -0.12 & 0.10 \end{bmatrix}$ and $\theta_2 = \begin{bmatrix} 0.09 & 0.08 \\ 0.10 & 0.05 \end{bmatrix}$, the maximum eigenvalue of the roots of the autoregressive polynomial is $\lambda_{max} = 0.41$. The third column represents the covariance of the innovations, ω. Columns 4-7 indicate the bias of the OLS PVAR estimator, $\hat{\theta}_j$; similarly, columns 8-11 indicate the bias of the bias corrected PVAR estimator, $\hat{\hat{\theta}}_j$, for $j = 1, 2$. The last 4 columns represent the ratio of the RMSE of $\hat{\hat{\theta}}_j$ and the RMSE of $\hat{\theta}_j$, for $j = 1, 2$. Values smaller than 1 indicate a better performance for the bias corrected estimator.

Table 4: Small sample performance of the panel VAR estimators

n	T	ω	Bias of $\hat{\theta}_1$; $\hat{\theta}_2$				Bias of $\hat{\hat{\theta}}_1$; $\hat{\hat{\theta}}_2$				RMSE ratio			
			$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\hat{\theta}}_1$		$\hat{\hat{\theta}}_2$		$\hat{\hat{\theta}}_1/\hat{\theta}_1$		$\hat{\hat{\theta}}_2/\hat{\theta}_2$	
100	5	0	-0.46	0.02	-0.43	-0.07	-0.27	-0.02	-0.41	-0.04	0.60	1.08	0.96	0.80
			0.01	-0.44	0.08	-0.42	0.05	-0.26	0.05	-0.39	1.38	0.60	0.81	0.92
100	10	0	-0.15	-0.01	-0.13	-0.06	-0.02	-0.06	-0.15	-0.03	0.32	1.92	1.10	0.66
			0.02	-0.15	0.07	-0.13	0.06	-0.03	0.04	-0.13	1.84	0.36	0.68	1.00
100	15	0	-0.09	-0.01	-0.07	-0.04	0.00	-0.05	-0.09	-0.02	0.36	1.92	1.20	0.65
			0.01	-0.09	0.05	-0.07	0.05	-0.01	0.02	-0.08	1.79	0.36	0.67	1.06
100	5	0.4	-0.42	-0.01	-0.37	-0.08	-0.22	-0.06	-0.36	-0.05	0.54	1.38	0.97	0.78
			0.00	-0.48	0.08	-0.48	0.04	-0.31	0.06	-0.44	1.28	0.65	0.83	0.92
100	10	0.4	-0.13	-0.03	-0.09	-0.08	0.00	-0.08	-0.11	-0.04	0.33	1.90	1.18	0.65
			0.01	-0.16	0.07	-0.17	0.06	-0.06	0.04	-0.16	1.78	0.45	0.71	0.98
100	15	0.4	-0.08	-0.02	-0.05	-0.05	0.02	-0.06	-0.07	-0.02	0.46	1.85	1.29	0.65
			0.01	-0.09	0.05	-0.10	0.05	-0.02	0.03	-0.10	1.75	0.41	0.71	1.02
100	5	0.8	-0.30	-0.16	-0.20	-0.22	-0.08	-0.23	-0.21	-0.16	0.41	1.37	1.05	0.79
			0.05	-0.57	0.20	-0.63	0.10	-0.43	0.16	-0.57	1.41	0.77	0.86	0.91
100	10	0.8	-0.08	-0.08	0.01	-0.16	0.06	-0.15	-0.02	-0.10	0.92	1.58	1.02	0.71
			0.03	-0.19	0.14	-0.25	0.09	-0.11	0.10	-0.23	1.68	0.65	0.79	0.93
100	15	0.8	-0.04	-0.05	0.02	-0.10	0.05	-0.09	-0.01	-0.07	1.10	1.58	0.90	0.70
			0.02	-0.11	0.09	-0.15	0.06	-0.05	0.06	-0.14	1.62	0.60	0.79	0.98
200	5	0	-0.46	0.02	-0.43	-0.07	-0.27	-0.02	-0.41	-0.04	0.59	1.10	0.95	0.72
			0.00	-0.44	0.08	-0.42	0.05	-0.26	0.05	-0.39	1.58	0.59	0.75	0.92
200	10	0	-0.15	-0.01	-0.13	-0.06	-0.02	-0.06	-0.14	-0.03	0.26	2.42	1.10	0.59
			0.02	-0.15	0.07	-0.13	0.06	-0.03	0.04	-0.13	2.18	0.31	0.63	1.00
200	15	0	-0.09	-0.01	-0.07	-0.04	0.00	-0.05	-0.09	-0.02	0.26	2.37	1.21	0.56
			0.01	-0.08	0.05	-0.07	0.05	-0.01	0.02	-0.08	2.17	0.27	0.60	1.06
200	5	0.4	-0.42	-0.01	-0.37	-0.08	-0.22	-0.06	-0.36	-0.05	0.53	1.61	0.97	0.71
			0.00	-0.48	0.08	-0.48	0.04	-0.31	0.06	-0.44	1.43	0.64	0.78	0.92
200	10	0.4	-0.13	-0.03	-0.09	-0.08	0.00	-0.08	-0.11	-0.04	0.25	2.21	1.19	0.60
			0.01	-0.16	0.07	-0.17	0.06	-0.06	0.04	-0.16	2.18	0.40	0.67	0.97
200	15	0.4	-0.08	-0.02	-0.05	-0.05	0.02	-0.06	-0.07	-0.02	0.37	2.20	1.33	0.57
			0.01	-0.09	0.05	-0.10	0.05	-0.02	0.03	-0.10	2.10	0.34	0.64	1.02
200	5	0.8	-0.30	-0.16	-0.21	-0.22	-0.08	-0.23	-0.21	-0.16	0.34	1.41	1.05	0.76
			0.04	-0.56	0.20	-0.63	0.10	-0.42	0.16	-0.57	1.58	0.76	0.85	0.91
200	10	0.8	-0.08	-0.08	0.02	-0.16	0.06	-0.15	-0.02	-0.10	0.89	1.67	1.00	0.69
			0.03	-0.19	0.14	-0.25	0.09	-0.11	0.10	-0.23	1.97	0.61	0.78	0.93
200	15	0.8	-0.04	-0.05	0.02	-0.10	0.05	-0.09	0.00	-0.07	1.13	1.70	0.81	0.67
			0.02	-0.11	0.09	-0.15	0.06	-0.05	0.07	-0.14	1.93	0.54	0.77	0.98

