The smallest eigenvalues of Hamming graphs, Johnson graphs and other distance-regular graphs with classical parameters

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Abstract

We prove a conjecture by Van Dam & Sotirov on the smallest eigenvalue of (distance-j) Hamming graphs and a conjecture by Karloff on the smallest eigenvalue of (distance-j) Johnson graphs. More generally, we study the smallest eigenvalue and the second largest eigenvalue in absolute value of the graphs of the relations of classical P- and Q-polynomial association schemes.

1 Introduction

In this paper we study the smallest eigenvalue as well as the second largest one in absolute value of the adjacency matrix of several important families of graphs, all belonging to the classical P- and Q-polynomial association schemes [2, Chapter 6].

The most well-known example of a P-polynomial association scheme is the Hamming scheme. We investigate the eigenvalues of the graphs that have the vectors in $\mathbb{F}_q^d$ as vertices and two vertices are adjacent if they have Hamming distance j. The smallest eigenvalues are important for determining the max-cut of certain graphs in the Hamming scheme. These graphs provide examples where the performance ratio of the Goemans-Williamson algorithm is tight [1]. The smallest eigenvalues are also used for determining the max-k-cut [6] and the chromatic number of the graphs in the Hamming scheme [6].

The second important scheme belonging to the family of P-polynomial association schemes is the Johnson scheme. Here the vertices are the d-subsets of $\{1, 2, \ldots, n\}$. We investigate the eigenvalues of the graph where two d-sets

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are adjacent if they differ in exactly j elements. As for the Hamming scheme, these graphs provide examples for which the performance ratio of the Goemans-Williamson algorithm is tight and their smallest eigenvalues are central for determining their max-cuts [20]. These graphs are also important for investigating subsets with exactly one forbidden intersection, a variation of the classical Erdős-Ko-Rado theorem due to Frankl and Füredi [18].

The other graphs under investigation are Grassmann graphs, dual polar graphs, and various forms graphs, most prominently the bilinear forms graphs. Again, the smallest eigenvalues can be used to investigate the max-cuts and intersecting families in these graphs. The P-polynomial graphs obtain their importance from various applications. For example, Grassmann graphs are of interest due to their applications in network coding theory [26] and their role in the recent proof of the 2-to-2-games conjecture [21].

In the following we give a short summary of our main results on the specific families.

1.1 Hamming graphs

Let $q \geq 2, d \geq 1$ be integers. Let Q be a set of size q. The Hamming scheme $H(d, q)$ is the association scheme with vertex set Q^d, and as relation the Hamming distance. The $d+1$ relation graphs $H(d, q, j)$, where $0 \leq j \leq d$, have vertex set Q^d, and two vectors of length d are adjacent when they differ in j places.

The eigenmatrix P of $H(d, q)$ has entries $P_{ij} = K_j(i)$, where

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h}.$$

The eigenvalues of the graph $H(d, q, j)$ are the numbers in column j of P, so are the numbers $K_j(i)$, $0 \leq i \leq d$. The graph $H(d, q, j)$ is regular of degree $K_j(0) = (q-1)^j \binom{d}{j}$, and this is the largest eigenvalue. Motivated by problems in semidefinite programming related to the max-cut of a graph, Van Dam & Sotirov [6] conjectured

Conjecture 1.1. *Let $q \geq 2$ and $j \geq d - \frac{d-1}{q}$ where j is even when $q = 2$. Then the smallest eigenvalue of $H(d, q, j)$ is $K_j(1)$.*

Alon & Sudakov [1] proved this for $q = 2$ and d large and j/d fixed. Dumer & Kapralova [13, Cor. 10], proved this for $q = 2$ and all d. Here we settle the full conjecture.

In most cases $K_j(1)$ is not only the smallest eigenvalue, but also the second largest eigenvalue in absolute value. The only exception is the case $d = 4$, $q = 3$: the P-matrix of $H(4, 3)$ is

$$P = \begin{pmatrix} 1 & 8 & 24 & 32 & 16 \\ 1 & 5 & 6 & -4 & -8 \\ 1 & 2 & -3 & -4 & 4 \\ 1 & -1 & -3 & 5 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

and the eigenvalues of $H(4, 3, 3)$ are -4, 5 and 32.

The binary case was already settled by Dumer & Kapralova. We give a short and self-contained proof.

Theorem 1.2. ([13, Cor. 10]) *Let* $q = 2$.
(i) If $j \neq d/2$, *then* $|K_j(i)| \leq |K_j(1)|$ *for all* i, $1 \leq i \leq d-1$.
(ii) If $j = d/2$, *then* $K_j(1) = 0$ *and* $|K_j(i)| \leq |K_j(2)|$ *for all* i, $1 \leq i \leq d-1$.

Corollary 1.3. *Let* $q = 2$ *and* $j \geq (d+1)/2$.
(i) One has $K_j(1) \leq K_j(i)$ *for all* i, $0 \leq i \leq d-1$.
(ii) One has $K_j(1) \leq K_j(d)$ *if and only if* j *is even or* $j = d$.

The nonbinary case is settled here.

Theorem 1.4. *Let* $q \geq 3$ *and* $d - \frac{d-1}{q} \leq j \leq d$.
(i) One has $K_j(1) \leq K_j(i)$ *for all* i, $0 \leq i \leq d$.
(ii) One has $|K_j(i)| \leq |K_j(1)|$ *for all* $i \geq 1$, *unless* $(q,d,i,j) = (3,4,3,3)$.

1.2 Johnson graphs

The Johnson graphs $J(n,d)$ are the graphs with as vertices the d-subsets of a fixed n-set, adjacent when they meet in a $(d-1)$-set. W.l.o.g. we assume $n \geq 2d$ (since $J(n,d)$ is isomorphic to $J(n,n-d)$), and then these graphs are distance-regular of diameter d. The eigenmatrix P has entries $P_{ij} = E_j(i)$, where

$$E_j(i) = \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{j} \binom{n-d-i+h}{n-d-j}.$$

For $0 \leq j \leq d$, the distance-j graphs $J(n,d,j)$ of the Johnson graph $J(n,d)$ are the graphs with the same vertex set as $J(n,d)$, where two vertices are adjacent when they have distance j in $J(n,d)$, that is, when they meet in a $(d-j)$-set. For $j = d$ this graph is known as the Kneser graph $K(n,d)$. Motivated by problems in semidefinite programming related to the max-cut of a graph, Karloff [20] conjectured in 1999 the following:

Conjecture 1.5. *Let* $n = 2d$ *and* $j > d/2$. *Then the smallest eigenvalue of* $J(n,d,j)$ *is* $E_j(1)$.

Here we prove this conjecture (Corollary 3.11), and more generally determine precisely in which cases $E_j(1)$ is the smallest eigenvalue of $J(n,d,j)$ (Theorem 3.10).

1.3 Graphs with classical parameters

For general information on distance-regular graphs, see [2]. In [2, §6.1], *graphs with classical parameters* (d,b,α,β) are defined as distance regular graphs of diameter d with parameters given by certain expressions in d,b,α,β (see Section 4 for details).

The concept of graphs with classical parameters unifies a number of families of distance-regular graphs, such as the Hamming graphs, Johnson graphs, Grassmann graphs, dual polar graphs, bilinear forms graphs, etc.

d	b	α	β	family
d	1	0	$q-1$	Hamming graphs $H(d,q)$
d	1	1	$n-d$	Johnson graphs $J(n,d)$, $n \ge 2d$
d	q	q	$q\left[{n-d \atop 1}\right]$	Grassmann graphs $G_q(n,d)$, $n \ge 2d$
d	q	0	q^e	dual polar graphs $C_q(d,e)$, $e = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$
d	q	$q-1$	q^e-1	bilinear forms graph $H_q(d,e)$
$\lfloor n/2 \rfloor$	q^2	q^2-1	$q^{2n-2d-1}-1$	alternating forms graphs $A_q(n)$
d	$-q$	$-q-1$	$-(-q)^d-1$	Hermitian forms graphs $Q_q(d)$

Below we give the asymptotic behavior of the eigenmatrix P of these graphs when d, b, α are fixed and β tends to infinity (Theorem 4.5). We also give a simple explicit expression for the eigenvalues P_{dj}, that perhaps has not been noticed before (Proposition 4.1).

Subsequently, we investigate each of the individual families, and determine smallest and second largest eigenvalues and/or other properties of the eigenvalues. Main results are Theorem 5.8 for the Grassmann graphs, Corollary 6.5 for the dual polar graphs, Theorem 7.5 for the bilinear forms graphs, Theorem 8.3 for the alternating forms graphs, and Theorem 9.5 for the Hermitian forms graphs.

2 The Hamming case

We prove the stated results for the Hamming graphs.

2.1 Identities

We collect some (well-known) identities used in the sequel.

The defining equation gives $K_j(i)$ as a polynomial in i of degree j with leading coefficient $(-q)^j/j!$. We give three expressions.

$$\begin{aligned} K_j(i) &= \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h} \\ &= \sum_{h=0}^{j} (-q)^h (q-1)^{j-h} \binom{i}{h} \binom{d-h}{j-h} \\ &= \sum_{h=0}^{j} (-1)^h q^{j-h} \binom{d-i}{j-h} \binom{d-j+h}{h} \end{aligned}$$

(see Delsarte [7, p. 39], and [8, (15)]).

One has the symmetry

$$K_j(i)/\tbinom{d}{j}(q-1)^j = K_i(j)/\tbinom{d}{i}(q-1)^i.$$

In particular, $K_j(i)$ and $K_i(j)$ have the same sign.

There is also the symmetry

$$K_{d-j}(i) = (-1)^{i-j}(q-1)^{d-i-j} K_j(d-i).$$

Proposition 2.1. *Let $i, j \ge 1$. Then*
$(q-1)(d-i)K_j(i+1) - (i + (q-1)(d-i) - qj)K_j(i) + iK_j(i-1) = 0.$

2.2 Proofs

The occurrence of $d - \frac{d-1}{q}$ in Conjecture 1.1 is explained by the following proposition. Where it refers to $K_j(1)$ or $K_j(2)$, it is assumed that $d \geq 1$ or $d \geq 2$.

Proposition 2.2. *Let $q \geq 2$ and $0 \leq j \leq d$.*
(i) $K_j(1) < 0$ if and only if $j \geq d - \frac{d-1}{q}$.
(ii) $K_j(2) = K_j(1)$ if and only if $j = 0$ or $j = d - \frac{d-1}{q}$.
(ii)′ $K_j(2) > K_j(1)$ if and only if $j > d - \frac{d-1}{q}$.
(iii) $K_j(2) = \frac{-1}{q-1} K_j(1)$ if and only if $j = (d-1)(1 - \frac{1}{q})$ or $j = d$.
(iv) Let $d - \frac{d-1}{q} \leq j \leq d$. Then $|K_j(2)| \leq |K_j(1)|$.

Proof. (i) Since $K_j(i)$ has the same sign as $K_i(j)$, this follows from $K_1(j) = (q-1)d - qj$.

(ii) Since $K_j(i) = \binom{d}{j}(q-1)^{j-i} K_i(j) / \binom{d}{i}$, the claim says that $K_2(j) = \frac{1}{2}(q-1)(d-1)K_1(j)$ precisely for the two specified values of j. But this condition is quadratic in j, and is up to a constant factor $j(j - d + \frac{d-1}{q}) = 0$.

(ii)′ Clear from (ii), since $K_2(j)$ has positive leading coefficient.

(iii) The condition is equivalent to $K_2(j) = -\frac{1}{2}(d-1)K_1(j)$. Again it is quadratic in j. Up to a constant factor it is $(j-d)(j - (d-1)(1 - \frac{1}{q})) = 0$.

(iv) We want to show that $|K_2(j)| \leq \frac{1}{2}(q-1)(d-1)|K_1(j)|$. Since $K_1(j) < 0$ this is the pair of conditions $K_2(j) - \frac{1}{2}(q-1)(d-1)K_1(j) \geq 0$ and $-K_2(j) - \frac{1}{2}(q-1)(d-1)K_1(j) \geq 0$.

The former is up to a positive constant factor equivalent to $j(j - d + \frac{d-1}{q}) \geq 0$.

For the latter it suffices to see that $-K_2(j) - \frac{1}{2}(d-1)K_1(j) \geq 0$. Up to a positive constant factor this is equivalent to $(j-d)(j - (d-1)(1 - \frac{1}{q})) \leq 0$. □

If $j = d - \frac{d-1}{q}$, then $K_1(j) = -1$, and $K_j(1) = -\frac{1}{d}\binom{d}{j}(q-1)^{j-1}$.

In order to prove Theorems 1.2 and 1.4, we need three lemmas.

Lemma 2.3. $|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j}$.

Proof. Since $\binom{d-i}{j-h} = 0$ unless $j - h \leq d - i$, we have

$$|K_j(i)| = |\textstyle\sum_h (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h}| \leq \sum_{h \geq i+j-d} (q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$$
$$\leq (q-1)^{d-i} \textstyle\sum_h \binom{i}{h}\binom{d-i}{j-h} = (q-1)^{d-i}\binom{d}{j}.$$ □

Lemma 2.4. *Let $1 < i < d$ and $d - \frac{d-1}{q} \leq j \leq d$. If $qj \leq 2(q-1)(d-i)$, then $|K_j(i+1)| \leq \max(|K_j(i-1)|,\ |K_j(i)|)$.*

Proof. Apply Proposition 2.1. Put $a = (q-1)(d-i)$. One has $aK_j(i+1) - (i - qj + a)K_j(i) + iK_j(i-1) = 0$. If $|K_j(i-1)| \leq M$ and $|K_j(i)| \leq M$, then $a|K_j(i+1)| \leq |i - qj + a|M + iM$, and the conclusion follows if $i + |i - qj + a| \leq a$. Now $qj - i - a > (q-2)i \geq 0$, so we need $qj \leq 2a$, and that was one of the hypotheses. □

For $q = 2$ the scheme is imprimitive, and the graphs $H(d, q, j)$ are bipartite for odd j, and disconnected for even j. One has the additional symmetry $K_j(d - i) = (-i)^j K_j(i)$.

