DMRG and periodic boundary conditions: a quantum information perspective

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We introduce a picture to analyze the density matrix renormalization group (DMRG) numerical method from a quantum information perspective. This leads us to introduce some modifications for problems with periodic boundary conditions in which the results are dramatically improved. The picture also explains some features of the method in terms of entanglement and teleportation.

The discovery and development of the DMRG method [1, 2] to treat quantum many–body systems has enabled us to analyze and understand the physical properties of certain condensed matter systems with unprecedent precision [3]. Originally envisioned for 1D systems with short–range interactions at zero temperatures, during the last years this method has been successfully extended to other situations [3]. Its mathematical foundations have been established [4, 5] in terms of the so–called matrix product states (MPS) [6] and by now there exists a coherent theoretical picture of DMRG.

At the same time, the field of Quantum Information Theory (QIT) has emerged to describe the properties of quantum many–body systems from a different point of view. A theory of entanglement has been established, and has allowed us to describe and understand phenomena like teleportation [7], and to use them in the fields of communication and computation [8]. Recently it has been shown that QIT may also shed some new light in our understanding of condensed matter systems [9, 10], and, in particular, in the DMRG method [11, 12].

In this work we analyze the standard DMRG method using a physical picture which underlies QIT concepts. The picture has its roots in the AKLT model [13] and allows us to understand why DMRG offers much poorer results for problems with periodic boundary conditions (PBC) than for those with open boundary conditions (OBC), something which was realized at the origin of DMRG [2]. It also gives a natural way of improving the method for problems with PBC, in which several orders of magnitude in accuracy can be gained. The importance of this result lies in the fact that physically PBC are strongly preferable over OBC as boundary effects are eliminated and finite size extrapolations can be performed for much smaller system sizes.

Let us start by reviewing the simplest version of the DMRG method for 1-D spin chains with OBC, which is typically represented as $B \bullet B$ [2, 15]. We denote by d the dimension of the Hilbert space corresponding to each spin, and by D the number of states kept by the DMRG method. We assume that the spins at the edges have dimension $d_0 \geq D$ [14]. At some particular step the chain is split into two blocks and one spin in between. The left block (L) contains spins $1, \ldots, M-1$, and the

right one (R) spins $M+1, \ldots, N$. Then a set of $D \times D$ matrices A^s are determined such that the state

$$|\Psi\rangle = \sum_{s=1}^{d} \sum_{\alpha,\beta=1}^{D} A_{\alpha,\beta}^s |\alpha\rangle_L \otimes |s\rangle_M \otimes |\beta\rangle_R, \qquad (1)$$

minimizes the energy. The states $|\alpha\rangle_{L,R}$ are orthonormal, and have been obtained in previous steps. They can be constructed using the recurrence relations

$$|\alpha\rangle_L = \sum_{\alpha'=1}^{D} \sum_{s=1}^{d} U_{\alpha,\alpha'}^{[M-1],s} |s\rangle_{M-1} \otimes |\alpha'\rangle_{L'}, \qquad (2)$$

where the block L' contains the spins $1, \ldots, M-2$. The new matrices $U^{[M],s}$ are determined from A^s and fulfill

$$\sum_{s=1}^{d} U^{[M],s} \left(U^{[M],s} \right)^{\dagger} = \mathbf{1}. \qquad (3)$$

For the blocks consisting of the edge spins alone, the $|\alpha\rangle$ are taken as the members of an orthonormal set.

In order to give a pictorial representation of the above procedure we introduce at site M two auxiliary D–level systems, a_M and b_M. The corresponding Hilbert spaces $H_{a,b}$ are spanned by two orthonormal bases $|\alpha\rangle_{a,b}$, respectively. We take L and a_M (and also R and b_M) in the (unnormalized) maximally entangled state

$$|\phi\rangle := \sum_{\alpha=1}^{D} |\alpha\rangle \otimes |\alpha\rangle, \qquad (4)$$

We can always write $|\Psi\rangle = P_M |\phi\rangle_{L,a_M} |\phi\rangle_{R,b_M}$, where P_M maps $H_a \otimes H_b \rightarrow H_M$, with H_M the space corresponding to the M–th spin and [cf. (1)]

$$P_M = \sum_{s=1}^{d} \sum_{\alpha,\beta=1}^{D} A_{\alpha,\beta}^s |s\rangle\langle \alpha, \beta|. \qquad (5)$$

