# Inverse source problems in fractional evolutionary PDE's

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Thesis submitted to Ghent University in candidature for the academic degree of Doctor of Philosophy in Mathematics

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Faculty of Sciences
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Academic year 2017-2018

# Dedications and acknowledgements

To mom, without you I wouldn't be. Your kindness, courage and persistence will never stop inspiring me.

To Betka, Veronika, Michal, Martin, it was an honor and great adventure to grow up with you.

Mišo and Jaro, thank you for being friends.

Professor Marián Slodička, let me express the gratitude for all your help, encouragement and knowledge you pass to me during the last four years. Thank you.

Gabriela and Karel, I value greatly your remarks and corrections to this thesis. Thank you for your time.

I would also like to thank to many others in my life, family, friends and colleagues, for being here for me and helping me always to move forward.

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# Summary

Many fields of science meet alongside each other in the study of *inverse problems.* Problems are of interest for mathematicians, engineers as well as applied scientists across various fields. The mathematical analysis of inverse problems holds its rightful place. Mathematicians are concerned with the questions of existence and uniqueness of a solution, its regularity, ill-posedness, regularization, numerical algorithms for gaining an approximate solution, the convergence of the numerical scheme, error estimates and many others.

The term *fractional evolution equations* is used for equations evolving in time containing a fractional derivative. We deal with the equations involving the fractional derivative in time and classic derivatives in space. This kind of equations may be derived from the continuous time random walk assuming the divergent waiting time and the finite jump length variance. This results in an equation that follows power time dependence of the mean square displacement, and it is considered to be a generalization of Brownian motion. Processes described by the equation don't follow Gaussian statistics; therefore, the Fick's second law fails to describe their behavior. The non-linear growth of the mean square displacement which follows the power-law pattern is an attribute of the anomalous diffusion processes, slow-diffusion as well as sub-ballistic super-diffusion, found in many complex systems.

In our thesis, we are interested in the inverse source problems in the fractional diffusion/wave equation, particularly, in the reconstruction of the time-dependent part of the source term which represents the evolution of the source in time. We study the existence of a solution together with its regularity. Our approach produces a numerical algorithm of which convergence is also examined, and numerical experiments are performed. We address uniqueness of the solution in every case. The problems we are solving differ in the considered equation, boundary conditions and additional measurement.

Our thesis consists of five chapters. In the first one, the mathematical background is presented. Subsequent chapters are original work based on four articles

of which three have been already published, and one has been submitted for publication, all in well-respected journals. The chapters are organized as follows.

In the first chapter, we provide a mathematical background that serves as a foundation for understanding of the successive chapters. The introduction contains a short summary of functional analysis used in the thesis. The basic general and particular functional spaces and concepts are introduced. In addition, the crucial theorems including various identities and inequalities are formulated. The central part of the chapter is focused on the notion of a partial differential equation and an inverse source problem. The final part is devoted to a concise introduction to the fractional calculus, the derivation of the fractional diffusion and wave equation, and the comparison of their solutions with their classical counterparts. We close the chapter with two preparatory lemmas.

Chapter 2 is devoted to the study of an inverse source problem in a semi-linear fractional diffusion equation with a non-linearity in the form of an time integral on the right-hand side of the equation. The interest lies in reconstruction of the time dependent part of the source term from the integral over-determination. The weak formulation of the problem is stated and by applying the measurement on the equation, we gain an additional equation for the solution. The resulting equations are discretized in time, the existence and uniqueness of the solution along the time slices is addressed, and a priori estimates are proven. The existence of the solution is obtained using the Rothe functions which converge to the solution of the problem. Moreover, the uniqueness of the solution is established. The chapter is concluded with numerical experiments, also addressing a possibility of noisy data. The entire chapter is based on the article [119] published in journal *Computers and Mathematics with Applications* with impact factor 1.53 in 2016.

In Chapter 3 we study the identification of the time-dependent part of a source in a fractional wave equation with a nonlinear term on the right hand side of the equation. The additional measurement is assumed to have the form of an integral over the part of the domain. The existence and uniqueness of the solution is obtained using the Rothe method similarly as in Chapter 1. A couple of numerical experiments is presented at the end of the chapter. The article [130], published in the journal *Applied Numerical Mathematics*, with impact factor 1.087 in year 2016, served as foundation for this chapter.

Chapter 4 deals with the inverse source problem in a linear fractional wave equation accompanied with a non-standard boundary condition. The condition is a fractional analogy of the well-known dynamical boundary condition because it contains a fractional partial derivative with respect to time. The problem is discretized, and the uniqueness and existence of a solution is addressed. The important part is the obtained error estimate. To support the theoretical results some numerical experiments are performed. This chapter is grounded in the article [131] which has been published in the aforementioned journal *Computers and Mathematics with Applications*.

The last chapter discusses the reconstruction of the time-dependent source term in the fractional wave equation where the noninvasive type of measurement is used, i.e. the measurement is in the form of an integral over a part of the boundary. The Rothe method is applied to gain the existence of the solution, and the uniqueness is obtained too. Numerical examples in 2D are provided. The chapter is based on the article [129], submitted for publication in *Journal of Computational and Applied Mathematics* with impact factor 1.357 in 2016.

Our thesis is concluded with the discussion over the results and some possibilities for future work.

# Samenvatting

Veel wetenschapsgebieden ontmoeten elkaar in de studie van *inverse problemen.* Deze problemen zijn van belang voor zowel wiskundigen, als ingenieurs en toegepaste wetenschappers in verschillende onderzoeksdomeinen. De wiskundige analyse van inverse problemen is een belangrijk onderdeel in hun studie. Wiskundigen houden zich bezig met belangrijke vragen omtrent het bestaan en uniciteit van een oplossing, de regulariteit van de oplossing, de slecht-gesteldheid van het vraagstuk, de ontwikkeling van numerieke algoritmen voor het bekomen van een benaderende oplossing, de convergentie van de algoritmen en de foutschattingen.

De term *fractionele-evolutievergelijkingen* wordt gebruikt voor vergelijkingen die in de tijd evolueren en een fractionele afgeleide bevatten. In dit proefschrift behandelen we vergelijkingen die een fractionele afgeleide naar de tijd en klassieke afgeleiden naar de ruimtelijke veranderlijke bevatten. Dit soort van vergelijkingen kunnen worden afgeleid vertrekkend vanuit een toevalsbeweging in continue tijd, in de veronderstelling dat de wachttijd divergent is en dat de spronglengte een eindige variantie heeft. Dit resulteert in een vergelijking die wordt beschouwd als een veralgemening van de Brownse beweging. Processen die door deze vergelijking worden beschreven volgen geen normale verdeling en daarom kan Fick's tweede wet hun gedrag niet beschrijven.

In dit proefschrift zijn we geïnteresseerd in inverse bronproblemen voor zowel de fractionele diffusie als fractionele golfvergelijking. Meer specifiek, we focussen op de reconstructie van het tijdsafhankelijke deel van de bronterm dat de evolutie van de bron weergeeft in de tijd. We bestuderen het bestaan van een oplossing samen met de regulariteit ervan. Onze aanpak levert een numeriek algoritme op waarvan de convergentie wordt onderzocht. Numerieke experimenten worden uitgevoerd. We behandelen ook de uniciteit van de oplossing voor elk van de bestudeerde problemen. De problemen verschillen op basis van de beschouwde vergelijking, de randvoorwaarden en de bijkomende meting die nodig is om de onbekende bronterm te reconstrueren.

De dissertatie bestaat uit vijf hoofdstukken. In het eerste hoofdstuk wordt de wiskundige achtergrond gepresenteerd. De andere hoofdstukken bevatten origineel werk gebaseerd op vier artikels waarvan er al twee zijn gepubliceerd, één is geaccepteerd voor publicatie en één is ingediend, dit in hoogstaande tijdschriften. De hoofdstukken zijn georganiseerd als volgt.

In het eerste hoofdstuk bieden we de wiskundige achtergrond aan die de basis vormt waarop dit proefschrift is gebaseerd. Deze achtergrond is noodzakelijk om de volgende hoofdstukken te kunnen begrijpen. Het begin van het hoofdstuk bevat een kort overzicht van de resultaten uit de functionaalanalyse die in het proefschrift worden gebruikt. Algemene en specifieke functieruimten en concepten worden geïntroduceerd. Cruciale stellingen worden vermeld. Ook belangrijke identiteiten en ongelijkheden worden behandeld. In het midden van het hoofdstuk bespreken we kort de begrippen partiële differentiaalvergelijking en invers bronprobleem. Het laatste deel is gewijd aan een beknopte inleiding op de fractionele calculus, de afleiding van de fractionele diffusie- en golfvergelijking en het vergelijken van hun oplossingen met de oplossingen van hun klassieke tegenhangers. We sluiten het hoofdstuk af met twee technische lemma's die de hoeksteen vormen van de analyse in de volgende hoofdstukken.

Hoofdstuk 2 is gewijd aan de studie van een invers bronprobleem in een semilineaire fractionele diffusievergelijking met een niet-lineariteit in de vorm van een tijdsintegraal in het rechterlid van de vergelijking. De interesse ligt in de reconstructie van het tijdsafhankelijke deel van de bronterm op basis van een integraalmeting over het volledige domein. De zwakke formulering van het probleem wordt opgesteld, en door de meting op de vergelijking toe te passen krijgen we een extra vergelijking waaraan de oplossing moet voldoen. De resulterende vergelijkingen worden in de tijd gediscretiseerd, het bestaan en de uniciteit van de oplossing op de verschillende tijdstippen wordt bestudeerd en er worden apriori afschattingen berekend. Het bestaan van de oplossing wordt verkregen met behulp van zogenaamde Rothefuncties die convergeren naar de oplossing van het probleem. Bovendien wordt ook de uniciteit van een oplossing onderzocht. Het hoofdstuk wordt afgesloten met numerieke experimenten, waarbij ook de invloed van fouten in de data op de oplossing wordt onderzocht. Dit hoofdstuk is gebaseerd op het artikel [119] gepubliceerd in het tijdschrift *Computers and Mathematics with Applications* (impactfactor 1,53 in 2016).

In Hoofdstuk 3 bestuderen we de identificatie van het tijdsafhankelijke deel van de bron in een fractionele golfvergelijking met niet-lineaire term in het rechterlid. De extra meting is een integraalmeting over een deel van het domein. Het bestaan en de uniciteit van de oplossing wordt verkregen met behulp van de Rothemethode. De analyse is vergelijkbaar met de aanpak gebruikt in Hoofdstuk 2. Aan het einde van dit hoofdstuk worden een aantal numerieke experimenten gepresenteerd. Het artikel [130] gepubliceerd in het tijdschrift *Applied Numerical Mathematics* (met impactfactor 1.087 in jaar 2016) diende als basis voor dit hoofdstuk.

Hoofdstuk 4 behandelt een inverse bronprobleem in een lineaire fractionele golfvergelijking vergezeld met een niet-klassieke randvoorwaarde. Deze randconditie is de fractionele analogie van de bekende dynamische randvoorwaarde omdat deze een fractionele afgeleide bevat met betrekking tot de tijd. Het probleem is gediscretiseerd in de tijd en de uniciteit en het bestaan van een oplossing wordt aangetoond. De belangrijkste bijdrage in dit hoofdstuk is de verkregen foutschatting. Ter ondersteuning van de theoretische resultaten worden enkele numerieke experimenten uitgevoerd. Dit hoofdstuk is gebaseerd op het artikel [131] dat is geaccepteerd voor publicatie in het tijdschrift *Computers and Mathematics with Applications*.

Het laatste hoofdstuk bespreekt de reconstructie van een tijdsafhankelijke bronterm in de fractionele golfvergelijking waarbij een niet-invasieve meting wordt gebruikt, d.w.z. de meting is in de vorm van integraal over een deel van de rand van het domein. De Rothemethode wordt opnieuw toegepast om het bestaan van de oplossing te verkrijgen en de uniciteit wordt ook verkregen. Numerieke experimenten in een tweedimensionale setting worden gepresenteerd. Het hoofdstuk is gebaseerd op het artikel [129], ingediend voor publicatie in het tijdschrift *Journal of Computational and Applied Mathematics* (impactfactor 1.357 in 2016).

Dit proefschrift wordt afgesloten met de discussie van de resultaten en enkele mogelijke perspectieven voor toekomstig onderzoek.

# Chapter 1

# Mathematical background

In this chapter, we summarize mostly without proofs the relevant theory on which the later chapters are based, and we give a brief introduction to the fractional calculus. In the first section, we have compiled same basic definitions from functional analysis, later in the second section, we proceed to the well-known theorems. Section 1.3 deals with the equalities and inequalities used in the proofs later. Next three sections are devoted to the general notion of partial differential equations, methods used for their inspection and inverse problems connected to them. In the last section of this chapter, we introduce the concept of fractional derivative and also some preliminary technical results, which are stated and proved. For most of the mathematical background that we present here, we refer to [2, 35, 92, 98, 99, 101, 107, 132, 147].

## 1.1 Basic definitions

In this section we define the notions of metric space, convergence, normed space, Banach and Hilbert space.

**Definition 1.1.1.** *A function $d : M \times M \to [0, \infty)$, where $M$ is a set is called a metric if for all $x, y, z \in M$ the following is satisfied*

*(i)* $d(x, y) \geq 0$*,*

*(ii)* $d(x, y) = d(y, x)$*,*

*(iii)* $d(x, y) = 0$ *if and only if* $x = y$*,*

*(iv)* $d(x,y) \le d(x,z) + d(z,y)$.

A couple $(M,d)$ is called a *metric* space. From now on, let $(M,d)$ be a metric space.

**Definition 1.1.2.** *A sequence $\{x_n\}_{n=1}^{\infty} \subset M$ is called Cauchy, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n, m \ge n_0$, it holds*

$$d(x_n, x_m) < \varepsilon.$$

**Definition 1.1.3.** *A sequence $\{x_n\}_{n=1}^{\infty} \subset M$ is said to converge (be convergent) to $x \in M$, denoted as*

$$x_n \to x,$$

*if*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

**Definition 1.1.4.** *A metric space $M$ is called complete if every Cauchy sequence converges in $M$.*

Let $X$ be a real linear space.

**Definition 1.1.5.** *A map $\|\cdot\| : X \to [0,\infty)$ is called a norm if*

*(i)* $\|x+y\| \le \|x\| + \|y\|$ *for all* $x, y \in X$,

*(ii)* $\|\lambda x\| = |\lambda|\,\|x\|$ *for all* $x \in X$, $\lambda \in \mathbb{R}$,

*(iii)* $\|x\| = 0$ *if and only if* $x = 0$.

The couple $(X, \|\cdot\|)$ is called a *normed* linear space. We will denote a normed linear space only by the set $X$ if it is clear with which norm it is coupled. For a better clarity, we will denote the norm affiliated to the space $X$ as $\|\cdot\|_X$ when necessary. There can be defined a metric as $d(x,y) = \|x-y\|$ in every normed space. A normed linear space $X$ is called a *Banach* space if $(X,d)$ with the metric defined in that way is complete.

**Definition 1.1.6.** *Let $H$ be a real linear space. A function $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is called an inner product if for every $x, y \in H$*

*(i)* $(x,y) = (y,x)$,

*(ii) the mapping* $x \mapsto (x,y)$ *is linear for each* $y \in H$,

*(iii)* $(x,x) \ge 0$,

*(iv)* $(x,x) = 0$ *if and only if* $x = 0$.

With the above defined inner product, we may associate the norm

$$\|x\| := (x,x)^{\frac{1}{2}} .$$

A Banach space endowed with an inner product and its associated norm is called a *Hilbert* space.

## 1.2 Functional analysis

In this section, we define an bounded linear functional, reflexive Banach space, weak convergence, compact set, absolute and Lipschitz continuity, weak derivative, Lebesgue and Sobolev spaces, Bochner integral, spaces involving time, and convolution. We also state the Eberlein-Šmuljan theorem, Hahn-Banach theorem, Riesz representation theorem, Lax-Milgram theorem, Arzelà-Ascoli theorem, Trace theorem, Lebesgue's dominated convergence theorem and some other theorem connected to the definitions.

Let $X$ be a Banach space.

**Definition 1.2.1.** *A mapping* $f : X \to \mathbb{R}$ *is called a bounded linear functional on* $X$ *iff*

*(i)* $\sup\limits_{\|x\|\leq 1} |f(x)| \leq C$,

*(ii)* $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ *for every* $\alpha, \beta \in \mathbb{R}$ *and* $x, y \in X$.

A set of all bounded linear functionals on a space $X$, endowed with the norm

$$\|f\| = \sup_{\|x\|\leq 1} |f(x)| ,$$

forms again a Banach space which is called the *dual* space of $X$ and denoted by $X^*$. We also introduce the notation $\langle f, x\rangle = f(x)$ for $f \in X^*$ and $x \in X$. We denote a dual space of $X^*$ (the second dual of the space $X$) as $X^{**}$. There exists a natural map $j : X \to X^{**}$ given by $j(x) = x^{**}$ where

$$\langle x^{**}, f\rangle = \langle f, x\rangle,$$

for all $f \in X^*$. This mapping is often called the canonical mapping. Here $\|x^{**}\| = \|x\|$, in another words $j$ is isometric.

**Definition 1.2.2.** *If the canonical mapping $j : X \to X^{**}$ defined above is surjective, then $X$ is called $reflexive$.*

**Definition 1.2.3.** *We say that a sequence $\{x_n\} \subset X$ is weakly convergent (converges weakly) denoted as*

$$x_n \rightharpoonup x,$$

*when*

$$\langle f, x_n \rangle \to \langle f, x \rangle$$

*for every $f \in X^*$.*

**Theorem 1.2.1** (Eberlein-Šmuljan theorem)**.** *For a Banach space $X$, the following is equivalent:*

*(i) $X$ is reflexive.*

*(ii) Every bounded sequence $\{x_n\} \subset X$ contains a weakly convergent subsequence.*

**Theorem 1.2.2** (Hahn-Banach theorem)**.** *Let $Y$ be a a linear subspace of a Banach space $X$ and*

$$f : Y \to \mathbb{R}$$

*be a bounded linear functional on $Y$. Then there exist a bounded linear extension*

$$\overline{f} : X \to \mathbb{R}$$

*with $\|\overline{f}\|_{X^*} = \|f\|_{Y^*}$.*

**Theorem 1.2.3** (Riesz representation theorem)**.** *Let $H$ be a real Hilbert space, with inner product $(\cdot, \cdot)$. For every $x^* \in H^*$ there exists a unique element $x \in H$ such that*

$$\langle x^*, y \rangle = (x, y) \quad \textit{for all } y \in H.$$

**Theorem 1.2.4** (Lax-Milgram theorem)**.** *Let $H$ be a real Hilbert space and $B : H \times H \to \mathbb{R}$ a bilinear mapping, for which there exist constants $C_1, C_2$ such that for every $x, y \in H$, it holds that*

*(i)*

$$|B[x, y]| \leq C_1 \|x\| \|y\|,$$

*(ii)*

$$C_2 \|x\|^2 \leq B[x, x].$$

*Assume also $f : H \to \mathbb{R}$ be a bounded linear functional on $H$. Then, there exists a unique $x \in H$ such that*

$$B[x, y] = \langle f, y \rangle$$

*for all $y \in H$.*

**Definition 1.2.4.** *A subset $M$ of a normed space $X$ is called compact if every sequence of points in $M$ has a subsequence converging in $X$ to an element of $M$. $M$ is called relatively compact if $\overline{M}$ is compact set.*

**Definition 1.2.5.** *Let $G$ be a nonempty bounded open set in $\mathbb{R}^n$ and $(Y, \|\cdot\|_Y)$ be a Banach space. By $C(\overline{G}, Y)$ we denote the set of all $u : \overline{G} \to Y$, which are continuous.*

Then, the set $C(\overline{G}, Y)$ furnished with the maximum norm defined as

$$\|u\| = \max_{x \in \overline{G}} \|u(x)\|_Y$$

form a Banach space. In the case when $Y = \mathbb{R}$, we write $C(\overline{G}, \mathbb{R}) = C(\overline{G})$.

**Theorem 1.2.5** (Arzelà-Ascoli theorem)**.** *The set $M \subset C(\overline{G}, Y)$ is relatively compact iff*

*(i) the set $\{u(x) : u \in M\}$ is relatively compact in $Y$ for all $x \in \overline{G}$,*

*(ii) for every $x \in \overline{G}$ and every $\varepsilon > 0$ there is a $\delta(\varepsilon, x) > 0$, independent of function $u$, such that*

$$\sup_{u \in M} \|u(x) - u(y)\|_Y < \varepsilon \quad \textit{whenever} \quad y \in \overline{G} \quad \textit{and} \quad |x - y| < \delta(\varepsilon, x).$$

**Remark 1.2.1.** *In the case when $Y = \mathbb{R}$, the condition $(i)$ in the theorem above can be changed to: there exists a constant $C$ such that for every $u \in M$ and $x \in G$, $|u(x)| \leq C$, compactly written $\sup_{u \in M} \sup_{x \in \overline{G}} |u(x)| < \infty$. Since every bounded sequence in $\mathbb{R}$ has a convergent subsequence in $\mathbb{R}$.*

**Theorem 1.2.6.** *Let $[a, b]$ be a finite interval of $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$. Then $f$ is absolutely continuous on $[a, b]$ if and only if there exist a Lebesgue integrable function $g : [a, b] \to \mathbb{R}$ such that*

$$f(x) = f(a) + \int_a^x g(t)\, dt \qquad \textit{for } x \in [a, b].$$

*Then $g = f'$ a.e. in $[a, b]$.*

**Definition 1.2.6.** *Let $n \in \mathbb{N}$, and $[a, b]$ be a finite interval of $\mathbb{R}$. We denote by $AC[a, b]$ the space of all functions $f$ which are absolutely continuous on $[a, b]$ and by $C^k[a, b]$ the space of all function which are continuously differentiable up to order $k \in \mathbb{N} \cup 0$. We define*

$$AC^n[a, b] = \left\{ f : [a, b] \to \mathbb{R}, f \in C^{n-1}[a, b] \text{ and } f^{(n-1)} \in AC[a, b] \right\}.$$

**Definition 1.2.7.** *Let $X, Y$ be Banach spaces. We say that a function $f : X \to Y$ is (global) Lipschitz continuous if there exists a real constant $C \geq 0$ such that for all $x_1, x_2 \in X$ it holds that*

$$\|f(x_1) - f(x_2)\|_Y \leq C \|x_1 - x_2\|_X .$$

Next, let $\Omega \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$ be open and nonempty.

**Definition 1.2.8.** *Let $p$ be a positive real number. The set of all measurable functions $u$ defined on $\Omega$ with*

$$\int_\Omega |u(x)|^p \, dx < \infty \tag{1.1}$$

*is denoted by $L^p(\Omega)$. In this set, all function that are equal almost everywhere are identified, so elements of $L^p(\Omega)$ are precisely the classes of equivalence. For convenience this distinction is ignored. For measurable function $u$, it is written $u \in L^p(\Omega)$ if $u$ satisfies (1.1), and $u = 0$ if $u$ is equal to $0$ almost everywhere in $\Omega$.*

It is clear that $L^p(\Omega)$ is a vector space; moreover, furnished with the norm defined as

$$\|u\|_{L^p(\Omega)} = \left( \int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, it forms a Banach space. Specially, for $p = 2$, one can naturally define a scalar product by

$$(u, v) = \int_\Omega u(x)v(x) \, \mathrm{d}x. \tag{1.2}$$

The set $L^2(\Omega)$ endowed with this scalar product is a Hilbert space.

**Definition 1.2.9.** *A measurable function $u$ defined on $\Omega$ is called essentially bounded on $\Omega$ if there exists a constant $C$*

$$\inf_{A \in \mathscr{N}} \sup_{x \in \Omega \setminus A} |u(x)| \leq C,$$

*where $\mathscr{N}$ is a set of all subset of $\Omega$ that have zero Lebesgue measure. Then, the lowest of such a $C$ is called the essential supremum of $u$ on $\Omega$ and is denoted by* $\operatorname{ess\,sup}_{x\in\Omega}|u(x)|$*. Moreover, we denote by $L^\infty(\Omega)$ the set of all essentially bounded functions on $\Omega$, again we identify all functions that are equal a.e. on $\Omega$ in the same way as above.*

The set $L^\infty(\Omega)$ with the norm defined as

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$$

forms a Banach space.

In the next theorem, we use the general notation long-established in the measure and integration theory, see [30]. Let $(\Omega, \Sigma, \mu)$ be a measurable space and $L^1(\mu) = L^1(\Omega, \Sigma, \mu)$.

**Theorem 1.2.7** (Lebesgue's dominated convergence theorem)**.** *Suppose $f_n, g_n, g \in L^1(\mu)$, $f_n \to f$ a.e., $|f_n| \le g_n$, $g_n \to g$ a.e. and*

$$\int_\Omega g_n d\mu \to \int_\Omega g d\mu.$$

*Then $f \in L^1(\mu)$ and*

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

*(In most typical applications of this theorem $g_n = g \in L^1(\mu)$ for all $n$.)*

**Definition 1.2.10.** *By $L^1_{loc}(\Omega)$ we denote the set of all locally integrable functions, thus, all measurable functions $u : \Omega \to \mathbb{R}$ such that for every compact subset $M$ of $\Omega$*

$$\int_M |u(x)|\ dx < \infty.$$

**Definition 1.2.11.** *For $u : \Omega \to \mathbb{R}$ set*

$$supp\ u := \overline{\{x : u(x) \neq 0\}}$$

*is called a support of $u$.*

**Definition 1.2.12.** *We denote by $C_0^\infty(\Omega)$ a set of all infinitely differentiable functions $u : \Omega \to \mathbb{R}$ with a compact support in $\Omega$.*

**Definition 1.2.13.** *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ where $\alpha_i$ are nonnegative integers for $1 \le i \le n$. We call $\alpha$ a multi-index.*

We denote by $x^\alpha$, with multi-index $\alpha$, the monomial $x^{\alpha_1}\cdots x^{\alpha_n}$, which has degree $|\alpha| = \sum_{i=1}^n \alpha_i$. Similarly, if $D^i = \frac{\partial}{\partial x_i}$, for $1 \le i \le n$, then $D^\alpha = D^{\alpha_1}\cdots D^{\alpha_n}$ is a differential operator of order $|\alpha|$.

**Definition 1.2.14.** *Let $u \in L^1_{loc}(\Omega)$ and $\alpha$ be a multi-index. If there exist a function $v_\alpha \in L^1_{loc}(\Omega)$ satisfying*

$$\int_\Omega u(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\varphi(x)\, dx$$

*for every $\varphi \in C_0^\infty(\Omega)$, then we call $v_\alpha$ an $\alpha$-th weak derivative of $u$.*

**Definition 1.2.15.** *Let $k \in \mathbb{Z}_0^+$ and $1 \le p \le \infty$. By $W^{k,p}(\Omega)$ we denote a set of all functions $u : \Omega \to \mathbb{R}$ such that for every multi-index $\alpha$, $|\alpha| \le k$, the weak derivative $D^\alpha u \in L^p(\Omega)$.*

The set $W^{k,p}(\Omega)$ equipped with a norm defined as

$$\|u\|_{W^{k,p}(\Omega)} = \left\{ \sum_{0\le|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right\}^{\frac{1}{p}} \qquad \text{if} \quad 1 \le p < \infty,$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{0\le|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)} \qquad \text{if} \quad p = \infty$$

forms a Banach space.

**Definition 1.2.16.** *By $W_0^{k,p}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

Spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ endowed with the above norms are called *Sobolev spaces*. For $p = 2$, we denote $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ equipped with the scalar product similarly defined as in (1.2) are Hilbert spaces.

**Theorem 1.2.8** (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with Lipschitz boundary $\partial\Omega$. If $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$, then there exists bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

*(i) $Tu = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,*

*(ii) $\|Tu\|_{L^p(\partial\Omega)} \le C\, \|u\|_{W^{1,p}(\Omega)}$, with $C = C(p, \Omega)$.*

*We call $Tu$ the trace of $u$ on $\partial\Omega$.*

Let $(I, \Sigma, \mu)$ be a space with $\sigma$-finite and complete measure, where $\Sigma$ is a $\sigma-$algebra of measurable sets on $I$, and $\mu : \Sigma \to [0, \infty)$ is a measure. Let $X$ be a Banach space and $f : I \to X$. We call $f$ *simple* if there exist a finite collection of disjoint sets with finite measure $M_1, .., M_k \in \Sigma$ and $x_1, ..., x_k$ such that $f = \sum_{i=1}^{k} \chi_{M_i} x_i$. We define the integral of simple function $f$ as $\int f = \int_I f d\mu := \sum_{i=1}^{k} \mu(M_i) x_i$. It holds $\left\| \int f \right\| \le \int \|f\|$, where the integral on the right hand side (r.h.s.) is the Lebesgue integral of the simple function $\|f\| : I \to [0, \infty)$. We call $f$ *measurable* if there exists a sequence of simple functions $\{f_k\}$ such that $f_k(t) \to f(t)$ for a.e. $t \in I$. Function $f$ is called *integrable* if there exist a sequence of simple functions $\{f_k\}$ such that $f_k(t) \to f(t)$ for a.e. $t \in I$ and $\int_I \|f_k - f\| \, d\mu \to 0$. We define the *Bochner integral* for an integrable function $f$ as $\int_I f d\mu := \lim_{k \to \infty} \int_I f_k d\mu$.

**Theorem 1.2.9.** *The measurable function $f : I \to X$ is (Bochner) integrable if and only if $\|f\|$ is (Lebesgue) integrable. If $f$ is integrable then it holds $\left\| \int f \right\| \le \int \|f\|$.*

**Theorem 1.2.10.** *Let $f : I \to X$ be an integrable function and $Y$ be a Banach space. If $A : X \to Y$ is a linear, continuous operator, then $Af : I \to Y$ is integrable and $\int Af = A(\int f)$.*

**Definition 1.2.17.** *Let $1 \le p \le \infty$. The space $L^p((0,T), X)$ consists of all measurable functions $w : [0,T] \to X$ such that*

$$\|w\|_{L^p((0,T),X)} = \left( \int_0^T \|w(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty \quad \text{if} \quad 1 \le p < \infty,$$

*and*

$$\|w\|_{L^\infty((0,T),X)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|w(t)\|_X < \infty.$$

**Definition 1.2.18.** *The space $C([0,T], X)$ is a space of all continuous functions $w : [0,T] \to X$ such that*

$$\|w\|_{C([0,T],X)} = \max_{t \in [0,T]} \|w(t)\|_X < \infty.$$

Assuming $X$ to be a Banach space, the spaces $L^p((0,T), X)$ and $C([0,T], X)$ equipped with the norms from the above definitions are also Banach spaces.

**Definition 1.2.19.** *For $w \in L^1((0,T), X)$, we define $v \in L^1((0,T), X)$ to be a weak derivative of $w$, writing*

$$w' = v,$$

*if for all test functions $\varphi \in C_0^\infty(0,T)$, it holds that*

$$\int_0^T \varphi'(t) w(t) \, dt = - \int_0^T \varphi(t) v(t) \, dt.$$

**Definition 1.2.20.** *We define the Sobolev space $W^{1,p}((0,T),X)$, for $1 \leq p \leq \infty$, as a space of all $w \in L^p\left((0,T),X\right)$ such that the weak derivative $w'$ exists and belongs to $L^p\left((0,T),X\right)$.*

The space $W^{1,p}((0,T),X)$ furnished with the norm defined as

$$\|w\|_{W^{1,p}((0,T),X)} = \left(\int_0^T \left(\|w(t)\|_X^p + \|w'(t)\|_X^p\right) \mathrm{d}t\right)^{\frac{1}{p}} < \infty \quad \text{if} \quad 1 \leq p < \infty$$

and

$$\|w\|_{W^{1,\infty}((0,T),X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \left(\|w(t)\|_X^p + \|w'(t)\|_X^p\right) < \infty \quad \text{if} \quad p = \infty$$

is a Banach space. In case when $p = 2$, we write $H^1((0,T),X) = W^{1,2}((0,T),X)$.

**Theorem 1.2.11.** *Let $1 \leq p \leq \infty$ and $w \in W^{1,p}((0,T),X)$. Then, the following is true*

*(i)* $w \in C\left([0,T],X\right)$ *(it might be necessary to redefined it on the set of zero measure),*

*(ii)* $w(t) = w(s) + \int_s^t w'(r)\, dr$ *for every* $s,t \in [0,T]$, $s \leq t$,

*(iii) it holds that*

$$\|w\|_{C([0,T],X)} \leq C(T)\,\|w\|_{W^{1,p}((0,T),X)}\,.$$

**Definition 1.2.21.** *Let $u, v : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$. We define a convolution of $u$ and $v$ at $x \in \mathbb{R}^d$ as*

$$(u * v)(x) := \int_{\mathbb{R}^n} u(x-y)v(y)dy,$$

*if the integral on the right hand side exists.*

Notice that for functions $u, v : [0,\infty] \to \mathbb{R}$, which we additionally define to be zero outside their definition domain, the integration limits in the definition of the convolution reduce so that we obtain

$$(u * v)(x) = \int_0^x u(x-y)v(y)dy \quad \text{for } x \in \mathbb{R},$$

we call the convolution on the positive half-line also the *Laplace convolution.*

## 1.3 Equalities and inequalities

**Theorem 1.3.1** (Young's inequality). *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \in \mathbb{R}$ then it holds*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q$$

*for $C_\varepsilon = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$.*

**Theorem 1.3.2** (Hölder's inequality). *Let $1 \leq p, q \leq \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, then it holds*

$$\int_\Omega |u(x)v(x)| \; dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} .$$

**Theorem 1.3.3** (Discrete Hölder's inequality). *Let $1 \leq p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{R}^n$ then it holds that*

$$\sum_{i=1}^{n} |a_i b_i| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}} .$$

**Theorem 1.3.4** (Cauchy-Schwarz inequality). *Let $x, y \in H$, then it holds*

$$|(x, y)| \leq \|x\| \|y\| .$$

**Theorem 1.3.5** (Grönwall's inequality (integral form)). *Let $u(t)$ be a nonnegative, summable function on $[0, T]$ which satisfies the integral inequality*

$$u(t) \leq C_1 + C_2 \int_0^t u(s) \, ds$$

*for a.e. $t$, where $C_1, C_2 \geq 0$. Then*

$$u(t) \leq C_1 \left(1 + C_2 t e^{C_2 t}\right)$$

*for a.e. $t \in [0, T]$.*

**Theorem 1.3.6** (Grönwall's inequality (discrete form)). *Let $\{a_i\}$, $\{B_i\}$ be sequences of nonnegative real numbers and $C \geq 0$. Let*

$$a_n \leq B_n + \sum_{i=1}^{n-1} C a_i$$

*for $n \in \mathbb{N}$. Then,*

$$a_n \leq B_n + e^{nC} \sum_{i=1}^{n-1} C B_i$$

*for $n \in \mathbb{N}$.*

**Theorem 1.3.7** (Abel's summation)**.** *Let $\{u_i\}$ be a subset of a Hilbert space, then*

$$2\sum_{i=1}^{n}(u_i, u_i - u_{i-1}) = \|u_n\|^2 - \|u_0\|^2 + \sum_{i=1}^{n}\|u_i - u_{i-1}\|^2$$

*for $n \in \mathbb{N}$.*

In the following theorem, we use the standard notation $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$ for the gradient operator, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ for the Laplace operator applied on $u$, and $\boldsymbol{\nu}$ denotes the outer normal unit vector on the boundary $\partial\Omega$.

**Theorem 1.3.8** (Green's identity)**.** *Let $u, v \in C^2(\overline{\Omega})$, then it holds that*

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x)\, dx = -\int_{\Omega} u(x)\Delta v(x)\, dx + \int_{\partial\Omega} u(x)\nabla u(x) \cdot \boldsymbol{\nu} dS.$$

Notice that the identity is valid also for functions from appropriate Sobolev spaces.

**Theorem 1.3.9** (Friedrichs's inequality)**.** *Let $\Omega$ be a bounded domain with Lipschitz boundary. Let $\Gamma \subset \partial\Omega$, $|\Gamma| \neq 0$. Then for $u \in W^{1,2}(\Omega)$, we have that*

$$\|u\|_{W^{1,2}(\Omega)} \leq C\left(\int_{\Gamma} |u(x)|^2\, dS + \int_{\Omega} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2\, dx\right)^{\frac{1}{2}}.$$

**Theorem 1.3.10** (Young's inequality for convolutions)**.** *Let $1 \leq p, q, r \leq \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

*and $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, $n \in \mathbb{N}$. It holds that*

$$\|u * v\|_{L^r(\mathbb{R}^n)} \leq \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}.$$

## 1.4 Partial differential equations

Many natural phenomena, which are interesting and important to understand, predict and control, can be described by equations containing physical quantities and their rate of change in space or time (partial derivative). Such processes can be found in various fields of science such as physics, chemistry, finance, biology, etc. From the enormous number of them we name for instance electromagnetism [56], viscoelasticity [21], deformation of solid bodies [126], heat transfer

[32], wave propagation [1], chemical kinetics [62], option pricing [14], fluid mechanics [12], etc. Equations containing partial derivatives of quantities are called *partial differential equations (PDEs)*. The solution to those equations is not a number as to algebraic equations but a function.

There does not exist the uniform mathematical definition of a partial differential equation containing all the possible cases for such an equation. In general, we can say that a partial differential equation is an equation involving an unknown function $u$ dependent on two or more variables and containing one or more partial derivatives of $u$. The following definition covers what we said above.

**Definition 1.4.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and $x = (x_1, \ldots, x_n) \in \Omega$, next, let's have an unknown function $u : \Omega \to \mathbb{R}$ that satisfies the following formula*

$$F\left(x_1, \ldots, x_n, u(x), D^1 u(x), \ldots, D^k u(x)\right) = 0, \quad \text{for } x \in \Omega \tag{1.3}$$

*where $k \geq 1$, $D^i$ is a vector containing all $i$-th order partial derivatives of $u$ and*

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^{n^{k-1}} \times \mathbb{R}^{n^k} \to \mathbb{R}.$$

*We call (1.3) a $k$-th order partial differential equation and $u$ is called a solution of partial differential equation (1.3).*

To find a solution of a partial differential equation means to find all functions $u$ satisfying the equation, or, if we fail to find an explicit prescription, then proving the existence and other properties of the solution.