Lemma 2.5. *Let $j < d/2$ and $0 < i < d$. Then $\binom{d-1}{j-1} \leq \sum_g \binom{i}{2g}\binom{d-i}{j-2g} \leq \binom{d-1}{j}$.*

We prove Lemma 2.5 in the proof of Theorem 1.2.

Theorem 1.2 ([13, Cor. 10]) *Let* $q = 2$.
(i) If $j \neq d/2$*, then* $|K_j(i)| \leq |K_j(1)|$ *for all* i*,* $1 \leq i \leq d-1$*.*
(ii) If $j = d/2$*, then* $K_j(1) = 0$ *and* $|K_j(i)| \leq |K_j(2)|$ *for all* i*,* $1 \leq i \leq d-1$*.*

Proof. (i) By the symmetry $K_{d-j}(i) = (-1)^i K_j(i)$ we may suppose $j < d/2$.

We prove Lemma 2.5 and part (i) of the theorem simultaneously. Since $K_j(i) = \sum_h (-1)^h \binom{i}{h}\binom{d-i}{j-h} = 2\sum_g \binom{i}{2g}\binom{d-i}{j-2g} - \binom{d}{j}$, and $K_j(1) = \binom{d-1}{j} - \binom{d-1}{j-1}$, both statements are equivalent for all i.

Prove the statement of the lemma by induction of d. The conclusion follows by adding the inequalities for $(d-1, j-1)$ and $(d-1, j)$, using that $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$, except possibly when $i = d-1$ or $j = (d-1)/2$. If $i = d-1$, the claim is that $\binom{d-1}{j-1} \leq \binom{d-1}{2[j/2]} \leq \binom{d-1}{j}$, which is true. Instead of treating $j = (d-1)/2$ we use symmetry and take $j = (d+1)/2$ and prove the statement in (i) by induction on i, using Proposition 2.2 (iv) and Lemma 2.4. Here we may suppose $2 \leq i \leq d/2$ by the symmetry $K_j(d-i) = (-1)^j K_j(i)$.

(ii) By symmetry, $K_j(i) = 0$ when $j = d/2$ and i is odd. The 3-term recurrence reduces to $(d-i)K_j(i+1) + iK_j(i-1) = 0$ for odd i, so that $K_j(2h) = (-1)^h \binom{d}{d/2}\binom{d/2}{h}/\binom{d}{2h}$ and $|K_j(2h)|$ decreases with increasing $2h \leq d/2$. □

Corollary 1.3 *Let* $q = 2$ *and* $j \geq (d+1)/2$.
(i) One has $K_j(1) \leq K_j(i)$ *for all* i*,* $0 \leq i \leq d-1$*.*
(ii) One has $K_j(1) \leq K_j(d)$ *if and only if* j *is even or* $j = d$*.*

Proof. Since $K_j(1) < 0$, part (i) follows from part (i) of the theorem, and part (ii) from $K_j(d) = (-1)^j K_j(0)$. □

Next, consider the nonbinary case.

Theorem 1.4 *Let* $q \geq 3$ *and* $d - \frac{d-1}{q} \leq j \leq d$.
(i) One has $K_j(1) \leq K_j(i)$ *for all* i*,* $0 \leq i \leq d$*.*
(ii) One has $|K_j(i)| \leq |K_j(1)|$ *for all* $i \geq 1$*, unless* $(q, d, i, j) = (3, 4, 3, 3)$*.*

If $(q, d, j) = (3, 4, 3)$ then $K_j(0) = 32$, $K_j(i) = -4$ for $i = 1, 2, 4$, and $K_j(3) = 5$.

Proof. Since $K_j(1) < 0$ (and $K_j(0)$ is the largest eigenvalue), part (i) follows from part (ii). The case $i = 2$ was handled in Proposition 2.2, so we may assume $i \geq 3$.

For $j = d$ one has $K_j(i) = (-1)^i (q-1)^{d-i}$, and the statement is true.

For $j = d-1$ one has $K_j(i) = (-1)^{i-1}(q-1)^{d-i-1}(qi-d)$ and $d \geq q+1$. To show the claim it suffices to show that $qi - d \leq (q-1)^{i-1}(d-q)$ $(*)$, and this follows from $q(i-1) - 1 \leq (q-1)^{i-1}$, unless $(q, i) = (3, 3)$, in which case $(*)$ still holds, unless $d = 4$.

So, we may assume $d - \frac{d-1}{q} \leq j \leq d-2$. This implies that $3 \leq q \leq (d-1)/2$.

If $qj \leq 2(q-1)(d-i+1)$ then we can apply Lemma 2.4 (and induction on i) to conclude that $|K_j(i)| \leq \max(|K_j(1)|, |K_j(2)|)$, and we are done. So, assume $qj > 2(q-1)(d-i+1)$.

One has $K_j(1) = (q-1)^{j-1}\binom{d}{j}(q-1-\frac{qj}{d})$, where the last factor is negative. From Lemma 2.3 we see that $|K_j(i)| \leq |K_j(1)|$ when $d \leq (q-1)^{i+j-d-1}(qj - (q-1)d)$.

Using $qj-(q-1)d \ge 1$ and $d-i+1 < \frac{qj}{2(q-1)} \le \frac{3}{4}j$ and $j \ge d - \frac{d-1}{q} \ge \frac{2}{3}d$ and $q \ge 3$ we see that it suffices to have $d^6 \le 2^d$, so $d \ge 30$ suffices. The finitely many d with $d < 30$ can be checked separately. □

2.3 Large q

Proposition 2.6. *For fixed d, let q be sufficiently large. Then $K_j(i)$ is positive for $i+j \le d$, and has sign $(-1)^{i+j-d}$ for $i+j \ge d$. For each $j > 0$, the smallest eigenvalue of $H(d,q,j)$ is $K_j(d-j+1)$.*

Proof. We have $K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h}$. When q tends to infinity, and d, j are fixed, this sum is dominated by its first nonzero term. So $K_j(i) \approx (q-1)^j \binom{d-i}{j}$ if $i+j \le d$, and $K_j(i) \approx (-1)^{j+i-d}(q-1)^{d-i}\binom{i}{j+i-d}$ if $i+j \ge d$. □

How large is 'sufficiently large'? The value $K_j(d-j+1)$ is the unique smallest eigenvalue of $H(d,q,j)$ for all j when $q \ge q_0(d)$.

d	2	3	4	5	6	7	8	9	10	12	14	16	18	20	30	40	50	60	100
q_0	2	3	4	5	7	9	12	15	18	26	35	45	57	70	156	277	433	623	1730

Lemma 2.7. *Suppose $q > \frac{1}{4}d^2 + 1$. Then*
(i) $K_j(i) > 0$ for $i \le d-j$,
(ii) $K_j(d-j+1) < 0$,
(iii) $|K_j(i)| < |K_j(d-j+1)|$ for $i > d-j+1$.

Proof. If $q > \frac{1}{4}d^2+1$, then the terms $(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$ decrease monotonically when h increases, so that the sign of $K_j(i)$ is that of the first nonzero term and the difference between $K_j(i)$ and the first nonzero term is smaller than the next term.

For $2 \le e \le j$ we have
$|K_j(d-j+e)| \le (q-1)^{j-e}\binom{d-j+e}{e} + (q-1)^{j-e-1}\binom{d+j+e}{e+1}(j-e)$ and
$|K_j(d-j+1)| \ge (q-1)^{j-1}(d-j+1) - (q-1)^{j-2}\binom{d-j+1}{2}(j-1)$, so that
$|K_j(d-j+e)|/(q-1)^{j-e-1} \le \binom{d-j+e}{e}(q-1+\frac{d-j}{e+1}(j-e)) \le \frac{4}{3}q\binom{d-j+e}{e}$
and $|K_j(d-j+1)| \ge \frac{1}{2}q(q-1)^{j-2}(d-j+1)$. So, it suffices to see
$\binom{d-j+e}{e} \le \frac{3}{8}(q-1)^{e-1}(d-j+1)$. This holds for $e \ge 3$, and for $e=2$, $j \ge 3$, and for $j=e=2$ we can drop the factor $\frac{4}{3}$, and the conclusion holds. □

2.4 Coincidences

A general matrix A in the Bose-Mesner algebra $\mathcal{A}$ of a d-class association scheme (see [2, Chapter 2] for a definition) will have $d+1$ distinct eigenvalues, and generate $\mathcal{A}$, in the sense that each element of $\mathcal{A}$ is a polynomial of degree at most d in A. Cases where some relation matrix A_j has fewer eigenvalues (and hence generates a proper subalgebra) are of interest.

Look at the Hamming scheme. For $q=2$, the main expected coincidences between the $P_{ij} = K_j(i)$ for fixed d and j are given in the following lemma.

Lemma 2.8. *Let $q=2$.*
(i) If j is even, then $P_{ij} = P_{d-i,j}$.
(ii) If $d=2j$, then $P_{ij} = 0$ for all odd i.

(iii) If $d = 2j - 1$, then $P_{2h-1,j} = P_{2h,j}$ for $1 \le h \le j - 1$.
(iv) If $j = d$, then $P_{ij} = (-1)^i$ for all i.

Proof. We only have to show (iii), and this follows from Proposition 2.2 (ii), and the 3-term recurrence given in Proposition (2.1). □

If $K_j(i) = 0$, then also $K_j(d-i) = 0$ and we have a further coincidence (when j is odd and $i \neq d/2$). Integral zeros of Krawtchouk polynomials play a role e.g. in the study of the existence of perfect codes or the invertibility of Radon transforms, and have been studied by many authors, cf. [4, 12, 15, 16, 22, 30, 31]. For $j = 1, 2, 3$ there are infinite families. For fixed $j \ge 4$ there are zeros only for finitely many d. Recall that $K_j(i) = 0$ if and only if $K_i(j) = 0$.

Lemma 2.9. ([4, Th. 4.6] and [12, Ex. 10]) *Let $q = 2$, $i \le d/2$, $j \le d/2$.*
(i) $K_1(i) = 0$ if and only if $d = 2i$.
(ii) $K_2(i) = 0$ if and only if $i = \binom{h}{2}$, $d = h^2$ for some integral $h \ge 3$.
(iii) $K_3(i) = 0$ if and only if $i = h(3h \pm 1)/2$, $d = 3h^2 + 3h + \frac{3}{2} \pm (h + \frac{1}{2})$ for some integral $h \ge 2$.
(iv) $K_{2h}(4h - 1) = 0$ if $d = 8h + 1$.

The family given last has $j = (d-3)/2$. There are also infinite families with $j = (d-t)/2$ for $t = 4, 5, 6, 8$ ([15]).

For arbitrary q there are fewer obvious coincidences.

Lemma 2.10. *Let $q \ge 2$.*
(i) If $j = 0$, then $P_{ij} = 1$ for all i.
(ii) If $j = 2$, then $P_{hj} = P_{ij}$ if and only if $h + i = 2(d-1)(1 - \frac{1}{q}) + 1$.
(iii) If $qj = (q-1)d + 1$, then $P_{1j} = P_{2j}$.

Proof. (i) The matrix $A_0 = I$ only has the single eigenvalue 1.
(ii) Note that $K_2(i)$ is quadratic in i.
(iii) This is what Proposition 2.2 (ii) says. □

We look for cases where some A_j has fewer distinct eigenvalues than expected (given the above lemmas), or just has few distinct eigenvalues. Below we list cases where $H(d, q, j)$ has precisely n distinct eigenvalues, while $d + 1 > n$, for $n = 3, 4, 5, 6$.

Conjecture 2.11. *If $H(d, q, j)$ is connected, it has more than $d/2$ distinct eigenvalues.*

2.4.1 Three distinct eigenvalues

If $H(d, q, j)$ has three distinct eigenvalues, it is strongly regular, or (in case $q = 2$ and j even) it is the disjoint union of two isomorphic connected components, both strongly regular.

For example, the P-matrix of $H(4, 3)$ was given above,

$$P = \begin{pmatrix} 1 & 8 & 24 & 32 & 16 \\ 1 & 5 & 6 & -4 & -8 \\ 1 & 2 & -3 & -4 & 4 \\ 1 & -1 & -3 & 5 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

and $H(4,3,3)$ is strongly regular with parameters $(v,k,\lambda,\mu) = (81,32,13,12)$ and spectrum $32^1\ 5^{32}\ (-4)^{48}$.

For $H(7,2)$ one gets

$$P = \begin{pmatrix} 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ 1 & 5 & 9 & 5 & -5 & -9 & -5 & -1 \\ 1 & 3 & 1 & -5 & -5 & 1 & 3 & 1 \\ 1 & 1 & -3 & -3 & 3 & 3 & -1 & -1 \\ 1 & -1 & -3 & 3 & 3 & -3 & -1 & 1 \\ 1 & -3 & 1 & 5 & -5 & -1 & 3 & -1 \\ 1 & -5 & 9 & -5 & -5 & 9 & -5 & 1 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & -1 \end{pmatrix}$$

and the graph $H(7,2,4)$ has two connected components, both isomorphic to the graph Δ on the 64 binary vectors of length 7 and even weight, adjacent when they differ in 4 places. The graph Δ is strongly regular with parameters $(v,k,\lambda,\mu) = (64,35,18,20)$ and spectrum $35^1\ 3^{35}\ (-5)^{28}$.

