On the valuations of the near polygon $\mathbb{H}_n$

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: bdb@cage.ugent.be

Abstract

We characterize the valuations of the near polygon $\mathbb{H}_n$ that are induced by classical valuations of the dual polar space $DW(2n-1,2)$ into which it is isometrically embeddable. An application to near $2n$-gons that contain $\mathbb{H}_n$ as a full subgeometry is given.

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1 Introduction

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with non-empty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near polygon* if the following three properties are satisfied:

(NP1) Every two distinct points are incident with at most one line.

(NP2) For every point x and every line L, there exists a unique point on L that is nearest to x with respect to the distance function $\mathrm{d}(\cdot,\cdot)$ in the collinearity graph Γ.

(NP3) The diameter of Γ is finite.

If d is the diameter of Γ, then the near polygon is called a *near $2d$-gon*. This paper is about two families of near polygons, the family $DW(2n-1,2)$, $n \geq 2$ of symplectic dual polar spaces over the field $\mathbb{F}_2$ and the family $\mathbb{H}_n$, $n \geq 2$ of near polygons that arise from matchings of complete graphs.

The main tool for studying near polygons that contain isometrically embedded full sub-near-polygons is that of valuations. In the literature, one can find different variants of the notion of valuation, but in the current paper, we will take the most basic definition. A *semi-valuation* of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a map $f : \mathcal{P} \to \mathbb{Z}$ with the property that every line L contains a unique point x_L such that $f(x) = f(x_L) + 1$ for every point x on L distinct from x_L. If the minimal value attained by f is equal to 0, then the semi-valuation is called a *valuation*. If x is a point of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, then the map $\mathcal{P} \to \mathbb{Z}; y \mapsto \mathrm{d}(x, y)$ is a valuation of $\mathcal{S}$, the so-called *classical valuation with center* x. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two near polygons such that $\mathcal{S}_1$ is a full subgeometry of $\mathcal{S}_2$, then every semi-valuation of $\mathcal{S}_2$ will *induce* a (semi-)valuation of $\mathcal{S}_1$.

Valuations seem to be the most valuable tool when it comes to studying and classifying near polygons that contain isometrically embedded full sub-near-polygons. For this reason, they form an indispensable tool for classifying so-called dense near polygons, as theoretical results of Shult & Yanushka [18] and Brouwer & Wilbrink [7] guarantee that such near polygons must have isometrically embedded sub-near-polygons (like quads, hexes, maxes, etc.). It is therefore no surprise that the very first successes of "valuation theory" were achieved in the classification of dense near polygons (more precisely, for octagons with three and four points per line). In more recent years, valuations have also been successful in the study of generalized polygons. They have been used to show that the Ree-Tits generalized octagon of order $(2, 4)$ is the unique generalized octagon of that order that contains a suboctagon of order $(2, 1)$, to show that the dual twisted triality hexagon of order $(2, 8)$ is the unique near hexagon that contains the split Cayley hexagon $H(2)^D$ as a proper isometrically embedded full subgeometry, and to show that there are no semi-finite generalized hexagons that contain a subhexagon of order $(2, 2)$. More details about these results can be found in [1, 14]. An overview of the most important results and applications of valuations till the year 2012 can be found in the survey paper [13]. A recent and exciting breakthrough was the fact that valuations have been used to construct new near polygons that are highly symmetric and closely related to finite simple groups [2]. In the latter paper, a chain of near polygons was described that was intimately related to the Suzuki chains of groups and graphs. In the recent work [3], valuations have been used to characterize these Suzuki chain near polygons.

The construction and characterization results obtained in [1, 2, 3, 14] all invoke valuation geometries. The *valuation geometry* of a near polygon $\mathcal{S}$ is a point-line geometry whose points are the valuations of $\mathcal{S}$ and whose lines are certain nice sets of mutually neighbouring valuations. Two valuations f_1 and f_2 of $\mathcal{S}$ are called *neighbouring* if there exists an $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$ for every point x of $\mathcal{S}$.

In the believe that we have not yet seen the full potential of valuations and that more classification results are still to come, we pursue our investigation of valuations in the present paper. From the eight basic classes of dense near polygons with three points per line described in [10, Chapter 6], there are seven whose valuations have been completely classified elsewhere in the literature. The remaining class consists of the near polygons $\mathbb{H}_n$, $n \geq 2$ and these are under investigation here.

Although we have not been successful in classifying all valuations of $\mathbb{H}_n$, we were still able to obtain the following partial classification. In order to understand this theorem, one should know that the near polygon $\mathbb{H}_n$ can be isometrically embedded as a full subgeometry in $DW(2n-1, 2)$ and that $\mathbb{H}_n$ has full subquadrangles isomorphic to $W(2) \cong \mathbb{H}_2$, the so-called $W(2)$-*quads* (see Section 2).

Theorem 1.1 *Suppose* $\mathbb{H}_n$ *is isometrically embedded into* $DW(2n-1, 2)$. *Then:*

(1) *The valuations of* $\mathbb{H}_n$ *induced by the classical valuations of* $DW(2n-1, 2)$ *are pre-*

cisely the valuations of $\mathbb{H}_n$ *for which all induced* $W(2)$*-quad valuations are classical. In fact, each such valuation of* $\mathbb{H}_n$ *is induced by precisely one classical valuation of* $DW(2n-1,2)$*.*

(2) *Let* x_1 *and* x_2 *be two distinct points of* $DW(2n-1,2)$ *and let* f_1 *and* f_2 *be the valuations of* $\mathbb{H}_n$ *induced by the classical valuations of* $DW(2n-1,2)$ *with centers* x_1 *and* x_2*. Then* f_1 *and* f_2 *are neighbouring if and only if* $\mathrm{d}(x_1,x_2)=1$*.*

If x is a point and Q is a $W(2)$-quad of a near polygon such that $\mathrm{d}(x,Q)=i$, then by [18, Proposition 2.6] the set of points of Q at distance i from x is either a singleton or an *ovoid* (which contains a unique point of each line of Q). The pair (x,Q) is called *classical* or *ovoidal* depending on whether the first or last case occurs.

Although Theorem 1.1 does not offer a complete classification of all valuations of $\mathbb{H}_n$, this result is certainly useful for studying near $2n$-gons $\mathcal{S}$ that contain $\mathbb{H}_n$ as an isometrically embedded subgeometry such that (x,Q) is classical for every point x of $\mathcal{S}$ and every $W(2)$-quad Q of $\mathbb{H}_n$. Every known such near polygon $\mathcal{S}$ is an isometric embedded full subgeometry of $DW(2n-1,2)$ and we conjecture that this is always the case. We did not succeed in proving this, but by relying on Theorem 1.1, we were able to prove the following.

Theorem 1.2 *Suppose the near polygon* $\mathbb{H}_n$ *is isometrically embedded as a full subgeometry in* $\Delta=DW(2n-1,2)$*. If* $\mathcal{S}$ *is a near* $2n$*-gon that contains an isometrically embedded copy* $\mathbb{H}'_n$ *of* $\mathbb{H}_n$ *such that every point* $W(2)$*-quad pair* (x,Q) *with* $Q\subseteq\mathbb{H}'_n$ *is classical, then every line of* $\mathcal{S}$ *is incident with precisely three points and there exists a map* θ *from the point set of* $\mathcal{S}$ *to the point set of* $\Delta=DW(2n-1,2)$ *satisfying the following:*

(1) θ *defines an isomorphism between* $\mathbb{H}'_n$ *and* $\mathbb{H}_n$*.*

(2) *If* $\{x,y,z\}$ *is a line of* $\mathcal{S}$*, then* $\{x^\theta,y^\theta,z^\theta\}$ *is a line of* $\Delta=DW(2n-1,2)$*.*

(3) *If* x *is a point of* $\mathcal{S}$ *and* y *is a point of* $\mathbb{H}'_n$*, then* $d_{\mathcal{S}}(x,y)=d_\Delta(x^\theta,y^\theta)$*. In particular, we have* $d_{\mathcal{S}}(x,\mathbb{H}'_n)=d_\Delta(x^\theta,\mathbb{H}_n)$*.*

Note that if the map θ in Theorem 1.2 is injective, then $\mathcal{S}$ can be regarded as a full subgeometry of $DW(2n-1,2)$.

