Non-classical hyperplanes of finite thick dual polar spaces

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Abstract

We obtain a classification of the non-classical hyperplanes of all finite thick dual polar spaces of rank at least 3 under the assumption that there are no ovoidal and semi-singular hex intersections. In view of the absence of known examples of ovoids and semi-singular hyperplanes in finite thick dual polar spaces of rank 3, the condition on the nonexistence of these hex intersections can be regarded as not very restrictive. As a corollary, we also obtain a classification of the non-classical hyperplanes of $DW(2n-1,q)$, q even. In particular, we obtain a complete classification of all non-classical hyperplanes of $DW(2n-1,q)$ if $q \in \{8, 32\}$.

1   Introduction

Suppose Π is a finite thick polar space of rank $n \geq 2$ which is fully embeddable in a projective space Σ. By Tits' classification of polar spaces, we then know that there is some prime power q such that Π is isomorphic to one of $W(2n-1,q)$, $H(2n-1,q^2)$, $H(2n,q^2)$, $Q(2n,q)$, $Q^-(2n+1,q)$. Each of these polar spaces is defined by a nonsingular quadric Q (of Witt index n) or a polarity ζ of Σ (see Table 1). The singular subspaces of Π are then those subspaces of Σ that are either contained in Q or totally isotropic with respect to ζ. We note that not all these polar spaces are nonisomorphic. Indeed, $Q(2n,q) \cong W(2n-1,q)$ if and only if q is even.

The dual polar space Δ associated with Π is the point-line geometry whose points and lines are the $(n-1)$- and $(n-2)$-dimensional subspaces of Π, with incidence being reverse containment. The distance $\mathrm{d}(x_1, x_2)$ between two points x_1 and x_2 of Δ is by convention the distance in the collinearity graph. This is a graph of diameter n. The notation of Δ is also given in Table 1.

We use the brackets $\langle \cdots \rangle$ to denote the smallest convex subspace of Δ that contains the objects they enclose. If x_1 and x_2 are two points of Δ, then $\langle x_1, x_2 \rangle$ is the unique

Polar space	Ambient space	Defining object	Dual polar space
$W(2n-1,q)$	$PG(2n-1,q)$	symplectic polarity	$DW(2n-1,q)$
$H(2n-1,q^2)$	$PG(2n-1,q^2)$	Hermitian polarity	$DH(2n-1,q^2)$
$H(2n,q^2)$	$PG(2n,q^2)$	Hermitian polarity	$DH(2n,q^2)$
$Q(2n,q)$	$PG(2n,q)$	parabolic quadric	$DQ(2n,q)$
$Q^-(2n+1,q)$	$PG(2n+1,q)$	elliptic quadric	$DQ^-(2n+1,q)$

Table 1: The finite thick (dual) polar spaces

convex subspace of diameter $d(x_1,x_2)$ containing x_1 and x_2. Such convex subspaces are called *quads* if $d(x_1,x_2)=2$, *hexes* if $d(x_1,x_2)=3$ and *maxes* if $d(x_1,x_2)=n-1$.

If F is a nonempty convex subspace, then we denote by $\widetilde{F}$ the point-line geometry induced on F by those points and lines that are contained in F. If the diameter δ of F is at most 1, then $\widetilde{F}$ is either a point or a line. If $\delta \geq 2$, then $\widetilde{F}$ is a dual polar space of rank δ of the same type as Δ (e.g., if $\Delta \cong DQ^-(2n+1,q)$, then $\widetilde{F} \cong DQ^-(2\delta+1,q)$).

A *hyperplane* of a point-line geometry $\mathcal{S}$ is a set of points, distinct from the whole point set, intersecting each line in either a singleton or the whole line. An *ovoid* of a point-line geometry is a set of points meeting each line in a singleton. So, ovoids are special cases of hyperplanes. If $e : \mathcal{S} \to \Sigma'$ is a full embedding of $\mathcal{S}$ into the projective space Σ' and γ is a hyperplane of Σ', then the set of all points of $\mathcal{S}$ that are mapped by e into γ is a hyperplane of $\mathcal{S}$. Any hyperplane of $\mathcal{S}$ which can be obtained in this way is said to *arise from e*. A hyperplane of $\mathcal{S}$ is called *classical* if it arises from some full projective embedding of $\mathcal{S}$. The point-line geometry $\mathcal{S}$ is said to be *fully embeddable* if it admits at least one full projective embedding. The dual polar spaces $DW(2n-1,q)$, $DH(2n-1,q^2)$, $DQ(2n,q)$ and $DQ^-(2n+1,q)$ are all fully embeddable, while $DH(2n,q^2)$ is not. So, all hyperplanes of $DH(2n,q^2)$ are non-classical.

Consider again the dual polar space Δ of rank $n \geq 2$. The set of points of Δ at distance at most $n-1$ from a distinguished point x is a hyperplane of Δ, called the *singular hyperplane with center x*. Suppose Δ is a dual polar space of rank 3, and X is a set of points of Δ at distance 3 from a distinguished point x such that every line at distance 2 from x has a unique point in common with X. Then $X \cup x^\perp$, where $x^\perp$ denotes the set of points at distance at most 1 from x, is a hyperplane of Δ, a so-called *semi-singular hyperplane (with center x)*.

Suppose F is a convex subspace of diameter $\delta \geq 0$ of Δ. If H is a hyperplane of Δ, then $H \cap F$ is either F or a hyperplane of $\widetilde{F}$. The maximal distance from a point of Δ to F is equal to $n-\delta$. If H_F is a hyperplane of $\widetilde{F}$, then the points of Δ at distance at most $n-\delta-1$ from F together with those points at distance $n-\delta$ from F which lie at distance $n-\delta$ from a (necessarily unique) point of H_F constitute a hyperplane of Δ, called the *extension* of H_F. Notice that if $\delta = n$, then the extension of H_F coincides with H_F itself. If $\delta = 0$, namely F is a singleton $\{x\}$ and $H_F = \emptyset$, then the extension of H_F is the singular hyperplane with center x. If H_F is a non-classical hyperplane of $\widetilde{F}$, then its extension necessarily is a non-classical hyperplane of Δ.

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All hyperplanes of the dual polar spaces $DQ(2n, q)$ (q odd) and $DH(2n - 1, q^2)$ are classical, see [4, Corollary 1.6] and [27, Main Theorem]. All known hyperplanes of the dual polar spaces $DW(2n - 1, q)$ and $DQ^-(2n + 1, q)$ are either classical or extensions of non-classical ovoids of quads, and all known hyperplanes of the dual polar space $DH(2n, q^2)$ are singular hyperplanes. None of these hyperplanes can intersect a given hex F in a semi-singular hyperplane or an ovoid of $\widetilde{F}$. In the present paper, we prove the following.