Cases with three eigenvalues (the connected graphs among these are strongly regular—we give the standard parameters (v,k,λ,μ)):

d	q	j	comment
4	2	2	2 copies of $\overline{4K_2}$
5	2	2	2 copies of the Clebsch graph
5	2	4	2 copies of the complement of the Clebsch graph
7	2	4	2 copies of $VO^+(6,2)$
4	3	2	$(81,24,9,6)$
4	3	3	$(81,32,13,12)$: $VO^+(4,3)$
3	4	2	$(64,27,10,12)$: $VO^-(6,2)$

More generally, if we take the Hamming scheme $H(d,q)$ with $q=4$, and call two distinct vertices adjacent if their distance is even, we obtain a strongly regular graph (as was observed in [19, Case III]), namely the graph $VO^{\pm}(2d,2)$, where the sign is $(-1)^d$. Indeed, the weight of a quaternary digit is given by the (elliptic) binary quadratic form $x_1^2 + x_1x_2 + x_2^2$. For $d=3$ this graph is $H(3,4,2)$.

2.4.2 Four/five/six distinct eigenvalues

In Table 1 below we list further cases in which $H(d,q,j)$ has fewer than $d+1$ distinct eigenvalues.

For example, the eigenmatrix of $H(7,3)$ is

$$P = \begin{pmatrix} 1 & 14 & 84 & 280 & 560 & 672 & 448 & 128 \\ 1 & 11 & 48 & 100 & 80 & -48 & -128 & -64 \\ 1 & 8 & 21 & 10 & -40 & -48 & 16 & 32 \\ 1 & 5 & 3 & -17 & -16 & 24 & 16 & -16 \\ 1 & 2 & -6 & -8 & 17 & 6 & -20 & 8 \\ 1 & -1 & -6 & 10 & 5 & -21 & 16 & -4 \\ 1 & -4 & 3 & 10 & -25 & 24 & -11 & 2 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & -1 \end{pmatrix}.$$

and we see coincidences in columns 2, 3, 5, 6.

d	q	j	comment
5	2	3	L2.8 (iii)
6	2	2,4	L2.8 (i)
7	2	2,6	L2.8 (i)
8	2	4	L2.8 (i),(ii)
11	2	6	L2.8 (i),(iii)
5	3	3	$P_{13} = P_{43}$ $P_{23} = P_{53}$
5	4	2	L2.10 (ii)
4	6	2	L2.10 (ii)

Four eigenvalues

d	q	j	comment
6	2	3	L2.8 (ii)
8	2	2,6	L2.8 (i)
9	2	2,4,6,8	L2.8 (i)
10	2	4	L2.8 (i) $P_{24} = P_{34}$
10	2	8	L2.8 (i) $P_{38} = P_{48}$
11	2	4	L2.8 (i) $P_{24} = P_{44}$
11	2	8	L2.8 (i) $P_{38} = P_{58}$
12	2	6	L2.8 (i),(ii)
15	2	8	L2.8 (i),(iii)
7	3	2	L2.10 (ii)
7	3	5	L2.10 (iii) $P_{35} = P_{65}$ $P_{55} = P_{75}$
5	4	3	$P_{33} = P_{53}$
5	4	4	L2.10 (iii)
6	5	2	L2.10 (ii)
5	6	3	$P_{23} = P_{53}$
5	8	2	L2.10 (ii)

Five eigenvalues

d	q	j	comment
7	2	3	$P_{13} = P_{53}$
9	2	5	L2.8 (iii)
10	2	2,6	L2.8 (i)
11	2	2,10	L2.8 (i)
12	2	4	L2.8 (i) $P_{24} = P_{64}$
12	2	8	L2.8 (i) $P_{28} = P_{68}$
16	2	8	L2.8 (i),(ii)
19	2	10	L2.8 (i),(iii)
7	3	3	$P_{23} = P_{53} = P_{63}$
7	3	6	$P_{26} = P_{36} = P_{56}$
7	4	2	L2.10 (ii)
7	4	4	$P_{24} = P_{64}$ $P_{54} = P_{74}$
6	5	3	$P_{43} = P_{63}$
6	5	5	L2.10 (iii)
7	6	2	L2.10 (ii)
6	10	2	L2.10 (ii)
6	10	3	$P_{43} = P_{63}$

Six eigenvalues

Table 1: Cases where $H(d, q, j)$ has fewer than $d + 1$ distinct eigenvalues

3 The Johnson case

The eigenvalues of $J(n, d, j)$ are $P_{ij} = E_j(i)$ $(0 \leq i, j \leq d)$. We give three expressions for the $E_j(i)$:

$$\begin{aligned} E_j(i) &= \sum_{h=0}^{j} (-1)^h \binom{i}{h} \binom{d-i}{j-h} \binom{n-d-i}{j-h} \\ &= \sum_{h=0}^{j} (-1)^{j-h} \binom{d-i}{h} \binom{d-h}{j-h} \binom{n-d-i+h}{h} \\ &= \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{j} \binom{n-d-i+h}{n-d-j} \end{aligned}$$

(see Delsarte [7, p. 48], and Karloff [20, Theorem 2.1]).

3.1 Identities

Using the second of the expressions given above for $E_j(i)$ we find the eigenvalues of the Kneser graph.

Proposition 3.1. (Lovász [23]) *The eigenvalues of the Kneser graph are* $P_{id} = (-1)^i \binom{n-d-i}{d-i} = (-1)^i \binom{n-d-i}{n-2d}$.

Proof. We use that $\binom{n+h}{h} = (-1)^h \binom{-n-1}{h}$ and $\sum_h \binom{a}{c-h} \binom{b}{h} = \binom{a+b}{c}$ and find $P_{id} = \sum_h (-1)^{d-h} \binom{d-i}{h} \binom{n-d+h-i}{h} = (-1)^d \sum_h \binom{d-i}{d-i-h} \binom{-n+d+i-1}{h}$
$= (-1)^d \binom{-n+2d-1}{d-i} = (-1)^i \binom{n-d-i}{d-i}$. $\square$

Let us write $E_j^{n,d}(i)$ instead of $E_j(i)$ when it is necessary to make the dependence on n and d explicit. Now we have the following induction.

Proposition 3.2. *Let $i, j \geq 1$. Then $E_j^{n+2,d+1}(i) = E_j^{n,d}(i-1) - E_{j-1}^{n,d}(i-1)$.*

Proof. Using $E_j^{n,d}(i) = \sum_h (-1)^h \binom{i}{h}\binom{d-i}{j-h}\binom{n-d-i}{j-h}$ one sees that the claim reduces to $\binom{i}{h} = \binom{i-1}{h} + \binom{i-1}{h-1}$. □

There is a symmetry if $n = 2d$.

Proposition 3.3. *If $n = 2d$, then $E_{d-j}(i) = (-1)^i E_j(i)$. In particular, if moreover $j = d/2$, i odd, then $E_j(i) = 0$.* □

3.2 Coincidences

The association scheme on the set X of partitions of a $2k$-set into two k-sets has $\lfloor \frac{1}{2}k+1 \rfloor$ relations R_j (mutual intersection sizes $0+k$, $1+(k-1)$, ..., $\lfloor \frac{1}{2}k \rfloor + \lceil \frac{1}{2}k \rceil$). If one picks a fixed element in the $2k$-set, one sees that (X, R_j) is isomorphic to the graph on the $(k-1)$-subsets of a $(2k-1)$-set, adjacent when they meet in either $j-1$ or $k-j-1$ points. Thus, in the Johnson scheme with $n = 2d+1$, the matrices $A_j + A_{d-j+1}$ have not more than $(d+3)/2$ distinct eigenvalues.

Proposition 3.4. *Let $n = 2d+1$ and $j = (d+1)/2$ and $0 < t < d/2$. Then $E_j^{n,d}(2t-1) = E_j^{n,d}(2t) = E_j^{n-1,d}(2t-1)$.* □

3.3 Negative $E_j(1)$

Let us write $e := n-d$ to make our formulas shorter and nicer.

Proposition 3.5. *Let $j > 0$. Then*
(i) $E_j(1) = 0$ if and only if $j = de/n$.
(ii) $E_j(1) < 0$ if and only if $j > de/n$,
(iii) $E_j(1) = E_j(2)$ if and only if $j(n-1) = de$.
(iv) $E_j(1) < E_j(2)$ if and only if $j(n-1) > de$.

Proof. (i)-(ii) We have $E_j(1) = (1 - \frac{jn}{de})\binom{d}{j}\binom{e}{j}$.

(iii)-(iv) Let $j > 1$. Writing out the expressions for $E_j(1)$ and $E_j(2)$, dividing by $\binom{d-2}{j-2}\binom{e-2}{j-2}$, multiplying by $j^2(j-1)^2$, and simplifying, we see that $E_j(1) \leq E_j(2)$ is equivalent to $j(n-1) \geq de$. (There is a factor $n-2$, but $i = 2$ occurs only for $d \geq 2$, $n \geq 4$.) For $j = 1$ we have $E_1(i) = (d-i)(e-i) - i = de - i(n-i+1)$, and $E_j(1) \leq E_j(2)$ is equivalent to $n \leq 2$, which is false. □

For $J(8,3)$ we have

$$P = \begin{pmatrix} 1 & 15 & 30 & 10 \\ 1 & 7 & -2 & -6 \\ 1 & 1 & -5 & 3 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

3.4 Auxiliary results

For any regular graph Γ with adjacency matrix A, the sum of the squares of the eigenvalues of Γ (i.e., of A) is the trace of A^2, which is vk, if Γ has v vertices and is regular of valency k. We apply this to $J(n,d,j)$, and find $vk_j = \sum_{i=0}^{d} m_i E_j(i)^2$, where $v = \binom{n}{d}$ is the number of vertices of $J(n,d)$, $k_j = \binom{d}{j}\binom{e}{j}$ is the valency of $J(n,d,j)$ (with $e := n-d$), and $m_i = \binom{n}{i} - \binom{n}{i-1}$ is the multiplicity of the i-th eigenvalue (cf. [2, 9.1.2]). It follows that $E_j(i)^2 \le vk_j/m_i$.

We need to estimate k_j close to its maximum value, and use Chvátal's tail inequality for the hypergeometric distribution.

Lemma 3.6. *Let $I = (\frac{de}{n} - \sqrt{d}, \frac{de}{n} + \sqrt{d})$. Then $\sum_{j\in I} k_j \ge \frac{8}{11}v$.*

Proof. Consider the random variable X that is j with probability k_j/v. It has expected value $E(X) = \frac{de}{n}$. According to Chvátal [5] (cf. [27]),

$$Pr(|X - E(X)| \ge td) \le 2\exp(-2t^2 d).$$

Choosing $t = d^{-1/2}$ yields the assertion, as $1 - 2\exp(-2) > \frac{8}{11}$. $\square$

Lemma 3.7. *Let $j_0 = \frac{de}{n}$, and let $j_0 \le j < j_0 + \frac{3}{2}$. If $\frac{de}{n-1} \le j < d$ and $i \ge 3$, then $|E_j(i)| \le |E_j(1)|$.*

Proof. We start with some observations that hold when d is not too small.

(1) Since $\frac{de}{n-1} \le j \le d-1$, we find $de \le (d-1)(d+e-1)$, that is, $e \le (d-1)^2$.

(2) Since n^3/d^2e^2 decreases with e for $e \le 2d$, and increases with e for $e \ge 2d$, it is maximal for $e = (d-1)^2$ (for $d \ge 7$), so that $n/j_0^2 \le (d^2-d+1)^3/d^2(d-1)^4 < 1 + \frac{3}{2d}$ (for $d \ge 10$).

(3) We show that $k_{j-1}/k_j < 3$ if $d \ge 10$. Indeed, $k_{j-1}/k_j = c_j/b_{j-1} = j^2/(d-j+1)(e-j+1)$ so that $k_{j-1}/k_j < 3$ is equivalent to $de - n(j-1) + (j-1)^2 - \frac{1}{3}j^2 > 0$. The LHS decreases with j, so it suffices to show this for $j = j_0 + \frac{3}{2}$. Since $de = nj_0$ we have to show $\frac{4}{3}j_0^2 \ge n+1$, and this follows from (2).

(4) We show that $v/k_j < \frac{1}{6}(n-5)$ for $n \ge 42$. According to Lemma 3.6, $\sum_{|\ell - j_0| < \sqrt{d}} k_\ell > \frac{8}{11}v$. Let k_{j_1} be the largest among the k_ℓ. Then $\lfloor j_0 \rfloor \le j_1 \le \lceil j_0 \rceil$ and $2\sqrt{d}\, k_{j_1} > \frac{8}{11}v$, that is, $v/k_{j_1} < \frac{11}{4}\sqrt{d}$. The index j differs at most 2 from j_1, and $j \ge j_1$. Since $k_{j-2}/k_{j-1} \le k_{j-1}/k_j < 3$ we have $k_{j_1}/k_j < 9$ and hence $v/k_j < \frac{99}{4}\sqrt{d}$. Our aim was $v/k_j < \frac{1}{6}(n-5)$, and since $n \ge 2d$ so that $n \ge \sqrt{2n}\sqrt{d}$ this follows from $n > 11255$. The finitely many cases with $42 \le n \le 11255$ were checked by computer.

