GLOBAL QUANTIZATION OF PSEUDO-DIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS, SU(2) AND 3-SPHERE

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ABSTRACT. Global quantization of pseudo-differential operators on compact Lie groups is introduced relying on the representation theory of the group rather than on expressions in local coordinates. Operators on the 3-dimensional sphere $\mathbb{S}^3$ and on group SU(2) are analysed in detail. A new class of globally defined symbols is introduced giving rise to the usual Hörmander's classes of operators $\Psi^m(G)$, $\Psi^m(\mathbb{S}^3)$ and $\Psi^m(\mathrm{SU}(2))$. Properties of the new class and symbolic calculus are analysed. Properties of symbols as well as L^2-boundedness and Sobolev L^2–boundedness of operators in this global quantization are established on general compact Lie groups.

CONTENTS

1. INTRODUCTION

In this paper we investigate a global quantization of operators on compact Lie groups. We develop a non-commutative analogue of the Kohn–Nirenberg quantization of pseudo-differential operators ([12]). The introduced matrix-valued full symbols turn out to have a number of unexpected properties. Among other things, the introduced approach provides a characterization of the Hörmander's class of pseudo-differential operators on compact Lie groups using a global quantization of operators

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relying on the representation theory rather than on the usual expressions in local coordinate charts. The cases of the 3-dimensional sphere $\mathbb{S}^3$ and Lie group SU(2) are analysed in detail and we show that pseudo-differential operators from Hörmander's classes Ψ^m on these spaces have matrix-valued symbols with a remarkable rapid off-diagonal decay property.

There have been many works aiming at the understanding of pseudo-differential operators on Lie groups, see e.g. work on left-invariant operators [26, 15, 6], convolution calculus on nilpotent Lie groups [14], L^2-boundedness of convolution operators related to the Howe's conjecture [11, 7], and many others. In particular, Theorem 3.1 allows x-dependence and also removes the decay condition on the symbol in the setting of general compact Lie groups (a possibility of the relaxation of decay conditions for derivatives of symbols with respect to the dual variable for the L^2-boundedness was conjectured in [11]).

The present research is inspired by M. Taylor's work [28], who used the exponential mapping to rely on pseudo-differential operators on the Lie algebra which can be viewed as the Euclidean space with the corresponding standard theory of pseudo-differential operators. However, the approach developed in this paper is different from that of [28, 29] since it relies on the group structure directly and thus we do not need to work in neighbourhoods of the neutral element and can approach global symbol classes directly.

As usual, $S^m_{1,0} \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ refers to the Euclidean space symbol class, defined by the symbol inequalities

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq C\ \langle\xi\rangle^{m-|\alpha|}, \tag{1.1}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and where constant C is independent of $x, \xi \in \mathbb{R}^n$ but may depend on α, β, p, m. On a compact Lie group G we define the class $\Psi^m(G)$ to be the usual Hörmander's class of pseudo-differential operators of order m. Thus, operator A belongs to $\Psi^m(G)$ if its integral kernel $K(x,y)$ is smooth outside the diagonal $x = y$ and if in (all) local coordinates operator A is a pseudo-differential operator on $\mathbb{R}^n$ with symbol $p(x,\xi)$ satisfying estimates (1.1). We refer to [9, 10] for the historic development of this subject.

It is a natural idea to build pseudo-differential operators out of smooth families of convolution operators on Lie groups. In this paper, we strive to develop the convolution approach into a symbolic quantization, which always provides a much more convenient framework for the analysis of operators. For this, our analysis of operators and their symbols is based on the representation theory of Lie groups. This leads to the description of the full symbols of pseudo-differential operators on Lie groups as sequences of matrices of growing sizes equal to dimensions of representations. Moreover, the analysis is not confined to neighborhoods of the neutral element since it does not rely on the exponential mapping and its properties. We also characterize, in terms of the introduced quantizations, standard Hörmander's classes $\Psi^m(G)$ on Lie groups. One of the advantages of the presented approach is that we obtain a notion of full (global) symbols compared with only principal symbols available in the standard theory via localizations.

To illustrate some ideas, let us now briefly formulate one of the outcomes of this approach in the case of the 3-dimensional sphere $\mathbb{S}^3$. Before that we note that if we have a closed simply-connected 3-dimensional manifold M, then by the recently resolved Poincaré conjecture there is a global diffeomorphism $M \simeq \mathbb{S}^3 \simeq \mathrm{SU}(2)$ that turns M into a Lie group with a group structure induced by $\mathbb{S}^3$ (or by SU(2)). Thus, we can use the approach developed in this paper to immediately obtain the corresponding global quantization of operators on M with respect to this induced group product. In fact, all the formulae remain completely the same since the unitary dual of SU(2) (or $\mathbb{S}^3$ in the quaternionic $\mathbb{R}^4$) is mapped by this diffeomorphism as well; for an example of this construction in the case of $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ see Section 11. The choice of the group structure on M may be not unique and is not canonical, but after using the machinery that we develop for SU(2), the corresponding quantization can be described entirely in terms of M, for an example see Theorem 1.1 for $\mathbb{S}^3$ and Theorem 10.4 for SU(2). In this sense, as different quantizations of operators exist already on $\mathbb{R}^n$ depending on the choice of the underlying structure (e.g. Kohn–Nirenberg quantization, Weyl quantizations, etc.), the possibility to choose different group products on M resembles this. In a subsequent paper we will carry out the detailed analysis of operators on homogeneous spaces and on higher dimensional spheres $\mathbb{S}^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$ viewed as homogeneous spaces. Although we do not have general analogues of the diffeomorphic Poincaré conjecture in higher dimensions, this will cover cases when M is a convex surface or a surface with positive curvature tensor, as well as more general manifolds in terms of their Pontryagin class, etc.

To fix the notation for the Fourier analysis on $\mathbb{S}^3$, let $t^l : \mathbb{S}^3 \to U(2l+1) \subset \mathbb{C}^{(2l+1)\times(2l+1)}$, $l \in \frac{1}{2}\mathbb{N}_0$, be a family of group homomorphisms, which are the irreducible continuous (and hence smooth) unitary representations of $\mathbb{S}^3$ when it is endowed with the SU(2) structure via the quaternionic product, see Section 11 for details. The Fourier coefficient $\widehat{f}(l)$ of $f \in C^\infty(\mathbb{S}^3)$ is defined by $\widehat{f}(l) = \int_{\mathbb{S}^3} f(x)\ t^l(x)^*\ \mathrm{d}x$, where the integration is performed with respect to the Haar measure, so that $\widehat{f}(l) \in \mathbb{C}^{(2l+1)\times(2l+1)}$. The corresponding Fourier series is given by

$$f(x) \;=\; \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)\ \mathrm{Tr}\left(\widehat{f}(l)\ t^l(x)\right).$$

Now, if $A : C^\infty(\mathbb{S}^3) \to C^\infty(\mathbb{S}^3)$ is a continuous linear operator, we define its full symbol as a mapping

$$(x,l) \mapsto \sigma_A(x,l), \quad \sigma_A(x,l) = t^l(x)^*(At^l)(x) \in \mathbb{C}^{(2l+1)\times(2l+1)}.$$

Then we have the representation of operator A in the form

$$Af(x) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)\ \mathrm{Tr}\left(t^l(x)\ \sigma_A(x,l)\ \widehat{f}(l)\right),$$

see Theorem 2.4. We also note that if

$$Af(x) = \int_{\mathbb{S}^3} K_A(x,y)\ f(y)\ \mathrm{d}y = \int_{\mathbb{S}^3} f(y)\ R_A(x, y^{-1}x)\ \mathrm{d}y,$$

where R_A is the right convolution kernel of A, then $\sigma_A(x,l) = \int_{\mathbb{S}^3} R_A(x,y)\ t^l(y)^*\ \mathrm{d}y$ by Theorem 2.5, where, as usual, the integration is performed with respect to the Haar measure with a standard distributional interpretation.

One of the arising fundamental questions is what condition on the matrix symbols σ_A characterize operators from Hörmander's class $\Psi^m(\mathbb{S}^3)$. For this, we introduce symbol class $S^m(\mathbb{S}^3)$. We write $\sigma_A \in S^m(\mathbb{S}^3)$ if the corresponding kernel $K_A(x,y)$ is smooth outside the diagonal $x=y$ and if we have the estimate

$$\left|\triangle_l^\alpha \partial_x^\beta \sigma_{A_u}(x,l)_{ij}\right| \leq C_{A\alpha\beta mN}\ (1+|i-j|)^{-N}(1+l)^{m-|\alpha|}, \tag{1.2}$$

for every $N\geq 0$, every $u\in\mathbb{S}^3$, and all multi-indices α,β, where symbol σ_{A_u} is the symbol of operator $A_u f = A(f\circ\varphi_u)\circ\varphi_u^{-1}$, where $\varphi_u(x)=xu$ is the quaternionic product. Symbols of A_u and A can be shown to be related by formula $\sigma_{A_u}(x,l) = t^l(u)^*\sigma_A(xu^{-1},l)\ t^l(u)$. We notice that imposing the same conditions on all symbols σ_{A_u} in (1.2) simply refers to the well-known fact that the class $\Psi^m(\mathbb{S}^3)$ should be in particular "translation"-invariant (i.e. invariant under the changes of variables induced by quanternionic products φ_u), namely that $A\in\Psi^m(\mathbb{S}^3)$ if and only if $A_u\in\Psi^m(\mathbb{S}^3)$, for all $u\in\mathbb{S}^3$. Condition (1.2) is the growth condition with respect to the *quantum number* l combined with a rather striking condition that matrices $\sigma_A(x,l)$ must have a rapid off-diagonal decay. We also write $\triangle_l^\alpha = \triangle_+^{\alpha_1}\triangle_-^{\alpha_2}\triangle_0^{\alpha_3}$, where operators $\triangle_+,\triangle_-,\triangle_0$ are discrete difference operators acting on matrices $\sigma_A(x,l)$ in variable l, and explicit formulae for them and their properties are given in Section 6. With this definition, we have the following characterization:

Theorem 1.1. *We have $A\in\Psi^m(\mathbb{S}^3)$ if and only if $\sigma_A\in S^m(\mathbb{S}^3)$.*

The proof of this theorem is based on the detailed analysis of pseudo-differential operators and their symbols on Lie group SU(2) where we can use its representation theory and geometric information to derive the corresponding characterization of pseudo-differential operators. We note that this approach works globally on the whole sphere, since the version of the Fourier analysis is different from the one in e.g. [24, 27, 25] which covers only a hemisphere, with singularities at the equator.

In our analysis on a Lie group G, at some point we have to make a choice whether to work with left- or right-convolution kernels. Since left-invariant operators on $C^\infty(G)$ correspond to right-convolutions $f\mapsto f*k$, once we decide to identify the Lie algebra $\mathfrak{g}$ of G with the left-invariant vector fields on G, it becomes most natural to work with right-convolution kernels in the sequel, and to define symbols as we do in Definition 2.3.

Finally, we mention that the more extensive analysis can be carried out in the case of commutative Lie groups. The main simplification in this case is that full symbols are just complex-valued scalars (as opposed to being matrix-valued in the non-commutative case) because the continuous irreducible unitary representations are all one-dimensional. In particular, we can mention the well-known fact that pseudo-differential operators $A\in\Psi^m(\mathbb{T}^n)$ on the n-torus can be globally characterised by conditions

$$\left|\triangle_\eta^\alpha \partial_x^\beta p(x,\eta)\right| \leq C\ (1+|\eta|)^{m-|\alpha|}, \tag{1.3}$$

for all $\eta\in\mathbb{Z}^n$, and all multi-indices $\alpha,\beta\in\mathbb{N}_0^n$, where difference operators $\triangle_\eta^\alpha = \triangle_{\eta_1}^{\alpha_1}\cdots\triangle_{\eta_n}^{\alpha_n}$ are defined by $\triangle_{\eta_j}p(x,\eta) = p(x,\eta+e_j)-p(x,\eta)$, $(e_j)_k=\delta_{jk}$, for all $1\leq j,k\leq n$, etc. If we denote by $S^m(\mathbb{T}^n\times\mathbb{Z}^n)$ the class of functions $p:\mathbb{T}^n\times\mathbb{Z}^n\to\mathbb{C}$ satisfying (1.3), then we have $OpS^m(\mathbb{T}^n\times\mathbb{Z}^n)=\Psi^m(\mathbb{T}^n)$, see e.g. [1, 13, 33, 16, 17] with different proofs, as well as numerical application of this description in e.g. [20, 21]. We note that in [17], more general symbol classes as well as analogues of Fourier integral operators on the torus and toroidal microlocal analysis were developed using the so-called toroidal quantization, which is the torus version of the quantization developed here.

It is also known that globally defined symbols of pseudo-differential operators can be introduced on manifolds in the presence of a connection which allows one to use a suitable globally defined phase function, see e.g. [35, 19, 23]. However, on a compact Lie groups the use of the groups structure allows one to develop a theory parallel to those of $\mathbb{R}^n$ and $\mathbb{T}^n$ in the sense that the Fourier analysis is well adopted to the underlying representation theory. Some elements of such theory were discussed in [32] and in the PhD thesis of the second author, and a consistent development from different points of view will eventually appear in [18].

The global quantization introduced in this paper provides a relatively easy to use approach to deal with problems on $\mathbb{S}^n$ (and on more general Lie groups) which depend on lower order terms of the symbol. Thus, applications to global hypoellipticity, global solvability and other problems in the global setting will appear in the sequel of this paper.

In this paper, the commutator of matrices $X, Y \in \mathbb{C}^{n\times n}$ will be denoted by $[X,Y] = XY - YX$. On SU(2), the conventional abbreviations in summation indices are

$$\sum_l = \sum_{l\in\frac{1}{2}\mathbb{N}_0}, \qquad \sum_l \sum_{m,n} = \sum_{l\in\frac{1}{2}\mathbb{N}_0} \ \sum_{|m|\leq l,\ l+m\in\mathbb{Z}} \ \sum_{|n|\leq l,\ l+n\in\mathbb{Z}},$$

where $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \cdots\}$. The space of all linear mappings from a finite dimensional vector space $\mathcal{H}$ to itself will be denoted by $\mathrm{End}(\mathcal{H})$. As usual, a mapping $U \in \mathcal{L}(\mathcal{H})$ is called unitary if $U^* = U^{-1}$ and the space of all unitary linear mappings on a finite dimensional inner product space $\mathcal{H}$ will be denoted by $\mathcal{U}(\mathcal{H})$.

2. Full symbols on general compact Lie groups

Let G be a compact Lie group, not necessarily just SU(2). Let us endow $\mathcal{D}(G) = C^\infty(G)$ with the usual test function topology. For a continuous linear operator $A : C^\infty(G) \to C^\infty(G)$, let $K_A, L_A, R_A \in \mathcal{D}'(G \times G)$ denote respectively the Schwartz, left-convolution and right-convolution kernels, i.e.

$$\begin{aligned}(2.1)\quad Af(x) &= \int_G K_A(x,y)\ f(y)\ \mathrm{d}y = \\ &= \int_G L_A(x, xy^{-1})\ f(y)\ \mathrm{d}y = \int_G f(y)\ R_A(x, y^{-1}x)\ \mathrm{d}y\end{aligned}$$

in the sense of distributions. To simplify the notation in the sequel, we will often write integrals in the sense of distributions, with a standard distributional interpretation. Notice that

$$R_A(x,y) = L_A(x, xyx^{-1}),$$

and that left-invariant operators on $C^\infty(G)$ correspond to right-convolutions $f \mapsto f * k$. Since we identify the Lie algebra $\mathfrak{g}$ of G with the left-invariant vector fields on G, it will be most natural to study right-convolution kernels in the sequel.

Let us begin with fixing the notation concerning Fourier series on a compact group G (for general background on the representation theory we refer to e.g. [8]). In the sequel, let $\mathrm{Rep}(G)$ denote the set of all strongly continuous irreducible unitary representations of G. In this paper, whenever we mention unitary representations (of a compact Lie group G), we always mean strongly continuous irreducible unitary representations, which are then also automatically smooth. Let $\widehat{G}$ denote the unitary dual of G, i.e. the set of equivalence classes of irreducible unitary representations from $\mathrm{Rep}(G)$. Let $[\xi] \in \widehat{G}$ denote the equivalence class of an irreducible unitary representation $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$; the representation space $\mathcal{H}_\xi$ is finite-dimensional since G is compact, and we set $\dim(\xi) = \dim \mathcal{H}_\xi$. We will always equip compact Lie groups with the Haar measure, i.e. the uniquely determined bi-invariant Borel regular probability measure. Let us define the Fourier coefficient $\widehat{f}(\xi) \in \mathrm{End}(\mathcal{H}_\xi)$ of $f \in L^1(G)$ by

$$\widehat{f}(\xi) := \int_G f(x)\ \xi(x)^*\ \mathrm{d}x; \tag{2.2}$$

more precisely,

$$(\widehat{f}(\xi)u, v)_{\mathcal{H}_\xi} = \int_G f(x)\,(\xi(x)^* u, v)_{\mathcal{H}_\xi}\,\mathrm{d}x = \int_G f(x)\,(u, \xi(x)v)_{\mathcal{H}_\xi}\,\mathrm{d}x$$

for all $u, v \in \mathcal{H}_\xi$, where $(\cdot,\cdot)_{\mathcal{H}_\xi}$ is the inner product of $\mathcal{H}_\xi$. Notice that $\xi(x)^* = \xi(x)^{-1} = \xi(x^{-1})$.

Remark 2.1. Let $U \in \mathrm{Hom}(\eta, \xi)$ be an intertwining isomorphism, i.e. let $U : \mathcal{H}_\eta \to \mathcal{H}_\xi$ be a bijective unitary linear mapping such that $U\eta(x) = \xi(x)U$ for every $x \in G$. Then we have

$$\widehat{f}(\eta) = U^{-1}\widehat{f}(\xi)\ U \in \mathrm{End}(\mathcal{H}_\eta). \tag{2.3}$$

Let us also consider the inner automorphisms

$$\phi_u = (x \mapsto u^{-1}xu) : G \to G,$$

where $u \in G$. If $\xi \in \mathrm{Rep}(G)$ then we also have

$$\begin{aligned} \widehat{f \circ \phi_u}(\xi) &= \int_G f(u^{-1}xu)\ \xi(x)^*\ \mathrm{d}x = \int_G f(x)\ \xi(uxu^{-1})^*\ \mathrm{d}x \\ &= \xi(u) \int_G f(x)\ \xi(x)^*\ \mathrm{d}x\ \xi(u)^* = \xi(u)\ \widehat{f}(\xi)\ \xi(u)^*. \end{aligned} \tag{2.4}$$

Remark 2.2. If $f, g \in L^1(G)$ then

$$\begin{aligned} \widehat{f * g}(\xi) &= \int_G f * g(x)\ \xi(x)^*\ \mathrm{d}x = \int_G \int_G f(xy^{-1})g(y)\ \mathrm{d}y\ \xi(x)^*\ \mathrm{d}x = \\ &= \int_G g(y)\ \xi(y)^* \int_G f(xy^{-1})\ \xi(xy^{-1})^*\ \mathrm{d}x\ \mathrm{d}y = \widehat{g}(\xi)\ \widehat{f}(\xi), \end{aligned}$$

which in general differs from $\widehat{f}(\xi)\ \widehat{g}(\xi)$. This order exchange is due to the definition of the Fourier coefficients, where we chose the integration of the function with respect to $\xi(x)^*$ instead of $\xi(x)$. This choice actually serves us well, as we chose to identify the Lie algebra $\mathfrak{g}$ with left-invariant vector fields on the Lie group G: namely, a left-invariant continuous linear operator $A : C^\infty(G) \to C^\infty(G)$ can be presented as a right-convolution operator $C_a = (f \mapsto f * a)$, resulting in convenient expressions like

$$\widehat{C_a C_b f} \;=\; \widehat{a}\,\widehat{b}\,\widehat{f}.$$

If $\xi : G \to \mathrm{U}(d)$ is an irreducible unitary matrix representation then $\widehat{f}(\xi) \in \mathbb{C}^{d\times d}$ in (2.2) has matrix elements

$$\widehat{f}(\xi)_{mn} = \int_G f(x)\ \overline{\xi(x)_{nm}}\ \mathrm{d}x \in \mathbb{C},\ 1 \le m, n \le d,$$

where the matrix elements are calculated with respect to the standard basis of $\mathbb{C}^d$. If here $f \in L^2(G)$ then $\widehat{f}(\xi)_{mn} = (f, \xi(x)_{nm})_{L^2(G)}$, and by the Peter–Weyl Theorem

$$f(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\xi(x)\ \widehat{f}(\xi)\right) = \sum_{[\xi]\in\widehat{G}} \dim(\xi) \sum_{m,n=1}^{\dim(\xi)} \xi(x)_{nm}\ \widehat{f}(\xi)_{mn} \tag{2.5}$$

for almost every $x \in G$, where the summation is understood so that from each class $[\xi] \in \widehat{G}$ we pick just (any) one representative $\xi \in [\xi]$. The choice of a representation from the same representation class is irrelevant due to formula (2.3) and the presence of the trace in (2.5).