### Evolution equations

In the case when a natural phenomenon that we model with a partial differential equation evolves in time, the variable which represent the time is denoted by $t$, and it is assumed to be bigger or equal to zero. When considered, the final time is denoted as $T$. The variable representing space and spatial domain are then denoted by $x$ and $\Omega$, respectively, with $\Gamma = \partial\Omega$. In literature, this kind of equations are called *evolution equations*. Likely, the most known examples are the heat equation

$$\partial_t u - \Delta u = f \tag{1.4}$$

and the wave equation

$$\partial_{tt} u - \Delta u = f. \tag{1.5}$$

One of possible approaches to evolution equations is the change of perspective. Instead of looking for the value of solution in point $(x, t)$, we may look for the "state"

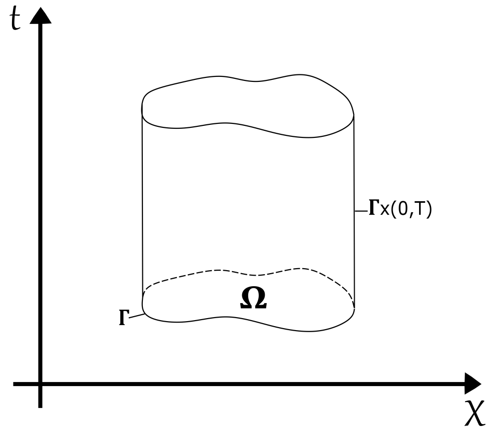



Figure 1.1: Space-time domain

of the system at every time $t$. Thus, in place of searching $u : \Omega \times (0, T) \to \mathbb{R}$, we rather look for mapping

$$u : (0, T) \to X,$$

where

$$t \to u(t),$$

and $X$ is a functional space, usually one of the Sobolev spaces. This change of perspective will naturally be reflected in the mathematical analysis of a problem. Above we have already defined spaces involving time which will take a role in the analysis. This approach also allows to consider regularity different for space and for time.

### Boundary and initial conditions

Often *boundary conditions* are accompanying an equation. These are conditions prescribing values which the solution of the equation should hold on the boundary of $\Omega$. Those conditions restricts the number of allowed solutions. The most standard boundary conditions are

(i) the Dirichlet boundary condition

$$u(x) = a(x) \quad \text{on} \;\; \Gamma,$$

(ii) the Neumann boundary condition

$$-\nabla u(x) \cdot \boldsymbol{\nu} = b(x) \quad \text{on} \;\; \Gamma,$$

(iii) the Robin boundary condition

$$\lambda_1 u(x) + \lambda_2 \nabla u(x) \cdot \boldsymbol{\nu} = c(x) \quad \text{on } \Gamma,$$

where $a, b, c$ are some functions, $\boldsymbol{\nu}$ is an outer normal vector to the boundary $\Gamma$, and $\lambda_1, \lambda_2 \neq 0$ are real numbers. It is clear that by the Dirichlet boundary condition, we prescribe a value of physical quantity on the boundary, which might be judged as slightly artificial from the physical point of view. The Neumann boundary condition seems to be more natural as it can be interpreted as the flux of a quantity coming from the domain to outside.

Besides the standard boundary conditions, there exists quite a big variety of others. For instance, there exists nonlinear version of all above boundary conditions. The nonlinear version for the Neumann boundary condition has form

$$f(x, u, \nabla u) = 0 \quad \text{on } \Gamma,$$

where $f$ is a real-valued continuous function, for example

$$\nabla u(x) \cdot \boldsymbol{\nu} = d(x)\sqrt{1 + \|\nabla u\|^2} \quad \text{on } \Gamma,$$

which is also known as the capillary boundary condition, see [11]. Other type of boundary condition may contain non-local terms such as an integral over the whole domain, for example

$$-\nabla u \cdot \boldsymbol{\nu} = \alpha u + \beta + \int_\Omega K u \,\mathrm{d}x \quad \text{on } \Gamma,$$

see [118].

In the situation when we consider an evolution equation the boundary condition for $t = 0$ is called an *initial boundary condition* and is prescribed separately from the rest of the boundary. There can exist also boundary conditions which contain a time-derivative of a solution. This kind of boundary conditions is called *dynamical boundary conditions*, for instance

$$-\partial_{tt} u(x,t) - \nabla u(x,t) \cdot \boldsymbol{\nu} = d(x,t) \quad \text{on } \Gamma \times (0,T),$$

that can be used to model a membrane that is vibrating on the part of the boundary [40].

### Well-posed problem

At first, one is naturally motivated to solve the problem for partial differential equation by finding an explicit formula for a solution which obeys boundary and initial conditions, if provided. Unfortunately, that is usually not an easy task. Therefore, the notion of well-posedness is introduced as it '*captures many of the desirable features of what it means to solve a PDE*'[35].

We say that a problem is $well-posed$ in the Hadamard sense if:

(i) the problem has a solution,

(ii) this solution is unique,

(iii) solution is continuously dependent on the data.

If one or more of the above conditions are not fulfilled, then we say that a problem is $ill-posed$. Notice that this informal definition is very general and might be applied for various problems not only the one for partial differential equations.

Very basic example of the well-posed problem is the integration of a function. On the other hand the differentiation is ill-posed problem as only a "small change" in data can produce a "big change" in the result.

### Classical and weak solution

In the upper section, we have mentioned a solution to a PDE, but we have not said what we precisely expect from a solution. Let's take the equation (1.4) for example. It seems unnecessary to ask the solution of that equation to be smooth. It should be satisfying that all spatial and temporal derivatives mentioned in the equation exist and are continuous. This solution is then called a *classical* solution.

Finding the classical solution can not be always achieved. For example, when we study PDEs modeling formation and propagation of shock waves it is reasonable to allow solutions that are not continuously differentiable or continuous. This brings us to the notion of a *weak* solution and $weak\ formulation$.

Let's firstly illustrate these notions on an example. Assume the simple Poisson equation

$$-\Delta u(x) = f(x) \quad \text{for} \quad x \in \Omega, \tag{1.6}$$

for some function $f \in C(\overline{\Omega})$, and let $u(x) = 0$ on the boundary $\Gamma$. Then a classical solution of that PDE would be a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (1.6) and the boundary condition $u = 0$ on $\Gamma$. If we now assume such a solution, multiply (1.6) with a function $\varphi \in C_0^\infty(\Omega)$ and integrate the whole equality over $\Omega$, we obtain

$$-\int_\Omega \Delta u(x)\varphi(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \quad \text{for} \quad \varphi \in C_0^\infty(\Omega). \tag{1.7}$$

Notice that although we assume the classical solution, the equation above and the boundary condition make sense also for $u \in H_0^1(\Omega) \cap H^2(\Omega)$. If we now apply the Green identity, Theorem 1.3.8, we gain

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) \, \mathrm{d}x = \int_\Omega f(x)\varphi(x) \, \mathrm{d}x \quad \text{for} \quad \varphi \in C_0^\infty(\Omega), \tag{1.8}$$

since $\varphi$ is vanishing on $\Gamma$, the boundary integral from the Green identity also vanishes. Ni, we can see that satisfactory condition for the integrals in the equation (1.8) to exist is $u, \varphi \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$. Thus, we can abandon the notion of the classical solution. Let $f \in L^2(\Omega)$, we say that the function $u \in H_0^1(\Omega)$ is a *weak solution* of (1.6) if the equation (1.8) holds for all $\varphi \in H_0^1(\Omega)$. We call the equation (1.8) a *weak* or *variational formulation* of (1.6). Functions $\varphi \in C_0^\infty(\Omega)$ are called *test functions*.

It is immediately clear that every classical solution is also a weak solution and it can be proven that every weak solution which has sufficient regularity is also a classical solution.

In general, we obtain the weak formulation of PDE if we follow the next steps [107]:

1. Multiply the PDE by a test function;
2. Integrate over $\Omega$;
3. Use the Green identity;
4. Involve boundary conditions, either by choosing proper function spaces or by substituting into the boundary integral appearing after application of Green's theorem.

Choosing the function space for the solution and for the test functions is highly dependent on the equation and the boundary conditions, and varies on a case-by-case basis. The evolution equations undergo the similar process to obtain the weak formulation, with the additional requirement, that the equation should hold in almost all (a.a.) $t \in (0, T)$, and additional assumptions on the initial conditions are needed.

## 1.5 Methods for solving PDEs

In this section, we briefly describe methods which we later use for proving the existence of solution of PDE and for computing the numerical approximation of

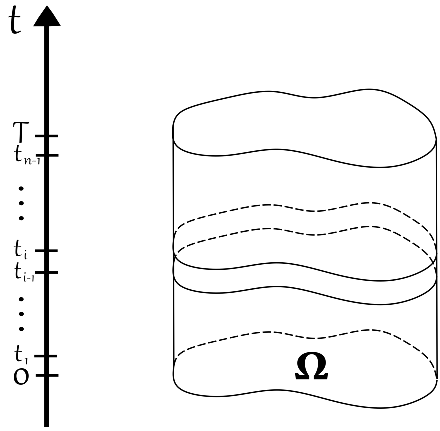



Figure 1.2: Time discretization

the solution. To prove the uniqueness of solution a typical approach is used, which consist of assuming at least two solutions of a problem and then proving that those two solutions are equal to each other.

### 1.5.1 Rothe's Method

To prove the existence of the solution of evolution PDE, we use a method proposed by Rothe [106], and later adopted and evolved by Ladyzhenskaya [70, 71], Rektorys [102], Kačur [57] and many others. The good description of the method can also be found in [107]. In this section, we briefly describe this method, which we later use on specific problems.

The first step of the method consists of the time discretization. Assuming $n$ to be an integer, we divide the domain $\Omega \times [0, T]$ into equidistant layers by the planes $t = i\tau$, $i = 1, ..., n$, see Figure 1.2. We introduce the equidistant (for the simplicity of notation) time-partitioning of the interval $[0, T]$ by the step $\tau = \frac{T}{n}$, for any $n \in \mathbb{N}$. The notation $t_i = i\tau$ is used for $i = 1, ..., n$. For any function $z$, we write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

We apply this discretization on the variational formulation of our problem by replacing the time derivative of the solution $u$ in the formulation by $\delta u_i$ and all the time dependent functions $z(t)$ by $z_i$. By this we gain an elliptic equation approximating our problem at time $t_i$, also called the discretized equation. Then we proceed in the following steps:

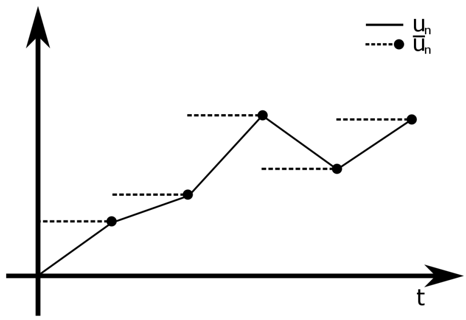



Figure 1.3: Rothe's functions

(a) Solving elliptic problem.
Applying the time discretization on the variational formulation leaves us with an elliptic problem on the every time line $t_i$. The existence and uniqueness of the problem are usually tackled by the Lax-Milgram theorem, see Theorem 1.2.4, or by the theory of monotone operators for nonlinear PDEs.

(b) A priori estimates for $u_i$.
One derives a priori estimates from the discretized variational formulation by choosing a suitable test function and by using proper inequalities. Those estimates, often called also the energy estimates serve later for proving the convergence.

(c) Introduction and convergence of the Rothe functions.
We define the Rothe functions (see Figure 1.3) in the following manner as the mappings $u_n, \overline{u}_n, \tilde{u}_n : [0,T] \to L^2(\Omega)$ with

$$u_n : t \mapsto \begin{cases} u_0, & t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i, & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases}$$

$$\overline{u}_n : t \mapsto \begin{cases} u_0, & t = 0 \\ u_i, & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases} \tag{1.9}$$

$$\tilde{u}_n : t \mapsto \begin{cases} u_0, & t \in [0, \tau] \\ \overline{u}_n(t - \tau), & t \in (t_{i-1}, t_i], \quad 2 \le i \le n, \end{cases}$$

Then thanks to a priori estimates, we are able to prove the convergence of subsequences of above functions to a function in appropriate spaces.

(d) Convergence of the approximation scheme.
Using the above functions, we are able to rewrite the discretized equation for the whole time-interval and prove its convergence to the variational formulation.

(e) Regularity of the solution.
With the help of a priori estimates, we can sometimes prove better quality of the weak solution or that the weak solution is also the classical solution.

There are two advantages of this method: besides getting the existence of the solution it also proposes an interesting algorithm for obtaining the numerical approximation, consisting of calculating the numerical solution on every time layer.

The important lemma addressing the convergence of the Rothe functions can be found in [57, Lemma 1.3.13]. We also state it here.

**Lemma 1.5.1.** *Let $V, Y$ be reflexive Banach spaces and let the imbedding $V \hookrightarrow Y$ be compact. If the estimates*

$$\int_I \left\| \frac{du_n(s)}{ds} \right\|_Y^2 ds \le C, \quad \|\overline{u}_n(t)\| \le C \quad \text{for all } t \in I$$

*hold for all $n \ge n_0 > 0$ then there exist $u \in C(I,Y) \cap L^\infty(I,V)$ with $\frac{du}{dt} \in L^2(I,Y)$ ($u$ is differentiable a.e. in $I$) and a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that*

$$u_{n_k} \to u \text{ in } C(I,Y),$$

$$u_{n_k}(t) \rightharpoonup u(t), \quad u_{n_k}(t) \rightharpoonup u(t) \text{ in } V \text{ for all } t \in I,$$

*and*

$$\frac{du_{n_k}}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(I,Y).$$

*Moreover, if*

$$\left\| \frac{du_n(t)}{dt} \right\|_Y \le C \quad \text{for all } n \ge n_0 \text{ and a.e. } t \in I,$$

*then $\frac{du}{dt} \in L^\infty(I,Y)$ and $u : I \to Y$ is Lipschitz continuous, i.e.*

$$\|u(t) - u(t')\|_Y \le C\,|t - t'| \quad \text{for all } t, t' \in I.$$

### 1.5.2 Finite element method

To get the numerical solution at time $t_i$ from the elliptic problems mentioned in the previous section, the finite element method (FEM) will be applied. The method was developed by engineers in the aircraft industry circa in the middle of the previous century. Since then ,the method was applied to solve various problems in the variety of scientific domains. The method was also intensively studied by mathematicians, therefore, there exists extensive literature about it; we may mention for example [15, 22, 41, 63, 90, 148], which is just a small selection.

The main idea of the method is quite straightforward, but there are details which may be complicated or extensive to write. The FEM is based on the division of the spatial domain into *finite elements* and approximating the solution of the problem by the finite set of basis functions. Then the original variational formulation problem transforms into a discrete problem where just the finite number of unknown coefficients is sought.

Assume $H$ to be a Hilbert space and $V$ its (closed) subspace. Let consider a general linear variational problem

$$a(u, v) = b(v) \qquad \forall\, v \in V, \tag{1.10}$$

where $u \in V$ is an unknown, $a$ is a bilinear form

$$a : V \times V \to \mathbb{R},$$

satisfying the assumptions from the Lax-Milgram theorem 1.2.4 and $b$ is a bounded linear functional

$$b : V \to \mathbb{R}.$$

Let $V_h$ be a finite-dimensional subspace of $V$. Then we discretize the variational problem (1.10) in the following way: we look for $u_h \in V_h \subset V$ such that

$$a(u_h, v) = b(v) \qquad \forall\, v \in V_h. \tag{1.11}$$

This formulation is often called the Galerkin approximation problem or discrete problem. Thanks to the assumptions on $a$ and $b$, the existence and uniqueness of the solution to the discrete variational problem is guaranteed by the Lax-Milgram theorem.

To understand better the relation between $u$ and $u_h$ let's subtract the two above formulations from each other, so, we obtain

$$a(u - u_h, v) = 0 \qquad \forall\, v \in V_h.$$

The relation is often called Galerkin orthogonality or orthogonality property. The equality may be simply interpreted when we assume $a$ to be symmetric. Then $(\cdot,\cdot)_a = a(\cdot,\cdot)$ defines an inner product on $V$, with an induced norm defined by $\|v\|_a = \sqrt{a(v, v)}$. Hence, the function $u_h$ is an orthogonal projection of $u$ onto the space $V_h$. The following theorem estimates the error between $u$ and $u_h$ in the space $V$.

**Theorem 1.5.1** (Céa). *Suppose that $u$ solves the variational problem (1.10). Then for the discrete problem (1.11) we have*

$$\|u - u_h\|_V \leq \frac{C_1}{C_2} \min_{v \in V_h} \|u - v\|_V ,$$

*where $C_1$ and $C_2$ are the same constants as in Theorem 1.2.4.*

Thus, the above theorem says that approximation error depends directly on the choice of the space $V_h$ but not on the choice of the basis functions of $V_h$.

Let $\{\phi_i\}_{i=1}^N$ be a basis in the discrete space $V_h$. We may assume

$$u_h = \sum_{j=1}^{N} U_j \phi_j ,$$

and substituting this together with $v = \phi_i$ for $i = 1, ..., N$ into (1.11) gives us

$$\sum_{j=1}^{N} U_j a(\phi_j, \phi_i) = b(\phi_i) \qquad i = 1, ..., N.$$

Introducing the notation

$$\begin{array}{rcll} A_{ij} & = & a(\phi_j, \phi_i), & i, j = 1, ..., N, \\ b_i & = & b(\phi_i), & \end{array} \tag{1.12}$$

we may rewrite this into

$$AU = b, \tag{1.13}$$

where the matrix $A = (A_{ij})_{i,j=1,...,N}$, $b = (b_1, .., b_N)$ and $U = (U_1, ..., U_N)$. Then to find the approximate solution $u_h$ means to find the solution $U$ to the linear system of equations (1.13). The matrix $A$ is called the $stiffness\ matrix$, its invertibility is assured by assuming the property $(ii)$ in the Lax-Milgram theorem for the bilinear form $a$.

In fact, the method of discretization we describe above is called in general the Galerkin method. In the method, the space $V_h$ may be chosen in many ways. If we choose $V_h$ to be the space of piecewise polynomial functions, then we talk about the finite element method. The following example shows one of the simplest choices for such a space.

**Example 1.5.1.** *Assume $\Omega = (0, 1)$ and choose $n$ points $x_i$ in this interval such that $0 = x_0 < x_1 < ... < x_n < x_{n+1} = 1$. Then we define*

$$\begin{array}{rl} V_h = & \{v : [0,1] \to \mathbb{R} : v \in C([0,1]), \\ & v|[x_i, x_{i+1}] \textit{ is linear for } i = 0, ..., n, \textit{ and } v(0) = v(1) = 0\}. \end{array}$$

Notice that in the above example the domain $\Omega$ is divided into the smaller subdomains. This is a common approach. To define $V_h$ the domain $\Omega$ is divided into a finite set of cells $T_h = \{T\}$ so that $\Omega = \cup_{T \in T_h} T$. The cells are typically intervals, triangles, quadrilaterals, tetrahedral or another simple polygonal shape. The division of the domain has its own rules, which is a separate topic, and we will omit it here. When we have the partitioning of the domain, then we define a local function space $V$ on each cell $T$. The cell $T$ with the space $V$ and a set of rules which describe the functions in $V$ is called a *finite element*. The function in $V_h$ then can be locally represented in terms of the local function space $V$. Next step is to patch those spaces together with help of so-called local-to-global mapping. Finally, the global function space $V_h$ is defined as a set of functions on $\Omega$ satisfying that for every $v \in V_h$ it holds $v|_T \in V_T$ for all $T$ in $T_h$ and some natural pairing conditions for the local-to-global mapping. This is a very dense description of how the spaces $V_h$ are constructed; however, we will omit details due to their extensiveness. One may find them in every book mentioned in the beginning of this section.

#### FEM solvers

The FEM was developed as an efficient way how to calculate the numerical solution to the PDEs with the help of computers. Since fifties there have been implemented many software packages on many platforms. Among the well established ones we can find for example deal.II, DUNE, FEniCS Project, FreeFem++, GetFEM++, ADINA, COMSOL Multiphysics, and many others.

The FEniCS Project software will be used in our numerical experiments later in our thesis. The FeniCS Project is an open-source computing platform for solving PDEs with high-level Python and C++ interfaces. It consists of a collection of components such as DOLFIN, FFC, FIAT, Instant, UFC, UFL, etc. each covering a certain area and together forming a robust tool. For further information visit the website of the project: www.fenicsproject.org or see [6, 61, 73].

## 1.6 Inverse problems in PDEs

The notion of inverse problems covers a set of various problems coming from a variety of science fields. Considering a partial differential equation with all necessary additional conditions, one is naturally interest in the problem of finding a solution of the PDE, we call this a direct problem. In the case when the PDE itself is unknown, then we talk about an inverse problem. Those kind of problems arise in fields as optics, radar acoustics, signal processing, medical imaging, computer vision, geophysics, oceanography, astronomy, machine learning, etc.

**Example 1.6.1** (The inverse problem of gravimetry). *Let $u$ be a gravitational field, which can be measured by the gravitational force $\nabla u$ and is generated by the mass distribution $f$. This field is a solution of the Poisson equation*

$$-\Delta u = f$$

*in $\mathbb{R}^3$, where $\lim_{|x|\to\infty} u(x) = 0$, we also assume $f$ to be zero outside a bounded domain $\Omega$. The inverse problem in gravimetry is to find the function $f$ given the gravitational force $\nabla u$ on $\Gamma$ which is a part of the boundary $\partial\Omega$. The problem is fundamental for gravitational navigation.*

**Example 1.6.2** (The inverse conductivity problem). *Assume the conductivity equation for electric potential $u$*

$$-div(c\nabla u) = 0 \quad \text{in } \Omega,$$

*with the boundary condition*

$$u = g_0 \quad \text{in } \partial\Omega,$$

*where $c$ is a scalar function. The inverse conductivity problem is to find the function $c$ given $g_0$ on the boundary $\partial\Omega$. The problem is a mathematical foundation to electrical impedance tomography, mine and rock detention, and the search for underground water.*

**Example 1.6.3** (Tomography). *The inverse problem is to find function $f$ given the integrals*

$$\int_\gamma f d\gamma$$

*over a family of manifolds.*

For a comprehensive summary of inverse problems in PDEs, we refer to [99], [51].

### 1.6.1 Inverse source problems

An interesting type of the inverse problem in the PDEs is finding a source term function or one of its components. By the source we usually understand the right side of a differential equation. Inverse gravimetry is a classical example of an inverse source problem. In the evolution equation the source function $F$ is often assumed to have a specific shape, for example

$$F(x,t) = f(x)h(t)$$

or

$$F(x,t) = f(x)h(x,t) + g(x,t),$$

where usually only one component is sought. In the first case, the function $f$ could be interpreted as the location of the source in space, and the function $h$ might describe the evolution of the source in the time.

In the inverse source problem, we are interested in finding a couple consisting of a solution of the equation and a source function. To identify both functions, it is necessary to have some additional information, so-called measurement. The measurement can take several forms. For finding a solely space-dependent part of the source one uses usually a space-dependent measurement, such as a state of system at the final time

$$u(x,T) = M(x), \quad x \in \Omega.$$

In case when we are interested in the reconstruction of a time-dependent part of the source, we use a time-dependent measurement too. One may consider two types of measurement: integral and point. In the integral measurement the solution is measured with the help of a sensor which makes a certain averaging over the domain. This can be mathematically represented by an integral

$$\int_\Omega \omega(x)u(x,t)\,\mathrm{d}x = m(t), \quad t \in [0,T]$$

or

$$\int_{\partial\Omega} \omega(x)u(x,t)dS = m(t), \quad t \in [0,T].$$

The function $\omega$ is usually assumed to have a compact support in $\Omega$, respectively, in $\Gamma$. That means we measure only through subdomain of $\Omega$, respectively, $\Gamma$. The measurement over the part of the boundary is called also non-invasive, since it does not require information from inside the domain. Another type of the measurement is a point measurement. The function is measured in the specific point $x_0 \in \Omega$, so we have

$$u(x_0,t) = m(t), \quad t \in [0,T].$$

There exist several approaches for solving the inverse source problems. The approaches differ for different type of measurement and also whether time-dependent or space-dependent is sought. We will be interested in the recovery of the time-dependent part of the source. A very common tool used for proving the existence and uniqueness of the problem is the Banach fixed point theorem, e.g. [133, 138]. We will use a method based on the application of the measurement on the equation and then on the elimination of the function in which we are interested. This provides us one more equation that defines the problem and change the inverse problem to a direct one.

## 1.7 Fractional derivative

In this section, we will give a brief introduction to the fractional derivative and fractional calculus. One can find a comprehensive summary for example in [28, 58, 89, 98].

The history of fractional calculus goes back to the 17th century; to the period when Newton and Leibniz were developing the foundation for the differential and integral calculus. Leibniz was the first who introduced in his correspondence with Newton the symbol

$$\frac{d^n}{\mathrm{d}x^n} f(x)$$

for the $n-th$ derivative of function $f$ at a point $x$, with the implicit assumption that the $n$ in the symbol belongs to $\mathbb{N}$. But Newton asked Leibniz the following question

$$\textit{“What does } \frac{d^n}{\mathrm{d}x^n} f(x) \textit{ mean if } n = \frac{1}{2} \textit{?”} \quad (1695)$$

to which Leibniz carefully replied:

*“This is an apparent paradox from which, one day,*
*useful consequences will be drawn...”*

Form this time on, the concept of fractional derivative was usually ignored or taken purely as a mathematical toy. This is suggested also by the fact that the first book, by Oldham and Spanier [93], devoted exclusively to the subject of fractional calculus was published in 1974.

Since then, however, a plethora of applications of fractional calculus appeared in various fields of science such as material theory [83, 120], viscoelastic materials [38, 112], anomalous processes [20, 84, 145], transport processes [10, 13, 52, 96], fluid flow phenomena [26], earthquakes [64, 68], solute transport [95, 114], chemistry [29, 47], wave propagation [36, 42, 43, 115], signal theory [7, 31], image processing [8, 24], biology [76], relaxation of polymers [39], electromagnetic theory [33, 34], thermodynamics [23, 121], mechanics [104, 105], astrophysics [72], finance [59, 69, 82, 137], control theory [18, 88, 97, 125], chaos and fractals [94, 144, 146], human behavior [5, 123] and many more.

The fractional calculus is based on the idea of generalizing of the derivative of order $n \in \mathbb{N}$ to the derivative of order $\alpha \in \mathbb{R}$. The name “fractional calculus” itself is a one of misnomers since it deals with integrals and derivatives of arbitrary order not just fractional one. There are some expectations from such a generalized differential operator $D^\alpha$. Among the basic ones belongs additivity of the derivative

$$D^\alpha D^\beta = D^{\alpha+\beta},$$

then the restriction of the fractional operator on the natural numbers should coincide with the classical derivative, so

$$D^\alpha = \frac{d^\alpha}{dx^\alpha} \quad \text{for } \alpha \in \mathbb{N},$$

and, of course, we expect

$$D^\alpha = I \quad \text{for } \alpha = 0.$$

The next question is how to construct such an operator. The motivation comes from the basic integration, as the inverse operation to the differentiation, and the definition of the fractional integral. Let's assume that $[a, b]$ is a finite interval of the real line $\mathbb{R}$, $\alpha \in \mathbb{R}^+$ and

$$n = \lfloor \alpha \rfloor + 1,$$

with $\lfloor \cdot \rfloor$ being the floor function. We define

$$F(x) := \int_a^x f(t)\,\mathrm{d}t,$$

for $f \in L^1(a, b)$, then we know that $F$ is differentiable and $F' = f$ a.e. in $[a, b]$. By simple repetition of the above integration, we get

$$\int_a^x \cdots \int_a^{t_2} f(t_1)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n = \frac{1}{(n-1)!}\int_a^x (x-\tau)^{n-1} f(\tau) d\tau =: I_a^n f(x).$$

Now, we want to replace $n \in \mathbb{N}$ for $\alpha \in \mathbb{R}_0^+$. The factorial in the denominator of the definition is replaced by its generalization, the Gamma function, defined as

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x}\,\mathrm{d}x, \quad z \in \mathbb{C},$$

thus by replacing, we get that

$$I_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau,$$

for $\alpha > 0$ and we additionally define

$$I_a^0 := I.$$

We call $I_a^\alpha f$ a *fractional integral* of function $f$.

**Lemma 1.7.1.** *Let $f \in L^1[a, b]$ and $\alpha > 0$. Then integral $I_a^\alpha f(x)$ exists for almost every $x \in [a, b]$. Moreover, the function $I_a^\alpha f$ itself is also an element of $L^1[a, b]$.*

**Property 1.7.1.** *Let $f \in L^1[a,b]$ and $\alpha, \beta > 0$. The following properties hold:*

$$\begin{array}{ll} I_a^\alpha I_a^\beta f = I_a^{\alpha+\beta} = I_a^\beta I_a^\alpha f & \text{a.e. on } [a,b], \\ D^n I_a^n f = f & \text{for } n \in \mathbb{N}, \\ D^n f = D^m I_a^{m-n} f & \text{for } n, m \in \mathbb{N}, m > n. \end{array}$$

**Example 1.7.1.** *Let $[a,b]$ be a finite interval of the real line, $y(t) = (t-a)^{\beta-1}$, and $\alpha, \beta > 0$ then*

$$(I_a^\alpha y)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\beta+\alpha-1}. \tag{1.14}$$

With the definition of the fractional integral one may proceed to the definition of fractional derivative. The definition combines the classical derivative and the fractional integral. There are two ways how to do it. We can either first use the fractional integration and than derivation or otherwise. Those approaches bring two slightly different results.

**Definition 1.7.1.** *We define the Riemann-Liouville fractional derivative of order $\alpha \geq 0$ as*

$$(D_a^\alpha y)(x) := (D^n I_a^{n-\alpha} y)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a. \tag{1.15}$$

**Definition 1.7.2.** *The Caputo fractional derivative of order $\alpha \geq 0$ is defined as*

$$({}^C D_a^\alpha y)(x) := (I_a^{n-\alpha} D^n y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a. \tag{1.16}$$

If $\alpha = m \in \mathbb{N}_0$, then ${}^C D_a^\alpha y$ and $D_a^\alpha y$ coincide with the classical derivative of $y$, in particular

$$(D_a^0 y)(x) = ({}^C D_a^0 y)(x) = y(x).$$

**Theorem 1.7.1.** *Let $\alpha \geq 0$. If $y \in AC^n[a,b]$ then the fractional derivatives $D_a^\alpha y$ and ${}^C D_a^\alpha y$ exist almost everywhere on $[a,b]$. Moreover, to following relation holds*

$$({}^C D_a^\alpha y)(x) = (D_a^\alpha y)(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. \tag{1.17}$$

We address the additivity of Riemann-Liouville fractional derivative in the next theorem. The additivity property is slightly more complicated; therefore, we state only the basic property for the Riemann-Liouville derivative. For more details look in [58].

**Property 1.7.2.** *Assume $\alpha, \beta \geq 0$. Moreover let $\phi \in L^1[a,b]$ and $f = I_a^{\alpha+\beta}\phi$, then*

$$D_a^\alpha D_a^\beta f = D^{\alpha+\beta} f.$$

The existence of the function $\phi$ is crucial, otherwise it could happen that $D_a^\alpha D_a^\beta f = D_a^\beta D_a^\alpha f \neq D^{\alpha+\beta} f$ or $D_a^\alpha D_a^\beta f \neq D_a^\beta D_a^\alpha f = D^{\alpha+\beta} f$.

**Example 1.7.2.** *Let $y(t) = (t-a)^{\beta-1}$, $\beta > 0$ then*

$$(D_a^\alpha y)(x) = \begin{cases} \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, & \text{if } \alpha-\beta \notin \mathbb{N} \\ & \text{or } \alpha-\beta \in \mathbb{N} \text{ and } \alpha-\beta \geq n, \\ 0, & \text{if } \beta = \alpha - j, j \in \{0, ..., n-1\}, \end{cases}$$

$$({}^C D_a^\alpha y)(x) = \begin{cases} \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq n+1 \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > n, \\ 0, & \text{if } \beta \in \{1, ..., n\}, \end{cases} \tag{1.18}$$

*in particular, notice the main difference in the fractional derivative of a constant*

$$\begin{aligned} (D_a^\alpha 1)(x) &= \frac{1}{\Gamma(1-\alpha)}(x-a)^{-\alpha}, \\ ({}^C D_a^\alpha 1)(x) &= 0. \end{aligned} \tag{1.19}$$

*Figure 1.4 shows the function $\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\cdot - a)^{\beta-\alpha-1}$ for various values of $\alpha$ and $\beta$.*

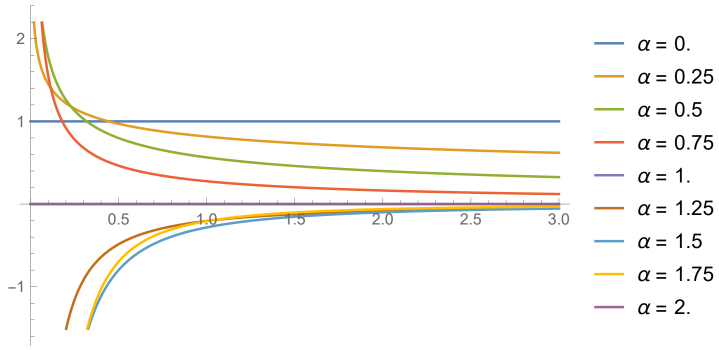
There are some basic differences between the Riemann-Liouville and the Caputo definition of fractional derivative. As we may see in the above example, the Caputo fractional derivative of constant is zero, what we would expect from a derivative, on the contrary, the Riemann-Liouville fractional derivative of a constant is a power function with a negative real exponent. The relation between both definition is given by (1.17). We can see that the definitions coincide if

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0.$$

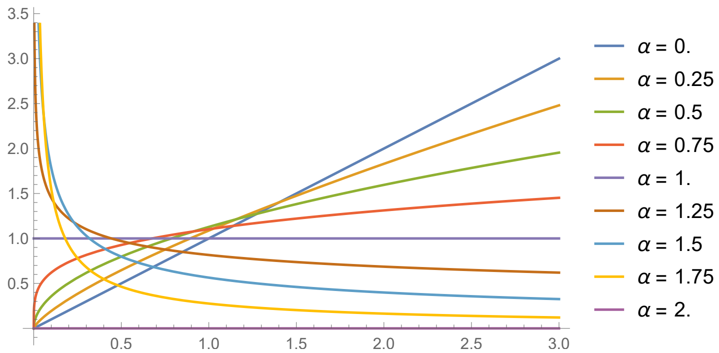
Assume basic fractional differential equations

$$(D_a^\alpha y)(t) = f(t), \qquad ({}^C D_a^\alpha y)(t) = f(t),$$

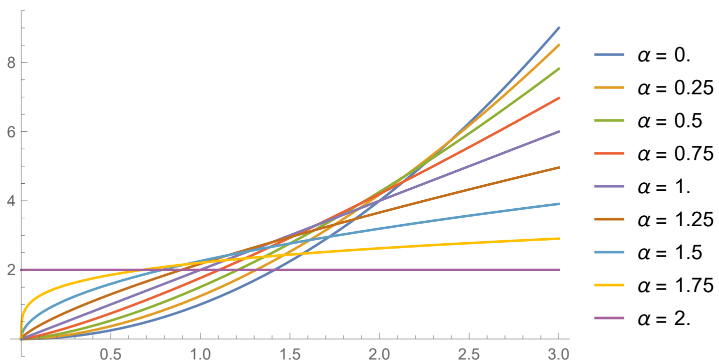
for $t \in (0,T)$ and $\alpha > 0$. We face different situations when we want to define the initial conditions for those equations. For the first equation the initial conditions




(a) $\beta = 1$




(b) $\beta = 2$




(c) $\beta = 3$

Figure 1.4: Function $\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}$ from Example 1.7.2, for $a = 0$ and various values of $\alpha$ and $\beta$

can be set in the terms of the fractional integrals as

$$(I_a^{n-\alpha}y)(0) = b_0, (D^1 I_a^{n-\alpha}y)(0) = b_1, ..., (D^{n-1} I_a^{n-\alpha}y)(0) = b_{n-1}.$$

Although, this is a mathematically correct way there seems to be no proper physical background for it. The derivative of a fractional integral does not have any known physical interpretation, therefore its use as the initial conditions is rather to be avoided. On the other hand, the initial conditions for the second equation, containing the Caputo fractional derivative, can be set in the form of the classical derivative as

$$y(0) = b_0, D^1 y(0) = b_1, ..., D^{n-1} y(0) = b_{n-1}.$$

It is one of the reasons why the Caputo fractional derivative occurs more in the evolutionary fractional differential equations.

The Caputo fractional derivative can be also rewritten as a convolution with a positive definite kernel. If we define

$$g_{n-\alpha}(t) = \frac{t^{n-\alpha}}{\Gamma(n-\alpha)}, \qquad t > 0,$$

we see that $g_{n-\alpha} \in L^1_{loc}(\mathbb{R})$, and then the definition of the Caputo fractional derivative can be rewritten as

$$^C D_a^\alpha y(t) = \left(g_{n-\alpha} * y^{(n)}\right)(t),$$

where $*$ stands for the convolution on the positive half-line, i.e.

$$(k * v)(t) = \int_0^t k(t-s) v(s)\,\mathrm{d}s. \tag{1.20}$$

We define the partial Caputo fractional derivative of order $\alpha \in (0,2)$ for the function $v$ defined on $(0,T) \times \mathbb{R}^d$ as

$$\partial_t^\alpha v(t) := \begin{cases} (g_{1-\alpha} * \partial_t v)(t), & \alpha \in (0,1), \\ (g_{2-\alpha} * \partial_{tt} v)(t), & \alpha \in (1,2), \\ \partial_t v(t), & \alpha = 1. \end{cases}$$

The above definition is also used for $v \in H^1((0,T),H)$ or $v \in H^2((0,T),H)$, where $H$ is a Hilbert space, and the Bochner integral is used in the convolution.

### Continuous time random walk

The derivation of the fractional diffusion equation is explained for instance in [3, 86]. The fractional diffusion equation was derived there using the continuous time random walk (CTRW) that can be considered to be a generalization of the Brownian motion, from which a classical diffusion equation may be derived. The derivation below is taken from [86].

In the Brownian walk the particle is assumed to jump in the constant discrete time step of length $\Delta t$ to one of its nearest neighbor positions in the square lattice (in 2D case) with the lattice distance $\Delta x$. The diffusion equation is then derived from the master equation [86].

In the CTRW model, the so-called jump probability density function (PDF) $\psi(x,t)$ is assumed. From this PDF the length of a given jump and the time between two jumps is possible to derive. The jump length PDF is given by

$$\lambda(x) = \int_0^\infty \psi(x,t)\,\mathrm{d}t,$$

and the waiting time PDF is obtained as

$$w(t) = \int_{-\infty}^\infty \psi(x,t)\,\mathrm{d}x.$$

Types of the CTRW processes can be characterized by the jump length variance and the characteristic waiting time defined by

$$\Sigma^2 = \int_{-\infty}^\infty \lambda(x)x^2\,\mathrm{d}x \text{ and } \quad T = \textstyle\int_0^\infty w(t)t\,\mathrm{d}t,$$

respectively. Assuming the jump length and waiting time to be independent random variable, the jump PDF takes the decoupled form

$$\psi(x,t) = \lambda(x)w(t).$$

Assuming this, the CTRW process is given by the equation

$$\eta(x,t) = \int_{-\infty}^\infty \int_0^\infty \eta(x',t')\psi(x-x',t-t')\,\mathrm{d}t'\,\mathrm{d}x' + \delta(x)\delta(t), \tag{1.21}$$

where $\eta(x,t)$ is a PDF of just having arrived at position $x$ at time $t$, and $\delta(x)$ is chosen to be an initial condition of the random walk. The equation (1.21) is formally equivalent to the generalized master equation [3]. One is then interested in the PDF of being in $x$ at the time $t$ which can be described by

$$W(x,t) = \int_0^t \eta(x,t')\Phi(t-t')\,\mathrm{d}t'. \tag{1.22}$$

where $\Phi(t)$ is a probability of no jump in the interval $(0,t)$ given by

$$\Phi(t) = 1 - \int_0^t w(t')\,\mathrm{d}t'.$$

So, $W(x,t)$ is then a PDF of arriving at the position $x$ at the time $t'$ and not having moved since. Applying the Fourier transformation on (1.22) in the $x-$direction and the Laplace transformation [25] in the $t-$direction brings

$$W(k,s) = \frac{1-w(s)}{s}\frac{W_0(k)}{1-\psi(k,s)}, \tag{1.23}$$

where $W(k,s)$ is a transformed $W(x,t)$, with the new variables $k$ and $s$ corresponding to $x$ and $t$, respectively, and $W_0(k)$ is the Fourier transform of the initial condition.