(5) We show that $E_j(i)^2 \le E_j(1)^2$ if $i \ge 3$. In the discussion above we found that $E_j(i)^2 \le vk_j/m_i$, where $m_i \ge m_3 = \frac{1}{6}n(n-1)(n-5)$. On the other hand, $E_j(1) = \binom{d}{j}\binom{e}{j}(1 - \frac{j}{j_0})$, and $j - j_0 \ge \frac{de}{n-1} - \frac{de}{n} = \frac{j_0}{n-1}$, so that $E_j(i)^2 \le E_j(1)^2$ will hold when $v/k_j \le \frac{1}{6}\frac{n(n-5)}{n-1}$. That was shown in (4). Earlier we needed $d \ge 10$ (or $n \ge 42$), but if $d \le 9$ then $n \le 73$, and these cases were checked by computer. $\square$

3.5 The smallest eigenvalue

It looks like $|E_j(1)|$ is the largest among the $|E_j(i)|$ $(1 \le i \le d)$ when j is not very close to the zero $\frac{de}{n}$ of $E_j(1)$ (viewed as polynomial in j), say at least when

$|j - \frac{de}{n}| \geq \frac{1}{4}$. If $|E_j(1)|$ is largest, and moreover $E_j(1) < 0$, then $E_j(1)$ is the smallest among the $E_j(i)$, $0 \leq i \leq d$. We prove below that this is the case if and only if $j \geq \frac{de}{n-1}$.

Lemma 3.8. *Let $(j-1)(n+1) \geq de$. Then $E_j(0) + |E_{j-1}(1)| + |E_j(1)| \leq E_{j-1}(0)$.*

Proof. Use $E_j(0) = \binom{d}{j}\binom{e}{j}$ and $E_j(1) = \binom{d}{j}\binom{e}{j}(1 - \frac{jn}{de})$ and $jn > de$ to see that the desired inequality is equivalent to $\frac{jn}{de}\frac{d-j+1}{j}\frac{e-j+1}{j} + |1 - \frac{(j-1)n}{de}| \leq 1$. If $(j-1)n \leq de$ we have to show that $(d-j+1)(e-j+1) \leq j(j-1)$, that is, $de \leq (j-1)(n+1)$, which is our hypothesis. If $(j-1)n \geq de$ we have to show that $(d-j+1)(e-j+1)+j(j-1) \leq \frac{2dej}{n}$, that is, $de-(j-1)(n-2j+1) \leq \frac{2dej}{n}$, that is, $de(n-2j) \leq (j-1)n(n-2j+1)$, which holds by hypothesis. $\square$

Since we know the eigenvalues of the Kneser graph, the case $j = d$ is immediate.

Proposition 3.9. *Let $d \geq 1$. The smallest eigenvalue of $K(n,d)$, and the second largest in absolute value, is $E_d(1)$.* $\square$

Theorem 3.10. *Let $j > 0$. Then $E_j(1)$ is the smallest eigenvalue of $J(n,d,j)$ if and only if $j(n-1) \geq de$. In this case $E_j(1)$ is also the second largest in absolute value among the eigenvalues of $J(n,d,j)$.*

Proof. By Proposition 3.5, if $E_j^{n,d}(1)$ is the smallest eigenvalue of $J(n,d,j)$, then $j(n-1) \geq de$, and $E_j^{n,d}(1) < 0$. We now show by induction on d that if $j(n-1) \geq de$, then $|E_j^{n,d}(i)| \leq |E_j^{n,d}(1)|$. If $j = d$ the statement follows from Proposition 3.9. If $\frac{de}{n-1} \leq j < \frac{de}{n-3}$, then (since $n \geq 2d$ and $d \geq 3$) $j_0 < j < j_0 + \frac{3}{2}$, where $j_0 = \frac{de}{n}$, and our claim holds by Lemma 3.7 if $i \geq 3$. We wish to show that if $j(n-1) \geq (d+1)(e+1)$, then $|E_j^{n+2,d+1}(i)| \leq |E_j^{n+2,d+1}(1)|$, that is, by Proposition 3.2, $|E_j^{n,d}(i-1) - E_{j-1}^{n,d}(i-1)| \leq |E_j^{n,d}(0) - E_{j-1}^{n,d}(0)|$. Now $j(n-1) \geq (d+1)(e+1)$ implies $(j-1)(n-1) \geq de$, and by induction, or trivially if $i = 2$, $|E_j^{n,d}(i-1)| \leq |E_j^{n,d}(1)|$ and $|E_{j-1}^{n,d}(i-1)| \leq E_{j-1}^{n,d}(1)|$ and our claim follows by Lemma 3.8. $\square$

Karloff [20] studied graphs $J(n,d,j)$ for the special case $n = 2d$. (His notation is $J(n,d,d-j)$ instead of our $J(n,d,j)$.) He proves ([20], Theorem 2.3) that $E_j(1)$ is the smallest eigenvalue of $J(n,d,j)$ when $d = n/2$ and $j \geq 5d/6$. He conjectures ([20], Conjecture 2.12) that $E_j(1)$ is the smallest eigenvalue of $J(n,d,j)$ when $d = n/2$ and $j > d/2$. This conjecture immediately follows from the above theorem.

Corollary 3.11. *If $j > d/2$, then the smallest eigenvalue of $J(2d,d,j)$, and the second largest in absolute value, is $E_j(1)$.* $\square$

For $n = 2d+1$ and $j = \frac{1}{2}d$ we have $E_j(2) = -\frac{d}{d-1}E_j(1)$ so that $|E_j(2)| > |E_j(1)|$.

3.6 Large n

Proposition 3.12. *For fixed d, let n be sufficiently large. Then $E_j(i)$ is positive for $i+j \leq d$, and has sign $(-1)^{i+j-d}$ for $i+j \geq d$. For each $j > 0$, the smallest eigenvalue of $J(n,d,j)$ is $E_j(d-j+1)$.*

Proof. We have $E_j(i) = \sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{d-i}{j-h}\binom{n-d-i}{j-h}$. When n tends to infinity, and d is fixed, this sum is dominated by its first nonzero term. So $E_j(i) \approx \binom{d-i}{j}\binom{n-d-i}{j}$ if $i+j \le d$, and $E_j(i) \approx (-1)^{i+j-d}\binom{i}{i+j-d}\binom{n-d-i}{d-i}$ if $i+j \ge d$. Also, for $i+j < d$, $E_j(i)/E_j(i+1) \approx \frac{(d-i)(n-d-i)}{(d-i-j)(n-d-i-j)} > 1$, so the $E_j(i)$ decrease in absolute value with increasing i. $\square$

For example, for $J(27,5)$:

$$P = \begin{pmatrix} 1 & 110 & 2310 & 15400 & 36575 & 26334 \\ 1 & 83 & 1176 & 4060 & 665 & -5985 \\ 1 & 58 & 451 & 60 & -1710 & 1140 \\ 1 & 35 & 60 & -400 & 475 & -171 \\ 1 & 14 & -66 & 104 & -71 & 18 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix}.$$

For $d = 5$ this is the smallest n with the described sign pattern. We have to go to $n = 34$ to get decreasing absolute values in the columns.

4 Graphs with classical parameters

Given a constant b, define

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_b = \begin{cases} 0 & \text{if } m < 0, \\ \dbinom{n}{m} & \text{if } b = 1, \\ \displaystyle\prod_{h=0}^{m-1} \frac{b^{n-h}-1}{b^{m-h}-1} & \text{otherwise.} \end{cases}$$

Graphs *with classical parameters* are distance-regular graphs with intersection numbers $b_i = (\left[{d \atop 1}\right] - \left[{i \atop 1}\right])(\beta - \alpha\left[{i \atop 1}\right])$ and $c_i = \left[{i \atop 1}\right](1 + \alpha\left[{i-1 \atop 1}\right])$ $(0 \le i \le d)$ (see [2, §6.1]). It follows that $k = \beta\left[{d \atop 1}\right]$ and $a_i = \left[{i \atop 1}\right](\beta - 1 + \alpha(\left[{d \atop 1}\right] - \left[{i \atop 1}\right] - \left[{i-1 \atop 1}\right]))$. In [2], Corollary 8.4.2, the eigenvalues of graphs with classical parameters are found to be $\theta_i = \left[{d-i \atop 1}\right](\beta - \alpha\left[{i \atop 1}\right]) - \left[{i \atop 1}\right]$ $(0 \le i \le d)$.

The base b is an integer different from 0, -1 ([2, 6.2.1]).

4.1 Identities

The P_{ij} follow from the recurrence $P_{i,j+1} = ((\theta_i - a_j)P_{ij} - b_{j-1}P_{i,j-1})/c_{j+1}$ and the starting values $P_{i0} = 1$, $P_{i1} = \theta_i$ (see [2, Chapter 4.1 (11)]). There is a simple explicit expression for the last row of the P matrix. It is independent of α and β.

Proposition 4.1. $P_{dj} = (-1)^j\left[{d \atop j}\right]b^{\binom{j}{2}}$.

Proof. Induction on j, using the recurrence. $\square$

Graphs with classical parameters are formally self-dual when $\alpha = b - 1$. If this is the case, then $P_{ij}/P_{0j} = P_{ji}/P_{0i}$ for all i, j, and the number of vertices is $v = (\beta+1)^d$. In this case, the above proposition can be translated to give the values of the last column of P.

Proposition 4.2. $P_{id}/P_{i+1,d} = 1 - (\beta+1)b^{-i}$. $\square$

4.2 Sign changes

The columns of the matrix P correspond to the graph distances on the distance-regular graph under consideration, and hence have a natural ordering. For general distance-regular graphs one is free to choose the ordering of the rows, corresponding to an ordering of the eigenspaces. According to [3], Proposition 11.6.2, the i-th row and the i-th column of P have exactly i sign changes if we order the rows according to descending real order on the θ_i.

Graphs with classical parameters are Q-polynomial, and hence have a natural ordering on the eigenspaces. Usually this is the order with descending θ_i, provided $b > 0$.

Proposition 4.3. *Suppose $b > 0$. Then $\theta_0 > \theta_1 > \ldots > \theta_d$ if and only if $\alpha \le b - 1$ or $\beta > \alpha\left[{d-1 \atop 1}\right] - b^{d-1}$. If this is the case, then the i-th row and the i-th column of P have exactly i sign changes ($0 \le i \le d$).*

Proof. We have to check that $\theta_i > \theta_{i+1}$, i.e., that $\beta > \alpha\left[{2i+1-d \atop 1}\right] - b^{2i+1-d}$ for $0 \le i \le d-1$. If $\alpha \le b-1$ then the strongest of these is the inequality for $i = 0$, but it is automatically satisfied since θ_0 is the graph valency. If $\alpha > b-1$ the strongest is the inequality for $i = d-1$, and we find the stated bound on β. □

The hypothesis of this proposition is satisfied for all families of graphs with classical parameters considered in this note, except for that of the Hermitian forms graphs, which have $b < 0$.

In many cases the sign pattern is forced.

Proposition 4.4. *If the i-th row and the i-th column of P have exactly i sign changes, and $P_{ij} > 0$ if $i + j \le d$, then P_{ij} has sign $(-1)^{i+j-d}$ if $i + j \ge d$.*

Proof. The only way to have i sign changes in P_{ij}, $d - i \le j \le d$ is to have P_{ij} and $P_{i,j+1}$ of opposite sign for all j, $d - i \le j \le d - 1$. □

4.3 Large β

In the theorem below we show for graphs with classical parameters (d, b, α, β) that if (d, b, α) is fixed and β is large, then $P_{d-j+1,j}$ is the smallest eigenvalue of the distance-j graph, and $|P_{1j}|$ is its second largest eigenvalue in absolute value. We also determine the sign pattern of the matrix P. This generalizes Propositions 2.6 and 3.12 above.

There are families of graphs with classical parameters with $b < 1$, such as the Hermitian forms graphs and the triality graphs. However, Metsch [24] showed that β is bounded as a function of (d, b, α) unless the graph is a Hamming, Johnson, Grassmann, or bilinear forms graph. It follows that $b \ge 1$ when β is unbounded.

Theorem 4.5. *For fixed (d, b, α), let β be sufficiently large. Then*
(i) $P_{ij} > 0$ for $i + j \le d$, and P_{ij} has sign $(-1)^{i+j-d}$ for $i + j \ge d$.
(ii) $P_{d-j+1,j} = \min\{P_{ij} \mid 0 \le i \le d\}$ for $j > 0$.
(iii) If $b \ge 1$, then $|P_{i+1,j}| < |P_{ij}|$ for $0 \le i \le d-1$.

Proof. For $|\beta| \to \infty$, we have $a_i \sim \left[{i \atop 1}\right]\beta$, hence $\beta > 0$ and $b + 1 \ge 0$ since $a_i \ge 0$ for $i = 1, 2$. By [2] (6.2.1), b is an integer different from $0, -1$, so $b \ge 1$.

(i) In order to prove this, one only has to prove the first part, then the second part follows by Propositions 4.3 and 4.4.

From the recurrence $P_{i,j+1} = ((\theta_i - a_j)P_{ij} - b_{j-1}P_{i,j-1})/c_{j+1}$ and $b_i \sim (\left[{d \atop 1}\right] - \left[{i \atop 1}\right])\beta$, and $c_i = O(1)$, and $a_i \sim \left[{i \atop 1}\right]\beta$, and $\theta_i \sim \left[{d-i \atop 1}\right]\beta$, it follows by induction that $P_{ij} \sim C_{ij}\beta^j$ for $i + j \leq d$ and some positive constants C_{ij}.