Definition 2.3 (Symbols of pseudo-differential operators on G)**.** Let $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$ be an irreducible unitary representation. The *symbol* of a linear continuous operator $A : C^\infty(G) \to C^\infty(G)$ at $x \in G$ and $\xi \in \mathrm{Rep}(G)$ is defined by $\sigma_A(x, \xi) = \widehat{r_x}(\xi) \in \mathrm{End}(\mathcal{H}_\xi)$, where

$$r_x(y) = R_A(x, y)$$

is the right convolution kernel of A as in (2.1). Hence

$$\sigma_A(x, \xi) = \int_G R_A(x, y)\ \xi(y)^*\ \mathrm{d}y \tag{2.6}$$

in the sense of distributions, and operator A can be represented by its symbol:

Theorem 2.4. *Let the symbol σ_A of a continuous linear operator $A : C^\infty(G) \to C^\infty(G)$ be defined as in Definition 2.3. Then*

$$Af(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\xi(x)\ \sigma_A(x, \xi)\ \widehat{f}(\xi)\right). \tag{2.7}$$

for every $f \in C^\infty(G)$ and $x \in G$.

Proof. Let us define a right-convolution operator $A_{x_0} \in \mathcal{L}(C^\infty(G))$ by kernel $R_A(x_0, y) = r_{x_0}(y)$, i.e. by

$$A_{x_0} f(x) := \int_G f(y)\ r_{x_0}(y^{-1}x)\ \mathrm{d}y = (f * r_{x_0})(x).$$

Thus $\sigma_{A_{x_0}}(x, \xi) = \widehat{r_{x_0}}(\xi) = \sigma_A(x_0, \xi)$, so that by (2.5) we have

$$A_{x_0} f(x) \;=\; \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\xi(x)\ \widehat{A_{x_0} f}(\xi)\right)$$

$$= \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\xi(x)\ \sigma_A(x_0,\xi)\ \widehat{f}(\xi)\right),$$

where we used that $\widehat{f * r_{x_0}} = \widehat{r_{x_0}}\widehat{f}$ by Remark 2.2. This implies the result, because $Af(x) = A_x f(x)$. $\square$

For a symbol σ_A, the corresponding operator A defined by (2.7) will be also denoted by $Op(\sigma_A)$.

Thus, if $\xi : G \to \mathrm{U}(\dim(\xi))$ are irreducible unitary matrix representations then

$$Af(x) \quad = \quad \sum_{[\xi]\in\widehat{G}} \dim(\xi) \sum_{m,n=1}^{\dim(\xi)} \xi(x)_{nm} \left(\sum_{k=1}^{\dim(\xi)} \sigma_A(x,\xi)_{mk}\ \widehat{f}(\xi)_{kn} \right).$$

Alternatively, setting $A\xi(x)_{mn} := (A(\xi_{mn}))(x)$, we have

$$\sigma_A(x,\xi)_{mn} = \sum_{k=1}^{\dim(\xi)} \overline{\xi_{km}(x)}\ (A\xi_{kn})(x), \tag{2.8}$$

$1 \le m,n \le \dim(\xi)$, which follows from the following theorem:

Theorem 2.5. *Let the symbol σ_A of a continuous linear operator $A : C^\infty(G) \to C^\infty(G)$ be defined as in Definition 2.3. Then*

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x). \tag{2.9}$$

Proof. Working with representations $\xi : G \to \mathrm{U}(\dim(\xi))$, we have

$$\begin{aligned}
\sum_{k=1}^{\dim(\xi)} \overline{\xi_{km}(x)}\ (A\xi_{kn})(x) \quad &= \quad \sum_k \overline{\xi_{km}(x)} \sum_{[\eta]\in\widehat{G}} \dim(\eta)\ \mathrm{Tr}\left(\eta(x)\ \sigma_A(x,\eta)\ \widehat{\xi_{kn}}(\eta)\right) \\
&= \quad \sum_k \overline{\xi_{km}(x)} \sum_{[\eta]\in\widehat{G}} \dim(\eta) \sum_{i,j,l} \eta(x)_{ij}\ \sigma_A(x,\eta)_{jl}\ \widehat{\xi_{kn}}(\eta)_{li} \\
&= \quad \sum_{k,j} \overline{\xi_{km}(x)}\ \xi(x)_{kj}\ \sigma_A(x,\xi)_{jn} \\
&= \quad \sigma_A(x,\xi)_{mn},
\end{aligned}$$

where if $\eta \in [\xi]$ in the sum, we take $\eta = \xi$, so that $\widehat{\xi_{kn}}(\eta)_{li} = \langle \xi_{kn}, \eta_{il} \rangle_{L^2}$, which equals $\frac{1}{\dim \xi}$ if $\xi = \eta, k = i$ and $n = l$, and zero otherwise. $\square$

Remark 2.6. The symbol of $A \in \mathcal{L}(C^\infty(G))$ is a mapping

$$\sigma_A : G \times \mathrm{Rep}(G) \to \bigcup_{\xi\in\mathrm{Rep}(G)} \mathrm{End}(\mathcal{H}_\xi),$$

where $\sigma_A(x,\xi) \in \mathrm{End}(\mathcal{H}_\xi)$ for every $x \in G$ and $\xi \in \mathrm{Rep}(G)$. However, it can be viewed as a *mapping on the space* $G \times \widehat{G}$. Indeed, let $\xi, \eta \in \mathrm{Rep}(G)$ be equivalent via an intertwining isomorphism $U \in \mathrm{Hom}(\xi,\eta)$: i.e. such that there exists a linear unitary bijection $U : \mathcal{H}_\xi \to \mathcal{H}_\eta$ such that $\eta(x)\ U = U\ \xi(x)$ for every $x \in G$, that is $\eta(x) = U\ \xi(x)\ U^*$. Then by Remark 2.1 we have $\widehat{f}(\eta) = U\ \widehat{f}(\xi)\ U^*$, and hence also

$$\sigma_A(x,\eta) = U\ \sigma_A(x,\xi)\ U^*.$$

Therefore, taking any representation from the same class $[\xi] \in \widehat{G}$ leads to the same operator A in view of the trace in formula (2.7). In this sense we may think that symbol σ_A is defined on $G \times \widehat{G}$ instead of $G \times \mathrm{Rep}(G)$.

Notice that if $A = (f \mapsto f * a)$ then $R_A(x,y) = a(y)$ and

$$\sigma_A(x,\xi) = \widehat{a}(\xi),$$

i.e. $\widehat{Af}(\xi) = \widehat{a}(\xi)\ \widehat{f}(\xi)$. Moreover, if $B = (f \mapsto b * f)$ then $L_B(x,y) = b(y)$, $R_B(x,y) = L_B(x, xyx^{-1}) = b(xyx^{-1})$, and by (2.4) we have

$$\sigma_B(x,\xi) = \xi(x)^*\ \widehat{b}(\xi)\ \xi(x).$$

Remark 2.7. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G, and let $n = \dim(G) = \dim(\mathfrak{g})$. By the exponential mapping $\exp : \mathfrak{g} \to G$, a neighbourhood of the neutral element $e \in G$ can be identified with a neighbourhood of $0 \in \mathfrak{g}$. Let $\mathcal{X}^m = S^m_{1\#} \subset S^m_{1,0}$ consist of the x-invariant symbols $(x,\xi) \mapsto p(\xi)$ in $S^m_{1,0}$ with the usual Fréchet space topology. A distribution $k \in \mathcal{D}'(G)$ with a sufficiently small support is said to belong to space $\widehat{\mathcal{X}^m}$ if sing supp$(k) \subset \{e\}$ and $\widehat{k} \in \mathcal{X}^m \subset C^\infty(\mathfrak{g}')$, where the Fourier transform $\widehat{k}$ is the usual Fourier transform on $\mathfrak{g} \cong \mathbb{R}^n$, and the dual space satisfies $\mathfrak{g}' \cong \mathbb{R}^n$ (and we are using the exponential coordinates for $k(y)$ when $y \approx e \in G$). If $k \in \widehat{\mathcal{X}^m}$ then the convolution operator

$$u \mapsto k * u, \quad k * u(x) = \int_G k(xy^{-1})\ u(y)\ \mathrm{d}y,$$

is said to belong to space $OP\mathcal{X}^m$, which is endowed with the natural Fréchet space structure obtained from $\mathcal{X}^m$. Formally, let $k(x,y) = k_x(y)$ be the left-convolution kernel of a linear operator $\mathcal{K} : C^\infty(G) \to C^\infty(G)$, i.e.

$$\mathcal{K}u(x) = \int_G k_x(xy^{-1})\ u(y)\ \mathrm{d}y.$$

In [28], M. E. Taylor showed that $\mathcal{K} \in \Psi^m(G)$ if and only if the mapping

$$(x \mapsto (u \mapsto k_x * u)) : G \to OP\mathcal{X}^m$$

is smooth; here naturally $u \mapsto k_x * u$ must belong to $OP\mathcal{X}^m$ for each $x \in G$.

In the sequel, we will need conjugation properties of symbols which we will now analyse for this purpose.

Definition 2.8. Let $\phi : G \to G$ be a diffeomorphism, $f \in C^\infty(G)$, $A : C^\infty(G) \to C^\infty(G)$ continuous and linear. Then the *ϕ-pushforwards* $f_\phi \in C^\infty(G)$ and $A_\phi : C^\infty(G) \to C^\infty(G)$ are defined by

$$\begin{aligned} f_\phi &:= f \circ \phi^{-1}, \\ A_\phi f &:= (A(f_{\phi^{-1}}))_\phi = A(f \circ \phi) \circ \phi^{-1}. \end{aligned}$$

Notice that

$$A_{\phi\circ\psi} = (A_\psi)_\phi\,.$$

From the local theory of pseudo-differential operators, it is well-known that $A \in \Psi^\mu(G)$ if and only if $A_\phi \in \Psi^\mu(G)$.

Definition 2.9. For $u \in G$, let $u_L, u_R : G \to G$ be defined by

$$u_L(x) := ux \quad \text{and} \quad u_R(x) := xu.$$

Then $(u_L)^{-1} = (u^{-1})_L$ and $(u_R)^{-1} = (u^{-1})_R$. The inner automorphism $\phi_u : G \to G$ defined in Remark 2.1 by $\phi_u(x) := u^{-1}xu$ satisfies $\phi_u = u_L^{-1} \circ u_R = u_R \circ u_L^{-1}$.

Proposition 2.10. *Let $u \in G$, $B = A_{u_L}$, $C = A_{u_R}$ and $F = A_{\phi_u}$. Then we have the following relations between symbols:*

$$\begin{aligned} \sigma_B(x,\xi) &= \sigma_A(u^{-1}x,\xi), \\ \sigma_C(x,\xi) &= \xi(u)^* \ \sigma_A(xu^{-1},\xi) \ \xi(u), \\ \sigma_F(x,\xi) &= \xi(u)^* \ \sigma_A(uxu^{-1},\xi) \ \xi(u). \end{aligned}$$

Especially, if $A = (f \mapsto f * a)$, i.e. $\sigma_A(x,\xi) = \widehat{a}(\xi)$, then

$$\begin{aligned} \sigma_B(x,\xi) &= \widehat{a}(\xi), \\ \sigma_C(x,\xi) &= \xi(u)^* \ \widehat{a}(\xi) \ \xi(u) = \sigma_F(x,\xi). \end{aligned}$$

Proof. We notice that $F = C_{(u^{-1})_L}$, so it suffices to consider only operators B and C. For operator $B = A_{u_L}$, we get

$$\int_G f(z) \ R_B(x, z^{-1}x) \ \mathrm{d}z = Bf(x) = A(f \circ u_L)(u_L^{-1}(x)) =$$
$$= \int_G f(uy) \ R_A(u^{-1}x, y^{-1}u^{-1}x) \ \mathrm{d}y = \int_G f(z) \ R_A(u^{-1}x, z^{-1}x) \ \mathrm{d}z,$$

so $R_B(x,y) = R_A(u^{-1}x, y)$, yielding $\sigma_B(x,\xi) = \sigma_A(u^{-1}x,\xi)$. For operator $C = A_{u_R}$, we get similarly $R_C(x,y) = R_A(xu^{-1}, uyu^{-1})$, yielding the result. $\square$

Let us finally record how push-forwards by translation affect vector fields.

Lemma 2.11. *Let $u \in G$, $Y \in \mathfrak{g}$ and let $E = D_Y : C^\infty(G) \to C^\infty(G)$ be defined by $D_Y f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(x \ \exp(tY))\big|_{t=0}$. Then*

$$E_{u_R} = E_{\phi_u} = D_{u^{-1}Yu},$$

i.e. $D_Y(f \circ u_R)(xu^{-1}) = D_Y(f \circ \phi_u)(uxu^{-1}) = D_{u^{-1}Yu} f(x)$.

Proof. We have

$$E_{u_R} f(x) = E(f \circ u_R)(xu^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t} \ (f \circ u_R)(xu^{-1}\exp(tY))\big|_{t=0} =$$
$$= \frac{\mathrm{d}}{\mathrm{d}t} \ f(xu^{-1}\exp(tY)u))\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \ f(x\exp(tu^{-1}Yu)\big|_{t=0} = D_{u^{-1}Yu} f(x).$$

Due to the left-invariance, $E_{u_L} = E$, so that $E_{\phi_u} = (E_{u_L^{-1}})_{u_R} = E_{u_R} = D_{u^{-1}Yu}$. $\square$

3. Boundedness of pseudo-differential operators on $L^2(G)$ and $H^s(G)$

In this section we will state some natural conditions on the symbol of an operator $A : C^\infty(G) \to C^\infty(G)$ to guarantee the boundedness on Sobolev spaces. The Sobolev space $H^s(G)$ of order $s \in \mathbb{R}$ can be defined via a smooth partition of unity of the closed manifold G.

The Hilbert–Schmidt inner product of $A, B \in \mathbb{C}^{m\times n}$ is

$$\langle A, B\rangle_{HS} := \mathrm{Tr}(B^* A) = \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{B_{ij}} A_{ij},$$

with the corresponding norm $\|A\|_{HS} := \langle A, A\rangle_{HS}^{1/2}$, and the operator norm

$$\|A\|_{op} := \sup\left\{\|Ax\|_{HS} :\ x \in \mathbb{C}^{n\times 1},\ \|x\|_{HS} \leq 1\right\} = \|A\|_{\ell^2\to\ell^2}.$$

Let $A, B \in \mathbb{C}^{n\times n}$. Then we have $\|AB\|_{HS} \leq \|A\|_{op}\, \|B\|_{HS}$. Moreover, we also have $\|A\|_{op} = \sup\{\|AX\|_{HS} :\ X \in \mathbb{C}^{n\times n},\ \|X\|_{HS} \leq 1\}$. By this, taking the Fourier transform and using Plancherel's formula (see e.g. [22]), we get

$$\|g \mapsto f * g\|_{\mathcal{L}(L^2(G))} = \|g \mapsto g * f\|_{\mathcal{L}(L^2(G))} = \sup_{\xi\in\mathrm{Rep}(G)} \|\widehat{f}(\xi)\|_{op}, \tag{3.1}$$

by Remark 2.2. We also note that $\|\widehat{f}(\xi)\|_{op} = \|\widehat{f}(\eta)\|_{op}$ if $[\xi] = [\eta] \in \widehat{G}$.

Let us first consider a condition on the symbol for the corresponding operator to be bounded on $L^2(G)$.

Theorem 3.1. *Let G be a compact Lie group of dimension n and let k be an integer such that $k > n/2$. Let A be an operator with symbol σ_A defined as in Definition 2.3. Assume that there is a constant C such that*

$$\|\partial_x^\alpha \sigma_A(x,\xi)\|_{op} \leq C$$

for all $x \in G$, all $\xi \in \mathrm{Rep}(G)$, and all $|\alpha| \leq k$, where $\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, and $\partial_1, \ldots, \partial_n$ are first-order differential operators corresponding to a basis of the Lie algebra of G. Then A is bounded from $L^2(G)$ to $L^2(G)$.

Proof. Let $Af(x) = (f * r_A(x))(x)$, where $r_A(x)(y) = R_A(x,y)$ is the right-convolution kernel of A. Let $A_y f(x) = (f * r_A(y))(x)$, so that $A_x f(x) = Af(x)$. Then

$$\|Af\|^2_{L^2(G)} = \int_G |A_x f(x)|^2 \, \mathrm{d}x \leq \int_G \sup_{y\in G} |A_y f(x)|^2 \, \mathrm{d}x,$$

and by an application of the Sobolev embedding theorem we get

$$\sup_{y\in G} |A_y f(x)|^2 \leq C \sum_{|\alpha|\leq k} \int_G |\partial_y^\alpha A_y f(x)|^2 \, \mathrm{d}y.$$

Therefore, using the Fubini theorem to change the order of integration, we obtain

$$\begin{aligned}
\|Af\|^2_{L^2(G)} &\leq C \sum_{|\alpha|\leq k} \int_G \int_G |\partial_y^\alpha A_y f(x)|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\leq C \sum_{|\alpha|\leq k} \sup_{y\in G} \int_G |\partial_y^\alpha A_y f(x)|^2 \, \mathrm{d}x \\
&= C \sum_{|\alpha|\leq k} \sup_{y\in G} \|\partial_y^\alpha A_y f\|^2_{L^2(G)} \\
&\leq C \sum_{|\alpha|\leq k} \sup_{y\in G} \|f \mapsto f * \partial_y^\alpha r_A(y)\|^2_{\mathcal{L}(L^2(G))} \|f\|^2_{L^2(G)}
\end{aligned}$$

$$\leq \; C \sum_{|\alpha|\leq k} \sup_{y\in G} \sup_{[\xi]\in\widehat{G}} \|\partial_y^\alpha \sigma_A(y,\xi)\|_{op}^2 \|f\|_{L^2(G)}^2,$$

where the last inequality holds due to (3.1). This completes the proof. ∎

Let $\mathscr{L}$ be the bi-invariant Laplacian of G, i.e. the Laplace-Beltrami operator corresponding to the unique (up to scaling) bi-invariant Riemannian metric of G. The Laplacian is symmetric and $I-\mathscr{L}$ is positive. Denote $\Xi = (I-\mathscr{L})^{1/2}$. Then $\Xi^s \in \mathcal{L}(C^\infty(G))$ and $\Xi^s \in \mathcal{L}(\mathcal{D}'(G))$ for every $s\in\mathbb{R}$. Let us define

$$(f,g)_{H^s(G)} = (\Xi^s f, \Xi^s g)_{L^2(G)} \quad (f,g\in C^\infty(G)).$$

The completion of $C^\infty(G)$ with respect to the norm $f\mapsto \|f\|_{H^s(G)} = (f,f)_{H^s(G)}^{1/2}$ gives us Sobolev space $H^s(G)$ of order $s\in\mathbb{R}$, which coincides with the Sobolev space obtained using any smooth partition of unity on the compact manifold G. Operator Ξ^r is a Sobolev space isomorphism $H^s(G)\to H^{s-r}(G)$ for every $r,s\in\mathbb{R}$. To formulate the corresponding boundedness result in Sobolev spaces, let us introduce some notation.