2 Preliminaries and useful results

2.1 The near polygon $\mathbb{H}_n$

The near polygon $\mathbb{H}_n$, $n\geq 2$, is defined as the point-line geometry whose points are the partitions of the set $X=\{1,2,\dots,2n+2\}$ in $n+1$ subsets of size 2 and whose lines are the partitions of X in one subset of size 4 and $n-1$ subsets of size 2. A point p is incident with a line L if and only if p (regarded as a partition) is a refinement of L. $\mathbb{H}_n$ is a near $2n$-gon with three points per line. The near polygon $\mathbb{H}_n$ was introduced in [6, Section 5] and its basic properties can be found in [10, Section 6.2]. Throughout this paper, we meet two families of full subgeometries of $\mathbb{H}_n$.

(1) Suppose $n \geq 3$ and Y is a subset of size 2 of X. Then the points of $\mathbb{H}_n$ that contain Y form a subspace of $\mathbb{H}_n$ on which the induced geometry is isomorphic to $\mathbb{H}_{n-1}$. We call these full subgeometries the $\mathbb{H}_{n-1}$-*subgeometries*.

(2) Suppose Π is a partition of X in one subset of size 6 and $n-2$ subsets of size 2. Then the points of $\mathbb{H}_n$ that refine the partition Π form a subspace on which the induced subgeometry is isomorphic to the generalized quadrangle $W(2) \cong \mathbb{H}_2$. We call these full subgeometries the $W(2)$-*quads*.

In the abstract theory of near polygons, *quads* are defined as non-empty convex subspaces on which the induced full subgeometries are (nondegenerate) generalized quadrangles [15]. The near $2n$-gon $\mathbb{H}_n$, $n \geq 3$ has two types of quads, the $W(2)$-quads defined above and the grid-quads (which are associated with partitions Π of X in two subsets of size 4 and $n-3$ subsets of size 2). The following facts are well-known, see e.g. [10, Section 6.2].

Lemma 2.1 (1) *Suppose M is an $\mathbb{H}_{n-1}$-subgeometry of $\mathbb{H}_n$, $n \geq 3$. Then $d(x, M) \leq 1$ for every point x of $\mathbb{H}_n$. Moreover, there exists a unique point $\pi_M(x) \in M$ such that $d(x, y) = d(x, \pi_M(x)) + d(\pi_M(x), y)$ for every point y of M.*

(2) *Suppose M is an $\mathbb{H}_{n-1}$-subgeometry and Q is a quad of $\mathbb{H}_n$, $n \geq 3$ that meets M, but is not contained in M. Then $Q \cap M$ is a line.*

Lemma 2.2 *There exists a partition of $\mathbb{H}_n$, $n \geq 3$ in $2n+1$ mutually disjoint $\mathbb{H}_{n-1}$-subgeometries.*

Proof. Consider the $2n+1$ $\mathbb{H}_{n-1}$-subgeometries corresponding to the subsets $\{1, i\}$ of X, where $i \in \{2, 3, \ldots, 2n+2\}$. ∎

2.2 The near polygon $DW(2n-1, 2)$

With a symplectic polarity ζ of $\mathrm{PG}(2n-1, 2)$, there is associated a polar space $W(2n-1, 2)$ in the sense of Tits [19, Chapter 7]. The points of $W(2n-1, 2)$ are the points of $\mathrm{PG}(2n-1, 2)$, while the singular subspaces of $W(2n-1, 2)$ are the subspaces of $\mathrm{PG}(2n-1, 2)$ that are totally isotropic with respect to ζ. With $W(2n-1, 2)$, there is associated a dual polar space. This is the point-line geometry whose points are the maximal singular subspaces of $W(2n-1, 2)$ (those of dimension $n-1$) and whose lines are the next-to-maximal singular subspaces of $W(2n-1, 2)$ (those of dimension $n-2$), with incidence being reverse containment. The dual polar space $DW(2n-1, 2)$ is a near $2n$-gon with three points per line. If x is a point of $DW(2n-1, 2)$, then $\Gamma_i(x)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from x, and $x^\perp := \{x\} \cup \Gamma_1(x)$.

If α is a singular subspace of $W(2n-1,2)$ of dimension $n-1-k$ where $k \in \{0,1,\dots,n\}$, then the set of all maximal singular subspaces of $W(2n-1,2)$ containing α is a convex subspace F_α of diameter k of $DW(2n-1,2)$. This correspondence between singular subspaces of $W(2n-1,2)$ and non-empty convex subspaces of $DW(2n-1,2)$ is bijective. We will say that α is the *singular subspace of* $W(2n-1,2)$ *corresponding to* F_α, or that F_α is *the convex subspace of* $DW(2n-1,2)$ *corresponding to* α. Convex subspaces of diameter 2 are called *quads* and those of diameter $n-1$ are called *maxes*. The convex subspaces through a given point x of $DW(2n-1,2)$, ordered by ordinary inclusion, define a projective space $Res(x)$ isomorphic to $\mathrm{PG}(n-1,2)$. Every two points x_1 and x_2 of $DW(2n-1,2)$ at distance k from each other are contained in a unique convex subspace $\langle x_1, x_2\rangle$ of diameter k.

Suppose F is a convex subspace of diameter k. If $k \geq 2$, then the full subgeometry $\widetilde{F}$ of $DW(2n-1,2)$ induced on F by the lines that have all their points in F is isomorphic to $DW(2k-1,2)$. In particular, if $k=2$, then F is a quad and $\widetilde{F} \cong DW(3,2) \cong W(2)$. By abuse of notation, we will often write $F \cong DW(2k-1,2)$ instead of $\widetilde{F} \cong DW(2k-1,2)$. The maximal distance from a point x of $DW(2n-1,2)$ to F is equal to $n-k$. Moreover, there exists a unique point $\pi_F(x) \in F$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_F(x)) + \mathrm{d}(\pi_F(x),y)$ for every point y of F. A non-empty convex subspace of a near polygon having the latter property is called *classical*.

Two non-empty convex subspaces F_1 and F_2 of $DW(2n-1,2)$ are called *parallel* if $\mathrm{d}(x_1,F_2) = \mathrm{d}(x_2,F_1) = \mathrm{d}(F_1,F_2)$ for every $x_1 \in F_1$ and every $x_2 \in F_2$. If F_1 and F_2 are two parallel convex subspaces of $DW(2n-1,2)$, then they have the same diameter and the map $F_i \to F_{3-i}; x \mapsto \pi_{F_{3-i}}(x)$ defines an isomorphism between $\widetilde{F_i}$ and $\widetilde{F_{3-i}}$ for every $i \in \{1,2\}$. Moreover, $\theta_1^{-1} = \theta_2$.

Consider the ambient projective space $\mathrm{PG}(2n-1,2)$ of $W(2n-1,2)$. A line $\{x_1,x_2,x_3\}$ of $\mathrm{PG}(2n-1,2)$ that is not a singular line of $W(2n-1,2)$ is called a *hyperbolic line* of $W(2n-1,2)$. If M_i with $i \in \{1,2,3\}$ is the max of $DW(2n-1,2)$ corresponding to the point x_i, then $\{M_1,M_2,M_3\}$ is called a *hyperbolic set of maxes*. This is a set of three mutually disjoint maxes such that every line meeting two of them also meets the third.

A *hyperplane* of a point-line geometry $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ is a proper subset of $\mathcal{P}$ that meets each line in either a singleton or the whole line. If x is a point of a near $2n$-gon $\mathcal{S}$ for which $\Gamma_n(x) \neq \emptyset$, then the set of points of $\mathcal{S}$ at distance at most $n-1$ from x is a hyperplane of $\mathcal{S}$, the so-called *singular hyperplane with center x*.

Lemma 2.3 *Let x_1 and x_2 be two distinct points of the dual polar space $DW(2n-1,2)$, $n \geq 2$, and let H_i, $i \in \{1,2\}$, denote the singular hyperplane of $DW(2n-1,2)$ with center x_i. Then the complement H_3 of the symmetric difference of H_1 and H_2 is a singular hyperplane of $DW(2n-1,2)$ if and only if $d(x_1,x_2) \in \{1,2\}$. If $d(x_1,x_2)=1$ and if x_3 is the center of the singular hyperplane H_3, then $\{x_1,x_2,x_3\}$ is a line of $DW(2n-1,2)$. If $d(x_1,x_2)=2$ and if x_3 is the center of the singular hyperplane H_3, then $\{x_1,x_2,x_3\}$ is a hyperbolic line of the quad $\langle x_1,x_2\rangle \cong W(2)$.*

Proof. (1) Suppose $\mathrm{d}(x_1,x_2)=1$, let x_3 be the third point of $DW(2n-1,2)$ on the line x_1x_2 and let H_3 denote the singular hyperplane with center x_3. We show that H_3

coincides with the complement of the symmetric difference of H_1 and H_2. So, we must show that an arbitrary point u of $DW(2n-1,2)$ is contained in either 1 or 3 of the sets H_1, H_2 and H_3, or equivalently, that u is opposite to either 0 or 2 of the points x_1, x_2 and x_3. But this follows from the fact that $DW(2n-1,2)$ is a near $2n$-gon.