(ii) Now we know that $P_{d-j+1,j} < 0$ for large β. By downward induction on j one sees that all P_{ij} with $j \geq d - i$ have the same degree m_i in β. (Indeed, let P_{id} have degree $m = m_i$ in β. Then $c_{d+1-h}P_{i,d+1-h} = (\theta_i - a_{d-h})P_{i,d-h} - b_{d-h-1}P_{i,d-h-1}$ applied for $h = 0, 1, ..., i - 1$ shows that $P_{i,d-h-1}$ has degree m in β since the LHS has degree (at most) m, the middle term precisely $m+1$ and the final term must cancel that highest term.) Since $P_{i,d-i}$ has degree $d-i$ this proves that $m_i = d - i$. It follows that $P_{ij} \sim D_{ij}\beta^{d-i}$ for $i + j \geq d$ and some nonzero constants D_{ij}. Thus, $P_{d-j+1,j}$ is the most negative in its column when β is large enough.

(iii) In the interval $d - j \leq i \leq d$ the P_{ij} have decreasing degrees $d - i$ in β and hence decrease in absolute value when β is sufficiently large. For the interval $0 \leq i \leq d - j$ the degree is always j, and we have to work a bit more.

Put (just here) $c_{d+1} = 1$. Define polynomials $F_j(x)$ for $-1 \leq j \leq d + 1$ by $F_{-1}(x) = 0$, $F_0(x) = 1$, $c_{j+1}F_{j+1}(x) = (x - a_j)F_j(x) - F_{j-1}(x)b_{j-1}$. Then each F_j has degree j in x (for $j \geq 0$), and $P_{ij} = F_j(\theta_i)$ $(0 \leq i, j \leq d)$. Finally, $F_{d+1}(\theta_i) = 0$ $(0 \leq i \leq d)$. The c_j are independent of β, but a_j and b_j and θ_i depend linearly on β. Consider the coefficient of β in $\theta_i = \left[{d-i \atop 1}\right](\beta - \alpha\left[{i \atop 1}\right]) - \left[{i \atop 1}\right]$ a linear expression in the variable $w = b^{-i}$ (if $b \neq 1$) or i (if $b = 1$). Then the coefficient of β^j in $F_j(\theta_i)$ is a degree j polynomial $g_j(w) = \prod_{h=0}^{j-1}(\left[{d-i \atop 1}\right] - \left[{h \atop 1}\right])$ that vanishes for $d - j + 1 \leq i \leq d$ and hence nowhere else. That means that $P_{ij} = F_j(\theta_i) \sim g_j(b^{-i})\beta^j$ (or $g_j(i)\beta^j$) is monotone in i, assuming $b \geq 1$. Since $\left[{d-i \atop 1}\right]$ decreases with increasing i, also P_{ij} does (for $0 \leq i \leq d - j$). $\square$

5 Grassmann graphs

The Grassmann graphs $G_q(n, d)$ are the graphs with as vertices the d-subspaces of an n-dimensional vector space over $\mathbb{F}_q$, adjacent when they meet in codimension 1. W.l.o.g. we assume $n \geq 2d$ (since $G_q(n, d)$ is isomorphic to $G_q(n, n-d)$), and then these graphs are distance-regular of diameter d. Let $G_q(n, d, j)$ be the distance-j graph of $G_q(n, d)$, where $0 \leq j \leq d$. The eigenvalues of $G_q(n, d, j)$ are $P_{ij} = G_j(i)$ $(0 \leq i \leq d)$, where

$$\begin{aligned} G_j(i) &= \sum_{h=0}^{j} (-1)^{j-h}\, q^{hi+\binom{j-h}{2}} \begin{bmatrix} d-i \\ h \end{bmatrix} \begin{bmatrix} d-h \\ j-h \end{bmatrix} \begin{bmatrix} n-d-i+h \\ h \end{bmatrix} \\ &= \sum_{h=0}^{i} (-1)^{i-h}\, q^{j(j-i+h)+\binom{i-h}{2}} \begin{bmatrix} i \\ h \end{bmatrix} \begin{bmatrix} d-h \\ j \end{bmatrix} \begin{bmatrix} n-d-i+h \\ n-d-j \end{bmatrix} \end{aligned}$$

(see Delsarte [9], Theorem 10, and Eisfeld [14], Theorem 2.7).

5.1 Identities

Proposition 5.1. $G_d(i) = (-1)^i q^{d(d-i)+\binom{i}{2}} \left[{n-d-i \atop d-i}\right]$. $\square$

Let us write $G_j^{n,d}(i)$ instead of $G_j(i)$ when it is necessary to make the dependence on n and d explicit. The analog of Proposition 3.2 is as follows.

Proposition 5.2. *Let $i, j \geq 1$. Then*

$$G_j^{n+2,d+1}(i) = q^j G_j^{n,d}(i-1) - q^{j-1} G_{j-1}^{n,d}(i-1).$$

Proof. Use the first formula for $G_j^{n,d}(i)$, and $\begin{bmatrix} n+1 \\ m \end{bmatrix} = q^m \begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ m-1 \end{bmatrix}$. $\square$

5.2 The smallest eigenvalue

In Theorem 5.8 we find the smallest among the eigenvalues of $G_q(n,d,j)$ (for $(n,q) \neq (2d,2)$). In Proposition 5.4 (ii) we determine the second largest in absolute value (in all cases).

The following lemma provides tools to estimate Gaussian coefficients, and their quotients.

Lemma 5.3.
(i) If $n \leq m$, $b > 1$, then $(b^n - 1)/(b^m - 1) \leq b^{n-m}$.
(ii) If $m \geq 1$, $b > 1$, then $(b^n - 1)/(b^m - 1) < b^{n-m+1}/(b-1)$.
(iii) If $b > 1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_b \geq b^{k(n-k)}$.
(iv) ([17, Lemma 37]) *If $0 < k < n$, $b > 1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_b \geq (1 + \frac{1}{b}) b^{k(n-k)}$.*
(v) ([17, Lemma 34]) *If $0 \leq k \leq n$, $b \geq 4$, then $\begin{bmatrix} n \\ k \end{bmatrix}_b < (1 + \frac{2}{b}) b^{k(n-k)}$.* $\square$

Proposition 5.4.
(i) $G_j(1) < 0$ if and only if $j = d$. $G_j(1)$ is never zero.
(ii) Let $i \geq 1$. Then $|G_j(i)| \leq |G_j(1)|$.
(iii) Let $j \geq 1$, $i + j \leq d$. Then $0 < G_{j-1}(i) < G_j(i)$ if not $q = 2$, $n = 2d$, $i + j = d$.
(iv) Let $(n,q) \neq (2d,2)$. Then $G_j(i)$ has sign $(-1)^{\max(0,i+j-d)}$.
(v) Among the $G_d(i)$ with $i \geq 0$, the smallest is $G_d(1)$.

Proof.
(i) This is immediate from the second expression for $G_j(i)$.
(ii) Using $G_j(0) = q^{j^2} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} e \\ j \end{bmatrix}$ and $G_j(1) = q^{j^2} \begin{bmatrix} d-1 \\ j \end{bmatrix} \begin{bmatrix} e \\ j \end{bmatrix} - q^{j(j-1)} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} e-1 \\ j-1 \end{bmatrix}$, where $e = n - d$, we see that $G_{j-1}(0) + |G_{j-1}(1)| + |qG_j(1)| \leq qG_j(0)$.
Now apply induction on d and i: $|G^{n+2,d+1}(i)| \leq |G_j^{n+2,d+1}(1)|$ follows from $q|G_j^{n,d}(i-1)| + |G_{j-1}^{n,d}(i-1)| \leq q|G_j^{n,d}(1)| + |G_{j-1}^{n,d}(1)| \leq qG_j^{n,d}(0) - G_{j-1}^{n,d}(0)$.
(iii) Induction on d. Positiveness follows from monotony since $G_0(i) = 1$. For $i = 0$ we have to show that $q^{j^2} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} e \\ j \end{bmatrix}$ increases with j, and it does, with the indicated exception. Now for $i > 0$, using $j + 1 \leq d$ and $q \geq 2$:
$G_{j+1}^{n+2,d+1}(i) - G_j^{n+2,d+1}(i) = q^{j+1} G_{j+1}^{n,d}(i-1) - 2q^j G_j^{n,d}(i-1) + q^{j-1} G_{j-1}^{n,d}(i-1) > 0$.
(iv) This follows by part (iii) and Propositions 4.3, 4.4.
(v) This follows by parts (i) and (ii). $\square$

Conjecture 5.5.
(i) If $(n,q) \neq (2d,2)$, then $|G_j(i+1)| < |G_j(i)|$ when $0 \leq i \leq d-1$.
(ii) If $(n,q) = (2d,2)$, then $G_j(d-j)$ is negative for $(d,j) = (5,3)$ and when $d \geq 6$, $2 \leq j \leq d-2$, and $G_j(d-j)$ is the smallest among the $G_j(i)$ when $d \geq 6$, $3 \leq j \leq d-2$.

We can prove part (i) for $q \ge 5$, but omit the details.

We show that $G_j(i)$ is well-approximated by its main term T.

Lemma 5.6. *If $i+j \le d$, let $s := 1$ and $T := q^{j^2} \begin{bmatrix} d-i \\ j \end{bmatrix} \begin{bmatrix} n-d \\ n-d-j \end{bmatrix}$. If $i+j \ge d$, let $s := i+j-d$ and $T := (-1)^s q^{j(d-i)+\binom{s}{2}} \begin{bmatrix} i \\ d-j \end{bmatrix} \begin{bmatrix} n-i-j \\ n-d-j \end{bmatrix}$. If $q \ge 3$ or $q = 2$, $n > 2d$, then*

$$\left| \frac{G_j(i)}{T} - 1 \right| < \frac{q^{2d+1-n}}{(q-1)^2}.$$

Proof. Let T_h be the term with index h in the second expression for $G_j(i)$, so that $T = T_m$ with $m = \min(i, d-j)$, and $0 \le h \le m$. This expression is alternating, and

$$\begin{aligned} \left| \frac{T_{h-1}}{T_h} \right| &= \left| -q^{-h+i-j} \frac{q^h - 1}{q^{i-h+1} - 1} \frac{q^{d-h+1} - 1}{q^{d-h-j+1} - 1} \frac{q^{j-i+h} - 1}{q^{n-d-i+h} - 1} \right| \\ &< \frac{q^{d+h+j+1-n}}{(q-1)^2} \le \frac{q^{2d+1-n}}{(q-1)^2} \end{aligned}$$

if $h \ge 1$. (Here we used Lemma 5.3 (ii) twice, and (i) once, using that $h \le n-d-i+h$.) If $q \ge 3$ or $q = 2$, $n > 2d$, then the right-hand side is less than 1, and the sum is alternating with decreasing terms, so that the difference between the main term and the sum is not larger than the second term. The main term is $T = T_m$, the maximal index that occurs. $\square$

Remark. For $q = 2$, $i \ge d-j+1$ we shall need a slightly sharper bound. Now $i - h + 1 \ge 2$ and in the inequalities in the proof and conclusion of the lemma we can bound by $q^{2d+2-n}/((q-1)(q^2-1))$.

Above the main term of the second expression for $G_j(i)$ was T_i (if $i+j \le d$) or T_{d-j} (if $i+j \ge d$). If $q = 2$, $n = 2d$, $i+j \ge d \ge 6$, and $3 \le j \le d-2$, the main term is T_{d-j-1}.

Lemma 5.7. *Let $n = 2d$, $q = 2$, $d \ge 13$, $5 \le j \le d-5$ and $d-j \le i < d$. Set $s := i+j-d+1$. Let $T := (-1)^s q^{j(d-i-1)+\binom{s}{2}} \begin{bmatrix} i \\ d-j-1 \end{bmatrix} \begin{bmatrix} j+1 \\ 1 \end{bmatrix} \begin{bmatrix} 2d-i-j-1 \\ d-j \end{bmatrix}$. Then $|G_j(i)| \le \frac{3}{2}|T|$. For $i = d-j$, $G_j(i)$ is negative, and $|G_j(i)| \ge 5|T|/171$.*

Proof. Let T_h be the term with index h in the second expression for $G_j(i)$, so that $T = T_{d-j-1}$ and $0 \le h \le \min(i, d-j)$. As in the proof of Lemma 5.6, we have

$$\left| \frac{T_{h-1}}{T_h} \right| = \left| -2^{-h+i-j} \frac{2^h - 1}{2^{i-h+1} - 1} \frac{2^{d-h+1} - 1}{2^{d-h-j+1} - 1} \frac{2^{j-i+h} - 1}{2^{d-i+h} - 1} \right|.$$

For $h \le d-j-1$ ($\le i-1$), we find using Lemma 5.3 (i) with $h \le d-i+h$,

$$\left| \frac{T_{h-1}}{T_h} \right| \le 2^{2i-h-j-d} \cdot \frac{2^{j+2}}{3} \cdot \frac{2^{j-2i+2h+1}}{3} = \frac{2^{h+j+3-d}}{9} \le \frac{4}{9}.$$

For $h = d-j$ and $i+j > d$ we find, using $i \le d-1$ and $5 \le j \le d-5$,

$$\left| \frac{T}{T_{d-j}} \right| = 2^{i-d} \frac{(2^{d-j} - 1)(2^{j+1} - 1)(2^{d-i} - 1)}{(2^{i+j-d+1} - 1)(2^{2d-i-j} - 1)} \ge \frac{31}{63}.$$

For $h = d - j$ and $i + j = d$, we find, using $5 \le j \le d - 5$ and $d \ge 13$,

$$\left|\frac{T}{T_{d-j}}\right| = 2^{-j}\frac{(2^{d-j}-1)(2^{j+1}-1)(2^j-1)}{2^d-1} \ge \frac{31 \cdot 63 \cdot 255}{32 \cdot 8191} > \frac{19}{10}.$$

If $i+j = d$, then $G_j(i) = \sum_{h=0}^{d-j} T_h$ is an alternating sum with terms increasing in absolute value up to $T = T_{d-j-1}$, and then decreasing again, hence $|\frac{G_j(i)}{T} - 1| \le \frac{4}{9} + \frac{10}{19} = 1 - \frac{5}{171} < 1$, so that $G_j(d-j)$ has the sign of T, i.e., is negative. For general i, if $G_j(i)$ has the same sign as T, then $|G_j(i)| \le |T|$. If $G_j(i)$ has the opposite sign, then $|G_j(i)| \le |T_{d-j}| - |T| + |T_{d-j-2}| \le (\frac{63}{31} - 1 + \frac{4}{9})|T| < \frac{3}{2}|T|$. $\square$

Theorem 5.8. *Let $1 \le j \le d$.*

(i) If $q \ge 3$ or $q = 2$, $n \ge 2d+1$, then the smallest eigenvalue of $G_q(n,d,j)$ is $G_j(d-j+1)$.