In the situation that the characteristic waiting time $T$ is divergent and the jump length variance $\Sigma^2$ is finite, one of the possibilities is to consider a long-tailed waiting time PDF in the form

$$w(t) \sim A_\alpha \left(\frac{\tau}{t}\right)^{1+\alpha},$$

where $0 < \alpha < 1$, $A_\alpha$ is a constant, $\tau$ is a scale with the dimension $[\tau] = s^{-\alpha}$. Next, we consider a Gaussian jump length PDF in the form

$$\lambda(x) = \frac{1}{(4\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{x^2}{4\sigma^2}\right),$$

where the scale $\sigma$ has the dimension $[\sigma] = cm^2$. The corresponding Laplace transformation of $w(t)$ is of the shape

$$w(s) \sim 1 - (s\tau)^\alpha,$$

and the Fourier transformation of $\lambda(x)$ takes the form

$$\lambda(k) \sim 1 - \sigma^2 k^2 + \mathcal{O}\left(k^4\right),$$

the concrete details are not of the interest here. Assuming this, the equation (1.23) becomes

$$W(k,s) = \frac{W_0(k)/s}{1 + K_\alpha s^{-\alpha} k^2}$$

in the $(k,s) \to (0,0)$ diffusion limit, where $K_\alpha$ is the so-called generalized diffusion constant. Applying the Fourier differentiation theorem and Laplace fractional integration theorem [28, 58] on the above algebraic relation give

$$W(x,t) - W_0(x) = I_0^\alpha K_\alpha \frac{\partial^2}{\partial x^2} W(x,t),$$

which contains the fractional integral on the right hand side, and after its differentiation with respect to time the fractional diffusion equation is derived

$$\frac{\partial}{\partial t}W(x,t) = D_0^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2}W(x,t). \tag{1.24}$$

The above equation may be rewritten as

$$D_0^\alpha W(x,t) - \frac{t^\alpha}{\Gamma(1-\alpha)}W_0(x) = K_\alpha \frac{\partial^2}{\partial x^2}W(x,t),$$

where we can see that the initial condition decays with negative power law, in the contrary to the exponential law decay in the classical diffusion. The equation may be rewritten also in terms of the Caputo fractional derivative

$$^C D_0^\alpha W(x,t) = K_\alpha \frac{\partial^2}{\partial x^2}W(x,t). \tag{1.25}$$

It can be calculated that the mean square displacement, denoted by $\langle x^2(t)\rangle$, is not linearly dependent on $t$, but it follows the power-law

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)}t^\alpha,$$

where the generalized diffusion constant is given by $K_\alpha = \sigma^2/\tau^\alpha$, with the two scales $\tau, \sigma$ leading to the dimension $[K_\alpha] = cm^2 s^{-\alpha}$.

To conclude, the time fractional derivative appears in the diffusion equation after assuming that the characteristic waiting time diverges and that the waiting time distribution has asymptotic behavior. On the other side considering the finite $T$ and, for instance, Poisson waiting time PDF would lead the classical diffusion equation. Also the limit $\alpha \to 1$ in (1.24) leads to the Fick law as expected [86].

The fractional wave equation, the equation (1.25) for $1 < \alpha < 2$, is closely studied for example in [85, 87, 113, 136]. In [87, 136], the additive two state process was combined with an asymptotic power-law waiting time distribution resulting in the fractional wave equation, with the mean square displacement $\langle x^2(t)\rangle \sim t^\alpha$. With the limit $\alpha \to 1$ the equation reduces to the Brownian motion, with $\langle x^2(t)\rangle \sim t$, and with $\alpha \to 2$ the wave equation is obtained, with $\langle x^2(t)\rangle \sim t^2$.

### Numerical comparison of solution for various values of $\alpha$

The comparison of the solution of a simple partial fractional differential equation containing the Caputo fractional derivative in time for various orders of fractional derivative is made in [4]. In the article, they study the equation in form

$$\partial_t^\alpha u = b^2 \Delta u, \quad 0 < x < L, \quad t > 0, \tag{1.26}$$

where $\alpha \in (0,2]$ is assumed together with the boundary and initial conditions

$$u(0,t) = u(L,t) = 0, \quad t \geq 0$$
$$u(x,0) = f(x), \quad 0 < x < L,$$
$$\partial_t u(x,0) = 0, \quad 0 < x < L, \text{ for } 1 < \alpha \leq 2.$$

The solution of this problem is obtained using the sine transformation and the Laplace transformation [25]. The explicit formula for the solution is given by

$$u(t,x) := \frac{2}{L} \sum_{1}^{\infty} E_\alpha(-b^2 a^2 n^2 t^\alpha) \sin(anx) \int_0^L f(s) \sin(ans)\, \mathrm{d}s, \tag{1.27}$$

where $a = \frac{\pi}{L}$, and $E_\alpha$ is a special function called Mittag-Leffler function defined in the complex plane by the power series

$$E_\beta := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, z \in \mathbb{C},$$

where $E_1(-z) = e^{-z}$ and $E_2(-z^2) = \cos z$. So, for $\alpha = 1, 2$, the formula (1.27) represents the solution of the diffusion and the wave equation. Taking the initial condition in the form

$$f(x) := \begin{cases} x, & x \in [0,1], \\ 2 - x, & x \in (1,2], \end{cases} \tag{1.28}$$

and $b = 1$, we depict the solution using the formula (1.27) for various values of $\alpha$ in Figure 1.5. On the pictures we can see that for $\alpha = \frac{1}{2}$ we get the slow diffusion behavior and for $\alpha = \frac{3}{2}$ the solution exhibits the diffusion-wave behavior. For the details and more numerical examples, see [4].

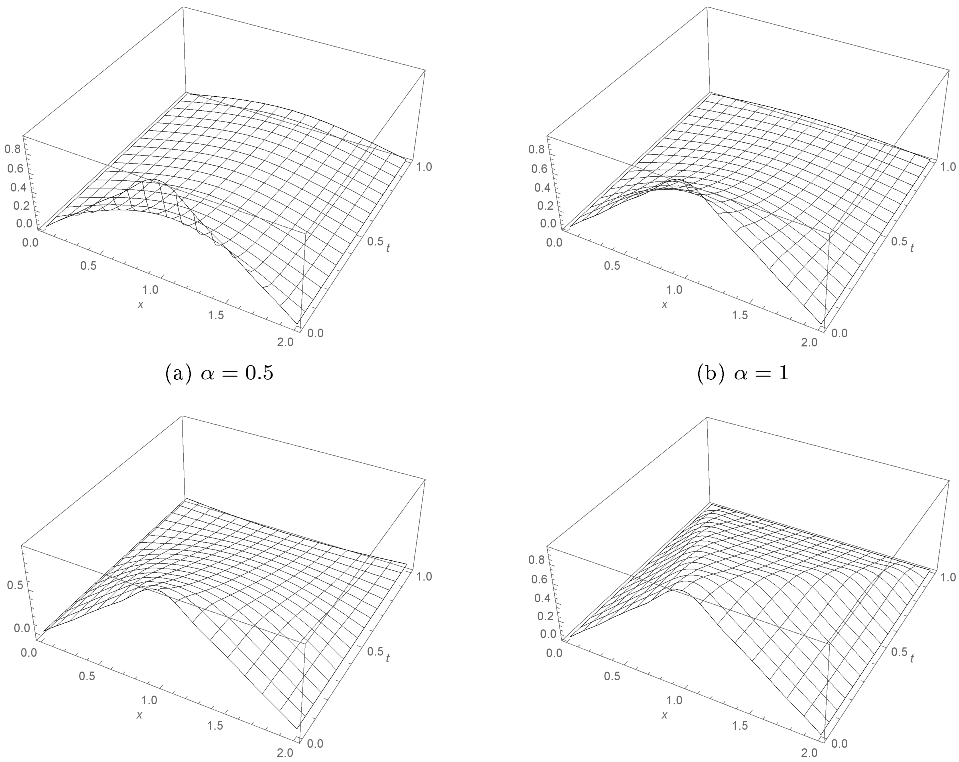
### Technical lemmas

We now prove the crucial technical lemma which will play a central role in the proofs in the following chapters.

**Lemma 1.7.2.** *Let $H$ be a real Hilbert space with a scalar product $(\cdot,\cdot)_H$ and the corresponding norm $\|\cdot\|_H$. Assume $T > 0$, $g \in L^1(0,T)$, $g' \in L^{1,loc}(0,T)$, $g' \leq 0$, $g \geq 0$. If $v : [0,T] \to H$ such that $v \in H^1((0,T),H)$ then*

$$\begin{aligned} \int_0^\xi \left( \frac{d}{dt} (g * v)(t), v(t) \right)_H dt &\geq \tfrac{1}{2} \left( g * \|v\|_H^2 \right)(\xi) + \tfrac{1}{2} \int_0^\xi g(t) \|v(t)\|_H^2 \, dt \\ &\geq \frac{g(T)}{2} \int_0^\xi \|v(t)\|_H^2 \, dt \end{aligned}$$

*for any $\xi \in [0,T]$.*




(a) $\alpha = 0.5$ (b) $\alpha = 1$

(c) $\alpha = 1.5$ (d) $\alpha = 2$

Figure 1.5: The approximate solution of (1.26) for various values of $\alpha$ with the initial condition (1.28), $L = 2, b = 1$, the solution is calculated from (1.27) using the first ten terms in the sum

*Proof.* Zacher [142, Lemma 2.3.2], [141, 143] has proved the following identity

$$\left(\frac{d}{dt}(k*v)(t), v(t)\right)_H = \tfrac{1}{2}\frac{d}{dt}\left(k*\|v\|_H^2\right)(t) + \tfrac{1}{2}k(t)\|v(t)\|_H^2 + \\ \tfrac{1}{2}\int_0^t [-k'(s)]\,\|v(t)-v(t-s)\|_H^2\,\mathrm{d}s \quad a.e.\ t\in(0,T),$$

which is valid for any $k \in H^{1,1}([0,T])$ and each $v \in L^2([0,T],H)$. Now, we replace $k$ by $g_n(s) := \min\{n, g(s)\}$. Thanks to the properties of $g$, it holds that

$$g_n'(s) \le 0, \qquad g_n(s) \to g(s) \qquad a.e.\ \text{in } [0,T].$$

Integration in time implies that

$$\begin{aligned}
&\int_0^\xi \left((g_n * \partial_t v)(t) + g_n(t)v(0), v(t)\right)_H \,\mathrm{d}t \\
&\qquad = \int_0^\xi \left(\frac{d}{dt}(g_n * v)(t), v(t)\right)_H \mathrm{d}t \\
&\qquad \ge \tfrac{1}{2}\left(g_n * \|v\|_H^2\right)(\xi) + \tfrac{1}{2}\int_0^\xi g_n(t)\,\|v(t)\|_H^2\,\mathrm{d}t.
\end{aligned} \tag{1.29}$$

Due to $v(0) \in H$ and $\partial_t v \in L^2((0,T),H)$, we see that

$$\begin{aligned}
&v(t) = v(0) + \int_0^t \partial_s v(s)\,\mathrm{d}s \\
&\qquad \Longrightarrow \|v(t)\|_H \le \|v(0)\|_H + \int_0^t \|\partial_s v(s)\,\mathrm{d}s\|_H \\
&\qquad\qquad \le \|v(0)\|_H + \sqrt{t}\sqrt{\int_0^t \|\partial_s v(s)\|_H^2\,\mathrm{d}s} \le C.
\end{aligned}$$

We successively deduce

$$
\begin{aligned}
&\left| \int_0^\xi \left( \left( g_n * \partial_t v \right)(t), v(t) \right)_H \right| \quad \mathrm{d}t \\
&\qquad \le \int_0^\xi \left\| \left( g_n * \partial_t v \right)(t) \right\|_H \|v(t)\|_H \, \mathrm{d}t \\
&\qquad \le \int_0^\xi \left( g_n * \|\partial_t v\|_H \right)(t) \|v(t)\|_H \, \mathrm{d}t \\
&\qquad \le \sqrt{\int_0^\xi \left( g_n * \|\partial_t v\|_H \right)^2 (t) \, \mathrm{d}t} \sqrt{\int_0^\xi \|v(t)\|_H^2 \, \mathrm{d}t} \\
&\qquad \le \int_0^\xi g_n(t) \, \mathrm{d}t \sqrt{\int_0^\xi \|\partial_t v(t)\|_H^2 \, \mathrm{d}t} \sqrt{\int_0^\xi \|v(t)\|_H^2 \, \mathrm{d}t} \\
&\qquad \le \int_0^\xi g(t) \, \mathrm{d}t \sqrt{\int_0^\xi \|\partial_t v(t)\|_H^2 \, \mathrm{d}t} \sqrt{\int_0^\xi \|v(t)\|_H^2 \, \mathrm{d}t},
\end{aligned}
$$

using the Cauchy inequality and Young's inequality for convolution. Applying the Lebesgue dominated theorem, we may pass to the limit $n \to \infty$ in (1.29) to get

$$
\begin{aligned}
&\int_0^\xi \left( \left( g * \partial_t v \right)(t) + v(0) g(t), v(t) \right)_H \quad \mathrm{d}t \\
&\qquad = \int_0^\xi \left( \frac{d}{dt} \left( g * v \right)(t), v(t) \right)_H \, \mathrm{d}t \\
&\qquad \ge \tfrac{1}{2} \left( g * \|v\|_H^2 \right)(t) + \tfrac{1}{2} \int_0^\xi g(t) \|v(t)\|_H^2 \, \mathrm{d}t \\
&\qquad \ge \frac{g(T)}{2} \int_0^\xi \|v(t)\|_H^2 \, \mathrm{d}t,
\end{aligned}
$$

which concludes the proof. □

The next technical lemma is a discrete analogy of Lemma 1.7.2. It plays a central role by establishing a priori estimates in the Rothe method further in this dissertation. Before we state the lemma, we define the discrete convolution by

$$
\left( K * v \right)_i := \sum_{k=1}^{i} K_{i+1-k} v_k \tau, \tag{1.30}
$$

where $\tau$ is the time step. Note that by this definition we avoid blow up problems if $K$ has a singularity at $t = 0$. Then we can calculate a difference for the discrete convolution as follows

$$
\delta \left( K * v \right)_i = \frac{\left( K * v \right)_i - \left( K * v \right)_{i-1}}{\tau} = K_1 v_i + \sum_{k=1}^{i-1} \delta K_{i+1-k} v_k \tau, \qquad i \ge 1, \tag{1.31}
$$

as

$$(K * v)_0 := 0$$

and we consider the sum to vanish per definition if, the upper bound of the sum is less than the lower bound.

**Lemma 1.7.3.** *Let $\{v_i\}_{i\in\mathbb{N}}$ and $\{K_i\}_{i\in\mathbb{N}}$ be sequences of real numbers. Assume that the sequence decreases, i.e. $K_i \le K_{i-1}$ for any $i$. Then*

$$2\delta\,(K*v)_i\, v_i \ge \delta\left(K*v^2\right)_i + K_i v_i^2, \qquad i\in\mathbb{N}. \tag{1.32}$$

*Proof.* We successively deduce that

$$\begin{aligned}
&\delta\left(K*v^2\right)_i + K_i v_i^2 \\
&\qquad \le \delta\left(K*v^2\right)_i + K_i v_i^2 - \sum_{k=1}^{i-1}\delta K_{i+1-k}\left(v_i - v_k\right)^2\tau \\
&\qquad \overset{(1.31)}{=} K_1 v_i^2 + \sum_{k=1}^{i-1}\delta K_{i+1-k} v_k^2\tau + K_i v_i^2 - \sum_{k=1}^{i-1}\delta K_{i+1-k}\left(v_i - v_k\right)^2\tau \\
&\qquad = \left(K_1 + K_i\right)v_i^2 + \sum_{k=1}^{i-1}\delta K_{i+1-k}\left[v_k^2 - \left(v_i - v_k\right)^2\right]\tau \\
&\qquad = \left(K_1 + K_i\right)v_i^2 + 2v_i\sum_{k=1}^{i-1}\delta K_{i+1-k} v_k\tau - v_i^2\sum_{k=1}^{i-1}\delta K_{i+1-k}\tau \\
&\qquad = \left(K_1 + K_i\right)v_i^2 + 2v_i\sum_{k=1}^{i-1}\delta K_{i+1-k} v_k\tau - (K_i - K_1)v_i^2 \\
&\qquad = 2K_1 v_i^2 + 2v_i\sum_{k=1}^{i-1}\delta K_{i+1-k} v_k\tau \\
&\qquad \overset{(1.31)}{=} 2\delta\,(K*v)_i\, v_i.
\end{aligned}$$

□

Summing up the inequality (1.32) for $i = 1, ..., j$, $j\in\mathbb{N}$, and multiplying by $\tau$, we get that

$$2\sum_{i=1}^{j}\delta\,(K*v)_i\, v_i\tau \ge \sum_{i=1}^{j}\delta\left(K*v^2\right)_i\tau + \sum_{i=1}^{j}K_i v_i^2\tau, \qquad j\in\mathbb{N},$$

which can be rewritten as

$$2\sum_{i=1}^{j}\delta\,(K*v)_i\, v_i\tau \ge \left(K*v^2\right)_j + \sum_{i=1}^{j}K_i v_i^2, \qquad j\in\mathbb{N}. \tag{1.33}$$

# Chapter 2

# An inverse source problem in a semilinear time-fractional diffusion equation

This chapter is based on the article [119], which has been already published in *Computers and Mathematics with Applications*.

## 2.1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with a Lipschitz boundary $\Gamma$ , cf. [67]. Consider a linear second order differential operator in the divergence form with space and time dependent coefficients

$$\begin{aligned} L(x,t)u &= \nabla \cdot (-\boldsymbol{A}(x,t)\nabla u - \boldsymbol{b}(x,t)u) + c(t)u, \\ \boldsymbol{A}(x,t) &= \left(a_{i,j}(x,t)\right)_{i,j=1,\ldots,d}, \\ \boldsymbol{b}(x,t) &= (b_1(x,t),\ldots,b_d(x,t)). \end{aligned}$$

We deal with a partial differential equation (PDE) with a fractional derivative in time

$$\left(g_{1-\beta} * \partial_t u(x)\right)(t) + L(x,t)u(x,t) = h(t)f(x) + \int_0^t F(x,s,u(x,s))\,\mathrm{d}s, \qquad (2.1)$$

for $x \in \Omega,\ t \in (0,T)$, where $g_{1-\beta}$ denotes the Riemann-Liouville kernel

$$g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \qquad t > 0,\ 0 < \beta < 1,$$

and $*$ stands for the convolution on the positive half-line defined by (1.20). Thus, the convolution term in (2.1) is the Caputo fractional derivative of order $\beta \in (0,1)$.

The governing PDE (2.1) is accompanied by the following initial and boundary conditions

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & x \in \Omega, \\ (-\boldsymbol{A}(x,t)\nabla u(x,t) - \boldsymbol{b}(x,t)u(x,t)) \cdot \boldsymbol{\nu} & = & g(x,t), & (x,t) \in \Gamma \times (0,T), \end{array} \tag{2.2}$$

where the symbol $\boldsymbol{\nu}$ denotes the outer normal vector assigned to the boundary $\Gamma$.

The integral term in the r.h.s. of (2.1) models memory effects with applications e.g. in elastoplasticity [103] or in the theory of reactive contaminant transport [27]. The solvability of forward fractional diffusion equations have been studied e.g. in [109, 141]. The *Inverse Source Problem* (ISP) studied in this chapter consists of finding a couple $(u(x,t), h(t))$ obeying (2.1), (2.2) and

$$\int_\Omega u(x,t)\,\mathrm{d}x = m(t), \qquad t \in [0,T]. \tag{2.3}$$

Determination of an unknown source is one of hot topics in inverse problems (IPs). There are many papers studying ISPs in parabolic or hyperbolic settings. If the source exclusively depends on the space variable, one needs an additional space measurement (e.g. solution at the final time), cf. [17, 44, 48, 50, 55, 99, 100, 108, 122, 127, 139]. For the solely time-dependent source a supplementary time-dependent measurement is needed, cf. [45, 46, 49, 99, 117]. This means that both kinds of ISPs need totally different additional data. ISPs for fractional diffusion equations become more popular in the last years. The recovery of a time dependent source in a fractional diffusion equation has been studied in [54, 109, 135]. Determination of a space dependent function in a fractional diffusion equation has been addressed in [53, 60, 124, 140]. The uniqueness of a solution to the inverse Cauchy problem for a fractional differential equation in a Banach space has been studied in [65]. The global existence in time of an ISP for a fractional integrodifferential equation by means of a fixed point method has been considered in [138].

The added value of this chapter relies on the global (in time) solvability of the ISP (2.1), (2.2), (2.3), and in the proposition of an interesting approximation scheme. We reformulate the ISP into an appropriate direct (non-local) formulation. We propose an variational technique based on elimination of $h$ from (2.1) by (2.3), which turns out to be possible for a sufficiently smooth solution. Then we prove the well-posedness of the problem. The proposed numerical scheme is based on a semi-discretization in time by Rothe's method, see the section 1.5.1. We show the existence of approximations at each time step of the time partitioning, and we derive suitable stability results. The convergence of approximations towards the exact solution is investigated in Theorem 2.3.1 in suitable function spaces. Finally, we present a numerical example supporting the obtained convergence results.

## 2.2 Uniqueness

Denote by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working at the boundary $\Gamma$, we use a similar notation, namely $(\cdot,\cdot)_\Gamma$, $L^2(\Gamma)$ and $\|\cdot\|_\Gamma$. In what follows $C, \varepsilon$ and $C_\varepsilon$ denote generic positive constants depending only on the given data, where $\varepsilon$ is a small one and $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$ is a large one. Different values of those constants in the same discussion are allowed.

We associate a bilinear form $\mathcal{L}$ with the differential operator $L$ as follows

$$(Lu,\varphi) = \mathcal{L}(u,\varphi) + (g,\varphi)_\Gamma, \qquad \forall \varphi \in H^1(\Omega),$$

i.e.

$$\mathcal{L}(t)\left(u(t),\varphi\right) = \left(\boldsymbol{A}(t)\nabla u(t) + \boldsymbol{b}(t)u(t), \nabla\varphi\right) + c(t)\left(u(t),\varphi\right).$$

Throughout the chapter we assume that

$$\begin{array}{llll} (a) & a_{i,j}, b_i : \overline{\Omega}\times[0,T] \to \mathbb{R}, & |a_{i,j}| + |b_i| \le C, & i,j = 1,\ldots,d, \\ (b) & 0 \le c(t) \le C, & \forall t \in [0,T], & \\ (c) & \mathcal{L}(t)\left(\varphi,\varphi\right) \ge C_0 \left\|\nabla\varphi\right\|^2, & \forall \varphi \in H^1(\Omega), & \forall t\in[0,T]. \end{array} \tag{2.4}$$

Integrating (2.1) over $\Omega$, applying the Green theorem and taking into account (2.3) we obtain

$$\left(g_{1-\beta} * m'\right)(t) + c(t)m(t) = h(t)\left(f,1\right) - \left(g(t),1\right)_\Gamma + \int_0^t \left(F(s,u(s)),1\right)\,\mathrm{d}s. \tag{MP}$$

Assuming that $(f,1) \neq 0$ we have

$$h(t) = \frac{\left(g_{1-\beta} * m'\right)(t) + c(t)m(t) + \left(g(t),1\right)_\Gamma - \displaystyle\int_0^t \left(F(s,u(s)),1\right)\,\mathrm{d}s}{(f,1)}. \tag{2.5}$$

The variational formulation of (2.1) and (2.2) reads as

$$
\begin{aligned}
&\left(\left(g_{1-\beta} * \partial_t u\right)(t), \varphi\right) + \mathcal{L}(t)\,(u(t), \varphi) \\
&\qquad = h(t)\,(f, \varphi) + \left(\int_0^t F(s, u(s))\,\mathrm{d}s, \varphi\right) - (g(t), \varphi)_\Gamma \qquad \text{(P)}
\end{aligned}
$$

for any $\varphi \in H^1(\Omega)$, a.a. $t \in [0,T]$ and $u(0) = u_0$. The relations (P) and (MP) represent the variational formulation of (2.1), (2.2) and (2.3).

Now, we are in a position to state uniqueness of solution to the ISP (P), (MP).

**Theorem 2.2.1** (uniqueness). *Let $f, u_0 \in L^2(\Omega)$, $\int_\Omega f \neq 0$, $m \in C^1([0,T])$, $F$ be a global Lipschitz continuous function in all variables. Assume (2.4) and $g \in C([0,T], L^2(\Gamma))$.*

*Then there exists at most one solution $(u,h)$ to the problem (P), (MP) obeying $u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$ with $\partial_t u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* Suppose that $(u_i, h_i)$, for $i = 1,2$ solve (P), (MP), and that they obey $u_i \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$ with $\partial_t u_i \in L^2\left((0,T), L^2(\Omega)\right)$, $h_i \in C([0,T])$. Set $u = u_1 - u_2$ and $h = h_1 - h_2$. Subtracting the corresponding variational formulations from each other, we obtain that

$$
\begin{aligned}
&\left(\left(g_{1-\beta} * \partial_t u\right)(t), \varphi\right) + \mathcal{L}(t)\,(u(t), \varphi) = \\
&\qquad h(t)\,(f, \varphi) + \left(\int_0^t \left[F(s, u_1(s)) - F(s, u_2(s))\right]\mathrm{d}s, \varphi\right) \qquad (2.6)
\end{aligned}
$$

and

$$
0 = h(t)\,(f, 1) + \left(\int_0^t \left[F(s, u_1(s)) - F(s, u_2(s))\right]\mathrm{d}s, 1\right). \qquad (2.7)
$$

We set $\varphi = u(t)$ in (2.6) and integrate in time over $(0, \xi)$. Taking into account (2.7), $u_0 = 0$ and $\left(g_{1-\beta} * \partial_t u\right)(t) = \partial_t \left(g_{1-\beta} * u\right)(t)$, we obtain

$$
\begin{aligned}
&\int_0^\xi \left(\partial_t \left(g_{1-\beta} * u\right)(t), u(t)\right)\mathrm{d}t + \int_0^\xi \mathcal{L}(t)\,(u(t), u(t))\,\mathrm{d}t \\
&\qquad = \int_0^\xi \frac{\int_0^t \left(F(s, u_2(s)) - F(s, u_1(s)), 1\right)\mathrm{d}s}{(f,1)}\,(f, u(t))\,\mathrm{d}t \\
&\qquad + \int_0^\xi \left(\int_0^t \left[F(s, u_1(s)) - F(s, u_2(s))\right]\mathrm{d}s, u(t)\right)\mathrm{d}t.
\end{aligned}
$$

The lower bound for the left hand side (l.h.s.) can be obtained from Lemma 1.7.2 and (2.4)

$$\int_0^\xi \left(\partial_t \left(g_{1-\beta} * u\right)(t), u(t)\right) \mathrm{d}t + \int_0^\xi \mathcal{L}(t)\left(u(t), u(t)\right) \mathrm{d}t$$
$$\geq \frac{g_{1-\beta}(T)}{2} \int_0^\xi \|u(t)\|^2 \mathrm{~d}t + C_0 \int_0^\xi \|\nabla u(t)\|^2 \mathrm{~d}t.$$

The upper bound of the r.h.s. can be achieved using the Cauchy and Young's inequalities in a standard way

$$\int_0^\xi \frac{\int_0^t \left(F(s, u_2(s)) - F(s, u_1(s)), 1\right) \mathrm{d}s}{(f, 1)} (f, u(t)) \mathrm{~d}t$$
$$+ \int_0^\xi \left(\int_0^t \left[F(s, u_1(s)) - F(s, u_2(s))\right] \mathrm{d}s, u(t)\right) \mathrm{d}t$$
$$\leq \varepsilon \int_0^\xi \|u(t)\|^2 \mathrm{~d}t + C_\varepsilon \int_0^\xi \int_0^t \|u(s)\|^2 \mathrm{~d}s \, \mathrm{d}t.$$

Assembling these estimates we arrive at

$$\left(\frac{g_{1-\beta}(T)}{2} - \varepsilon\right) \int_0^\xi \|u(t)\|^2 \mathrm{~d}t + C_0 \int_0^\xi \|\nabla u(t)\|^2 \mathrm{~d}t \leq C_\varepsilon \int_0^\xi \int_0^t \|u(s)\|^2 \mathrm{~d}s \, \mathrm{d}t.$$

Fixing a sufficiently small positive $\varepsilon$ and applying the Grönwall lemma [9], we conclude that $u = 0$ a.e. in $\Omega \times (0, T)$. Finally, the relation (2.7) ensures that $h = 0$ a.e. in $(0, T)$. □

## 2.3 Time discretization

Rothe [106] introduced a simple time-discretization method for parabolic problems. By now it grew up to a powerful technique for solving both linear and nonlinear evolutionary (scalar or vectorial) equations, cf. e.g. [57, 102, 116]. Using a simple discretization in time (backward Euler), a time-dependent problem is approximated by a sequence of elliptic problems, which have to be solved successively with increasing $t_i$. Solutions of these steady-state settings approximate the transient solution at the points of the time partitioning. The advantage of Rothe's method is twofold: next to the existence and possible uniqueness of a solution to the original problem, also a numerical algorithm is contained in this approach.

For ease of explanation, we consider an equidistant time-partitioning of the time frame $[0, T]$ with a step $\tau = T/n$, for any $n \in \mathbb{N}$. We use the notation $t_i = i\tau$ and for any function $z$ we write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}. \tag{2.8}$$

We remind the reader the definition (1.30) defining the discretized convolution

$$(K * v)_i := \sum_{k=1}^{i} K_{i+1-k} v_k \tau,$$

and also it holds

$$\delta (K * v)_i = \frac{(K * v)_i - (K * v)_{i-1}}{\tau} = K_1 v_i + \sum_{k=1}^{i-1} \delta K_{i+1-k} v_k \tau, \qquad i \geq 1. \quad (2.9)$$

Similarly, we may write

$$\delta (K * v)_i = K_i v_0 + \sum_{k=1}^{i} \delta v_k K_{i+1-k} \tau = K_i v_0 + (K * \delta v)_i\,, \qquad i \geq 1. \qquad (2.10)$$

Consider a system with unknowns $(u_i, h_i)$ for $i = 1, \ldots, n$. At time $t_i$ we approximate (P) by

$$\left((g_{1-\beta} * \delta u)_i\,, \varphi\right) + \mathcal{L}_i\,(u_i, \varphi) = h_i\,(f, \varphi) + \left( \sum_{k=1}^{i} F(t_k, u_{k-1})\tau, \varphi \right) - (g_i, \varphi)_\Gamma \quad (\mathrm{DP}i)$$

and (MP) by

$$(g_{1-\beta} * m')_i + c_i m_i = h_i\,(f, 1) + \left( \sum_{k=1}^{i} F(t_k, u_{k-1})\tau, 1 \right) - (g_i, 1)_\Gamma\,. \qquad (\mathrm{DMP}i)$$

Please note that (DMP$i$) and (DP$i$) are linear in $u_i$ and $h_i$, respectively, and both relations are decoupled. Thus for a given $i \in \{1, \ldots, n\}$, we first determine $h_i$ from (DMP$i$) and then we solve (DP$i$). Afterwards, we increase $i$ to $i+1$. The pseudo algorithm for computing the solution reads as

---

**Require:** $\Omega, \mathcal{L}, f, F, g, m$
1: $i \leftarrow 1$
2: **while** $i \leq n$ **do**
3: $\quad h_i \leftarrow$ Solve: (DMP$i$)
4: $\quad u_i \leftarrow$ Solve: (DP$i$)
5: $\quad i \leftarrow i + 1$
6: **return** $\{h_1, u_1\}, \ldots, \{h_n, u_n\}$

---

In the next lemma, we prove the existence and uniqueness of the solution along every time line. Decoupling the equations in the system at every time step enables the effective application of the Lax-Milgram lemma on the elliptic problem and gaining the result.

**Lemma 2.3.1.** *Let $f, u_0 \in L^2(\Omega)$, $\int_\Omega f \neq 0$, $m \in C^1([0,T])$, $F$ be a global Lipschitz continuous function in all variables. Assume (2.4) and $g \in C([0,T], L^2(\Gamma))$. Then for each $i \in \{1,\ldots,n\}$, there exists a unique couple $(u_i, h_i) \in H^1(\Omega) \times \mathbb{R}$ solving (DPi) and (DMPi).*

*Proof.* Resolving (DMP$i$) for $h_i$, we get

$$h_i = \frac{(g_{1-\beta} * m')_i + c_i m_i + (g_i, 1)_\Gamma - \left(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, 1\right)}{(f,1)} \in \mathbb{R} \tag{2.11}$$

if $u_{k-1} \in L^2(\Omega)$ for $0 \leq k \leq i$. The relation (DP$i$) can be rewritten as

$$\begin{aligned} g_{1-\beta}(\tau)\,(u_i,\varphi) + \mathcal{L}_i\,(u_i,\varphi) \quad &= h_i\,(f,\varphi) + \left(\sum_{k=1}^{i} F(t_k,u_{k-1})\tau, \varphi\right) - (g_i,\varphi)_\Gamma \\ &- \sum_{k=1}^{i-1} g_{1-\beta}(t_{i-k+1})\,(\delta u_k,\varphi)\,\tau + g_{1-\beta}(\tau)\,(u_{i-1},\varphi)\,. \end{aligned}$$

The l.h.s. represents an elliptic, continuous and bilinear form in $H^1(\Omega) \times H^1(\Omega)$. If $u_0, \ldots, u_{i-1} \in L^2(\Omega)$, then the r.h.s. is a linear bounded functional on $H^1(\Omega)$. The existence and uniqueness of $u_i \in H^1(\Omega)$ follows from the the Lax-Milgram lemma 1.2.4. □

### Energy estimates

Now, we start with a basic energy estimate for $u_i$ and $h_i$. Additionally, we introduce the following notation

$$\left(g_{1-\beta} * \|u\|^2\right)_j = \sum_{k=1}^{j} g_{1-\beta}(t_{j+1-k})\,\|u_k\|^2\,\tau.$$

**Lemma 2.3.2.** *Let the assumptions of Lemma 2.3.1 be fulfilled. Then there exist positive constants $C$ and $\tau_0$ such that for any $0 < \tau < \tau_0$, we have that*

*(i)* $\max\limits_{1\leq j\leq n} \left(g_{1-\beta} * \|u\|^2\right)_j + \sum\limits_{i=1}^{n} g_{1-\beta}(t_i)\,\|u_i\|^2\,\tau + \sum\limits_{i=1}^{n} \|u_i\|^2_{H^1(\Omega)}\,\tau \leq C,$

*(ii)* $\max\limits_{1\leq j\leq n} |h_j| \leq C.$

*Proof.* Starting from (2.11), we see that

$$|h_i| \leq C + C \sum_{k=1}^{i} \|F(t_k, u_{k-1})\| \, \tau \leq C + C \sum_{k=0}^{i-1} \|u_k\| \, \tau. \tag{2.12}$$

Set $\varphi = u_i\tau$ in (DP$i$) and sum the result up for $i = 1, \ldots, j$ to have

$$\begin{aligned}
&\sum_{i=1}^{j} \left( (g_{1-\beta} * \delta u)_i \, , u_i \right) \tau + \sum_{i=1}^{j} \mathcal{L}_i \left( u_i, u_i \right) \tau \\
&\qquad = \sum_{i=1}^{j} h_i \left( f, u_i \right) \tau + \sum_{i=1}^{j} \left( \sum_{k=1}^{i} F(t_k, u_{k-1})\tau, u_i \right) \tau - \sum_{i=1}^{j} \left( g_i, u_i \right)_\Gamma \tau.
\end{aligned} \tag{2.13}$$

Using (2.10) and Lemma 1.7.3, we see that

$$\begin{aligned}
&\sum_{i=1}^{j} \left( (g_{1-\beta} * \delta u)_i \, , u_i \right) \tau \\
&\overset{(2.10)}{=} \sum_{i=1}^{j} \left( \delta \left( g_{1-\beta} * u \right)_i - g_{1-\beta}(t_i) u_0, u_i \right) \tau \\
&= \sum_{i=1}^{j} \left( \delta \left( g_{1-\beta} * u \right)_i \, , u_i \right) \tau - \sum_{i=1}^{j} g_{1-\beta}(t_i) \left( u_0, u_i \right) \tau \\
&\geq \tfrac{1}{2} \left( g_{1-\beta} * \|u\|^2 \right)_j + \tfrac{1}{2} \sum_{i=1}^{j} g_{1-\beta}(t_i) \left\| u_i \right\|^2 \tau - \sum_{i=1}^{j} g_{1-\beta}(t_i) \left( u_0, u_i \right) \tau \\
&\geq \tfrac{1}{2} \left( g_{1-\beta} * \|u\|^2 \right)_j + \tfrac{1}{2} \sum_{i=1}^{j} g_{1-\beta}(t_i) \left\| u_i \right\|^2 \tau - \sum_{i=1}^{j} g_{1-\beta}(t_i) \left\| u_0 \right\| \left\| u_i \right\| \tau \\
&\geq \tfrac{1}{2} \left( g_{1-\beta} * \|u\|^2 \right)_j + \left( \tfrac{1}{2} - \varepsilon \right) \sum_{i=1}^{j} g_{1-\beta}(t_i) \left\| u_i \right\|^2 \tau - C_\varepsilon \sum_{i=1}^{j} g_{1-\beta}(t_i) \tau \\
&\geq \tfrac{1}{2} \left( g_{1-\beta} * \|u\|^2 \right)_j + \left( \tfrac{1}{2} - \varepsilon \right) \sum_{i=1}^{j} g_{1-\beta}(t_i) \left\| u_i \right\|^2 \tau - C_\varepsilon \\
&\geq \tfrac{1}{2} \left( g_{1-\beta} * \|u\|^2 \right)_j + \left( \frac{1}{4} - \varepsilon \right) \sum_{i=1}^{j} g_{1-\beta}(t_i) \left\| u_i \right\|^2 \tau + \frac{g_{1-\beta}(T)}{4} \sum_{i=1}^{j} \left\| u_i \right\|^2 \tau - C_\varepsilon,
\end{aligned}$$

where, in the last step, we estimated from below a part of the second term and gained a suitable estimate for the sum of $\|u_i\|^2$ without the weight function. Next, by ellipticity assumption, we may write

$$\sum_{i=1}^{j} \mathcal{L}_i \left( u_i, u_i \right) \tau \geq C_0 \sum_{i=1}^{j} \left\| \nabla u_i \right\|^2 \tau.$$

The first term on the r.h.s. of (2.13) can be readily estimated using the Cauchy and Young's inequalities and taking into account (2.12)

$$\begin{aligned}\left|\sum_{i=1}^{j} h_i\left(f, u_i\right)\tau\right| &\leq \varepsilon \sum_{i=1}^{j}\left\|u_i\right\|^2\tau + C_\varepsilon \sum_{i=1}^{j} h_i^2\tau \\ &\leq \varepsilon \sum_{i=1}^{j}\left\|u_i\right\|^2\tau + C_\varepsilon + C_\varepsilon \sum_{i=1}^{j}\sum_{k=0}^{i-1}\left\|u_k\right\|^2\tau^2.\end{aligned} \tag{2.14}$$

For the second term on the r.h.s. of (2.13), we again apply the Cauchy and Young's inequalities to get

$$\left|\sum_{i=1}^{j}\left(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, u_i\right)\tau\right| \leq \varepsilon \sum_{i=1}^{j}\left\|u_i\right\|^2\tau + C_\varepsilon + C_\varepsilon \sum_{i=1}^{j}\sum_{k=0}^{i-1}\left\|u_k\right\|^2\tau^2.$$

The last term of (2.13) can be handled similarly involving the Cauchy and Young inequalities and the trace theorem

$$\left|\sum_{i=1}^{j}\left(g_i, u_i\right)_\Gamma\tau\right| \leq \varepsilon \sum_{i=1}^{j}\left\|u_i\right\|_\Gamma^2\tau + C_\varepsilon \leq \varepsilon \sum_{i=1}^{j}\left\|u_i\right\|^2\tau + \varepsilon \sum_{i=1}^{j}\left\|\nabla u_i\right\|^2\tau + C_\varepsilon.$$

Putting all estimates together we arrive at

$$\begin{aligned}\frac{1}{2}\left(g_{1-\beta} * \left\|u\right\|^2\right)_j + \left(\frac{1}{4} - \varepsilon\right)\sum_{i=1}^{j} g_{1-\beta}(t_i)\left\|u_i\right\|^2\tau + \left(\frac{g_{1-\beta}(T)}{4} - \varepsilon\right)\sum_{i=1}^{j}\left\|u_i\right\|^2\tau \\ + \left(C_0 - \varepsilon\right)\sum_{i=1}^{j}\left\|\nabla u_i\right\|^2\tau \leq C_\varepsilon + C_\varepsilon \sum_{i=1}^{j}\sum_{k=0}^{i-1}\left\|u_k\right\|^2\tau^2.\end{aligned}$$

Fixing a sufficiently small $0 < \varepsilon < 1$ and using Grönwall's argument, we get for $0 < \tau < \tau_0$ that

$$\left(g_{1-\beta} * \left\|u\right\|^2\right)_j + \sum_{i=1}^{j} g_{1-\beta}(t_i)\left\|u_i\right\|^2\tau + \sum_{i=1}^{j}\left\|u_i\right\|_{H^1(\Omega)}^2\tau \leq C.$$

This together with (2.12) imply

$$|h_i| \leq C.$$

$\square$

We shall need a compatibility condition, i.e. we assume that (2.1) is fulfilled at $t = 0$, i.e. (P) holds true for $t = 0$. Therefore we may also put $t = 0$ in (MP),

$$\mathcal{L}(0)\left(u_0, \varphi\right) = h_0\left(f, \varphi\right) - \left(g_0, \varphi\right)_\Gamma, \qquad \forall \varphi \in H^1(\Omega) \tag{2.15}$$

which allows us to define $h_0$ as follows

$$h_0 = \frac{c_0 m_0 + (g_0, 1)_\Gamma}{(f, 1)}. \tag{2.16}$$

We adopt the following notation

$$\left(g_{1-\beta} * \|\delta u\|^2\right)_j = \sum_{k=1}^{j} g_{1-\beta}(t_{j+1-k}) \|\delta u_k\|^2 \tau.$$

**Lemma 2.3.3.** *Let the assumptions of Lemma 2.3.1 be fulfilled. Moreover, assume (2.15),(2.16), $u_0 \in H^1(\Omega)$, $g \in C^1([0,T], L^2(\Gamma))$, $m \in C^2([0,T])$, $\partial_t c \in L^\infty(0,T)$ and $\partial_t a_{i,j}, \partial_t b_i \in L^\infty(\Omega \times (0,T))$ for all $i, j = 1, \dots, d$. Then there exist positive constants $C$ and $\tau_0$ such that for any $0 < \tau < \tau_0$ we have*

*(i)* $\max_{1 \le j \le n} \left(g_{1-\beta} * \|\delta u\|^2\right)_j + \sum_{i=1}^{n} g_{1-\beta}(t_i) \|\delta u_i\|^2 \tau + \sum_{i=1}^{n} \|\delta u_i\|^2_{H^1(\Omega)} \tau \le C,$

*(ii)* $|\delta h_i| \le C + C t_i^{-\beta}$ *for any* $i = 1, \dots, n$.

