On multipliers on compact Lie groups *

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Abstract

[1] In this note we announce L^p multiplier theorems for invariant and non-invariant operators on compact Lie groups in the spirit of the well-known Hörmander-Mikhlin theorem on $\mathbb{R}^n$ and its variants on tori $\mathbb{T}^n$. Applications are given to the mapping properties of pseudo-differential operators on L^p-spaces and to a-priori estimates for non-hypoelliptic operators.

1. Introduction.

Let G be a compact Lie group of dimension n, with identity 1 and the unitary dual $\widehat{G}$. The following considerations are based on the group Fourier transform

$$\mathscr{F}\phi = \widehat{\phi}(\xi) = \int_G \phi(x)\xi(x)^* \mathrm{d}x, \qquad \phi(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\widehat{\phi}(\xi)) = \mathscr{F}^{-1}[\widehat{\phi}] \tag{1}$$

defined in terms of equivalence classes $[\xi]$ of irreducible unitary representations $\xi : G \to \mathrm{U}(d_\xi)$ of dimension d_ξ. The Peter–Weyl theorem on G implies in particular that this pair of transforms is inverse to each other and that the Plancherel identity

$$\|\phi\|_2^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \|\widehat{\phi}(\xi)\|_{HS}^2 =: \|\widehat{\phi}\|_{\ell^2(\widehat{G})} \tag{2}$$

holds true for all $\phi \in L^2(G)$. Here $\|\widehat{\phi}(\xi)\|_{HS}^2 = \operatorname{Tr}(\widehat{\phi}(\xi)\widehat{\phi}(\xi)^*)$ denotes the Hilbert–Schmidt norm of matrices. The Fourier inversion statement (1) is valid for all $\phi \in \mathcal{D}'(G)$ and the Fourier series converges in $C^\infty(G)$ provided ϕ is smooth. It is further convenient to denote $\langle\xi\rangle = \max\{1, \lambda_\xi\}$, where $-\lambda_\xi^2$ is the eigenvalue of the Laplace-Beltrami (Casimir) operator acting on the matrix coefficients associated to

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the representation ξ. The Sobolev spaces can be characterised by Fourier coefficients as

$$\phi \in H^s(G) \quad \Longleftrightarrow \quad \langle\xi\rangle^s \widehat{\phi}(\xi) \in \ell^2(\widehat{G}),$$

where $\ell^2(\widehat{G})$ is defined as the space of matrix-valued sequences such that the sum on the right-hand side of (2) is finite.

In the following we consider continuous linear operators $A : C^\infty(G) \to \mathcal{D}'(G)$, which can be characterised by their symbol

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x) \tag{3}$$

which is a function on $G \times \widehat{G}$ taking matrices from $\mathbb{C}^{d_\xi \times d_\xi}$ as values. As a consequence of (1) we obtain that for any given $\phi \in C^\infty(G)$ the distribution $A\phi \in \mathcal{D}'(G)$ satisfies

$$A\phi(x) = \sum\nolimits_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\sigma_A(x,\xi)\widehat{\phi}(\xi)). \tag{4}$$

We denote the operator A defined by a symbol σ_A as $\operatorname{op}(\sigma_A)$. This quantisation and its properties have been consistently developed in [7] and we refer to it for details. We speak of a *Fourier multiplier* if the symbol $\sigma_A(x,\xi)$ is independent of the first argument. This is equivalent to requiring that A commutes with left translations. It is evident from the Plancherel identity that such an operator is L^2-bounded if and only if $\sup_{[\xi]\in\widehat{G}} \|\sigma_A(\xi)\|_{op} < \infty$, where $\|\cdot\|_{op}$ denotes the operator norm on the inner-product space $\mathbb{C}^{d_\xi}$.