(ii) If $(n,q) = (2d,2)$, and $7 \le j \le d-5$, then the smallest eigenvalue of $G_q(n,d,j)$ is $G_j(d-j)$.

Proof. (i) The case $j = d$ is handled in Proposition 5.4, so we may assume $j < d$. The smallest among the $G_j(i)$ is negative, and hence i is one of the values $d-j+1+2t$ where $t \ge 0$. First consider the case $q \ge 3$. We compare $G_j(i)$ with $G_j(i+2)$. By Lemma 5.6, both are approximated by their main term T with an error that is not larger than $\frac{3}{4}T$. Let T, T', T'' be the main terms for $G_j(i), G_j(i+1), G_j(i+2)$. Then $|G_j(i+2)|/|G_j(i)| \le (\frac{7}{4}|T''|)/(\frac{1}{4}|T|) = 7|T''|/|T|$. Now

$$\frac{|T'|}{|T|} = q^{i-d}\frac{q^{i+1}-1}{q^{i-d+j+1}}\frac{q^{d-i}-1}{q^{n-i-j}-1} < \frac{q^{d+i-n+1}}{q-1}$$

using Lemma 5.3 (i), (ii), since $d - i \le n - i - j$. It follows that $|T''|/|T| < (q^{2d+2i-2n+3})/(q-1)^2$. Since $i+2 \le d$ and $n \ge 2d$ we have $2d+2i-2n+3 \le -1$ and $|G_j(i+2)|/|G_j(i)| \le 7|T''|/|T| < 7/12 < 1$, as desired.

For $q = 2$, $n \ge 2d+1$ we use the remark following Lemma 5.6 and find $|G_j(i+2)| \le \frac{5}{3}|T''|$ and $|G_j(i)| \ge \frac{1}{3}|T|$, so that $|G_j(i+2)|/|G_j(i)| \le 5|T''|/|T| < 5/8 < 1$, as desired.

(ii) The cases with $d < 13$ can be checked by computer, so we may assume $d \ge 13$. The smallest among the $G_j(i)$ is negative, so has $i \ge d-j$ by Proposition 5.4 (iii). The value $G_j(d-j)$ is negative. We show that it has maximal absolute value among the $G_j(i)$ with $i \ge d-j$.

Let T and T' be the main terms of $G_j(d-j)$ and $G_j(i)$, where $i < d$. By Lemma 5.7, $|G_j(d-j)| \ge \frac{5}{171}|T|$ and $G_j(i) \le \frac{3}{2}|T'|$. Then $|G_j(i)/G_j(d-j)| \le \frac{3 \cdot 171}{2 \cdot 5}|T'|/|T|$. Now, as $d \ge 5$, for $i = d-j+1$ we have

$$\frac{|T'|}{|T|} = 2^{-j+1}\frac{\begin{bmatrix}d-j+1\\2\end{bmatrix}\begin{bmatrix}d-2\\j-2\end{bmatrix}}{\begin{bmatrix}d-j\\1\end{bmatrix}\begin{bmatrix}d-1\\j-1\end{bmatrix}} = 2^{-j+1}\frac{2^{d-j+1}-1}{3}\frac{2^{j-1}-1}{2^{d-1}-1} < \frac{2^{-j+2}}{3} \cdot \frac{16}{15}.$$

As $\frac{2^{-j+2}}{3} \cdot \frac{16}{15} < \frac{2 \cdot 5}{3 \cdot 171}$ for $j \ge 7$, $|G_j(d-j+1)| < |G_j(d-j)|$. Now, let $d-j+1 \le i \le d-2$. Let T' and T'' be the main terms of $G_j(i)$ and $G_j(i+1)$. Then

$$\frac{|T''|}{|T'|} = 2^{i-d+1}\frac{2^{i+1}-1}{2^{i+j-d+2}-1}\frac{2^{d-i-1}-1}{2^{2d-i-j-1}-1} \le \frac{4 \cdot 64}{3 \cdot 63}2^{i-d} < 1.$$

Hence, $|G_j(i)| < |G_j(d-j)|$ for $d-j+2 \le i \le d-1$. Lemma 5.7 excludes $i = d$, so we need to treat that case separately. By Proposition 4.1, $G_j(d) = (-1)^j \begin{bmatrix} d \\ j \end{bmatrix} q^{\binom{j}{2}}$, and hence

$$\frac{|G_j(d)|}{|T|} = \frac{2^{\binom{j}{2}} \begin{bmatrix} d \\ j \end{bmatrix}}{2^{j(j-1)} \begin{bmatrix} d-j \\ 1 \end{bmatrix} \begin{bmatrix} j+1 \\ 1 \end{bmatrix} \begin{bmatrix} d-1 \\ j-1 \end{bmatrix}} \le \frac{2^{-\binom{j}{2}}}{(2^{j+1}-1)(2^j-1)} < 2^{-j}.$$

Hence, $|G_j(d)| < |G_j(d-j)|$. $\square$

6 Dual polar graphs

Let q be a prime power. There are six types of finite classical polar spaces, $C_d(q)$, $B_d(q)$, $D_d(q)$, ${}^2D_{d+1}(q)$, ${}^2A_{2d}(q)$, and ${}^2A_{2d-1}(q)$ with associated parameter (in the same order) $e = 1, 1, 0, 2, 1/2, 3/2$ (see [2, §9.4]). In the cases ${}^2A_{2d}(q)$ and ${}^2A_{2d-1}(q)$ the parameter q is the square of a prime power. The dual polar graphs $C_q(d, e)$ are the graphs with as vertices the maximal subspaces of a polar space of rank d with parameter e over $\mathbb{F}_q$, adjacent when they meet in codimension 1. These graphs are distance-regular of diameter d. The eigenmatrix P has entries $P_{ij} = C_j(i)$, where

$$C_j(i) = \sum_{h=\max(i-j,0)}^{\min(d-j,i)} (-1)^{i-h}\, q^{\binom{i-h}{2}+\binom{j-i+h}{2}+(j-i+h)e} \begin{bmatrix} d-i \\ d-j-h \end{bmatrix} \begin{bmatrix} i \\ h \end{bmatrix}.$$

This formula was taken from Vanhove [32, Theorem 4.3.6]. An expression in terms of q-Krawtchouk polynomials was given in Stanton [28, Thm. 5.4].

6.1 Identities

Let us write $C_j^d(i)$ instead of $C_j(i)$ when it is necessary to make the dependence on d explicit.

Proposition 6.1. *(i) If $0 \le i \le d$, then $C_j^{d+1}(i) = q^{d+e-i} C_{j-1}^d(i) + C_j^d(i)$.*
(ii) If $1 \le i \le d+1$, then $C_j^{d+1}(i) = -q^{i-1} C_{j-1}^d(i-1) + C_j^d(i-1)$. $\square$

Since these two values are equal, one also has $C_j(i-1) = C_j(i) + q^{i-1}C_{j-1}(i-1) + q^{d+e-i}C_{j-1}(i)$.

We have $C_1(i) = q^e \begin{bmatrix} d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $C_d(i) = (-1)^i q^{\binom{d}{2}+(d-i)(e-i)}$, and see that for $j = 1$ and for $j = d$ the sequence $|C_j(i)|$ $(0 \le i \le d)$ is unimodal, with smallest element $|C_j(i)|$ for $i = \lfloor (d+e+1)/2 \rfloor$, largest element $C_j(0)$ and second largest element $|C_j(d)|$ if $e \le 1$, and $|C_j(1)|$ if $e > 1$. This is what we try to prove for all j.

There are small exceptions. E.g. for $(q, d, e) = (2, 5, 1)$ the $j = 4$ column of P is not unimodal, and the $j = 2$ column has its minimum earlier:

$$P = \begin{pmatrix} 1 & 62 & 1240 & 9920 & 31744 & 32768 \\ 1 & 29 & 250 & 680 & 64 & -1024 \\ 1 & 11 & 16 & -76 & -80 & 128 \\ 1 & -1 & -20 & 20 & 64 & -64 \\ 1 & -13 & 40 & 20 & -176 & 128 \\ 1 & -31 & 310 & -1240 & 1984 & -1024 \end{pmatrix}.$$

More generally, if $(q,e) = (2,1)$, then $|C_{d-1}(2)| > |C_{d-1}(1)| = q^{\binom{d-1}{2}}$ for all $d \geq 2$, and the sequence $|C_{d-1}(i)|$ is not unimodal for $(q,e) = (2,1)$, $d \geq 5$.

For $e = 1$ we have the coincidence $|C_d(1)| = |C_d(d)|$. More generally, $|C_d(i)| = |C_d(d+e-i)|$ for integral e and $e \leq i \leq d$.

For $e = 0$ the graphs $C_q(d,e)$ are bipartite, and we have $C_j(d-i) = (-1)^j C_j(i)$.

6.2 The smallest eigenvalue

The following conjecture is a variation of Lemma 47 in [17] where the authors investigated the sum of the relations $\{d-j, d-j+1, \ldots, d\}$ instead of just the jth relation.

Conjecture 6.2. *The sequence $|C_j(i)|$ (j fixed, $0 \leq i \leq d$) is unimodal if not $(q,e) = (2,1)$ and not $(q,e,j) = (2,2,d-4)$, $8 \leq d \leq 12$. If it is unimodal with minimum at i_0, and $i_1 = \lfloor (d+e+1)/2 \rfloor$, then $i_0 = i_1$ for $e = 0, \frac{1}{2}, \frac{3}{2}$, and $|i_0 - i_1| \leq 1$ for $e = 1, 2$, except that $i_0 = i_1 - 2$ for $(q,e,j,d) = (2,1,3,4), (2,2,3,7)$.*

Conjecture 6.3. *The index $i_{\min}$ of the smallest among the $C_j(i)$ (j fixed, $0 \leq i \leq d$) is*

$$i_{\min} = \begin{cases} 1 & \text{if } j = d \text{ and } (j \text{ is even or } e \geq 1) \\ d & \text{if } j \text{ is odd and } (j < d \text{ or } e \leq 1) \\ \lfloor (d-j+2)/2 \rfloor & \text{if } j \text{ is even, } e = 0 \\ (d-j+2)/2 & \text{if } j \text{ and } d \text{ are even, } e = \frac{1}{2} \text{ or } e = 1 \\ (d+j-1)/2 & \text{if } j \text{ is even, } d \text{ is odd, } e = \frac{1}{2} \text{ or } e = 1 \\ (d+j)/2 & \text{if } j \text{ and } d \text{ are even, } e = \frac{3}{2} \text{ or } e = 2 \\ (d-j+3)/2 & \text{if } j \text{ is even, } d \text{ is odd, } e = \frac{3}{2} \text{ or } e = 2 \end{cases}$$

except that when $q = 2$ and $e = 2$ and d is even and $j \geq d-4$ one finds $i_{\min} = 2$ for $j = d-2$, $d \geq 6$ and $i_{\min} = 3$ for $j = d-4$, $d \geq 14$.

We show the second case of this conjecture in Corollary 6.5. We can show the conjecture for some more cases if $q \geq 11$, but omit the details.

Proposition 6.4. *Let $1 \leq j \leq d$.*
(i) $C_j(1) < 0$ if and only if $j = d$ or $(j,e) = (d-1,0)$.
(ii) Let $d \geq 3$. Then $|C_j(2)| \leq |C_j(1)|$ unless $(q,j,e) = (2,d-1,1)$.
(iii) Let $1 \leq i \leq d$. Then $|C_j(i)| \leq |C_j(d)|$ if $i \geq 2$ or $e \leq 1$.
(iv) $|C_j(1)| \leq |C_j(d)|$ if $e \leq 1$ with equality only if $(j,e) = (d,1)$.

Proof. (i) This is immediate from $C_j(1) = q^{\binom{j}{2}+je}\begin{bmatrix} d-1 \\ j \end{bmatrix} - q^{\binom{j-1}{2}+(j-1)e}\begin{bmatrix} d-1 \\ j-1 \end{bmatrix}$.