Simulations are based on 5000 replications for model 3 presented in Table 1, $\theta_1 = \begin{bmatrix} 0.30 & -0.20 \\ 0.15 & 0.20 \end{bmatrix}$ and $\theta_2 = \begin{bmatrix} 0.15 & -0.10 \\ 0.10 & 0.15 \end{bmatrix}$, the maximum eigenvalue of the roots of the autoregressive polynomial is $\lambda_{max} = 0.59$. The third column represents the covariance of the innovations, ω. Columns 4-7 indicate the bias of the OLS PVAR estimator, $\hat{\theta}_j$; similarly, columns 8-11 indicate the bias of the bias corrected PVAR estimator, $\hat{\hat{\theta}}_j$, for $j = 1, 2$. The last 4 columns represent the ratio of the RMSE of $\hat{\hat{\theta}}_j$ and the RMSE of $\hat{\theta}_j$, for $j = 1, 2$. Values smaller than 1 indicate a better performance for the bias corrected estimator.

Table 5: Small sample performance of the panel VAR estimators

n	T	ω	Bias of $\hat{\theta}_1$; $\hat{\theta}_2$				Bias of $\hat{\hat{\theta}}_1$; $\hat{\hat{\theta}}_2$				RMSE ratio			
			$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\hat{\theta}}_1$		$\hat{\hat{\theta}}_2$		$\hat{\hat{\theta}}_1/\hat{\theta}_1$		$\hat{\hat{\theta}}_2/\hat{\theta}_2$	
100	5	0	-0.53	0.00	-0.48	0.10	-0.33	0.05	-0.47	0.07	0.64	1.41	0.97	0.80
			0.02	-0.42	-0.05	-0.44	-0.05	-0.23	0.00	-0.43	1.31	0.56	0.83	0.97
100	10	0	-0.16	0.02	-0.12	0.09	-0.03	0.08	-0.14	0.05	0.34	2.24	1.13	0.62
			-0.02	-0.12	-0.08	-0.10	-0.13	-0.01	-0.01	-0.12	3.04	0.36	0.46	1.11
100	15	0	-0.08	0.01	-0.06	0.06	0.01	0.06	-0.08	0.03	0.37	2.25	1.30	0.56
			-0.02	-0.07	-0.06	-0.05	-0.10	0.00	0.01	-0.06	3.18	0.42	0.49	1.21
100	5	0.4	-0.55	-0.02	-0.55	0.12	-0.37	0.03	-0.53	0.09	0.69	1.18	0.96	0.83
			-0.07	-0.35	-0.08	-0.37	-0.16	-0.15	-0.01	-0.36	1.72	0.47	0.71	1.00
100	10	0.4	-0.17	0.00	-0.17	0.10	-0.05	0.07	-0.17	0.06	0.42	2.11	1.04	0.65
			-0.06	-0.09	-0.09	-0.05	-0.17	0.02	-0.01	-0.07	2.39	0.49	0.43	1.33