In fact, we can proceed in the same way at any other site $k \neq 1, M, N$ by defining two auxiliary systems a_k and b_k and a map Q_k defined as in (5) but with the matrices U instead of the A. For the edge spins 1 and N we define a single auxiliary system b_1 and a_N, respectively and define
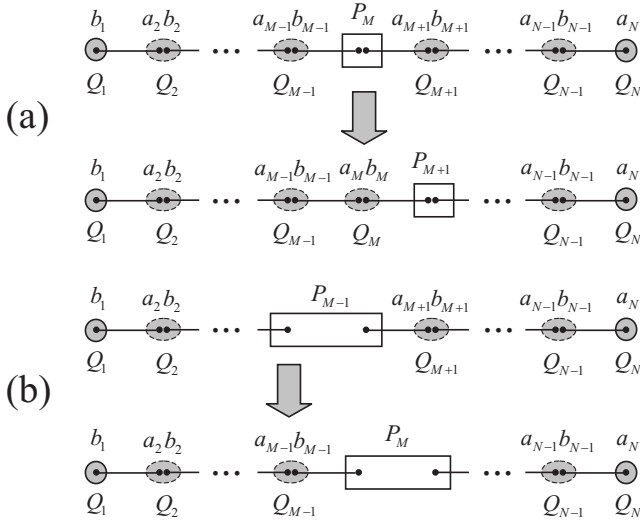
FIG. 1: Schematic picture of the DMRG method for the $B \bullet B$ (a) and the $B \bullet \bullet B$ (b) configurations. Horizontal lines represent maximally entangled states $|\phi\rangle$, the ellipses and circles (squares) the operators Q (P) which map the auxiliary system into the physical ones.

accordingly the operators $Q_{1,N}$ which now map $H_{b,a} \rightarrow H_{1,N}$. Thus, the state Ψ is then obtained by applying the operators $Q_1 \ldots, P_M, \ldots Q_N$ to the set of maximally entangled states ϕ between the auxiliary systems b_k and a_{k+1} ($k = 1, \ldots, N-1$) [see Fig. 1(a)].

The DMRG procedure can be now represented as follows. At location M, one finds an operator P_M acting on the subsystems a_M and b_M by determining the matrices A^s. From them, one obtains the operator Q_M and goes to the next step at location $M+1$. One proceeds in the same way, moving to the right, until one reaches the location N. At that point, one starts moving to the left until one reaches the location 1 at which point it moves again to the right. The procedure is continued until a fixed point for the energy is reached, something which always occurs since the energy is a monotonically decreasing function of the step number. This proves that DMRG with the $B \bullet B$ is a *variational method which always converges*.

The more standard scenario ($B \bullet \bullet B$) is represented in Fig. 1(b). The operator P_M acts on the auxiliary subsystems a_M and b_{M+1} and maps $H_a \otimes H_b \rightarrow H_M \otimes H_{M+1}$. In this picture [for both configurations, Figs. 1 (a,b)] it is very clear that the two edge spins are treated on a very different footing since they are represented by a single auxiliary system which is not entangled to any other.

In the case of a problem with PBC a slight modification of the scheme is used [2]. The idea is to still separate the system into two blocks and two spins as before but now with the configuration $B \bullet B\bullet$. This ensures the sparseness of the matrices one has to diagonalize and thus it increases the speed of the algorithm [2]. One can draw the diagram corresponding to this procedure
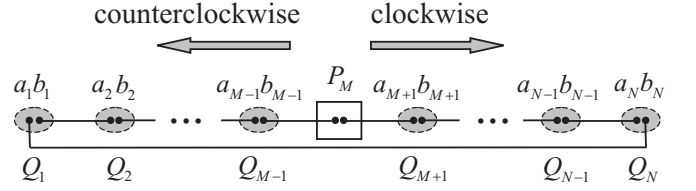

FIG. 2: Proposed configurations for the case of PBC. One may also use two spins instead of one

in a similar way as in Fig. 1. The important point is that still there are always two sites (left most and right most of both blocks B) which are treated differently since they are represented by a single auxiliary spin which is not entangled to any other. In our opinion, this is the reason of the poor performance of the DMRG method for problems with PBC.