In the book [7], as well as in the paper [8] the authors gave a characterisation of Hörmander type pseudo-differential operators on G in terms of their matrix-valued symbols. The symbol classes, as well as the multiplier theorems given below, depend on the so-called difference operators acting on moderate sequences of matrices, i.e., on elements of

$$\Sigma(\widehat{G}) = \{\sigma : \xi \mapsto \sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi} : \|\sigma(\xi)\|_{op} \lesssim \langle\xi\rangle^N \text{ for some } N\}.$$

A difference operator Q of order ℓ is defined in terms of a corresponding function $q \in C^\infty(G)$, which vanishes to (at least) ℓth order in the identity element $1 \in G$ via

$$Q\sigma = \mathscr{F}\left(q(x)\mathscr{F}^{-1}\sigma\right). \tag{5}$$

Note, that $\sigma \in \Sigma(\widehat{G})$ implies $\mathscr{F}^{-1}\sigma \in \mathcal{D}'(G)$ and therefore the multiplication with a smooth function is well-defined. The main idea of introducing such operators is that applying differences to symbols of Calderon–Zygmund operators brings an improvement in the behaviour of $\mathrm{op}(Q\sigma)$ since we multiply the integral kernel of $\mathrm{op}(\sigma)$ by a function vanishing on its singular set.

Different collections of difference operators have been explored in [8] in the pseudo-differential setting. Difference operators of particular interest arise from matrix-coefficients of representations. For a fixed irreducible representation ξ_0 we define the (matrix-valued) difference operator ${}_{\xi_0}\mathbb{D} = ({}_{\xi_0}\mathbb{D}_{ij})_{i,j=1,\ldots,d_{\xi_0}}$ corresponding to the matrix elements of the matrix-valued function $\xi_0(x) - \mathrm{I}$, with $q_{ij}(x) = \xi_0(x)_{ij} - \delta_{ij}$ in (5), δ_{ij} the Kronecker delta. If the representation is fixed, we omit the index ξ_0. For a sequence of difference operators of this type, $\mathbb{D}_1 = {}_{\xi_1}\mathbb{D}_{i_1j_1}, \mathbb{D}_2 = {}_{\xi_2}\mathbb{D}_{i_2j_2}, \ldots, \mathbb{D}_k = {}_{\xi_k}\mathbb{D}_{i_kj_k}$, with $[\xi_m] \in \widehat{G}$, $1 \le i_m, j_m \le d_{\xi_m}$, $1 \le m \le k$, we define $\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1} \cdots \mathbb{D}_k^{\alpha_k}$. In the sequel we will work with a collection Δ_0 of representations chosen as follows. Let $\widetilde{\Delta_0}$ be the collection of the irreducible components of the adjoint representation, so that $\mathrm{Ad} = (\dim Z(G))1 \oplus \bigoplus_{\xi \in \widetilde{\Delta_0}} \xi$, where ξ are irreducible representations and 1 is the trivial one-dimensional representation. In the case when the centre $Z(G)$ of the group is nontrivial, we extend the collection $\widetilde{\Delta_0}$ to some collection Δ_0 by adding to $\widetilde{\Delta_0}$ a family of irreducible representations such that their direct sum is nontrivial on $Z(G)$, and such that the function

$$\rho^2(x) = \sum\nolimits_{[\xi]\in\Delta_0} (d_\xi - \mathrm{Tr}\,\xi(x)) \ge 0$$

(which vanishes only in $x = 1$) would define the square of some distance function on

G near the identity element. Such an extension is always possible, and we denote by Δ_0 any such extension; in the case of the trivial centre we do not have to take an extension and we set $\Delta_0 = \widetilde{\Delta_0}$. We denote further by $\mathbb{A}$ the second order difference operator associated to $\rho^2(x)$, $\mathbb{A} = \mathscr{F}\rho^2(x)\mathscr{F}^{-1}$. In the sequel, when we write $\mathbb{D}^\alpha$, we can always assume that it is composed only of ${}_{\xi_m}\mathbb{D}_{i_m j_m}$ with $[\xi_m] \in \Delta_0$.

2. Main results.

The following condition (6) is a natural relaxation from the L^p-boundedness of zero order pseudo-differential operators to a multiplier theorem and generalises the Hörmander–Mikhlin ([5, 6], [4]) theorem to arbitrary groups.

Theorem 1 *Denote by $\varkappa$ be the smallest even integer larger than $\frac{1}{2}\dim G$. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be left-invariant. Assume that its symbol σ_A satisfies*

$$\|\mathbb{A}^{\varkappa/2}\sigma_A(\xi)\|_{op} \leq C\langle\xi\rangle^{-\varkappa} \qquad \text{and} \qquad \|\mathbb{D}^\alpha\sigma_A(\xi)\|_{op} \leq C_\alpha\langle\xi\rangle^{-|\alpha|} \tag{6}$$

for all multi-indices α with $|\alpha| \leq \varkappa - 1$, and for all $[\xi] \in \widehat{G}$. Then the operator A is of weak[2] type $(1,1)$ and L^p-bounded for all $1 < p < \infty$.