Let $\xi\in\mathrm{Rep}(G)$. Given $v,w\in\mathcal{H}_\xi$, the function $\xi^{vw}: G\to\mathbb{C}$ defined by

$$\xi^{vw}(x) := \langle \xi(x)v, w\rangle_{\mathcal{H}_\xi}$$

is not only continuous but even C^∞-smooth. Let $\mathrm{span}(\xi)$ denote the linear span of $\{\xi^{vw}:\ v,w\in\mathcal{H}_\xi\}$. If $\xi\sim\eta$ then $\mathrm{span}(\xi)=\mathrm{span}(\eta)$; consequently, we may write

$$\mathrm{span}[\xi] := \mathrm{span}(\xi)\subset C^\infty(G).$$

It follows that $-\mathscr{L}\xi^{vw}(x) = \lambda_{[\xi]}\xi^{vw}(x)$, where $\lambda_{[\xi]}\geq 0$, and we denote

$$\langle\xi\rangle = (1+\lambda_{[\xi]})^{1/2}. \tag{3.2}$$

We note that $\sigma_{\mathscr{L}}(x,\xi) = -\lambda_{[\xi]} I_{\dim\xi}$, where $I_{\dim\xi}$ is the identity mapping on $\mathcal{H}_\xi$.

Now we can formulate the main result on Sobolev space boundedness:

Theorem 3.2. *Let G be a compact Lie group of dimension n. Let A be an operator with symbol σ_A defined as in Definition 2.3. Assume that there are constants $\mu, C_\alpha\in\mathbb{R}$ such that*

$$\|\partial_x^\alpha \sigma_A(x,\xi)\|_{op} \leq C_\alpha\ \langle\xi\rangle^\mu$$

holds for all $x\in G$, $\xi\in\mathrm{Rep}(G)$ and all multi-indices α, where $\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ is as in Theorem 3.1. Then A is bounded from $H^s(G)$ to $H^{s-\mu}(G)$, for all $s\in\mathbb{R}$.

Remark 3.3. We shall prove this theorem later in Section 8, after introducing tools for symbolic calculus. However, notice that we may easily obtain a special case of this result with $s=\mu$. Namely, if σ_A is as in Theorem 3.2, then

$$\left\|\partial_x^\alpha \left(\sigma_A(x,\xi)\langle\xi\rangle^{-\mu}\right)\right\|_{op} \leq C_\alpha$$

for every multi-index α. Here $\sigma_A(x,\xi)\langle\xi\rangle^{-\mu} = \sigma_{A\circ\Xi^{-\mu}}(x,\xi)$, and thus Theorem 3.1 implies that $A\circ\Xi^{-\mu}$ is bounded on $L^2(G)$, so that $A\in\mathcal{L}(H^\mu(G), L^2(G))$.

4. Preliminaries on SU(2)

We study the compact group SU(2) defined by

$$\mathrm{SU}(2) = \left\{ u \in \mathbb{C}^{2\times 2} : \ \det(u) = 1 \text{ and } u^* u = I \right\},$$

where $e = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ is the identity matrix. Matrix $u \in \mathbb{C}^{2\times 2}$ belongs to SU(2) if and only if it is of the form $u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$, where $|\alpha|^2 + |\beta|^2 = 1$. We will now fix the notation concerning the representations of SU(2). Let us identify $z = (z_1, z_2) \in \mathbb{C}^2$ with matrix $z = \begin{pmatrix} z_1 & z_2 \end{pmatrix} \in \mathbb{C}^{1\times 2}$, and let $\mathbb{C}[z_1, z_2]$ be the space of two-variable polynomials $f : \mathbb{C}^2 \to \mathbb{C}$. Consider mappings

$$t^l : \mathrm{SU}(2) \to \mathrm{GL}(V_l), \quad (t^l(u) f)(z) = f(zu),$$

where $l \in \frac{1}{2}\mathbb{N}_0$ may be called the quantum number, and where V_l is the $(2l+1)$-dimensional subspace of $\mathbb{C}[z_1, z_2]$ containing the homogeneous polynomials of order $2l \in \mathbb{N}_0$, i.e.

$$V_l = \left\{ f \in \mathbb{C}[z_1, z_2] : \ f(z_1, z_2) = \sum_{k=0}^{2l} a_k z_1^k z_2^{2l-k}, \quad \{a_k\}_{k=0}^{2l} \subset \mathbb{C} \right\}.$$

Then the family $\{t^l\}_{l \in \frac{1}{2}\mathbb{N}_0}$ is the family of irreducible unitary representations of SU(2) such that any other irreducible unitary representation of SU(2) is equivalent to one of t^l. The collection $\{q_{lk} : \ k \in \{-l, -l+1, \cdots, +l-1, +l\}\}$ is a basis for the representation space V_l, where

$$q_{lk}(z) = \frac{z_l^{l-k} z_2^{l+k}}{\sqrt{(l-k)!(l+k)!}}.$$

Let us give the matrix elements $t^l_{mn}(u)$ of $t^l(u)$ with respect to this basis, where (4.1) is well-known and (4.2) follows from it.

Proposition 4.1. *Let* $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then*

$$t^l_{mn}(u) = \left(\frac{\mathrm{d}}{\mathrm{d}z_1} \right)^{l-m} \left(\frac{\mathrm{d}}{\mathrm{d}z_2} \right)^{l+m} \frac{(z_1 a + z_2 c)^{l-n} (z_1 b + z_2 d)^{l+n}}{\sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}}, \tag{4.1}$$

where

$$P^l_{mn}(x) = c^l_{mn} \frac{(1-x)^{(n-m)/2}}{(1+x)^{(m+n)/2}} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{l-m} \left[(1-x)^{l-n} (1+x)^{l+n} \right]$$

with

$$c^l_{mn} = 2^{-l} \frac{(-1)^{l-n} \ \mathrm{i}^{n-m}}{\sqrt{(l-n)! \ (l+n)!}} \sqrt{\frac{(l+m)!}{(l-m)!}}.$$

Moreover, we have

$$t^l_{mn}(u) = \sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}} \times \tag{4.2}$$

$$\times \sum_{i=\max\{0,n-m\}}^{\min\{l-n,l-m\}} \frac{(l-n)!(l+n)!}{i!(l-n-i)!(l-m-i)!(n+m+i)!} a^i b^{l-m-i} c^{l-n-i} d^{n+m+i}.$$

On a compact group G, a function $f : G \to \mathbb{C}$ is called a trigonometric polynomial if its translates span the finite-dimensional vector space, i.e. if

$$\dim \operatorname{span} \left\{ (x \mapsto f(y^{-1}x)) : G \to \mathbb{C} \mid y \in G \right\} < \infty.$$

A trigonometric polynomial can be expressed as a linear combination of matrix elements of irreducible unitary representations. Thus a trigonometric polynomial is continuous, and on a Lie group even C^∞-smooth. Moreover, trigonometric polynomials form an algebra with the usual pointwise multiplication. On SU(2), actually,

$$t^{l'}_{m'n'}\, t^l_{mn} = \sum_{k=|l-l'|}^{l+l'} C^{ll'(l+k)}_{m'm(m'+m)}\; C^{ll'(l+k)}_{n'n(n'+n)}\; t^{l+k}_{(m'+m)(n'+n)},$$

where $C^{ll'(l+k)}_{m'm(m'+m)}$ are Clebsch-Gordan coefficients, for which there are explicit formulae, see e.g. [34]. Now we are going to give basic multiplication formulae for trigonometric polynomials $t^l_{mn} : \mathrm{SU}(2) \to \mathbb{C}$; for general multiplication of trigonometric polynomials, one can use these formulae iteratively.

Theorem 4.2. *Let*

$$\begin{pmatrix} t_{--} & t_{-+} \\ t_{+-} & t_{++} \end{pmatrix} \equiv t^{1/2} = \begin{pmatrix} t^{1/2}_{-1/2,-1/2} & t^{1/2}_{-1/2,+1/2} \\ t^{1/2}_{+1/2,-1/2} & t^{1/2}_{+1/2,+1/2} \end{pmatrix}$$

and denote $x^{\pm} := x \pm 1/2$ for $x \in \mathbb{R}$. Then

$$\begin{aligned}
t^l_{mn} t_{--} &= \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}\, t^{l^+}_{m^-n^-} + \frac{\sqrt{(l+m)(l+n)}}{2l+1}\, t^{l^-}_{m^-n^-}, \\
t^l_{mn} t_{++} &= \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}\, t^{l^+}_{m^+n^+} + \frac{\sqrt{(l-m)(l-n)}}{2l+1}\, t^{l^-}_{m^+n^+}, \\
t^l_{mn} t_{-+} &= \frac{\sqrt{(l-m+1)(l+n+1)}}{2l+1}\, t^{l^+}_{m^-n^+} - \frac{\sqrt{(l+m)(l-n)}}{2l+1}\, t^{l^-}_{m^-n^+}, \\
t^l_{mn} t_{+-} &= \frac{\sqrt{(l+m+1)(l-n+1)}}{2l+1}\, t^{l^+}_{m^+n^-} - \frac{\sqrt{(l-m)(l+n)}}{2l+1}\, t^{l^-}_{m^+n^-}.
\end{aligned}$$

These formulae imply, in particular, that expressions similar to these will appear naturally in the developed quantization of operators on SU(2).

5. Left-invariant differential operators on SU(2)

Let us analyse first-order partial differential operators on SU(2) from the point of view of pseudo-differential operators and their global quantization. Homomorphisms $\omega : \mathbb{R} \to \mathrm{SU}(2)$ are called one-parametric subgroups, and they are of the form $\omega = (t \mapsto \exp(tY))$ for $Y = \omega'(0) \in \mathfrak{su}(2)$. As usual, we identify the Lie algebra $\mathfrak{su}(2)$ with the left-invariant vector fields on SU(2), by associating $Y \in \mathfrak{su}(2)$ to the left-invariant operator $D_Y : C^\infty(\mathrm{SU}(2)) \to C^\infty(\mathrm{SU}(2))$ defined by

$$D_Y f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(x\ \exp(tY))\Big|_{t=0}. \tag{5.1}$$

Remark 5.1. Notice first that vector field $\mathrm{i}D_Y$ is symmetric on an arbitrary G:

$$(\mathrm{i}D_Y f, g)_{L^2(G)} = \int_G (\mathrm{i}D_Y f)(x)\,\overline{g(x)}\,\mathrm{d}x = -\mathrm{i}\int_G f(x)\,\overline{D_Y g(x)}\,\mathrm{d}x = (f, \mathrm{i}D_Y g)_{L^2(G)}\,.$$

Hence it is always possible to choose a representative $\xi \in \mathrm{Rep}(G)$ from each $[\xi] \in \widehat{G}$ such that $\sigma_{\mathrm{i}D_Y}(x,\xi)$ is a diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{\dim(\xi)} \end{pmatrix}$, with diagonal entries $\lambda_j \in \mathbb{R}$, which follows because symmetric matrices can be diagonalised by unitary matrices. Notice that then also $[\sigma_{\mathrm{i}D_Y}, \sigma_A](x,\xi)_{mn} = (\lambda_m - \lambda_n)\ \sigma_A(x,\xi)_{mn}$.

In the case of SU(2), we will simplify the notation writing $\widehat{f}(l)$ instead of $\widehat{f}(t^l)$, etc., since we can take a representative t^l in each equivalence class in $\widehat{\mathrm{SU}(2)}$.

Definition 5.2. Let us define one-parametric subgroups $\omega_1, \omega_2, \omega_3 : \mathbb{R} \to \mathrm{SU}(2)$ by

$$\begin{aligned} \omega_1(t) &= \begin{pmatrix} \cos(t/2) & \mathrm{i}\sin(t/2) \\ \mathrm{i}\sin(t/2) & \cos(t/2) \end{pmatrix}, \\ \omega_2(t) &= \begin{pmatrix} \cos(t/2) & -\sin(t/2) \\ \sin(t/2) & \cos(t/2) \end{pmatrix}, \\ \omega_3(t) &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t/2} \end{pmatrix}. \end{aligned}$$

Let $Y_j := \omega_j'(0)$, i.e.

$$Y_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Y_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y_3 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

Matrices Y_1, Y_2, Y_3 constitute a basis for the real vector space $\mathfrak{su}(2)$. Notice that

$$[Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2.$$

Let us define differential operators $D_j := D_{Y_j}$.

We note that matrices $\frac{2}{\mathrm{i}} Y_j$, $j = 1, 2, 3$, are known as Pauli (spin) matrices in physics. It can be also noted that $\mathfrak{k} = \mathrm{span}\{Y_3\}$ and $\mathfrak{p} = \mathrm{span}\{Y_1, Y_2\}$ form a Cartan pair of the Lie algebra $\mathfrak{su}(2)$.

Proposition 5.3. *Let $w_j = \omega_j(\pi/2)$ and $t \in \mathbb{R}$. Then*

$$w_1\ \omega_2(t)\ w_1^{-1} = \omega_3(t),\ w_2\ \omega_3(t)\ w_2^{-1} = \omega_1(t),\ w_3\ \omega_1(t)\ w_3^{-1} = \omega_2(t).$$

The differential versions of these formulae are

$$w_1\ Y_2\ w_1^{-1} = Y_3,\ w_2\ Y_3\ w_2^{-1} = Y_1,\ w_3\ Y_1\ w_3^{-1} = Y_2.$$

The proof is straightforward and follows simply by multiplying these matrices.

Proposition 5.4. *We have*

$$(D_3)_{(w_1)_R} = D_2,\ (D_1)_{(w_2)_R} = D_3,\ (D_2)_{(w_3)_R} = D_1.$$

Symbols of operators D_1, D_2 can be turned to that of D_3 by taking suitable conjugations:

$$\sigma_{D_1}(x,l) = t^l(w_2)\ \sigma_{D_3}(x,l)\ t^l(w_2)^*, \tag{5.2}$$

$$\sigma_{D_2}(x,l) = t^l(w_1)^*\ \sigma_{D_3}(x,l)\ t^l(w_1). \tag{5.3}$$

Moreover, if $D \in \mathfrak{su}(2)$ there is $u \in \mathrm{SU}(2)$ such that $\sigma_D(l) = t^l(u)^*\ \sigma_{D_3}(l)\ t^l(u)$.

Proof. Combining Lemma 2.11 with Proposition 5.3, we see that $(D_3)_{(w_1)_R} = D_2$, $(D_1)_{(w_2)_R} = D_3$ and $(D_2)_{(w_3)_R} = D_1$. Since D_1, D_2, D_3 are left-invariant operators, their symbols $\sigma_{D_j}(x,l)$ do not depend on $x \in G$, and by Proposition 2.10 we obtain (5.2) and (5.3). The last statement follows from Proposition 2.10 since D is a rotation of D_3. □

Although operators D_j have meaning as derivatives with respect to $\frac{\mathrm{i}}{2}$ Pauli matrices, it will be technically simpler for us to work with their linear combinations (see Remark 5.9, also for the explanation of the terminology), which we will now define.

Definition 5.5. Let us define left-invariant first-order partial differential operators $\partial_+, \partial_-, \partial_0 : C^\infty(\mathrm{SU}(2)) \to C^\infty(\mathrm{SU}(2))$, called creation, annihilation, and neutral operators, respectively, by

$$\begin{cases} \partial_+ := \mathrm{i}D_1 - D_2, \\ \partial_- := \mathrm{i}D_1 + D_2, \\ \partial_0 := \mathrm{i}D_3, \end{cases} \quad \text{i.e.} \quad \begin{cases} D_1 = \frac{-\mathrm{i}}{2}\left(\partial_- + \partial_+\right) \\ D_2 = \frac{1}{2}\left(\partial_- - \partial_+\right), \\ D_3 = -\mathrm{i}\partial_0. \end{cases}$$

Remark 5.6. The Laplacian $\mathscr{L}$ satisfies $\mathscr{L} = D_1^2 + D_2^2 + D_3^2$ and $[\mathscr{L}, D_j] = 0$ for every $j \in \{1,2,3\}$. Notice that it can be expressed as $\mathscr{L} = -\partial_0^2 - (\partial_+\partial_- + \partial_-\partial_+)/2$. Operators $\partial_+, \partial_-, \partial_0$ satisfy $[\partial_0, \partial_+] = \partial_+, [\partial_-, \partial_0] = \partial_-, [\partial_+, \partial_-] = 2\partial_0$.

Theorem 5.7. *We have*

$$\begin{aligned} \partial_+ t^l_{mn} &= -\sqrt{(l-n)(l+n+1)}\ t^l_{m,n+1}, \\ \partial_- t^l_{mn} &= -\sqrt{(l+n)(l-n+1)}\ t^l_{m,n-1}, \\ \partial_0 t^l_{mn} &= n\ t^l_{mn}, \\ \mathscr{L} t^l_{mn} &= -l(l+1)\ t^l_{mn}. \end{aligned}$$

Proof. Formulae for $\partial_+, \partial_-, \partial_0$ follow from calculations in [34, p. 141-142] . Since

$$\mathscr{L} = -\partial_0^2 - (\partial_+\partial_- + \partial_-\partial_+)/2,$$

we get

$$\begin{aligned} \mathscr{L} t^l_{mn} &= -n^2\ t^l_{mn} + \frac{1}{2}\Big(\sqrt{(l+n)(l-n+1)}\ \partial_+ t^l_{m,n-1} \\ &\qquad + \sqrt{(l-n)(l+n+1)}\ \partial_- t^l_{m,n+1}\Big) \\ &= \frac{-1}{2}\Big(2n^2 + \sqrt{(l+n)(l-n+1)}\sqrt{(l-(n-1))(l+(n-1)+1)} \end{aligned}$$

$$
\begin{aligned}
&\quad +\sqrt{(l-n)(l+n+1)}\sqrt{(l+(n+1))(l-(n+1)+1)}\Big)\, t^l_{mn} \\
&= \frac{-1}{2}\left(2n^2+(l+n)(l-n+1)+(l-n)(l+n+1)\right) t^l_{mn} \\
&= \frac{-1}{2}\left(2n^2+2(l^2-n^2)+(l+n)+(l-n)\right) t^l_{mn} \\
&= -l(l+1)\ t^l_{mn}.
\end{aligned}
$$

□

We can now calculate symbols of $\partial_+, \partial_-, \partial_0$ and of the Laplacian $\mathscr{L}$.