(2) Suppose $\mathrm{d}(x_1,x_2)=2$ and let $Q \cong W(2)$ be the unique quad through x_1 and x_2. Let x_3 denote the unique point of Q distinct from x_1 and x_2 which is collinear with every point of $\Gamma_1(x_1)\cap\Gamma_1(x_2)$, and let H_3 denote the singular hyperplane of $DW(2n-1,2)$ with center x_3. Note that $\{x_1,x_2,x_3\}$ is a hyperbolic line of $Q \cong W(2)$. For every $i \in \{1,2,3\}$, put $U_i := x_i^\perp \cap Q$. We show that H_3 coincides with the complement of the symmetric difference of H_1 and H_2. Since Q is classical in $DW(2n-1,2)$, this is equivalent with showing that U_3 coincides with the complement of the symmetric difference of U_1 and U_2 (in Q). The latter claim is easily verified by a direct inspection in $Q \cong W(2)$.

(3) Suppose $n=3$ and $\mathrm{d}(x_1,x_2)=3$. Let H be the hyperplane of $DW(5,2)$ which is the complement of the symmetric difference of H_1 and H_2. Then it is known (see e.g. Cooperstein [9, proof of Proposition 2.1]) that $\widetilde{H}$ is isomorphic to the split Cayley generalized hexagon of order 2. The hyperplane H is therefore called a *hexagonal hyperplane* of $DW(5,2)$. So, H is not a singular hyperplane.

(4) Suppose now that $\delta := \mathrm{d}(x_1,x_2) \geq 3$. We prove by downwards induction on $i \in \{0,1,\ldots,\delta\}$ that there exists a convex subspace F_i of diameter i such that

- $\mathrm{d}(x_1,\pi_{F_i}(x_1)) = \mathrm{d}(x_2,\pi_{F_i}(x_2)) = n-i$;
- $\mathrm{d}(\pi_{F_i}(x_1),\pi_{F_i}(x_2)) = i$.

Suppose first that $i=\delta$. In the dual polar space $DW(2n-1,2)$, there exists a point y at maximal distance n from x_2 such that x_1 is on a shortest path from x_2 to y. So, $\mathrm{d}(y,x_1)=n-\delta$ and $\mathrm{d}(y,x_2)=n$. Recall that the convex subspaces of $DW(2n-1,2)$ through y define an $(n-1)$-dimensional projective space $Res(y)$. The convex subspace $\langle x_1,y\rangle$ corresponds to an $(n-1-\delta)$-dimensional subspace α of $Res(y)$. Let F_δ denote a convex subspace of diameter δ through y such that the $(\delta-1)$-dimensional subspace β of $Res(y)$ corresponding to F_δ is disjoint from α. Since $\pi_{F_\delta}(x_1)$ is on a shortest path between x_1 and y, the convex subspace $\langle \pi_{F_\delta}(x_1),y\rangle$ is contained in both F_δ and $\langle x_1,y\rangle$. Hence, $y=\pi_{F_\delta}(x_1)$ and $\mathrm{d}(x_1,\pi_{F_\delta}(x_1)) = \mathrm{d}(x_1,y) = n-\delta$. If z is a point of F_δ at distance δ from $\pi_{F_\delta}(x_2)$, then from $n \geq \mathrm{d}(x_2,z) = \mathrm{d}(x_2,\pi_{F_\delta}(x_2)) + \mathrm{d}(\pi_{F_\delta}(x_2),z) = \mathrm{d}(x_2,\pi_{F_\delta}(x_2))+\delta$, it follows that $\mathrm{d}(x_2,\pi_{F_\delta}(x_2)) \leq n-\delta$. From $n = \mathrm{d}(x_2,y) = \mathrm{d}(x_2,\pi_{F_\delta}(x_1)) = \mathrm{d}(x_2,\pi_{F_\delta}(x_2)) + \mathrm{d}(\pi_{F_\delta}(x_2),\pi_{F_\delta}(x_1)) \leq n-\delta+\delta = n$, it follows that $\mathrm{d}(x_2,\pi_{F_\delta}(x_2)) = n-\delta$ and $\mathrm{d}(\pi_{F_\delta}(x_1),\pi_{F_\delta}(x_2)) = \delta$.

Suppose $i<\delta$. By the induction hypothesis, there exists a convex subspace F_{i+1} of diameter $i+1$ of $DW(2n-1,2)$ satisfying $\mathrm{d}(x_1,\pi_{F_{i+1}}(x_1)) = \mathrm{d}(x_2,\pi_{F_{i+1}}(x_2)) = n-i-1$ and $\mathrm{d}(\pi_{F_{i+1}}(x_1),\pi_{F_{i+1}}(x_2)) = i+1$. Put $x_1' := \pi_{F_{i+1}}(x_1)$ and $x_2' := \pi_{F_{i+1}}(x_2)$. Now, let L_1 denote a line of F_{i+1} through the point x_1' and let y_1 denote the unique point of L_1 at distance $\mathrm{d}(x_1',x_2')-1=i$ from x_2'. Let L_2 denote a line of F_{i+1} through x_2' not contained in $\langle y_1,x_2'\rangle$ and let y_2 denote the unique point of L_2 at distance $\mathrm{d}(x_1',x_2')-1=i$ from x_1'. Let z_i, $i\in\{1,2\}$, denote the unique point of the line L_i distinct from x_i' and y_i. Recall that for every point u of L_j, $j\in\{1,2\}$, there exists a unique point on L_{3-j} nearest to u. Using this it is straightforward to verify that $\mathrm{d}(L_1, L_2) = \mathrm{d}(x_1', y_2) = \mathrm{d}(x_2', y_1) = \mathrm{d}(z_1, z_2) = i$. Put $F_i := \langle z_1, z_2 \rangle$. Since F_i does not contain the points x_1' and x_2', we have $\pi_{F_i}(x_1) = z_1$ and $\pi_{F_i}(x_2) = z_2$. So, $\mathrm{d}(x_1, \pi_{F_i}(x_1)) = \mathrm{d}(x_1, \pi_{F_{i+1}}(x_1)) + \mathrm{d}(\pi_{F_{i+1}}(x_1), z_1) = n - i - 1 + 1 = n - i$, $\mathrm{d}(x_2, \pi_{F_i}(x_2)) = \mathrm{d}(x_2, \pi_{F_{i+1}}(x_2)) + \mathrm{d}(\pi_{F_{i+1}}(x_2), z_2) = n - i - 1 + 1 = n - i$ and $\mathrm{d}(\pi_{F_i}(x_1), \pi_{F_i}(x_2)) = \mathrm{d}(z_1, z_2) = i$.

Suppose now that the complement of the symmetric difference of H_1 and H_2 is a singular hyperplane H_3 with center x_3. Put $F := F_3$ and $H_i' := F \cap H_i$ for every $i \in \{1, 2, 3\}$. Recall that F is classical in $DW(2n-1, 2)$. If $\mathrm{d}(x_3, \pi_F(x_3)) < n - 3$, then $F \subset H_3$ and hence $H_3' = F$. If $\mathrm{d}(x_3, \pi_F(x_3)) = n - 3$, then $H_3' = H_3 \cap F$ is the singular hyperplane of $\widetilde{F}$ with center $\pi_F(x_3)$. Since $\mathrm{d}(x_1, \pi_F(x_1)) = \mathrm{d}(x_2, \pi_F(x_2)) = n - 3$, the hyperplanes H_1' and H_2' of $\widetilde{F}$ are singular hyperplanes having the points $\pi_F(x_1)$ and $\pi_F(x_2)$ as respective centers. Now, H_3' equals the complement of the symmetric difference of H_1' and H_2'. Since $\mathrm{d}(\pi_F(x_1), \pi_F(x_2)) = 3$, H_3' should be a hexagonal hyperplane of $\widetilde{F}$ (recall (3)). But that is impossible since H_3' is either F or a singular hyperplane of $\widetilde{F}$. ∎

2.3 Isometric embeddings of $\mathbb{H}_n$ in $DW(2n-1, 2)$

With a *full isometric embedding* of a point-line geometry $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ into a point-line geometry $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$, we mean an injective map $\theta : \mathcal{P}_1 \to \mathcal{P}_2$ that maps lines of $\mathcal{S}_1$ to lines of $\mathcal{S}_2$ such that $\mathrm{d}_{\mathcal{S}_1}(x, y) = \mathrm{d}_{\mathcal{S}_2}(x^\theta, y^\theta)$ for all $x, y \in \mathcal{P}_1$. By [6, Section 5], there exists a (full) isometric embedding of $\mathbb{H}_n$ into $DW(2n-1, 2)$. By [12], such an embedding is even unique, up to isomorphism. If $\mathbb{H}_n$ is isometrically embedded into $DW(2n-1, 2)$, then every two points x and y of $\mathbb{H}_n$ are contained in a unique convex subspace F of diameter k of $\mathbb{H}_n$ and a unique convex subspace $\overline{F}$ of diameter k of $DW(2n-1, 2)$. Moreover, the points of F are precisely those points of $\mathbb{H}_n$ that are contained in $\overline{F}$ ([12, Proposition 2.5]).