We now give some particular applications of Theorem 1. The selection is not complete and indicates a few applications which could be derived from the main result. Full proofs can be found in [9].

Theorem 2 *Assume that $\sigma_A \in \mathscr{S}^0_\rho(G)$, i.e., by definition, it satisfies inequalities*

$$\|\mathbb{D}^\alpha\sigma_A(\xi)\|_{op} \leq C_\alpha\langle\xi\rangle^{-\rho|\alpha|},$$

for some $\rho \in [0,1]$ and all α. Then A defines a bounded operator from[3] $W^{p,r}(G)$ to $L^p(G)$ for $r = \varkappa(1-\rho)|\frac{1}{p} - \frac{1}{2}|$, $\varkappa$ as in Theorem 1 and $1 < p < \infty$.

[2]The operator A is said to be of weak type $(1,1)$ if there exists a constant $C > 0$ such that for all $\lambda > 0$ and $u \in L^1(G)$ the inequality $\mu\{x \in G : |Au(x)| > \lambda\} \leq C\|u\|_{L^1(G)}/\lambda$ holds true, where μ is the Haar measure on G.

[3] Here $W^{p,r}(G)$ stands for the Sobolev space consisting of all distributions f such that $(I - \mathscr{L})^{r/2} f \in L^p(G)$, where $\mathscr{L}$ is a Laplacian (Laplace-Betrami operator, Casimir element) on G.

The previous statement applies in particular to the parametrices constructed in [8]. We will give two examples on the group $\mathrm{SU}(2) \cong \mathbb{S}^3$. Let D_1, D_2 and D_3 be an orthonormal basis of $\mathfrak{su}(2)$. Then both, the sub-Laplacian $\mathcal{L}_s = \mathrm{D}_1^2 + \mathrm{D}_2^2$ as well as the 'heat' operator $\mathcal{H} = \mathrm{D}_3 - \mathrm{D}_1^2 - \mathrm{D}_2^2$ have a parametrix from[4] $\mathrm{op}\mathscr{S}^{-1}_{\frac12}(\mathbb{S}^3)$ and therefore the sub-elliptic estimates

$$\|u\|_{W^{p,1-|\frac1p-\frac12|}(\mathbb{S}^3)} \le C_p \|\mathcal{L}_s u\|_{L^p(\mathbb{S}^3)} \qquad \text{and} \qquad \|u\|_{W^{p,1-|\frac1p-\frac12|}(\mathbb{S}^3)} \le C_p \|\mathcal{H} u\|_{L^p(\mathbb{S}^3)} \quad (7)$$

are valid for all $1 < p < \infty$. The following statement concerns operators which are neither locally invertible nor locally hypoelliptic.

Corollary 3 *Let X be a left-invariant real vector field on G. Then there exists a discrete exceptional set $\mathscr{C} \subset \mathrm{i}\mathbb{R}$, such that for any complex number $c \notin \mathscr{C}$ the operator $X + c$ is invertible with inverse in $\mathrm{op}\mathscr{S}^0_0(G)$. Consequently, the inequality*

$$\|f\|_{L^p(G)} \le C_p \|(X+c)f\|_{W^{p,\varkappa|\frac1p-\frac12|}(G)}$$

holds true for all $1 < p < \infty$ and all functions f from that Sobolev space, with $\varkappa$ as above.

For the particular case $G = \mathrm{SU}(2)$, the exceptional set coincides with the spectrum of the skew-selfadjoint realisation of X suitably normalised with respect to the Killing norm, e.g., $\mathscr{C} = \mathrm{i}\frac12\mathbb{Z}$ if $X = \mathrm{D}_3$.