Theorem 1.1 (1) *Suppose H is a hyperplane of $DW(2n - 1, q)$, $n \geq 3$, such that for no hex F of $DW(2n - 1, q)$, the intersection $H \cap F$ is a semi-singular hyperplane of $\widetilde{F}$. Then H is either classical or the extension of a non-classical ovoid of a quad.*

(2) *Suppose H is a hyperplane of $DQ^-(2n + 1, q)$, $n \geq 3$, such that for no hex F of $DQ^-(2n + 1, q)$, the intersection $H \cap F$ is an ovoid or a semi-singular hyperplane of $\widetilde{F}$. Then H is either classical or the extension of a non-classical ovoid of a quad.*

(3) *Suppose H is a hyperplane of $DH(2n, q^2)$, $n \geq 3$, such that for no hex F of $DH(2n, q^2)$, the intersection $H \cap F$ is an ovoid or a semi-singular hyperplane of $\widetilde{F}$. Then H is either a singular hyperplane or the extension of an ovoid of a quad.*

In view of the absence of known examples of semi-singular hyperplanes and ovoids in any finite thick dual polar space of rank 3, the assumption on the nonexistence of semi-singular and ovoidal hex intersections could be regarded as not very restrictive.

It is known that the dual polar space $DW(5, q)$ does not have semi-singular hyperplanes if q is even or if q is a prime, see [11, Corollary 3.10 & Theorem 3.11]. For q prime however, it is already known that every hyperplane of $DW(2n - 1, q)$ is classical, see [10, Corollary, p. 1385]. These facts and Theorem 1.1(1) thus imply the following.

Corollary 1.2 *Let q be a prime power for which the dual polar space $DW(5, q)$ has no semi-singular hyperplanes. Then every hyperplane of $DW(2n - 1, q)$, $n \geq 2$, is either classical or the extension of a non-classical ovoid of a quad. In particular, this holds if q is even.*

There exists a complete classification of all ovoids of the generalized quadrangle $Q(4, q) \cong DW(3, q)$ if $q \in \{2, 4, 8, 16, 32\}$, see [1, 13, 14, 15, 16, 17, 21]. If $q \in \{2, 4, 16\}$ then every ovoid of $Q(4, q)$ is classical, but for these values of q it is already known that every hyperplane of $DW(2n - 1, q)$, $n \geq 2$, is classical, see [10, Corollary, p. 1385]. If $q \in \{8, 32\}$, then every ovoid of $Q(4, q)$ is either a classical ovoid or a so-called Tits ovoid. So, Corollary 1.2 implies the following.

Corollary 1.3 *Let $q \in \{8, 32\}$ and $n \geq 2$. Then every non-classical hyperplane of $DW(2n - 1, q)$ is the extension of a Tits ovoid of a quad of $DW(2n - 1, q)$.*

The author is only aware of one nonexistence result regarding ovoids and semi-singular hyperplanes in elliptic dual polar spaces of rank 3, namely the dual polar space $DQ^-(7,2)$ cannot have ovoids. The nonexistence of these ovoids is a consequence of the fact that the collinearity graph of the $O^-(8,2)$ quadric is not geometrisable, see [18] and [29, p. 160]. As far as the author knows, the existence of ovoids and semi-singular hyperplanes has been ruled out for only one Hermitian dual polar space of type $DH(6,q^2)$, namely the dual polar space $DH(6,4)$. The nonexistence of these ovoids and semi-singular hyperplanes is a consequence of the nonexistence of ovoids in the generalized quadrangle $DH(4,4)$, or equivalently, the nonexistence of spreads in $H(4,4)$ (computer result of Andries Brouwer from the early 80's). By Theorem 1.1(3), we then know that every hyperplane of $DH(2n,4)$, $n \geq 3$, is singular. But the latter fact was basically already known (it is an immediate consequence of [3, Theorem 1.1]).

2   SDPS-sets and SDPS-hyperplanes

With every (thick) polar space Π of rank $n \geq 1$, there is associated a (thick) dual polar space Δ of rank n. A dual polar space of rank 2 is a generalized quadrangle and a dual polar space of rank 1 is a line. By convention, a dual polar space of rank 0 is a point (no lines).

Suppose Δ is a dual polar space of rank n. If x is a point of Δ, then the convex subspaces through x, ordered by ordinary inclusion, define a projective space $\mathsf{Res}(x)$ of dimension $n-1$. If x is a point of Δ and $i \in \mathbb{N}$, then $\Delta_i(x)$ denotes the set of points at distance i from x. If F is a nonempty convex subspace of Δ, then for every point x of Δ, there exists a unique point $\pi_F(x) \in F$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_F(x)) + \mathrm{d}(\pi_F(x),y)$ for every $y \in F$. So, for every point x and every line L, there exists a unique point (namely $\pi_L(x)$) on L nearest to x (implying that Δ is a so-called *near polygon*).

Suppose Δ is a thick dual polar space of rank $2n$, $n \in \mathbb{N}$. A nonempty set X of points of Δ is called an *SDPS-set* if it satisfies the following properties.

- No two distinct points of X are on the same line.

- If a quad Q contains two distinct points of X then it intersects X in an ovoid of $\widetilde{Q}$.

- The partial linear space Δ' whose points are the elements of X and whose lines are the quad intersections of size at least two (natural incidence) is a dual polar space of rank n.

- If x_1 and x_2 are two points of X, then the distance between x_1 and x_2 in the geometry Δ is twice the distance between these points in the geometry Δ'.

- Every line of Δ meeting X is contained in a (necessarily unique) quad that intersects X in at least two points.

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An SDPS-set of a dual polar space of rank 0 consists of the unique point of that geometry, while an SDPS-set of a thick generalized quadrangle is just an ovoid of that geometry. The word SDPS is an abbreviation of *Sub Dual Polar Space* and refers to the fact that Δ' can be regarded as a sub dual polar space of Δ. SDPS-sets were introduced in [12]. In [12] (finite case) and [6, Chapter 5] (general case), it was shown that if X is an SDPS-set of a thick dual polar space Δ of rank $2n$, then the maximal distance from a point of Δ to X is equal to n. Moreover, the set of points of Δ at distance at most $n - 1$ from X is a hyperplane of Δ. Any hyperplane of a thick dual polar space that can be obtained in this way is called an *SDPS-hyperplane*. The following proposition is the main result of [7].