100	15	0.4	-0.09	0.01	-0.08	0.06	-0.01	0.06	-0.09	0.03	0.38	2.18	1.16	0.59
			-0.03	-0.05	-0.07	-0.01	-0.12	0.02	0.00	-0.03	2.69	0.66	0.44	1.37
100	5	0.8	-0.55	-0.07	-0.69	0.20	-0.39	-0.02	-0.67	0.18	0.74	0.88	0.97	0.89
			-0.34	-0.14	-0.12	-0.24	-0.45	0.07	-0.01	-0.28	1.30	0.68	0.62	1.14
100	10	0.8	-0.14	-0.04	-0.26	0.15	-0.05	0.02	-0.25	0.11	0.56	0.94	0.97	0.73
			-0.14	-0.03	-0.13	0.03	-0.26	0.09	-0.02	0.00	1.75	1.76	0.45	0.78
100	15	0.8	-0.08	-0.02	-0.13	0.09	-0.01	0.04	-0.13	0.06	0.57	1.35	1.03	0.68
			-0.07	-0.02	-0.09	0.04	-0.15	0.05	-0.01	0.02	2.01	1.39	0.48	0.66
200	5	0	-0.53	0.00	-0.48	0.10	-0.33	0.05	-0.46	0.07	0.64	1.66	0.97	0.77
			0.02	-0.42	-0.05	-0.44	-0.05	-0.23	0.00	-0.43	1.47	0.55	0.71	0.97
200	10	0	-0.16	0.02	-0.12	0.09	-0.03	0.08	-0.14	0.05	0.28	2.76	1.14	0.58
			-0.02	-0.12	-0.08	-0.10	-0.13	-0.01	-0.01	-0.12	3.64	0.27	0.34	1.11
200	15	0	-0.09	0.01	-0.06	0.06	0.01	0.06	-0.08	0.03	0.27	2.76	1.33	0.51
			-0.02	-0.07	-0.05	-0.05	-0.10	0.00	0.01	-0.06	3.95	0.32	0.37	1.24
200	5	0.4	-0.55	-0.02	-0.55	0.12	-0.37	0.03	-0.53	0.09	0.68	1.22	0.96	0.81
			-0.07	-0.35	-0.08	-0.36	-0.16	-0.15	-0.01	-0.36	1.91	0.45	0.57	1.00
200	10	0.4	-0.17	0.01	-0.17	0.10	-0.05	0.07	-0.17	0.06	0.38	2.76	1.04	0.62
			-0.06	-0.09	-0.09	-0.05	-0.17	0.02	-0.01	-0.07	2.56	0.40	0.33	1.38
200	15	0.4	-0.09	0.01	-0.08	0.06	-0.01	0.06	-0.09	0.03	0.28	2.80	1.16	0.55
			-0.03	-0.05	-0.07	-0.01	-0.12	0.02	0.00	-0.03	2.97	0.57	0.33	1.59
200	5	0.8	-0.55	-0.07	-0.69	0.20	-0.39	-0.02	-0.66	0.17	0.73	0.77	0.96	0.88
			-0.34	-0.14	-0.12	-0.24	-0.45	0.07	-0.01	-0.28	1.31	0.61	0.48	1.14
200	10	0.8	-0.14	-0.04	-0.26	0.15	-0.05	0.03	-0.25	0.10	0.48	0.89	0.96	0.72
			-0.14	-0.03	-0.13	0.03	-0.26	0.09	-0.02	0.00	1.79	2.14	0.36	0.66
200	15	0.8	-0.08	-0.02	-0.13	0.09	-0.01	0.04	-0.13	0.06	0.44	1.50	1.02	0.65
			-0.07	-0.02	-0.09	0.04	-0.15	0.05	-0.01	0.02	2.11	1.61	0.37	0.57