The Hörmander multiplier theorem [4], although formulated in $\mathbb{R}^n$, has a natural analogue on the torus $\mathbb{T}^n$. The assumtions in Theorem 1 on the top order difference brings a refinement of the toroidal multiplier theorem, at least for some dimensions. If $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the set Δ_0 can be chosen to consist of $2n$ functions $\mathrm{e}^{\pm 2\pi \mathrm{i} x_j}$, $1 \le j \le n$. Consequently, we have that $\rho^2(x) = 2n - \sum_{j=1}^n \left(\mathrm{e}^{2\pi\mathrm{i}x_j} + \mathrm{e}^{-2\pi\mathrm{i}x_j}\right)$ in (),

[4] The class $\mathrm{op}\mathscr{S}^{-1}_{\frac12}(\mathbb{S}^3)$ is defined as the class of operators with symbols σ_A satisfying the inequalities $\|\mathbb{D}^\alpha \sigma_A(\xi)\|_{op} \le C_\alpha \langle\xi\rangle^{-1-|\alpha|/2}$.

and hence $\mathbb{A}\sigma(\xi) = 2n\sigma(\xi) - \sum_{j=1}^{n} (\sigma(\xi + e_j) + \sigma(x - e_j))$, where $\xi \in \mathbb{Z}^n$ and e_j is its jth unit basis vector in $\mathbb{Z}^n$.

A (translation) invariant operator A and its symbol σ_A are related by $\sigma_A(k) = \mathrm{e}^{-2\pi \mathrm{i} x\cdot k}(A\mathrm{e}^{2\pi \mathrm{i} x\cdot k}) = (A\mathrm{e}^{2\pi \mathrm{i} x\cdot k})|_{x=0}$ and $A\phi(x) = \sum_{k\in\mathbb{Z}^n} e^{2\pi \mathrm{i} x\cdot k}\sigma_A(k)\widehat{\phi}(k)$. Thus, it follows from Theorem 1 that, for example on $\mathbb{T}^3$, a translation invariant operator A is bounded on $L^p(\mathbb{T}^3)$ provided that there is a constant $C > 0$ such that $|\sigma_A(k)| \leq C$, $|k||\sigma_A(k + e_j) - \sigma_A(k)| \leq C$ and

$$|k|^2|\sigma_A(k) - \tfrac{1}{6}\sum_{j=1}^{3} (\sigma_A(k + e_j) + \sigma_A(k - e_j))\,| \leq C, \tag{8}$$

for all $k \in \mathbb{Z}^3$ and all (three) unit vectors e_j, $j = 1, 2, 3$. Here we do not make assumptions on all second order differences in (8), but only on one of them.

Finally, Theorem 1 also implies a boundedness statement for operators of form (4). Let for this ∂_{x_j}, $1 \leq j \leq n$, be a collection of left invariant first order differential operators corresponding to some linearly independent family of the left-invariant vector fields on G. As usual, we denote $\partial_x^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$.

Theorem 4 *Denote by $\varkappa$ be the smallest even integer larger than $\frac{n}{2}$, n the dimension of the group G. Let $1 < p < \infty$ and let $l > \frac{n}{p}$ be an integer. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a linear continuous operator such that its matrix symbol σ_A satisfies*

$$\|\partial_x^\beta \mathbb{D}^\alpha \sigma_A(x, \xi)\|_{op} \leq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}$$

for all multi-indices α, β with $|\alpha| \leq \varkappa$ and $|\beta| \leq l$, for all $x \in G$ and $[\xi] \in \widehat{G}$. Then the operator A is bounded on $L^p(G)$.

3. Discussion. 1. The conditions are needed for the week type $(1, 1)$ property. Interpolation allows to reduce assumptions on the number of differences for L^p-boundedness. The result generalises the corresponding statements in the case of the group SU(2) in [1], [2], also presented in [3].

2. Examples similar to (7) can be given for arbitrary compact Lie groups. The assumptions of Theorem 2 concerning the numbers of difference operators can be relaxed to the same as those in Theorem 1.

3. If the operator $A \in \Psi^0(G)$ is the usual pseudo-differential operator of Hörmander type of order 0 on G (i.e. in all local coordinate it belongs to Hörmander class $\Psi^0(\mathbb{R}^n)$), it was shown in [7] the estimates (4) hold for all α, β. The converse is also true. Namely, if estimates (4) hold for all α, β, then $A \in \Psi^0(G)$, cf. [8].

4. Noncommutative matrix quantisation (3)–(4) has a full symbolic calculus (compositions, adjoints, parametrix, etc.), which have been established in the monograph [7].

5. On SU(2) the operators corresponding to our difference operators but defined explicitly in terms of the Clebsch-Gordan coefficients have been used in [2, 3]. The general definition (5), the main tool in the present investigation, has been introduced and analysed in [7] and [8].

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