Provably Stable Local Application of Crank-Nicolson Time Integration to the FDTD Method with Nonuniform Gridding and Subgridding

A. Van Londersele, D. De Zutter, and D. Vande Ginste
Department of Information Technology (INTEC)
Electromagnetics (EM) Group
Ghent University/imec/IDLab, Belgium
arne.vanlondersele@ugent.be, daniel.dezutter@ugent.be, dries.vandeginste@ugent.be

***Abstract*—This contribution removes some doubts about the stability issues associated with the local and anisotropic use of Crank-Nicolson (CN) time integration in Finite-Difference Time-Domain (FDTD) simulations with spatial irregularities such as nonuniformity and subgridding. Due to the lack of space, only the most significant steps in the stability analysis are treated here. Intermediate steps as well as numerical examples and implementation details will be provided during the presentation.**

I. Introduction

To tackle the present-day multiscale electromagnetic problems, Finite-Difference Time-Domain (FDTD) solvers must provide a minimum degree of spatial flexibility, which is typically offered by nonuniform gridding and subgridding techniques. Both techniques allow varying cell sizes that are needed to efficiently resolve the multiscale geometry, but the former preserves the overall tensor-product structure of the grid, whereas the latter supports edge termination as to yield nested tensor-product grids. Although subgridding definitely uses less memory, the more intricate memory organization can cause it to be actually less efficient than nonuniform grids in terms of CPU time depending on the type of problem. Apart from reducing the number of spatial samples, the FDTD algorithm can be further optimized by reducing the number of time samples. However, the time step cannot exceed a certain stability upper bound, better known as the Courant limit, which is proportional to the smallest cell size occurring inside the grid. Conventional subgridding techniques typically assign a local time step, matching the local stability constraint, to each subgrid. This approach virtually always suffers from late-time instability and, although being explicit, the high number of iterations inside each subgrid can be less efficient than applying a (partially) implicit method, such as the Crank-Nicolson (CN) scheme and its Hybrid Implicit-Explicit (HIE) derivatives. Despite some interesting research on this topic, e.g., [1,2], no rigorous stability analysis has been proposed, until the recent publishing of [3], where the local and anisotropic application of CN, Newmark-beta and leapfrog alternating-direction implicit (ADI) schemes to nonuniform grids are thoroughly discussed in terms of stability. Here, we first summarize the key findings of [3] and then extend the stability analysis to a very general class of subgridding schemes. This should provide the reader with the necessary insights on how to combine explicit leapfrog and implicit CN FDTD methods in a stable way, not merely based on intuition.

II. Nonuniform Gridding

With the notations copied from [3, eqns. (2)-(15)], the hybrid explicit-leapfrog implicit-CN FDTD update equation for a general lossless inhomogeneous medium discretized on a nonuniform tensor-product grid is,

$$\begin{bmatrix} \frac{1}{\Delta t} D_\varepsilon & -\frac{1}{2}(I+\mathcal{P})\,\hat{\mathcal{V}}\,\mathcal{C}\,\hat{\mathcal{W}} \\ \frac{1}{2}\mathcal{V}\,\mathcal{C}^T\mathcal{W}\,(I-\mathcal{P}) & \frac{1}{\Delta t} D_\mu \end{bmatrix} x|^{n+1} = \begin{bmatrix} \frac{1}{\Delta t} D_\varepsilon & \frac{1}{2}(I-\mathcal{P})\,\hat{\mathcal{V}}\,\mathcal{C}\,\hat{\mathcal{W}} \\ -\frac{1}{2}\mathcal{V}\,\mathcal{C}^T\mathcal{W}\,(I+\mathcal{P}) & \frac{1}{\Delta t} D_\mu \end{bmatrix} x|^{n} + s|^{n} \,, \tag{1}$$

with s the source vector and x the field vector,

$$x|^{n} = \begin{bmatrix} \mathcal{P}\,e(n\,\Delta t) + (I-\mathcal{P})\,e((n-0.5)\,\Delta t) \\ h((n-0.5)\Delta t) \end{bmatrix} \,, \tag{2}$$

and $\mathcal{P}$ the diagonal matrix with elements,

$$[\mathcal{P}]_{i,i} = \begin{cases} 1 & \text{if } e_i \text{ is updated explicitly} \\ 0 & \text{if } e_i \text{ is updated implicitly} \end{cases} \,. \tag{3}$$