(ii) We can assume $1 < j < d$ as we did already show the claim for $j = 1$ and $j = d$. Rename d to $d+1$, so that $d \geq 2$ and $j \leq d$. We have $C_j^{d+1}(1) = q^{\binom{j}{2}+je}\begin{bmatrix} d \\ j \end{bmatrix} - q^{\binom{j-1}{2}+(j-1)e}\begin{bmatrix} d \\ j-1 \end{bmatrix}$, and $C_j^{d+1}(2) = C_j^d(1) - qC_{j-1}^d(1)$ by Proposition 6.1 (ii). Dividing the expression $|C_j^{d+1}(2)| \leq |C_j^{d+1}(1)|$ by $q^{\binom{j-1}{2}+(j-1)e}\begin{bmatrix} d-1 \\ j-1 \end{bmatrix}$ and simplifying yields the claim.

(iii) Note that $C_j(d) = (-1)^j q^{\binom{j}{2}}\begin{bmatrix} d \\ d-j \end{bmatrix}$ has alternating sign. Use induction on d. By Proposition 6.1 (i) and (ii),

$$\begin{aligned} |C_j^{d+1}(i)| &= |q^{d+e-i}C_{j-1}^d(i) + C_j^d(i)| \\ &\leq |q^d C_{j-1}^d(d)| + |C_j^d(d)| = |C_j^{d+1}(d+1)|. \end{aligned}$$

(iv) This is immediate from the expressions for $C_j(1)$ and $C_j(d)$. $\square$

Corollary 6.5. *Let $d \geq 3$ and $1 \leq j \leq d$. Then*
(i) $|C_j(1)| = \max\{|C_j(i)| : 1 \leq i \leq d)$ if $e > 1$ or $(j,e) = (d,1)$.
(ii) $|C_j(d)| = \max\{|C_j(i)| : 1 \leq i \leq d)$ if $e \leq 1$.
(iii) If $j < d$ is odd, then $C_j(d) = \min\{C_j(i) : 0 \leq i \leq d\}$.

Proof. We only have to show (iii). Here we only have to show that $C_j(d)$ is negative. This follows from Proposition 4.1. $\square$

7 Bilinear forms graphs

The bilinear forms graphs $H_q(d,e)$ are the graphs with as vertices $d \times e$ matrices over $\mathbb{F}_q$, adjacent when the difference has rank 1. W.l.o.g. we assume $d \leq e$. The eigenmatrix P has entries $P_{ij} = B_j(i)$, where

$$B_j(i) = \sum_{h=0}^{j} (-1)^{j-h} \, q^{eh+\binom{j-h}{2}} \begin{bmatrix} d-h \\ d-j \end{bmatrix} \begin{bmatrix} d-i \\ h \end{bmatrix}$$

(Delsarte [10], Theorem A2).

The valencies here are $k_j = B_j(0) = \left[{d \atop j}\right] \left[{e \atop j}\right] \prod_{h=1}^{j} (q^j - q^{j-h})$ ([2], p. 281).

The eigenvalues of $H_q(d,e)$ are $\theta_i = (q^{d+e-i} - q^d - q^e + 1)/(q-1)$.

The scheme is self-dual, so that $P_{ij}/P_{0j} = P_{ji}/P_{0i}$, and P_{ij} and P_{ji} have the same sign.

7.1 Identities

Let us write $B_j^{d,e}(i)$ instead of $B_j(i)$ when it is necessary to make the dependence on d and e explicit.

Proposition 7.1. (Delsarte [10, Proof of Theorem A2])

$$B_j^{d,e}(i) - B_j^{d,e}(i+1) = q^{d+e-i-1} B_{j-1}^{d-1,e-1}(i).$$

Proposition 7.2. (Stanton, [29, Prop. 1(ii),(iii)])
(i) $(q^{d-j+1} - 1) B_j^{d+1,e}(i) = (q^{d+1} - q^i) B_j^{d,e}(i) + (q^i - 1) B_j^{d,e}(i-1)$.
(ii) $(q^{e-j+1} - 1) B_j^{d,e+1}(i) = (q^{e+1} - q^i) B_j^{d,e}(i) + (q^i - 1) B_j^{d,e}(i-1)$.

7.2 Negative $B_j(1)$

For the bilinear forms graphs the $i = 1$ row of P has only a single negative value.

Proposition 7.3.
(i) $B_j(1) < 0$ if and only if $j = d$, and otherwise $B_j(1) > 0$.
(ii) $B_d(1)$ is the smallest eigenvalue of the distance-d graph, and the second largest in absolute value.

Proof. (i) This follows from $B_1(i) = (q^{d+e-i} - q^d - q^e + 1)/(q-1)$ and $e \geq d$ and the fact that $B_1(i)$ and $B_i(1)$ have the same sign.

(ii) Proposition 4.1 gives $B_j(d) = (-1)^j \left[{d \atop j}\right] q^{\binom{j}{2}}$, and it follows that $B_d(i) = (k_d/k_i)(-1)^i \left[{d \atop i}\right] q^{\binom{i}{2}}$. The claim follows using $k_i = \left[{d \atop i}\right] \left[{e \atop i}\right] q^{\binom{i}{2}} \prod_{h=1}^{i} (q^h - 1)$. $\square$

Lemma 7.4. *Let $1 \le j \le d-1$ and either $j \le d-2$ or $q > 2$ or $q = 2, e > d$. Then $|B_j(2)| \le |B_j(1)|$. If $j = d-1$ and $q = 2$ and $e = d$, then $|B_j(2)|/|B_j(1)| = (2^{d-1}+1)/(2^{d-1}-1)$.*

Proof. Find $B_j(1)$ and $B_j(2)$ from $B_1(i)$ and $B_2(i)$ and the relation $P_{ij}/k_j = P_{ji}/k_i$. Abbreviate $q^n - 1$ with $[n]$. One gets

$$\frac{B_j(2)}{B_j(1)} = \frac{q[d][d-1] - (q+1)q^e[d-1][d-j] + q^{2e}[d-j][d-j-1]}{q[d-1][e-1](q^{d+e-j} - q^d - q^e + 1)}.$$

The numerator is of the form $A - B + C$ where $B \ge A \ge 0$ and $C \ge 0$. If $j \le d-3$, or $j = d-2$, $q > 2$, or $j = d-2$, $q = 2$, $e > d$, then $C \ge B$. Now estimate the numerator with C and find that $|\frac{B_j(2)}{B_j(1)}| \le 1$. The same conclusion follows by direct computation in the case $j = d-2$, $q = 2$, $e = d$. This leaves the case $j = d-1$ (with $C = 0$). Again treat the cases $q > 2$ and $q = 2$, $e > d$ separately and find the same conclusion. □

As the scheme is self-dual, so that $P_{ij}/k_j = P_{ji}/k_i$, the recurrence $c_{j+1}P_{i,j+1} = (\theta_i - a_j)P_{ij} - b_{j-1}P_{i,j-1}$ implies $b_iP_{i+1,j} = (\theta_j - a_i)P_{ij} - c_iP_{i-1,j}$. In our case this gives (after multiplication by $q-1$)

$$\begin{aligned} &q^{2i}[d-i][e-i]B_j(i+1) \\ &= \left(q^e[d-j] - [d] - [i](q^e + q^d - q^i - q^{i-1} - 1)\right) B_j(i) - q^{i-1}[i]B_j(i-1), \end{aligned}$$

again with the abbreviation $[n] = q^n - 1$.

Theorem 7.5. *For $q \ge 4$, $|B_j(1)| \ge |B_j(i)|$ for $1 \le i \le d$, $0 \le j \le d$.*

Proof. For $j = 0$ the claim is trivial, so we assume $j \ge 1$. By Propostion 7.3 we can assume $j < d$. Now $|B_j(i)| \le |B_j(1)|$ follows by induction on i, starting with Lemma 7.4 for $i = 2$, and using the recurrence for $i > 2$. We have to show that $\max(|q^e[d-j] - [d]|,\ [i](q^e + q^d - q^i - q^{i-1} - 1)) + q^{i-1}[i] \le q^{2i}[d-i][e-i]$, and that is easily checked, assuming $q \ge 4$. □

Conjecture 7.6. *For $q \ge 3$, or $q = 2$ and $d \ne e$, $B_j(d-j+1)$ is the smallest eigenvalue in the distance-j graph for $1 \le j \le d$.*

Let $b_{i,j}(h)$ be the exponent of q in the h-th term of the expression for $B_j(i)$ if we approximate $\begin{bmatrix} n \\ k \end{bmatrix}$ with $q^{k(n-k)}$. That is, let

$$b_{i,j}(h) = h(d+e-i-h) + (d-j)(j-h) + \binom{j-h}{2}.$$

Let $h_0 = e - i + \frac{1}{2}$. Then the quadratic expression $b_{i,j}(h)$ is maximal for $h = h_0$, and $b_{i,j}(h_0 + x) = b_{i,j}(h_0) - \frac{1}{2}x^2$. Let $h_{\max} = \min(j, d-i)$. The terms occurring in the sum have indices h with $h \le h_{\max} < h_0$, so the term with largest index has largest exponent.

Lemma 7.7. *Let $q \ge 4$ and put $s := b_{i,j}(h_{\max})$. We have*

$$\frac{5}{9}q^s < |B_j(i)| < \frac{13}{4}q^s.$$

Proof. The expression for $B_j(i)$ is an alternating series with terms decreasing in absolute value after the first, so we can estimate $B_j(i)$ by the main term with an error not larger than the second term. $\square$

Proposition 7.8. *Let $q \geq 4$. The sign of $B_j(i)$ is $(-1)^{\max(0,i+j-d)}$. The smallest among the $B_j(i)$ for fixed j is $B_j(d-j+1)$.*

Proof. The sign of $B_j(i)$ is that of the main term. The negative terms are $B_j(d-j+1+2t)$. Increasing i by 2 (from $d-j+1+2t$ to $d-j+3+2t$) means decreasing s by at least 2, and since $\frac{5}{9}q^2 > \frac{13}{4}$ that decreases the absolute value. So $B_j(d-j+1)$ is most negative. $\square$

8 Alternating forms graphs

The alternating forms graphs $A_q(n)$ are the graphs with as vertices the skew symmetric $n \times n$ matrices over $\mathbb{F}_q$ with zero diagonal, adjacent when the difference has rank 2.

Let $d = \lfloor n/2 \rfloor$. The graph $A_q(n)$ is distance-regular with diameter d. The eigenmatrix P has entries $P_{ij} = A_j(i)$, where

$$A_j(i) = \sum_{h=0}^{j} (-1)^{j-h} \, q^{(j-h)(j-h-1)} \, q^{hm} \begin{bmatrix} d-h \\ d-j \end{bmatrix}_b \begin{bmatrix} d-i \\ h \end{bmatrix}_b .$$

Here the Gaussian coefficients have base $b = q^2$ and $m = n(n-1)/(2d) = 2n-2d-1$ so that $\{m, 2d\} = \{n-1, n\}$ and m is odd (Delsarte [11, (15)]).

The valencies here are $k_j = A_j(0) = q^{j(j-1)} \prod_{i=0}^{2j-1}(q^{n-i}-1)/\prod_{i=1}^{j}(q^{2i}-1)$.

The eigenvalues of $A_q(n)$ are $\theta_i = (q^{2n-2i-1} - q^n - q^{n-1} + 1)/(q^2-1)$.

The scheme is self-dual, so that $P_{ij}/P_{0j} = P_{ji}/P_{0i}$, and P_{ij} and P_{ji} have the same sign.

8.1 Identities

Let us write $A_j^n(i)$ instead of $A_j(i)$ when it is necessary to make the dependence on n explicit.

Proposition 8.1. (Delsarte [11, (66)]) $A_j^n(i) = A_j^n(i-1) - q^{2n-2i-1} A_{j-1}^{n-2}(i-1)$.

Proposition 8.2. *$A_d(i) = -(q^{m-2i}-1)A_d(i+1)$ for $0 \leq i \leq d-1$.* $\square$

8.2 The smallest and the second largest eigenvalue

We determine the smallest eigenvalue, and the second largest in absolute value, for the distance-j graphs of $A_q(n)$.

Theorem 8.3. *Let $1 \leq j \leq d$.*
(i) $\min_{0 \leq i \leq d} A_j(i) = A_j(d-j+1)$.
(ii) $\max_{1 \leq i \leq d} |A_j(i)| = |A_j(1)|$.
(iii) Let $0 \leq i \leq d-1$, $1 \leq j \leq d$. Then:
a) $|A_j(i)| < |A_j(i+1)|$ if and only if $(q, n, i) = (2, 2d, d-1)$ and $1 \leq j \leq d-1$.
b) $|A_j(i)| = |A_j(i+1)|$ if and only if $(q, n, i) = (2, 2d, d-1)$ and $j = d$.
c) In all other cases $|A_j(i)| > |A_j(i+1)|$.

The proof of this theorem is given below.