Simulations are based on 5000 replications for model 4 presented in Table 1, $\theta_1 = \begin{bmatrix} 0.40 & 0.15 \\ -0.30 & 0.30 \end{bmatrix}$ and $\theta_2 = \begin{bmatrix} 0.25 & 0.10 \\ -0.20 & 0.20 \end{bmatrix}$, the maximum eigenvalue of the roots of the autoregressive polynomial is $\lambda_{max} = 0.75$. The third column represents the covariance of the innovations, ω. Columns 4-7 indicate the bias of the OLS PVAR estimator, $\hat{\theta}_j$; similarly, columns 8-11 indicate the bias of the bias corrected PVAR estimator, $\hat{\hat{\theta}}_j$, for $j = 1, 2$. The last 4 columns represent the ratio of the RMSE of $\hat{\hat{\theta}}_j$ and the RMSE of $\hat{\theta}_j$, for $j = 1, 2$. Values smaller than 1 indicate a better performance for the bias corrected estimator.

Table 6: Small sample performance of the panel VAR estimators

n	T	ω	Bias of $\hat{\theta}_1$; $\hat{\theta}_2$				Bias of $\hat{\hat{\theta}}_1$; $\hat{\hat{\theta}}_2$				RMSE ratio			
			$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\hat{\theta}}_1$		$\hat{\hat{\theta}}_2$		$\hat{\hat{\theta}}_1/\hat{\theta}_1$		$\hat{\hat{\theta}}_2/\hat{\theta}_2$	
100	5	0	-0.56	-0.01	-0.50	0.03	-0.34	0.02	-0.52	0.01	0.62	1.15	1.04	0.94
			-0.02	-0.53	-0.03	-0.49	-0.06	-0.32	-0.01	-0.49	1.37	0.61	0.89	1.01
100	10	0	-0.17	0.01	-0.16	0.06	0.01	0.09	-0.23	0.01	0.30	2.57	1.41	0.50
			-0.03	-0.16	-0.07	-0.15	-0.11	-0.01	0.00	-0.20	2.59	0.29	0.48	1.28
100	15	0	-0.09	0.01	-0.08	0.05	0.05	0.09	-0.15	-0.01	0.60	3.15	1.80	0.46
			-0.02	-0.09	-0.06	-0.07	-0.10	0.02	0.01	-0.12	3.10	0.42	0.49	1.52
100	5	0.4	-0.57	-0.03	-0.54	0.05	-0.36	0.00	-0.55	0.03	0.64	1.00	1.03	0.88
			-0.07	-0.48	-0.06	-0.44	-0.12	-0.26	-0.02	-0.46	1.44	0.56	0.79	1.03
100	10	0.4	-0.17	-0.01	-0.20	0.08	-0.01	0.07	-0.25	0.02	0.32	2.07	1.26	0.49
			-0.06	-0.14	-0.08	-0.11	-0.15	0.03	0.00	-0.16	2.29	0.39	0.44	1.51
100	15	0.4	-0.10	0.00	-0.10	0.06	0.03	0.08	-0.16	0.00	0.47	2.87	1.50	0.41
			-0.04	-0.08	-0.07	-0.04	-0.12	0.04	0.01	-0.09	2.74	0.68	0.44	2.01
100	5	0.8	-0.55	-0.09	-0.65	0.13	-0.36	-0.05	-0.66	0.12	0.67	0.89	1.01	0.90
			-0.25	-0.33	-0.12	-0.34	-0.32	-0.09	-0.04	-0.39	1.27	0.39	0.63	1.14
100	10	0.8	-0.13	-0.07	-0.30	0.15	-0.01	0.02	-0.32	0.08	0.55	0.77	1.08	0.61
			-0.14	-0.07	-0.13	-0.01	-0.26	0.11	-0.01	-0.10	1.84	1.36	0.40	2.26