Theorem 5.8. *We have*

$$
\begin{aligned}
\sigma_{\partial_+}(x,l)_{mn} &= -\sqrt{(l-n)(l+n+1)}\ \delta_{m,n+1} = -\sqrt{(l-m+1)(l+m)}\ \delta_{m-1,n}, \\
\sigma_{\partial_-}(x,l)_{mn} &= -\sqrt{(l+n)(l-n+1)}\ \delta_{m,n-1} = -\sqrt{(l+m+1)(l-m)}\ \delta_{m+1,n}, \\
\sigma_{\partial_0}(x,l)_{mn} &= n\ \delta_{mn} = m\ \delta_{mn}, \\
\sigma_{\mathscr{L}}(x,l)_{mn} &= -l(l+1)\ \delta_{mn},
\end{aligned}
$$

where δ_{mn} is the Kronecker delta: $\delta_{mn}=1$ for $m=n$ and, $\delta_{mn}=0$ otherwise.

Proof. Let $e\in \mathrm{SU}(2)$ be the neutral element of SU(2) and let t^l be a unitary matrix representation of SU(2). First we note that

$$\delta_{mn} = t^l(e)_{mn} = t^l(x^{-1}x)_{mn} = \sum_k t^l(x^{-1})_{mk}\ t^l(x)_{kn} = \sum_k \overline{t^l(x)_{km}}\ t^l(x)_{kn}.$$

Similarly, $\delta_{mn} = \sum_k t^l(x)_{mk}\ \overline{t^l(x)_{nk}}$. From this, formulae (2.9)-(2.8), and Theorem 5.7 we get

$$
\begin{aligned}
\sigma_{\partial_+}(x,l)_{mn} &= \sum_k \overline{t^l_{km}(x)}\ \left(\partial_+ t^l_{kn}\right)(x) \\
&= -\sqrt{(l-n)(l+n+1)}\sum_k \overline{t^l_{km}(x)}\ t^l_{k,n+1}(x) \\
&= -\sqrt{(l-n)(l+n+1)}\ \delta_{m,n+1},
\end{aligned}
$$

and the case of $\sigma_{\partial_-}(x,l)$ is analogous. Finally,

$$\sigma_{\partial_0}(x,l)_{mn} = \sum_k \overline{t^l_{km}(x)}\ \left(\partial_0 t^l_{kn}\right)(x) = n\sum_k \overline{t^l_{km}(x)}\ t^l_{k,n}(x) = n\ \delta_{m,n},$$

and similarly for $\mathscr{L}$, completing the proof. □

Remark 5.9. Notice that $\sigma_{\partial_0}(x,l)$ and $\sigma_{\mathscr{L}}(x,l)$ are diagonal matrices. The non-zero elements reside just above the diagonal of $\sigma_{\partial_+}(x,l)$, and just below the diagonal of $\sigma_{\partial_-}(x,l)$. Because of this operators ∂_0, ∂_+ and ∂_- may be called neutral, creation and annihilation operators, respectively, and this explains our preference to work with them rather than with D_j's, which have more non-zero entries.

6. Differences for symbols on SU(2)

In this section we describe difference operators on SU(2) leading to symbol inequalities for symbols introduced in Definition 2.3. From Proposition 4.1 and Theorem 4.2 we recall the notation

$$t^{1/2} = \begin{pmatrix} t_{--} & t_{-+} \\ t_{+-} & t_{++} \end{pmatrix} = \begin{pmatrix} t^{1/2}_{-1/2,-1/2} & t^{1/2}_{-1/2,+1/2} \\ t^{1/2}_{+1/2,-1/2} & t^{1/2}_{+1/2,+1/2} \end{pmatrix}.$$

Definition 6.1. For $q \in C^\infty(\mathrm{SU}(2))$ and $f \in \mathcal{D}'(\mathrm{SU}(2))$, let $\triangle_q \widehat{f}(l) := \widehat{qf}(l)$. We shall use abbreviations $\triangle_+ = \triangle_{q_+}$, $\triangle_- = \triangle_{q_-}$ and $\triangle_0 = \triangle_{q_0}$, where

$$\begin{aligned} q_- &:= t_{-+} = t^{1/2}_{-1/2,+1/2}, \\ q_+ &:= t_{+-} = t^{1/2}_{+1/2,-1/2}, \\ q_0 &:= t_{--} - t_{++} = t^{1/2}_{-1/2,-1/2} - t^{1/2}_{+1/2,+1/2}. \end{aligned}$$

Thus each trigonometric polynomial $q_+, q_-, q_0 \in C^\infty(\mathrm{SU}(2))$ vanishes at the neutral element $e \in \mathrm{SU}(2)$. In this sense trigonometric polynomials $q_- + q_+, q_- - q_+, q_0$ on SU(2) are analogues of polynomials x_1, x_2, x_3 in the Euclidean space $\mathbb{R}^3$.

The aim now is to define difference operators acting on symbols. For this purpose we may only look at symbols independent of x corresponding to right invariant operators since the following construction is independent of x. Thus, let $a = a(\xi)$ be a symbol as in Definition 2.3. It follows that $a = \widehat{s}$ for some right-convolution kernel $s \in \mathcal{D}'(\mathrm{SU}(2))$ so that operator $Op(a)$ is given by

$$Op(a)f = f * s.$$

Let us define "difference operators" $\triangle_+, \triangle_-, \triangle_0$ acting on symbol a by

$$\triangle_+ a := \widehat{q_+\, s}, \tag{6.1}$$

$$\triangle_- a := \widehat{q_-\, s}, \tag{6.2}$$

$$\triangle_0 a := \widehat{q_0\, s}. \tag{6.3}$$

We note that this construction is analogous to the one producing usual derivatives in $\mathbb{R}^n$ or difference operators on the torus $\mathbb{T}^n$ (see [17] for details). On SU(2), to analyse the structure of these difference operators, we first need to know how to multiply functions t^l_{mn} by q_+, q_-, q_0, and the necessary formulae are given in Theorem 4.2.

Let us now derive explicit expressions for the first order difference operators $\triangle_+$, $\triangle_-$, $\triangle_0$ defined in (6.1)-(6.3). To abbreviate the notation, we will also write $a^l_{nm} = a(x,l)_{nm}$, even if symbol $a(x,l)$ depends on x, keeping in mind that the following theorem holds pointwise in x.

Theorem 6.2. *The difference operators are given by*

$$\begin{aligned} (\triangle_- a)^l_{nm} &= \frac{\sqrt{(l-m)(l+n)}}{2l+1} a^{l^-}_{n^- m^+} - \frac{\sqrt{(l+m+1)(l-n+1)}}{2l+1} a^{l^+}_{n^- m^+}, \\ (\triangle_+ a)^l_{nm} &= \frac{\sqrt{(l+m)(l-n)}}{2l+1} a^{l^-}_{n^+ m^-} - \frac{\sqrt{(l-m+1)(l+n+1)}}{2l+1} a^{l^+}_{n^+ m^-}, \end{aligned}$$

$$\begin{aligned}(\triangle_0 a)^l_{nm} &= \frac{\sqrt{(l-m)(l-n)}}{2l+1} a^{l^-}_{n^+m^+} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1} a^{l^+}_{n^+m^+} - \\ &\quad - \frac{\sqrt{(l+m)(l+n)}}{2l+1} a^{l^-}_{n^-m^-} - \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1} a^{l^+}_{n^-m^-},\end{aligned}$$

where $k^\pm = k \pm \frac{1}{2}$, *and satisfy commutator relations*

$$[\triangle_0, \triangle_+] = [\triangle_0, \triangle_-] = [\triangle_-, \triangle_+] = 0. \tag{6.4}$$

Proof. Equalities (6.4) follow immediately from (6.1)–(6.3). We can abbreviate $a(x,l)$ by $a(l)$ since none of the arguments in the proof will act on the variable x. Recall that by (2.6) we have

$$a(x,l)_{nm} = a^l_{nm} = \widehat{s}(l)_{nm} = \int_{\mathrm{SU}(2)} s(y)\ \overline{t^l_{mn}(y)}\ \mathrm{d}y,$$

and

$$s(x) = \sum_l (2l+1)\ \mathrm{Tr}\left(a(x,l)\ t^l(x)\right) = \sum_l (2l+1) \sum_{m,n} a^l_{nm}\ t^l_{mn}. \tag{6.5}$$

In the calculation below we will not worry about boundaries of summations keeping in mind that we can always view finite matrices as infinite ones simply by extending them be zeros. Recalling that $q_- = t_{-+}$ and using Theorem 4.2, we can calculate

$$\begin{aligned} q_-\ s &= \sum_l (2l+1) \sum_{m,n} a^l_{nm}\ q_-\ t^l_{mn} \\ &= \sum_l \sum_{m,n} a^l_{nm} \left[t^{l^+}_{m^-n^+} \sqrt{(l-m+1)(l+n+1)} - t^{l^-}_{m^-n^+} \sqrt{(l+m)(l-n)} \right] \\ &= \sum_l \sum_{m,n} t^l_{mn} \left[a^{l^-}_{n^-m^+} \sqrt{(l-m)(l+n)} - a^{l^+}_{n^-m^+} \sqrt{(l+m+1)(l-n+1)} \right]. \end{aligned}$$

Since $\triangle_- a = \widehat{q_- s}$, we obtain the desired formula for $\triangle_-$. The calculation for $\triangle_+$ is analogous. Finally, for $\triangle_0$, we calculate

$$\begin{aligned} q_0\ s &= \sum_l (2l+1) \sum_{m,n} a^l_{nm}\ q_0\ t^l_{mn} \\ &= \sum_l \sum_{m,n} a^l_{nm} \Big[t^{l^+}_{m^-n^-} \sqrt{(l-m+1)(l-n+1)} + t^{l^-}_{m^-n^-} \sqrt{(l+m)(l+n)} \\ &\quad - t^{l^+}_{m^+n^+} \sqrt{(l+m+1)(l+n+1)} - t^{l^-}_{m^+n^+} \sqrt{(l-m)(l-n)} \Big] \\ &= \sum_l \sum_{m,n} t^l_{mn} \Big[a^{l^-}_{n^+m^+} \sqrt{(l-m)(l-n)} + a^{l^+}_{n^+m^+} \sqrt{(l+m+1)(l+n+1)} \\ &\quad - a^{l^-}_{n^-m^-} \sqrt{(l+m)(l+n)} - a^{l^+}_{n^-m^-} \sqrt{(l-m+1)(l-n+1)} \Big]. \end{aligned}$$

From this we obtain the desired formula for $\triangle_0$ and the proof of Theorem 6.2 is complete. ☐

Let us now calculate higher order differences of symbol $a\sigma_{\partial_0}$ which will be needed in the sequel.

Theorem 6.3. *For any $\alpha \in \mathbb{N}_0^3$, we have the formula*

$$\left[\triangle_+^{\alpha_1}\triangle_-^{\alpha_2}\triangle_0^{\alpha_3}(a\sigma_{\partial_0})\right]_{nm}^l = \\ = (m-\alpha_1/2+\alpha_2/2)\left[\triangle_+^{\alpha_1}\triangle_-^{\alpha_2}\triangle_0^{\alpha_3}a\right]_{nm}^l + \alpha_3\left[\overline{\triangle_0}\triangle_+^{\alpha_1}\triangle_-^{\alpha_2}\triangle_0^{\alpha_3-1}a\right]_{nm}^l,$$

where $\overline{\triangle_0}$ is given by

$$\begin{aligned}(\overline{\triangle_0}a)_{nm}^l &= \frac{1}{2}\left[\frac{\sqrt{(l-m)(l-n)}}{2l+1}a_{n^+m^+}^{l^-} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}a_{n^+m^+}^{l^+} + \right.\\ &\quad\left. + \frac{\sqrt{(l+m)(l+n)}}{2l+1}a_{n^-m^-}^{l^-} + \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}a_{n^-m^-}^{l^+}\right],\end{aligned}$$

and satisfies $[\triangle_0, \overline{\triangle_0}] = 0$.

Proof. First we observe that we have

$$(a\ \sigma_{\partial_0})_{nm}^l = \sum_k a_{nk}^l\ k\ \delta_{km} = m\ a_{nm}^l.$$

Then using Theorem 6.2, we get

$$\begin{aligned}\triangle_-(a\sigma_{\partial_0})_{nm}^l &= \frac{\sqrt{(l-m)(l+n)}}{2l+1}m^+a_{n^-m^+}^{l^-} - \frac{\sqrt{(l+m+1)(l-n+1)}}{2l+1}m^+a_{n^-m^+}^{l^+}\\ &= (m^+\triangle_-a)_{nm}^l,\end{aligned}$$

and we can abbreviate this by writing $\triangle_-(a\sigma_{\partial_0}) = m^+\triangle_-a$. Further, we have

$$\begin{aligned}&\triangle_-(\triangle_-(a\sigma_{\partial_0}))_{nm}^l =\\ &= \frac{\sqrt{(l-m)(l+n)}}{2l+1}\left[\triangle_-(a\sigma_{\partial_0})\right]_{n^-m^+}^{l^-} - \frac{\sqrt{(l+m+1)(l-n+1)}}{2l+1}\left[\triangle_-(a\sigma_{\partial_0})\right]_{n^-m^+}^{l^+}\\ &= \frac{\sqrt{(l-m)(l+n)}}{2l+1}(m+1)(\triangle_-a)_{n^-m^+}^{l^-} - \frac{\sqrt{(l+m+1)(l-n+1)}}{2l+1}(m+1)(\triangle_-a)_{n^-m^+}^{l^+}\\ &= (m+1)(\triangle_-^2a)_{nm}^l.\end{aligned}$$

Continuing this calculation we can obtain

$$\left[\triangle_-^k(a\sigma_{\partial_0})\right]_{nm}^l = (m+k/2)(\triangle_-^ka)_{nm}^l. \tag{6.6}$$

By Theorem 6.2 we also have

$$\begin{aligned}&[\triangle_+(\triangle_-(a\sigma_{\partial_0}))]_{nm}^l =\\ &= \left[\triangle_+\left(m^+\triangle_-a\right)\right]_{nm}^l\\ &= \frac{\sqrt{(l+m)(l-n)}}{2l+1}\left(m^+\triangle_-a\right)_{n^+m^-}^{l^-} - \frac{\sqrt{(l-m+1)(l+n+1)}}{2l+1}\left(m^+\triangle_-a)\right)_{n^+m^-}^{l^+}\\ &= m(\triangle_+\triangle_-a)_{nm}^l.\end{aligned}$$

By induction we then get

$$\left[\triangle_+^{k_1}\triangle_-^{k_2}(a\sigma_{\partial_0})\right]_{nm}^l = (m-k_1/2+k_2/2)(\triangle_+^{k_1}\triangle_-^{k_2}a)_{nm}^l. \tag{6.7}$$

The situation with $\triangle_0$ is more complicated because there are more terms. Using Theorem 6.2 we have

$$\begin{aligned}
&\triangle_0(a\sigma_{\partial_0})^l_{nm} = \\
= \;& \frac{\sqrt{(l-m)(l-n)}}{2l+1}(ma)^{l^-}_{n^+m^+} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}(ma)^{l^+}_{n^+m^+} - \\
& -\frac{\sqrt{(l+m)(l+n)}}{2l+1}(ma)^{l^-}_{n^-m^-} - \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}(ma)^{l^+}_{n^-m^-} \\
= \;& \frac{\sqrt{(l-m)(l-n)}}{2l+1}m^+a^{l^-}_{n^+m^+} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}m^+a^{l^+}_{n^+m^+} - \\
& -\frac{\sqrt{(l+m)(l+n)}}{2l+1}m^-a^{l^-}_{n^-m^-} - \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}m^-a^{l^+}_{n^-m^-} \\
= \;& m(\triangle_0 a)^l_{nm} + \frac{1}{2}\left[\frac{\sqrt{(l-m)(l-n)}}{2l+1}a^{l^-}_{n^+m^+} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}a^{l^+}_{n^+m^+} + \right. \\
& \left. +\frac{\sqrt{(l+m)(l+n)}}{2l+1}a^{l^-}_{n^-m^-} + \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}a^{l^+}_{n^-m^-}\right] \\
= \;& m(\triangle_0 a)^l_{nm} + (\overline{\triangle_0}a)^l_{nm},
\end{aligned}$$

where $\overline{\triangle_0}$ is a weighted averaging operator given by

$$\begin{aligned}
(\overline{\triangle_0}a)^l_{nm} = \;& \frac{1}{2}\left[\frac{\sqrt{(l-m)(l-n)}}{2l+1}a^{l^-}_{n^+m^+} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}a^{l^+}_{n^+m^+} + \right. \\
& \left. +\frac{\sqrt{(l+m)(l+n)}}{2l+1}a^{l^-}_{n^-m^-} + \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}a^{l^+}_{n^-m^-}\right].
\end{aligned}$$

We want to find a formula for $\triangle_0^k$, and for this we first calculate

$$\begin{aligned}
&\left[\triangle_0(\overline{\triangle_0}a)\right]^l_{nm} = \\
= \;& \frac{\sqrt{(l-m)(l-n)}}{2l+1}(\overline{\triangle_0}a)^{l^-}_{n^+m^+} + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}(\overline{\triangle_0}a)^{l^+}_{n^+m^+} - \\
& -\frac{\sqrt{(l+m)(l+n)}}{2l+1}(\overline{\triangle_0}a)^{l^-}_{n^-m^-} - \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}(\overline{\triangle_0}a)^{l^+}_{n^-m^-} \\
= \;& \frac{\sqrt{(l-m)(l-n)}}{2l+1}\frac{1}{2}\left[\frac{\sqrt{(l^--m^+)(l^--n^+)}}{2l^-+1}a^{l^{--}}_{n^{++}m^{++}} + \right. \\
& +\frac{\sqrt{(l^-+m^++1)(l^-+n^++1)}}{2l^-+1}a^{l^{-+}}_{n^{++}m^{++}} + \\
& \left. +\frac{\sqrt{(l^-+m^+)(l^-+n^+)}}{2l^-+1}a^{l^{--}}_{n^{+-}m^{+-}} + \frac{\sqrt{(l^--m^++1)(l^--n^++1)}}{2l^-+1}a^{l^{-+}}_{n^{+-}m^{+-}}\right] + \\
& +\frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l^++1}\left[\sqrt{(l^+-m^+)(l^+-n^+)}a^{l^{+-}}_{n^{++}m^{++}} + \right.
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{(l^+ + m^+ + 1)(l^+ + n^+ + 1)}a^{l^{++}}_{n^{++}m^{++}} + \\
&+ \sqrt{(l^+ + m^+)(l^+ + n^+)}a^{l^{+-}}_{n^{+-}m^{+-}} + \sqrt{(l^+ - m^+ + 1)(l^+ - n^+ + 1)}a^{l^{++}}_{n^{+-}m^{+-}}\Big] - \\
&-\frac{\sqrt{(l+m)(l+n)}}{2l+1}\frac{1}{2}\frac{1}{2l^- + 1}\Big[\sqrt{(l^- - m^-)(l^- - n^-)}a^{l^{--}}_{n^{-+}m^{-+}} + \\
&+\sqrt{(l^- + m^- + 1)(l^- + n^- + 1)}a^{l^{-+}}_{n^{-+}m^{-+}} + \\
&+ \sqrt{(l^- + m^-)(l^- + n^-)}a^{l^{--}}_{n^{--}m^{--}} + \sqrt{(l^- - m^- + 1)(l^- - n^- + 1)}a^{l^{-+}}_{n^{--}m^{--}}\Big] - \\
&-\frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l^+ + 1}\Big[\sqrt{(l^+ - m^-)(l^+ - n^-)}a^{l^{+-}}_{n^{-+}m^{-+}} + \\
&+\sqrt{(l^+ + m^- + 1)(l^+ + n^- + 1)}a^{l^{++}}_{n^{-+}m^{-+}} + \\
&+ \sqrt{(l^+ + m^-)(l^+ + n^-)}a^{l^{+-}}_{n^{--}m^{--}} + \sqrt{(l^+ - m^- + 1)(l^+ - n^- + 1)}a^{l^{++}}_{n^{--}m^{--}}\Big].
\end{aligned}
$$