The method we propose is very clear in terms of this picture (Fig. 2). One has to substitute *at all sites k* the spin by *two* auxiliary systems a_k and b_k of dimension D, with b_k and a_{k+1} (with $a_{N+1} := a_1$) in a maximally entangled states and find the maps $P_k : H_a \otimes H_b \rightarrow H_k$ which lead to a state

$$|\Psi\rangle = P_1 \otimes P_2 \ldots P_{N-1} \otimes P_N |\phi\rangle^{\otimes N}, \qquad (6)$$

with the minimal energy. This minimization can be performed in a similar way to the one used in the standard DMRG method. Before showing how to do this in practice, we derive some formulas in terms of these operators. We write

$$P_k = \sum_{s=1}^{d} |s\rangle\langle\varphi_s^{[k]}|, \quad \langle\varphi_s^{[k]}| = \sum_{\alpha,\beta} B_{\alpha,\beta}^{[k],s} \langle\alpha,\beta|. \qquad (7)$$

Thus, the problem is solved once the states φ (or equivalently, the matrices B) are determined. Note that starting from these states, it is possible to calculate expectation values of products of local observables [4], since

$$\langle\Psi|O_1 \ldots O_N|\Psi\rangle = \mathrm{Tr}\left(E_{O_1}^{[1]} \ldots E_{O_N}^{[N]}\right), \qquad (8)$$

where

$$E_O^{[k]} = \sum_{s,s'=1}^{d} \langle s|O|s'\rangle B^{[k],s} \otimes \left(B^{[k],s}\right)^*. \qquad (9)$$

Thus, the main idea to perform the minimization is very simple. Given the Hamiltonian H describing the system, one chooses one site M and writes the energy as

$$E = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\langle\psi^{[M]}|H_M|\psi^{[M]}\rangle}{\langle\psi^{[M]}|N_M|\psi^{[M]}\rangle}, \qquad (10)$$

where $|\psi^{[M]}\rangle = \oplus_s |\psi_s^{[M]}\rangle$ is a vector built by concatenating the $\psi_s^{[M]}$, and N_M and H_M are $d \times D^2$ hermitian

square matrices which are built using the vectors $\psi_s^{[k]}$ at $k \neq M$. For example, $N_M = \oplus_s N_0$ is a block diagonal matrix with identical blocks N_0 which has matrix elements $(N_0)_{(\alpha,\alpha'),(\beta,\beta')} = (\tilde{N}_0)_{(\alpha,\beta),(\alpha',\beta')}$, with

$$\tilde{N}_0 = E_{\mathbf{1}}^{[M+1]} \ldots E_{\mathbf{1}}^{[N]} E_{\mathbf{1}}^{[1]} \ldots E_{\mathbf{1}}^{[M-1]}. \tag{11}$$

Thus, at this step the operator P_M is found by solving the generalized eigenvalue problem

$$H_M |\psi^{[M]}\rangle = \lambda N_M |\psi^{[M]}\rangle, \tag{12}$$

with λ minimum, which in turns gives the energy at this step. Then one chooses another site and proceeds in the same way until the energy converges. At the end we have all the P_k and can evaluate all expectation values.

The above method is not very efficient numerically. First, the matrix N_0 may be ill conditioned. Second, one stores many matrices ($\sim N^2$) and performs many matrix multiplications ($\sim N^2$) at each step. Now we explain how one can make the method much more efficient.

Let assume that we have a set of spins in a ring. The idea is to determine operators P_k in a clockwise order (first P_1, then P_2, until P_{N-1}), then improve them following a counterclockwise ordering (from P_N to P_2), then again clockwise, until the fixed point is reached. At each step, a normalization condition similar to (3) is imposed, depending on whether we are in a clockwise or counterclockwise cycle, which makes the matrix N_M well behaved. On the other hand, at each step only the operators which are strictly needed in later steps are calculated in an efficient way and stored.