100	15	0.8	-0.07	-0.04	-0.17	0.11	0.02	0.05	-0.19	0.04	0.74	1.21	1.17	0.49
			-0.07	-0.04	-0.10	0.03	-0.18	0.08	0.00	-0.03	2.16	1.54	0.41	1.03
100	5	0	-0.56	-0.01	-0.50	0.03	-0.34	0.02	-0.52	0.01	0.61	1.20	1.04	0.83
			-0.02	-0.53	-0.03	-0.49	-0.06	-0.32	-0.01	-0.49	1.57	0.61	0.80	1.01
100	10	0	-0.17	0.01	-0.16	0.06	0.01	0.09	-0.23	0.01	0.22	3.33	1.42	0.38
			-0.03	-0.16	-0.07	-0.15	-0.11	-0.01	0.00	-0.20	3.05	0.21	0.36	1.29
100	15	0	-0.09	-0.02	0.01	-0.09	0.05	-0.10	0.09	0.02	0.57	3.75	4.14	0.35
			-0.08	-0.06	0.05	-0.07	-0.15	0.01	-0.01	-0.12	1.84	0.41	0.36	1.54
100	5	0.4	-0.57	-0.04	-0.54	0.05	-0.36	0.00	-0.55	0.03	0.63	0.91	1.03	0.81
			-0.07	-0.48	-0.05	-0.44	-0.12	-0.26	-0.02	-0.46	1.54	0.55	0.67	1.03
100	10	0.4	-0.17	-0.01	-0.20	0.08	-0.01	0.07	-0.25	0.02	0.24	2.63	1.27	0.40
			-0.06	-0.14	-0.08	-0.10	-0.15	0.03	0.00	-0.16	2.46	0.33	0.32	1.53
100	15	0.4	-0.09	0.00	-0.10	0.06	0.03	0.08	-0.16	0.00	0.41	3.86	1.51	0.31
			-0.04	-0.07	-0.07	-0.04	-0.12	0.05	0.01	-0.09	3.00	0.67	0.36	2.19
100	5	0.8	-0.55	-0.08	-0.65	0.13	-0.36	-0.05	-0.65	0.11	0.66	0.80	1.01	0.88
			-0.25	-0.33	-0.11	-0.34	-0.33	-0.09	-0.04	-0.39	1.28	0.34	0.53	1.14
100	10	0.8	-0.13	-0.07	-0.30	0.15	-0.01	0.02	-0.32	0.08	0.41	0.63	1.07	0.58
			-0.14	-0.07	-0.13	-0.01	-0.26	0.11	-0.01	-0.10	1.89	1.43	0.30	2.92
100	15	0.8	-0.07	-0.04	-0.17	0.11	0.02	0.05	-0.19	0.04	0.62	1.24	1.17	0.44
			-0.07	-0.04	-0.10	0.03	-0.18	0.08	0.00	-0.03	2.28	1.70	0.31	1.04

Simulations are based on 5000 replications for model 5 presented in Table 1, $\theta_1 = \begin{bmatrix} 0.60 & 0.10 \\ -0.10 & 0.50 \end{bmatrix}$ and $\theta_2 = \begin{bmatrix} 0.20 & 0.10 \\ -0.10 & 0.20 \end{bmatrix}$, the maximum eigenvalue of the roots of the autoregressive polynomial is $\lambda_{max} = 0.83$. The third column represents the covariance of the innovations, ω. Columns 4-7 indicate the bias of the OLS PVAR estimator, $\hat{\theta}_j$; similarly, columns 8-11 indicate the bias of the bias corrected PVAR estimator, $\hat{\hat{\theta}}_j$, for $j = 1, 2$. The last 4 columns represent the ratio of the RMSE of $\hat{\hat{\theta}}_j$ and the RMSE of $\hat{\theta}_j$, for $j = 1, 2$. Values smaller than 1 indicate a better performance for the bias corrected estimator.


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