From this we get

$$
\begin{aligned}
&\left[\triangle_0(\overline{\triangle_0}a)\right]^l_{nm} = \\
= \; &\frac{\sqrt{(l-m)(l-n)}}{2l+1}\frac{1}{2}\frac{1}{2l}\Big[\sqrt{(l-m-1)(l-n-1)}a^{l^{--}}_{n^{++}m^{++}} + \\
&+\sqrt{(l+m+1)(l+n+1)}a^{l}_{n^{++}m^{++}} + \\
&+ \sqrt{(l+m)(l+n)}a^{l^{--}}_{nm} + \sqrt{(l-m)(l-n)}a^{l}_{nm}\Big] + \\
&+\frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l+2}\Big[\sqrt{(l-m)(l-n)}a^{l}_{n^{++}m^{++}} + \\
&+\sqrt{(l+m+2)(l+n+2)}a^{l^{++}}_{n^{++}m^{++}} + \\
&+ \sqrt{(l+m+1)(l+n+1)}a^{l}_{nm} + \sqrt{(l-m+1)(l-n+1)}a^{l^{++}}_{nm}\Big] - \\
&-\frac{\sqrt{(l+m)(l+n)}}{2l+1}\frac{1}{2}\frac{1}{2l}\Big[\sqrt{(l-m)(l-n)}a^{l^{--}}_{nm} + \\
&+\sqrt{(l+m)(l+n)}a^{l}_{nm} + \\
&+ \sqrt{(l+m-1)(l+n-1)}a^{l^{--}}_{n^{--}m^{--}} + \sqrt{(l-m+1)(l-n+1)}a^{l}_{n^{--}m^{--}}\Big] - \\
&-\frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l+2}\Big[\sqrt{(l-m+1)(l-n+1)}a^{l}_{nm} + \\
&+\sqrt{(l+m+1)(l+n+1)}a^{l^{++}}_{nm} + \\
&+ \sqrt{(l+m)(l+n)}a^{l}_{n^{--}m^{--}} + \sqrt{(l-m+2)(l-n+2)}a^{l^{++}}_{n^{--}m^{--}}\Big],
\end{aligned}
$$

and we can note that here pairs of terms with $a^{l^{--}}_{nm}$, $a^{l^{++}}_{nm}$ cancel, and also four terms with a^l_{nm} cancel in view of the identity

$$
\frac{(l-m)(l-n)}{(2l+1)(2l)} + \frac{(l+m+1)(l+n+1)}{(2l+1)(2l+2)} - \frac{(l+m)(l+n)}{(2l+1)(2l)} - \frac{(l-m+1)(l-n+1)}{(2l+1)(2l+2)}
$$

$$= \frac{-2l(m+n)}{(2l+1)(2l)} + \frac{(2l+2)(m+n)}{(2l+1)(2l+2)} = 0.$$

Calculating in the other direction, we get

$$\begin{aligned}
&\left[\overline{\triangle_0}(\triangle_0 a)\right]^l_{nm} = \\
= \;& \frac{1}{2}\frac{\sqrt{(l-m)(l-n)}}{2l+1}(\triangle_0 a)^{l^-}_{n^+m^+} + \frac{1}{2}\frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}(\triangle_0 a)^{l^+}_{n^+m^+} + \\
& + \frac{1}{2}\frac{\sqrt{(l+m)(l+n)}}{2l+1}(\triangle_0 a)^{l^-}_{n^-m^-} + \frac{1}{2}\frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}(\triangle_0 a)^{l^+}_{n^-m^-} \\
= \;& \frac{\sqrt{(l-m)(l-n)}}{2l+1}\frac{1}{2}\frac{1}{2l^-+1}\Big[\sqrt{(l^- - m^+)(l^- - n^+)}\,a^{l^{--}}_{n^{++}m^{++}} + \\
& + \sqrt{(l^- + m^+ + 1)(l^- + n^+ + 1)}\,a^{l^{-+}}_{n^{++}m^{++}} - \\
& - \sqrt{(l^- + m^+)(l^- + n^+)}\,a^{l^{--}}_{n^{+-}m^{+-}} - \sqrt{(l^- - m^+ + 1)(l^- - n^+ + 1)}\,a^{l^{-+}}_{n^{+-}m^{+-}}\Big] + \\
& + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l^+ + 1}\Big[\sqrt{(l^+ - m^+)(l^+ - n^+)}\,a^{l^{+-}}_{n^{++}m^{++}} + \\
& + \sqrt{(l^+ + m^+ + 1)(l^+ + n^+ + 1)}\,a^{l^{++}}_{n^{++}m^{++}} - \\
& - \sqrt{(l^+ + m^+)(l^+ + n^+)}\,a^{l^{+-}}_{n^{+-}m^{+-}} - \sqrt{(l^+ - m^+ + 1)(l^+ - n^+ + 1)}\,a^{l^{++}}_{n^{+-}m^{+-}}\Big] + \\
& + \frac{\sqrt{(l+m)(l+n)}}{2l+1}\frac{1}{2}\frac{1}{2l^- + 1}\Big[\sqrt{(l^- - m^-)(l^- - n^-)}\,a^{l^{--}}_{n^{-+}m^{-+}} + \\
& + \sqrt{(l^- + m^- + 1)(l^- + n^- + 1)}\,a^{l^{-+}}_{n^{-+}m^{-+}} - \\
& - \sqrt{(l^- + m^-)(l^- + n^-)}\,a^{l^{--}}_{n^{--}m^{--}} - \sqrt{(l^- - m^- + 1)(l^- - n^- + 1)}\,a^{l^{-+}}_{n^{--}m^{--}}\Big] + \\
& + \frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l^+ + 1}\Big[\sqrt{(l^+ - m^-)(l^+ - n^-)}\,a^{l^{+-}}_{n^{-+}m^{-+}} + \\
& + \sqrt{(l^+ + m^- + 1)(l^+ + n^- + 1)}\,a^{l^{++}}_{n^{-+}m^{-+}} - \\
& - \sqrt{(l^+ + m^-)(l^+ + n^-)}\,a^{l^{+-}}_{n^{--}m^{--}} - \sqrt{(l^+ - m^- + 1)(l^+ - n^- + 1)}\,a^{l^{++}}_{n^{--}m^{--}}\Big].
\end{aligned}$$

From this we get

$$\begin{aligned}
&\left[\overline{\triangle_0}(\triangle_0 a)\right]^l_{nm} = \\
= \;& \frac{\sqrt{(l-m)(l-n)}}{2l+1}\frac{1}{2}\frac{1}{2l}\Big[\sqrt{(l-m-1)(l-n-1)}\,a^{l^{--}}_{n^{++}m^{++}} + \\
& + \sqrt{(l+m+1)(l+n+1)}\,a^{l}_{n^{++}m^{++}} - \\
& - \sqrt{(l+m)(l+n)}\,a^{l^{--}}_{nm} - \sqrt{(l-m)(l-n)}\,a^{l}_{nm}\Big] + \\
& + \frac{\sqrt{(l+m+1)(l+n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l+2}\Big[\sqrt{(l-m)(l-n)}\,a^{l}_{n^{++}m^{++}} + \\
& + \sqrt{(l+m+2)(l+n+2)}\,a^{l^{++}}_{n^{++}m^{++}} - \\
& - \sqrt{(l+m+1)(l+n+1)}\,a^{l}_{nm} - \sqrt{(l-m+1)(l-n+1)}\,a^{l^{++}}_{nm}\Big] +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sqrt{(l+m)(l+n)}}{2l+1}\frac{1}{2}\frac{1}{2l}\Big[\sqrt{(l-m)(l-n)}a^{l--}_{nm}+\\
&+\sqrt{(l+m)(l+n)}a^{l}_{nm}-\\
&-\sqrt{(l+m-1)(l+n-1)}a^{l--}_{n--m--}-\sqrt{(l-m+1)(l-n+1)}a^{l}_{n--m--}\Big]+\\
&+\frac{\sqrt{(l-m+1)(l-n+1)}}{2l+1}\frac{1}{2}\frac{1}{2l+2}\Big[\sqrt{(l-m+1)(l-n+1)}a^{l}_{nm}+\\
&+\sqrt{(l+m+1)(l+n+1)}a^{l++}_{nm}-\\
&-\sqrt{(l+m)(l+n)}a^{l}_{n--m--}-\sqrt{(l-m+2)(l-n+2)}a^{l++}_{n--m--}\Big]
\end{aligned}
$$

and we can note that here terms a^l_{nm}, a^{l--}_{nm} and a^{l++}_{nm} cancel again. From these calculations we obtain

$$\overline{\triangle_0}\triangle_0 a=\triangle_0\overline{\triangle_0}a.$$

Then we can easily see that

$$\triangle_0^2(ma)=\triangle_0(m\triangle_0 a+\overline{\triangle_0}a)=m\triangle_0^2 a+2\overline{\triangle_0}\triangle_0 a,$$

and, moreover,

$$\triangle_0^k(ma)=m\triangle_0^k a+k\overline{\triangle_0}\triangle_0^{k-1}a.$$

Let us now apply this to (6.7). Using commutativity of $\triangle_0,\triangle_+$ and $\triangle_-$ from Theorem 6.2, we get

$$
\begin{aligned}
&\left[\triangle_+^{k_1}\triangle_-^{k_2}\triangle_0^{k_3}(a\sigma_{\partial_0})\right]^l_{nm}=\\
=\;&\left[\triangle_0^{k_3}\triangle_+^{k_1}\triangle_-^{k_2}(a\sigma_{\partial_0})\right]^l_{nm}\\
=\;&\left[\triangle_0^{k_3}\left((m-k_1/2+k_2/2)\triangle_+^{k_1}\triangle_-^{k_2}a\right)\right]^l_{nm}\\
=\;&\left[\triangle_0^{k_3}\left(m\triangle_+^{k_1}\triangle_-^{k_2}a\right)\right]^l_{nm}-\left[\triangle_0^{k_3}\left((k_1/2-k_2/2)\triangle_+^{k_1}\triangle_-^{k_2}a\right)\right]^l_{nm}\\
=\;&m\left[\triangle_0^{k_3}\triangle_+^{k_1}\triangle_-^{k_2}a\right]^l_{nm}+k_3\left[\overline{\triangle_0}\triangle_0^{k_3-1}\triangle_+^{k_1}\triangle_-^{k_2}a\right]^l_{nm}-\\
&-(k_1/2-k_2/2)\left[\triangle_0^{k_3}\triangle_+^{k_1}\triangle_-^{k_2}a\right]^l_{nm}\\
=\;&(m-k_1/2+k_2/2)\left[\triangle_+^{k_1}\triangle_-^{k_2}\triangle_0^{k_3}a\right]^l_{nm}+k_3\left[\overline{\triangle_0}\triangle_+^{k_1}\triangle_-^{k_2}\triangle_0^{k_3-1}a\right]^l_{nm},
\end{aligned}
$$

completing the proof. □

We now collect some properties of first-order differences.

Theorem 6.4. *We have*

$$\sigma_I=\triangle_+\sigma_{\partial_+}=\triangle_-\sigma_{\partial_-}=\triangle_0\sigma_{\partial_0}. \tag{6.8}$$

If $\mu,\nu\in\{+,-,0\}$ are such that $\mu\neq\nu$, then

$$\triangle_\mu\sigma_{\partial_\nu}=0, \tag{6.9}$$

and for every $\nu\in\{+,-,0\}$, we have

$$\triangle_\nu\sigma_I(x)=0. \tag{6.10}$$

Moreover, if $\mathscr{L}$ is the bi-invariant Laplacian, then

$$\triangle_+\sigma_{\mathscr{L}}=-\sigma_{\partial_-},\quad \triangle_-\sigma_{\mathscr{L}}=-\sigma_{\partial_+},\quad \triangle_0\sigma_{\mathscr{L}}=-2\sigma_{\partial_0}. \tag{6.11}$$

Proof. Let us prove (6.8). From Theorem 4.2 we get an expression for $q_+ t^l_{mn} = t_{+-} t^l_{mn}$, which is used in the following calculation together with (6.5) and Theorem 5.8:

$$
\begin{aligned}
q_+ \ s_{\partial_+} &= q_+ \sum_l (2l+1) \sum_{m,n} \sigma_{\partial_+}(l)_{mn} \ t^l_{nm} \\
&= \sum_l \sum_n \sigma_{\partial_+}(l)_{n+1,n} \ (2l+1) \ q_+ \ t^l_{n,n+1} \\
&= \sum_l \sum_n - \left(\sqrt{(l-n)(l+n+1)} \right)^2 \left(t^{l^+}_{n^+ n^+} - t^{l^-}_{n^+ n^+} \right) \\
&= \sum_l (2l+1) \sum_k t^l_{kk} \\
&= s_I .
\end{aligned}
$$

Hence $\triangle_+ \sigma_{\partial_+} = \sigma_I$. Similarly, we can show that $\triangle_- \sigma_{\partial_-} = \sigma_I$ and that $\triangle_0 \sigma_{\partial_0} = \sigma_I$. Let us now prove (6.9). We have

$$
\begin{aligned}
q_+ \ s_{\partial_-} &= q_+ \sum_l (2l+1) \sum_{m,n} \sigma_{\partial_-}(l)_{mn} \ t^l_{nm} \\
&= \sum_l \sum_n \sigma_{\partial_-}(l)_{n-1,n} \ (2l+1) \ q_+ \ t^l_{n,n-1} \\
&= \sum_l \sum_n - \sqrt{(l+n)(l-n+1)} \\
&\qquad \Big(\sqrt{(l+n+1)(l-n+2)} \ t^{l+1/2}_{n+1/2,n-3/2} \\
&\qquad - \sqrt{(l-n)(l+n-1)} \ t^{l-1/2}_{n+1/2,n-3/2} \Big) \\
&= \sum_l \sum_n t^{l+1/2}_{n+1/2,n-3/2} \\
&\qquad \Big(- \sqrt{(l+n+1)(l-n+2)} \sqrt{(l-n+1)(l+n)} \\
&\qquad + \sqrt{(l+n)(l-n+1)} \sqrt{(l+n+1)(l-n+2)} \Big) \\
&= 0.
\end{aligned}
$$

Analogously, one can readily show the rest of (6.9). Let us now prove (6.10). We have

$$
\begin{aligned}
q_- \ s_I &= q_- \sum_l (2l+1) \sum_{m,n} t^l_{mn} \\
&= \sum_l \sum_n (2l+1) \ q_- \ t^l_{nn} \\
&= \sum_l \sum_n \left(\sqrt{(l-n+1)(l+n+1)} \ t^{l^+}_{n^-,n^+} - \sqrt{(l+n)(l-n)} \ t^{l^-}_{n^-,n^+} \right) \\
&= \sum_l \sum_n t^{l^-}_{n^-,n^+} \left(\sqrt{(l-n)(l+n)} - \sqrt{(l+n)(l-n)} \right)
\end{aligned}
$$

$$= \quad 0.$$

Analogously, we have $q_+ \ s_I = q_0 \ s_I = 0$ which proves proves (6.10). Let us finally prove (6.11). Since

$$\sigma_{\mathscr{L}}(x,l)_{mn} = -l(l+1)\ \delta_{mn}$$

by Theorem 5.8, we get

$$\begin{aligned}
q_- \ s_{-\mathscr{L}} &= q_- \ \sum_l (2l+1) \sum_{m,n} \sigma_{-\mathscr{L}}(x,l)_{mn}\ t^l_{nm} \\
&= \sum_l (2l+1) \sum_n l(l+1)\ q_-\ t^l_{nn}(y) \\
&= \sum_l \sum_n l(l+1) \Big(+\sqrt{(l-n+1)(l+n+1)}\ t^{l+}_{n^-,n^+} \\
&\qquad\qquad -\sqrt{(l+n)(l-n)}\ t^{l-}_{n^-,n^+} \Big) \\
&= \sum_l \sum_n t^{l+}_{n^-,n^+} \Big(+l(l+1)\sqrt{(l-n+1)(l+n+1)} \\
&\qquad\qquad -(l+1)(l+2)\sqrt{(l+n+1)(l-n+1)} \Big) \\
&= \sum_l \sum_n -2(l+1)\sqrt{(l+n+1)(l-n+1)}\ t^{l+}_{n^-,n^+} \\
&= \sum_l (2l+1) \sum_n -\sqrt{(l+n)(l-n+1)}\ t^l_{n-1,n} \\
&= s_{\partial_+}.
\end{aligned}$$

Analogously, one can readily show that $q_+ \ s_{-\mathscr{L}} = s_{\partial_-}$ and that $q_0 \ s_{-\mathscr{L}} = 2s_{\partial_0}$, completing the proof. $\square$

Remark 6.5. In Theorem 6.4 we applied the differences on the symbols of specific differential operators on SU(2). In general, on a compact Lie group G, a difference operator of order $|\gamma|$ applied to a symbol of a partial differential operator of order N gives a symbol of order $N-|\gamma|$. More precisely, let

$$D = \sum_{|\alpha|\le N} c_\alpha(x)\ \partial_x^\alpha$$

be a partial differential operator with coefficients $c_\alpha \in C^\infty(G)$. For $q \in C^\infty(G)$ such that $q(e)=0$, we define difference operator $\triangle_q$ acting on symbols by

$$\triangle_q \widehat{f}(\xi) := \widehat{qf}(\xi).$$

Then we obtain

$$\triangle_q \sigma_D(x,\xi) = \sum_{|\alpha|\le N} c_\alpha(x) \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (-1)^{|\beta|}\ (\partial_x^\beta q)(e)\ \sigma_{\partial_x^{\alpha-\beta}}(x,\xi),$$

which is a symbol of a partial differential operator of order at most $N-1$.

7. Taylor expansion on Lie groups

As Taylor polynomial expansions are useful in obtaining symbolic calculus on $\mathbb{R}^n$, we would like to have analogous expansions on group G. Here, Taylor expansion formula on G will be obtained by embedding G into some $\mathbb{R}^m$, using the Taylor expansion formula in $\mathbb{R}^m$, and then restricting it back to G.