Proposition 2.1 ([7]) *The following statements are equivalent for a hyperplane H of a thick dual polar space Δ of rank at least 3.*

- *H is the extension of an SDPS-hyperplane of a convex subspace of even diameter of Δ.*

- *For every hex F of Δ, the intersection $H \cap F$ is either F, a singular hyperplane of $\widetilde{F}$ or the extension of an ovoid of a quad of $\widetilde{F}$.*

The following proposition is taken from [6, Theorem 5.31], but its proof heavily relies on results of [23].

Proposition 2.2 ([6, 23]) (1) *Let Δ be one of the dual polar spaces $DW(4n - 1, q)$, $DQ^-(4n + 1, q)$ where $n \geq 2$, and let X be an SDPS-set of Δ. Then for every quad Q of Δ containing at least two points of X, the ovoid $Q \cap X$ of $\widetilde{Q}$ is classical.*

(2) *The dual polar space $DH(4n, q^2)$ does not have SDPS-sets if $n \geq 2$.*

In the following two propositions, two ovoids O and O' of the respective generalized quadrangles $\widetilde{Q}$ and $\widetilde{Q'}$ are called *isomorphic* if there exists an isomorphism from $\widetilde{Q}$ to $\widetilde{Q'}$ mapping O to O'.

Proposition 2.3 *Let X be an SDPS-set of a thick dual polar space Δ of rank $2n \geq 2$. Let H be the SDPS-hyperplane of Δ associated to X. Suppose Q is a quad of Δ such that $Q \cap H$ is an ovoid of $\widetilde{Q}$. Then there exists a quad Q' of Δ intersecting X in at least two points such that the ovoid $Q \cap H$ of $\widetilde{Q}$ is isomorphic to the ovoid $Q' \cap X$ of $\widetilde{Q'}$.*

Proof. We will prove the proposition by induction on n. Clearly, the proposition holds if $n = 1$ (take $Q' = Q$). So, suppose $n > 1$. Let F denote a max of Δ containing Q. Then by [6, Lemmas 5.42 and 5.44], the set $X_1 := F \cap X$ is an SDPS-set of a convex subspace F_1 of diameter $2n - 2$ of $\widetilde{F}$. Moreover, by [6, Lemma 5.45] the hyperplane $F \cap H$ of $\widetilde{F}$ is the extension of the SDPS-hyperplane H_1 of $\widetilde{F_1}$ associated with the SDPS-set X_1. As Q is ovoidal, it cannot meet F_1. Every point x of Q is collinear with the point $\pi_{F_1}(x)$ of F_1. The set $Q_1 := \pi_{F_1}(Q) = \{\pi_{F_1}(x) \mid x \in Q\}$ is a quad of $\widetilde{F_1}$ and π_{F_1} realizes an isomorphism

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between $\widetilde{Q}$ and $\widetilde{Q_1}$. So, the ovoid $Q \cap H$ of $\widetilde{Q}$ is isomorphic to the ovoid $Q_1 \cap H_1$ of $\widetilde{Q_1}$. By the induction hypothesis, there exists a quad Q' intersecting X_1 (and hence also X) in at least two points such that the ovoid $Q_1 \cap H_1$ of $\widetilde{Q_1}$ is isomorphic to the ovoid $Q' \cap X_1 = Q' \cap X$ of $\widetilde{Q'}$. $\qquad\square$

Proposition 2.4 *Let Δ be a thick dual polar space, F a convex subspace of even diameter of Δ, X an SDPS-set in $\widetilde{F}$ and H_F the SDPS-hyperplane of $\widetilde{F}$ associated with X. Let H denote the hyperplane of Δ which arises by extending H_F. If Q is a quad of Δ such that $Q \cap H$ is an ovoid of $\widetilde{Q}$, then there exists a quad Q' of $\widetilde{F}$ intersecting X in at least two points such that the ovoid $Q \cap H$ of $\widetilde{Q}$ is isomorphic to the ovoid $Q' \cap X$ of $\widetilde{Q'}$.*

Proof. Let n denote the rank of Δ and 2δ the rank of $\widetilde{F}$. Then the maximal distance from a point of Δ to F is equal to $n - 2\delta$. Every point at distance at most $n - 2\delta - 1$ from F lies in H. Suppose Q contains a point x at distance at most $n - 2\delta - 1$ from F. Then there exists a max M of Δ containing x and F. Every point of M lies at distance at most $n - 2\delta - 1$ from F and hence belongs to H. If we look at the projective space $\mathsf{Res}(x)$, then we see that $Q \cap M \subseteq H$ is either Q or a line, in contradiction with the fact that $Q \cap H$ is an ovoid of $\widetilde{Q}$. Hence, every point of Q lies at distance $n - 2\delta$ from F. Let $y \in Q$. Then $\pi_F(y)$ and Q are contained in a unique convex subspace F' of diameter $n - 2\delta + 2$. Since $\langle y, \pi_F(y) \rangle \cap F = \{\pi_F(y)\}$ and $\langle y, \pi_F(y) \rangle$ has diameter $n - 2\delta$, the intersection of F' and F is a quad Q_1 (look at the projective space $\mathsf{Res}(\pi_F(y))$). The map $z \mapsto \pi_F(z)$ defines an isomorphism between $\widetilde{Q}$ and $\widetilde{Q_1}$. The intersection $Q_1 \cap H_F$ is equal to $\pi_F(Q \cap H)$ and hence is an ovoid of $\widetilde{Q_1}$ isomorphic to the ovoid $Q \cap H$ of $\widetilde{Q}$. By Proposition 2.3, there exists a quad Q' of $\widetilde{F}$ intersecting X in at least two points such that the ovoid $Q_1 \cap H_F$ of $\widetilde{Q_1}$ is isomorphic to the ovoid $Q' \cap X$ of $\widetilde{Q'}$. $\qquad\square$

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. So, let Δ be one of the dual polar spaces $DW(2n-1,q)$, $DQ^-(2n+1,q)$, $DH(2n,q^2)$ of rank $n \geq 3$, and let H be a hyperplane of Δ. We make the following assumptions.

(I) For no hex F of Δ, the intersection $F \cap H$ is a semi-singular hyperplane of $\widetilde{F}$.

(II) If $\Delta \in \{DQ^-(2n+1,q), DH(2n,q^2)\}$, then for no hex F of Δ, the intersection $F \cap H$ is an ovoid of $\widetilde{F}$.

The dual polar space $DW(5,q)$ has no ovoids, see [5], [19, Proposition 2.8] and [28, Theorem 3.2]. So, also if $\Delta = DW(2n-1,q)$ there will be no hexes F for which $F \cap H$ is an ovoid of $\widetilde{F}$. We will also make the following assumption.

(III) H is non-classical.

In view of what we need to prove, it then suffices to prove that H is the extension of a non-classical ovoid of a quad or (only if $\Delta = DH(2n,q^2)$) a singular hyperplane.