For $(q,n) = (2,4)$ we have

$$P = \begin{pmatrix} 1 & 35 & 28 \\ 1 & 3 & -4 \\ 1 & -5 & 4 \end{pmatrix}.$$

Let $a_{i,j}(h)$ be the exponent of q in the h-th term of the expression for $A_j(i)$ if we approximate $\left[{n \atop k}\right]_b$ with $q^{2k(n-k)}$. Then

$$\begin{aligned} a_{i,j}(h) &= (j-h)(j-h-1) + hm + 2(d-j)(j-h) + 2h(d-i-h) \\ &= -h^2 + h(m+1-2i) + j(2d-j-1). \end{aligned}$$

This quadratic function of h is maximal for $h_0 = \frac{m+1}{2} - i$. The nonzero terms in the expression for $A_j(i)$ have indices h with $0 \le h \le \min(d-i, j)$. Since $h_0 = d-i$ if n is even, and $h_0 = d-i+1$ if n is odd, the term with the largest exponent is the one with index $\min(d-i, j)$.

Proposition 8.4. *If $i + j \le d$, then*

$$0 \le 1 - \frac{A_j(i)}{q^{jm}\left[{d-i \atop j}\right]_b} < \frac{2}{q^{m+2-2i-2j}}.$$

In particular, $A_j(i) > 0$.

Proof. Use that $a_{i,j}(h_0-x) = a_{i,j}(h_0) - x^2$. If $i+j \le d$, then $\min(d-i,j) = j$. The sum is alternating, and since $b = q^2 \ge 4$ and $(1+2q^{-2})^2 < q^3$ it follows from Lemma 5.3 (iii,v) that terms after the first (reading down from largest h) decrease in size, and the difference between $A_j(i)$ and the first term is not larger than the second term. (That is, $A_j(i) = T_0 - T_1 + T_2 - \cdots$ where all T_ℓ have the same sign, and $|T_1| \ge |T_2| \ge \cdots$. Our conclusion will be $A_j(i) = T_0 - \gamma T_1$ with $0 \le \gamma \le 1$, that is, $1 - \frac{T_1}{T_0} \le \frac{A_j(i)}{T_0} \le 1$.) Estimate the absolute value of second term divided by the first, using Lemma 5.3 (ii), by

$$\frac{\left[{d-j+1 \atop 1}\right]\left[{d-i \atop j-1}\right]}{q^m \left[{d-i \atop j}\right]} = q^{-m} \frac{b^{d-j+1}-1}{b-1} \frac{b^j - 1}{b^{d-i-j+1}-1} < q^{-m+2i+2j-2} \frac{q^4}{(q^2-1)^2}.$$

If n is odd, $m+2-2i-2j = 2d+3-2i-2j \ge 3$. If n is even, $m+2-2i-2j = 2d+1-2i-2j \ge 1$. In both cases, the RHS of the inequality is less than 1. $\square$

Proposition 8.5. *If $s := i+j-d \ge 0$, then*

$$0 \le 1 - \frac{A_j(i)}{(-1)^s q^{s(s-1)+(d-i)m}\left[{i \atop d-j}\right]} \le \frac{\left[{i+1 \atop d-j}\right]\left[{d-i \atop 1}\right]}{q^{m-2s}\left[{i \atop d-j}\right]} < \frac{q^3}{(q^2-1)^2} \frac{1}{q^{2n-4d}} < 1.$$

In particular, $A_j(i)$ has sign $(-1)^s$.

Proof. If $i + j \ge d$, then $\min(d-i, j) = d-i$. Again the difference between $A_j(i)$ and the first term is not larger than the second term. Estimate the absolute value of second term divided by the first, using Lemma 5.3 (ii), by

$$\frac{\left[{i+1 \atop d-j}\right]\left[{d-i \atop 1}\right]}{q^{m-2s}\left[{i \atop d-j}\right]} = \frac{(b^{d-i}-1)(b^{i+1}-1)}{q^{m-2s}(b-1)(b^{s+1}-1)} < \frac{q^{-m+2d+2}}{(q^2-1)^2}.$$

Finally, $m - 2d = 2n - 4d - 1$. $\square$

Proof of Theorem 8.3. First of all, by Proposition 8.2 all statements are true for $j = d$, so we may suppose $1 \le j \le d - 1$.

Next, prove part (iiic). We have $A_j^n(i+1) = A_j^n(i) - q^{2n-2i-3} A_{j-1}^{n-2}(i)$.

If $i + j + 1 \le d$, then each of $A_j^n(i+1)$, $A_j^n(i)$, $A_{j-1}^{n-2}(i)$ is positive, and $|A_j^n(i+1)| < |A_j^n(i)|$ follows from $0 < A_j^n(i+1) < A_j^n(i)$.

If $i + j \ge d$, use the (strong form of the) second proposition to find

$$\left|\frac{A_j(i+1)}{A_j(i)}\right| < \frac{q^{2s-m}\left[{i+1 \atop d-j}\right]}{\left(1 - \dfrac{\left[{i+1 \atop d-j}\right]\left[{d-i \atop 1}\right]}{q^{m-2s}\left[{i \atop d-j}\right]}\right)\left[{i \atop d-j}\right]} = \frac{1}{q^{m-2s}\dfrac{b^{s+1}-1}{b^{i+1}-1} - \dfrac{b^{d-i}-1}{b-1}}$$
$$= \frac{(b-1)(b^{i+1}-1)}{q^{m-2s}(b-1)(b^{s+1}-1) - (b^{d-i}-1)(b^{i+1}-1)}$$
$$< \frac{(b-1)b^{i+1}}{(b-1)^2 q^m - b^{d+1}}.$$

If n is odd, then $m = 2d + 1$, and the RHS is at most $\frac{3}{14}$ (since $i \le d - 1$ and $q \ge 2$). If n is even, then $m = 2d - 1$. Now if $q \ge 3$ then the RHS is at most $\frac{24}{37}$. If $q = 2$ and $i \le d - 3$ then the RHS is at most $\frac{3}{8}$. For $q = 2$ and $i = d - 2$ we use the sharper form of the last inequality. The claim $|A_j(i+1)/A_j(i)| < 1$ follows from $q^{m-2s}\dfrac{b^{s+1}-1}{b^{i+1}-1} - \dfrac{b^{d-i}-1}{b-1} > 1$, which is true since $b^d > b^{d-s} - 12$. That proves part (iiic).

Part (iiib) is the case $(q, n, i) = (2, 2d, d-1)$ of Proposition 8.2.

Part (iiia) follows from $-\frac{A_j(d-1)}{A_j(d)} = q^{m-2j+2}\frac{b^j-1}{b^d-1} - 1$. This is larger than 1, unless $q = 2$ and n is even.

That proves part (iii). Now part (ii) follows, except in the case $(q, n) = (2, 2d)$. We show that in this case $|A_j(d-2)| > |A_j(d)|$. Indeed, $|A_j(d)| = q^{j(j-1)}\left[{d \atop d-j}\right]_b$ and $|A_j(d-2)| > (1-\gamma)q^{(j-2)(j-3)+2m}\left[{d-2 \atop d-j}\right]_b$, where $\gamma < \frac{8}{9}$ and the desired inequality follows from Lemma 5.3 (iii),(v).

Finally part (i) follows, since the smallest among the $A_j(i)$ is the first one that is negative. $\square$

9 Hermitian forms graphs

The Hermitian forms graphs $Q_q(d)$ are the graphs with as vertices the Hermitian $d \times d$ matrices over $\mathbb{F}_{q^2}$, adjacent when the difference has rank 1.

The graph $Q_q(d)$ is distance-regular with diameter d. The eigenmatrix P has entries $P_{ij} = Q_j(i)$, where

$$Q_j(i) = (-1)^j \sum_{h=0}^{j} (-q)^{\binom{j-h}{2}+hd} \begin{bmatrix} d-h \\ d-j \end{bmatrix}_b \begin{bmatrix} d-i \\ h \end{bmatrix}_b.$$

Here the Gaussian coefficients have base $b = -q$. This formula was taken from Schmidt [25]. An expression in terms of q-Krawtchouk polynomials was given in Stanton [29].

The eigenvalues of $Q_q(d)$ are $\theta_i = ((-q)^{2d-i} - 1)/(q+1)$.

The scheme is self-dual, so that $P_{ij}/P_{0j} = P_{ji}/P_{0i}$, and P_{ij} and P_{ji} have the same sign.

9.1 Identities

Let us write $Q_j^d(i)$ instead of $Q_j(i)$ when it is necessary to make the dependency on d explicit.

Proposition 9.1. ([25, Lemma 7]) $Q_j^d(i) = Q_j^d(i-1) + (-q)^{2d-i} Q_{j-1}^{d-1}(i-1)$.

9.2 The smallest and the second largest eigenvalue

Conjecture 9.2. *(i) If j is odd, then $Q_j(1) \le Q_j(i)$ for $0 \le i \le d$.*

(ii) If j is even, $j \ge 2$, then $Q_j(d-j+2) \le Q_j(i)$ for $0 \le i \le d$.

Conjecture 9.3. *Let $d \ge 3$. Then $|Q_j(i)| < |Q_j(1)|$ for $2 \le i \le d$.*

In the following we prove both conjectures for $q \ge 4$.

Let $q_{i,j}(h)$ be the exponent of q in the h-th term of the expression for $Q_j(i)$ if we approximate $|\left[{n \atop k}\right]_{-q}|$ with $q^{k(n-k)}$. Then

$$q_{i,j}(h) = (d-j)(j-h) + h(d-i-h) + (j-h)(j-h-1)/2 + hd.$$

Let $h_0 = d - i + \frac{1}{2}$. Then the quadratic expression $q_{i,j}(h)$ is maximal for $h = h_0$, and $q_{i,j}(h_0 + x) = q_{i,j}(h_0) - \frac{1}{2}x^2$. Let $h_{\max} = \min(j, d-i)$. The terms occurring in the sum have indices h with $h \le h_{\max} < h_0$, so the term with the largest index has the largest exponent.

Proposition 9.4. *Let $d \ge 2$, $j \ge 1$ and $q \ge 4$. Set $S = S(i) := \left[{d-i \atop j}\right](-q)^{jd}$ if $d - i \ge j$ and $S = S(i) := \left[{i \atop d-j}\right](-q)^{\binom{i+j-d}{2} + (d-i)d}$ otherwise. Then*

$$\left|Q_j(i) - (-1)^j S\right| \le \frac{11}{27}|S|.$$

In particular, the sign of $Q_j(i)$ is the sign of $(-1)^j S$.

Proof. If we divide the absolute value of the h-th term in the expression by the absolute value of the $(h-1)$-th term in the expression, then we obtain, using $1 \le h \le \min(j, d-i)$, and (for $m > 0$)

$$\begin{array}{cccl} (1-q^{-m})q^m & \le |b^m - 1| \le & q^m & \text{if } m \text{ is even}, \\ q^m & \le |b^m - 1| \le & (1+q^{-m})q^m & \text{if } m \text{ is odd}, \end{array}$$

and $q \ge 4$, that

$$\left| \frac{b^{j-h+1}-1}{b^{d-h+1}-1} \cdot \frac{b^{d-i-h+1}-1}{b^h - 1} \cdot b^{h-j+d} \right| \ge \frac{(1-q^{-2})^2}{(1+q^{-1})(1+q^{-3})} q^{d-i-h+1} \ge \frac{9}{13} q^a$$

where $a = 1$ if $h = d - i$, and $a = 2$ otherwise. Then (again using $q \ge 4$)

$$\left|Q_j(i) - (-1)^j S\right| \le \frac{13}{36} \sum_{h \ge 0} 10^{-h} \, |S| \le \frac{11}{27}|S|.$$

This shows the assertion. $\square$

Theorem 9.5. *Let $j \geq 1$ and $q \geq 4$.*

(i) Let $d \geq 3$. Then $|Q_j(i+1)| < |Q_j(i)|$ for $0 \leq i \leq d-1$.

(ii) If j is odd, then $Q_j(1) \leq Q_j(i)$ for $0 \leq i \leq d$.

(iii) If j is even, then $Q_j(d-j+2) \leq Q_j(i)$ for $0 \leq i \leq d$.

Proof. (i) By Proposition 9.4, we have $|Q_j(i)| \geq \frac{16}{27}|S(i)|$ and $|Q_j(i+1)| \leq \frac{38}{27}|S(i+1)|$. We have to show that $|S(i)|/|S(i+1)| > \frac{19}{8}$. If $i+j \leq d-1$,

$$\frac{|S(i)|}{|S(i+1)|} = \left| \frac{\left[{d-i \atop j}\right]}{\left[{d-i-1 \atop j}\right]} \right| = \left| \frac{b^{d-i}-1}{b^{d-i-j}-1} \right| > \frac{1-q^{-1}}{1+q^{-1}}\, q^j > \frac{19}{8}.$$

If $i+j \geq d$,

$$\frac{|S(i)|}{|S(i+1)|} = \left| \frac{\left[{i \atop d-j}\right]}{\left[{i+1 \atop d-j}\right]} b^{-i-j+2d} \right| = \left| \frac{b^{i+j-d+1}-1}{b^{i+1}-1}\, b^{-i-j+2d} \right| > \frac{1-q^{-1}}{1+q^{-1}}\, q^{d-i} > \frac{19}{8}.$$

(ii) and (iii) By Proposition 9.4 and part (i), we only have to find the smallest i for which $(-1)^j S(i)$ is negative. The sign of $(-1)^j \left[{d-i \atop j}\right] b^{jd}$ is positive for j even, and $(-1)^{jd+d-i} = (-1)^i$ for j odd. This proves part (ii). The sign of $(-1)^j \left[{i \atop d-j}\right] (-q)^{\binom{i+j-d}{2}+(d-i)d}$ where j is even, is $(-1)^{\binom{i+j-d}{2}}$, hence is positive for $i = d-j+1$ and negative for $i = d-j+2$. This shows (iii). $\square$

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