Lemma 2.4 *Suppose $\mathbb{H}_n$ is isometrically embedded into $DW(2n-1, 2)$, and let M_1 and M_2 be two disjoint $\mathbb{H}_{n-1}$-subgeometries of $\mathbb{H}_n$. Then $\overline{M_1}$ and $\overline{M_2}$ are two disjoint maxes of $DW(2n-1, 2)$.*

Proof. This is a special case of Proposition 2.7 of De Bruyn [12]. ∎

Lemma 2.5 *Suppose $\mathbb{H}_n$ is isometrically embedded into $DW(2n-1, 2)$. Then for every quad Q of $DW(2n-1, 2)$, there exists a $W(2)$-quad Q' of $\mathbb{H}_n$ parallel with Q.*

Proof. We will prove this by induction on the diameter n of $DW(2n-1, 2)$.

Suppose first that $n = 2$. Then $\mathbb{H}_2 \cong DW(3, 2) \cong W(2)$. The claim is obvious as there is only one quad in $DW(3, 2)$.

Suppose therefore that $n \geq 3$ and that the claim of the lemma holds for every isometric embedding of $\mathbb{H}_{n'}$ into $DW(2n'-1, 2)$, where $n' \in \{2, 3, \ldots, n-1\}$. Let Q be an arbitrary quad of $DW(2n-1, 2)$. By Lemma 2.2, there exists a collection $M_1, M_2, \ldots, M_{2n+1}$ of mutually disjoint $\mathbb{H}_{n-1}$-subgeometries of $\mathbb{H}_n$ partitioning its point set. For every $i \in \{1, 2, \ldots, 2n+1\}$, let $\overline{M_i}$, $i \in \{1, 2, \ldots, 2n+1\}$, denote the unique max of $DW(2n-1, 2)$

containing M_i. Then $\widetilde{M_i} \cong \mathbb{H}_{n-1}$ is isometrically embedded into $\widetilde{\overline{M_i}} \cong DW(2n-3,2)$. By Lemma 2.4, $\{\overline{M_1}, \overline{M_2}, \ldots, \overline{M_{2n+1}}\}$ is a set of $2n+1$ mutually disjoint maxes of $DW(2n-1,2)$.

We prove that there exists a $j \in \{1,2,\ldots,2n+1\}$ such that $\overline{M_j}$ is disjoint from Q. If this would not be the case, then by Lemma 2.1(2) each $\overline{M_i}$, $i \in \{1,2,\ldots,2n+1\}$, intersects Q in at least a line. This would imply that $15 = |Q| \geq (2n+1)\cdot 3 \geq 21$, a contradiction.

So, let $j \in \{1,2,\ldots,2n+1\}$ such that $\overline{M_j} \cap Q = \emptyset$. Then $\pi_{\overline{M_j}}(Q)$ is a $W(2)$-quad of $\widetilde{\overline{M_j}} \cong DW(2n-3,2)$. By the induction hypothesis, there exists a $W(2)$-quad Q' of $\widetilde{M_j} \cong \mathbb{H}_{n-1}$ such that $\pi_{\overline{M_j}}(Q)$ and Q' are parallel quads of $\widetilde{\overline{M_j}}$. Since $\overline{M_j}$ is a classical convex subspace of $DW(2n-1,2)$, it is now readily seen that also the quads Q and Q' need to be parallel. ■

The dual polar space $DW(2n-1,2)$, $n \geq 2$, has a nice full projective embedding ϵ in a projective space $\mathrm{PG}(V)$, where V is some vector space of dimension $\binom{2n}{n} - \binom{2n}{n-2}$ over $\mathbb{F}_2$, see e.g. Cooperstein [8, Proposition 5.1]. This embedding is known as the *Grassmann embedding*. If Π is a hyperplane of $\mathrm{PG}(V)$, then the set of all points of $DW(2n-1,2)$ that are mapped into Π by ϵ is a hyperplane of $DW(2n-1,2)$, a so-called *hyperplane of* $DW(2n-1,2)$ *arising from the Grassmann embedding*.

Lemma 2.6 *Suppose $\mathbb{H}_n$ is isometrically embedded into $DW(2n-1,2)$. Then for every hyperplane H of $\mathbb{H}_n$, there exists a unique hyperplane H' of $DW(2n-1,2)$ arising from the Grassmann embedding such that $H \subseteq H'$. For this hyperplane H', we have $H = \mathbb{H}_n \cap H'$.*

Proof. Let $\epsilon_1 : DW(2n-1,2) \to \Sigma_1$ denote the Grassmann embedding of the dual polar space $DW(2n-1,2)$. Then ϵ_1 induces an embedding ϵ_2 of $\mathbb{H}_n$ into a subspace Σ_2 of Σ_1. By [5, Section 3] and [6, Section 5], $\Sigma_2 = \Sigma_1$ and ϵ_2 is isomorphic to the so-called universal embedding of $\mathbb{H}_n$. This means by [16, Corollary 2, p. 180] that there exists a unique hyperplane Π of $\Sigma := \Sigma_1 = \Sigma_2$ such that $H = \epsilon_2^{-1}(\epsilon_2(X) \cap \Pi) = \epsilon_1^{-1}(\epsilon_1(X) \cap \Pi)$, where X is the point set of $\mathbb{H}_n$. Now, put $H'' := \epsilon_1^{-1}(\epsilon_1(\mathcal{P}) \cap \Pi)$, where $\mathcal{P}$ is the set of points of $DW(2n-1,2)$. Then H'' is a hyperplane of $DW(2n-1,2)$ arising from the Grassmann embedding such that $H \subseteq H'' \cap X \subsetneq X$. By [4, Theorem 7.3] and [17, Lemma 6.1], the hyperplane H of $\mathbb{H}_n$ must be a maximal proper subspace, implying that $H = X \cap H''$. The maximality of H also implies that $\epsilon_1(H)$ generates the subspace Π.

Conversely, suppose that H' is a hyperplane of $DW(2n-1,2)$ arising from the Grassmann embedding such that $H \subseteq H'$. Let Π' be the unique hyperplane of Σ such that $H' := \epsilon_1^{-1}(\epsilon_1(\mathcal{P}) \cap \Pi')$. As $H \subseteq H'$ and $\epsilon_1(H)$ generates Π, we should have $\Pi' = \Pi$, i.e. $H' = H''$. ■

2.4 Semi-valuations of near polygons

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near polygon having only lines of size 3. Suppose also that $f_1 : \mathcal{P} \to \mathbb{Z}$ and $f_2 : \mathcal{P} \to \mathbb{Z}$ are two maps such that $|f_1(x) - f_2(x)| \leq 1$ for every point $x \in \mathcal{P}$. If $f_1(x) = f_2(x)$, then we define $f_1 \diamond f_2(x) := f_1(x) - 1 = f_2(x) - 1$. If $|f_1(x) - f_2(x)| = 1$, then we define $f_1 \diamond f_2(x) := \max\{f_1(x), f_2(x)\}$. Clearly, $f_2 \diamond f_1 = f_1 \diamond f_2$. Notice also that $|f_1(x) - f_1 \diamond f_2(x)|, |f_2(x) - f_1 \diamond f_2(x)| \leq 1$ for every point x of $\mathcal{S}$, and that $(f_1 \diamond f_2) \diamond f_1 = f_2$ and $(f_1 \diamond f_2) \diamond f_2 = f_1$. The following lemma was proved in [11, Proposition 2.4].