The normalization condition is based on the following fact. Given the state Ψ, characterized by matrices B, if we substitute $B^{[M],s} \to B^{[M],s} X := U^{[M],s}$ and $B^{[M+1],s} \to X^{-1} B^{[M+1],s}$, where X is a nonsingular matrix, we obtain the same state. Analogously, we can substitute $B^{[M],s} \to Y B^{[M],s} := V^{[M]}$ and $B^{[M-1],s} \to B^{[M-1],s} Y^{-1}$. We choose X in the clockwise cycles to impose (3) and Y in the counterclockwise ones to impose

$$\sum_{s=1}^{d} \left(V^{[M],s}\right)^{\dagger} \left(V^{[M],s}\right) = \mathbf{1}. \tag{13}$$

Thus, at the point of determining the operator P_M,

$$|\Psi\rangle = Q_1 \otimes \ldots Q_{M-1} \otimes P_M \otimes \tilde{Q}_{M+1} \ldots \otimes \tilde{Q}_N |\phi\rangle^{\otimes N} \tag{14}$$

where Q_k and $\tilde{Q}_k$ are defined as in (7) but with U and V instead of B, respectively. Thus, the operators X and Y are all of them moved over, such that they are now included in those corresponding to P_M. It can be easily shown that these conditions on the operator U (V) are equivalent to imposing that $E_{\mathbf{1}}$ has the maximally entangled state $|\phi\rangle$ as right (left) eigenvector with eigenvalue 1. This is immediately reflected in the fact that the matrix N_M is better behaved, which makes the problem numerically stable.

Let us now illustrate how the procedure works with simplest nearest neighbor Hamiltonian $\sigma_z^{[k]} \sigma_z^{[k+1]}$, namely the Ising Model. Let us assume that we are running the optimization of the operators clockwise and that we want to determine P_M. So far, in previous steps, apart from the matrices U and V, we have stored: (a) For each $k < M$, the following four operators:

$$r_k := E_{\mathbf{1}}^{[1]} E_{\mathbf{1}}^{[2]} \ldots E_{\mathbf{1}}^{[k-2]} E_{\mathbf{1}}^{[k-1]}, \tag{15a}$$

$$s_k := E_{\sigma_z}^{[1]} E_{\mathbf{1}}^{[2]} \ldots E_{\mathbf{1}}^{[k-2]} E_{\mathbf{1}}^{[k-1]}, \tag{15b}$$

$$t_k := E_{\mathbf{1}}^{[1]} E_{\mathbf{1}}^{[2]} \ldots E_{\mathbf{1}}^{[k-2]} E_{\sigma_z}^{[k-1]}, \tag{15c}$$

$$h_k := \sum_{n=1}^{k-2} E_{\mathbf{1}}^{[1]} E_{\mathbf{1}}^{[2]} \ldots E_{\sigma_z}^{[n]} E_{\sigma_z}^{[n+1]} \ldots E_{\mathbf{1}}^{[k-2]} E_{\mathbf{1}}^{[k-1]}. \tag{15d}$$

(b) For each $k > M$ other four similar operators which contain products from $E^{[k]}$ to $E^{[N]}$. With them, one can build H_M and N_0 by few matrix multiplications and thus determine P_M by solving (12). From it, Q_M is determined. Then, we construct $r_{M+1}, s_{M+1}, t_{M+1}$ and h_{M+1} starting from r_M, s_M, t_M and h_M [16]. We continue in the same vein, finding four matrices at each step, and storing them, until we reach N. Then we start moving counterclockwise and start constructing the corresponding four matrices at each step. Notice that in order to construct the matrices H_M and N_0 we will have to use the stored matrices (15) which were determined when we were moving clockwise. Thus, with this procedure we have to store of the order of $4N$ matrices of dimension D^2 (apart from the matrices U, V and the last B's) but the number of operations per step is independent of N. At the end, when we have reached the fixed point, we can determine the expectation value of any operator by using (8) and determining the required matrices using (9). Note that if the problem has translational symmetry, then all these evaluations are even simpler.

We have applied the above method to the spin $1/2$ Heisenberg chain. We have plotted in Fig. 3 the energies obtained as a function of D and compared them with those obtained by the standard DMRG method with OBC and PBC. From the figure it is clear that the accuracies we obtain are comparable with those obtained with DMRG for problems with OBC but much better than for PBC. We have determined the errors by comparing with the exact results [17]. In the insert of Fig. 3 we have plotted the local bond strength $\langle \mathbf{S}^{[k]} \mathbf{S}^{[k+1]} \rangle$ as a function of k. As expected, the result is independent of the position k, as opposed to what occurs with OBC.