The FDTD system is exponentially stable if the poles of the z-domain transfer matrix are not located outside the unit disk. Besides, it is also polynomially stable if the repeated poles on the unit circle have linearly independent eigenvectors. An appropriate change of basis shows that the inverse of the transfer matrix belonging to (1) is algebraically similar to,

$$T(z) = \begin{bmatrix} (z-1)I & -z\,\mathcal{I}_2\,\tilde{\mathcal{C}} \\ \tilde{\mathcal{C}}^T\mathcal{I}_1 & (z-1)I \end{bmatrix} \,, \tag{4}$$

with *implicitization operators*,

$$[\mathcal{I}_1]_{i,i} = \begin{cases} 1 & \text{if } e_i \text{ is updated explicitly} \\ (z+1)/2 & \text{if } e_i \text{ is updated implicitly} \end{cases} \,, \tag{5}$$

$$[\mathcal{I}_2]_{i,i} = \begin{cases} 1 & \text{if } e_i \text{ is updated explicitly} \\ (1+z^{-1})/2 & \text{if } e_i \text{ is updated implicitly} \end{cases} \,, \tag{6}$$

and modifield curl,

$$\tilde{\mathcal{C}} = \Delta t \left(D_{\varepsilon}^{-1} \hat{\mathcal{V}}\,\mathcal{W}\right)^{1/2} \mathcal{C} \left(D_{\mu}^{-1}\,\mathcal{V}\,\hat{\mathcal{W}}\right)^{1/2} . \tag{7}$$

Hence, the FDTD system is exponentially stable if the roots of the characteristic equation $\det(T(z)) = 0$ satisfy $|z| \leq 1$. A separate treatment of static and dynamic solutions together with a partitioned-matrix rule translates the characterstic equation to,

$$\det\left(z^{-1}(z-1)^2 I_{n_h} + \tilde{\mathcal{C}}^T \mathcal{I}_1\,\mathcal{I}_2\,\tilde{\mathcal{C}}\right) = 0 . \tag{8}$$

Note that this is the wave equation in the z-space. Now, substitution of the bilinear transformation $z = (\zeta - 1)/(\zeta + 1)$ allows to interpret (8) as a linear eigenvalue problem. More specifically,

$$\zeta^2\ \tilde{\mathcal{C}}^T \tilde{\mathcal{C}}\ v = \left(\tilde{\mathcal{C}}^T \mathcal{P}\,\tilde{\mathcal{C}} - 4 I_{n_h}\right)\ v . \tag{9}$$

Left-multiplying (9) by the hermitian transpose of v and subsequently subtracting/adding the hermitian-transposed equation, yields respectively,

$$\mathrm{Im}(\zeta^2)\ \|\tilde{\mathcal{C}}\,v\|_2^2 = 0 , \tag{10}$$

$$\mathrm{Re}(\zeta^2)\ \|\tilde{\mathcal{C}}\,v\|_2^2 = \|\mathcal{P}\,\tilde{\mathcal{C}}\,v\|_2^2 - 4\|v\|_2^2 . \tag{11}$$

As (10) and (11) should hold for any non-zero vector v, the condition $|z| \leq 1$, which is equivalent to $\mathrm{Re}(\zeta) \geq 0$, is satisfied if and only if $\mathrm{Re}(\zeta^2) \leq 0$. A more careful analysis shows that this can only occur for ζ lying on the imaginary axis or, equivelantly, z lying on the unit circle. In other words, the FDTD system has been proven to be stable if $\mathrm{Re}(\zeta^2) \leq 0$, which upon inspection of (11) yields,

$$\|\mathcal{P}\,\tilde{\mathcal{C}}\|_2 = \max_{v \neq 0} \frac{\|\mathcal{P}\,\tilde{\mathcal{C}}\,v\|_2}{\|v\|_2} \leq 2 . \tag{12}$$

As shown in [3], the limit on Δt imposed by (12) is the exact Courant limit, but in order to avoid polynomial growth the strict inequality should hold. It is clear from (12), that the operator $\mathcal{P}$ can be tuned as to eliminate the smallest cell sizes from the stability limit at the cost of implicit CN computations.