Lemma 2.7 ([11]) *Suppose $\mathcal{S}$ is a near polygon having only lines of size 3 and f_1, f_2 are two semi-valuations of $\mathcal{S}$ such that $|f_1(x) - f_2(x)| \leq 1$ for every point x of $\mathcal{S}$. Then also $f_1 \diamond f_2$ is a semi-valuation of $\mathcal{S}$.*

For a proof of the following result, see Lemma 2.2 of [1].

Lemma 2.8 ([1]) *Let $\mathcal{S}$ and $\mathcal{S}'$ be two near polygons such that $\mathcal{S}$ is an isometrically embedded subgeometry of $\mathcal{S}'$. For every point x of $\mathcal{S}'$ and for every point y of $\mathcal{S}$, we define $f_x(y) := d(x, y)$. Then:*

(1) *For every point x of $\mathcal{S}'$, the map f_x is a semi-valuation of $\mathcal{S}$.*

(2) *If x_1 and x_2 are two collinear points of $\mathcal{S}'$, then f_{x_1} and f_{x_2} are two neighbouring semi-valuations of $\mathcal{S}$.*

(3) *If $L = \{x, y, z\}$ is a line of size 3 of $\mathcal{S}'$, then $f_x \diamond f_y = f_z$.*

3 Proof of Theorem 1.1(1)

In this section, we suppose that $\mathbb{H}_n$ is isometrically embedded into $DW(2n-1, 2)$.

Lemma 3.1 *For every point x of $DW(2n-1, 2)$, there exists a point of $\mathbb{H}_n$ at maximal distance n from x.*

Proof. Let y be a point of $\mathbb{H}_n$ at maximal distance δ from x, and let F denote the unique convex subspace of diameter δ of $DW(2n-1, 2)$ containing x and y. Then $F \cap \mathbb{H}_n$ is a convex subspace of $\mathbb{H}_n$ whose diameter δ' is at most δ. Suppose $\delta \leq n - 1$. Then also $\delta' \leq n - 1$ and so there exists a line L of $\mathbb{H}_n$ through y not contained in $F \cap \mathbb{H}_n$. In particular, L is not contained in F and thus contains a point at distance $\delta + 1$ from x, in contradiction with the maximality of $\mathrm{d}(x, y)$. We must thus have that $\delta = n$. ■

If x is a point of $DW(2n-1, 2)$, then the classical valuation of $DW(2n-1, 2)$ with center x induces a valuation g_x of $\mathbb{H}_n$. For every point y of $\mathbb{H}_n$, we have $g_x(y) = \mathrm{d}(x, y) - m_x$, where $m_x := \mathrm{d}(x, \mathbb{H}_n)$. Lemma 3.1 then implies the following.

Corollary 3.2 *If M_x is the maximal value attained by g_x, then $m_x + M_x = n$.*

Lemma 3.3 *Every $W(2)$-quad valuation induced by g_x is classical.*

Proof. Let Q be a $W(2)$-quad of $\mathbb{H}_n$. Then Q is also a $W(2)$-quad of $DW(2n-1,2)$ and so is classical in $DW(2n-1,2)$. It follows that there exists a unique point $x' \in Q$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,x') + \mathrm{d}(x',y)$ for every $y \in Q$. The latter implies that $g_x(y) = g_x(x') + \mathrm{d}(x',y)$ for every $y \in Q$, i.e. the valuation of $\widetilde{Q}$ induced by g_x is a classical with center equal to x'. ■

Lemma 3.4 *If x_1 and x_2 are two distinct points of $DW(2n-1,2)$, then $g_{x_1} \neq g_{x_2}$.*

Proof. For every $i \in \{1,2\}$, let H_i denote the singular hyperplane of $DW(2n-1,2)$ with center x_i. By Lemma 3.1, $H_i \cap \mathbb{H}_n$ is a hyperplane H_i' of $\mathbb{H}_n$. Since H_1, H_2 arise from the Grassmann embedding and $H_1 \neq H_2$, we must have $H_1' \neq H_2'$ by Lemma 2.6. As H_i' with $i \in \{1,2\}$ is the set of points of $\mathbb{H}_n$ with non-maximal g_{x_i}-value, we must have that $g_{x_1} \neq g_{x_2}$. ■

Let M_1 be the $\mathbb{H}_{n-1}$-subgeometry of $\mathbb{H}_n$, $n \geq 3$ corresponding to the pair $\{1,2\}$, let M_2 be the $\mathbb{H}_{n-1}$-subgeometry of $\mathbb{H}_n$ corresponding to the pair $\{1,3\}$ and let M_3 denote the $\mathbb{H}_{n-1}$-subgeometry of $\mathbb{H}_n$ corresponding to the pair $\{2,3\}$. Then M_1, M_2, M_3 are mutually disjoint and every line meeting two of M_1, M_2, M_3 also meets the third. For every $i \in \{1,2,3\}$, $\widetilde{M_i} \cong \mathbb{H}_{n-1}$ is isometrically embedded into $\widetilde{\overline{M_i}} \cong DW(2n-3,2)$. Note also that $\{\overline{M_1}, \overline{M_2}, \overline{M_3}\}$ is a hyperbolic set of maxes of $DW(2n-1,2)$.

Lemma 3.5 *Suppose g_1 and g_2 are two semi-valuations of $\mathbb{H}_n$, $n \geq 3$ for which all induced $W(2)$-quad valuations are classical. If $g_1(x) = g_2(x)$ for all $x \in M_1 \cup M_2 \cup M_3$, then $g_1 = g_2$.*

Proof. We still need to prove that $g_1(x) = g_2(x)$ for every point x of $\mathbb{H}_n$ not contained in $M_1 \cup M_2 \cup M_3$. We will rely on Lemma 2.1. For every $i \in \{1,2,3\}$, let x_i denote the unique point of M_i collinear with x. Then $\mathrm{d}(x_1,x_2) = 2$ and the quad $Q := \langle x_1, x_2 \rangle$ intersects each M_i in a line L_i. The quad Q intersects $M_1 \cup M_2 \cup M_3$ in the 3×3-grid $G = L_1 \cup L_2 \cup L_3$ and contains the additional point x, showing that Q is a $W(2)$-quad. There are now two cases to consider.

(1) g_1 (and hence g_2) takes three values in G. Then there exists a point $u \in G$ such that $g_1(y) = g_2(y) = g_1(u) + \mathrm{d}(u,y)$ for every $y \in G$. The fact that all induced $W(2)$-quad valuations are classical then implies that $g_1(y) = g_2(y) = g_1(u) + \mathrm{d}(u,y)$ for every $y \in Q$. In particular, $g_1(x) = g_2(x)$.

(2) g_1 (and hence g_2) takes two values in G. Then there exists an ovoid $\{u_1,u_2,u_3\}$ of G such that $g_1(u_1) = g_1(u_2) = g_1(u_3) = g_1(v) - 1$ for every $v \in G \setminus \{u_1,u_2,u_3\}$. If $u \in Q$ denotes the unique point of Q collinear with u_1, u_2 and u_3, then the fact that all induced $W(2)$-quad valuations are classical implies that $g_1(y) = g_2(y) = g_1(u) + d(u,y)$ for every $y \in Q$. In particular, $g_1(x) = g_2(x)$. ■

Suppose now that f is a valuation of $\mathbb{H}_n$ with the property that every induced $W(2)$-quad valuation is classical. By Lemma 3.4 we then know that f is induced by at most one classical valuation of $DW(2n-1,2)$. So, if we are able to prove that there exists a point

x of $DW(2n-1,2)$ and an $\epsilon \in \mathbb{Z}$ such that $f(y) = \mathrm{d}(x,y)+\epsilon$ for every point y of $\mathbb{H}_n$, then we would have shown the validity of Theorem 1.1(1). We will prove this by induction on n, the case $n = 2$ being trivial. So, suppose $n \geq 3$. For every $i \in \{1,2,3\}$ and for every point x of M_1, we define $f_i(x) := f(\pi_{M_i}(x))$. Then f_1, f_2 and f_3 are three semi-valuations of $\widetilde{M_1} \cong \mathbb{H}_{n-1}$ with the property that all induced $W(2)$-quad valuations are classical. Note that $\{x, \pi_{M_2}(x), \pi_{M_3}(x)\}$ is a line for every $x \in M_1$. So, $|f_1(x) - f_2(x)| \leq 1$ for every point x of M_1 and $f_1 \diamond f_2 = f_3$. For every $\delta \in \mathbb{Z}$ and every $i \in \{1,2,3\}$, the map $f_i + \delta : x \mapsto f_i(x) + \delta$ also is a semi-valuation of M_1. We denote by $[f_i]$ the set of all semi-valuations of M_1 that arise in this way. We now distinguish two cases.