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Suppose Q is a quad of Δ. Then $Q \cap H$ is either Q or a hyperplane of $\widetilde{Q}$. So, one of the following possibilities must then occur: (1) $Q \subseteq H$; (2) $Q \cap H$ consists of all points of Q collinear with or equal to a distinguished point; (3) $Q \cap H$ is a proper subquadrangle of $\widetilde{Q}$; (4) $Q \cap H$ is an ovoid of $\widetilde{Q}$. The quad Q is called *deep, singular, subquadrangular* or *ovoidal (with respect to H)* depending on whether case (1), (2), (3) or (4) occurs. A quad Q of Δ is called *bad ovoidal (with respect to H)* if $Q \cap H$ is a non-classical ovoid of $\widetilde{Q}$. If $\Delta \in \{DQ^-(2n+1, q), DH(2n, q^2)\}$, then every quad of Δ has order (s, t) with $t < s$ and hence Δ cannot have proper full subquadrangles by [20, 2.2.2(i)]. So, in this case, there cannot exist quads that are subquadrangular with respect to the hyperplane. Pralle [22] obtained a classification of hyperplanes of thick dual polar spaces of rank 3 that do not have subquadrangular quads. The following result follows from this classification and a few other results on uniform hyperplanes obtained in [19, 25]. (With a *uniform hyperplane*, we mean a hyperplane which admits, besides deep quads, only one other type of quad.)

Proposition 3.1 ([19, 22, 25]) *Let $\Delta_1 \in \{DW(5, q), DQ^-(7, q), DH(6, q^2)\}$ and let H_1 be a hyperplane of Δ_1 not admitting subquadrangular quads. Then H_1 is one of the following:*

   (i) a singular hyperplane;
   (ii) an ovoid;
   (iii) a semi-singular hyperplane;
   (iv) the extension of an ovoid of a quad;
   (v) (only if q is even) a hexagonal hyperplane of $DW(5, q)$;
   (vi) a hexagonal hyperplane of $DQ^-(7, q)$.

The dual polar space $DQ(6, q)$ (which is isomorphic to $DW(5, q)$ if q is even) admits so-called *hexagonal hyperplanes* ([25]). If H is a hexagonal hyperplane of $DQ(6, q)$, then the points and lines contained in H define a so-called split-Cayley generalized hexagon. A hexagonal hyperplane of $DQ(6, q)$ only admits singular quads. In fact, every point x of a hexagonal hyperplane H is contained in a unique (singular) quad Q_x such that $Q_x \cap H = x^\perp \cap Q_x$. All hexagonal hyperplanes of $DQ(6, q)$ are classical by [8, 27].

   The polar space $Q^-(7, q)$ has hyperplane sections of type $Q(6, q)$ on which a dual polar space of type $DQ(6, q)$ can be defined. By [22, Theorem 3], every hexagonal hyperplane of $DQ(6, q)$ gives rise to a hyperplane of the dual polar space $DQ^-(7, q)$ associated with $Q^-(7, q)$. Every hyperplane of $DQ^-(7, q)$ that can be obtained is this way is called a *hexagonal hyperplane* of $DQ^-(7, q)$. If H is a hexagonal hyperplane of $DQ^-(7, q)$, then there are no points x for which $x^\perp \subseteq H$ and every quad is singular or ovoidal with respect to H. All hexagonal hyperplanes of $DQ^-(7, q)$ are classical, see [8, Proposition 4.6].

By relying on Proposition 3.1, we can prove the following.

Lemma 3.2 *If $\Delta = DH(2n, q^2)$, then every hex F intersects H in either F, a singular hyperplane of $\widetilde{F}$ or the extension of an ovoid of a quad of $\widetilde{F}$.*

Proof. Obviously, $F \cap H$ is either F or a hyperplane of $\widetilde{F}$. Suppose the latter occurs. As $\widetilde{F} \cong DH(6, q^2)$, no quad of $\widetilde{F}$ has proper subquadrangles and so no quad of $\widetilde{F}$ is subquadrangular with respect to $F \cap H$. One of the cases (i), (ii), (iii) or (iv) of Proposition 3.1 should therefore occur for the hyperplane $F \cap H$ of H. But the cases (ii) and (iii) cannot occur by Assumptions (I) and (II). $\qquad\square$

Our aim will be to prove a similar result for the dual polar spaces $DW(2n - 1, q)$ and $DQ^-(2n + 1, q)$. This will ultimately be realized in Corollary 3.14.

Lemma 3.3 (1) *If* $\Delta = DW(2n - 1, q)$, *then* $q \neq 2$.

(2) *If there exists no bad ovoidal quad, then* $\Delta = DH(2n, q^2)$ *and* H *is a singular hyperplane of* Δ.

Proof. (1) Since the dual polar space $DW(2n-1, 2)$ is fully embeddable, every hyperplane of $DW(2n - 1, 2)$ must be classical by [24, Corollary 2, p. 180]. As H is non-classical (Assumption (III)), we must have that $q \neq 2$ if $\Delta = DW(2n - 1, q)$.

(2) If $\Delta = \{DW(2n - 1, q), DQ^-(2n + 1, q)\}$, then the fact that there are no bad ovoidal quads implies that the hyperplane is classical by [8, Theorem 1.4] and [10, Main Theorem]. So, we may suppose that $\Delta = DH(2n, q^2)$. As the generalized quadrangle $DH(4, q^2)$ is nonembeddable, the nonexistence of bad ovoidal quads implies that there are no ovoidal quads at all. As there are also no subquadrangular quads, every quad is either deep or singular, that means that H is locally singular using the terminology of [19]. By [3, Theorem 1.1], this implies that the hyperplane H must be singular. $\qquad\square$

So, in the sequel, we may also assume the following.

(IV) There exists at least one bad ovoidal quad.

In view of what we need to prove, it will then suffice to prove that H is the extension of a non-classical ovoid of a quad.

The following proposition was shown in [11, Corollary 1.3].