III. Subgridding

Consider a nonuniform coarse grid that is locally overlapped by a nonuniform subgrid, whose outer edges coincide with coarse primary edges. If the overlapped coarse part would be filled with perfect magnetic conductors, which cannot reduce the time step limit for trivial reasons, the resulting scheme is equivalent to a conventional grid stitching scheme without overlap. Hence, stability with overlap is a sufficient condition for stability without overlap. This insight allows us to detach coarse curl $\mathcal{C}_{\mathrm{c}}$, fine curl $\mathcal{C}_{\mathrm{f}}$ and coupling operator $\mathcal{S}$ from each other, such that, for the subgridding scheme with overlap, (4) translates to,

$$T(z) = \begin{bmatrix} (z-1)I & -z\,\tilde{\mathcal{C}}_{\mathrm{c}} & & -s\,z\,\mathcal{I}_{2,\mathrm{s}}\,\tilde{\mathcal{S}}^T \\ \tilde{\mathcal{C}}_{\mathrm{c}}^T & (z-1)I & & \\ & & (z-1)I & -z\,\mathcal{I}_{2,\mathrm{f}}\,\tilde{\mathcal{C}}_{\mathrm{f}} \\ \tilde{\mathcal{S}}\,\mathcal{I}_{1,\mathrm{s}} & & \tilde{\mathcal{C}}_{\mathrm{f}}^T \mathcal{I}_{1,\mathrm{f}} & (z-1)I \end{bmatrix} , \tag{13}$$

where implicitization operators of the form (5)–(6) were added to the subgrid and the coupling. Note that the construction of (13) requires some symmetry relation between the coarse-to-fine and fine-to-coarse coupling operators as well as some nonuniformity rescaling. Also, the scalar s is a normalization factor. More details will be given during the presentation. Following the same procedure as for the nonuniform gridding, we end up with a quadratic eigenvalue problem $(\zeta^2\,A + \zeta\,B + C)\,v = 0$, with

$$A = A^T = \begin{bmatrix} \frac{1}{4}\tilde{\mathcal{C}}_{\mathrm{c}}\,\tilde{\mathcal{C}}_{\mathrm{c}}^T & 0 \\ 0 & \frac{1}{4}\tilde{\mathcal{C}}_{\mathrm{f}}^T\,\tilde{\mathcal{C}}_{\mathrm{f}} \end{bmatrix} , \tag{14}$$

$$B = -B^T = \begin{bmatrix} 0 & -\frac{\sqrt{s}}{2}\,\tilde{\mathcal{S}}_i^T \\ \frac{\sqrt{s}}{2}\,\tilde{\mathcal{S}}_i & 0 \end{bmatrix} , \tag{15}$$

$$C = C^T = \begin{bmatrix} I - \frac{1}{4}\tilde{\mathcal{C}}_{\mathrm{c}}\,\tilde{\mathcal{C}}_{\mathrm{c}}^T & \frac{\sqrt{s}}{2}\,\mathcal{P}_{\mathrm{s}}\,\tilde{\mathcal{S}}_i^T \\ \frac{\sqrt{s}}{2}\,\tilde{\mathcal{S}}_i\,\mathcal{P}_{\mathrm{s}} & I - \frac{1}{4}\tilde{\mathcal{C}}_{\mathrm{f}}^T\mathcal{P}_{\mathrm{f}}\,\tilde{\mathcal{C}}_{\mathrm{f}} \end{bmatrix} . \tag{16}$$

An eigenvalue problem with this particular symmetry and with $A \succeq 0$ has roots ζ in the right half-plane if and only if $C \succeq 0$. With the decomposition $C = I - D$, the required positive semi-definiteness gives rise to the condition $\lambda_{\max}(D) \leq 1$. Separating on-diagonal from off-diagonal blocks by means of the triangle inequality finally yields,

$$\frac{1}{4}\max\left(\ \|\tilde{\mathcal{C}}_{\mathrm{c}}\|_2^2\ ,\ \|\mathcal{P}_{\mathrm{f}}\,\tilde{\mathcal{C}}_{\mathrm{f}}\|_2^2\ \right) + \frac{\sqrt{s}}{2}\|\tilde{\mathcal{S}}_i\,\mathcal{P}_{\mathrm{s}}\|_2^2 \leq 1 . \tag{17}$$

By updating the subgrid and the coupling updates implicitly, the coarse-grid Courant limit is retrieved.

IV. Conclusion

The exact time step limit is provided for nonuniform grids with local and anistropic application of CN time integration. A similar limit is shown to exist for general subgridding schemes. However, this limit is a sufficient but not a necessary upper bound for Δt. The concept of a z-domain implicitization operator in combination with the bilinear transformation can be extended to other stability proofs. For example, a spatial analog may be used to find the exact stability condition for the hybrid staggered-collocated methods proposed in [4,5], which apply the CN scheme both in space and time.

References

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