(I) Suppose $[f_1] = [f_2]$. Then $[f_1] = [f_2] = [f_3]$. As $f_1 \diamond f_2 = f_3$, there exists a $j \in \{1,2,3\}$ such that $f_j + 1 = f_{j+1} = f_{j+2}$, where the additions in the subindices happen modulo 3. By the induction hypothesis, there exists a unique point $x_j \in \overline{M_1}$ and a unique $\epsilon \in \mathbb{Z}$ such that $f_j(x) = \mathrm{d}(x_j, x) + \epsilon$ for every $x \in M_1$. Now, put $x^* := \pi_{\overline{M_j}}(x_j)$. Since $\{\overline{M_1}, \overline{M_2}, \overline{M_3}\}$ is a hyperbolic set of maxes of $DW(2n-1,2)$ and $f_{j+1} = f_{j+2} = f_j + 1$, we have $f(x) = \mathrm{d}(x^*, x) + \epsilon$ for every $x \in M_1 \cup M_2 \cup M_3$. By Lemma 3.3, the map $x \mapsto \mathrm{d}(x^*, x) + \epsilon$ also defines a semi-valuation of $\mathbb{H}_n$ for which all induced $W(2)$-quad valuations are classical. By Lemma 3.5, we thus have that $f(x) = \mathrm{d}(x^*, x) + \epsilon$ for every point x of $\mathbb{H}_n$.

(II) Suppose $[f_1] \neq [f_2]$. Then $[f_1]$, $[f_2]$ and $[f_3]$ are mutually distinct. For every $i \in \{1,2,3\}$, let H_i denote the set of all points of M_1 having non-maximal f_i-value. Then H_1, H_2 and H_3 are hyperplanes of $\widetilde{M_1}$. Since $f_1 \diamond f_2 = f_3$ and $[f_1]$, $[f_2]$, $[f_3]$ are mutually distinct, we know from [11, Proposition 2.14] that $H_3 = H_1 * H_2 := M_1 \setminus (H_1 \Delta H_2)$, where $H_1 \Delta H_2$ denotes the symmetric difference of H_1 and H_2. By the induction hypothesis, there exists for every $i \in \{1,2,3\}$ a unique $x_i \in \overline{M_1}$ and a unique $\epsilon_i \in \mathbb{Z}$ such that $f_i(x) = \mathrm{d}(x_i, x) + \epsilon_i$ for every $x \in M_1$. For every $i \in \{1,2,3\}$, let $\overline{H_i}$ denote the singular hyperplane of $\overline{M_1}$ with center x_i. As $\overline{H_1}$, $\overline{H_2}$ arise from the Grassmann embedding, also $\overline{H_1} * \overline{H_2} := \overline{M_1} \setminus (\overline{H_1} \Delta \overline{H_2})$ arises from the Grassmann embedding. By Lemma 3.1, we have $H_i = \overline{H_i} \cap M_1$. Since $H_3 = H_1 * H_2$, we thus have that $H_3 = (\overline{H_1} * \overline{H_2}) \cap M_1$. Since $\overline{H_3}$ and $\overline{H_1} * \overline{H_2}$ are two hyperplanes of $\overline{M_1}$ arising from the Grassmann embedding intersecting M_1 in H_3, we must have $\overline{H_3} = \overline{H_1} * \overline{H_2}$ by Lemma 2.6. By Lemma 2.3, we then know that $\{x_1, x_2, x_3\}$ is either a line of $\overline{M_1}$ or a hyperbolic line of a quad of $\overline{M_1}$.

We show that the latter case cannot occur. Suppose $\{x_1, x_2, x_3\}$ is a hyperbolic line of a quad Q of $\overline{M_1}$. By Lemma 2.5, there exists a $W(2)$-quad R of M_1 parallel with Q. Put $y_i := \pi_R(x_i)$, $i \in \{1,2,3\}$, and $\delta := \mathrm{d}(Q,R)$. We have $f_1(y_1) = \mathrm{d}(x_1, y_1) + \epsilon_1 = \delta + \epsilon_1$ and $f_2(y_1) = \mathrm{d}(x_2, y_1) + \epsilon_2 = \mathrm{d}(x_2, y_2) + \mathrm{d}(y_2, y_1) + \epsilon_2 = \delta + 2 + \epsilon_2$. As $|f_1(y_1) - f_2(y_1)| \leq 1$, we see that $\epsilon_2 < \epsilon_1$. By reversing the roles of y_1 and y_2, we would also have that $\epsilon_1 < \epsilon_2$, an obvious contradiction. We conclude that $\{x_1, x_2, x_3\}$ is a line of $\overline{M_1}$.

Now, by Lemma 3.1, there exists a point x of $M_1 \cong \mathbb{H}_{n-1}$ at distance $n-1$ from a point of $\{x_1, x_2, x_3\}$. Without loss of generality, we may suppose that $\mathrm{d}(x, x_1) = n-2$ and $\mathrm{d}(x, x_2) = \mathrm{d}(x, x_3) = n-1$. The convex subspace $\langle x, x_1 \rangle$ intersects M_1 in a convex subspace of M_1 of diameter at most $n-2$. So, there exists a line L of $M_1 \cong \mathbb{H}_{n-1}$ through x not contained in $\langle x, x_1 \rangle \cap M_1$, i.e. not contained in $\langle x, x_1 \rangle$. This line necessarily is parallel with $\{x_1, x_2, x_3\}$. Let y_i with $i \in \{1, 2, 3\}$ denote the unique point of L nearest to x_i. Since $f_1(y_1) = \mathrm{d}(x_1, y_1) + \epsilon_1 = n-2+\epsilon_1$ and $f_2(y_1) = \mathrm{d}(x_2, y_1) + \epsilon_2 = \mathrm{d}(x_2, y_2) + \mathrm{d}(y_2, y_1) + \epsilon_2 = n-1+\epsilon_2$, it follows that $\epsilon_2 \leq \epsilon_1$. Reversing the roles of y_1 and y_2, we see that also $\epsilon_1 \leq \epsilon_2$. Hence, $\epsilon_1 = \epsilon_2$. By symmetry, we can thus conclude that $\epsilon_1 = \epsilon_2 = \epsilon_3$. Now, let Q denote the unique $W(2)$-quad of $DW(2n-1, 2)$ through $\{x_1, x_2, x_3\}$ meeting $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$ in lines, and let x^* denote the unique point of Q collinear with each point of the ovoid $\{x_1, \pi_{\overline{M_2}}(x_2), \pi_{\overline{M_3}}(x_3)\}$ of the 3×3-grid $Q \cap (\overline{M_1} \cup \overline{M_2} \cup \overline{M_3})$. Then $x^* \notin \overline{M_1} \cup \overline{M_2} \cup \overline{M_3}$ and so $f(x) = f_i(\pi_{\overline{M_1}}(x)) = \mathrm{d}(x_i, \pi_{\overline{M_1}}(x)) + \epsilon_i = \mathrm{d}(\pi_{\overline{M_i}}(x_i), x) + \epsilon_i = \mathrm{d}(x^*, x) + \epsilon_i - 1$ for every $i \in \{1, 2, 3\}$ and every $x \in M_i$. As $\epsilon_1 = \epsilon_2 = \epsilon_3$, it thus follows that $f(x) = \mathrm{d}(x^*, x) + \epsilon_1 - 1$ for every $x \in M_1 \cup M_2 \cup M_3$. By Lemma 3.5, this again implies that $f(x) = \mathrm{d}(x^*, x) + \epsilon_1 - 1$ for every point x of $\mathbb{H}_n$.

In each of the cases (I) and (II) above, we have seen that f is induced by a classical valuation of $DW(2n-1, 2)$, finishing the proof of Theorem 1.1(1).

4 Proof of Theorem 1.1(2)

In this section, we suppose that $\mathbb{H}_n$ is isometrically embedded in $DW(2n-1, 2)$.