*Proof.* Subtract (2.16) from (2.11) for $i = 1$, divide the result by $\tau$ to get

$$\delta h_1 = \frac{h_1 - h_0}{\tau} = \frac{m_1 \delta c_1 + c_0 \delta m_1 + g_{1-\beta}(\tau) m'(\tau) + (\delta g_1, \varphi)_\Gamma - (F(\tau, u_0), 1)}{(f, 1)}.$$

Thus, we have $|\delta h_1| \le C + C t_1^{-\beta}$.

Now, applying the $\delta$-operation on (2.11) for $i \ge 2$, we deduce that

$$\begin{aligned} \delta h_i &= \frac{1}{(f,1)} \left(\delta(c_i m_i) + \delta\left(g_{1-\beta} * m'\right)_i + (\delta g_i, 1)_\Gamma - (F(t_i, u_{i-1}), 1)\right) \\ &\overset{(2.10)}{=} \frac{1}{(f,1)} \Big( m_i \delta c_i + c_{i-1} \delta m_i + g_{1-\beta}(t_i) m'(0) + \sum_{k=1}^{i} \delta m'_k g_{1-\beta}(t_{i+1-k}) \tau \\ &\quad + (\delta g_i, 1)_\Gamma - (F(t_i, u_{i-1}), 1) \Big). \end{aligned}$$

That is why the following relation holds true

$$|\delta h_i| \le C t_i^{-\beta} + C + C \|u_{i-1}\|, \qquad \forall i \ge 1. \tag{2.17}$$

Clearly

$$z_i w_i - z_{i-1} w_{i-1} = z_i (w_i - w_{i-1}) + (z_i - z_{i-1}) w_{i-1}.$$

Therefore

$$\delta\left(\mathcal{L}_i(u_i, \varphi)\right) = \mathcal{L}_i(\delta u_i, \varphi) + (\delta \mathcal{L})_i (u_{i-1}, \varphi),$$

where

$$(\delta\mathcal{L})_i(u,\varphi) = (\delta\boldsymbol{A}_i\nabla u + \delta\boldsymbol{b}_i u, \nabla\varphi) + \delta c_i(u,\varphi), \qquad \forall u,\varphi \in H^1(\Omega).$$

Taking a difference of (DP$i$), we get

$$\left(\delta\left(g_{1-\beta} * \delta u\right)_i, \varphi\right) + \mathcal{L}_i(\delta u_i, \varphi) + (\delta\mathcal{L})_i(u_{i-1},\varphi) = \delta h_i(f,\varphi) + (F(t_i,u_{i-1}),\varphi) - (\delta g_i, \varphi)_\Gamma. \quad (2.18)$$

This difference can be taken for $i \geq 2$. When $i = 1$ we subtract (2.15) from (DP$i$) for $i = 1$ (please note that $(g_{1-\beta} * \delta u)_0 = 0$). Set $\varphi = \delta u_i \tau$ in (2.18) and sum the result up for $i = 1, \ldots, j$ to obtain

$$\sum_{i=1}^{j}\left(\delta\left(g_{1-\beta} * \delta u\right)_i, \delta u_i\right)\tau + \sum_{i=1}^{j}\mathcal{L}_i(\delta u_i,\delta u_i)\tau + \sum_{i=1}^{j}(\delta\mathcal{L})_i(u_{i-1},\delta u_i)\tau = \sum_{i=1}^{j}\delta h_i(f,\delta u_i)\tau + \sum_{i=1}^{j}(F(t_i,u_{i-1}),\delta u_i)\tau - \sum_{i=1}^{j}(\delta g_i,\delta u_i)_\Gamma\tau. \quad (2.19)$$

Using Lemma 1.7.3, we see that

$$\begin{aligned}\sum_{i=1}^{j}\left(\delta\left(g_{1-\beta} * \delta u\right)_i, \delta u_i\right)\tau &\geq \tfrac{1}{2}\left(g_{1-\beta} * \|\delta u\|^2\right)_j + \tfrac{1}{2}\sum_{i=1}^{j} g_{1-\beta}(t_i)\|\delta u_i\|^2\tau \\ &\geq \tfrac{1}{2}\left(g_{1-\beta} * \|\delta u\|^2\right)_j + \frac{1}{4}\sum_{i=1}^{j} g_{1-\beta}(t_i)\|\delta u_i\|^2\tau + \frac{g_{1-\beta}(T)}{4}\sum_{i=1}^{j}\|\delta u_i\|^2\tau.\end{aligned}$$

The ellipticity assumption yields

$$\sum_{i=1}^{j}\mathcal{L}_i(\delta u_i,\delta u_i)\tau \geq C_0\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau.$$

We involve Lemma 2.3.2 and $u_0 \in H^1(\Omega)$ to get

$$\left|\sum_{i=1}^{j}(\delta\mathcal{L})_i(u_{i-1},\delta u_i)\tau\right| \leq C\sum_{i=1}^{j}\|u_{i-1}\|_{H^1(\Omega)}\|\delta u_i\|_{H^1(\Omega)}\tau \leq \varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2_{H^1(\Omega)}\tau + C_\varepsilon.$$

The first term on the r.h.s. of (2.19) can be readily estimated using the Cauchy and Young's inequalities and taking into account (2.17)

$$\begin{aligned}\left|\sum_{i=1}^{j} \delta h_i \left(f, \delta u_i\right) \tau\right| &\leq C \sum_{i=1}^{j} |\delta h_i| \, \|\delta u_i\| \, \tau \\ &\overset{(2.17)}{\leq} C \sum_{i=1}^{j} \left(g_{1-\beta}(t_i) + 1 + \|u_{i-1}\|\right) \|\delta u_i\| \, \tau \\ &\leq \varepsilon \sum_{i=1}^{j} g_{1-\beta}(t_i) \, \|\delta u_i\|^2 \, \tau + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \, \tau + C_\varepsilon.\end{aligned}$$

For the second term on the r.h.s. of (2.19), we again apply the Cauchy and Young's inequalities to get

$$\left|\sum_{i=1}^{j} \left(F(t_i, u_{i-1}), \delta u_i\right) \tau\right| \leq \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \, \tau + C_\varepsilon.$$

The last term of (2.19) can be handled similarly involving the Cauchy and Young's inequalities and the trace theorem

$$\left|\sum_{i=1}^{j} \left(\delta g_i, \delta u_i\right)_\Gamma \tau\right| \leq \varepsilon \sum_{i=1}^{j} \|\delta u_i\|_\Gamma^2 \, \tau + C_\varepsilon \leq \varepsilon \sum_{i=1}^{j} \|\delta u_i\|_{H^1(\Omega)}^2 \, \tau + C_\varepsilon.$$

Collecting all estimates above, we may write

$$\begin{aligned}\frac{1}{2} \left(g_{1-\beta} * \|\delta u\|^2\right)_j &+ \left(\frac{1}{4} - \varepsilon\right) \sum_{i=1}^{j} g_{1-\beta}(t_i) \, \|\delta u_i\|^2 \, \tau \\ &+ \left(\frac{g_{1-\beta}(T)}{4} - \varepsilon\right) \sum_{i=1}^{j} \|\delta u_i\|^2 \, \tau + (C_0 - \varepsilon) \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \, \tau \leq C_\varepsilon.\end{aligned}$$

Fixing a sufficiently small $0 < \varepsilon < 1$, we obtain that

$$\left(g_{1-\beta} * \|\delta u\|^2\right)_j + \sum_{i=1}^{j} g_{1-\beta}(t_i) \, \|\delta u_i\|^2 \, \tau + \sum_{i=1}^{j} \|\delta u_i\|_{H^1(\Omega)}^2 \, \tau \leq C.$$

This together with (2.17) imply

$$|\delta h_i| \leq C g_{1-\beta}(t_i) + C + C \, \|u_{i-1}\| \leq C t_i^{-\beta} + C.$$

□

**Rothes' functions and existence theorem**

Now, let us introduce the following piecewise linear Rothe's functions in time $u_n, \overline{u}_n, \tilde{u}_n : [0,T] \to L^2(\Omega)$

$$
\begin{aligned}
u_n : t &\mapsto \begin{cases} u_0, & t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i, & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases} \\
\overline{u}_n : t &\mapsto \begin{cases} u_0, & t = 0 \\ u_i, & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases} \\
\tilde{u}_n : t &\mapsto \begin{cases} u_0, & t \in [0, \tau] \\ \overline{u}_n(t - \tau), & t \in (t_{i-1}, t_i], \quad 2 \le i \le n. \end{cases}
\end{aligned}
\tag{2.20}
$$

Analogously, we define $h_n, \overline{h}_n$, $\overline{F}_n$, $\overline{\mathcal{L}}_n$, $\overline{g}_n$, $\overline{g_{1-\beta}}_n$ and $\overline{m'}_n$. Now, we can rewrite (DP$i$) and (DMP$i$) on the whole time frame as (for $t \in (t_{i-1}, t_i]$)

$$
\begin{aligned}
&\left(\left(\overline{g_{1-\beta}}_n * \partial_t u_n\right)(t_i), \varphi\right) + \overline{\mathcal{L}}_n(t)\left(\overline{u}_n(t), \varphi\right) \\
&\qquad = \overline{h}_n(t)\,(f, \varphi) + \left(\int_0^{t_i} \overline{F}_n(s, \tilde{u}_n(s))\,\mathrm{d}s, \varphi\right) - \left(\overline{g}_n(t), \varphi\right)_\Gamma
\end{aligned}
\tag{DP}
$$

and

$$
\left(\overline{g_{1-\beta}}_n * \overline{m'}_n\right)(t_i) = \overline{h}_n(t)\,(f, 1) + \left(\int_0^{t_i} \overline{F}_n(s, \tilde{u}_n(s))\,\mathrm{d}s, 1\right) - \left(\overline{g}_n(t), 1\right)_\Gamma . \tag{DMP}
$$

We are in a position to prove the existence of a variational solution to (P) and (MP). We do so by showing the convergence of the Rothe functions and also by showing the convergence of the (DP), (DMP) to (P), (MP).

**Theorem 2.3.1** (existence of a solution)**.** *Let $f \in L^2(\Omega)$, $u_0 \in H^1(\Omega)$, $\int_\Omega f \neq 0$, $m \in C^2([0,T])$, and $g \in C^1([0,T], L^2(\Gamma))$. Suppose that $F$ is a global Lipschitz continuous function in all variables. Assume (2.4), (2.15), (2.16), $\partial_t c \in L^\infty[0,T]$ and $\partial_t a_{i,j}, \partial_t b_i \in L^\infty(\overline{\Omega \times (0,T)})$ for all $i,j = 1, \ldots, d$.*

*Then there exists a solution $(u,h)$ to (P), (MP) obeying $u \in C\left([0,T], H^1(\Omega)\right)$ with $\partial_t u \in L^2\left((0,T), H^1(\Omega)\right)$, $h \in C([0,T])$.*

*Proof.* The estimate from Lemma 2.3.3 $(ii)$ implies for $t \in (t_{i-1}, t_i]$ that

$$
|h_n'(t)| = |\delta h_i| \le C t_i^{-\beta} + C \le C t^{-\beta} + C.
$$

Therefore

$$
\begin{aligned}
|h_n(t+\varepsilon)-h_n(t)| = \left|\int_t^{t+\varepsilon} h_n'(x)\,\mathrm{d}x\right| &\leq C\int_t^{t+\varepsilon}\left(x^{-\beta}+1\right)\,\mathrm{d}x \\
&\leq C\frac{(t+\varepsilon)^{1-\beta}-t^{1-\beta}}{1-\beta}+\varepsilon=\mathcal{O}\left(\varepsilon^{1-\beta}\right),
\end{aligned}
$$

which yields the equi-continuity of the sequence $\{h_n\}$. Lemma 2.3.2 also guarantees the equi-boundedness of $\{h_n\}$. By means of the Arzelà-Ascoli theorem 1.2.5, we get compactness of $h_n$ in $C([0,T])$.

Lemma 2.3.3 says that $\|\overline{u}_n(t)\|_{H^1(\Omega)}+\int_0^T\|\partial_t u_n\|^2_{H^1(\Omega)}\leq C$. Due to the compact embedding $H^1(\Omega)\Subset L^2(\Omega)$, we may invoke Lemma 1.5.1 to claim the existence of $u\in C\left([0,T],L^2(\Omega)\right)\cap L^\infty\left((0,T),H^1(\Omega)\right)$, which is time-differentiable a.e. in $[0,T]$, and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (denoted by the same symbol again) such that

$$
\begin{cases}
u_n \to u & \text{in} \quad C\left([0,T],L^2(\Omega)\right), & (2.21\text{a})\\
u_n(t) \rightharpoonup u(t) & \text{in} \quad H^1(\Omega), \quad \forall t\in[0,T], & (2.21\text{b})\\
\overline{u}_n(t) \rightharpoonup u(t) & \text{in} \quad H^1(\Omega), \quad \forall t\in[0,T], & (2.21\text{c})\\
\partial_t u_n \rightharpoonup \partial_t u & \text{in} \quad L^2\left((0,T),L^2(\Omega)\right). & (2.21\text{d})
\end{cases}
$$

Reflexivity of the space $L^2\left((0,T),H^1(\Omega)\right)$ together with Lemma 2.3.3 also give

$$
\partial_t u_n \rightharpoonup \partial_t u, \qquad \text{in} \qquad L^2\left((0,T),H^1(\Omega)\right)
$$

and

$$
\begin{aligned}
u(t)-u(s) &= \int_s^t \partial_t u(z)\,\mathrm{d}z \Longrightarrow \\
&\|u(t)-u(s)\|_{H^1(\Omega)}\leq\sqrt{|t-s|}\sqrt{\int_0^T\|\partial_t u(z)\|^2_{H^1(\Omega)}\,\mathrm{d}z}\leq C\sqrt{|t-s|}.
\end{aligned}
$$

Taking into account $u_0\in H^1(\Omega)$, we get $u\in C\left([0,T],H^1(\Omega)\right)$. Further, it holds

$$
\int_0^T\|\tilde{u}_n-\overline{u}_n\|^2=\mathcal{O}\left(\tau^2\right).
$$

We are allowed to write for $t\in(t_{i-1},t_i]$

$$
\left|\left(\overline{g_{1-\beta}}_n * \overline{m'}_n\right)(t_i)-\left(\overline{g_{1-\beta}}_n * \overline{m'}_n\right)(t)\right|
$$

$$
\begin{aligned}
&\leq \left| \int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s)\overline{m'}_n(s)\,\mathrm{d}s \right| \\
&\quad + \left| \int_0^t \left( \overline{g_{1-\beta}}_n(t_i - s) - \overline{g_{1-\beta}}_n(t - s) \right) \overline{m'}_n(s)\,\mathrm{d}s \right| \\
&\leq C \int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s)\,\mathrm{d}s + C \int_0^t \left| \overline{g_{1-\beta}}_n(t_i - s) - \overline{g_{1-\beta}}_n(t - s) \right| \,\mathrm{d}s.
\end{aligned}
$$

The pointwise convergence $\overline{g_{1-\beta}}_n \to g_{1-\beta}$ and $\overline{m'}_n \to m'$ in $(0,T)$ and the Lebesgue dominated theorem yield

$$
\left( \overline{g_{1-\beta}}_n * \overline{m'}_n \right)(t_i) \to \left( g_{1-\beta} * m' \right)(t) \qquad \text{for } n \to \infty.
$$

Based on the considerations above, we may pass to the limit $n \to \infty$ in (DMP) to arrive at (MP). The process is straightforward, therefore we omit further details.

It remains to show that the couple $(u,h)$ also obey (P). We successively deduce that

$$
\begin{aligned}
&\left\| \left( \overline{g_{1-\beta}}_n * \partial_t u_n \right)(t_i) - \left( \overline{g_{1-\beta}}_n * \partial_t u_n \right)(t) \right\| \\
&\qquad \leq \left\| \int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s)\partial_t u_n(s)\,\mathrm{d}s \right\| \\
&\qquad\quad + \left\| \int_0^t \left( \overline{g_{1-\beta}}_n(t_i - s) - \overline{g_{1-\beta}}_n(t - s) \right) \partial_t u_n(s)\,\mathrm{d}s \right\| \\
&\qquad \leq \int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s) \left\| \partial_t u_n(s) \right\| \,\mathrm{d}s \\
&\qquad\quad + \int_0^t \left| \overline{g_{1-\beta}}_n(t_i - s) - \overline{g_{1-\beta}}_n(t - s) \right| \left\| \partial_t u_n(s) \right\| \,\mathrm{d}s.
\end{aligned}
$$

The first term on the r.h.s. can be estimated as follows

$$
\begin{aligned}
&\int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s) \left\| \partial_t u_n(s) \right\| \,\mathrm{d}s \\
&\qquad \leq \sqrt{\int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s)\,\mathrm{d}s} \sqrt{\int_t^{t_i} \overline{g_{1-\beta}}_n(t_i - s) \left\| \partial_t u_n(s) \right\|^2 \,\mathrm{d}s} \\
&\qquad \leq \sqrt{\tau^{1-\beta}} \sqrt{\int_0^{t_i} \overline{g_{1-\beta}}_n(t_i - s) \left\| \partial_t u_n(s) \right\|^2 \,\mathrm{d}s} \\
&\qquad \leq C\sqrt{\tau^{1-\beta}}. \qquad\qquad \text{(Lemma 2.3.3)}
\end{aligned}
$$

Using the Lebesgue dominated theorem, we find that

$$
\lim_{n\to\infty} \int_0^t \left| \overline{g_{1-\beta}}_n(t_i - s) - \overline{g_{1-\beta}}_n(t - s) \right| \left\| \partial_t u_n(s) \right\| \,\mathrm{d}s = 0.
$$

By the Cauchy, Hölder and Young's inequalities, we have

$$\begin{aligned}&\left|\int_0^\xi \left(\left[\overline{g_{1-\beta}}_n - g_{1-\beta}\right] * (\partial_t u_n, \varphi)\right)(t)\,\mathrm{d}t\right| \\ &\qquad \le \int_0^\xi \left|\overline{g_{1-\beta}}_n - g_{1-\beta}\right|\,\mathrm{d}t \sqrt{\int_0^\xi \left\|\partial_t u_n\right\|^2\,\mathrm{d}t}\sqrt{\int_0^\xi \left\|\varphi\right\|^2\,\mathrm{d}t} \le C\left\|\varphi\right\|.\end{aligned}$$

The pointwise convergence $\overline{g_{1-\beta}}_n \to g_{1-\beta}$ in $(0,T)$ and the Lebesgue dominated theorem imply that

$$\lim_{n\to\infty}\left|\int_0^\xi \left(\left[\overline{g_{1-\beta}}_n - g_{1-\beta}\right] * (\partial_t u_n, \varphi)\right)(t)\,\mathrm{d}t\right| = 0.$$

Using

$$\partial_t\left(g_{1-\beta} * u_n\right)(t) = \left(g_{1-\beta} * \partial_t u_n\right)(t) + g_{1-\beta}(t)u_n(0),$$

we see that

$$\int_0^\xi \left(g_{1-\beta} * (\partial_t u_n, \varphi)\right)(t)\,\mathrm{d}t = \left(g_{1-\beta} * (u_n, \varphi)\right)(\xi) - (u_0, \varphi)\int_0^\xi g_{1-\beta}(t)\,\mathrm{d}t.$$

Passing to the limit for $n \to \infty$ and taking into account (2.21a), we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^\xi \left(g_{1-\beta} * (\partial_t u_n, \varphi)\right)(t)\,\mathrm{d}t &= \left(g_{1-\beta} * (u, \varphi)\right)(\xi) - \int_0^\xi g_{1-\beta}(t)\,\mathrm{d}t\,(u_0, \varphi) \\ &= \int_0^\xi \left(g_{1-\beta} * (\partial_t u, \varphi)\right)(t)\,\mathrm{d}t \\ &= \int_0^\xi \left(\left(g_{1-\beta} * \partial_t u\right)(t), \varphi\right)\,\mathrm{d}t.\end{aligned}$$

In order to check that $(u,h)$ solve (P), we start from (DP), which we integrate in time over $(0,\xi)$. Then (based on considerations above) we may pass to the limit $n \to \infty$ to get

$$\begin{aligned}&\int_0^\xi \left[\left(\left(g_{1-\beta} * \partial_t u\right)(t), \varphi\right) + \mathcal{L}(t)\left(u(t), \varphi\right)\right]\mathrm{d}t \\ &\qquad = \int_0^\xi \left[h(t)\left(f, \varphi\right) + \left(\int_0^t F(s, u(s))\,\mathrm{d}s, \varphi\right) - \left(g(t), \varphi\right)_\Gamma\right]\mathrm{d}t. \qquad (2.22)\end{aligned}$$

Differentiation with respect to $\xi$ brings us to the desired result.

$\square$

The convergences of Rothe's functions towards the weak solution (P)-(MP) (as stated in the proof of Theorem 2.3.1) have been shown for a subsequence. Note that taking into account Theorem 2.2.1, we see that the whole sequence of Rothe's functions converge against the weak solution.

**Remark 1.** *Section 2.2 addressed the uniqueness and Section 2.3 showed the existence of a solution to the inverse source problem (2.1)-(2.3). We would like to point out that the same technique can be applied for solving a direct problem (2.1)-(2.2) if $h(t)$ is known.*

## 2.4 Numerical Experiments

In this section, we test the above-mentioned numerical scheme to approximate the solution of (P)-(MP), which is based on (DP$i$) and (DMP$i$). Numerical results are presented and discussed.

### 2.4.1 Exact data

We consider problem (P)-(MP) for $\Omega = (0.5, 3)$, $T = 3$ and $\beta = 0.5$ with

$$\begin{aligned} \mathcal{L}(u, \varphi) &= (\nabla u, \nabla \varphi), \\ f(x) &= \sin x, \\ F(x,t,u) &= -4tu \exp\left(1 - \frac{u^2}{\sin^2 x}\right), \end{aligned}$$

along with the initial and boundary conditions

$$\begin{aligned} u_0(x) &= 2\sin x, \\ g(0.5, t) &= (t^2+2)\cos\tfrac{1}{2}, \\ g(3,t) &= (t^2+2)\cos 3, \end{aligned}$$

where the time-dependent measurement is

$$m(t) = \left(\cos\frac{1}{2} - \cos 3\right)\left(t^2+2\right).$$

One can easily verify that functions

$$u(x,t) = \left(t^2+2\right)\sin x$$

and

$$h(t) = \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} + t^2 - \exp\left(1 - (t^2-2)^2\right) + e^{-3} + 2$$

solve the given problem.

To get a solution of (DP$i$), the domain $\Omega$ is uniformly divided into 50 subintervals. The solution $u_i$ is calculated using a finite element method with Lagrange polynomials of the second order used as basis functions. Calculations were made several times for various values of $\tau$. The algorithm was implemented in Python using the FEniCS Project.

Fig. 2.1 (a), (b) show a numerical approximation of functions $h$ and $u(T)$ for $\tau = 0.01$ together with exact $h$ and $u(T)$, respectively. Fig. 2.1 (c), (d) display relative error of $h$ and $u$ in time, respectively, for $\tau = 0.01$, which shows that the numerical accuracy is fair. Please note that the reconstruction of $u$ is more accurate then the reconstruction of $h$.

Maximal relative errors in time of $h$ and $u$ for different values of $\tau$ are depicted in Fig. 2.1 (e), (f), respectively. The linear regression lines plotted through data points are given by $0.39529 \log_{10} \tau - 0.71487$ for the error of $h$ and $0.99983 \log_{10} \tau - 0.47112$ for the error of $u$.

### 2.4.2 Noisy data

As the measured data usually contain some amount of noise, the question of dealing with noisy data is interesting. The algorithm we proposed in the theoretical part works with the first derivative of the measurement $m$. When dealing with real data, the continuous derivative can hardly be expected due to the present noise. Using for example the finite difference in such a situation for an approximation of the derivative is practically useless as it might just enlarge the noise, and the result is often unusable [19]. To avoid this it is necessary to use some kind of "smoothing or filtering" of noisy data. In [91] there is a mollification used on this purpose, minimization of an appropriate functional was used in [19]. We use the nonlinear least square method to get a sufficiently smooth function approximating the noisy data, and afterwards we deal with this smooth approximation in our algorithm.

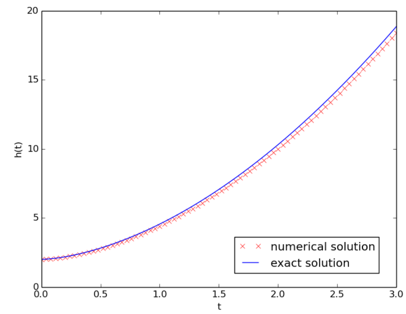
We consider the same example as in Section 2.4.1. The noisy measurement has the following form

$$m_\epsilon(t) = m(t) + \epsilon\delta m_{max}, \tag{2.23}$$

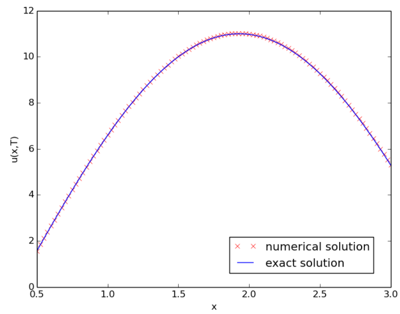
where $\epsilon$ is a small parameter, $\delta$ is the Gaussian distributed noise with the mean and standard deviation equal to 0 and 1, respectively, and $m_{max}$ is a maximal value of $m$ on the interval $[0, T]$. First, we look for a function $m_{app}$ that approximates $m$ and has the form

$$m_{app}(t) = \alpha t^\beta + \gamma. \tag{2.24}$$

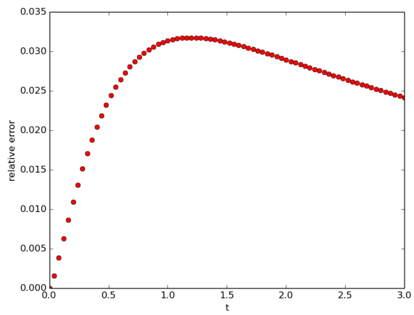
We use then the function $m_{app}$ as the measurement in our algorithm. All other settings from the previous experiment remain the same. The exact, noisy and



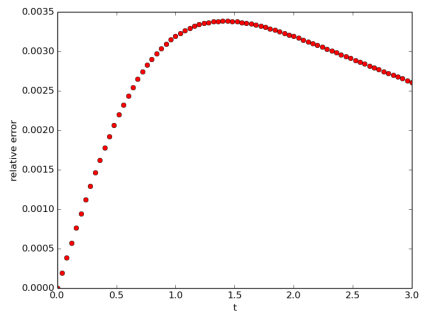

(a) Reconstruction of $h$ together with exact $h$.



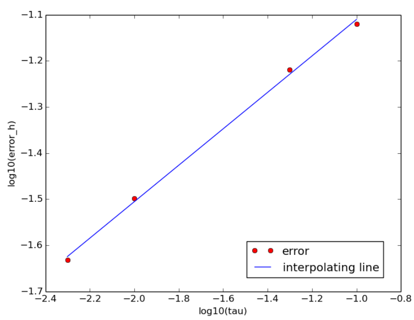

(b) Reconstruction of $u(T)$ together with exact $u(T)$.



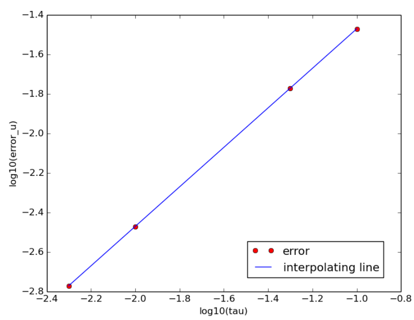

(c) Relative error $\frac{|h_{approx}(t) - h_{exact}(t)|}{|h_{exact}(t)|}$.




(d) Relative error $\frac{||u_{approx}(t) - u_{exact}(t)||}{||u_{exact}(t)||}$.



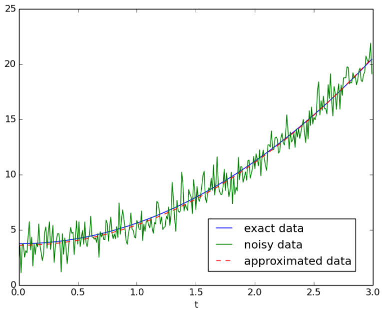

(e) Logarithm of maximal relative error in time of $h$ for different values of $\tau$. Slope of the line is 0.39529.



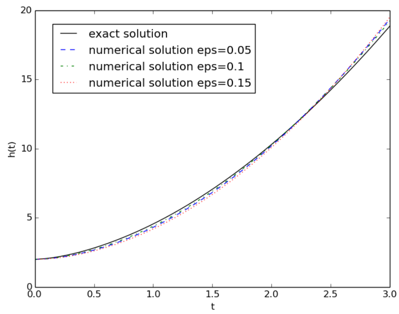

(f) Logarithm of maximal relative error in time of $u$ for different values of $\tau$. Slope of the line is 0.99983.

Figure 2.1: The results of the reconstruction algorithm. In (a)-(d) $\tau = 0.01$.

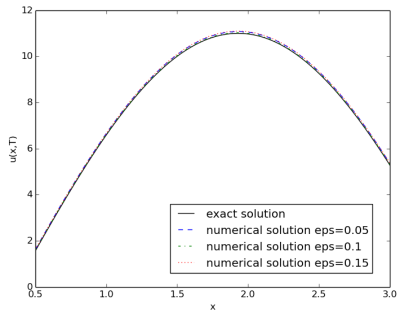
approximated data computed with the nonlinear least square method can be seen on Fig. 2.2. The reconstruction of $h$ and $u(T)$ is presented in Fig. 2.2 (a), (b), respectively. Finally, the relative errors in time are depicted in Fig. 2.2 (c), (d).



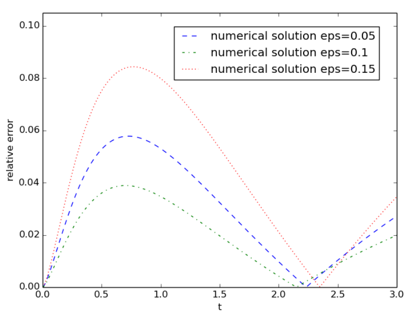

(a) Exact and noisy data for $\epsilon = 0.05$. Approximating curve has the form $m_{app}(t) = 1.8444t^{2.0234} + 3.5918$.



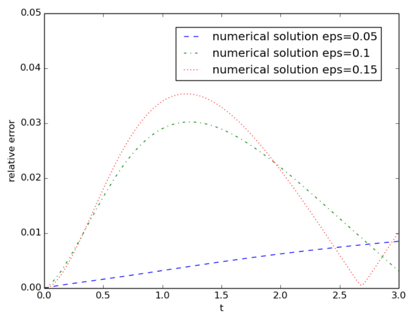

(b) Reconstruction of $h$ together with exact $h$.




(c) Reconstruction of $u(T)$ together with exact $u(T)$.




(d) Relative error $\frac{|h_{approx}(t)-h_{exact}(t)|}{|h_{exact}(t)|}$.




(e) Relative error $\frac{||u_{approx}(t)-u_{exact}(t)||}{||u_{exact}(t)||}$.

Figure 2.2: The result of reconstruction of $h$ and $u$ for noisy data with a various amount of noise $\epsilon$ and $\tau = 0.01$

# Chapter 3

# Recognition of a time-dependent source in a time-fractional wave equation

The content of this chapter is based on the article [130], which has been already published in the journal: *Applied Numerical Mathematics*.

## 3.1 Introduction

Consider a partial differential equation (PDE) with a fractional derivative in time $t$

$$
\begin{aligned}
&\left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \Delta u(x,t) \\
&\qquad = h(t) f(x) + F(x,t,u(x,t)), \qquad x \in \Omega,\ t \in (0,T), \qquad (3.1)
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary $\Gamma$, $T > 0$, and $g_{2-\beta}$ is the Riemann-Liouville kernel given by

$$
g_{2-\beta}(t) = \frac{t^{1-\beta}}{\Gamma(2-\beta)}, \qquad t > 0,\ 1 < \beta < 2.
$$

We supplement the governing PDE (3.1) with the following initial and boundary conditions

$$
\begin{array}{rll}
u(x,0) & = u_0(x), & x \in \Omega, \\
\partial_t u(x,0) & = v_0(x), & x \in \Omega, \\
-\nabla u(x,t) \cdot \boldsymbol{\nu} & = g(x,t), & (x,t) \in \Gamma \times (0,T),
\end{array} \qquad (3.2)
$$

where the symbol $\boldsymbol{\nu}$ denotes the outer normal vector assigned to the boundary $\Gamma$.

The *Inverse Source Problem* (ISP) studied in this chapter consists of finding a couple $(u(x,t),h(t))$ obeying (3.1), (3.2) and

$$\int_\Omega u(x,t)\omega(x)\,\mathrm{d}x = m(t), \qquad t \in [0,T], \tag{3.3}$$

where the weight function $\omega$ is just a space-dependent function. Usually $\omega$ is chosen to be a function with compact support in $\Omega$, and then this type of measurement represents the weighted average of $u$ on a subdomain of $\Omega$.

The fractional wave equation is used for example to model the propagation of diffusive waves in viscoelastics solids [77, 81]. The uniqueness and existence of a solution to the direct Cauchy problem for a fractional diffusion-wave equation has been studied in [110]. In [80], a fundamental solution of Cauchy problem is expressed using the Laplace transform. More about the direct fractional wave problem can be found in [75, 78, 79]. The recovery of a time dependent source in a fractional integrodifferential wave equation by means of the Banach fixed point theorem has been studied in [133, 138]. In [138], the time dependent source is reconstructed using a time trace at point $x_0 \in \Omega$. In [133], two measurements in the form of integral over the subdomain were used to identify the time- dependent source and convolution kernel. In both articles zero Dirichlet boundary condition is considered. To the best of the our knowledge, there are no articles considering the Neumann boundary condition in the problem of time-dependent source identification in the fractional wave equation. Moreover, we design also a numerical scheme for reconstruction.