Proposition 3.4 ([11]) *Suppose* $n = 3$ *and* $q \neq 2$. *Let* H_1 *be a hyperplane of* $DW(5, q)$, *let* x *be a point of* H_1 *and let* $\mathcal{L}$ *be the set of lines through* x *contained in* H_1. *Then* $\mathcal{L}$, *regarded as a set of points of* $\mathsf{Res}(x) \cong \mathrm{PG}(2, q)$, *is one of the following:* (1) *the empty set;* (2) *a point;* (3) *a line;* (4) *the union of two distinct lines;* (5) *a nonsingular conic;* (6) *the whole set of points. If* H_1 *is not a semi-singular hyperplane of* $DW(5, q)$, *then possibility* (1) *cannot occur.*

Proposition 3.5 *Let* H_1 *be a hyperplane of* $DQ^-(7, q)$, x *a point of* H_1 *and* $\mathcal{L}$ *the set of lines through* x *contained in* H_1. *Then* $\mathcal{L}$, *regarded as a set of points of* $\mathsf{Res}(x) \cong \mathrm{PG}(2, q)$, *is one of the following:* (1) *the empty set;* (2) *a point;* (3) *a line;* (4) *the whole set of points. If* H_1 *is not an ovoid nor a semi-singular hyperplane of* $DQ^-(7, q)$, *then possibility* (1) *cannot occur.*

Proof. We first show that $\mathcal{L}$ is a subspace of $\mathsf{Res}(x)$. To that end, consider two distinct lines L_1 and L_2 of $\mathcal{L}$ and denote by Q the unique quad through L_1 and L_2. We need to show that every line of Q through x belongs to $\mathcal{L}$. Since Q is not subquadrangular, it is either singular or deep. So, as $L_1 \cup L_2 \subseteq H$, every line of Q through x must be contained in H, as we needed to prove.

Suppose now that H_1 is not an ovoid nor a semi-singular hyperplane. Then H_1 is either a singular hyperplane, the extension of an ovoid of a quad or a hexagonal hyperplane by Proposition 3.1. For each of these three possibilities, one can verify that possibility (1) cannot occur. This is obvious for singular hyperplanes and extensions of ovoids of quads. For hexagonal hyperplanes, this follows for instance from [9, Lemma 2.6]. $\qquad\square$

Lemma 3.6 *Let $\Delta \in \{DW(2n-1,q), DQ^-(2n+1,q)\}$. Let Q be a bad ovoidal quad and let F be a hex through Q. Then the hyperplane $F \cap H$ of $\widetilde{F}$ is the extension of a non-classical ovoid of a quad of $\widetilde{F}$.*

Proof. Suppose first that $\Delta = DW(2n-1,q)$. Since $Q \cap H$ is a non-classical ovoid of $\widetilde{Q}$, the hyperplane $F \cap H$ of $\widetilde{F}$ cannot be a classical hyperplane of $\widetilde{F} \cong DW(5,q)$. By [11, Theorem 1.1], we then know that the hyperplane $F \cap H$ of $\widetilde{F}$ is either the extension of a non-classical ovoid of a quad or a semi-singular hyperplane of $\widetilde{F}$. The latter possibility cannot occur by Assumption (I).

Suppose next that $\Delta = DQ^-(2n+1,q)$. Again we must have that $F \cap H$ is a non-classical hyperplane of $\widetilde{F}$. As singular hyperplanes of $DQ^-(7,q)$ do not admit ovoidal quads and hexagonal hyperplanes are classical, there is only one remaining possibility by Proposition 3.1 and Assumptions (I) and (II), namely $F \cap H$ should be the extension of a (necessarily non-classical) ovoid of a quad. $\qquad\square$

Lemma 3.7 *Suppose $\Delta \in \{DW(7,q), DQ^-(9,q)\}$ (so, $n = 4$) and Q is a bad ovoidal quad. Let x be an arbitrary point of $Q \cap H$. Then there exists a unique quad Q_x through x such that the lines through x contained in H are precisely the lines through x contained in Q_x. The quad Q_x is moreover deep.*

Proof. Let $\mathcal{L}$ denote the set of lines through x contained in H. If F is a hex through Q, then by Lemma 3.6 $F \cap H$ is the extension of an ovoid of a quad of $\widetilde{F}$ and hence there exists a unique line through x contained in $F \cap H$. Since there are $q + 1$ hexes through Q, we have $|\mathcal{L}| = q + 1$.

We prove that $\mathcal{L}$ is a line of $\mathsf{Res}(x) \cong \mathrm{PG}(3,q)$. Suppose that this is not the case. Then there are three lines $L_1, L_2, L_3 \in \mathcal{L}$ that are not contained in a common quad. Consider the hex $F = \langle L_1, L_2, L_3 \rangle$. The hex F defines a subplane α of $\mathsf{Res}(x)$ isomorphic to $\mathrm{PG}(2,q)$. By Proposition 3.5 applied to the hyperplane $H \cap F$ of $\widetilde{F}$, we see that the case $\Delta = DQ^-(9,q)$ is not possible. So, we should have $\Delta = DW(7,q)$. By Proposition 3.4, the $q+1$ lines through x contained in H must then define a nonsingular conic in the subplane α of $\mathsf{Res}(x)$. The hex F does not contain Q and hence intersects Q in a line L. Suppose that either q is odd or (q is even and L is not the nucleus of the nonsingular conic $\mathcal{L}$). Then there exists a quad $Q_1 \subseteq F$ through L that does not contain any line

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of the set $\mathcal{L}$. If we choose Q_1 in such a way, then the hex $\langle Q, Q_1 \rangle$ through Q does not contain any line of $\mathcal{L}$ which is impossible. We must therefore have that q is even and L is the nucleus of $\mathcal{L}$. Let $L' \subseteq F$ be a line through x not contained in $\mathcal{L} \cup \{L\}$, let Q' be the quad $\langle L, L' \rangle$ and let F' be the hex $\langle Q, Q' \rangle$. Then $F \cap F' = Q'$. By Lemma 3.6, the hyperplane $F' \cap H$ of $\widetilde{F'}$ is the extension of a non-classical ovoid of a quad of $\widetilde{F'}$ and hence there exists a bad ovoidal quad Q'' through L' contained in F'. Since L is the nucleus of $\mathcal{L}$, the intersection $x^\perp \cap Q' \cap H$ is a line and hence Q' must be a singular quad. Hence $Q'' \neq Q'$ and $Q'' \cap F = L'$ (as $F \cap F' = Q'$). We can now repeat the above arguments with Q replaced by Q''. A contradiction is then easily obtained, taking into account that L' is this time not the nucleus of the conic $\mathcal{L}$. So, we can conclude that $\mathcal{L}$ is a line of $\mathsf{Res}(x)$. This means that there exists a quad Q_x through x such that $\mathcal{L}$ coincides with the set of lines through x contained in Q_x.