Lemma 4.1 *Let x_1 and x_2 be two distinct collinear points of $DW(2n-1, 2)$ and let f_i with $i \in \{1, 2\}$ denote the valuation of $\mathbb{H}_n$ induced by the classical valuation of $DW(2n-1, 2)$ with center x_i. Then f_1 and f_2 are neighbouring.*

Proof. There exist $\epsilon_1, \epsilon_2 \in \mathbb{Z}$ such that $f_1(x) = \mathrm{d}(x_1, x) + \epsilon_1$ and $f_2(x) = \mathrm{d}(x_2, x) + \epsilon_2$ for every point x of $\mathbb{H}_n$. We have $|f_1(x) - f_2(x) + \epsilon_2 - \epsilon_1| = |\,\mathrm{d}(x_1, x) - \mathrm{d}(x_2, x)| \leq \mathrm{d}(x_1, x_2) = 1$ for every point x of $\mathbb{H}_n$, showing that f_1 and f_2 are neighbouring. ■

In the sequel of this section, we suppose that x_1 and x_2 are two points of $DW(2n-1, 2)$ such that the valuations f_1 and f_2 are neighbouring, where f_i with $i \in \{1, 2\}$ is the valuation of $\mathbb{H}_n$ induced by the classical valuation of $DW(2n-1, 2)$ with center x_i. We shall prove that x_1 and x_2 are collinear.

If $f_1 = f_2$, then $x_1 = x_2$ by Lemma 3.4. We will therefore suppose that $f_1 \neq f_2$. Then let $g_1 \in [f_1]$ and $g_2 \in [f_2]$ such that $|g_1(x) - g_2(x)| \leq 1$ for every point x of $\mathbb{H}_n$. Put $g_3 := g_1 \diamond g_2$. By Lemma 2.7, we then know that g_3 is a semi-valuation of $\mathbb{H}_n$. We now show that all $W(2)$-quad valuations induced by g_3 are classical. So, suppose Q is a $W(2)$-quad. Put $y_1 := \pi_Q(x_1)$ and $y_2 := \pi_Q(x_2)$. Since Q is classical in $DW(2n-1, 2)$, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Z}$ such that $g_1(x) = \mathrm{d}(y_1, x) + \epsilon_1$ and $g_2(x) = \mathrm{d}(y_2, x) + \epsilon_2$ for every $x \in Q$. We have $|g_1(x) - g_2(x)| = |\,\mathrm{d}(y_1, x) - \mathrm{d}(y_2, x) + \epsilon_1 - \epsilon_2| \leq 1$ for every point x of Q. Putting x equal to y_1 and y_2, we respectively find that $|\epsilon_1 - \epsilon_2 - \mathrm{d}(y_1, y_2)| \leq 1$ and $|\epsilon_1 - \epsilon_2 + \mathrm{d}(y_1, y_2)| \leq 1$. So, we have $\mathrm{d}(y_1, y_2) \neq 2$. If $\mathrm{d}(y_1, y_2) = 1$, then necessarily $\epsilon_1 = \epsilon_2$, and we see that $g_3(x) = \mathrm{d}(x, y_3) + \epsilon_3$ for every point x of Q, where y_3 is the third point on the line y_1y_2 and $\epsilon_3 := \epsilon_1 = \epsilon_2$. If $\mathrm{d}(y_1, y_2) = 0$, i.e. $y_1 = y_2$, then we have that $|\epsilon_1 - \epsilon_2| \leq 1$. In this case, we have that $g_3(x) = \mathrm{d}(x, y_1) + \epsilon_3$ for every point x of Q, where

$\epsilon_3 := \epsilon_1 - 1 = \epsilon_2 - 1$ if $\epsilon_1 = \epsilon_2$ and $\epsilon_3 = \max\{\epsilon_1, \epsilon_2\}$ if $\epsilon_1 \neq \epsilon_2$. In any case, we see that the valuation of Q induced by g_3 is classical.

Since all $W(2)$-quad valuations induced by g_3 are classical, we know from Theorem 1.1(1) that there exists a unique point x_3 of $DW(2n-1,2)$ such that g_3 is induced by the classical valuation of $DW(2n-1,2)$ with center x_3. For every $i \in \{1,2,3\}$, let H_i' denote the singular hyperplane of $DW(2n-1,2)$ with center x_i and let H_i denote the hyperplane of $\mathbb{H}_n$ consisting of all points having non-maximal g_i-value. By Lemma 3.1, $H_i = H_i' \cap \mathbb{H}_n$ for every $i \in \{1,2,3\}$. By [11, Proposition 2.14], the fact that $g_3 = g_1 \diamond g_2$ and $[g_1] \neq [g_2] \neq [g_3] \neq [g_1]$ implies that $H_3 = H_1 * H_2$. Hence, $H_3 = (H_1' * H_2') \cap \mathbb{H}_n$. Since both H_3' and $H_1' * H_2'$ are two hyperplanes of $DW(2n-1,2)$ arising from the Grassmann embedding intersecting $\mathbb{H}_n$ in H_3, we know from Lemma 2.6 that $H_3' = H_1' * H_2'$. By Lemma 2.3, we then know that $\{x_1, x_2, x_3\}$ is either a line or a hyperbolic line of a quad of $DW(2n-1,2)$.

We show that the latter case cannot occur. Suppose $\{x_1, x_2, x_3\}$ is a hyperbolic line of a quad Q of $DW(2n-1,2)$. By Lemma 2.5, there exists a $W(2)$-quad R of $\mathbb{H}_n$ parallel with Q. Put $y_i := \pi_R(x_i)$, $i \in \{1,2,3\}$, and $\delta := \mathrm{d}(Q,R)$. There exist $\epsilon_1, \epsilon_2 \in \mathbb{Z}$ such that $g_1(x) = \mathrm{d}(x,x_1) + \epsilon_1$ and $g_2(x) = \mathrm{d}(x,x_2) + \epsilon_2$ for all points x of $\mathbb{H}_n$. In particular, we have $g_1(y_1) = \mathrm{d}(x_1,y_1) + \epsilon_1 = \delta + \epsilon_1$ and $g_2(y_1) = \mathrm{d}(x_2,y_1) + \epsilon_2 = \mathrm{d}(x_2,y_2) + \mathrm{d}(y_2,y_1) + \epsilon_2 = \delta + 2 + \epsilon_2$. As $|g_1(y_1) - g_2(y_1)| \leq 1$, we see that $\epsilon_2 < \epsilon_1$. By reversing the roles of y_1 and y_2, we would also have that $\epsilon_1 < \epsilon_2$, an obvious contradiction.

We conclude that $\{x_1, x_2, x_3\}$ is a line of $DW(2n-1,2)$. So, the points x_1 and x_2 are collinear as we needed to prove. This finishes the proof of Theorem 1.2.

5 Proof of Theorem 1.2

In this section, we suppose again that $\mathbb{H}_n$ is isometrically embedded into $\Delta = DW(2n-1,2)$. We suppose that $\mathcal{S}$ is a near $2n$-gon that contains an isometrically embedded copy $\mathbb{H}_n'$ of $\mathbb{H}_n$ such that every pair (x,Q) with x a point of $\mathcal{S}$ and Q a $W(2)$-quad of $\mathbb{H}_n'$ is classical. To ease notation, we will assume that $\mathbb{H}_n' = \mathbb{H}_n$.

Let $\mathcal{F}$ denote the set of all valuations of $\mathbb{H}_n$ for which all induced $W(2)$-quad valuations are classical. We denote by Γ the graph with vertex set $\mathcal{F}$, where two distinct elements $f_1, f_2 \in \mathcal{F}$ are adjacent whenever they are neighbouring. By Theorem 1.1, we then know that Γ is isomorphic to the collinearity graph of $DW(2n-1,2)$. Denote by $\mathcal{F}' \subseteq \mathcal{F}$ the set of all classical valuations of $\mathbb{H}_n$.

For every point x of $DW(2n-1,2)$, the classical valuation of $DW(2n-1,2)$ with center x will induce a valuation g_x of $\mathbb{H}_n$. If x is a point of $\mathbb{H}_n$, then g_x is the classical valuation of $\mathbb{H}_n$ with center x. By Theorem 1.1, we know the following.

Lemma 5.1 (1) *The map $x \mapsto g_x$ defines an isomorphism between the collinearity graph of $DW(2n-1,2)$ and the graph Γ.*

(2) *For every point x of $\Delta = DW(2n-1,2)$, we have $d_\Delta(x,\mathbb{H}_n) = d_\Gamma(g_x,\mathcal{F}')$.*

The following is an immediate consequence of Corollary 3.2 and Lemma 5.1.