The aim of this chapter is to prove the uniqueness and global existence of the weak solution of the ISP.

The chapter is organized as follows. In the second section, we introduce variational formulation of the ISP. In Section 3 we suggest a numerical scheme based on the Rothe method of in time semi-discretization. We prove the existence of the approximate solutions along the time slices and prove some a priori estimates. Convergence of the Rothe functions towards the solution of the ISP is shown in Section 4. In the last section, we present a numerical experiment to support our result.

## 3.2 Uniqueness

Multiplying (3.1) by the function $\omega$, integrating over $\Omega$, applying the Green theorem and using (3.3), we obtain

$$(g_{2-\beta} * m'')(t) + (\nabla u(t), \nabla\omega) = h(t)(f,\omega) - (g(t),\omega)_\Gamma + (F(t,u(t)),\omega). \quad \text{(MP2)}$$

Assuming $(f,\omega) \neq 0$, we get

$$h(t) = \frac{(g_{2-\beta} * m'')(t) + (\nabla u(t), \nabla\omega) + (g(t),\omega)_\Gamma - (F(t,u(t)),\omega)}{(f,\omega)}. \tag{3.4}$$

Similarly multiplying (3.1) by a function $\varphi \in H^1(\Omega)$ and using Green's theorem, we obtain the variational formulation of (3.1) and (3.2), which reads as

$$((g_{2-\beta} * \partial_{tt}u)(t), \varphi) + (\nabla u(t), \nabla\varphi) = h(t)(f,\varphi) + (F(t,u(t)),\varphi) - (g(t),\varphi)_\Gamma, \quad \text{(P2)}$$

for any $\varphi \in H^1(\Omega)$, a.a. $t \in [0,T]$ and $u(0) = u_0$, $\partial_t u(0) = v_0$. The relations (P2) and (MP2) represent the variational formulation of the ISP (3.1), (3.2) and (3.3).

The next theorem deals with the uniqueness of the solution in the appropriate spaces.

**Theorem 3.2.1** (uniqueness). *Let $f, v_0 \in L^2(\Omega), u_0, \omega \in H^1(\Omega)$, $\int_\Omega f\omega \neq 0$, $m \in C^2([0,T])$, $F$ be a global Lipschitz continuous function in all variables and $g \in C([0,T], L^2(\Gamma))$. Then there exists at most one solution $(u,h)$ to the (P2), (MP2) obeying $u \in C\left([0,T], H^1(\Omega)\right)$, $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), H^1(\Omega)\right)$ with $\partial_{tt}u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* Assume that there exists two solutions $(u_1,h_1), (u_2,h_2)$ of the (P2), (MP2) obeying assumptions from the theorem. Set $u = u_1 - u_2$ and $h = h_1 - h_2$. Then the pair $(u,h)$ solves the following problem

$$(\nabla u(t), \nabla\omega) = h(t)(f,\omega) + (F(t,u_1(t)) - F(t,u_2(t)),\omega), \tag{3.5}$$

and

$$\begin{aligned} &((g_{2-\beta} * \partial_{tt}u)(t), \varphi) + (\nabla u(t), \nabla\varphi) \\ &\qquad = h(t)(f,\varphi) + (F(t,u_1(t)) - F(t,u_2(t)),\varphi), \end{aligned} \tag{3.6}$$

for every $\varphi \in H^1(\Omega)$, a.a. $t \in [0,T]$ and $u(0) = 0$, $\partial_t u(0) = 0$. Let take $\varphi = \partial_t u(t)$ and integrate the relation (3.6) over the interval $(0,\xi)$, where $\xi \in (0,T]$. By using the fact that $\partial_t u = 0$, we get $(g_{2-\beta} * \partial_{tt}u)(t) = \partial_t (g_{2-\beta} * \partial_t u)(t)$ and together with assumption that $\int_\Omega f\omega \neq 0$, we obtain from (3.5) and (3.6) that

$$
\begin{aligned}
\int_0^\xi \left(\partial_t \left(g_{2-\beta} * \partial_t u\right)(t), \partial_t u(t)\right) \mathrm{d}t + \frac{1}{2}\left\|\nabla u(\xi)\right\|^2 \\
= \int_0^\xi \frac{(\nabla u(t), \nabla \omega) - (F(t, u_1(t)) - F(t, u_2(t)), \omega)}{(f, \omega)} (f, \partial_t u(t)) \mathrm{d}t \\
+ \int_0^\xi (F(t, u_1(t)) - F(t, u_2(t)), \partial_t u(t)) \mathrm{d}t.
\end{aligned}
\tag{3.7}
$$

To gain the lower bound of the l.h.s. we use Lemma 1.7.2

$$
\begin{aligned}
\int_0^\xi \left(\partial_t \left(g_{2-\beta} * \partial_t u\right)(t), \partial_t u(t)\right) \mathrm{d}t + \frac{1}{2}\left\|\nabla u(\xi)\right\|^2 \\
\geq \frac{g_{2-\beta}(T)}{2} \int_0^\xi \left\|\partial_t u(t)\right\|^2 \mathrm{d}t + \frac{1}{2}\left\|\nabla u(\xi)\right\|^2.
\end{aligned}
$$

Using the Cauchy and Young inequalities together with the Lipschitz continuity of $F$, we acquire the upper bound of the r.h.s. in (3.7)

$$
\begin{aligned}
\int_0^\xi \frac{(\nabla u(t), \nabla \omega) - (F(t, u_1(t)) - F(t, u_2(t)), \omega)}{(f, \omega)} (f, \partial_t u(t)) \mathrm{d}t \\
+ \int_0^\xi (F(t, u_1(t)) - F(t, u_2(s)), \partial_t u(t)) \mathrm{d}t \\
\leq \varepsilon \int_0^\xi \left\|\partial_t u(t)\right\|^2 \mathrm{d}t + C_\varepsilon \int_0^\xi \left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) \mathrm{d}t.
\end{aligned}
$$

Combination of the both estimates gives us

$$
\begin{aligned}
\left(\frac{g_{2-\beta}(T)}{2} - \varepsilon\right) \int_0^\xi \left\|\partial_t u(t)\right\|^2 \mathrm{d}t + \frac{1}{2}\left\|\nabla u(\xi)\right\|^2 \\
\leq C_\varepsilon \int_0^\xi \left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) \mathrm{d}t.
\end{aligned}
$$

We choose a fixed, sufficiently small $\varepsilon$. By the estimate $\|u(\xi)\|^2 \leq \int_0^\xi \left\|\partial_t u(t)\right\|^2 \mathrm{d}t$, we get

$$
\|u(\xi)\|^2 + \|\nabla u(\xi)\|^2 \leq C \int_0^\xi \left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) \mathrm{d}t.
$$

Now, we can apply Grönwall's lemma and obtain estimate

$$
\|u(\xi)\|^2 + \|\nabla u(\xi)\|^2 \leq 0, \tag{3.8}
$$

which hold for $\xi \in [0,T]$. It follows directly from (3.8) that $u = 0$ a.e. in $[0,T] \times \Omega$. Moreover, using this together with the Lipschitz continuity of $F$ in (3.5), it is easily seen that $h = 0$ a.e. in $[0,T]$. So, we get $u_1 = u_2$ and $h_1 = h_2$. □

## 3.3 Time discretization

We introduce the equidistant time-partitioning of the interval $[0, T]$ by the step $\tau = \frac{T}{n}$, for any $n \in \mathbb{N}$. We use notation $t_i = i\tau$, for $i = 1, ..., n$, and $z_i, \delta z_i$ for any function $z$ defined in the previous chapter (see (2.8)) as the value at the point $t_i$ and an $i$-th difference, respectively, moreover we write

$$\delta^2 z_i = \frac{\delta z_i - \delta z_{i-1}}{\tau}.$$

On the $i-$th time-layer, we approximate the solution of (P2), (MP2) by $(u_i, h_i)$ that solves

$$\left(\left(g_{2-\beta} * \delta^2 u\right)_i, \varphi\right) + (\nabla u_i, \nabla \varphi) = h_i\,(f, \varphi) + (F(t_i, u_{i-1}), \varphi) - (g_i, \varphi)_\Gamma\,, \quad (\text{DP}i)$$

for $\varphi \in H^1(\Omega)$, with $\delta u_0 := v_0$ and

$$(g_{2-\beta} * m'')_i + (\nabla u_{i-1}, \nabla \omega) = h_i\,(f, \omega) + (F(t_i, u_{i-1}), \omega) - (g_i, \omega)_\Gamma\,. \quad (\text{DMP}i)$$

To compute the solution of those equations for given $i$, we first find $h_i$ from (DMP$i$) and then calculate $u_i$ from (DP$i$). Afterward, we increase $i$ to $i + 1$.

Following lemma deals with the existence and uniqueness of the solution $(u_i, h_i)$ on every time-layer.

**Lemma 3.3.1.** *Let $f, v_0 \in L^2(\Omega)$, $u_0, \omega \in H^1(\Omega)$, $\int_\Omega f\omega \neq 0$, $m \in C^2([0, T])$, $g \in C([0, T], L^2(\Gamma))$ and $F$ be a global Lipschitz continuous function in all variables. Then for each $i \in \{1, \ldots, n\}$, there exists a unique couple $(u_i, h_i) \in H^1(\Omega) \times \mathbb{R}$ solving (DPi) and (DMPi) for every $\varphi \in H^1(\Omega)$.*

*Proof.* The requirement on $f$ and $\omega$ that $\int_\Omega f\omega \neq 0$ gives us

$$h_i = \frac{(g_{2-\beta} * m'')_i + (\nabla u_{i-1}, \nabla \omega) + (g_i, \omega)_\Gamma - (F(t_i, u_{i-1}), \omega)}{(f, \omega)} \in \mathbb{R}. \quad (3.9)$$

The equation (DP$i$) can be written as

$$\begin{aligned} &\frac{1}{\tau} g_{2-\beta}(\tau)\,(u_i, \varphi) + (\nabla u_i, \nabla \varphi) = h_i\,(f, \varphi) + (F(t_i, u_{i-1}), \varphi) - (g_i, \varphi)_\Gamma \\ &- \sum_{k=1}^{i-1} g_{2-\beta}(t_{i+1-k})\left(\delta^2 u_k, \varphi\right)\tau + \frac{1}{\tau} g_{2-\beta}(\tau)\,(u_{i-1}, \varphi) + g_{2-\beta}(\tau)\,(\delta u_{i-1}, \varphi)\,. \quad (3.10) \end{aligned}$$

The expression on the l.h.s. is a bilinear, elliptic, bounded form defined on $H^1(\Omega) \times H^1(\Omega)$. If $u_0, \ldots, u_{i-1}, v_0 \in L^2(\Omega)$, then the r.h.s. can be seen as a linear, bounded functional on $H^1(\Omega)$. The Lax-Milgram lemma 1.2.4 implies the existence of the unique solution in $H^1(\Omega)$ of (3.10). $\square$

The next goal is to establish some estimates of $u_i$, $h_i$ in appropriate norms.

**Lemma 3.3.2.** *Under the assumptions of Lemma 3.3.1, if moreover it holds that $g \in C^1\left([0,T], L^2(\Gamma)\right)$, then there exist positive constants $C$ (independent of $n$) such that*

$$(i)\ \max_{0\le i\le n}\left(g_{2-\beta} * \|\delta u\|^2\right)_i + \sum_{i=1}^n g_{2-\beta}(t_i)\left\|\delta u_i\right\|^2\tau + \sum_{i=1}^n \left\|\delta u_i\right\|^2\tau + \max_{0\le i\le n}\|u_i\|^2_{H^1(\Omega)} + \sum_{i=1}^n \left\|\nabla(u_i-u_{i-1})\right\|^2 \le C,$$

$$(ii)\ \max_{0\le i\le n}|h_i| \le C.$$

*Proof.* We start by estimating $h_i$ from (3.9)

$$|h_i| \le C\left(1+\|\nabla u_{i-1}\| + \|F(t_i,u_{i-1})\|\right) \le C\left(1+\|\nabla u_{i-1}\|+\|u_{i-1}\|\right). \tag{3.11}$$

Now, setting $\varphi = \delta u_i \tau$ in the relation (DP$i$), we get that

$$\begin{aligned}\left(\left(g_{2-\beta} * \delta^2 u\right)_i, \delta u_i\tau\right) &+ \left(\nabla u_i, \nabla\delta u_i\tau\right)\\ &= h_i\left(f,\delta u_i\tau\right) + \left(F(t_i,u_{i-1}),\delta u_i\tau\right) - \left(g_i,\delta u_i\tau\right)_\Gamma.\end{aligned} \tag{3.12}$$

We sum equations up over $i=1,\dots,j$, $j\in\{1,\dots,n\}$ and use the relation (2.10) to obtain

$$\begin{aligned}\sum_{i=1}^j\left(\delta\left(g_{2-\beta}*\delta u\right)_i,\delta u_i\right)\tau &+ \sum_{i=1}^j\left(\nabla u_i,\nabla u_i-\nabla u_{i-1}\right)\\ &= \sum_{i=1}^j h_i\left(f,\delta u_i\right)\tau + \sum_{i=1}^j\left(F(t_i,u_{i-1}),\delta u_i\right)\tau\\ &- \sum_{i=1}^j\left(g_i,\delta u_i\right)_\Gamma\tau + \sum_{i=1}^j g_{2-\beta}(t_i)\left(v_0,\delta u_i\right)\tau.\end{aligned} \tag{3.13}$$

To gain the lower bound of the l.h.s., we use Lemma 1.7.3 and Abel's summation

(Theorem 1.3.7) in the following way

$$
\begin{aligned}
&\sum_{i=1}^{j}\left(\delta\left(g_{2-\beta}*\delta u\right)_i, \delta u_i\right)\tau+\sum_{i=1}^{j}\left(\nabla u_i, \nabla u_i-\nabla u_{i-1}\right) \\
&\geq \frac{1}{2}\sum_{i=1}^{j}\delta\left(g_{2-\beta}*\|\delta u\|^2\right)_i\tau+\frac{1}{2}\sum_{i=1}^{j}g_{2-\beta}(t_i)\left\|\delta u_i\right\|^2\tau \\
&+\frac{1}{2}\left\|\nabla u_j\right\|^2-\frac{1}{2}\left\|\nabla u_0\right\|^2+\frac{1}{2}\sum_{i=1}^{j}\left\|\nabla u_i-\nabla u_{i-1}\right\|^2 \\
&\geq \frac{1}{2}\left(g_{2-\beta}*\|\delta u\|^2\right)_j+\frac{1}{4}\sum_{i=1}^{j}g_{2-\beta}(t_i)\left\|\delta u_i\right\|^2\tau \\
&+\frac{g_{2-\beta}(T)}{4}\sum_{i=1}^{j}\left\|\delta u_i\right\|^2\tau+\frac{1}{2}\left\|\nabla u_j\right\|^2-C+\frac{1}{2}\sum_{i=1}^{j}\left\|\nabla u_i-\nabla u_{i-1}\right\|^2 .
\end{aligned}
\tag{3.14}
$$

To estimate the first term of the r.h.s. of (3.13), we use Cauchy's and Young's inequality

$$
\begin{aligned}
\left|\sum_{i=1}^{j}h_i\left(f,\delta u_i\right)\tau\right| &\overset{(3.11)}{\leq} C_\varepsilon\sum_{i=1}^{j}\left|h_i\right|^2\tau+\varepsilon\sum_{i=1}^{j}\left\|\delta u_i\right\|^2\tau \\
&\leq C_\varepsilon+C_\varepsilon\sum_{i=0}^{j-1}\left(\left\|u_i\right\|^2+\left\|\nabla u_i\right\|^2\right)\tau+\varepsilon\sum_{i=1}^{j}\left\|\delta u_i\right\|^2\tau.
\end{aligned}
\tag{3.15}
$$

The second term can be estimated by employing Cauchy's and Young's inequality together with Lipschitz's continuity of $F$

$$
\begin{aligned}
\left|\sum_{i=1}^{j}\left(F(t_i,u_{i-1}),\delta u_i\right)\tau\right| &\leq C\sum_{i=1}^{j}\left(1+\left\|u_{i-1}\right\|\right)\left\|\delta u_i\right\|\tau \\
&\leq C_\varepsilon+C_\varepsilon\sum_{i=1}^{j}\left\|u_{i-1}\right\|^2\tau+\varepsilon\sum_{i=1}^{j}\left\|\delta u_i\right\|^2\tau.
\end{aligned}
\tag{3.16}
$$

We can rewrite third term as follows

$$
\sum_{i=1}^{j}\left(g_i,\delta u_i\right)_\Gamma\tau=-\sum_{i=1}^{j-1}\left(\delta g_i,u_i\right)_\Gamma\tau+\left(g_j,u_j\right)_\Gamma-\left(g_1,u_0\right)_\Gamma,
$$

then we apply the same inequalities as in the previous estimate and the trace theorem to obtain the upper bound of this term

$$
\begin{aligned}
&\left|\sum_{i=1}^{j} (g_i, \delta u_i)_\Gamma \, \tau\right| \\
&\qquad \leq \sum_{i=1}^{j-1} \|\delta g_i\|_\Gamma \, \|u_i\|_\Gamma \, \tau + \|g_j\|_\Gamma \, \|u_j\|_\Gamma + \|g_1\|_\Gamma \, \|u_0\|_\Gamma \\
&\qquad \leq C_\varepsilon + C \sum_{i=1}^{j-1} \|u_i\|_\Gamma^2 \, \tau + \varepsilon \, \|u_j\|_\Gamma^2 \\
&\qquad \leq C_\varepsilon + C \sum_{i=1}^{j-1} (\, \|u_i\|^2 + \|\nabla u_i\|^2 \,)\tau + \varepsilon\Big( \|u_j\|^2 + \|\nabla u_j\|^2 \Big).
\end{aligned}
\tag{3.17}
$$

For the last term in (3.13), we get estimate

$$
\begin{aligned}
\left|\sum_{i=1}^{j} g_{2-\beta}(t_i) \, (v_0, \delta u_i) \, \tau\right| &\leq \sum_{i=1}^{j} g_{2-\beta}(t_i) \, \|v_0\| \, \|\delta u_i\| \, \tau \\
&\leq C_\varepsilon + \varepsilon \sum_{i=1}^{j} g_{2-\beta}(t_i) \, \|\delta u_i\|^2 \, \tau.
\end{aligned}
\tag{3.18}
$$

Putting estimates (3.12)-(3.18) together, using the inequality

$$
\|u_j\|^2 \leq C \left( 1 + \sum_{i=1}^{j} \|\delta u_i\|^2 \, \tau \right)
$$

and choosing sufficiently small $\varepsilon$ give us

$$
\begin{aligned}
&\Big(g_{2-\beta} * \|\delta u\|^2\Big)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i) \, \|\delta u_i\|^2 \, \tau + \sum_{i=1}^{j} \|\delta u_i\|^2 \, \tau + \|u_j\|^2 \\
&\quad + \|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C \left( 1 + \sum_{i=1}^{j-1} (\, \|u_i\|^2 + \|\nabla u_i\|^2 \,)\tau \right).
\end{aligned}
\tag{3.19}
$$

Finally, we use Grönwall's lemma to obtain

$$
\begin{aligned}
&\Big(g_{2-\beta} * \|\delta u\|^2\Big)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i) \, \|\delta u_i\|^2 \, \tau \\
&\qquad + \sum_{i=1}^{j} \|\delta u_i\|^2 \, \tau + \|u_j\|^2 + \|\nabla u_j\|^2 + \frac{1}{2} \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C,
\end{aligned}
\tag{3.20}
$$

and thereby from (3.11), we get that

$$|h_i| \le C.$$

□

In the next lemma, we will need to assume a so called compatibility condition at time $t = 0$ (i.e. the initial condition obeys the boundary conditions and the equation (3.1)), i.e. we assume that

$$(\nabla u_0, \nabla\varphi) = h_0\,(f,\varphi) + (F(0,u_0),\varphi) - (g_0,\varphi)_\Gamma\,, \qquad \forall\varphi \in H^1(\Omega), \tag{3.21}$$

which enables us to define $h_0$ as follows

$$h_0 = \frac{(\nabla u_0, \nabla\omega) + (g_0,\omega)_\Gamma - (F(0,u_0),\omega)}{(f,\omega)}. \tag{3.22}$$

**Lemma 3.3.3.** *Under the assumptions of Lemma 3.3.1, if moreover $v_0 \in H^1(\Omega)$, $m \in C^3([0,T])$, $g \in C^2\left([0,T], L^2(\Gamma)\right)$, and (3.21) holds, then there exist positive constants $C$ (independent of $n$) such that*

*(i)* $$\max_{0\le i\le n}\left(g_{2-\beta} * \left\|\delta^2 u\right\|^2\right)_i + \sum_{i=1}^n g_{2-\beta}(t_i)\left\|\delta^2 u_i\right\|^2\tau + \sum_{i=1}^n \left\|\delta^2 u_i\right\|^2\tau + \max_{0\le i\le n}\|\delta u_i\|^2_{H^1(\Omega)} + \sum_{i=1}^n \|\nabla\delta u_i - \nabla\delta u_{i-1}\|^2 \le C,$$

*(ii)* $|\delta h_i| \le C\left(1 + g_{2-\beta}(t_i)\right).$

*Proof.* Subtracting (3.22) from $h_1$ and dividing by $\tau$, we obtain

$$\delta h_1 = \frac{1}{(f,\omega)}\left(g_{2-\beta}(t_1)m_1'' + (\delta g_1,\omega)_\Gamma - \frac{1}{\tau}\left(F(t_1,u_0) - F(0,u_0),\omega\right)\right),$$

and consequently

$$|\delta h_1| \le C(1 + g_{2-\beta}(t_1)). \tag{3.23}$$

Further, for $i \ge 2$

$$\delta h_i \overset{(2.10)}{=} \frac{1}{(f,\omega)}\Bigg(g_{2-\beta}(t_i)m_0'' + \left(g_{2-\beta} * \delta m''\right)_i + (\nabla\delta u_{i-1}, \nabla\omega) + (\delta g_i,\omega)_\Gamma - \frac{1}{\tau}\left(F(t_i,u_{i-1}) - F(t_{i-1},u_{i-2}),\omega\right)\Bigg),$$

which can be estimated as

$$|\delta h_i| \leq C \left(1 + g_{2-\beta}(t_i) + \|\nabla \delta u_{i-1}\| + \|\delta u_{i-1}\|\right). \tag{3.24}$$

Next, we subtract relations (3.21) and (DP$i$) for $i = 1$ from each other to get

$$\begin{aligned}&\left(\delta \left(g_{2-\beta} * \delta^2 u\right)_1, \varphi\right) \tau + \left(\nabla \delta u_1, \nabla \varphi\right) \tau \\ &\qquad = \delta h_1 \left(f, \varphi\right) \tau + \left(F(t_1, u_0) - F(0, u_0), \varphi\right) - \left(\delta g_1, \varphi\right)_\Gamma \tau,\end{aligned} \tag{3.25}$$

where $\left(g_{2-\beta} * \delta u\right)_0 = 0$, according to the definition. We also take the difference of (DP$i$) for $i \geq 2$ to find

$$\begin{aligned}&\left(\delta \left(g_{2-\beta} * \delta^2 u\right)_i, \varphi\right) \tau + \left(\nabla \delta u_i, \nabla \varphi\right) \tau \\ &= \delta h_i \left(f, \varphi\right) \tau + \left(F(t_i, u_{i-1}) - F(t_{i-1}, u_{i-2}), \varphi\right) - \left(\delta g_i, \varphi\right)_\Gamma \tau.\end{aligned} \tag{3.26}$$

We set $\varphi = \delta^2 u_1$ in (3.25) and $\varphi = \delta^2 u_i$ in (3.26). By summing up (3.25) and (3.26) for $i = 2, \ldots, j$, $j \in \{1, \ldots, n\}$, we obtain that

$$\begin{aligned}&\sum_{i=1}^{j} \left(\delta \left(g_{2-\beta} * \delta^2 u\right)_i, \delta^2 u_i\right) \tau + \sum_{i=1}^{j} \left(\nabla \delta u_i, \nabla \delta^2 u_i\right) \tau \\ &\qquad = \sum_{i=1}^{j} \delta h_i \left(f, \delta^2 u_i\right) \tau + \left(F(t_1, u_0) - F(0, u_0), \delta^2 u_1\right) \\ &\qquad + \sum_{i=2}^{j} \left(F(t_i, u_{i-1}) - F(t_{i-1}, u_{i-2}), \delta^2 u_i\right) - \sum_{i=1}^{j} \left(\delta g_i, \delta^2 u_i\right)_\Gamma \tau.\end{aligned} \tag{3.27}$$

We estimate first term of the l.h.s. in (3.27) using Lemma 1.7.3

$$\begin{aligned}&\sum_{i=1}^{j} \left(\delta \left(g_{2-\beta} * \delta^2 u\right)_i, \delta^2 u_i\right) \tau \\ &\qquad \geq \frac{1}{2} \sum_{i=1}^{j} \delta \left(g_{2-\beta} * \left\|\delta^2 u\right\|^2\right)_i \tau + \frac{1}{2} \sum_{i=1}^{j} g_{2-\beta}(t_i) \left\|\delta^2 u_i\right\|^2 \tau \\ &\qquad \geq \frac{1}{2} \left(g_{2-\beta} * \left\|\delta^2 u\right\|^2\right)_j + \frac{1}{4} \sum_{i=1}^{j} g_{2-\beta}(t_i) \left\|\delta^2 u_i\right\|^2 \tau \\ &\qquad\qquad + \frac{g_{2-\beta}(T)}{4} \sum_{i=1}^{j} \left\|\delta^2 u_i\right\|^2 \tau.\end{aligned} \tag{3.28}$$

The second term of the l.h.s in (3.27) can be rewritten as

$$\sum_{i=1}^{j} (\nabla\delta u_i, \nabla\delta u_i - \nabla\delta u_{i-1})$$
$$= \frac{1}{2}\left(\|\nabla\delta u_j\|^2 - \|\nabla v_0\|^2 + \sum_{i=1}^{j}\|\nabla\delta u_i - \nabla\delta u_{i-1}\|^2\right). \quad (3.29)$$

In the following estimations of terms on the r.h.s. of (3.27), we always use combination of Cauchy's and Young's inequalities. Applying them on the first term together with estimates (3.23), (3.24) gives

$$\begin{aligned}
&\left|\sum_{i=1}^{j} \delta h_i \left(f, \delta^2 u_i\right)\tau\right| \\
&\qquad \le C\sum_{i=1}^{j}(1 + g_{2-\beta}(t_i) + \|\nabla\delta u_{i-1}\| + \|\delta u_{i-1}\|)\left\|\delta^2 u_i\right\|\tau \\
&\qquad \le C_\varepsilon + \varepsilon\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau + \varepsilon\sum_{i=1}^{j} g_{2-\beta}(t_i)\left\|\delta^2 u_i\right\|^2\tau \qquad (3.30)\\
&\qquad\quad + C_\varepsilon\sum_{i=1}^{j}(\,\|\nabla\delta u_{i-1}\|^2 + \|\delta u_{i-1}\|^2\,)\tau.
\end{aligned}$$

To estimate the terms containing $F$, we use the Lipschitz continuity to obtain

$$\begin{aligned}
&\left(F(t_1, u_0) - F(0, u_0), \delta^2 u_i\right) + \sum_{i=2}^{j}\left(F(t_i, u_{i-1}) - F(t_{i-1}, u_{i-2}), \delta^2 u_i\right) \\
&\qquad \le C\sum_{i=1}^{j}(1 + \|\delta u_{i-1}\|)\left\|\delta^2 u_i\right\|\tau \qquad (3.31)\\
&\qquad \le C_\varepsilon + C_\varepsilon\sum_{i=1}^{j}\|\delta u_{i-1}\|^2\tau + \varepsilon\sum_{i=1}^{j}\left\|\delta^2 u_i\right\|^2\tau.
\end{aligned}$$

The last term in (3.27) is rewritten and estimated using the trace theorem in the following way

$$\begin{aligned}
&\left|\sum_{i=1}^{j}\left(\delta g_i, \delta^2 u_i\right)_\Gamma\tau\right| = \left|(\delta g_j, \delta u_j)_\Gamma - (\delta g_1, v_0)_\Gamma - \sum_{i=1}^{j-1}\left(\delta^2 g_{i+1}, \delta u_i\right)_\Gamma\tau\right| \\
&\qquad \le C_\varepsilon + \varepsilon\left(\|\delta u_j\|^2 + \|\nabla\delta u_j\|^2\right) + C\sum_{i=1}^{j-1}\left(\|\delta u_i\|^2 + \|\nabla\delta u_i\|^2\right)\tau. \qquad (3.32)
\end{aligned}$$

Getting all estimates together, using inequality $\|\delta u_j\|^2 \le C\left(1+\sum_{i=1}^j \left\|\delta^2 u_i\right\|^2 \tau\right)$ and choosing a sufficiently small $\varepsilon$, we obtain

$$\left(g_{2-\beta} * \left\|\delta^2 u\right\|^2\right)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \left\|\delta^2 u_i\right\|^2 \tau + \sum_{i=1}^j \left\|\delta^2 u_i\right\|^2 \tau + \|\delta u_j\|^2_{H^1(\Omega)} \\ + \sum_{i=1}^j \|\nabla \delta u_i - \nabla \delta u_{i-1}\|^2 \le C\left(1+\sum_{i=1}^{j-1} \|\delta u_j\|^2_{H^1(\Omega)} \tau\right). \quad (3.33)$$

Finally, employing Grönwall's argument, we get

$$\left(g_{2-\beta} * \left\|\delta^2 u\right\|^2\right)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \left\|\delta^2 u_i\right\|^2 \tau \quad + \sum_{i=1}^j \left\|\delta^2 u_i\right\|^2 \tau + \|\delta u_j\|^2_{H^1(\Omega)} \\ + \sum_{i=1}^j \|\nabla \delta u_i - \nabla \delta u_{i-1}\|^2 \le C,$$

and consequently

$$|\delta h_i| \le C\left(1 + g_{2-\beta}(t_i)\right).$$

$\square$

In the same way as in Section 2.3, we introduce piecewise linear interpolations in time $u_n, \overline{u}_n, \tilde{u}_n : [0,T] \to L^2(\Omega)$. Moreover, we define the linear interpolations for the difference of $u_i$ as $v_n, \overline{v}_n : [0,T] \to L^2(\Omega)$

$$v_n : t \mapsto \begin{cases} v_0, & t = 0 \\ \delta u_{i-1} + (t - t_{i-1})\delta^2 u_i, & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases} \\ \overline{v}_n : t \mapsto \begin{cases} v_0, & t = 0 \\ \delta u_i, & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases} \quad (3.34)$$

also known as Rothe's functions. Analogously, we define $h_n, \overline{h}_n$, $\overline{F}_n$, $\overline{g}_n$, $\overline{g_{2-\beta}}_n$ and $\overline{m''}_n$. The goal is to prove that the corresponding above defined Rothe's functions converge to the solution $(u, h)$. We rewrite (DP$i$) and (DMP$i$) for the whole time frame in terms of Rothe's functions

$$\left(\left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t_i), \varphi\right) + \left(\nabla \overline{u}_n(t), \nabla\varphi\right) \\ = \overline{h}_n(t)\left(f,\varphi\right) + \left(\overline{F}_n(t, \tilde{u}_n(t))\,\mathrm{d}s, \varphi\right) - \left(\overline{g}_n(t), \varphi\right)_\Gamma, \quad \text{(DP)}$$

and

$$\left(\overline{g_{2-\beta}}_n * \overline{m'}_n\right)(t_i) + (\nabla \tilde{u}_n(t), \nabla \omega)$$
$$= \overline{h}_n(t)\,(f,\omega) + \left(\overline{F}_n(t, \tilde{u}_n(t)), \omega\right) - (\overline{g}_n(t), \omega)_\Gamma\,, \quad \text{(DMP)}$$

for $t \in (t_{i-1}, t_i]$.

**Theorem 3.3.1** (existence of a solution)**.** *Let $f \in L^2(\Omega)$, $u_0, v_0, \omega \in H^1(\Omega)$, $\int_\Omega f\omega \neq 0$, $m \in C^3([0,T])$, and $g \in C^2([0,T], L^2(\Gamma))$. Suppose that $F$ is a global Lipschitz continuous function in all variables and (3.21) holds true.*

*Then there exists a solution $(u,h)$ to (P2), (MP2) obeying $u \in C\left([0,T], H^1(\Omega)\right)$ with $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$, $\partial_{tt} u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* From Lemma 3.3.3 estimate $(ii)$, we obtain

$$|h_n'(t)| = |\delta h_i| \leq C t_i^{-\beta} + C \leq C t^{-\beta} + C,$$

for $t \in (t_{i-1}, t_i]$, which leads to

$$|h_n(t+\varepsilon) - h_n(t)| = \left| \int_t^{t+\varepsilon} h_n'(s)\,\mathrm{d}s \right| \leq C \int_t^{t+\varepsilon} \left(s^{1-\beta} + 1\right)\,\mathrm{d}s$$
$$\leq C \frac{(t+\varepsilon)^{2-\beta} - t^{2-\beta}}{2-\beta} + \varepsilon C = \mathcal{O}\left(\varepsilon^{2-\beta}\right).$$

This implies the equi-continuity of the sequence $\{h_n\}$. The estimate $(ii)$ in Lemma 3.3.2 brings the equi-boundedness of $\{h_n\}$. The Arzelà-Ascoli theorem 1.2.5 gives us compactness of $\{h_n\}$ in $C([0,T])$.

From Lemma 3.3.2 and Lemma 3.3.3, we get moreover the following estimate

$$\max_{t\in[0,T]} \|\overline{u_n}(t)\|^2_{H^1(\Omega)} + \max_{t\in[0,T]} \|\partial_t u_n(t)\|^2 \leq C.$$

This together with compact embedding $H^1(\Omega) \Subset L^2(\Omega)$ and Lemma 1.5.1 gives us the existence of a function $u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$ with $\partial_t u \in L^\infty\left((0,T), L^2(\Omega)\right)$ and the subsequences of $\{u_n\}_{n\in\mathbb{N}}$ and $\{\overline{u}_n\}_{n\in\mathbb{N}}$ (for simplicity of notation denoted by the same symbol) such that

$$\begin{cases} u_n \to u, & \text{in} \quad C\left([0,T], L^2(\Omega)\right) & \text{(3.35a)} \\ u_n(t) \rightharpoonup u(t), & \text{in} \quad H^1(\Omega), \quad \forall t \in [0,T] & \text{(3.35b)} \\ \overline{u}_n(t) \rightharpoonup u(t), & \text{in} \quad H^1(\Omega), \quad \forall t \in [0,T] & \text{(3.35c)} \\ \partial_t u_n \rightharpoonup \partial_t u, & \text{in} \quad L^2\left((0,T), L^2(\Omega)\right). & \text{(3.35d)} \end{cases}$$

In addition, the reflexivity of $L^2\left((0,T), H^1(\Omega)\right)$ and Lemma 3.3.3 yield

$$\partial_t u_n \rightharpoonup \partial_t u, \qquad \text{in} \qquad L^2\left((0,T), H^1(\Omega)\right),$$

and then

$$
\begin{aligned}
&u(t) - u(s) = \int_s^t \partial_t u(r)\,\mathrm{d}r \Longrightarrow \\
&\|u(t) - u(s)\|_{H^1(\Omega)} \leq \sqrt{|t-s|}\sqrt{\int_0^T \|\partial_t u(r)\|^2_{H^1(\Omega)}\,\mathrm{d}r} \leq C\sqrt{|t-s|}.
\end{aligned}
$$

With the fact that $u_0 \in H^1(\Omega)$, we get $u \in C\left([0,T], H^1(\Omega)\right)$. Similarly, we get an estimate for sequences $\{v_n\}_{n\in\mathbb{N}}$, $\{\overline{v}_n\}_{n\in\mathbb{N}}$, utilizing Lemma 3.3.3

$$
\max_{t\in[0,T]} \|\overline{v}_n(t)\|^2_{H^1(\Omega)} + \int_0^T \|\partial_t v_n(t)\|^2\,\mathrm{d}t \leq C,
$$

which with the same argument as above gives us existence of $v \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$ with $\partial_t v \in L^2\left((0,T), L^2(\Omega)\right)$ and the subsequences of $\{v_n\}_{n\in\mathbb{N}}$ and $\{\overline{v}_n\}_{n\in\mathbb{N}}$ (again denoted by the same symbol) such that

$$
\begin{cases}
v_n \to v, & \text{in} \quad C\left([0,T], L^2(\Omega)\right) & \text{(3.36a)}\\
v_n(t) \rightharpoonup v(t), & \text{in} \quad H^1(\Omega), \quad \forall t \in [0,T] & \text{(3.36b)}\\
\overline{v}_n(t) \rightharpoonup v(t), & \text{in} \quad H^1(\Omega), \quad \forall t \in [0,T] & \text{(3.36c)}\\
\partial_t v_n \rightharpoonup \partial_t v, & \text{in} \quad L^2\left((0,T), L^2(\Omega)\right). & \text{(3.36d)}
\end{cases}
$$

Since $\overline{v}_n = \partial_t u_n$, the relation between $u$ and $v$ is established after passing to the limit for $n \to \infty$ in the identity

$$
(u_n(t) - u_0, \varphi) = \int_0^t (\overline{v}_n(s), \varphi)\,ds, \qquad \text{for } \varphi \in L^2(\Omega),
$$

and obtaining

$$
(u(t) - u_0, \varphi) = \int_0^t (v(s), \varphi)\,ds, \qquad \text{for } \varphi \in L^2(\Omega).
$$

It is immediately clear that $v(t) = \partial_t u(t)$ a.e. in $[0,T]$.

It remains to prove that the pair $(u,h)$ obeys (MP2), (P2) for every $\varphi \in H^1(\Omega)$. First, it holds that

$$
\int_0^T \|\tilde{u}_n(t) - \overline{u}_n(t)\|^2\,\mathrm{d}t = \mathcal{O}\left(\tau^2\right).
$$

Further,

$$\begin{aligned}
&\left|\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t)\right| \\
&\quad \le \left|\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\overline{m''}_n(s)\,\mathrm{d}s\right| \\
&\qquad + \left|\int_0^t \left(\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right)\overline{m''}_n(s)\,\mathrm{d}s\right| \\
&\quad \le C\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\,\mathrm{d}s + C\int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right|\,\mathrm{d}s.
\end{aligned}$$

As $\overline{g_{2-\beta}}_n \to g_{2-\beta}$ in $(0,T)$ pointwise, the Lebesgue dominated theorem gives

$$\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) \to \left(g_{2-\beta} * m''\right)(t).$$

With this we can pass to the limit in (DMP), for $n \to \infty$, to obtain (MP2).