Let $U \subset \mathbb{R}^m$ be an open neighbourhood of some point $\vec{e} \in \mathbb{R}^m$. The Nth order Taylor polynomial $P_N f : \mathbb{R}^m \to \mathbb{C}$ of $f \in C^\infty(U)$ at $\vec{e}$ is given by

$$P_N f(\vec{x}) = \sum_{\alpha\in\mathbb{N}_0^m:\ |\alpha|\leq N} \frac{1}{\alpha!}\,(\vec{x}-\vec{e})^\alpha\,\partial_x^\alpha f(\vec{e}).$$

Then the remainder $E_N f := f - P_N f$ satisfies

$$E_N f(\vec{x}) = \sum_{|\alpha|=N+1} (\vec{x}-\vec{e})^\alpha\, f_\alpha(\vec{x})$$

for some functions $f_\alpha \in C^\infty(U)$. In particular,

$$E_N f(\vec{x}) = \mathcal{O}(\|\vec{x}-\vec{e}\|^{N+1}) \quad \text{as} \quad \vec{x}\to\vec{e}.$$

Let G be a compact Lie group; we would like to approximate a smooth function $u : G \to \mathbb{C}$ using a Taylor polynomial type expansion nearby the neutral element $e \in G$. We may assume that G is a closed subgroup of $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$, the group of real invertible $(n\times n)$-matrices, and thus a closed submanifold of the Euclidean space of dimension $m = n^2$. This embedding of G into $\mathbb{R}^m$ will be denoted by $x \mapsto \vec{x}$, and the image of G under this embedding will be still denoted by G. Also, if $x \in G$, we may still write x for $\vec{x}$ to simplify the notation. Let $U \subset \mathbb{R}^m$ be a small enough open neighbourhood of $G \subset \mathbb{R}^m$ such that for each $\vec{x} \in U$ there exists a unique nearest point $p(\vec{x}) \in G$ (with respect to the Euclidean distance). For $u \in C^\infty(G)$ define $f \in C^\infty(U)$ by

$$f(\vec{x}) := u(p(\vec{x})).$$

The effect is that f is constant in the directions perpendicular to G. As above, we may define the Euclidean Taylor polynomial $P_N f : \mathbb{R}^m \to \mathbb{C}$ at $e \in G \subset \mathbb{R}^m$. Let us define $P_N u : G \to \mathbb{C}$ as the restriction,

$$P_N u := P_N f|_G.$$

We call $P_N u \in C^\infty(G)$ a *Taylor polynomial of u of order N at $e \in G$*. Then for $x \in G$, we have

$$u(x) - P_N u(x) = \sum_{|\alpha|=N+1} u_\alpha(x)\,(x-e)^\alpha$$

for some functions $u_\alpha \in C^\infty(G)$, where we set $(x-e)^\alpha := (\vec{x}-\vec{e})^\alpha$. Taylor polynomials on G are given by

$$P_N u(x) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\,(x-e)^\alpha\,\partial_x^{(\alpha)} u(e),$$

where we set $\partial_x^{(\alpha)} u(e) := \partial_x^\alpha f(\vec{e})$.

Let us now consider especially $G = \mathrm{SU}(2)$. Recall the quaternionic identification

$$(x_0\mathbf{1} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mapsto (x_0,x_1,x_2,x_3)) : \mathbb{H} \to \mathbb{R}^4.$$

Moreover, there is the identification $\mathbb{H} \supset \mathbb{S}^3 \cong \mathrm{SU}(2)$,

$$\vec{x} = (x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 + \mathrm{i}x_3 & x_1 + \mathrm{i}x_2 \\ -x_1 + \mathrm{i}x_2 & x_0 - \mathrm{i}x_3 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x.$$

Hence we identify $(1, 0, 0, 0) \in \mathbb{R}^4$ with the neutral element of $\mathrm{SU}(2)$. Notice that

$$\begin{aligned} q_+(x) &= x_{12} = x_1 + \mathrm{i}x_2, \\ q_-(x) &= x_{21} = -x_1 + \mathrm{i}x_2, \\ q_0(x) &= x_{11} - x_{22} = 2\mathrm{i}x_3. \end{aligned}$$

A function $u \in C^\infty(\mathbb{S}^3)$ can be extended to $f \in C^\infty(U) = C^\infty(\mathbb{R}^4 \setminus \{0\})$ by

$$f(\vec{x}) := u(\vec{x}/\|\vec{x}\|).$$

Then we obtain $P_N u \in C^\infty(\mathbb{S}^3)$,

$$P_N u(\vec{x}) := \sum_{|\alpha| \le N} \frac{1}{\alpha!}\ (\vec{x} - \vec{e})^\alpha\ \partial_x^\alpha f(\vec{e}),$$

where $\vec{e} = (1, 0, 0, 0)$. Expressing this in terms of $x \in \mathrm{SU}(2)$, we obtain Taylor polynomials for $x \in \mathrm{SU}(2)$:

$$P_N u(x) = \sum_{|\alpha| \le N} \frac{1}{\alpha!}\ (x - e)^\alpha\ \partial_x^{(\alpha)} u(e),$$

where we write $\partial_x^{(\alpha)} u(e) = \partial_x^\alpha f(\vec{e})$, and where

$$\begin{aligned} (x - e)^\alpha = (\vec{x} - \vec{e})^\alpha = (x_0 - 1)^{\alpha_1} x_1^{\alpha_2} x_2^{\alpha_3} x_3^{\alpha_4} = \\ = \left(\frac{x_{11} + x_{22}}{2} - 1\right)^{\alpha_1} \left(\frac{x_{12} - x_{21}}{2}\right)^{\alpha_2} \left(\frac{x_{12} + x_{21}}{2\mathrm{i}}\right)^{\alpha_3} \left(\frac{x_{11} - x_{22}}{2\mathrm{i}}\right)^{\alpha_4}. \end{aligned}$$

8. Properties of global pseudo-differential symbols

In this section, we study the global symbols of pseudo-differential operators on compact Lie groups. We also derive elements of the calculus in more general classes of symbols, and prove the Sobolev boundedness Theorem 3.2.

As explained in Section 7, smooth functions on a group G can be approximated by Taylor polynomial type expansions. More precisely, there exist partial differential operators $\partial_x^{(\alpha)}$ of order $|\alpha|$ on G such that for every $u \in C^\infty(G)$ we have

(8.1)
$$u(x) = \sum_{|\alpha| \le N} \frac{1}{\alpha!}\ q_\alpha(x^{-1})\ \partial_x^{(\alpha)} u(e) + \sum_{|\alpha| = N+1} q_\alpha(x^{-1})\ u_\alpha(x) \sim \sum_{\alpha \ge 0} \frac{1}{\alpha!}\ q_\alpha(x^{-1})\ \partial_x^{(\alpha)} u(e)$$

in a neighbourhood of $e \in G$, where $u_\alpha \in C^\infty(G)$, and $q_\alpha \in C^\infty(G)$ satisfy $q_{\alpha+\beta} = q_\alpha q_\beta$. Moreover, here $q_0 \equiv 1$, and $q_\alpha(e) = 0$ if $|\alpha| \ge 1$. Let us define difference operators $\triangle_\xi^\alpha$ acting on Fourier coefficients by $\triangle_\xi^\alpha \widehat{f}(\xi) := \widehat{q_\alpha f}(\xi)$. Notice that $\triangle_\xi^{\alpha+\beta} = \triangle_\xi^\alpha \triangle_\xi^\beta$.

Remark 8.1. The technical choice of writing $q_\alpha(x^{-1})$ in (8.1) is dictated by our desire to make asymptotic formulae in Theorems 8.3 and 8.4 look similar to the familiar Euclidean formulae, and by an obvious freedom in selecting among different forms of Taylor polynomials q_α. For example, on SU(2), if we work with operators $\Delta_+, \Delta_-, \Delta_0$ defined in (6.1)-(6.3), we can choose the form of the Taylor expansion (8.1) adapted to functions q_+, q_-, q_0. Here we can observe that $q_+(x^{-1}) = -q_-(x)$, $q_-(x^{-1}) = -q_+(x)$, $q_0(x^{-1}) = -q_0(x)$, so that for $|\alpha| = 1$ functions $q_\alpha(x)$ and $q_\alpha(x^{-1})$ are linear combinations of q_+, q_-, q_0.

Let $\{Y_j\}_{j=1}^{\dim(G)}$ be a basis for the Lie algebra of G, and let ∂_j be the left-invariant vector fields corresponding to Y_j. For $\beta \in \mathbb{N}_0^n$, let us denote $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

For a compact closed manifold M, let $\mathcal{A}_0^m(M)$ denote the set of those continuous linear operators $A : C^\infty(M) \to C^\infty(M)$ which are bounded from $H^m(M)$ to $L^2(M)$. Recursively define $\mathcal{A}_{k+1}^m(M) \subset \mathcal{A}_k^m(M)$ such that $A \in \mathcal{A}_k^m(M)$ belongs to $\mathcal{A}_{k+1}^m(M)$ if and only if $[A, D] = AD - DA \in \mathcal{A}_k^m(M)$ for every smooth vector field D on M. Now we will use a variant of the commutator characterization of pseudo-differential operators (see e.g. [2, 3, 4, 5, 30]), but we will need the following Sobolev space version proved in [31], assuring that the behaviour of commutators in Sobolev spaces characterizes pseudo-differential operators:

Theorem 8.2. *A continuous linear operator $A : C^\infty(M) \to C^\infty(M)$ belongs to $\Psi^m(M)$ if and only if $A \in \bigcap_{k=0}^\infty \mathcal{A}_k^m(M)$.*

In such characterization on a compact Lie group $M = G$, it suffices to consider vector fields of the form $D = M_\phi \partial_x$, where $M_\phi f := \phi f$ is multiplication by $\phi \in C^\infty(G)$, and ∂_x is left-invariant. Notice that

$$[A, M_\phi \partial_x] = M_\phi \; [A, \partial_x] + [A, M_\phi] \; \partial_x,$$

where $[A, M_\phi]f = A(\phi f) - \phi A f$. Hence we need to consider compositions $M_\phi A$, AM_ϕ, $A \circ \partial_x$ and $\partial_x \circ A$. First, we observe that

$$\sigma_{M_\phi A}(x,\xi) = \phi(x) \; \sigma_A(x,\xi), \tag{8.2}$$

$$\sigma_{A \circ \partial_x}(x,\xi) = \sigma_A(x,\xi) \; \sigma_{\partial_x}(x,\xi), \tag{8.3}$$

$$\sigma_{\partial_x \circ A}(x,\xi) = \sigma_{\partial_x}(x,\xi) \; \sigma_A(x,\xi) + (\partial_x \sigma_A)(x,\xi), \tag{8.4}$$

where $\sigma_{\partial_x}(x,\xi)$ is independent of $x \in G$. Here (8.4) follows by the Leibnitz formula:

$$\begin{aligned} \partial_x \circ Af(x) &= \partial_x \sum_{[\xi] \in \widehat{G}} \dim(\xi) \; \mathrm{Tr}\left(\xi(x) \; \sigma_A(x,\xi) \; \widehat{f}(\xi)\right) \\ &= \sum_{[\xi] \in \widehat{G}} \dim(\xi) \; \mathrm{Tr}\left((\partial_x \xi)(x) \; \sigma_A(x,\xi) \; \widehat{f}(\xi)\right) \\ &\quad + \sum_{[\xi] \in \widehat{G}} \dim(\xi) \; \mathrm{Tr}\left(\xi(x) \; \partial_x \sigma_A(x,\xi) \; \widehat{f}(\xi)\right). \end{aligned}$$

Next we claim that we have the fomula

$$\sigma_{AM_\phi}(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \; \triangle_\xi^\alpha \sigma_A(x,\xi) \; \partial_x^{(\alpha)} \phi(x), \tag{8.5}$$

where $\partial_x^{(\alpha)}$ are certain partial differential operators of order $|\alpha|$. This will follow from the following general composition formula:

Theorem 8.3. *Let $m_1, m_2 \in \mathbb{R}$ and $\rho > \delta \geq 0$. Let $A, B : C^\infty(G) \to C^\infty(G)$ be continuous and linear, their symbols satisfying*

$$\begin{aligned} \left\| \triangle_\xi^\alpha \sigma_A(x,\xi) \right\|_{op} &\leq C_\alpha \ \langle \xi \rangle^{m_1 - \rho|\alpha|}, \\ \left\| \partial_x^\beta \sigma_B(x,\xi) \right\|_{op} &\leq C_\beta \ \langle \xi \rangle^{m_2 + \delta|\beta|}, \end{aligned}$$

for all multi-indices α and β, uniformly in $x \in G$ and $[\xi] \in \widehat{G}$. Then

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \ (\triangle_\xi^\alpha \sigma_A)(x,\xi) \ \partial_x^{(\alpha)} \sigma_B(x,\xi), \tag{8.6}$$

where the asymptotic expansion means that for every $N \in \mathbb{N}$ we have

$$\left\| \sigma_{AB}(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \ (\triangle_\xi^\alpha \sigma_A)(x,\xi) \ \partial_x^{(\alpha)} \sigma_B(x,\xi) \right\|_{op} \leq C_N \langle \xi \rangle^{m_1 + m_2 - (\rho - \delta)N}.$$

Proof. First,

$$\begin{aligned} ABf(x) &= \int_G (Bf)(xz) \ R_A(x, z^{-1}) \ \mathrm{d}z \\ &= \int_G \int_G f(xy^{-1}) \ R_B(xz, yz) \ \mathrm{d}y \ R_A(x, z^{-1}) \ \mathrm{d}z, \end{aligned}$$

where we use the standard distributional interpretation of integrals. Hence

$$\begin{aligned} \sigma_{AB}(x,\xi) &= \int_G R_{AB}(x,y) \ \xi(y)^* \ \mathrm{d}y \\ &= \int_G \int_G R_A(x, z^{-1}) \ \xi(z^{-1})^* \ R_B(xz, yz) \ \xi(yz)^* \ \mathrm{d}z \ \mathrm{d}y \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} \int_G \int_G R_A(x, z^{-1}) \ q_\alpha(z^{-1}) \ \xi(z^{-1})^* \ \partial_x^{(\alpha)} R_B(x, yz) \ \xi(yz)^* \ \mathrm{d}z \ \mathrm{d}y \\ &\quad + \sum_{|\alpha|=N} \int_G \int_G R_A(x, z^{-1}) \ q_\alpha(z^{-1}) \ \xi(z^{-1})^* \ u_\alpha(x, yz) \ \xi^*(yz) \ \mathrm{d}z \ \mathrm{d}y \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} \ (\triangle_\xi^\alpha \sigma_A)(x,\xi) \ \partial_x^{(\alpha)} \sigma_B(x,\xi) + \sum_{|\alpha|=N} (\triangle_\xi^\alpha \sigma_A)(x,\xi) \ \widehat{u_\alpha}(x,\xi). \end{aligned}$$

Now the statement follows because we have $\|\widehat{u_\alpha}(x,\xi)\|_{op} \leq C \langle \xi \rangle^{m_1 + \delta N}$ since $u_\alpha(x,y)$ is the remainder in the Taylor expansion of $R_B(x,y)$ in x only and so it satisfies similar estimates to those of σ_B with respect to ξ. This completes the proof. □

Before discussing symbol classes, let us complement Theorem 8.3 with a result about adjoint operators:

Theorem 8.4. *Let $m \in \mathbb{R}$ and $\rho > \delta \geq 0$. Let $A : C^\infty(G) \to C^\infty(G)$ be continuous and linear, with symbol σ_A satisfying*

$$\left\| \triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi) \right\|_{op} \leq C_\alpha \ \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}, \tag{8.7}$$

for all multi-indices α, uniformly in $x \in G$ and $[\xi] \in \widehat{G}$. Then the symbol of A^* is

$$\sigma_{A^*}(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \ \triangle_\xi^\alpha \partial_x^{(\alpha)} \sigma_A(x,\xi)^*, \tag{8.8}$$

where the asymptotic expansion means that for every $N \in \mathbb{N}$ we have

$$\left\| \triangle_\xi^\gamma \partial_x^\beta \left(\sigma_A(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \ \triangle_\xi^\alpha \partial_x^{(\alpha)} \sigma_A(x,\xi)^* \right) \right\|_{op} \leq C_N \langle \xi \rangle^{m-(\rho-\delta)N-\rho|\gamma|+\delta|\beta|}.$$

Remark 8.5. We note that if we impose conditions of the type (8.7) on both symbols σ_A, σ_B in Theorem 8.3, we also get the asymptotic expansion (8.6) with the remainder estimate as in Theorem 8.4.

Proof of Theorem 8.4. First we observe that writing $A^* g(y) = \int_G g(x) R_{A^*}(y, x^{-1}y) \, \mathrm{d}x$, we get the relation $R_{A^*}(y, x^{-1}y) = \overline{R_A(x, y^{-1}x)}$ between kernels, which means that $R_{A^*}(x,v) = \overline{R_A(xv^{-1}, v^{-1})}$. From this we find

$$\begin{aligned}
\sigma_{A^*}(x,\xi) &= \int_G R_{A^*}(x,v) \ \xi(v)^* \ \mathrm{d}v \\
&= \int_G \overline{R_A(xv^{-1}, v^{-1})} \ \xi(v)^* \ \mathrm{d}v \\
&= \sum_{|\alpha|<N} \frac{1}{\alpha!} \int_G q_\alpha(v) \ \partial_x^{(\alpha)} \overline{R_A(x, v^{-1})} \ \xi(v)^* \mathrm{d}v + \mathcal{R}_N(x,\xi) \\
&= \sum_{|\alpha|<N} \frac{1}{\alpha!} \ \triangle_\xi^\alpha \partial_x^{(\alpha)} \sigma_A(x,\xi)^* + \mathcal{R}_N(x,\xi),
\end{aligned}$$

where the last formula for the asymptotic expansion follows in view of

$$\sigma_A(x,\xi)^* = \left(\int_G R_A(x,v) \ \xi^*(v) \ \mathrm{d}v \right)^* = \int_G \overline{R_A(x, v^{-1})} \ \xi^*(v) \ \mathrm{d}v,$$

and estimate for the remainder $\mathcal{R}_N(x,\xi)$ follows by an argument similar to that in the proof of Theorem 8.3. □

On the way to characterize the usual Hörmander's classes $\Psi^m(G)$ in Theorem 9.2, we need some properties concerning symbols of pseudo-differential operators.

Lemma 8.6. *Let $A \in \Psi^m(G)$. Then there exists a constant $C < \infty$ such that*

$$\|\sigma_A(x,\xi)\|_{op} \leq C \langle \xi \rangle^m$$

for all $x \in G$ and $\xi \in \mathrm{Rep}(G)$. Also, if $u \in G$ and if B is an operator with symbol $\sigma_B(x,\xi) = \sigma_A(u,\xi)$, then $B \in \Psi^m(G)$.

Proof. First, $B \in \Psi^m(G)$ follows from the local theory of pseudo-differential operators, by studying $Bf(x) = \int_G K_A(u, ux^{-1}y)\ f(y)\ \mathrm{d}y$. Hence the right-convolution operator B is bounded from $H^s(M)$ to $H^{s-m}(G)$, implying $\|\sigma_A(u,\xi)\| \leq C\langle\xi\rangle^m$. □

Lemma 8.7. *Let $A \in \Psi^m(G)$. Then $Op(\triangle_\xi^\alpha \partial_x^\beta \sigma_A) \in \Psi^{m-|\alpha|}(G)$ for all α, β.*

Proof. First, given $A \in \Psi^m(G)$, let us define $\sigma_B(x,\xi) = \triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)$. We must show that $B \in \Psi^{m-|\alpha|}(G)$. If here $|\beta| = 0$, we obtain

$$Bf(x) = \int_G f(xy^{-1})\ q_\alpha(y)\ R_A(x,y)\ \mathrm{d}y = \int_G q_\alpha(y^{-1}x)\ K_A(x,y)\ f(y)\ \mathrm{d}y.$$

Moving to local coordinates, we need to study

$$\tilde{B}f(x) = \int_{\mathbb{R}^n} \phi(x,y)\ K_{\tilde{A}}(x,y)\ f(y)\ \mathrm{d}y,$$

where $\tilde{A} \in \Psi^m(\mathbb{R}^n)$ with $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the kernel $K_{\tilde{A}}$ being compactly supported. Let us calculate the symbol of $\tilde{B}$:

$$\begin{aligned} \sigma_{\tilde{B}}(x,\xi) &= \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}2\pi(y-x)\cdot\xi}\ \phi(x,y)\ K_{\tilde{A}}(x,y)\ \mathrm{d}y \\ &\sim \sum_{\gamma\geq 0} \frac{1}{\gamma!}\ \partial_z^\gamma \phi(x,z)|_{z=x} \int_G \mathrm{e}^{\mathrm{i}2\pi(y-x)\cdot\xi}\ (y-x)^\gamma\ K_{\tilde{A}}(x,y)\ \mathrm{d}y \\ &= \sum_{\gamma\geq 0} \frac{1}{\gamma!}\ \partial_z^\gamma \phi(x,z)|_{y=x}\, D_\xi^\gamma \sigma_{\tilde{A}}(x,\xi). \end{aligned}$$

This shows that $\tilde{B} \in \Psi^m(\mathbb{R}^n)$. We obtain $Op(\triangle_\xi^\alpha \sigma_A) \in \Psi^{m-|\alpha|}(G)$ if $A \in \Psi^m(G)$.