Corollary 5.2 *Suppose f is a valuation of $\mathbb{H}_n$ for which all induced $W(2)$-quad valuations are classical, and let M denote the maximal value attained by f. Then $d_\Gamma(f,\mathcal{F}')+M=n$.*

By Lemma 2.8(1), every point x of $\mathcal{S}$ will induce a valuation f_x of $\mathbb{H}_n$. Since every pair (x,Q) with x a point of $\mathcal{S}$ and Q a quad of $\mathbb{H}_n$ is classical, this valuation has the property that all induced $W(2)$-quad valuations are classical, i.e. $f_x \in \mathcal{F}$. Since $\mathbb{H}_n$ is isometrically embedded in both $\mathcal{S}$ and $DW(2n-1,2)$, we know that for every point x of $\mathbb{H}_n$, the valuations f_x and g_x are equal to the classical valuation of $\mathbb{H}_n$ with center x.

Lemma 5.3 *For every valuation $f \in \mathcal{F}$ and every point y of $\mathbb{H}_n$, we have $d_\Gamma(f,f_y) = d_\Gamma(f,\mathcal{F}')+f(y)$.*

Proof. Let x denote the unique point of $DW(2n-1,2)$ for which $f=g_x$. By Lemma 5.1, $\mathrm{d}_\Gamma(f,f_y)=\mathrm{d}_\Gamma(g_x,g_y)=\mathrm{d}_\Delta(x,y)=\mathrm{d}_\Delta(x,\mathbb{H}_n)+g_x(y)=\mathrm{d}_\Gamma(g_x,\mathcal{F}')+g_x(y)=\mathrm{d}_\Gamma(f,\mathcal{F}')+f(y)$. ∎

Lemma 5.4 *Let x be a point of $\mathcal{S}$. Then $d_\mathcal{S}(x,\mathbb{H}_n)=d_\Gamma(f_x,\mathcal{F}')$ and there exists a point of $\mathbb{H}_n$ at distance n from x.*

Proof. Let M denote the maximal value attained by f_x. Then the maximal distance d from a point of $\mathbb{H}_n$ to x is equal to $\mathrm{d}_\mathcal{S}(x,\mathbb{H}_n)+M$. By Lemma 2.8(2), we have that $\mathrm{d}_\mathcal{S}(x,\mathbb{H}_n)\geq \mathrm{d}_\Gamma(f_x,\mathcal{F}')$. Hence, $d \geq \mathrm{d}_\Gamma(f_x,\mathcal{F}')+M$. By Corollary 5.2, we have $\mathrm{d}_\Gamma(f_x,\mathcal{F}')+M=n$. As $n \geq d$, we then have that $d=n$ and that $\mathrm{d}_\mathcal{S}(x,\mathbb{H}_n)=\mathrm{d}_\Gamma(f_x,\mathcal{F}')$. ∎

Lemma 5.5 *For every point x of $\mathcal{S}$ and for every point y of $\mathbb{H}_n$, we have $d_\mathcal{S}(x,y) = d_\Gamma(f_x,f_y)$.*

Proof. By Lemmas 5.3 and 5.4, we have $\mathrm{d}_\mathcal{S}(x,y)=\mathrm{d}_\mathcal{S}(x,\mathbb{H}_n)+f_x(y)=\mathrm{d}_\Gamma(f_x,\mathcal{F}')+f_x(y)=\mathrm{d}_\Gamma(f_x,f_y)$. ∎

Lemma 5.6 *Every line L of $\mathcal{S}$ contains precisely three points.*

Proof. Let $x \in L$, let y be a point of $\mathbb{H}_n$ at maximal distance n from x (see Lemma 5.4) and let z be the unique point of L nearest to y. Then $\mathrm{d}_\mathcal{S}(y,z)=\mathrm{d}_\mathcal{S}(y,L)=n-1$. Let z' be the unique point of $DW(2n-1,2)$ such that $g_{z'}=f_z$. By Lemmas 5.1(1) and 5.5, we have $\mathrm{d}_\Delta(y,z')=\mathrm{d}_\Gamma(g_y,g_{z'})=\mathrm{d}_\Gamma(f_y,f_z)=\mathrm{d}_\mathcal{S}(y,z)=n-1$. The convex subspace $\langle y,z'\rangle$ of $DW(2n-1,2)$ intersects $\mathbb{H}_n$ in a convex subspace of $\mathbb{H}_n$ of diameter at most $n-1$, showing that there exists a line of $\mathbb{H}_n$ through y not contained in $\langle y,z'\rangle$. Such a line contains a point u at distance n from z'. By Lemmas 5.1(1) and 5.5, we have $\mathrm{d}_\mathcal{S}(u,z)=\mathrm{d}_\Gamma(f_u,f_z)=\mathrm{d}_\Gamma(g_u,g_{z'})=\mathrm{d}_\Delta(u,z')=n$.

Now, put $a_1 := y$, $a_2 := u$ and let a_3 denote the third point on the line a_1a_2. For every $i \in \{1,2,3\}$, let b_i be the unique point of L nearest to a_i. Then $b_1=z$ and $\mathrm{d}_\mathcal{S}(a_1,b_1)=\mathrm{d}_\mathcal{S}(a_1,L)=n-1$. As $\mathrm{d}_\mathcal{S}(z,u)=n$, we have $b_2 \neq b_1$ and $\mathrm{d}_\mathcal{S}(a_2,b_2)=\mathrm{d}_\mathcal{S}(a_2,L)=n-1$. If we would have $\mathrm{d}_\mathcal{S}(a_3,b_3)\leq n-2$, then $\mathrm{d}_\mathcal{S}(a_1,b_3),\mathrm{d}_\mathcal{S}(a_2,b_3)\leq n-1$ by the triangle inequality, and we would have $b_1=b_3=b_2$, an obvious contradiction. So, $\mathrm{d}_\mathcal{S}(a_3,b_3)=\mathrm{d}_\mathcal{S}(a_3,L)=n-1$. Hence, $\mathrm{d}_\mathcal{S}(a_1a_2,L)=n-1$ and every point b of L has distance $n-1$ from a unique point a of a_1a_2. The correspondence $b \mapsto a$ is bijective and so the lines L and a_1a_2 should contain the same number of points, namely 3. ■

Lemma 5.7 *If x and y are two distinct collinear points of $\mathcal{S}$, then $f_x \neq f_y$.*

Proof. Let z denote the third point of the line xy. Suppose that $f_x = f_y$. Then Lemma 2.8(3) implies that $f_x = f_y = f_z$. Let u be a point of $\mathbb{H}_n$ for which $f_x(u) = f_y(u) = f_z(u) = 0$ and let M denote the maximal value attained by $f_x = f_y = f_z$. As $\mathbb{H}_n$ contains points at distance n from x by Lemma 5.4, we have $\mathrm{d}_{\mathcal{S}}(x,u) + M = \mathrm{d}_{\mathcal{S}}(x, \mathbb{H}_n) + M = n$, i.e. $\mathrm{d}_{\mathcal{S}}(x,u) = n - M$. A similar argument shows that $\mathrm{d}_{\mathcal{S}}(y,u) = \mathrm{d}_{\mathcal{S}}(z,u) = n - M$. But that is impossible, as it would imply that u has the same distance from each point of L. ■

By Lemma 5.1(1), we can identify each point x of $\Delta = DW(2n-1,2)$ with its corresponding valuation $g_x \in \mathcal{F}$. Then the map $x \mapsto f_x$ will induce a map θ from the point set $\mathcal{P}$ of $\mathcal{S}$ to the point set of $DW(2n-1,2)$, i.e. for every point x of $\mathcal{S}$, x^θ denotes the unique point of $DW(2n-1,2)$ for which $g_{x^\theta} = f_x$. If x is a point of $\mathbb{H}_n$, then both g_x and f_x are equal to the classical valuation of $\mathbb{H}_n$ with center x, implying that θ fixes all points of $\mathbb{H}_n$. By Lemmas 2.8(2), 5.1(1) and 5.7, θ maps distinct collinear points of $\mathcal{S}$ to distinct collinear points of $DW(2n-1,2)$. So, θ maps each line of $\mathcal{S}$ to a collection of three mutually collinear points of $DW(2n-1,2)$, i.e. to a line of $DW(2n-1,2)$. By Lemmas 5.1(1) and 5.5, we have that $d_{\mathcal{S}}(x,y) = \mathrm{d}_\Gamma(f_x, f_y) = \mathrm{d}_\Gamma(g_{x^\theta}, g_{y^\theta}) = \mathrm{d}_\Delta(x^\theta, y^\theta)$ for every point x of $\mathcal{S}$ and every point y of $\mathbb{H}_n$. This finishes the proof of Theorem 1.2.

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