Next, we deduce for $t \in (t_{i-1}, t_i]$

$$\begin{aligned}
&\left|\int_0^\xi \left(\left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t), \varphi\right)\,\mathrm{d}t\right| \\
&\qquad \le \int_0^\xi \left|\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\left(\partial_t v_n(s), \varphi\right)\,\mathrm{d}s\right| dt \\
&\qquad + \int_0^\xi \left|\int_0^t \left(\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right)\left(\partial_t v_n(s), \varphi\right)\,\mathrm{d}s\right|\,\mathrm{d}t \\
&\qquad \le \int_0^\xi \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\left\|\partial_t v_n(s)\right\|\left\|\varphi\right\|\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad + \int_0^\xi \int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right|\left\|\partial_t v_n(s)\right\|\left\|\varphi\right\|\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{3.37}$$

The first term in (3.37) is estimated using Hölder's inequality and Lemma 3.3.3 as follows

$$\begin{aligned}
&\int_0^\xi \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\left\|\partial_t v_n(s)\right\|\left\|\varphi\right\|\,\mathrm{d}s\,\mathrm{d}t \\
&\le \left\|\varphi\right\|\int_0^\xi \sqrt{\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\,\mathrm{d}s}\sqrt{\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\left\|\partial_t v_n(s)\right\|^2\,\mathrm{d}s}\,\mathrm{d}t \\
&\le \left\|\varphi\right\|\sqrt{\tau^{2-\beta}}\int_0^\xi \sqrt{\int_0^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\left\|\partial_t v_n(s)\right\|^2\,\mathrm{d}s}\,\mathrm{d}t \\
&\le C\left\|\varphi\right\|\sqrt{\tau^{2-\beta}}.
\end{aligned}$$

The upper bound for the second term in (3.37) is obtained by switching the order of integration and using Hölder's inequality

$$\begin{aligned}
&\int_0^\xi \int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \left\|\partial_t v_n(s)\right\| \left\|\varphi\right\| \,\mathrm{d}s\,\mathrm{d}t\\
&\le \left\|\varphi\right\| \int_0^\xi \int_s^\xi \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \left\|\partial_t v_n(s)\right\| \,\mathrm{d}t\,\mathrm{d}s\\
&\le \left\|\varphi\right\| \int_0^\xi \left\|\partial_t v_n(s)\right\| \int_s^\xi \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \,\mathrm{d}t\,\mathrm{d}s\\
&\le \left\|\varphi\right\| \sqrt{\int_0^\xi \left\|\partial_t v_n(s)\right\|^2 \,\mathrm{d}s} \sqrt{\int_0^\xi \left(\int_s^\xi \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \,\mathrm{d}t\right)^2 \,\mathrm{d}s}\\
&\le C \left\|\varphi\right\|.
\end{aligned}$$

Due to the pointwise convergence $\overline{g_{2-\beta}}_n \to g_{2-\beta}$ in $(0,T)$ and the Lebesgue's dominated convergence theorem, we get that

$$\lim_{n\to\infty} \left| \int_0^\xi \left( \left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t), \varphi \right) \,\mathrm{d}t \right| = 0.$$

Next, an application of the Cauchy, Hölder and Young inequalities yields

$$\begin{aligned}
&\left| \int_0^\xi \left( \left[\overline{g_{2-\beta}}_n - g_{2-\beta}\right] * (\partial_t v_n, \varphi) \right)(t) \,\mathrm{d}t \right|\\
&\le \int_0^\xi \left|\overline{g_{2-\beta}}_n(t) - g_{2-\beta}(t)\right| \,\mathrm{d}t \sqrt{\int_0^\xi \left\|\partial_t v_n(t)\right\|^2 \,\mathrm{d}t} \sqrt{\int_0^\xi \left\|\varphi\right\|^2 \,\mathrm{d}t} \le C \left\|\varphi\right\|.
\end{aligned}$$

Again, using Lebesgue's dominated convergence theorem brings us to

$$\lim_{n\to\infty} \left| \int_0^\xi \left( \left[\overline{g_{2-\beta}}_n - g_{2-\beta}\right] * (\partial_t v_n, \varphi) \right)(t) \,\mathrm{d}t \right| = 0.$$

Thanks to

$$\begin{aligned}
&\left| \int_0^\xi \left( g_{2-\beta} * (\partial_t v_n, \varphi) \right)(t) \,\mathrm{d}t \right|\\
&\qquad\le \int_0^\xi g_{2-\beta}(t) \,\mathrm{d}t \sqrt{\int_0^\xi \left\|\partial_t v_n(t)\right\|^2 \,\mathrm{d}t} \sqrt{\int_0^\xi \left\|\varphi\right\|^2 \,\mathrm{d}t}\\
&\qquad\le C \left\|\partial_t v_n\right\|_{L^2((0,T),L^2(\Omega))} \left\|\varphi\right\|,
\end{aligned}$$

we can see the estimated integral as the linear bounded functional on the space $L^2\left((0,T), L^2(\Omega)\right)$, and using (3.36d), we arrive to

$$\lim_{n\to\infty} \int_0^\xi \left( g_{2-\beta} * (\partial_t v_n, \varphi) \right)(t) \,\mathrm{d}t = \int_0^\xi \left( g_{2-\beta} * (\partial_t v, \varphi) \right)(t) \,\mathrm{d}t.$$

Now, integrating (DP) in time over $(0,\xi)$ and passing to the limit $n \to \infty$, we get

$$\begin{aligned}&\int_0^\xi \left(\left(g_{2-\beta} * \partial_{tt} u\right)(t), \varphi\right)\,\mathrm{d}t + \int_0^\xi \left(\nabla u(t), \nabla\varphi\right)\,\mathrm{d}t\\ &= \int_0^\xi h(t)\,(f,\varphi)\,\mathrm{d}t + \int_0^\xi \left(F(t,u(t)),\varphi\right)\,\mathrm{d}t - \int_0^\xi \left(g(t),\varphi\right)_\Gamma\,\mathrm{d}t,\end{aligned} \tag{3.38}$$

using above estimates, convergences and relations. Differentiation of (3.38) with respect to $\xi$ yields (P2), which concludes the proof.

□

In the proof in Theorem 3.3.1, we proved that the subsequence of the Rothe functions converges to the solution of the problem. Theorem 3.2.1 implies that the whole sequence converges to the solution. Moreover, it can be seen from the proof that $u(\cdot)$ is also Lipschitz continuous with respect to the norm in $H^1(\Omega)$.

## 3.4 Numerical Experiments

### 3.4.1 Exact data

We present two numerical experiments based on the algorithm presented above. Experiments differ in the function $\omega$. While in the first experiment, we set $\omega = 1$, in the second one we choose $\omega$ to be a function with compact support in $\Omega$. We consider 1D model with the domain $\Omega = (1.6; 4.5)$, $T = 3$ and $\beta = 1.3$. Further

$$\begin{aligned} f(x) &= \cos x,\\ F(x,t,u) &= -4tu\exp\left(1 - \frac{u^2}{\cos^2 x}\right),\end{aligned}$$

and we set initial and boundary conditions

$$\begin{aligned} u_0(x) &= 2\cos x,\\ v_0(x) &= 0,\\ g(1.6,t) &= (t^3+2)\sin 1.6,\\ g(4.5,t) &= -(t^3+2)\sin 4.5.\end{aligned}$$

As mentioned in the beginning of section, firstly, we consider the additional measurement in form

$$m(t) = \int_\Omega u(x,t)\omega_1(x)\,\mathrm{d}x = (\sin 4.5 - \sin 1.6)\left(t^3+2\right),$$

where $\omega_1(x) = 1$. In the second case, we use

$$m(t) = \int_\Omega u(x,t)\omega_2(x)\,\mathrm{d}x = 2\left(-\cos 4 - \cos 2 + \sin 4 - \sin 2\right)\left(t^3+2\right),$$

where

$$\omega_2(x) = \begin{cases} 1-(x-3)^2, & |x-3| \le 1, \\ 0, & |x-3| \ge 1. \end{cases}$$

In both cases, it is easy to verify that functions

$$u(x,t) = \left(t^3+2\right)\cos x$$

and

$$h(t) = \frac{6}{(2-\beta)(3-\beta)\Gamma(2-\beta)} t^{3-\beta} + t^3 - t(t^3+2)\exp\left(1-(t^3+2)^2\right) + 2$$

solve the given problem.

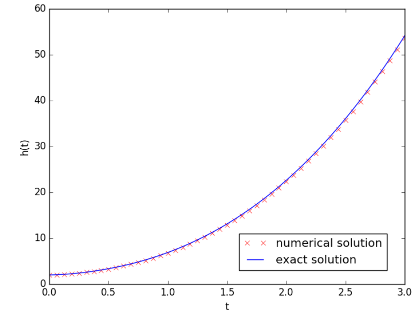
We use Python and the FEniCS Project [73] for the implementation of algorithm. The domain $\omega$ is divided in to 50 sub-intervals, and solution $u_i$ is found by using the Lagrange basis functions of order 2. We calculated numerical solution for several values of $\tau$.

Starting with $\omega_1$, on the Fig. 3.1 (a), (b) we can see the exact solution and numerical approximation of $h$ and $u(T)$, respectively. Relative errors of $h$ and $u$ developing in time are showed of Fig. 3.1 (c), (d), respectively. Decay of relative errors for decreasing $\tau$ is depicted on Fig. 3.1 (e), (f). Fig. 3.2 shows the same for $\omega_2$. The linear regression lines plotted through data points in Fig. 3.1 are given by $0.2665\log_2\tau - 4.1607$ for the error of $h$ and $1.0259\log_\tau + 0.3968$ for the error of $u$. In Fig. 3.2, the lines are given by $0.6012\log_2\tau - 1.5274$ for the error of $h$ and $1.0158\log\tau - 0.2128$ for the error of $u$.

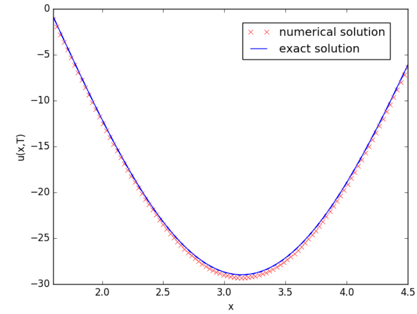
### 3.4.2 Noisy data

The proposed algorithm make use of a second derivative of the measurement, which seems to be a major limitation. We use the nonlinear least square method on the noisy data to obtain a function in a specific shape which is smooth enough to use in the algorithm.

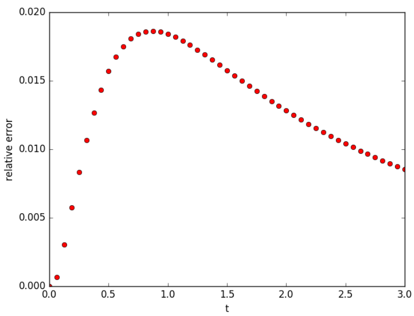
Again as in Chapter 3, we model noisy measurement in our experiment by adding the scaled Gaussian distributed noise, with the mean and standard deviation equal to 0 and 1, respectively, to the exact measurement $m$, so that the noisy measurement takes the form (2.23), where scaling $\epsilon$ will take values $0.05, 0.1, 0.15$.



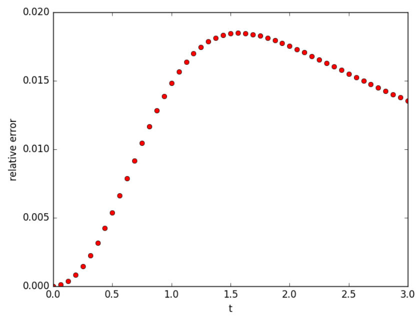

(a) Reconstruction of $h$ together with exact $h$.



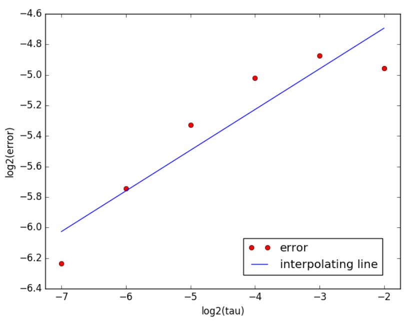

(b) Reconstruction of $u(T)$ together with exact $u(T)$.



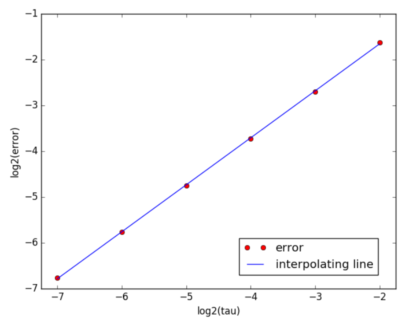

(c) Relative error $\frac{|h_{approx}(t) - h_{exact}(t)|}{|h_{exact}(t)|}$.




(d) Relative error $\frac{||u_{approx}(t) - u_{exact}(t)||}{||u_{exact}(t)||}$.

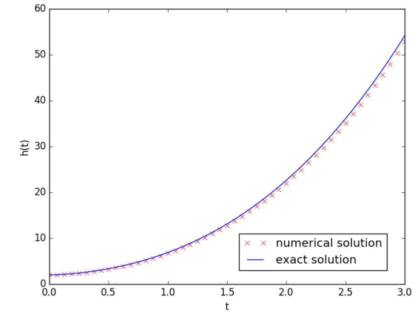



(e) Logarithm of maximal relative error in time of $h$ for different values of $\tau$. Slope of the line is 0.266.

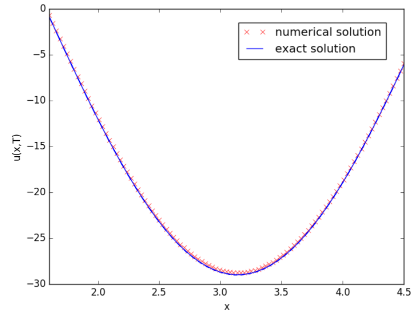



(f) Logarithm of maximal relative error in time of $u$ for different values of $\tau$. Slope of the line is 1.0259.

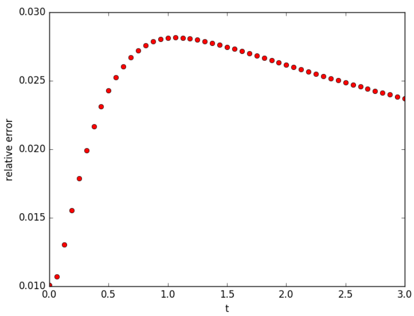
Figure 3.1: The results of the reconstruction algorithm for $\tau = 0.015625$ and $\omega_1$



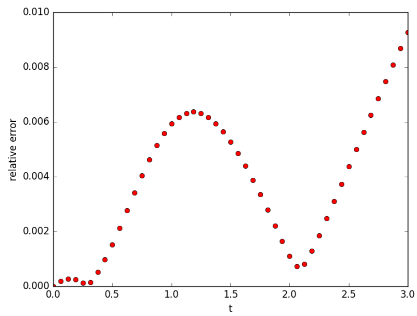

(a) Reconstruction of $h$ together with exact $h$.



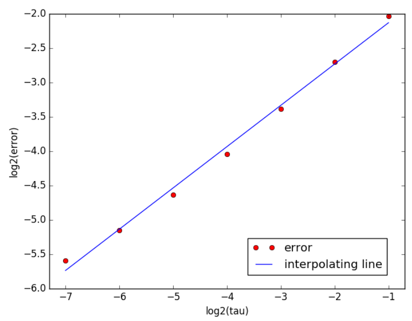

(b) Reconstruction of $u(T)$ together with exact $u(T)$.



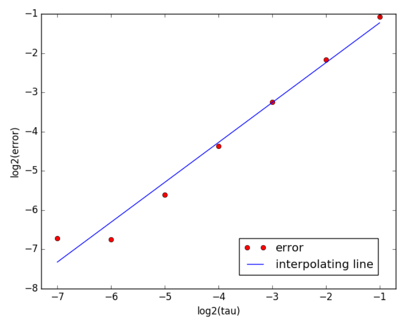

(c) Relative error $\frac{|h_{approx}(t)-h_{exact}(t)|}{|h_{exact}(t)|}$.




(d) Relative error $\frac{||u_{approx}(t)-u_{exact}(t)||}{||u_{exact}(t)||}$.




(e) Logarithm of maximal relative error in time of $h$ for different values of $\tau$. Slope of the line is 0.6012.

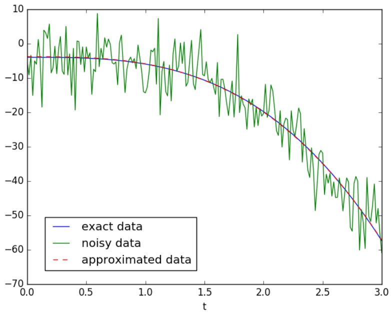



(f) Logarithm of maximal relative error in time of $u$ for different values of $\tau$. Slope of the line is 1.0158.

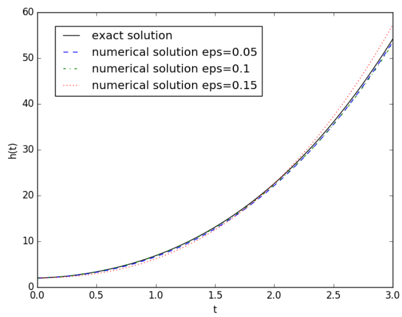
Figure 3.2: The results of the reconstruction algorithm for $\tau = 0.015625$ and $\omega_2$

As mention above our ‘smoothing’ consists of using the nonlinear least square method on $m_\epsilon$ in order to find function in the form (2.24), and then we use it in the algorithm instead of exact measurement $m$.

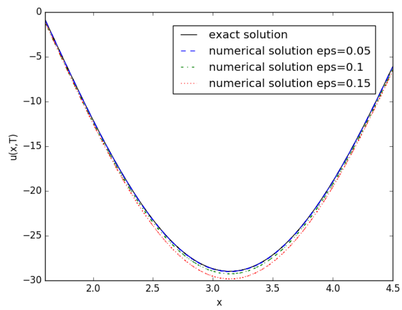
In the experiment, we use the same setting as in Section 3.4.1 with function $\omega_1$ and corresponding measurement. Results can be seen on Fig. 3.3. The exact measurement together with noisy data and approximation of noisy data is shown on Fig. 3.3 (a). Comparison of exact and approximated solution can be seen on Fig. 3.3 (b), (c), and on Fig. 3.3 (d), (e) we see the relative error of that solution.



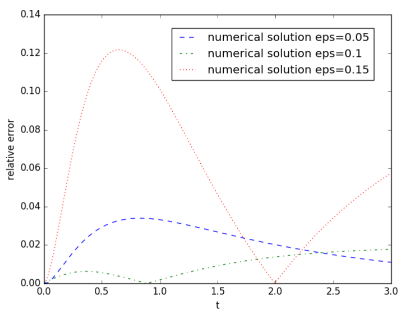

(a) Exact and noisy data for $\epsilon = 0.1$. Approximating curve has the form $m_{app}(t) = -2.0801t^{2.9542} - 3.7789$.



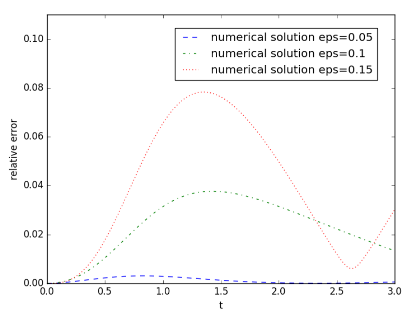

(b) Reconstruction of $h$ together with exact $h$.




(c) Reconstruction of $u(T)$ together with exact $u(T)$.




(d) Relative error $\frac{|h_{approx}(t) - h_{exact}(t)|}{|h_{exact}(t)|}$.




(e) Relative error $\frac{||u_{approx}(t) - u_{exact}(t)||}{||u_{exact}(t)||}$.

Figure 3.3: The result of reconstruction of $h$ and $u$ for noisy data with a various amount of noise $\epsilon$ and $\tau = 0.015625$

# Chapter 4

# A source identification problem in a time-fractional wave equation with a dynamical boundary condition

This chapter is based on the article [131], which was published in the journal *Computers and Mathematics with Applications*.

## 4.1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be bounded with the Lipschitz boundary $\Gamma$ and $T > 0$, we study the equation

$$\left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \Delta u(x,t) = h(t) f(x), \qquad x \in \Omega,\ t \in (0,T), \tag{4.1}$$

The equation (4.1) is accompanied with the following initial and boundary conditions

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & x \in \Omega, \\ \partial_t u(x,0) & = & v_0(x), & x \in \Omega, \\ u(x,t) & = & 0, & (x,t) \in \Gamma_D \times (0,T), \\ -\left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \nabla u(x,t) \cdot \boldsymbol{\nu} & = & \sigma(x,t), & (x,t) \in \Gamma_N \times (0,T), \end{array} \tag{4.2}$$

where we assume $\Gamma_D \cap \Gamma_N = \emptyset$, $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma$, $|\Gamma_D| > 0$, and $\boldsymbol{\nu}$ is a outer normal vector on $\Gamma$. The boundary condition we consider on the part of the boundary $\Gamma_N$ is called the dynamical boundary condition.

The inverse source problem (ISP) we are interested in here consists of finding the couple $(u, h)$. It is necessary to possess an additional measurement to accomplish this in the following form

$$\int_\Omega u(x,t)\omega(x)\,\mathrm{d}x = m(t), \qquad t \in [0,T], \tag{4.3}$$

where the function $\omega$ is solely space dependent. Frequently, it is assumed, $\omega$ is with compact support in $\Omega$, then the measurement (4.3) can be interpreted as the weighted average over the sub-domain of $\Omega$ [99].

The equation (4.1) without the source term is studied in [113], where authors derived explicit expression for solution through the corresponding Green's functions in terms of Fox functions and provided probabilistic interpretation of the equation in one-dimensional case. The equation without the source term is also studied in [85, 87].

The hyperbolic equation accompanied with the dynamical boundary condition for 1D space can model a viscoelastic rod with a mass attached to its free tip, see [16]. According [40] such a boundary condition can also occur in modeling a flexible membrane with boundary affected by vibration only in a region. In [37] the dynamical boundary condition is derived including the influence of the heavy frame in the modeling of small vertical oscillation of flexible membrane. The direct problem for the fractional diffusion equation with the dynamical boundary condition was studied in [66]. The dynamical boundary condition in (4.2) containing the fractional derivative is a generalization of the dynamical boundary condition containing the classical derivative as in [128]. The boundary condition with a convolution term containing the solution can be found in [74].

The chapter is organized as follows. In the second section, we introduce some notation used in the chapter and state the variational formulation of our problem. We reformulate our problem into a direct one by applying the measurement on the equation (4.1) and gaining the second equation for the couple $(u, h)$. In the third section, the uniqueness of the inverse problem is addressed in the appropriate spaces. In the fourth section, the time discretization is introduced, the existence of the solutions along each of the slices is shown, and the a priori estimates are proven. We then define the Rothe functions and state the existence theorem in which we prove the convergence of those functions to the solution of our problem. The error estimate is presented in the fifth section. In the last part, we present a couple of numerical experiments. The solution is calculated for various values of time step and different measurement functions. We also present calculation with a possible treatment of the noisy data.

## 4.2 Reformulation of problem

Next we introduce the functional space

$$\boldsymbol{V} = \left\{\varphi : \Omega \to \mathbb{R}, \varphi = 0 \text{ on } \Gamma_D, \|\varphi\| + \|\nabla\varphi\| + \|\Delta\varphi\| + \|\nabla\varphi\cdot\boldsymbol{\nu}\|_{\Gamma_N} < \infty\right\}.$$

The space $\boldsymbol{V}$ furnished with the norm $\|\cdot\|_{\boldsymbol{V}}$ that is induced by the scalar product $(\phi,\psi)_{\boldsymbol{V}} = (\phi,\psi) + (\nabla\phi,\nabla\psi) + (\Delta\phi,\Delta\psi) + (\nabla\phi\cdot\boldsymbol{\nu},\nabla\psi\cdot\boldsymbol{\nu})_{\Gamma_N}$ is a Hilbert space.

We proceed to the reformulation of the problem. Multiplying (4.1) by $\omega$ and integrating over the domain $\Omega$, we get

$$\left(g_{2-\beta} * m''\right)(t) - (\Delta u(t),\omega) = h(t)\,(f,\omega)\,, \tag{MP}$$

which is called the measured equation or measured problem. If $(f,\omega)\neq 0$, we may eliminate the time-dependent source and get

$$h(t) = \frac{\left(g_{2-\beta} * m''\right)(t) - (\Delta u(t),\omega)}{(f,\omega)}. \tag{4.4}$$

Similarly, we multiply (4.1) by $-\Delta\varphi$, where $\varphi\in\boldsymbol{V}$, and use the Green theorem to obtain

$$\begin{aligned}\left(\left(g_{2-\beta} * \partial_{tt}\nabla u\right)(t),\nabla\varphi\right) - \left(\left(g_{2-\beta} * \partial_{tt} u\right)(t),\nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N} + (\Delta u(t),\Delta\varphi)&\\ = h(t)\left((\nabla f,\nabla\varphi) - (f,\nabla\varphi\cdot\boldsymbol{\nu})_{\Gamma}\right).&\end{aligned} \tag{4.5}$$

Since $\varphi\in V$, we cannot say anything about $\nabla\varphi\cdot\boldsymbol{\nu}$ on $\Gamma_D$; therefore, the second term on the right hand side might not be properly defined. Hence, we assume $f = 0$ on $\Gamma$, which can be interpreted as the restriction of the source location on a interior part of the domain. Using this and the boundary condition on $\Gamma_N$, we gain the variational formulation for the strong solution

$$\begin{aligned}\left(\left(g_{2-\beta} * \partial_{tt}\nabla u\right)(t),\nabla\varphi\right) + (\nabla u(t)\cdot\boldsymbol{\nu},\nabla\varphi\cdot\boldsymbol{\nu})_{\Gamma_N} + (\Delta u(t),\Delta\varphi)&\\ = h(t)\,(\nabla f,\nabla\varphi) - (\sigma(t),\nabla\varphi\cdot\boldsymbol{\nu})_{\Gamma_N}.&\end{aligned} \tag{P}$$

We are looking for the couple $(u,h)$ solving the coupled relations (P) and (MP) for any $\varphi\in\boldsymbol{V}$, a.a. $t\in[0,T]$ and $u(0)=u_0$, $\partial_t u(0) = v_0$.

## 4.3 Uniqueness

With this we may proceed to the uniqueness theorem.

**Theorem 4.3.1** (uniqueness). *Let $|\Gamma_D| > 0, f \in H_0^1(\Omega), u_0 \in \boldsymbol{V}, v_0, \omega \in L^2(\Omega)$, $(f,\omega) \neq 0$, $m \in C^2([0,T])$, $F$ be a global Lipschitz continuous function in all variables and $\sigma \in C([0,T], L^2(\Gamma_N))$. Then there exists at most one solution $(u,h)$ to the (P), (MP) obeying $u \in C([0,T], \boldsymbol{V})$, $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), \boldsymbol{V}\right)$ with $\partial_{tt} u \in L^2\left((0,T), H^1(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* Let assume that $(u_1,h_1), (u_2,h_2)$ are two solution of the (P), (MP), such that they obey the presumptions from the theorem. Define $u = u_1 - u_2$ and $h = h_1 - h_2$, which are then a solution of the slightly different problem

$$-\left(\Delta u(t), \omega\right) = h(t)\left(f, \omega\right), \tag{4.6}$$

and

$$\begin{aligned}\left(\left(g_{2-\beta} * \partial_{tt} \nabla u\right)(t), \nabla \varphi\right) + \left(\nabla u(t) \cdot \boldsymbol{\nu}, \nabla \varphi \cdot \boldsymbol{\nu}\right)_{\Gamma_N} + \left(\Delta u(t), \Delta \varphi\right)& \\ = h(t)\left(\nabla f, \nabla \varphi\right),&\end{aligned} \tag{4.7}$$

for every $\varphi \in \boldsymbol{V}$, a.a. $t \in [0,T]$ and $u(0) = 0$, $\partial_t u(0) = 0$. Since $(f,\omega) \neq 0$, we may eliminate $h$ from (4.6) and substitute to (4.7). Next, we set $\varphi = \partial_t u(t)$ and integrate over $(0,\xi)$, for $\xi \in (0,T]$, to obtain

$$\begin{aligned}\int_0^\xi \left(\partial_t\left(g_{2-\beta} * \partial_t \nabla u\right)(t), \partial_t \nabla u(t)\right) \mathrm{d}t + \frac{1}{2}\left\|\nabla u(\xi) \cdot \boldsymbol{\nu}\right\|_{\Gamma_N}^2 \mathrm{d}t + \frac{1}{2}\left\|\Delta u(\xi)\right\|^2& \\ = \int_0^\xi -\frac{\left(\Delta u(t), \omega\right)}{(f,\omega)}\left(\nabla f, \partial_t \nabla u(t)\right) \mathrm{d}t,&\end{aligned} \tag{4.8}$$

where for the first term on the l.h.s we used the relationship

$$\left(g_{2-\beta} * \partial_{tt} u\right)(t) = \partial_t\left(g_{2-\beta} * \partial_t u\right)(t),$$

as $\partial_t u(0) = 0$. The l.h.s. of (4.8) can be estimated using Lemma 1.7.2, and r.h.s. of (4.8) is estimated using the Cauchy and Young inequalities so that we gain

$$\begin{aligned}\frac{g_{2-\beta}(T)}{2}\int_0^\xi \left\|\partial_t \nabla u(t)\right\|^2 \mathrm{d}t + \frac{1}{2}\left\|\nabla u(\xi) \cdot \boldsymbol{\nu}\right\|_{\Gamma_N}^2 + \frac{1}{2}\left\|\Delta u(\xi)\right\|^2& \\ \leq \varepsilon \int_0^\xi \left\|\partial_t \nabla u(t)\right\|^2 \mathrm{d}t + C_\varepsilon \int_0^\xi \left\|\Delta u(t)\right\|^2 \mathrm{d}t.&\end{aligned} \tag{4.9}$$

Choosing an appropriate $\varepsilon > 0$, we get that

$$\int_0^\xi \|\partial_t \nabla u(t)\|^2 \, \mathrm{d}t + \|\nabla u(\xi)\cdot\boldsymbol{\nu}\|^2_{\Gamma_N} + \|\Delta u(\xi)\|^2 \le C\int_0^\xi \|\Delta u(t)\|^2 \, \mathrm{d}t.$$

Due to the estimate $\|\nabla u(\xi)\|^2 \le \int_0^\xi \|\partial_t \nabla u(t)\|^2 \, \mathrm{d}t$, we obtain that

$$\|\nabla u(\xi)\|^2 + \|\nabla u(\xi)\cdot\boldsymbol{\nu}\|^2_{\Gamma_N} + \|\Delta u(\xi)\|^2 \le C\int_0^\xi \|\Delta u(t)\|^2 \, \mathrm{d}t.$$

Finally, we apply the Grönwall lemma and get

$$\|\nabla u(\xi)\|^2 + \|\nabla u(\xi)\cdot\boldsymbol{\nu}\|^2_{\Gamma_N} + \|\Delta u(\xi)\|^2 \le 0, \tag{4.10}$$

which holds for any $\xi \in [0,T]$. Since $|\Gamma_D| > 0$, we may use the Friedrichs inequality, Theorem 1.3.9, $\|u(t)\| \le C\|\nabla u(t)\|$, which implies that $u = 0$ a.e. in $\Omega\times[0,T]$. Furthermore, we can conclude from (4.6) that $h = 0$ a.e. in $[0,T]$. $\square$

## 4.4 Existence

Let divide the interval $[0,T]$ by the step $\tau = \frac{T}{n}$, for any $n \in \mathbb{N}$. We introduce $t_i = i\tau$, for $i = 1,...,n$, and for any function $z$ we define $z_i$, $\delta z_i$, $\delta^2 z_i$ as in the previous chapters. Using this notation, we may define the approximate solution along the time slice $(u_i, h_i)$ as the solution of discretized equation of (P) and (MP). We get a system of equations

$$\begin{aligned}\left(\left(g_{2-\beta} * \delta^2 \nabla u\right)_i, \nabla\varphi\right) + (\nabla u_i\cdot\boldsymbol{\nu}, \nabla\varphi\cdot\boldsymbol{\nu})_{\Gamma_N} + (\Delta u_i, \Delta\varphi)&\\ = h_i(\nabla f, \nabla\varphi) - (\sigma_i, \nabla\varphi\cdot\boldsymbol{\nu})_{\Gamma_N}, &\end{aligned} \tag{DPi}$$

for all $\varphi \in \boldsymbol{V}$, with $u(0) = u_0$, $\partial_t u(0) = v_0$, and

$$\left(g_{2-\beta} * m''\right)_i - (\Delta u_{i-1}, \omega) = h_i(f, \omega), \tag{DMPi}$$

for $i \in \{1,...,n\}$ with $n \in \mathbb{N}$.

The next lemma deals with the existence of the solution of the above coupled equations for every $i$.

**Lemma 4.4.1.** *Let $u_0 \in \boldsymbol{V}$, $v_0 \in H^1(\Omega)$, $f \in H^1_0(\Omega)$, $\omega \in L^2(\Omega)$, $(f,\omega) \neq 0$, $m \in C^2([0,T])$, $\sigma \in C([0,T], L^2(\Gamma_N))$. Then for each $i \in \{1,\ldots,n\}$ there exists a unique couple $(u_i, h_i) \in \boldsymbol{V}\times\mathbb{R}$ solving (DPi) and (DMPi) for every $\varphi \in \boldsymbol{V}$.*

*Proof.* Since we assume $(f,\omega)\neq 0$, we can eliminate $h_i$ from (DMPi) to obtain

$$h_i = \frac{\left(g_{2-\beta} * m''\right)_i - (\Delta u_{i-1},\omega)}{(f,\omega)}.$$

When $u_{i-1}\in \boldsymbol{V}$, then $h_i\in\mathbb{R}$. We can rewrite (DPi) for $i=1$ into

$$\begin{aligned}
&\frac{1}{\tau} g_{2-\beta}(\tau)\left(\nabla u_1, \nabla\varphi\right) + \left(\nabla u_1\cdot\boldsymbol{\nu}, \nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N} + \left(\Delta u_1,\Delta\varphi\right)\\
&= h_1\left(\nabla f,\nabla\varphi\right) - \left(\sigma_1,\nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N} + \frac{1}{\tau} g_{2-\beta}(\tau)\left(\nabla u_0,\nabla\varphi\right) + g_{2-\beta}(\tau)\left(\nabla v_0,\nabla\varphi\right).
\end{aligned} \tag{4.11}$$

The l.h.s. of this equation can be understood as a bounded bilinear form on $\boldsymbol{V}$ and it also holds

$$\frac{1}{\tau} g_{2-\beta}(\tau)\left(\nabla u_1,\nabla u_1\right) + \left(\nabla u_1\cdot\boldsymbol{\nu},\nabla u_1\cdot\boldsymbol{\nu}\right)_{\Gamma_N} + \left(\Delta u_1,\Delta u_1\right) \geq C(\tau)\left\|u_1\right\|^2_{\boldsymbol{V}}.$$

Moreover, the r.h.s. can be seen as the linear bounded functional on $\boldsymbol{V}$. Therefore, according the Lax-Milgram theorem 1.2.4, there exist a unique $u_1\in\boldsymbol{V}$ solving (4.11). The similar as above can be done for the rest of $i\in\{1,...,n\}$.

□

**Lemma 4.4.2.** *Under the assumptions of Lemma 4.4.1, if moreover it holds that $\sigma\in C^1([0,T],L^2(\Gamma_N))$, then there exists a positive constant $C$ (independent of $n$) such that*

$$\begin{aligned}
(i)\quad & \max_{0\le i\le n}\left(g_{2-\beta} * \left\|\nabla\delta u\right\|^2\right)_i + \sum_{i=1}^n g_{2-\beta}(t_i)\left\|\nabla\delta u_i\right\|^2\tau + \sum_{i=1}^n \left\|\delta u_i\right\|^2_{H^1(\Omega)}\tau\\
&+ \max_{0\le i\le n}\left\|\nabla u_i\cdot\boldsymbol{\nu}\right\|^2_{\Gamma_N} + \max_{0\le i\le n}\left\|\Delta u_i\right\|^2 + \sum_{i=1}^n\left\|\Delta u_i - \Delta u_{i-1}\right\|^2\\
&+ \sum_{i=1}^n \left\|\nabla u_i\cdot\boldsymbol{\nu} - \nabla u_{i-1}\cdot\boldsymbol{\nu}\right\|^2 \le C,\\
(ii)\quad & \max_{0\le i\le n}|h_i| \le C.
\end{aligned} \tag{4.12}$$

*Proof.* We set $\varphi=\delta u_i\tau$ in (DPi) and sum it up for $1\le i\le j$, $j\in\{1,...,n\}$, to get

$$\sum_{i=1}^j\left(\left(g_{2-\beta} * \delta^2\nabla u\right)_i,\nabla\delta u_i\right)\tau + \sum_{i=1}^j\left(\nabla u_i\cdot\boldsymbol{\nu},\nabla\delta u_i\cdot\boldsymbol{\nu}\right)_{\Gamma_N}\tau + \sum_{i=1}^j\left(\Delta u_i,\Delta\delta u_i\right)\tau$$

$$= \sum_{i=1}^{j} h_i \left(\nabla f, \nabla \delta u_i\right) \tau - \sum_{i=1}^{j} \left(\sigma_i, \nabla \delta u_i \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau. \quad (4.13)$$

For the first term on the l.h.s. in (4.13), we use (2.10) to rewrite it and Lemma 1.7.3 to estimate the lower bound in the following manner

$$\begin{aligned}
&\sum_{i=1}^{j} \left(\left(g_{2-\beta} * \delta^2 \nabla u\right)_i, \nabla \delta u_i\right) \tau \\
&= \sum_{i=1}^{j} g_{2-\beta}(t_i) \left(\nabla \delta u_0, \nabla \delta u_i\right) \tau + \sum_{i=1}^{j} \left(\delta \left(g_{2-\beta} * \delta \nabla u\right)_i, \nabla \delta u_i\right) \tau \\
&\geq \sum_{i=1}^{j} g_{2-\beta}(t_i) \left(\nabla \delta u_0, \nabla \delta u_i\right) \tau \\
&\quad + \frac{1}{2} \sum_{i=1}^{j} \delta \left(g_{2-\beta} * \|\nabla \delta u\|^2\right)_i \tau + \frac{1}{2} \sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau \\
&\geq \sum_{i=1}^{j} g_{2-\beta}(t_i) \left(\nabla \delta u_0, \nabla \delta u_i\right) \tau + \frac{1}{2} \left(g_{2-\beta} * \|\nabla \delta u\|^2\right)_j \\
&\qquad + \frac{1}{4} \sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \frac{g_{2-\beta}(T)}{4} \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau. \quad (4.14)
\end{aligned}$$

On the second and third term on the l.h.s. in (4.13), we apply Theorem 1.3.7. The Cauchy and Young inequalities are used on the first term on the r.h.s in (4.14) and on the first term on the r.h.s in (4.14), coming from the use of (2.10); they are also used on the second term on the r.h.s in (4.13), after rewriting it as

$$\begin{aligned}
-\sum_{i=1}^{j} \left(\sigma_i, \nabla \delta u_i \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau = \sum_{i=1}^{j-1} \left(\delta \sigma_{i+1}, \nabla u_i \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau - \left(\sigma_j, \nabla u_j \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \\
+ \left(\sigma_1, \nabla u_0 \cdot \boldsymbol{\nu}\right)_{\Gamma_N}.
\end{aligned}$$

By realizing the above steps, we acquire

$$\begin{aligned}
&\frac{1}{2} \left(g_{2-\beta} * \|\nabla \delta u\|^2\right)_j + \frac{1}{4} \sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \frac{g_{2-\beta}(T)}{4} \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau \\
&+ \frac{1}{2} \|\nabla u_j \cdot \boldsymbol{\nu}\|_{\Gamma_N}^2 + \frac{1}{2} \|\Delta u_j\|^2 + \frac{1}{2} \sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 + \frac{1}{2} \sum_{i=1}^{j} \|\nabla u_i \cdot \boldsymbol{\nu} - \nabla u_{i-1} \cdot \boldsymbol{\nu}\|^2
\end{aligned}$$

$$\leq C\Bigg(1 + C_\varepsilon \sum_{i=1}^{j} |h_i|^2 \tau + C_\varepsilon \sum_{i=1}^{j-1} \|\nabla u_i \cdot \boldsymbol{\nu}\|_{\Gamma_N}^2 \tau + \varepsilon \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau + \varepsilon \|\nabla u_j \cdot \boldsymbol{\nu}\|_{\Gamma_N}^2 + \varepsilon \sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau\Bigg).$$

Next, we may estimate $h_i$ from (DMPi) by

$$|h_i| \leq C\left(1 + \|\Delta u_{i-1}\|\right),$$

then if we choose the appropriate $\varepsilon > 0$, we are prepared to use the discrete Grönwall lemma and obtain

$$\frac{1}{2}\left(g_{2-\beta} * \|\nabla \delta u\|^2\right)_j + \frac{1}{4}\sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \frac{g_{2-\beta}(T)}{4}\sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau + \frac{1}{2}\|\nabla u_i \cdot \boldsymbol{\nu}\|_{\Gamma_N}^2 + \frac{1}{2}\|\Delta u_j\|^2 + \frac{1}{2}\sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 + \frac{1}{2}\sum_{i=1}^{j} \|\nabla u_i \cdot \boldsymbol{\nu} - \nabla u_{i-1} \cdot \boldsymbol{\nu}\|^2 \leq C.$$

Since we have $\Gamma_D = 0$ and $|\Gamma_D| > 0$, we can use the Friedrich inequality to estimate

$$\sum_{i=1}^{j} \|\delta u_i\|^2 \tau \leq C \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau.$$

With this we arrive to the estimate $(i)$ from the lemma and consequently also to $(ii)$. □

For the next lemma, we need to additionally define $h_0$ from (MP) as

$$h_0 = -\frac{(\Delta u_0, \omega)}{(f, \omega)},$$

and assume that the following compatibility condition holds

$$(\nabla u_0 \cdot \boldsymbol{\nu}, \nabla \varphi \cdot \boldsymbol{\nu})_{\Gamma_N} + (\Delta u_0, \Delta \varphi) = h_0 (\nabla f, \nabla \varphi) - (\sigma_0, \nabla \varphi \cdot \boldsymbol{\nu})_{\Gamma_N}, \tag{4.15}$$

for every $\varphi \in \boldsymbol{V}$.