Next we show that $B = Op(\partial_x^\beta \sigma_A) \in \Psi^m(G)$. We may assume that $|\beta| = 1$. Left-invariant vector field ∂_x^β is a linear combination of terms of the type $c(x)D_x$, where $c \in C^\infty(G)$ and D_x is right-invariant. By the previous considerations on $\tilde{B}$, we may remove $c(x)$ here, and consider only $C = Op(D_x\sigma_A)$. Since $R_A(x,y) = K_A(x, xy^{-1})$, we get

$$D_x R_A(x,y) = (D_x + D_z) K_A(x,z)|_{z=xy^{-1}}\,,$$

leading to

$$Cf(x) = \int_G f(xy^{-1})\ D_x R_A(x,y)\ \mathrm{d}y = \int_G f(y)\ (D_x + D_y) K_A(x,y)\ \mathrm{d}y.$$

Thus, we study local operators of the form

$$\tilde{C}f(x) = \int_{\mathbb{R}^n} f(y) \left(\phi(x,y)\partial_x^\beta + \psi(x,y)\partial_y^\beta\right) K_{\tilde{A}}(x,y)\ \mathrm{d}y,$$

where the kernel of $\tilde{A} \in \Psi^m(\mathbb{R}^n)$ has compact support, $\phi, \psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $\phi(x,x) = \psi(x,x)$ for every $x \in \mathbb{R}^n$. Let $\tilde{C} = \tilde{D} + \tilde{E}$, where

$$\begin{aligned} \tilde{D}f(x) &= \int_{\mathbb{R}^n} f(y)\ \phi(x,y) \left(\partial_x^\beta + \partial_y^\beta\right) K_{\tilde{A}}(x,y)\ \mathrm{d}y, \\ \tilde{E}f(x) &= \int_{\mathbb{R}^n} f(y)\ \left(\psi(x,y) - \phi(x,y)\right)\ \partial_y^\beta K_{\tilde{A}}(x,y)\ \mathrm{d}y. \end{aligned}$$

By the above considerations about $\tilde{B}$, we may assume that $\phi(x,y) \equiv 1$ here, and obtain $\sigma_{\tilde{D}}(x,\xi) = \partial_x^\beta \sigma_{\tilde{A}}(x,\xi)$. Thus $\tilde{D} \in \Psi^m(\mathbb{R}^n)$. Moreover,

$$\tilde{E}f(x) \;\sim\; \sum_{\gamma\geq 0} \frac{1}{\gamma!}\; \partial_z^\gamma\left(\psi(x,z)-\phi(x,z)\right)\big|_{z=x} \int_{\mathbb{R}^n} f(y)\;(y-x)^\gamma\; \partial_y^\beta K_{\tilde{A}}(x,y)\;\mathrm{d}y,$$

yielding

$$\sigma_{\tilde{E}}(x,\xi) \;\sim\; \sum_{\gamma\geq 0} c_\gamma(x)\; \partial_\xi^\gamma\left(\xi^\beta\;\sigma_{\tilde{A}}(x,\xi)\right)$$

for some functions $c_\gamma \in C^\infty(\mathbb{R}^n)$ for which $c_0(x) \equiv 0$. Since $|\beta| = 1$, this shows that $\tilde{E} \in \Psi^m(\mathbb{R}^n)$. Thus $Op(\partial_x^\beta \sigma_A) \in \Psi^m(G)$ if $A \in \Psi^m(G)$. □

Lemma 8.8. *Let $A \in \Psi^m(G)$ and let $D : C^\infty(G) \to C^\infty(G)$ be a smooth vector field. Then $Op(\sigma_A\sigma_D) \in \Psi^{m+1}(G)$ and $Op([\sigma_A,\sigma_D]) \in \Psi^m(G)$.*

Proof. For simplicity, we may assume that $D = M_\phi \partial_x$, where ∂_x is left-invariant and $\phi \in C^\infty(G)$. Now

$$\sigma_A(x,\xi)\;\sigma_D(x,\xi) = \phi(x)\;\sigma_A(x,\xi)\;\sigma_{\partial_x}(\xi) = \sigma_{M_\phi A\circ\partial_x}(x,\xi),$$

and it is well-known that $M_\phi A \circ \partial_x \in \Psi^{m+1}(G)$. Thus $Op(\sigma_A\sigma_D) \in \Psi^{m+1}(G)$. Next,

$$\begin{aligned}
\sigma_D(x,\xi)\;\sigma_A(x,\xi) &= \phi(x)\;\sigma_{\partial_x}(\xi)\;\sigma_A(x,\xi) \\
&\overset{(8.4)}{=} \phi(x)\left(\sigma_{\partial_x\circ A}(x,\xi) - (\partial_x\sigma_A)(x,\xi)\right) \\
&= \sigma_{M_\phi\circ\partial_x\circ A}(x,\xi) - \phi(x)\;(\partial_x\sigma_A)(x,\xi).
\end{aligned}$$

From this we see that

$$\begin{aligned}
Op([\sigma_A,\sigma_D]) &= M_\phi A\circ\partial_x - M_\phi\partial_x\circ A + M_\phi\;Op(\partial_x\sigma_A) \\
&= M_\phi[A,\partial_x] + M_\phi\;Op(\partial_x\sigma_A).
\end{aligned}$$

Here $Op(\partial_x\sigma_A) \in \Psi^m(G)$ by Lemma 8.7. Hence $Op([\sigma_A,\sigma_D])$ belongs to $\Psi^m(G)$ by the known properties of pseudo-differential operators. □

Finally, let us prove the Sobolev space boundedness of pseudo-differential operators given in Theorem 3.2.

Proof of Theorem 3.2. Observing the continuous mapping $\Xi^s : H^s(G) \to L^2(G)$, we have to prove that operator $\Xi^{s-\mu}\circ A\circ\Xi^{-s}$ is bounded from $L^2(G)$ to $L^2(G)$. Let us denote $B = A\circ\Xi^{-s}$, so that the symbol of B satisfies $\sigma_B(x,\xi) = \langle\xi\rangle^{-s}\sigma_A(x,\xi)$ for all $x \in G$ and $\xi \in \mathrm{Rep}(G)$, where $\langle\xi\rangle$ is defined in (3.2). Since $\Xi^{s-\mu} \in \Psi^{s-\mu}(G)$, by (3.1) and Lemma 8.7 its symbol satisfies

$$\|\triangle_\xi^\alpha \sigma_{\Xi^{s-\mu}}(x,\xi)\|_{op} \leq C'_\alpha\;\langle\xi\rangle^{s-\mu-|\alpha|}. \tag{8.9}$$

Now we can observe that the asymptotic formula in Theorem 8.3 works for the composition $\Xi^{s-\mu}\circ B$ in view of (8.9), and we obtain

$$\partial_x^\beta \sigma_{\Xi^{s-\mu}\circ B}(x,\xi) \;\sim\; \sum_{\alpha\geq 0}\frac{1}{\alpha!}\left(\triangle_\xi^\alpha\sigma_{\Xi^{s-\mu}}(x,\xi)\right)\;\langle\xi\rangle^{-s}\;\partial_x^{(\alpha)}\partial_x^\beta\sigma_A(x,\xi).$$

It follows that

$$\left\|\partial_x^\beta\sigma_{\Xi^{s-\mu}\circ B}(x,\xi)\right\|_{op} \leq C''_\beta,$$

so that $\Xi^{s-\mu}\circ B$ is bounded on $L^2(G)$ by Theorem 3.1. This completes the proof. $\square$

9. Symbol classes on compact Lie groups

The goal of this section is to describe the pseudo-differential symbol inequalities on compact Lie groups that yield Hörmander's classes $\Psi^m(G)$. Combined with asymptotic expansion (8.6) for composing operators, Theorem 8.2 motivates defining the following symbol classes $\Sigma^m(G)=\bigcap_{k=0}^\infty \Sigma^m_k(G)$, that we will show to characterize Hörmander's class $\Psi^m(G)$.

Definition 9.1. Let $m\in\mathbb{R}$. We denote $\sigma_A\in\Sigma^m_0(G)$ if

$$\text{sing supp}\,(y\mapsto R_A(x,y))\subset\{e\} \tag{9.1}$$

and if

$$\|\triangle_\xi^\alpha\partial_x^\beta\sigma_A(x,\xi)\|_{op}\leq C_{A\alpha\beta m}\ \langle\xi\rangle^{m-|\alpha|}, \tag{9.2}$$

for all $x\in G$, all multi-indices α,β, and all $\xi\in\mathrm{Rep}(G)$, where $\langle\xi\rangle$ is defined in (3.2). Then we say that $\sigma_A\in\Sigma^m_{k+1}(G)$ if and only if

$$\sigma_A \in \Sigma^m_k(G), \tag{9.3}$$

$$\sigma_{\partial_j}\sigma_A-\sigma_A\sigma_{\partial_j} \in \Sigma^m_k(G), \tag{9.4}$$

$$(\triangle_\xi^\gamma\sigma_A)\ \sigma_{\partial_j} \in \Sigma^{m+1-|\gamma|}_k(G), \tag{9.5}$$

for all $|\gamma|>0$ and $1\leq j\leq\dim(G)$. Let

$$\Sigma^m(G)=\bigcap_{k=0}^\infty\Sigma^m_k(G).$$

Let us denote $A\in Op\Sigma^m(G)$ if and only if $\sigma_A\in\Sigma^m(G)$.

Theorem 9.2. *Let G be a compact Lie group and let $m\in\mathbb{R}$. Then $A\in\Psi^m(G)$ if and only if $\sigma_A\in\Sigma^m(G)$, i.e. $Op\Sigma^m(G)=\Psi^m(G)$.*

Proof. First, applying Theorem 8.3 to $\sigma_A\in\Sigma^m_{k+1}(G)$, we notice that $[A,D]\in Op\Sigma^m_k(G)$ for any smooth vector field $D:C^\infty(G)\to C^\infty(G)$. Consequently, if here $A\in Op\Sigma^m(G)$ then also $[A,D]\in Op\Sigma^m(G)$. By Remark 3.3, $Op\Sigma^m(G)\subset\mathcal{L}(H^m(G),L^2(G))$. Hence Theorem 8.2 implies $Op\Sigma^m(G)\subset\Psi^m(G)$.

Conversely, we have to show that $\Psi^m(G)\subset Op\Sigma^m(G)$. This follows by Lemma 8.6, and Lemmas 8.7 and 8.8. More precisely, let $A\in\Psi^m(G)$. Then we have

$$\begin{aligned} Op(\triangle_\xi^\alpha\partial_x^\beta\sigma_A) &\in \Psi^{m-|\alpha|}(G),\\ Op\left([\sigma_{\partial_j},\sigma_A]\right) &\in \Psi^m(G),\\ Op\left((\triangle_\xi^\gamma\sigma_A)\sigma_{\partial_j}\right) &\in \Psi^{m+1-|\gamma|}(G).\end{aligned}$$

Moreover, $\|\sigma_A(x,\xi)\|\leq C\langle\xi\rangle^m$ by Lemma 8.6, and the singular support $y\mapsto R_A(x,y)$ is contained in $\{e\}\subset G$. This completes the proof. $\square$

Corollary 9.3. *The set $\Sigma^m(G)$ is invariant under x-freezings, x-translations and ξ-conjugations. More precisely, if $(x,\xi)\mapsto\sigma_A(x,\xi)$ belongs to $\Sigma^m(G)$ and $u\in G$ then also the following symbols belong to $\Sigma^m(G)$:*

$$(x,\xi)\ \mapsto\ \sigma_A(u,\xi), \tag{9.6}$$

$$(x,\xi) \mapsto \sigma_A(ux,\xi), \tag{9.7}$$
$$(x,\xi) \mapsto \sigma_A(xu,\xi), \tag{9.8}$$
$$(x,\xi) \mapsto \xi(u)^*\,\sigma_A(x,\xi)\,\xi(u). \tag{9.9}$$

Proof. The symbol classes $\Sigma^m(G)$ are defined by conditions (9.1)-(9.5), which are checked for points $x \in G$ fixed (with constants uniform in x). Therefore it follows that $\Sigma^m(G)$ is invariant under the x-freezing (9.6), and under the left and right x-translations (9.7),(9.8). The x-freezing property (9.6) would have followed also from Lemma 8.6 and Theorem 9.2. From the general theory of pseudo-differential operators it follows that $A \in \Psi^m(G)$ if and only if the ϕ-pullback A_ϕ belongs to the same class $\Psi^m(G)$, where $A_\phi f = A(f\circ\phi)\circ\phi^{-1}$. This, combined with the x-translation invariances and Proposition 2.10, implies the conjugation invariance in (9.9). □

From Theorem 9.2 and Lemma 8.7 we also obtain:

Corollary 9.4. *If $\sigma_A \in \Sigma^m(G)$ then $\triangle_\xi^\alpha \partial_x^\beta \sigma_A \in \Sigma^{m-|\alpha|}(G)$.*

10. Symbol classes on SU(2)

Let us now turn to the analysis on SU(2). In this section we derive a much simpler symbolic characterization of pseudo-differential operators on SU(2) than the one given in Definition 9.1. First we summarize the approach in the case of SU(2) also simplifying the notation in this case.

By the Peter-Weyl theorem $\{\sqrt{2l+1}\,t^l_{nm} :\ l \in \frac{1}{2}\mathbb{N}_0,\ -l \le m,n \le l,\ l-m, l-n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathrm{SU}(2))$, where t^l were defined in Section 4, and thus $f \in C^\infty(\mathrm{SU}(2))$ has a Fourier series representation

$$f = \sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1) \sum_m \sum_n \widehat{f}(l)_{mn}\ t^l_{nm},$$

where the Fourier coefficients are computed by

$$\widehat{f}(l)_{mn} := \int_{\mathrm{SU}(2)} f(g)\ \overline{t^l_{nm}(g)}\ \mathrm{d}g = \langle f, t^l_{nm}\rangle_{L^2(\mathrm{SU}(2))},$$

so that $\widehat{f}(l) \in \mathbb{C}^{(2l+1)\times(2l+1)}$. We recall that in the case of SU(2), we simplify the notation writing $\widehat{f}(l)$ instead of $\widehat{f}(t^l)$, etc.

Let $A : C^\infty(\mathrm{SU}(2)) \to C^\infty(\mathrm{SU}(2))$ be a continuous linear operator and let $R_A \in \mathcal{D}'(\mathrm{SU}(2)\times\mathrm{SU}(2))$ be its right-convolution kernel, i.e.

$$Af(x) = \int_{\mathrm{SU}(2)} f(y)\ R_A(x, y^{-1}x)\ \mathrm{d}y = (f * R_A(x,\cdot))(x)$$

in the sense of distributions. According to Definition 2.3, by the symbol of A we mean the sequence of matrix-valued mappings

$$(x \mapsto \sigma_A(x,l)) : \mathrm{SU}(2) \to \mathbb{C}^{(2l+1)\times(2l+1)},$$

where $2l \in \mathbb{N}_0$, obtained from

$$\sigma_A(x,l)_{mn} = \int_{\mathrm{SU}(2)} R_A(x,y)\ \overline{t^l_{nm}(y)}\ \mathrm{d}y.$$

That is, $\sigma_A(x,l)$ is the l^{th} Fourier coefficient of the function $y \mapsto R_A(x,y)$. Then by Theorem 2.4 we have

$$\begin{aligned} Af(x) &= \sum_l (2l+1) \operatorname{Tr}\left(t^l(x)\ \sigma_A(x,l)\ \widehat{f}(l) \right) \\ &= \sum_l (2l+1) \sum_{m,n} t^l(x)_{nm} \left(\sum_k \sigma_A(x,l)_{mk}\ \widehat{f}(l)_{kn} \right). \end{aligned}$$

Alternatively, by Theorem 2.5 we have

$$\sigma_A(x,l) = t^l(x)^* \ \left(At^l\right)(x),$$

that is

$$\sigma_A(x,l)_{mn} = \sum_k \overline{t^l_{km}(x)} (At^l_{kn})(x).$$

In the case of SU(2), quantity $\langle t^l \rangle$ for $\xi = t^l$ in (3.2) can be calculated as

$$\langle t^l \rangle = (1+\lambda_{[t^l]})^{1/2} = (1+l(l+1))^{1/2}$$

in view of Theorem 5.7, and Definition 9.1 becomes:

Definition 10.1. We write that symbol $\sigma_A \in \Sigma^m_0(\mathrm{SU}(2))$ if

$$\operatorname{sing\,supp}\left(y \mapsto R_A(x,y)\right) \subset \{e\} \tag{10.1}$$

and if

$$\left\| \triangle_l^\alpha \partial_x^\beta \sigma_A(x,l) \right\|_{\mathbb{C}^{2l+1}\to\mathbb{C}^{2l+1}} \le C_{A\alpha\beta m}\ (1+l)^{m-|\alpha|} \tag{10.2}$$

for all $x \in G$, all multi-indices α, β, and $l \in \frac{1}{2}\mathbb{N}_0$. Here $\triangle_l^\alpha = \triangle_0^{\alpha_1} \triangle_+^{\alpha_2} \triangle_-^{\alpha_3}$ and $\partial_x^\beta = \partial_0^{\beta_1}\partial_+^{\beta_2}\partial_-^{\beta_3}$. Moreover, $\sigma_A \in \Sigma^m_{k+1}(\mathrm{SU}(2))$ if and only if

$$\sigma_A \in \Sigma^m_k(\mathrm{SU}(2)), \tag{10.3}$$

$$[\sigma_{\partial_j}, \sigma_A] = \sigma_{\partial_j}\sigma_A - \sigma_A\sigma_{\partial_j} \in \Sigma^m_k(\mathrm{SU}(2)), \tag{10.4}$$

$$\left(\triangle_l^\gamma \sigma_A\right) \sigma_{\partial_j} \in \Sigma^{m+1-|\gamma|}_k(\mathrm{SU}(2)), \tag{10.5}$$

for all $|\gamma| > 0$ and $j \in \{0,+,-\}$. Let

$$\Sigma^m(\mathrm{SU}(2)) = \bigcap_{k=0}^\infty \Sigma^m_k(\mathrm{SU}(2)),$$

so that by Theorem 9.2 we have $Op\Sigma^m(\mathrm{SU}(2)) = \Psi^m(\mathrm{SU}(2))$.

Remark 10.2. We would like to provide a more direct definition for $\Sigma^m(\mathrm{SU}(2))$, without resorting to classes $\Sigma^m_k(\mathrm{SU}(2))$. Condition (10.2) is just an analogy of the usual symbol inequalities. Conditions (10.1) and (10.3) are straightforward. We may have difficulties with differences $\triangle_l^\alpha$, but derivatives ∂_x^β do not cause problems; if we want, we may assume that the symbols are constant in x. By the definition of operators $\triangle_l^\alpha$ and ∂_x^β we also have the following properties:

$$\triangle_l^\alpha \partial_x^\beta \sigma_A(x,l) = \partial_x^\beta \triangle_l^\alpha \sigma_A(x,l),$$

$$\partial_j \left(\sigma_A(x,l)\ \sigma_B(x,l)\right) = \left(\partial_j \sigma_A(x,l)\right) \sigma_B(x,l) + \sigma_A(x,l)\ \partial_j \sigma_A(x,l),$$

$$\partial_y^\beta \left(\sigma_A(x,l)\ \sigma_B(y,l)\ \sigma_C(z,l)\right) = \sigma_A(x,l) \left(\partial_y^\beta \sigma_B(y,l)\right) \sigma_C(z,l).$$

We now give another, simpler characterization of pseudo-differential operators.