Next, we show that the quad Q_x is deep. Suppose to the contrary that Q_x is not deep. Let L be an arbitrary line through x contained in Q_x. By Lemma 3.6, the hex $\langle Q, L \rangle$ intersects H in the extension of an ovoid of a quad Q'. So, if we put $L \cap Q' = \{x'\}$, then we see that every point of $L \setminus \{x'\}$ is contained in a bad ovoidal quad. By the previous paragraph, we then know that through every point of $L \setminus \{x, x'\}$ there are precisely $q + 1$ lines that are contained in H. Since Q_x is not deep, only one of these $q + 1$ lines, namely L, is contained in Q_x. Now, there are $q^2 + q$ quads through L distinct from Q_x. If R is one of these quads, then $x^\perp \cap R \cap H$ is the line L and hence the quad R must be singular. So, the $q^2 + q$ quads through L distinct from Q_x define $q^3 + q^2$ lines that are contained in H and meet $L \setminus \{x\}$ in a singleton. As each of the $q - 1$ points of $L \setminus \{x, x'\}$ is contained in q lines that meet $L \setminus \{x\}$ in a singleton, the point x' is contained in precisely $q^3 + q^2 - (q - 1)q + 1 = q^3 + q + 1$ lines that are contained in H (including L) and precisely q^2 lines that are not contained in H. Let K be a line through x' not contained in H. As there are $q^2 + q + 1$ quads through K, there must exist a quad S through K with the property that every line through x' contained in S and distinct from K must be contained in H. This is only possible when S is a subquadrangular quad. As $DQ^-(5, q)$ does not have proper subquadrangles, we should have $\Delta = DW(7, q)$ and $S \cap H$ is a $(q + 1) \times (q + 1)$-grid of $\widetilde{S} \cong DW(3, q) \cong Q(4, q)$. The fact that every line of S through x' distinct from K is contained in H then implies that $q = 2$, in contradiction with Lemma 3.3(1). $\qquad \square$

Lemma 3.8 *Suppose $\Delta \in \{DW(2n - 1, q), DQ^-(2n + 1, q)\}$. Then every point x of Δ not contained in H is contained in a bad ovoidal quad.*

Proof. Let Q be a bad ovoidal quad. Let y_1 be an arbitrary point of $Q \setminus H$. Since the complement of H is connected ([2, Theorem 7.3], [26, Lemma 6.1]), there exists a path $y_1, y_2, \ldots, y_k = x$ of $k \geq 1$ points in the complement of H. We prove by induction on $i \in \{1, 2, \ldots, k\}$ that the point y_i is contained in some bad ovoidal quad Q_i. Obviously, this holds if $i = 1$ (take $Q_1 = Q$). Suppose now that the claim holds for a certain $i \in \{1, 2, \ldots, k - 1\}$, and let Q_i be a bad ovoidal quad through y_i. If y_{i+1} is contained in Q_i, then the claim holds for $i + 1$ (just take $Q_{i+1} = Q_i$). Suppose therefore that y_{i+1} is

not contained in Q_i and consider the hex $F = \langle y_{i+1}, Q_i \rangle$. By Lemma 3.6, the intersection $F \cap H$ is the extension of a non-classical ovoid of a quad of $\widetilde{F}$. This implies that y_{i+1} is contained in some bad ovoidal quad Q_{i+1}. $\qquad\square$

Lemma 3.9 *Suppose* $\Delta = DW(2n-1, q)$. *Then there cannot exist a subquadrangular quad* Q_1 *and a bad ovoidal quad* Q_2 *that intersect in a line.*

Proof. Suppose the contrary and consider the hex $F = \langle Q_1, Q_2 \rangle$. Since Q_2 is a bad ovoidal quad, the intersection $F \cap H$ must be the extension of a non-classical ovoid of a quad of $\widetilde{F}$ by Lemma 3.6. The latter however implies that no subquadrangular quad can be contained in F. $\qquad\square$

Lemma 3.10 *Suppose* $\Delta = DW(2n-1, q)$. *Then there cannot exist subquadrangular quads.*

Proof. Suppose Q is a subquadrangular quad. By Lemmas 3.8 and 3.9, there should exist a bad ovoidal quad intersecting $Q \setminus H$ in a singleton and so we should have that $n \geq 4$. We will now derive a contradiction. It suffices to deal with the case $n = 4$. Indeed, if $n > 4$ then the reasoning below applied to any convex subspace of diameter 4 containing Q and a bad ovoidal quad meeting $Q \setminus H$ in a singleton would yield the desired contradiction.

Step 1. *If R is a quad intersecting Q in a singleton $\{x\}$ not contained in H, then R is bad ovoidal.*
PROOF. By Lemmas 3.8 and 3.9, there exists a bad ovoidal quad R_1 meeting Q in the singleton $\{x\}$. Now, consider $\mathsf{Res}(x) \cong \mathrm{PG}(3, q)$. The quads Q, R and R_1 define lines α, β and β_1 in $\mathsf{Res}(x)$. The graph whose vertices are the lines of $\mathsf{Res}(x) \cong \mathrm{PG}(3, q)$ disjoint from α, with two vertices adjacent whenever the corresponding lines meet in a point is connected. So, when proving the above-mentioned claim, it suffices to consider the case where the lines β and β_1 meet in a singleton, that means, the case where the quads R and R_1 meet in a line. Consider the hex $F := \langle R, R_1 \rangle$. This hex meets Q in a line M through x. Let y denote the unique point of H on this line. Since R_1 is a bad ovoidal quad, the hyperplane $F \cap H$ of $\widetilde{F}$ is the extension of a non-classical ovoid of a quad R_2 of $\widetilde{F}$ (see Lemma 3.6). Since the line M is not contained in a bad ovoidal quad (Lemma 3.9), the quad R_2 should contain the point y. So, the quads R and R_2 are disjoint, implying that $R \cap H$ is the extension of a non-classical ovoid of $\widetilde{R}$, i.e. R is bad ovoidal. (qed)

Step 2. *Let F be a hex meeting Q in a line L not contained in H, and let x be the unique point in $L \cap H$. Then $F \cap H$ is the extension of a non-classical ovoid of a quad R of $\widetilde{F}$. This quad R intersects Q in the singleton $\{x\}$, and the lines through x contained in $H \cap F$ are precisely the lines through x contained in R.*
PROOF. By Step 1, there exists a bad ovoidal quad in F meeting Q in a singleton belonging to $L \setminus \{x\}$. So, by Lemma 3.6, $F \cap H$ is the extension of a non-classical ovoid O of a quad R of $\widetilde{F}$. By Lemma 3.9, $\widetilde{F}$ cannot have bad ovoidal quads through the line L. So, the line L should meet R and R should contain the point x. Since L is not contained

in H, $R \cap Q = \{x\}$ and $x \notin O$. Since $x \notin O$, the lines through x contained in $H \cap F$ are precisely the lines through x contained in R. (qed)

Now, let x^* be a fixed point of $Q \cap H$. By considering a hex through x^* meeting Q in a line not contained in H, we see by Step 2 that there should exist a deep quad Q_{x^*} through x^* for which $Q_{x^*} \cap Q = \{x^*\}$. Let L_1 and L_2 be the two lines through x^* contained in $Q \cap H$.