**Lemma 4.4.3.** *Under the assumptions of Lemma 4.4.1, if moreover* $v_0 \in \boldsymbol{V}$, $m \in C^3([0,T])$, $\sigma \in C^2([0,T], L^2(\Gamma_N))$ *and (4.15) holds, then there exists a positive constant* $C$ *(independent of* $n$*) such that*

$$
\begin{aligned}
(i) \quad & \max_{0\le i\le n} \left(g_{2-\beta} * \left\|\nabla \delta^2 u\right\|^2\right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \left\|\nabla \delta^2 u_i\right\|^2 \tau + \sum_{i=1}^n \left\|\delta^2 u_i\right\|^2_{H^1(\Omega)} \tau \\
& + \max_{0\le i\le n} \left\|\nabla \delta u_i \cdot \boldsymbol{\nu}\right\|^2_{\Gamma_N} + \max_{0\le i\le n} \left\|\Delta \delta u_i\right\|^2 + \sum_{i=1}^n \left\|\Delta \delta u_i - \Delta \delta u_{i-1}\right\|^2 \\
& + \sum_{i=1}^n \left\|\nabla \delta u_i \cdot \boldsymbol{\nu} - \nabla \delta u_{i-1} \cdot \boldsymbol{\nu}\right\|^2 \le C, \\
(ii) \quad & |\delta h_i| \le C\left(1 + g_{2-\beta}(t_i)\right).
\end{aligned}
\tag{4.16}
$$

*Proof.* First we estimate the difference of the $h_i$, for $i = 1$, we get

$$
|\delta h_1| \le \left|\frac{g_{2-\beta}(t_1) m_1'' \tau}{(f,\omega)}\right| \le C g_{2-\beta}(t_1),
$$

and for $i \ge 2$, we see that

$$
\begin{aligned}
|\delta h_i| \le C \left|\frac{g_{2-\beta}(t_i) m_0'' + \left(g_{2-\beta} * \delta m''\right)_i - \left(\delta \Delta u_{i-1}, \omega\right)}{(f,\omega)}\right| & \\
\le C\left(1 + g_{2-\beta}(t_i) + \left\|\Delta \delta u_{i-1}\right\|\right).&
\end{aligned}
$$

Next, we make the difference of two consecutive discretized equations (DPi)

$$
\begin{aligned}
\left(\delta \left(g_{2-\beta} * \delta^2 \nabla u\right)_i, \nabla \varphi\right) \tau + \left(\nabla \delta u_i \cdot \boldsymbol{\nu}, \nabla \varphi \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau + \left(\Delta \delta u_i, \Delta \varphi\right) \tau & \\
= \delta h_i \left(\nabla f, \nabla \varphi\right) \tau - \left(\delta \sigma_i, \nabla \varphi \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau.&
\end{aligned}
$$

We set $\varphi = \delta^2 u_i$ and sum those equations up for $1 \le i \le j$, $j \in \{1, ..., n\}$, to obtain

$$
\begin{aligned}
& \sum_{i=1}^j \left(\delta \left(g_{2-\beta} * \delta^2 \nabla u\right)_i, \nabla \delta^2 u_i\right) \tau + \sum_{i=1}^j \left(\nabla \delta u_i \cdot \boldsymbol{\nu}, \nabla \delta^2 u_i \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau \\
& \quad + \sum_{i=1}^j \left(\Delta \delta u_i, \Delta \delta^2 u_i\right) \tau = \sum_{i=1}^j \delta h_i \left(\nabla f, \nabla \delta^2 u_i\right) \tau - \sum_{i=1}^j \left(\delta \sigma_i, \nabla \delta^2 u_i \cdot \boldsymbol{\nu}\right)_{\Gamma_N} \tau.
\end{aligned}
$$

Terms are estimated analogously as in Lemma 4.4.2 except the first term on the r.h.s., which is estimated in following way

$$
\begin{aligned}
\sum_{i=1}^{j} \delta h_i \left(\nabla f, \nabla \delta^2 u_i\right) \tau &\leq C \sum_{i=1}^{j} |\delta h_i| \left\|\nabla \delta^2 u_i\right\| \tau \\
&\leq C \sum_{i=1}^{j} \left(1 + g_{2-\beta}(t_i) + \|\Delta \delta u_{i-1}\|\right) \left\|\nabla \delta^2 u_i\right\| \tau \\
&\leq C_\varepsilon + \varepsilon \sum_{i=1}^{j} \left\|\nabla \delta^2 u_i\right\|^2 \tau + C_\varepsilon \sum_{i=1}^{j} g_{2-\beta}(t_i) \tau \\
&\quad + \varepsilon \sum_{i=1}^{j} g_{2-\beta}(t_i) \left\|\nabla \delta^2 u_i\right\|^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|\Delta \delta u_{i-1}\|^2 \tau,
\end{aligned}
$$

using Cauchy and Young inequalities. Choosing an appropriate $\varepsilon > 0$, using the Grönwall lemma and Friedrich inequality bring us to the results in the lemma. $\square$

In the next step, we define piecewise constant or linear interpolations in time as $u_n, \overline{u}_n, \tilde{u}_n : [0,T] \to \boldsymbol{V}$ and $v_n, \overline{v}_n : [0,T] \to \boldsymbol{V}$ with prescription (2.20) and (3.34), respectively. With those definitions, we may rewrite (DPi) and (DMPi) into

$$
\begin{aligned}
&\left(\left(\overline{g_{2-\beta}}_n * \nabla \partial_t v_n\right)(t_i), \nabla \varphi\right) + \left(\nabla \overline{u}_n(t) \cdot \boldsymbol{\nu}, \nabla \varphi \cdot \boldsymbol{\nu}\right)_{\Gamma_N} + \left(\Delta \overline{u}_n(t), \Delta \varphi\right) \\
&\qquad = \overline{h}_n(t) \left(\nabla f, \nabla \varphi\right) - \left(\overline{\sigma}_n(t), \nabla \varphi \cdot \boldsymbol{\nu}\right)_{\Gamma_N}, \qquad \text{(DP)}
\end{aligned}
$$

and

$$
\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) + \left(\Delta \tilde{u}_n(t), \omega\right) = \overline{h}_n(t) \left(f, \omega\right), \qquad \text{(DMP)}
$$

respectively, for $t \in (t_{i-1}, t_i]$.

In the following theorem, we prove the convergence of the above sequences of functions to the function $u$ in the appropriate spaces and the convergence of the (DP) and (DMP) to the (P) and (MP), respectively.

**Theorem 4.4.1** (existence of a solution). *Let $f \in H_0^1(\Omega)$, $\omega \in L^2(\Omega)$, $u_0, v_0 \in \boldsymbol{V}$, $(f, \omega) \neq 0$, $m \in C^3([0,T])$, $\sigma \in C^2([0,T], L^2(\Gamma_N))$ and suppose that (4.15) holds true.*

*Then there exists a solution $(u, h)$ to the (P), (MP) obeying $u \in C([0,T], \boldsymbol{V})$ with $\partial_t u \in C\left([0,T], H^1(\Omega)\right) \cap L^\infty((0,T), \boldsymbol{V})$, $\partial_{tt} u \in L^2\left((0,T), H^1(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* First, we will prove the uniform equi-continuity of the sequence $\{h_n\}$, using

Lemma 4.4.3 $(ii)$ we obtain

$$\begin{aligned}|h_n(t)-h_n(s)| = \left|\int_s^t h_n'(r)\,\mathrm{d}r\right| &\leq C\int_s^t \left(r^{2-\beta}+1\right)\,\mathrm{d}r\\ &\leq C\frac{t^{2-\beta}-s^{2-\beta}}{2-\beta}+\varepsilon C = C(\varepsilon^{2-\beta}+\varepsilon),\end{aligned}$$

for $s,t \in [0,T]$, $s \leq t$, such that $|t-s| \leq \varepsilon$, where $C$ is independent from $n$. Lemma 4.4.2 $(ii)$ gives us the equi-boundedness of $\{h_n\}$. Using the Arzelà-Ascoli theorem 1.2.5, we get the convergence of subsequence of $\{h_n\}$ in $C([0,T])$ to some $h \in C([0,T])$. For the sequences $\{u_n\}$ and $\{\overline{u}_n\}$ the estimate

$$\max_{t\in[0,T]} \|\overline{u}_n(t)\|^2_{\boldsymbol{V}} + \max_{t\in[0,T]} \|\partial_t u_n(t)\|^2 \leq C$$

is obtained from Lemma 4.4.2 and Lemma 4.4.3. According to the compact embedding $\boldsymbol{V} \Subset L^2(\Omega)$, we can use Lemma 1.5.1, which brings us the existence of $u$ belonging to $C\left([0,T],L^2(\Omega)\right) \cap L^\infty\left((0,T),\boldsymbol{V}\right)$ with $\partial_t u \in L^\infty\left((0,T),L^2(\Omega)\right)$ and the subsequence of $\{u_n\}$, $\{\overline{u}_n\}$ (indexed again by $n$) such that

$$\begin{cases} u_n \to u, & \text{in} \quad C\left([0,T],L^2(\Omega)\right) & (4.17\text{a})\\ u_n(t) \rightharpoonup u(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t\in(0,T) & (4.17\text{b})\\ \overline{u}_n(t) \rightharpoonup u(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t\in(0,T) & (4.17\text{c})\\ \partial_t u_n \rightharpoonup \partial_t u, & \text{in} \quad L^2\left((0,T),L^2(\Omega)\right). & (4.17\text{d})\end{cases}$$

Since the space $L^2\left((0,T),\boldsymbol{V}\right)$ is reflexive from the estimate $(ii)$ from Lemma 4.4.3, we obtain

$$\partial_t u_n \rightharpoonup \partial_t u, \qquad \text{in} \qquad L^2\left((0,T),\boldsymbol{V}\right),$$

and consequently

$$\|u(t)-u(s)\|_{\boldsymbol{V}} \leq \sqrt{|t-s|}\sqrt{\int_0^T \|\partial_t u(r)\|^2_{\boldsymbol{V}}\,\mathrm{d}r} \leq C\sqrt{|t-s|}.$$

Since $u_0 \in \boldsymbol{V}$, we obtain $u \in C\left([0,T],\boldsymbol{V}\right)$. Similarly, as above form Lemma 4.4.3 we obtain the estimate

$$\max_{t\in[0,T]} \|\overline{v}_n(t)\|^2_{\boldsymbol{V}} + \int_0^T \|\partial_t v_n(t)\|^2\,\mathrm{d}t \leq C,$$

which implies existence of $v \in C\left([0,T],L^2(\Omega)\right) \cap L^\infty\left((0,T),\boldsymbol{V}\right)$ together with $\partial_t v \in L^2\left((0,T),L^2(\Omega)\right)$ to which a subsequence of $\{v_n\}$, $\{\overline{v}_n\}$ (indexed again by

$n$) converges in following way

$$\begin{cases} v_n \to v, & \text{in} \quad C\left([0,T], L^2(\Omega)\right) & \text{(4.18a)} \\ v_n(t) \rightharpoonup v(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t \in (0,T) & \text{(4.18b)} \\ \overline{v}_n(t) \rightharpoonup v(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t \in (0,T) & \text{(4.18c)} \\ \partial_t v_n \rightharpoonup \partial_t v, & \text{in} \quad L^2\left((0,T), L^2(\Omega)\right). & \text{(4.18d)} \end{cases}$$

Furthermore, the estimate $\int_0^T \|\partial_t v_n(t)\|^2_{H^1(\Omega)} \, \mathrm{d}t \leq C$ from Lemma 4.4.3 and the reflexivity of the space $L^2\left((0,T), H^1(\Omega)\right)$ give

$$\partial_t v_n \rightharpoonup \partial_t v, \qquad \text{in} \qquad L^2\left((0,T), H^1(\Omega)\right),$$

and then

$$\|v(t) - v(s)\|_{H^1(\Omega)} \leq \sqrt{|t-s|}\sqrt{\int_0^T \|\partial_t v(r)\|^2_{H^1(\Omega)} \, \mathrm{d}r} \leq C\sqrt{|t-s|}.$$

So by assuming $v_0 \in H^1(\Omega)$, we get $v \in C\left([0,T], H^1(\Omega)\right)$. There is a connection between $u$ and $v$ which we can see after we pass the limit $n \to \infty$ in

$$(u_n(t) - u_0, \varphi) = \int_0^t (\overline{v}_n(s), \varphi) \, ds \qquad \text{for } \varphi \in L^2(\Omega),$$

and get

$$(u(t) - u_0, \varphi) = \int_0^t (v(s), \varphi) \, ds \qquad \text{for } \varphi \in L^2(\Omega).$$

So, it holds $v(t) = \partial_t u(t)$ a.e. in $[0,T]$.

The next step is to show that the couple $(u,h)$ solves (MP) and (P) for all $\varphi \in \boldsymbol{V}$. Hence, we need to proof the convergence of (DMP) and (DP) to (MP) and (P), respectively. We start with (DMP), first, we may estimate

$$\begin{aligned} &\left|\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t)\right| \\ &\leq \left|\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\overline{m''}_n(s) \, \mathrm{d}s\right| + \left|\int_0^t \left(\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right) \overline{m''}_n(s) \, \mathrm{d}s\right| \\ &\leq C \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \, \mathrm{d}s + C \int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \mathrm{d}s. \end{aligned}$$

From the pointwise convergence of $\overline{g_{2-\beta}}_n$ to $g_{2-\beta}$ in $(0,T)$ and the Lebesgue dominated convergence theorem, we obtain convergence

$$\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) \to (g_{2-\beta} * m'')(t).$$

Furthermore, the estimate $\max_{0\le i\le n} \|\Delta\delta u_i\|^2 \le C$ from Lemma 4.4.3 $(i)$ yields

$$\int_0^T \|\Delta\tilde{u}_n(t) - \Delta\overline{u}_n(t)\| \,\mathrm{d}t = \mathcal{O}(\tau),$$

from the same lemma part $(ii)$, we obtain the similar fact that

$$\int_0^T \left|\overline{h}_n(t) - h_n(t)\right| \,\mathrm{d}t = \mathcal{O}(\tau).$$

With the above arguments and convergences, we may proceed to the next step. Before passing to the limit, we integrate the whole equality (DMP) over $(0,\xi)$, for $\xi \in (0,T]$. Next, we pass to the limit $n\to\infty$ and differentiate, which bring us to (MP) for a.a. $t\in[0,T]$.

We advance with limiting to the equality (DP). The most interesting part is the first term on the l.h.s.; to pass the limit the following estimate is needed

$$\begin{aligned}
&\left|\int_0^\xi \left(\left(\overline{g_{2-\beta}}_n * \partial_t\nabla v_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \partial_t\nabla v_n\right)(t), \nabla\varphi\right) \mathrm{d}t\right| \\
&\qquad\le \int_0^\xi \left|\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\left(\partial_t\nabla v_n(s), \nabla\varphi\right) \mathrm{d}s\right| dt \\
&\qquad\quad + \int_0^\xi \left|\int_0^t \left(\overline{g_{2-\beta}}_n(t_i-s) - \overline{g_{2-\beta}}_n(t-s)\right)\left(\partial_t\nabla v_n(s), \nabla\varphi\right) \mathrm{d}s\right| \mathrm{d}t \\
&\qquad\le \int_0^\xi \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i-s)\,\|\partial_t\nabla v_n(s)\|\,\|\nabla\varphi\| \,\mathrm{d}s\,\mathrm{d}t \\
&\qquad\quad + \int_0^\xi \int_0^t \left|\overline{g_{2-\beta}}_n(t_i-s) - \overline{g_{2-\beta}}_n(t-s)\right| \|\partial_t\nabla v_n(s)\|\,\|\nabla\varphi\| \,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

The first term in the inequality above may be estimated using Hölder's inequality and Lemma 4.4.3 as follows

$$\begin{aligned}
&\int_0^\xi \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i-s)\,\|\partial_t\nabla v_n(s)\|\,\|\nabla\varphi\| \,\mathrm{d}s\,\mathrm{d}t \\
&\le \|\nabla\varphi\| \int_0^\xi \sqrt{\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i-s)\,\mathrm{d}s} \sqrt{\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i-s)\,\|\partial_t\nabla v_n(s)\|^2 \,\mathrm{d}s\,\mathrm{d}t} \\
&\le \|\nabla\varphi\| \sqrt{\tau^{2-\beta}} \int_0^\xi \sqrt{\int_0^{t_i} \overline{g_{2-\beta}}_n(t_i-s)\,\|\partial_t\nabla v_n(s)\|^2 \,\mathrm{d}s\,\mathrm{d}t} \\
&\le C\,\|\nabla\varphi\| \sqrt{\tau^{2-\beta}},
\end{aligned} \tag{4.19}$$

The second term is estimated in the following manner

$$
\begin{aligned}
&\int_0^\xi \int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \|\partial_t \nabla v_n(s)\| \, \|\nabla\varphi\| \, \mathrm{d}s \, \mathrm{d}t \\
&\le \|\nabla\varphi\| \int_0^\xi \int_s^\xi \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \|\partial_t \nabla v_n(s)\| \, \mathrm{d}t \, \mathrm{d}s \\
&\le \|\nabla\varphi\| \int_0^\xi \|\partial_t \nabla v_n(s)\| \int_s^\xi \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \mathrm{d}t \, \mathrm{d}s \\
&\le \|\nabla\varphi\| \sqrt{\int_0^\xi \|\partial_t \nabla v_n(s)\|^2 \, \mathrm{d}s} \sqrt{\int_0^\xi \left(\int_s^\xi \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \mathrm{d}t\right)^2 \mathrm{d}s} \\
&\le C \|\nabla\varphi\|,
\end{aligned} \tag{4.20}
$$

where switching the order of integration was done, the Hölder inequality and estimate form Lemma 4.4.3 were used. Since $\overline{g_{2-\beta}}_n \to g_{2-\beta}$ in $(0,T)$ poitnwise with the above estimates we obtain

$$
\lim_{n\to\infty} \left| \int_0^\xi \left( \left(\overline{g_{2-\beta}}_n * \partial_t \nabla v_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \partial_t \nabla v_n\right)(t), \nabla\varphi \right) \mathrm{d}t \right| = 0,
$$

by applying the Lebesgue dominated convergence theorem. Next, with the use of the Cauchy, Hölder and Young inequalities, we get

$$
\begin{aligned}
&\left| \int_0^\xi \left( \left[\overline{g_{2-\beta}}_n - g_{2-\beta}\right] * (\partial_t \nabla v_n, \nabla\varphi) \right)(t) \, \mathrm{d}t \right| \\
&\le \int_0^\xi \left|\overline{g_{2-\beta}}_n(t) - g_{2-\beta}(t)\right| \mathrm{d}t \sqrt{\int_0^\xi \|\partial_t \nabla v_n(t)\|^2 \, \mathrm{d}t} \sqrt{\int_0^\xi \|\nabla\varphi\|^2 \, \mathrm{d}t} \le C \|\nabla\varphi\|,
\end{aligned}
$$

which allows us to use the Lebesgue convergence theorem to gain

$$
\lim_{n\to\infty} \left| \int_0^\xi \left( \left[\overline{g_{2-\beta}}_n - g_{2-\beta}\right] * (\partial_t \nabla v_n, \nabla\varphi) \right)(t) \, \mathrm{d}t \right| = 0.
$$

Last estimate necessary for passing to the limit is

$$
\begin{aligned}
&\left| \int_0^\xi \left( g_{2-\beta} * (\partial_t \nabla v_n, \nabla\varphi) \right)(t) \, \mathrm{d}t \right| \\
&\qquad \le \int_0^\xi g_{2-\beta}(t) \, \mathrm{d}t \sqrt{\int_0^\xi \|\partial_t \nabla v_n(t)\|^2 \, \mathrm{d}t} \sqrt{\int_0^\xi \|\nabla\varphi\|^2 \, \mathrm{d}t}.
\end{aligned}
$$

We consider the estimated integral as a linear bounded functional on the space $L^2\left((0,T),H^1(\Omega)\right)$, using the weak convergence of $\partial_t v_n$ to $\partial_t v$ in that space brings us

$$\lim_{n\to\infty}\int_0^\xi \left(g_{1-\beta}*\left(\partial_t\nabla v_n,\nabla\varphi\right)\right)(t)\,\mathrm{d}t=\int_0^\xi \left(g_{1-\beta}*\left(\partial_t\nabla v,\nabla\varphi\right)\right)(t)\,\mathrm{d}t.$$

In the final step, we integrate (DP) in time over $(0,\xi)$, for $\xi\in(0,T]$, with the above estimates and convergences we pass to the limit $n\to\infty$ to obtain

$$\int_0^\xi \left(\left(g_{2-\beta}*\partial_{tt}\nabla u\right)(t),\nabla\varphi\right)\,\mathrm{d}t+\int_0^\xi \left(\nabla u(t)\cdot\boldsymbol{\nu},\nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N}\,\mathrm{d}t+\int_0^\xi \left(\Delta u(t),\Delta\varphi\right)\,\mathrm{d}t$$
$$=\int_0^\xi h(t)\left(\nabla f,\nabla\varphi\right)\,\mathrm{d}t-\int_0^\xi \left(\sigma(t),\nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N}\,\mathrm{d}t.$$

Differentiation with respect to $\xi$ gives us (P). □

Note that the estimate $\sum_{i=1}^n g_{2-\beta}(t_i)\left\|\nabla\delta^2 u_i\right\|^2\tau\le C$ is essential for proving the convergence of the Rothe function to the solution.

## 4.5 Error estimate

**Theorem 4.5.1** (error estimate)**.** *Under the assumptions of Theorem 4.4.1 then there exists a positive constant $C$ (independent of $n$) such that*

$$\int_0^T\left|h(t)-\overline{h}_n(t)\right|^2\,dt\le C\tau^{2-\beta}\tag{4.21}$$

*and*

$$\|\nabla u(t)\cdot\boldsymbol{\nu}-\nabla u_n(t)\cdot\boldsymbol{\nu}\|^2_{\Gamma_N}+\|\Delta u(t)-\Delta u_n(t)\|^2$$
$$+\int_0^T\|\partial_t\nabla u(t)-\nabla v_n(t)\|^2\,dt\le C\tau^{2-\beta}.\tag{4.22}$$

*Proof.* First, we state an estimate for some differences of Rothe's functions

$$\int_0^T\|\Delta u_n(t)-\Delta\tilde{u}_n(t)\|^2\,\mathrm{d}t+\int_0^T\|\Delta u_n(t)-\Delta\overline{u}_n(t)\|^2\,\mathrm{d}t$$

$$+\int_0^T \|\nabla v_n(t) - \nabla \overline{v}_n(t)\|^2 \,\mathrm{d}t + \int_0^T \|\nabla u_n(t)\cdot\boldsymbol{\nu} - \nabla \overline{u}_n(t)\cdot\boldsymbol{\nu}\|_{\Gamma_N}^2 \,\mathrm{d}t$$
$$+\int_0^T \|\sigma_n(t) - \overline{\sigma}_n(t)\|^2 \,\mathrm{d}t \le C\tau^2.$$

We also remind that $\overline{v}_n = \partial_t u_n$ a.e. in $(0,T)$. It holds for the convolution kernel $g_{2-\beta}$ that

$$\left\|g_{2-\beta} - \overline{g_{2-\beta}}_n\right\|_{L^1(0,T)} \le C\tau^{2-\beta}.$$

Above estimates will be used through the whole proof; we will also assume $\tau$ small enough. Next, from the equations (MP) and (DMP), we calculate

$$\begin{aligned}
&\int_0^\xi \left|h(t) - \overline{h}_n(t)\right|^2 \,\mathrm{d}t \\
&\le \int_0^\xi \left| \frac{\left(g_{2-\beta} * m''\right)(t) - \left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) - \left(\Delta u(t) - \Delta\tilde{u}_n(t), \omega\right)}{(f,\omega)} \right|^2 \,\mathrm{d}t \\
&\le C\Bigg( \int_0^\xi \left|\left(g_{2-\beta} * (m'' - \overline{m''}_n)\right)(t)\right|^2 \,\mathrm{d}t \\
&\quad + \int_0^\xi \left|\left(\left(g_{2-\beta} - \overline{g_{2-\beta}}_n\right) * \overline{m''}_n\right)(t)\right|^2 \,\mathrm{d}t \\
&\quad + \int_0^\xi \left|\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t) - \left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i)\right|^2 \,\mathrm{d}t \\
&\quad + \int_0^\xi \|\Delta u(t) - \Delta\tilde{u}_n(t)\|^2 \,\mathrm{d}t \Bigg) \\
&\le C\Bigg( \|g_{2-\beta}\|_{L^1(0,T)}^2 \left\|m'' - \overline{m''}_n\right\|_{L^2(0,T)}^2 \\
&\quad + \left\|g_{2-\beta} - \overline{g_{2-\beta}}_n\right\|_{L^1(0,T)}^2 \left\|\overline{m''}_n\right\|_{L^2(0,T)}^2 \\
&\quad + \int_0^\xi \left( \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\,\mathrm{d}s \right)^2 \,\mathrm{d}t \\
&\quad + \int_0^\xi \left( \int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \,\mathrm{d}s \right)^2 \,\mathrm{d}t \\
&\quad + \int_0^\xi \|\Delta u(t) - \Delta\tilde{u}_n(t)\|^2 \,\mathrm{d}t \Bigg) \\
&\le C\left( \tau^2 + \tau^{(2-\beta)2} + \int_0^\xi \|\Delta u(t) - \Delta\tilde{u}_n(t)\|^2 \,\mathrm{d}t \right)
\end{aligned}$$

$$\leq C\left(\tau^{(2-\beta)2}+\int_0^\xi \|\Delta u(t)-\Delta\tilde{u}_n(t)\|^2\,\mathrm{d}t\right),$$

where we used

$$\begin{aligned}
&\int_0^t \left|\overline{g_{2-\beta}}_n(t_i-s)-\overline{g_{2-\beta}}_n(t-s)\right|\,\mathrm{d}s\\
&=\int_0^{t_{i-1}}\left(\overline{g_{2-\beta}}_n(t-s)-\overline{g_{2-\beta}}_n(t_i-s)\right)\mathrm{d}s\\
&\quad+\int_{t_{i-1}}^t\left(\overline{g_{2-\beta}}_n(t-s)-\overline{g_{2-\beta}}_n(t_i-s)\right)\mathrm{d}s\\
&\leq\int_0^{t_{i-1}}\left(\overline{g_{2-\beta}}_n(t_{i-1}-s)-\overline{g_{2-\beta}}_n(t_i-s)\right)\mathrm{d}s+C\int_{t_{i-1}}^t\overline{g_{2-\beta}}_n(t-s)\,\mathrm{d}s\\
&\leq C\int_0^{t_{i-1}}(t_{i-1}-s)^{1\beta}-(t_{i+1}-s)^{1\beta}\,\mathrm{d}s+C\int_{t_{i-1}}^t\overline{g_{2-\beta}}_n(t-s)\,\mathrm{d}s\\
&\leq C(t_{i-1}^{2-\beta}-t_{i+1}^{2-\beta}+\tau^{2-\beta})+C(t_i-t_{i-1})^{2-\beta}\\
&\leq C\tau^{2-\beta},
\end{aligned}$$

for $t\in(t_{i-1},t_i]$. We can estimate

$$\begin{aligned}
\int_0^\xi \|\Delta u(t)-\Delta\tilde{u}_n(t)\|^2\,\mathrm{d}t
&\leq C\int_0^\xi \|\Delta u(t)-\Delta u_n(t)\|^2\,\mathrm{d}t+C\int_0^\xi \|\Delta u_n(t)-\Delta\tilde{u}_n(t)\|^2\,\mathrm{d}t\\
&\leq C\left(\tau^2+\int_0^\xi \|\Delta u(t)-\Delta u_n(t)\|^2\,\mathrm{d}t\right),
\end{aligned}$$

so finally we get

$$\int_0^\xi \left|h(t)-\overline{h}_n(t)\right|^2\,\mathrm{d}t\leq C\left(\tau^{(2-\beta)2}+\int_0^\xi \|\Delta u(t)-\Delta u_n(t)\|^2\,\mathrm{d}t\right). \tag{4.23}$$

Subtracting (DP) from (P), we obtain

$$\begin{aligned}
&\left(\left(g_{2-\beta}*\partial_{tt}\nabla u\right)(t)-\left(\overline{g_{2-\beta}}_n*\nabla\partial_t v_n\right)(t_i),\nabla\varphi\right)\\
&\quad+\left(\nabla u(t)\cdot\boldsymbol{\nu}-\nabla\overline{u}_n(t)\cdot\boldsymbol{\nu},\nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N}+\left(\Delta u(t)-\Delta\overline{u}_n(t),\Delta\varphi\right)\\
&\qquad=\left(h(t)-\overline{h}_n(t)\right)(\nabla f,\nabla\varphi)-\left(\sigma(t)-\overline{\sigma}_n(t),\nabla\varphi\cdot\boldsymbol{\nu}\right)_{\Gamma_N},
\end{aligned} \tag{4.24}$$

choosing $\varphi=\partial_t(u(t)-u_n(t))$ and integrating the whole equation over $(0,\xi)$, $\xi\in(0,T]$, we get

$$
\begin{aligned}
&\int_0^\xi \big( \left( g_{2-\beta} * \partial_{tt} \nabla u \right)(t) - \left( \overline{g_{2-\beta}}_n * \nabla \partial_t v_n \right)(t_i), \nabla(\partial_t(u(t) - u_n(t))) \big) \, \mathrm{d}t \\
&\qquad + \int_0^\xi \left( \nabla u(t) \cdot \boldsymbol{\nu} - \nabla \overline{u}_n(t) \cdot \boldsymbol{\nu}, \nabla(\partial_t(u(t) - u_n(t))) \cdot \boldsymbol{\nu} \right)_{\Gamma_N} \, \mathrm{d}t \\
&\qquad + \int_0^\xi \left( \Delta u(t) - \Delta \overline{u}_n(t), \Delta(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t \\
&\qquad = \int_0^\xi \left( h(t) - \overline{h}_n(t) \right) \left( \nabla f, \nabla(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t \\
&\qquad - \int_0^\xi \left( \sigma(t) - \overline{\sigma}_n(t), \nabla(\partial_t(u(t) - u_n(t))) \cdot \boldsymbol{\nu} \right)_{\Gamma_N} \, \mathrm{d}t.
\end{aligned}
\tag{4.25}
$$

The third term on the l.h.s of (4.25) can be rewritten as

$$
\begin{aligned}
&\int_0^\xi \left( \Delta u(t) - \Delta \overline{u}_n(t), \Delta(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t \\
&= \int_0^\xi \left( \Delta u(t) - \Delta u_n(t), \Delta(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t \\
&+ \int_0^\xi \left( \Delta u_n(t) - \Delta \overline{u}_n(t), \Delta(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t \\
&= \frac{1}{2} \left\| \Delta u(\xi) - \Delta u_n(\xi)) \right\|^2 + \int_0^\xi \left( \Delta u_n(t) - \Delta \overline{u}_n(t), \Delta(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t.
\end{aligned}
$$

Next, we can estimate by Lemma 4.4.3

$$
\begin{aligned}
&\int_0^\xi \left( \Delta u_n(t) - \Delta \overline{u}_n(t), \Delta(\partial_t(u(t) - u_n(t))) \right) \, \mathrm{d}t \\
&\qquad \leq \int_0^\xi \left\| \Delta u_n(t) - \Delta \overline{u}_n(t) \right\| \left\| \Delta(\partial_t(u(t) - u_n(t))) \right\| \, \mathrm{d}t \leq C\tau.
\end{aligned}
\tag{4.26}
$$

We can estimate the second term on the l.h.s. and the second term on the r.h.s. of (4.25) in a similar manner. The first term on the l.h.s. in (4.25) may be rewritten as

$$
\begin{aligned}
&\int_0^\xi \big( \left( g_{2-\beta} * \partial_{tt} \nabla u \right)(t) - \left( \overline{g_{2-\beta}}_n * \nabla \partial_t v_n \right)(t_i), \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \big) \, \mathrm{d}t \\
&\quad = \int_0^\xi \left( \left( g_{2-\beta} * \partial_{tt} \nabla u \right)(t) - \left( g_{2-\beta} * \nabla \partial_t v_n \right)(t), \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \right) \, \mathrm{d}t \\
&\quad + \int_0^\xi \big( \left( g_{2-\beta} * \nabla \partial_t v_n \right)(t) - \left( \overline{g_{2-\beta}}_n * \nabla \partial_t v_n \right)(t), \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \big) \, \mathrm{d}t
\end{aligned}
$$

$$+\int_0^{\xi}\left(\left(\overline{g_{2-\beta}}_n*\nabla\partial_t v_n\right)(t)-\left(\overline{g_{2-\beta}}_n*\nabla\partial_t v_n\right)(t_i),\partial_t\nabla u(t)-\partial_t\nabla u_n(t)\right)\,\mathrm{d}t. \tag{4.27}$$

Here, the first term in (4.27) can be rewritten and then estimated by using Lemma 1.7.2 as

$$\begin{aligned}
&\int_0^{\xi}\left(\left(g_{2-\beta}*\partial_{tt}\nabla u\right)(t)-\left(g_{2-\beta}*\nabla\partial_t v_n\right)(t),\partial_t\nabla u(t)-\partial_t\nabla u_n(t)\right)\,\mathrm{d}t\\
&\qquad=\int_0^{\xi}\left(\partial_t\left(g_{2-\beta}*(\partial_t\nabla u-\nabla v_n)\right)(t),\partial_t\nabla u(t)-\nabla v_n(t)\right)\,\mathrm{d}t\\
&\qquad\quad+\int_0^{\xi}\left(\left(g_{2-\beta}*(\partial_{tt}\nabla u-\partial_t\nabla v_n)\right)(t),\nabla v_n(t)-\partial_t\nabla u_n(t)\right)\,\mathrm{d}t\\
&\qquad\geq C\int_0^{\xi}\left\|\partial_t\nabla u(t)-\nabla v_n(t)\right\|^2\,\mathrm{d}t\\
&\qquad\quad+\int_0^{\xi}\left(\left(g_{2-\beta}*(\partial_{tt}\nabla u-\partial_t\nabla v_n)\right)(t),\nabla v_n(t)-\partial_t\nabla u_n(t)\right)\,\mathrm{d}t,
\end{aligned}$$

where the second term is estimated as

$$\begin{aligned}
&\int_0^{\xi}\left(\left(g_{2-\beta}*(\partial_{tt}\nabla u-\partial_t\nabla v_n)\right)(t),\nabla v_n(t)-\partial_t\nabla u_n(t)\right)\,\mathrm{d}t\\
&\leq\left(\int_0^{\xi}\left\|\left(g_{2-\beta}*(\partial_{tt}\nabla u-\partial_t\nabla v_n)\right)(t)\right\|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^{\xi}\left\|\nabla v_n(t)-\partial_t\nabla u_n(t)\right\|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\leq C\tau.
\end{aligned}$$

The upper bound of the second term in (4.27) is obtained by using Young's inequality for convolutions in the following way

$$\int_0^{\xi}\left(\left(g_{2-\beta}*\nabla\partial_t v_n\right)(t)-\left(\overline{g_{2-\beta}}_n*\nabla\partial_t v_n\right)(t),\partial_t\nabla u(t)-\partial_t\nabla u_n(t)\right)\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^\xi \left( \left( (g_{2-\beta} - \overline{g_{2-\beta}}_n) * \nabla \partial_t v_n \right)(t), \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \right) \mathrm{d}t \\
&\le C_\varepsilon \int_0^\xi \left\| \left( (g_{2-\beta} - \overline{g_{2-\beta}}_n) * \nabla \partial_t v_n \right)(t) \right\|^2 \mathrm{d}t \\
&\quad + \varepsilon \int_0^\xi \left\| \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \right\|^2 \mathrm{d}t \\
&\le C_\varepsilon \left\| g_{2-\beta} - \overline{g_{2-\beta}}_n \right\|^2_{L^1(0,T)} \left\| \nabla \partial_t v_n \right\|^2_{L^2((0,T),L^2(\Omega))} \\
&\quad + \varepsilon \int_0^\xi \left\| \nabla \partial_t u(t) - \nabla v_n(t) \right\|^2 \mathrm{d}t \\
&\quad + \varepsilon \int_0^\xi \left\| \nabla v_n(t) - \nabla \partial_t u_n(t) \right\|^2 \mathrm{d}t \\
&\le C_\varepsilon \tau^{(2-\beta)2} + \varepsilon \int_0^\xi \left\| \nabla \partial_t u(t) - \nabla v_n(t) \right\|^2 \mathrm{d}t
\end{aligned}
$$

We get the estimate for the last term in (4.27) similarly as estimates (4.19), (4.20) in the proof of Theorem 4.4.1

$$
\begin{aligned}
&\int_0^\xi \left( \left( \overline{g_{2-\beta}}_n * \nabla \partial_t v_n \right)(t) - \left( \overline{g_{2-\beta}}_n * \nabla \partial_t v_n \right)(t_i), \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \right) \mathrm{d}t \\
&\le \int_0^\xi \Bigg( \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \left\| \partial_t \nabla v_n(s) \right\| \mathrm{d}s \\
&+ \int_0^t \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \left\| \partial_t \nabla v_n(s) \right\| \mathrm{d}s \Bigg) \left\| \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \right\| \mathrm{d}t \\
&\le C_\varepsilon \int_0^\xi \left( \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \left\| \partial_t \nabla v_n(s) \right\| \mathrm{d}s \right)^2 \mathrm{d}t \\
&+ C_\varepsilon \int_0^\xi \left( \int_0^t \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \left\| \partial_t \nabla v_n(s) \right\| \mathrm{d}s \right)^2 \mathrm{d}t \\
&+ \varepsilon \int_0^\xi \left\| \partial_t \nabla u(t) - \partial_t \nabla u_n(t) \right\|^2 \mathrm{d}t \\
&\le C_\varepsilon \tau^{2-\beta} + \varepsilon \int_0^\xi \left\| \nabla \partial_t u(t) - \nabla v_n(t) \right\|^2 \mathrm{d}t + \varepsilon \int_0^\xi \left\| \nabla v_n(t) - \nabla \partial_t u_n(t) \right\|^2 \mathrm{d}t \\
&\le C_\varepsilon \tau^{2-\beta} + \varepsilon \int_0^\xi \left\| \nabla \partial_t u(t) - \nabla v_n(t) \right\|^2 \mathrm{d}t.
\end{aligned}
$$

The first term on the r.h.s of (4.25) is estimated from above as

$$
\int_0^\xi \left( h(t) - \overline{h}_n(t) \right) \left( \nabla f, \nabla(\partial_t(u(t) - u_n(t))) \right) \mathrm{d}t
$$

$$
\begin{aligned}
&\leq C_\varepsilon \int_0^\xi \left|h(t) - \overline{h}_n(t)\right|^2 \,\mathrm{d}t + \varepsilon \int_0^\xi \|\nabla \partial_t u(t) - \nabla v_n(t)\|^2 \,\mathrm{d}t \\
&\quad + \varepsilon \int_0^\xi \|\nabla v_n(t) - \nabla \partial_t u_n(t)\|^2 \,\mathrm{d}t \\
&\leq C_\varepsilon \tau^{(2-\beta)2} + C_\varepsilon \int_0^\xi \|\Delta u(t) - \Delta u_n(t)\|^2 \,\mathrm{d}t + \varepsilon \int_0^\xi \|\nabla \partial_t u(t) - \nabla v_n(t)\|^2 \,\mathrm{d}t.
\end{aligned}
$$

With the above estimates and choosing an appropriate $\varepsilon > 0$, we arrive at

$$
\begin{aligned}
\int_0^\xi \|\partial_t \nabla u(t) - \nabla v_n(t)\|^2 \,\mathrm{d}t + \|\nabla u(\xi) \cdot \boldsymbol{\nu} - \nabla u_n(\xi) \cdot \boldsymbol{\nu}\|_{\Gamma_N}^2 + \|\Delta u(\xi) - \Delta u_n(\xi)\|^2 \\
\leq C \left( \tau^{2-\beta} + \int_0^\xi \|\Delta u(t) - \Delta u_n(t)\|^2 \,\mathrm{d}t \right),
\end{aligned}
$$

finally, applying the Grönwall lemma, we get

$$
\begin{aligned}
\int_0^\xi \|\partial_t \nabla u(t) - \nabla v_n(t)\|^2 \,\mathrm{d}t + \|\nabla u(\xi) \cdot \boldsymbol{\nu} - \nabla u_n(\xi) \cdot \boldsymbol{\nu}\|_{\Gamma_N}^2 \\
+ \|\Delta u(\xi) - \Delta u_n(\xi)\|^2 \leq C\tau^{2-\beta}.
\end{aligned}
$$

□

## 4.6 Numerical Experiments

We present here a couple of numerical experiments. The first two calculate the solution from the exact measurement. They differ in the measurement function. More concrete, we take two different choices for the function $\omega$, which will represent either measurement trough the whole domain or just trough its part. The second experiment is a possible approach to the noisy measurement.