Definition 10.3 (Symbol classes on SU(2))**.** For $u \in \mathrm{SU}(2)$, denote $A_u f := A(f \circ \phi) \circ \phi^{-1}$, where $\phi(x) = xu$; then (by Proposition 2.10)

$$\begin{aligned} R_{A_u}(x,y) &= R_A(xu^{-1}, uyu^{-1}), \\ \sigma_{A_u}(x,l) &= t^l(u)^* \ \sigma_A(xu^{-1}, l) \ t^l(u). \end{aligned}$$

The symbol class $S^m(\mathrm{SU}(2))$ consists of the symbols σ_A of those operators $A \in \mathcal{L}(C^\infty(\mathrm{SU}(2)))$ for which $(y \mapsto R_A(x,y)) \subset \{e\}$ and for which

$$\left| \triangle_l^\alpha \partial_x^\beta \sigma_{A_u}(x,l)_{ij} \right| \le C_{A\alpha\beta mN} \ \langle i-j \rangle^{-N} \ (1+l)^{m-|\alpha|} \tag{10.6}$$

uniformly in $x, u \in \mathrm{SU}(2)$, for every $N \ge 0$, all $l \in \frac{1}{2}\mathbb{N}_0$, every multi-indices $\alpha, \beta \in \mathbb{N}_0^3$, and for all matrix column/row numbers i, j. Thus, the constant in (10.6) may depend on A, α, β, m and N, but not on x, u, l, i, j.

We now formulate the main theorem of this section:

Theorem 10.4. *Operator $A \in \mathcal{L}(C^\infty(\mathrm{SU}(2)))$ belongs to $\Psi^m(\mathrm{SU}(2))$ if and only if $\sigma_A \in S^m(\mathrm{SU}(2))$. Moreover, we have the equality of symbol classes $S^m(\mathrm{SU}(2)) = \Sigma^m(\mathrm{SU}(2))$.*

In fact, we need to prove only the equality of symbol classes $S^m(\mathrm{SU}(2)) = \Sigma^m(\mathrm{SU}(2))$, from which the first part of the theorem would follow by Theorem 9.2. In the process of proving this equality, we establish a number of auxiliary results.

Remark 10.5. By Corollary 9.4, if $\sigma_A \in \Sigma^m(\mathrm{SU}(2))$ then $\triangle_l^\gamma \partial_x^\delta \sigma_A \in \Sigma^{m-|\delta|}(\mathrm{SU}(2))$. Let us show the analogous result for $S^m(\mathrm{SU}(2))$.

Lemma 10.6. *If $\sigma_A \in S^m(\mathrm{SU}(2))$ then $\sigma_B = \triangle_l^\gamma \partial_x^\delta \sigma_A \in S^{m-|\gamma|}(\mathrm{SU}(2))$.*

Proof. First, let $|\gamma| = 1$. Then $\triangle_l^\gamma \widehat{f}(l) = \widehat{qf}(l)$ for some

$$q \in Pol_1(\mathrm{SU}(2)) := \mathrm{span}\left\{ t_{ij}^{1/2} : \ i,j \in \{-1/2, +1/2\} \right\}$$

for which $q(e) = 0$. Let $r(y) := q(uyu^{-1})$. Then $r \in Pol_1(\mathrm{SU}(2))$, because

$$t_{ij}^{1/2}(uyu^{-1}) = \sum_{k,m} t_{ik}^{1/2}(u) \ t_{km}^{1/2}(y) \ t_{mj}^{1/2}(u^{-1}).$$

Moreover, we have $r(e) = 0$. Hence $\widehat{f}(l) \mapsto \widehat{rf}(l)$ is a linear combination of difference operators $\triangle_0, \triangle_+, \triangle_-$ because $\{f \in Pol^1(\mathrm{SU}(2)) : \ f(e) = 0\}$ is a three-dimensional vector space spanned by q_0, q_+, q_-. Now let $\gamma \in \mathbb{N}_0^3$ and $\sigma_B = \triangle_l^\gamma \partial_x^\delta \sigma_A$. We have

$$\begin{aligned} \triangle_l^\alpha \partial_x^\beta \sigma_{B_u}(x,l) &= \triangle_l^\alpha \partial_x^\beta \left(t^l(u)^* \ \sigma_B(xu^{-1}, l) \ t^l(u) \right) \\ &= \triangle_l^\alpha \partial_x^\beta \left(t^l(u)^* \left(\triangle_l^\gamma \partial_x^\delta \sigma_A(xu^{-1}, l) \right) t^l(u) \right) \\ &= \sum_{|\gamma'| = |\gamma|} \lambda_{u,\gamma'} \ \triangle_l^{\alpha+\gamma'} \partial_x^{\beta+\delta} \ \sigma_{A_u}(x,l), \end{aligned}$$

for some scalars $\lambda_{u,\gamma'} \in \mathbb{C}$ depending only on $u \in \mathrm{SU}(2)$ and multi-indices $\gamma' \in \mathbb{N}_0^3$. $\square$

Remark 10.7. Let D be a left-invariant vector field on SU(2). From the very definition of the symbol classes $\Sigma^m(\mathrm{SU}(2)) = \bigcap_{k=0}^{\infty} \Sigma_k^m(\mathrm{SU}(2))$, it is evident that $[\sigma_D, \sigma_A] \in \Sigma^m(\mathrm{SU}(2))$ if $\sigma_A \in \Sigma^m(\mathrm{SU}(2))$. We shall next prove the similar invariance for $S^m(\mathrm{SU}(2))$.

Lemma 10.8. *Let D be a left-invariant vector field on* SU(2)*. Let $\sigma_A \in S^m(\mathrm{SU}(2))$. Then $[\sigma_D, \sigma_A] \in S^m(\mathrm{SU}(2))$ and $\sigma_A\ \sigma_D \in S^{m+1}(\mathrm{SU}(2))$.*

Proof. For $D \in \mathfrak{su}(2)$ we write $D = \mathrm{i}E$, so that $E \in \mathrm{i}\ \mathfrak{su}(2)$. By Proposition 5.4 there is some $u \in \mathrm{SU}(2)$ such that $\sigma_E(l) = t^l(u)^*\ \sigma_{\partial_0}(l)\ t^l(u)$. Now, we have

$$
\begin{aligned}
[\sigma_E, \sigma_A](l) &= t^l(u)^*\ \left[\sigma_{\partial_0}(l), t^l(u)\ \sigma_A(x,l)\ t^l(u)^*\right]\ t^l(u) \\
&= \left[\sigma_{\partial_0}, \sigma_{A_{u^{-1}}}\right]_u (l).
\end{aligned}
$$

Next, notice that $S^m(\mathrm{SU}(2))$ is invariant under the mappings $\sigma_B \mapsto \sigma_{B_u}$ and $\sigma_B \mapsto [\sigma_{\partial_0}, \sigma_B]$; here $[\sigma_{\partial_0}, \sigma_B](l)_{ij} = (i-j)\ \sigma_B(l)_{ij}$. Finally,

$$
\begin{aligned}
\sigma_A(x,l)\ \sigma_E(l) &= t^l(u)^*\ t^l(u)\ \sigma_A(x,l)\ t^l(u)^*\ \sigma_{\partial_0}(l)\ t^l(u) \\
&= \left(\sigma_{A_{u^{-1}}}(x,l)\ \sigma_{\partial_0}(l)\right)_u .
\end{aligned}
$$

Just like in the first part of the proof, we see that $\sigma_A\ \sigma_D$ belongs to $S^{m+1}(\mathrm{SU}(2))$ since $\sigma_B\ \sigma_{\partial_0} \in S^{m+1}(\mathrm{SU}(2))$ if $\sigma_B \in S^m(\mathrm{SU}(2))$, by Theorem 6.3. □

Proof of Theorem 10.4. We have to show that $S^m(\mathrm{SU}(2)) = \Sigma^m(\mathrm{SU}(2))$, so that theorem would follow from Theorem 9.2. Both classes $S^m(\mathrm{SU}(2))$ and $\Sigma^m(\mathrm{SU}(2))$ require the singular support condition $(y \mapsto R_A(x,y)) \subset \{e\}$, so we do not have to consider this; moreover, the x-dependence of the symbol is not essential here, and therefore we abbreviate $\sigma_A(l) := \sigma_A(x,l)$. First, let us show that $\Sigma^m(\mathrm{SU}(2)) \subset S^m(\mathrm{SU}(2))$. Take $\sigma_A \in \Sigma^m(\mathrm{SU}(2))$. Then also $\sigma_{A_u} \in \Sigma^m(\mathrm{SU}(2))$ (either by the well-known properties of pseudodifferential operators and Theorem 9.2, or by checking directly that the definition of the classes $\Sigma_k^m(\mathrm{SU}(2))$ is conjugation-invariant). Let us define $c_N(B)$ by

$$
\sigma_{c_N(B)}(l)_{ij} := (i-j)^N\ \sigma_B(l)_{ij}.
$$

Now $\sigma_{c_N(A_u)} \in \Sigma^m(\mathrm{SU}(2))$ for every $N \in \mathbb{Z}^+$, because $\sigma_{A_u} \in \Sigma^m(\mathrm{SU}(2))$ and

$$
[\sigma_{\partial_0}, \sigma_B](l)_{ij} = (i-j)\ \sigma_B(l)_{ij}.
$$

This implies the "rapid off-diagonal decay" of σ_{A_u}:

$$
|\sigma_{A_u}(x,l)_{ij}| \le C_{AmN}\ \langle i-j\rangle^{-N}\ (1+l)^m,
$$

implying the norm comparability

$$
\|\cdots \sigma_{A_u}(l)\|_{op} \sim \sup_{i,j} |\cdots \sigma_{A_u}(l)_{ij}| \tag{10.7}
$$

in view of Lemma 12.2 in Appendix. Moreover, $\triangle_l^\alpha \partial_x^\beta \sigma_{A_u} \in \Sigma^{m-|\alpha|}(\mathrm{SU}(2))$ by Corollary 9.4, so that we obtain the symbol inequalities (10.6) from (9.2). Thereby $\Sigma^m(\mathrm{SU}(2)) \subset S^m(\mathrm{SU}(2))$.

Now we have to show that $S^m(\mathrm{SU}(2)) \subset \Sigma^m(\mathrm{SU}(2))$. Again, we may exploit the norm comparabilities (10.7): thus clearly $S^m(\mathrm{SU}(2)) \subset \Sigma_0^m(\mathrm{SU}(2))$. Consequently, $S^m(\mathrm{SU}(2)) \subset \Sigma_k^m(\mathrm{SU}(2))$ for all $k \in \mathbb{Z}^+$, due to Lemmas 10.6 and 10.8. □

11. Pseudo-differential operators on manifolds and on $\mathbb{S}^3$

In this section we discuss how the introduced constructions are mapped by global diffeomorphisms and give an example of this in the case of SU(2) and $\mathbb{S}^3$, proving Theorem 1.1.

Let $\Phi : G \to M$ be a diffeomorphism from a compact Lie group G to a smooth manifold M. Such diffeomorphisms can be obtained for large classes of compact manifolds by the Poincaré conjecture type results. For example, if $\dim M = 3$ it is now known that such Φ exists for any closed simply-connected manifold.

Let us endow M with the natural Lie group structure induced by Φ, i.e. with the group multiplication $((x, y) \mapsto x \cdot y) : M \times M \to M$ defined by

$$x \cdot y := \Phi\left(\Phi^{-1}(x)\ \Phi^{-1}(y)\right).$$

Spaces $C^\infty(G)$ and $C^\infty(M)$ are isomorphic via mappings

$$\begin{aligned} \Phi_* : C^\infty(G) \to C^\infty(M), &\qquad f \mapsto f_\Phi = f \circ \Phi^{-1}, \\ \Phi^* : C^\infty(M) \to C^\infty(G), &\qquad g \mapsto g_{\Phi^{-1}} = g \circ \Phi. \end{aligned}$$

The Haar integral on M is now given by

$$\int_M g\ \mathrm{d}\mu_M \equiv \int_M g\ \mathrm{d}x := \int_G g \circ \Phi\ \mathrm{d}\mu_G,$$

because for instance

$$\begin{aligned} &\int_M g(x \cdot y)\ \mathrm{d}x = \int_M g(\Phi(\Phi^{-1}(x)\ \Phi^{-1}(y)))\ \mathrm{d}x = \\ &\quad = \int_G (g \circ \Phi)(\Phi^{-1}(x)\ \Phi^{-1}(y))\ \mathrm{d}(\Phi^{-1}(x)) = \int_G (g \circ \Phi)(z)\ \mathrm{d}z = \int_M g(x)\ \mathrm{d}x. \end{aligned}$$

Moreover, $\Phi_* : C^\infty(G) \to C^\infty(M)$ extends to a linear unitary bijection $\Phi_* : L^2(\mu_G) \to L^2(\mu_M)$:

$$\int_M g(x)\ \overline{h(x)}\ \mathrm{d}x = \int_G (g \circ \Phi)\ (\overline{h \circ \Phi})\ \mathrm{d}\mu_G.$$

Notice also that there is an isomorphism

$$\Phi_* : \mathrm{Rep}(G) \to \mathrm{Rep}(M), \quad \xi \mapsto \Phi_*(\xi) = \xi \circ \Phi$$

of irreducible unitary representations. Thus $\widehat{G} \cong \widehat{M}$ in this sense. This immediately implies that the whole construction of symbols of pseudo-differential operators on M is equivalent to that on G.

Let us now apply this construction to the isomorphism $\mathbb{S}^3 \cong \mathrm{SU}(2)$. First recall the quaternion space $\mathbb{H}$ which is the associative $\mathbb{R}$-algebra with a vector space basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where $\mathbf{1} \in \mathbb{H}$ is the unit and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1} = \mathbf{ijk}.$$

Mapping $x = (x_m)_{m=0}^3 \mapsto x_0\mathbf{1} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ identifies $\mathbb{R}^4$ with $\mathbb{H}$. In particular, the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ is a multiplicative group. A bijective homomorphism $\Phi^{-1} : \mathbb{S}^3 \to \mathrm{SU}(2)$ is defined by

$$x \mapsto \Phi^{-1}(x) = \begin{pmatrix} x_0 + \mathrm{i}x_3 & x_1 + \mathrm{i}x_2 \\ -x_1 + \mathrm{i}x_2 & x_0 - \mathrm{i}x_3 \end{pmatrix},$$

and its inverse $\Phi : \mathrm{SU}(2) \to \mathbb{S}^3$ gives rise to the global quantisation of pseudo-differential operators on $\mathbb{S}^3$ induced by that on $\mathrm{SU}(2)$, as shown in the beginning of this section. This, combined with Theorem 10.4, proves Theorem 1.1.

12. Appendix on infinite matrices

In this section we discuss infinite matrices. The main conclusion that we need is that the operator-norm and the l^∞-norm are equivalent for matrices arising as full symbols of pseudo-differential operators in $\Psi^m(\mathrm{SU}(2))$.

Definition 12.1. Let $\mathbb{C}^{\mathbb{Z}}$ denote the space of complex sequences $x = (x_j)_{j\in\mathbb{Z}}$. A *matrix* $A \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ is presented as an infinite table $A = \left(A_{ij}\right)_{i,j\in\mathbb{Z}}$. As usual, we set $\langle x, y\rangle_{\ell^2} = \sum_{j\in\mathbb{Z}} x_j \overline{y_j}$, $\|x\|_{l^2} = \langle x, x\rangle_{\ell^2}^{1/2}$, and $\|A\|_{l^2\to l^2} = \sup\{\|Ax\|_{l^2} : \|x\|_{l^2} \leq 1\}$ provided that the sums $(Ax)_i = \sum_{j\in\mathbb{Z}} A_{ij}\, x_j$ converge absolutely. For each $k \in \mathbb{Z}$, let us define $A(k) \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ by

$$A(k)_{ij} = \begin{cases} A_{ij}, & \text{if } i-j = k, \\ 0, & \text{if } i-j \neq k. \end{cases}$$

A matrix $A \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ will be said to *decay (rapidly) off-diagonal* if

$$|A_{ij}| \leq c_{Ar}\ \langle i-j\rangle^{-r} \tag{12.1}$$

for every $i, j \in \mathbb{Z}$ and $r \in \mathbb{N}$, where constants $c_{Ar} < \infty$ depend on r, A, but not on i, j. The set of off-diagonally decaying matrices is denoted by $\mathcal{D}$.

Lemma 12.2. *Let* $A \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ *and* $\|A\|_{\ell^\infty} = \sup_{i,j\in\mathbb{Z}} |A_{ij}|$. *Then*

$$\|A\|_{\ell^\infty} \leq \|A\|_{op}.$$

Moreover, if $|A_{ij}| \leq c\langle i-j\rangle^{-r}$ *for some* $r > 1$ *then for* $c' = c\sum_{k\in\mathbb{Z}}\langle k\rangle^{-r}$ *we have*

$$\|A\|_{op} \leq c'\ \|A\|_{\ell^\infty}.$$

Proof. Let $\delta_i = (\delta_{ij})_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. Then $A_{ij} = \langle A\delta_j, \delta_i\rangle_{\ell^2}$. The first claim then follows from the Cauchy–Schwarz inequality:

$$|A_{ij}| = \left|(A\delta_j, \delta_i)_{\ell^2}\right| \leq \|A\|_{op}.$$

Next, since $A = \sum_{k\in\mathbb{Z}} A(k)$, we get

$$\|A\|_{op} \leq \sum_{k\in\mathbb{Z}} \|A(k)\|_{op} = \sum_{k\in\mathbb{Z}} \sup_j |A(k)_{j+k,j}| \leq \sum_{k\in\mathbb{Z}} c\langle k\rangle^{-r}.$$

From this we directly see that if $\|A\|_{\ell^\infty} \geq 1$ then $\|A\|_{op} \leq c'\ \|A\|_{\ell^\infty}$. By the linearity of the norms, this concludes the proof. □

Proposition 12.3. *Let* $A, B \in \mathcal{D}$. *Then* $AB \in \mathcal{D}$.

Proof. Matrices $A, B \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ in general cannot be multiplied, but here there is no problem as $A, B \in \mathcal{D}$, so that the matrix element $(AB)_{ik}$ is estimated by

$$\sum_j |A_{ij}|\,|B_{jk}| \leq c_{Ar}c_{Bs}\sum_j \langle i-j\rangle^{-r}\langle j-k\rangle^{s} \overset{\text{Peetre ineq.}}{\leq} c_{Ar}c_{Bs}\sum_j \langle i-k\rangle^{-r}\langle k-j\rangle^{|r|}\langle j-k\rangle^{s},$$

which converges if $|r| + s < -1$. This shows that $AB \in \mathcal{D}$. □

Altogether, we obtain the following

Theorem 12.4. *$\mathcal{D} \subset \mathcal{L}(\ell^2)$ is a unital involutive algebra. Moreover, for $A \in \mathcal{D}$, norms $\|A\|_{op}$ and $\|A\|_{\ell^\infty}$ are equivalent.*

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