Step 3. *If L is a line through x^* contained in H, then L is contained in either $\langle Q_{x^*}, L_1 \rangle$ or $\langle Q_{x^*}, L_2 \rangle$.*
PROOF. Suppose L is not contained in $\langle Q_{x^*}, L_1 \rangle$ nor in $\langle Q_{x^*}, L_2 \rangle$. Then consider the hex $F = \langle Q_{x^*}, L \rangle$. This hex intersects Q in a line M through x^* that is not contained in H. By Step 2, all lines through x^* contained in $F \cap H$ are contained in a certain quad through x^*. But that is impossible. All lines through x^* contained in Q_{x^*} are contained in H and the line L itself is also contained in H. (qed)

Now, let M be a line of Q through x^* distinct from L_1 and L_2. Let F be a hex through M not containing Q and distinct from $\langle M, Q_{x^*} \rangle$. Then $Q_{x^*} \cap F$ is a line contained in H. By Step 2, there exists a quad R through x^* contained in F such that the lines through x^* contained in $F \cap H$ are precisely the lines through x^* contained in R. Observe that the line $Q_{x^*} \cap F$ is contained in R. By Step 3, the quad R is contained in either $\langle Q_{x^*}, L_1 \rangle$ or $\langle Q_{x^*}, L_2 \rangle$. Without loss of generality, we may suppose that R is contained in $\langle Q_{x^*}, L_1 \rangle$. Observe now the following: (i) Q_{x^*} and R are two distinct quads through x^* contained in the hex $\langle Q_{x^*}, L_1 \rangle$ and none of these quads contains the line L_1; (ii) every line of Q_{x^*} through x^* is contained in H; (iii) every line of R through x^* is contained in H; (iv) the line L_1 is contained in H. By Lemma 3.3(1) and Proposition 3.4, this implies that every line of $\langle Q_{x^*}, L_1 \rangle$ through x^* is contained in H. Also, by Proposition 3.4, there exists a line $L_2' \neq L_2$ of $\langle Q_{x^*}, L_2 \rangle$ through x^* that is contained in H, but not in Q_{x^*}. Now, let F' be a hex through $\langle M, L_2' \rangle$ not containing Q. Then by Step 2, there exists a quad R' of $\widetilde{F'}$ through x^* such that the lines through x^* contained in $F' \cap H$ are precisely the lines through x^* contained in R'. But that is impossible. Every line through x^* contained in the quad $\langle Q_{x^*}, L_1 \rangle \cap F'$ is contained in H, and the line L_2' as well is also contained in H. So, we have our desired contradiction. $\square$

Lemma 3.11 *Suppose $\Delta \in \{DW(2n-1, q), DQ^-(2n+1, q)\}$. Let x be a point of H and let $\mathcal{L}$ denote the set of lines through x contained in H. Then $\mathcal{L}$ is a subspace of co-dimension at most 2 of $\mathsf{Res}(x) \cong \mathrm{PG}(n-1, q)$.*

Proof. We first show that $\mathcal{L}$ is a subspace of $\mathsf{Res}(x)$. Let L_1 and L_2 be two arbitrary distinct lines of $\mathcal{L}$ and let Q denote the unique quad through these lines. The quad Q cannot be ovoidal nor subquadrangular and hence has to be deep or singular. In any case, all lines of Q through x are contained in H and hence belong to $\mathcal{L}$. So, $\mathcal{L}$ should be a subspace of $\mathsf{Res}(x)$.

We show that the co-dimension of $\mathcal{L}$ as a subspace of $\mathsf{Res}(x)$ is at most 2. If this would not be the case then there would exist a hex F through x such that no line of $\mathcal{L}$ is

contained in F. But then the hyperplane $H \cap F$ of $\widetilde{F}$ would violate Propositions 3.4 and 3.5. $\qquad\square$

Lemma 3.12 *Suppose that either $\Delta = DW(2n-1, q)$ with q even or $\Delta = DQ^-(2n+1, q)$. Then there cannot exist a quad Q and a hex F for which the following hold:*
- *$F \cap H$ is a hexagonal hyperplane of $\widetilde{F}$;*
- *$Q \cap H$ is a non-classical ovoid of $\widetilde{Q}$;*
- *$Q \cap F$ is a line.*

Proof. Consider the convex subspace $F' := \langle Q, F \rangle$ of diameter 4. Then $F' \cap H$ is a non-classical hyperplane of $\widetilde{F'}$. Let x be the unique point of $Q \cap F \cap H$. By Lemma 3.7, there exists a unique quad Q_x through x such that the lines through x contained in $F' \cap H$ are precisely the lines through x contained in Q_x.

Suppose first that Q_x is contained in F. As $H \cap F$ is a hexagonal hyperplane of $\widetilde{F}$, the quad Q_x must be singular, but that is not possible by Lemma 3.7 which implies that the quad Q_x is deep.

Suppose next that Q_x is not contained in F. Then $\Delta = DQ^-(2n+1, q)$ and $\widetilde{F} = DQ^-(7, q)$. Let Q' be a quad through x contained in F intersecting Q_x in $\{x\}$ and Q in the line $Q \cap F$. Then Q' is an ovoidal quad and as the hexagonal hyperplane $F \cap H$ of $\widetilde{F} \cong DQ^-(7, q)$ is classical, the ovoid $Q' \cap H$ of $\widetilde{Q'}$ is also classical. Now, consider the hex $F' = \langle Q, Q' \rangle$. By Lemma 3.6, the hyperplane $F' \cap H$ of $\widetilde{F'}$ must be the extension of a non-classical ovoid of a quad of $\widetilde{F'}$. That is not possible as the hex F' contains ovoidal quads that are not bad (namely the quad Q'). $\qquad\square$

Lemma 3.13 *Suppose that either $\Delta = DW(2n-1, q)$ with q even or $\Delta = DQ^-(2n+1, q)$. Then there cannot exist a hex F such that $F \cap H$ is a hexagonal hyperplane of $\widetilde{F}$.*

Proof. Suppose F is a hex such that $F \cap H$ is a hexagonal hyperplane of $\widetilde{F}$. By Lemmas 3.8 and 3.12, there should exist a bad ovoidal quad intersecting $F \setminus H$ in a singleton and so we should have $n \geq 5$. We will now derive a contradiction. It suffices to deal with the case $n = 5$. Indeed, if $n > 5$, then the reasoning below applied to any convex subspace of diameter 5 containing F and a bad ovoidal quad meeting $F \setminus H$ in a singleton would yield the desired contradiction.