Finally we show that the picture introduced here may be valuable to understand the properties of states Ψ in terms of the language and tools developed in the field of QIT. First, one can easily see that the entropy of the block formed by systems $(k_0, k_0 + 1, \ldots, k_1)$ is bounded by $2 \log_2(D)$, as this block is connected to the rest only via a_{k_0} and b_{k_1+1}, and thus the rank of the reduced density operator for the block is bounded by the product
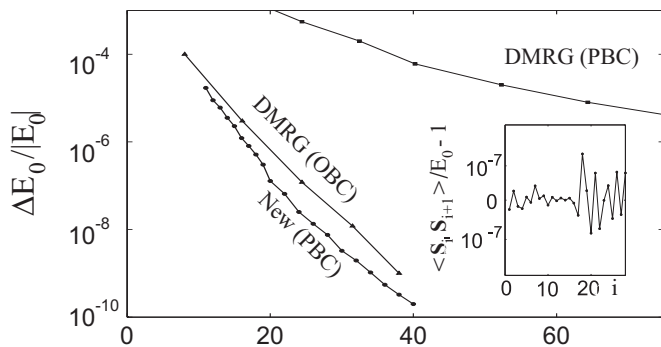
FIG. 3: (Left) Comparison between DMRG (squares) [2] and the new method (circles) for PBC, and $N = 28$. For reference the DMRG results [2] for the Heisenberg chain with OBC (triangles) are also shown. (Insert) Variation of the local bond strength from the average along the chain, calculated with the new method and $D = 40$.
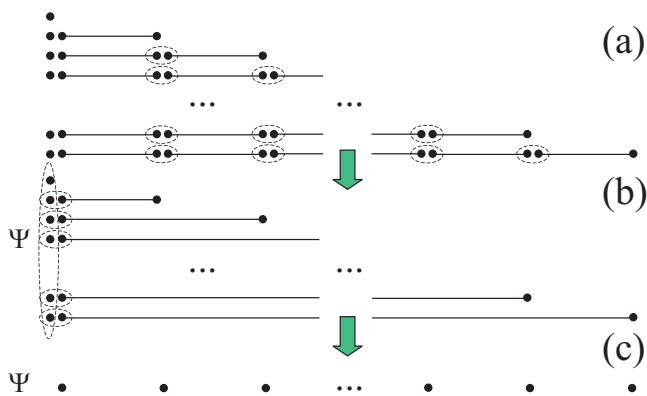

FIG. 4: General states can be expressed in the form (6).

of the dimensions of the corresponding Hilbert spaces. Secondly, it allows us to show that any state can be written in the form (6) (MPS [4, 6]) if we choose $D = d^N$ (actually, $D = d^{\lfloor \frac{N}{2} \rfloor}$ is sufficient). We consider a_k and b_k as composed of d–level subsystems, $a_k^1, \ldots, a_k^N$ and $b_k^1, \ldots, b_k^N$, respectively, and write $|\phi\rangle$ as a tensor product of maximally entangled states ϕ_d between b_k^l and a_{k+1}^l. For $k = 2, \ldots, N$, we choose the operators $P_k = \mathbf{1}_{a_k^k} \otimes \langle \eta_k |$ where $|\eta_k\rangle$ is a state for all particles but a_k^k, and contains $|\phi_d\rangle$ for each pair a_k^l-b_k^l ($l > k$) and $|0\rangle$ for the rest. The action of P_k is to teleport the entangled pairs such that at the end one has one entangled pairs between the first system and all the rest (Fig. 4), while leaving all the other auxiliary particles in $|0\rangle$. Finally, the operator P_1 is the product of two operators. The first acts on particles a_1 and transforms $|0\rangle^N \rightarrow |\Psi\rangle$. The second is $\mathbf{1}_{a_1^1} \otimes \langle \eta_1 |$, where $|\eta_1\rangle = |0\rangle_{b_1} \otimes |\phi_d\rangle^{\otimes N-1}$. This operator first pre-

pares the desired state Ψ in particles a_1 and then uses the available entangled pairs to teleport it to the the rest of the particles.

In summary, we have given a pictorial view of the DMRG method and have identified the reason of its poor performance for problems with PBC. Our picture immediately leads to a modified version of the DMRG method which dramatically improves the results. This is done at the expenses of no longer using sparse matrices, something which limits its applications. Nevertheless, we believe that the method may allow us to treat problems in condensed matter systems which so far have been difficult to tackle with the standard DMRG method. In any case, the present work illustrates how the developments made in QIT during the last years may prove useful in other branches of Physics.

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