Step 1. *If R is a quad intersecting F in a singleton $\{x\}$ not contained in H, then R is bad ovoidal.*
PROOF. By Lemmas 3.8 and 3.12, there exists a bad ovoidal quad R_1 meeting F in the singleton $\{x\}$. Now, consider $\mathsf{Res}(x) \cong \mathrm{PG}(4, q)$. The hex F determines a plane α of $\mathsf{Res}(x)$ and the quads R and R_1 define lines β and β_1 in $\mathsf{Res}(x)$. The graph whose vertices are the lines of $\mathsf{Res}(x) \cong \mathrm{PG}(4, q)$ disjoint from α, with two vertices adjacent whenever the corresponding lines meet in a point is connected. So, when proving the above-mentioned claim, it suffices to consider the case where the lines β and β_1 meet in a singleton, that means, the case where the quads R and R_1 meet in a line. Consider the hex $F' := \langle R, R_1 \rangle$. This hex meets F in a line M through x. Let y denote the unique point of H on M. Since R_1 is a bad ovoidal quad, the hyperplane $F' \cap H$ of $\widetilde{F'}$ is the

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extension of a non-classical ovoid of a quad R_2 of $\widetilde{F}$. Since the line M is not contained in a bad ovoidal quad (Lemma 3.12), the quad R_2 should contain the point y. So, the quads R and R_2 are disjoint, implying that R is a bad ovoidal quad. (qed)

Step 2. *Let $x \in F \cap H$ and let $\mathcal{L}$ denote the set of lines through x contained in H. Then $\mathcal{L}$ is a hyperplane of $\mathsf{Res}(x)$.*

PROOF. Suppose that this is not the case. Then by Lemma 3.11, $\mathcal{L}$ is a subspace of co-dimension 2 of $\mathsf{Res}(x)$. That implies that there exists a quad Q_1 through x such that $Q_1 \cap F = \{x\}$ and no line of Q_1 through x is contained in H. Now, let M be a line of F through x not contained in H and consider the hex $F' := \langle Q_1, M \rangle$. Then $F' \cap F = M$. By Step 1, there exist bad ovoidal quads that are contained in F'. So, $F' \cap H$ must be the extension of a non-classical ovoid of a quad Q_2 of $\widetilde{F'}$. Since M is not contained in a bad ovoidal quad, the quad Q_2 contains the point x. The quad Q_2 is deep with respect to H implying that the intersection $Q_1 \cap Q_2$ (which is at least a line) is also contained in H. This is in contradiction with the fact that no line through x is contained in $Q_1 \cap H$. (qed)

Let x^* be an arbitrary point of $F \cap H$, let $\mathcal{L}^*$ denote the set of lines through x^* contained in H. Then $\mathcal{L}^*$ is a hyperplane of $\mathsf{Res}(x^*)$. Let G^* denote the unique convex subspace of diameter 4 through x^* containing all lines of $\mathcal{L}^*$. Then G^* cannot contain F as $(x^*)^\perp \cap F$ is not contained in the hexagonal hyperplane $H \cap F$ of $\widetilde{F}$. So, $G^* \cap F$ is a quad.

Step 3. *Let R be a quad of $\widetilde{G^*}$ through x^* such that $R \cap F = \{x^*\}$. Then R is deep with respect to H.*

PROOF. Let M be a line of F through x^* not contained in H and consider the hex $F' := \langle R, M \rangle$. Then $F' \cap F = M$. By Step 1, there exist bad ovoidal quads that are contained in F'. So, $F' \cap H$ must be the extension of a non-classical ovoid of a quad Q_2 of $\widetilde{F'}$. Since M is not contained in a bad ovoidal quad, the quad Q_2 must contain x^* and hence must coincide with R. This implies that R is deep. (qed)

Step 4. *Let R be a quad of $\widetilde{G^*}$ through x^* meeting F in a line L. Then R is deep with respect to H.*

PROOF. Suppose R is not deep with respect to H. Then R must be singular and $R \cap H = (x^*)^\perp \cap R$. Consider a hex F' of $\widetilde{G^*}$ through R for which $F' \cap F = L$. Then $F' \cap H$ is a hyperplane of $\widetilde{F'}$. If K is a line of F' meeting $\Delta_3(x^*)$ and R, then $K \cap H$ is a singleton $\{u^*\}$ contained in $\Delta_3(x^*)$. Let R' be the unique quad through u^* meeting L. Every quad of $\widetilde{F'}$ through x^* not containing L is deep by Step 3, implying that every line of F' through u^* not contained in R' is contained in H. Propositions 3.4, 3.5 and the fact that $q \neq 2$ if $\Delta = DW(2n-1, q)$ then imply that every line of F' through u^* is contained in H. In particular, this holds for the unique line through u^* meeting R. This is not compatible with the fact that R is singular. (qed)

Let Q_{x^*} denote the quad $G^* \cap F$. As $H \cap F$ is a hexagonal hyperplane of $\widetilde{F}$, the quad Q_{x^*} cannot be deep and hence has to be singular. So, $Q_{x^*} \cap H = (x^*)^\perp \cap Q_{x^*}$. Let F' denote a hex through Q_{x^*} contained in G^*. Let K be a line of $\widetilde{F'}$ meeting Q_{x^*} and $\Delta_3(x^*)$. Then

$K \cap H$ is a singleton $\{u^*\}$ contained in $\Delta_3(x^*)$. Every quad of $\widetilde{F'}$ through x^* distinct from Q_{x^*} is deep, implying that K is the unique line of $\widetilde{F'}$ through u^* not contained in H. This is impossible by Propositions 3.4 and 3.5. $\qquad\square$

The following is an immediate consequence of Proposition 3.1 and Lemmas 3.10 and 3.13.

Corollary 3.14 *Let* $\Delta \in \{DW(2n-1,q), DQ^-(2n+1,q)\}$. *Then every hex of* Δ *intersects* H *in either* F, *a singular hyperplane of* $\widetilde{F}$ *or the extension of an ovoid of a quad of* $\widetilde{F}$.

The following proposition finishes the proof of Theorem 1.1.

Proposition 3.15 *The hyperplane* H *is the extension of a non-classical ovoid of a quad of* Δ.

Proof. By Proposition 2.1, Lemma 3.2 and Corollary 3.14, H must be the extension of an SDPS-hyperplane H_F of a convex subspace F of even diameter 2δ. Let X be the SDPS-set of $\widetilde{F}$ associated with H_F. By Assumption (IV), there exists a bad ovoidal quad Q. By Proposition 2.4, there exists a quad Q' of $\widetilde{F}$ such that $Q' \cap X$ is a non-classical ovoid of $\widetilde{Q'}$. By Proposition 2.2, this is only possible when F is a quad. So, $H_F = X$ is a non-classical ovoid of $\widetilde{F}$ and H is the extension of this non-classical ovoid. $\qquad\square$

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