We use the algorithm arising from the time discretization. The solution couple $(u_i, h_i)$ on the $i-$th time layer is calculated from (DMPi) and (DPi), in this order, and then we move to the next time level.

In the experiment, we assume $x \in \Omega = (0, \pi)$, $T = 3$ and $\beta = 1.3$, $\Gamma_D = \{\pi\}$, $\Gamma_N = \{0\}$, next

$$f(x) = \sin x,$$

accompanying boundary and initial conditions take the form

$$
\begin{aligned}
u_0(x) &= \sin x, \\
v_0(x) &= 0, \\
\sigma(0,t) &= t^3 - t^2 + 5.
\end{aligned}
$$

The first measurement is given by

$$m_1(t) = \int_\Omega u(x,t)\omega_1(x)\,\mathrm{d}x = 12t - 4,$$

where

$$\omega_1(x,y) = 1,$$

in the second we assume

$$m_2(t) = \int_\Omega u(x,t)\omega_2(x)\,\mathrm{d}x = 6t - 2,$$

with

$$\omega_2(x,y) = \begin{cases} 1, & x \in [0, \frac{\pi}{2}], \\ 0, & otherwise. \end{cases}$$

One can easily calculate that

$$\begin{aligned} u(x,t) &= (t^3 - t^2 + 5)\sin x, \\ h(t) &= \frac{6}{(2-\beta)(3-\beta)\Gamma(2-\beta)} t^{3-\beta} - \frac{4}{(2-\beta)\Gamma(2-\beta)} t^{2-\beta} + t^3 - t^2 + 5 \end{aligned}$$

are the exact solution of the problem given by above data.

The algorithm is implemented in Python, where we use the finite element library DOLFIN from the FEniCS Project [73]. The domain is divided into 50 cells and we use Lagrange basis functions of order 2. To avoid numerical complications we formulate the problem as the mixed one defining the new unknown $v = \nabla u$.

### 4.6.1 Exact data

For both measurements, we calculate the solution for a couple of time steps $\tau$. On the Fig. 4.1 (a)-(e) we see the reconstruction of $h$ for $\tau = 0.015625$, the evolution of relative errors and the decay of relative errors for decreasing $\tau$ for $\omega_1$. The interpolating lines in (d) and (e) take the shape $1.0103 log_2 \tau - 0.5738$ for error of $u$ and $0.6871 \log_2 \tau - 1.9117$ for $h$. The same is depicted on Fig. 4.2 for $\omega_2$. The interpolating lines in (d) and (e) take shape $0.9952 log_2 \tau - 0.5145$ for error of $u$ and $0.6942 \log_2 \tau - 1.7849$ for $h$. The relative errors depicted in Figures 4.1, 4.2(d), (e) are calculated as

$$error_u = \frac{\max\limits_t \|u_{exact}(t) - u_{app}(t)\|_{L^2(\Omega)}}{\max\limits_t \|u_{exact}(t)\|_{L^2(\Omega)}},$$

for the relative error of $u$ and

$$error_h = \frac{\sqrt{\int_0^T \left|h_{exact}(t) - h_{app}(t)\right|^2 \, \mathrm{d}t}}{\sqrt{\int_0^T \left|h_{exact}(t)\right|^2 \, \mathrm{d}t}},$$

for the relative error of $h$.

We assumed $\beta = 1.3$ in the calculations. Theorem 4.5.1 implies that $error_u^2 \leq C\tau^{0.7}$ and $error_h^2 \leq C\tau^{0.7}$. The slopes of the interpolation lines in the error decay pictures should correspond to $\frac{(2-\beta)}{2}$. The reason is that the errors from the theorem are squared. The denominators in the calculated errors influence just the intercept value in the interpolation line. Then, according the calculations for $\omega_1$, it should hold that

$$\|u_{exact}(t) - u_{app}(t)\|^2_{L^2(\Omega)} \leq C\tau^{2.0206},$$

and

$$\int_0^T \left|h_{exact}(t) - h_{app}(t)\right|^2 \, \mathrm{d}t \leq C\tau^{1.3742},$$

which agree with the error estimate form Theorem 4.5.1 since $\tau^{2.0206}$ and $\tau^{1.3742}$ is smaller then $\tau^{0.7}$ for small $\tau$.

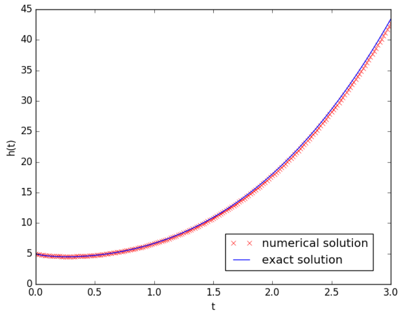
### 4.6.2 Noisy data

The noisy measurement for this calculation is modeled as previously. We apply the least square method on $m_\epsilon(t)$ to obtain a function in the form

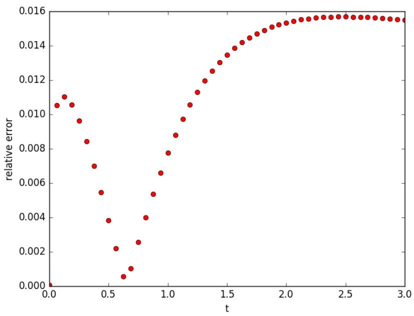
$$m_{app}(t) = at^3 + bt^2 + ct + d,$$

which approximate $m_\epsilon(t)$ and is smooth enough to be used in the algorithm.

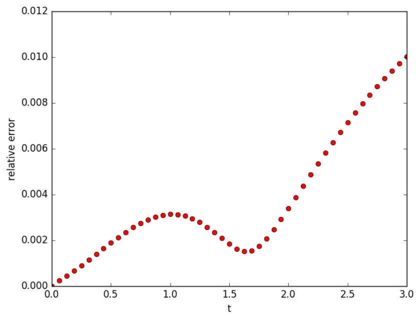
We can see the original function $m$, the noisy measurement $m_\epsilon(t)$ and its approximation on the Fig. 4.3 (a). The reconstruction of the source term for several values of $\varepsilon$ can be seen on the Fig. 4.3 (b). We see the corresponding relative error in time for $h$ and $u$ on the Fig. 4.3 (c), (d).



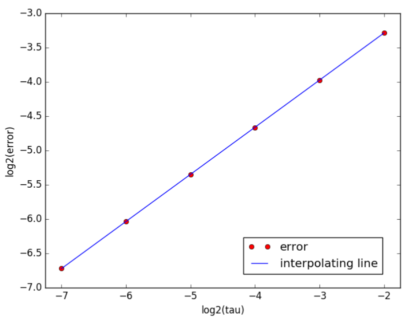

(a) Reconstruction of $h$ together with exact $h$.



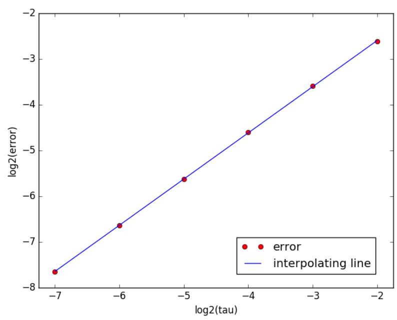

(b) Relative error $\frac{|h_{approx}(t)-h_{exact}(t)|}{|h_{exact}(t)|}$.




(c) Relative error $\frac{||u_{approx}(t)-u_{exact}(t)||}{||u_{exact}(t)||}$.

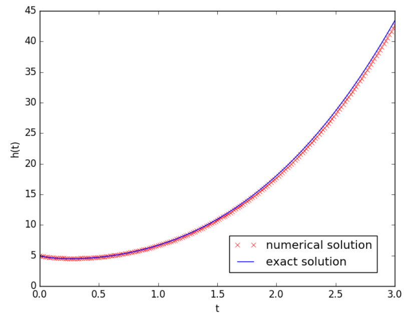



(d) Logarithm of integral relative error in time of $h$ for different values of $\tau$. Slope of the line is 0.6871.

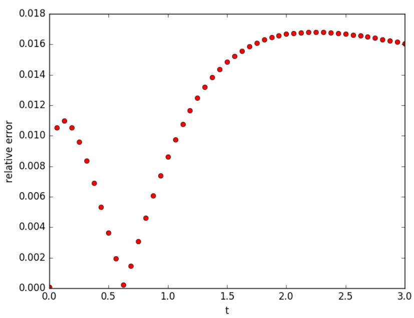



(e) Logarithm of maximal relative error in time of $u$ for different values of $\tau$. Slope of the line is 1.0103.

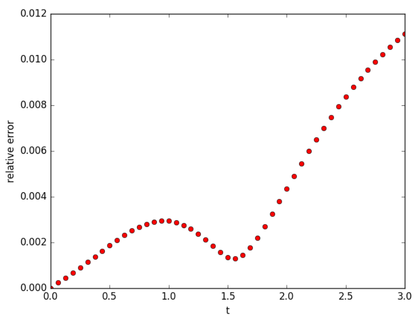
Figure 4.1: The results of the reconstruction algorithm for $\tau = 0.015625$ and $\omega_1$.



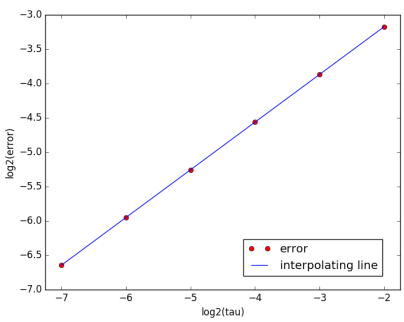

(a) Reconstruction of $h$ together with exact $h$.



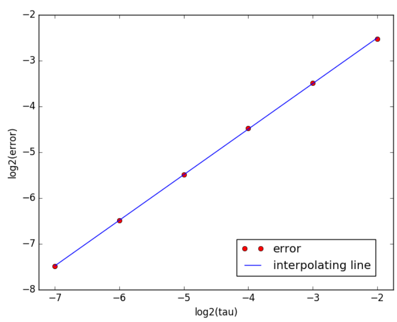

(b) Relative error $\frac{|h_{approx}(t)-h_{exact}(t)|}{|h_{exact}(t)|}$.




(c) Relative error $\frac{||u_{approx}(t)-u_{exact}(t)||}{||u_{exact}(t)||}$.

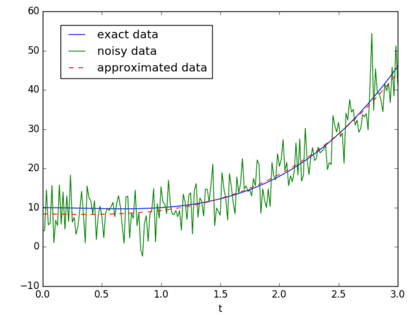



(d) Logarithm of integral relative error in time of $h$ for different values of $\tau$. Slope of the line is 0.6942.

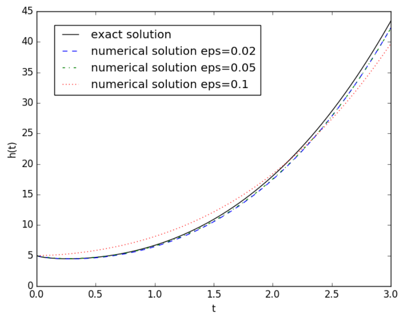



(e) Logarithm of maximal relative error in time of $u$ for different values of $\tau$. Slope of the line is 0.9952.

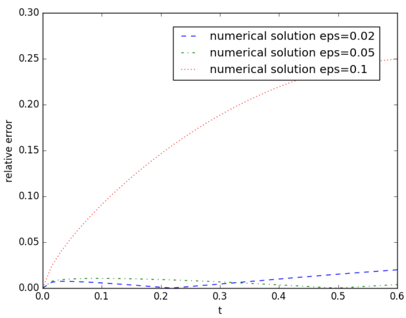
Figure 4.2: The results of the reconstruction algorithm for $\tau = 0.015625$ and $\omega_2$



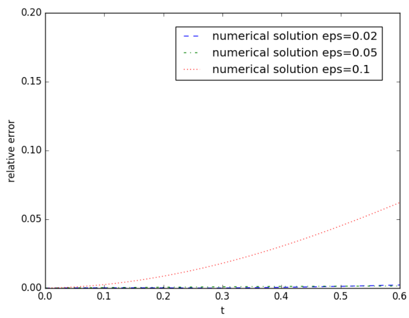

(a) Exact and noisy data for $\epsilon = 0.1$. Approximating curve has the form $m_{app}(t) = 1.9777t^3 - 2.0064t^2 + 7.4573t + 1.0013$.




(b) Reconstruction of $h$ together with exact $h$.




(c) Relative error $\frac{|h_{approx}(t)-h_{exact}(t)|}{|h_{exact}(t)|}$.




(d) Relative error $\frac{||u_{approx}(t)-u_{exact}(t)||}{||u_{exact}(t)||}$.

Figure 4.3: The results of the reconstruction algorithm for noisy data for various amount of noise $\epsilon$ and $\tau = 0.015625$.

# Chapter 5

# Identification of a source in a fractional wave equation from a boundary measurement

This chapter is based on the article [129], which has been already submitted to *Journal of Computational and Applied Mathematics* for publication.

## 5.1 Introduction

In this article, we are interested in the following fractional wave equation accompanied with standard initial condition and the Neumann boundary condition

$$\left\{ \begin{array}{rcll} \left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \Delta u(x,t) & = & h(t)f(x) + F(x,t), & x \in \Omega,\ t \in (0,T), \\ u(x,0) & = & u_0(x), & x \in \Omega, \\ \partial_t u(x,0) & = & v_0(x), & x \in \Omega, \\ -\nabla u(x,t) \cdot \boldsymbol{\nu} & = & \gamma(x,t), & (x,t) \in \Gamma \times (0,T), \end{array} \right. \tag{5.1}$$

where $\Omega \subset \mathbb{R}^d$ is bounded with the Lipschitz boundary $\Gamma$, $T > 0$ and $g_{2-\beta}$ is the Riemann-Liouville kernel.

The *Inverse Source Problem* (ISP) we are interested in here consists of identifying a couple $(u(x,t), h(t))$ obeying (5.1) and

$$\int_\Gamma u(x,t)\omega(x)dS = m(t), \qquad t \in [0,T], \tag{5.2}$$

where $\omega$ is a solely space-dependent function, many times chosen to have a compact support in $\Gamma$. This type of measurement is often called non-invasive as opposed to the measurements which take place inside the considered domain.

In Chapter 3, we have dealt with the similar equation but the measurement was taken over a subset of $\Omega$. The added value of this chapter relies on using the non-invasive measurement in the form of the integral over the part of the boundary. The approach, we take, will demand the estimates for the Laplacian of $u$ on the boundary, which was not necessary in Chapter 3.

This chapter is organized as follows. In the short second section, we reformulate our problem. In the third section we introduce the time-discretization, prove some useful a priori estimates, introduce the Rothe functions and at the end prove the existence of a solution. The last section deals with the uniqueness of the solution in appropriate spaces.

## 5.2 Reformulation of the problem

Without the loss of generality, we may assume that $F = 0$ and $\gamma = 0$. This follows from the superposition principle, which is valid for all linear systems. Then the solution of (5.1) can be written as $u = v + w$, where

$$\left\{\begin{array}{rcll} \left(g_{2-\beta} * \partial_{tt} v(x)\right)(t) - \Delta v(x,t) & = & F(x,t), & x \in \Omega,\ t \in (0,T), \\ v(x,0) & = & u_0(x), & x \in \Omega, \\ \partial_t v(x,0) & = & v_0(x), & x \in \Omega, \\ -\nabla v(x,t) \cdot \boldsymbol{\nu} & = & \gamma(x,t), & (x,t) \in \Gamma \times (0,T), \end{array}\right. \tag{5.3}$$

and

$$\left\{\begin{array}{rcll} \left(g_{2-\beta} * \partial_{tt} w(x)\right)(t) - \Delta w(x,t) & = & h(t)f(x), & x \in \Omega,\ t \in (0,T), \\ w(x,0) & = & 0, & x \in \Omega, \\ \partial_t w(x,0) & = & 0, & x \in \Omega, \\ -\nabla w(x,t) \cdot \boldsymbol{\nu} & = & 0, & (x,t) \in \Gamma \times (0,T). \end{array}\right. \tag{5.4}$$

Thus, instead of $(u,h)$ the new couple $(w,h)$ has to be found and measurement needs to be modified to

$$\int_\Gamma w(x,t)\omega(x)dS = m(t) - \int_\Gamma v(x,t)\omega(x)dS =: \tilde{m}(t), \qquad t \in [0,T]. \tag{5.5}$$

From now on, we will denote the new sought couple $(w,h)$ and the measurement function $\tilde{m}$ again by $(u,h)$ and $m$, respectively.

Next, we reformulate our problem into two coupled equations using the measurement and the variational formulation of (5.4). Taking the first equation of (5.4) and multiplying it by $\omega$ and integrating over the boundary $\Gamma$ we get

$$\left(g_{2-\beta} * m''\right)(t) - (\Delta u(t), \omega)_\Gamma = h(t)\,(f, \omega)_\Gamma\,. \tag{MP}$$

if we assume that $(f,\omega)_\Gamma \neq 0$, we may eliminate $h$ in the following manner

$$h(t) = \frac{\left(g_{2-\beta} * m''\right)(t) - (\Delta u(t), \omega)_\Gamma}{(f,\omega)_\Gamma}. \tag{5.6}$$

By multiplying the first equation of (5.4) by $\varphi \in H^1(\Omega)$ integrating over $\Omega$ and using the Green theorem, we obtain the weak formulation, thus, it holds

$$\left(\left(g_{2-\beta} * \partial_{tt} u\right)(t), \varphi\right) + (\nabla u(t), \nabla\varphi) = h(t)\,(f,\varphi)\,, \tag{P}$$

for any $\varphi \in H^1(\Omega)$, a.a. $t \in [0,T]$. Hence, in the reformulated inverse source problem, we are interested in finding a couple $(u,h)$ which solves the equations (P) and (MP) with $u(0) = 0$, $\partial_t u(0) = 0$.

## 5.3 Existence

We divide the interval $[0,T]$ into $n$ equidistant pieces, for $n \in \mathbb{N}$, and define a time step as $\tau = \frac{T}{n}$, for $i = 1, ..., n$, then for any function $z$ we define notation for the value at point $t_i$ and the first and second difference as in the previous chapters.

We approximate the solution of (P), (MP) on the $i-$th time-layer, for $i \geq 1$, by $(u_i, h_i)$ which solves

$$\left(\left(g_{2-\beta} * \delta^2 u\right)_i, \varphi\right) + (\nabla u_i, \nabla\varphi) = h_i\,(f,\varphi)\,, \tag{DP$i$}$$

for $\varphi \in H^1(\Omega)$, with $\delta u_0 := 0$ and

$$\left(g_{2-\beta} * m''\right)_i - (\Delta u_{i-1}, \omega)_\Gamma = h_i\,(f,\omega)_\Gamma\,. \tag{DMP$i$}$$

Next, we define set

$$\boldsymbol{V} = \{\varphi : \Omega \to \mathbb{R};\, \|\varphi\| + \|\nabla\varphi\| + \|\Delta\varphi\| + \|\nabla\Delta\varphi\| < \infty\}$$

which equipped with the norm $\|\varphi\|_{\boldsymbol{V}} = \left(\|\varphi\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 + \|\nabla\Delta\varphi\|^2\right)^{\frac{1}{2}}$ is Hilbert space compactly embedded in $L^2(\Omega)$. Since there occurs $\Delta u_i$ in (DMP$i$), we need to control it on the boundary which leads us in looking for the solution in the space $\boldsymbol{V}$. Following lemma handles the existence of the unique couple $(u_i, h_i)$ on every time slice.

**Lemma 5.3.1.** *Let $f \in H^1(\Omega)$, $\omega \in L^2(\Gamma)$, $(f,\omega)_\Gamma \neq 0$ and $m \in C^2([0,T])$. Then for each $i \in \{1,\dots,n\}$, there exists a unique couple $(u_i, h_i) \in \boldsymbol{V} \times \mathbb{R}$ solving (DPi) and (DMPi) for every $\varphi \in H^1(\Omega)$.*

*Proof.* Assuming $(f,\omega)_\Gamma \neq 0$ and $u_i \in \boldsymbol{V}$, we can write

$$h_i = \frac{(g_{2-\beta} * m'')_i - (\Delta u_{i-1}, \omega)_\Gamma}{(f,\omega)_\Gamma} \in \mathbb{R}.$$

The equation (DP$i$) can be rewritten such that all $u_k$'s with $k \leq i-1$ are placed on the right hand side of the equation, so we get

$$\frac{1}{\tau} g_{2-\beta}(\tau)\,(u_i,\varphi) + (\nabla u_i, \nabla\varphi) = h_i\,(f,\varphi) - \sum_{k=1}^{i-1} g_{2-\beta}(t_{i+1-k})\left(\delta^2 u_k, \varphi\right)\tau$$
$$+ \frac{1}{\tau} g_{2-\beta}(\tau)\,(u_{i-1},\varphi) + g_{2-\beta}(\tau)\,(\delta u_{i-1},\varphi)\,. \quad (5.7)$$

When $u_1, ..., u_{i-1} \in L^2(\Omega)$, then, with the assumptions on $f$, $u_0$ and $v_0$, the r.h.s. of the equation can be seen as a linear bounded functional on $H^1(\Omega)$, moreover, the l.h.s. of the equation is a bounded bilinear form

$$B[u_i,\varphi] := \frac{1}{\tau} g_{2-\beta}(\tau)\,(u_i,\varphi) + (\nabla u_i, \nabla\varphi)\,,$$

on $H^1(\Omega) \times H^1(\Omega)$ with $B[\varphi,\varphi] \geq C\,\|\varphi\|^2_{H^1(\Omega)}$. Using the Lax-Milgram lemma iteratively, we can conclude that there exist unique $u_i \in H^1(\Omega)$ solving (DP$i$). Now, we want to prove that $u_i \in \boldsymbol{V}$. Looking again at the equation (DP$i$), the term $(\nabla u_i, \nabla\varphi)$ can be understood as a realization of a linear bounded functional on $H^1(\Omega)$. From the Hahn-Banach theorem there exists an extension of that functional on $L^2(\Omega)$ with the same norm. The Riesz theorem says that this extension can be represented uniquely by a function from $L^2(\Omega)$, we denote this function as $-\Delta u_i$. We may write

$$-\left(\Delta u_i, \varphi\right) = h_i\,(f,\varphi) - \left(\left(g_{2-\beta} * \delta^2 u\right)_i, \varphi\right), \quad (5.8)$$

for every $\varphi \in L^2(\Omega)$, so,

$$-\Delta u_i = h_i f - \left(g_{2-\beta} * \delta^2 u\right)_i \in L^2(\Omega),$$

and using the assumptions of the lemma and applying the gradient on this equality leads to

$$-\nabla\Delta u_i = h_i \nabla f - \left(g_{2-\beta} * \nabla\delta^2 u\right)_i \in L^2(\Omega). \quad (5.9)$$

□

Now, we can work properly with $(u_i, h_i)$. Our next aim is to gain some estimates of them.

**Lemma 5.3.2.** *Under the assumptions of Lemma 5.3.1 there exists a positive constant $C$ (independent of $n$) such that*

$$\left(g_{2-\beta} * \|\delta u\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i)\,\|\delta u_i\|^2\,\tau + \sum_{i=1}^{j}\|\delta u_i\|^2\,\tau + \|u_j\|^2_{H^1(\Omega)} + \sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2 \le C\sum_{i=1}^{j} h_i^2\tau, \quad (5.10)$$

*for every $j \in 1, ..., n$, $n \in \mathbb{N}$.*

*Proof.* Let $\varphi = \delta u_i$ in (DP$i$), using the equality (2.10) we get that

$$\left(\delta\left(g_{2-\beta} * \delta u\right)_i, \delta u_i\right) + \left(\nabla u_i, \nabla \delta u_i\right) = h_i\left(f, \delta u_i\right).$$

Multiplying the equality by $\tau$ and summing it up for $i = 1...j$, we obtain

$$\sum_{i=1}^{j}\left(\delta\left(g_{2-\beta} * \delta u\right)_i, \delta u_i\right)\tau + \sum_{i=1}^{j}\left(\nabla u_i, \nabla u_i - \nabla u_{i-1}\right) = \sum_{i=1}^{j} h_i\left(f, \delta u_i\right)\tau. \quad (5.11)$$

Next, we use Lemma 1.7.3 for the first term on the l.h.s. of (5.11) and the Abel summation 1.3.7 for the second term. Moreover, the Young inequality is used on the r.h.s. in (5.11), hence, we get

$$\frac{1}{2}\left(g_{2-\beta} * \|\delta u\|^2\right)_j + \frac{1}{4}\sum_{i=1}^{j} g_{2-\beta}(t_i)\,\|\delta u_i\|^2\,\tau + \frac{g_{2-\beta}(T)}{4}\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau + \frac{1}{2}\|\nabla u_j\|^2 + \frac{1}{2}\sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2 \le C_\varepsilon \sum_{i=1}^{j} h_i^2\tau + \varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau.$$

Choosing suitable $\varepsilon > 0$, we derive

$$\left(g_{2-\beta} * \|\delta u\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i)\,\|\delta u_i\|^2\,\tau + \sum_{i=1}^{j}\|\delta u_i\|^2\,\tau + \|\nabla u_j\|^2 + \sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2 \le C\sum_{i=1}^{j} h_i^2\tau.$$

□

**Lemma 5.3.3.** *Under the assumptions of Lemma 5.3.1 there exists a positive constant $C$ (independent of $n$) such that*

$$\left(g_{2-\beta} * \|\nabla \delta u\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau + \|\Delta u_j\|^2 + \sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 \leq C \sum_{i=1}^{j} h_i^2 \tau,$$

*for every $j \in 1, ..., n$, $n \in \mathbb{N}$.*

*Proof.* To gain the estimate from the lemma we start with the equation (5.8) from the proof of Lemma 5.3.1. We set $\varphi = -\Delta \delta u_i$, which is justified since $u_i \in \boldsymbol{V}$, for $0 \leq i \leq j$,

$$\left(\left(g_{2-\beta} * \delta^2 u\right)_i, -\Delta \delta u_i\right) + (\Delta u_i, \Delta \delta u_i) = h_i\left(f, -\Delta \delta u_i\right).$$

This can be rewritten as

$$\left(\delta\left(g_{2-\beta} * \nabla \delta u\right)_i, \nabla \delta u_i\right) + (\Delta u_i, \Delta \delta u_i) = h_i\left(\nabla f, \nabla \delta u_i\right),$$

multiplying by $\tau$ and summing up for $1 \leq i \leq j$, we get

$$\sum_{i=1}^{j} \left(\delta\left(g_{2-\beta} * \nabla \delta u\right)_i, \nabla \delta u_i\right) \tau + \sum_{i=1}^{j} (\Delta u_i, \Delta \delta u_i) \tau = \sum_{i=1}^{j} h_i\left(\nabla f, \nabla \delta u_i\right) \tau.$$

This can be estimated in the similar way as in the previous lemma, with the help of Lemma 1.7.3, the Abel summation, the Cauchy and Young inequalities, we obtain that

$$\left(g_{2-\beta} * \|\nabla \delta u\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau + \|\Delta u_j\|^2 + \sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 \leq C_\varepsilon \sum_{i=1}^{j} h_i^2 \tau + \varepsilon \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau,$$

the estimate from the lemma is acquired by choosing an appropriate $\varepsilon > 0$. □

**Lemma 5.3.4.** *Under the assumptions of Lemma 5.3.1, if moreover $f \in H^2(\Omega)$ and $\nabla f \cdot \boldsymbol{\nu} = 0$ on $\Gamma$ then there exists a positive constant $C$ (independent of $n$) such that*

$$\left(g_{2-\beta} * \|\Delta\delta u\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i)\,\|\Delta\delta u_i\|^2\,\tau + \sum_{i=1}^{j} \|\Delta\delta u_i\|^2\,\tau + \|\nabla\Delta u_j\|^2$$
$$+ \sum_{i=1}^{j} \|\nabla\Delta u_i - \nabla\Delta u_{i-1}\|^2 \le C \sum_{i=1}^{j} h_i^2 \tau,$$

*for every* $j \in 1, ..., n,\ n \in \mathbb{N}$*.*

*Proof.* Starting from (5.9), we multiply the equality by $-\nabla\delta\Delta u_i$, integrate over the domain $\Omega$ and get

$$\left(\left(g_{2-\beta} * \nabla\delta^2 u\right)_i, -\nabla\delta\Delta u_i\right) + \left(\nabla\Delta u_i, \nabla\delta\Delta u_i\right) = h_i\left(\nabla f, -\nabla\delta\Delta u_i\right).$$

The equalities for $1 \le i \le j$, $j \in 1, ..., n$, are multiplied by $\tau$ and summed up to obtain

$$\sum_{i=1}^{j} \left(\left(g_{2-\beta} * \nabla\delta^2 u\right)_i, -\nabla\delta\Delta u_i\right)\tau + \sum_{i=1}^{j} \left(\nabla\Delta u_i, \nabla\delta\Delta u_i\right)\tau = \sum_{i=1}^{j} h_i\left(\nabla f, -\nabla\delta\Delta u_i\right)\tau. \tag{5.12}$$

Next, the first term on the l.h.s. of (5.12) is rewritten using the Green theorem and (2.10), then Lemma 1.7.3 is applied. For the second term on the l.h.s. the Abel summation is used. For the r.h.s of the equality (5.12), first the Green theorem is applied, and then the Cauchy and Young inequality are used to acquire

$$\left(g_{2-\beta} * \|\delta\Delta u\|^2\right)_i + \sum_{i=1}^{j} g_{2-\beta}(t_i)\,\|\delta\Delta u_i\|^2\,\tau + \sum_{i=1}^{j} \|\delta\Delta u_i\|^2\,\tau + \|\nabla\Delta u_j\|^2$$
$$+ \sum_{i=1}^{j} \|\nabla\Delta u_i - \nabla\Delta u_{i-1}\|^2 \le C_\varepsilon \sum_{i=1}^{j} h_i^2 \tau + \varepsilon \sum_{i=1}^{j} \|\delta\Delta u_i\|^2\,\tau, \tag{5.13}$$

choosing the appropriate $\varepsilon > 0$ leads us to the estimate from the lemma. □

The next lemma aggregates the results of Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.4 in to the final estimate.

**Lemma 5.3.5.** *Under the assumptions of Lemma 5.3.4, there exists a positive constants* $C$ *(independent of* $n$*) such that*

$$
\begin{aligned}
(i)\quad & \max_{0\le i\le n}\left(g_{2-\beta}*\|\delta u\|^2\right)_i+\sum_{i=1}^n g_{2-\beta}(t_i)\|\delta u_i\|^2\tau+\sum_{i=1}^n\|\delta u_i\|^2\tau\\
&+\max_{0\le i\le n}\|u_i\|^2_{H^1(\Omega)}+\sum_{i=1}^n\|\nabla u_i-\nabla u_{i-1}\|^2+\max_{0\le i\le n}\left(g_{2-\beta}*\|\nabla\delta u\|^2\right)_i\\
&+\sum_{i=1}^n g_{2-\beta}(t_i)\|\nabla\delta u_i\|^2\tau+\sum_{i=1}^n\|\nabla\delta u_i\|^2\tau+\max_{0\le i\le n}\|\Delta u_i\|^2_{H^1(\Omega)}\\
&+\sum_{i=1}^n\|\Delta u_i-\Delta u_{i-1}\|^2+\max_{0\le i\le n}\left(g_{2-\beta}*\|\Delta\delta u\|^2\right)_i+\sum_{i=1}^n g_{2-\beta}(t_i)\|\Delta\delta u_i\|^2\tau\\
&+\sum_{i=1}^n\|\Delta\delta u_i\|^2\tau+\sum_{i=1}^n\|\nabla\Delta u_i-\nabla\Delta u_{i-1}\|^2\le C.\\
(ii)\quad & \max_{0\le i\le n}|h_i|\le C
\end{aligned}
$$

*Proof.* Starting from the equation (DMP$i$), we can estimate

$$
|h_i|=C\left(1+\|\Delta u_{i-1}\|_\Gamma\right)\le C\left(1+\|\Delta u_{i-1}\|+\|\nabla\Delta u_{i-1}\|\right), \tag{5.14}
$$

where the inequality comes from the trace theorem. By summing all estimates from Lemma 5.3.2- Lemma 5.3.4 up and using (5.14), we are prepared to use the discrete Grönwall lemma to obtain the inequality $(i)$ and consequently also $(ii)$. □

In following set of a priori estimates, we will work with a difference of the discretized equations; therefore, we additionally need to define

$$
h_0=0. \tag{5.15}
$$

**Lemma 5.3.6.** *Under the assumptions of Lemma 5.3.1 there exists a positive constant $C$ (independent of $n$) such that*

$$
\begin{aligned}
&\left(g_{2-\beta}*\left\|\delta^2u\right\|^2\right)_j+\sum_{i=1}^j g_{2-\beta}(t_i)\left\|\delta^2u_i\right\|^2\tau+\sum_{i=1}^j\left\|\delta^2u_i\right\|^2\tau\\
&\qquad+\|\delta u_j\|^2_{H^1(\Omega)}+\sum_{i=1}^j\|\nabla\delta u_i-\nabla\delta u_{i-1}\|^2\le C\sum_{i=1}^j|\delta h_i|\left\|\delta^2u_i\right\|\tau,
\end{aligned}
$$

*for every $j\in 1,\ldots,n$, $n\in\mathbb{N}$.*

*Proof.* Subtracting equation (DP$i$) for $i-1$ from the one for $i$ and dividing by $\tau$ gives us

$$
\left(\delta\left(g_{2-\beta}*\delta^2u\right)_i,\varphi\right)+\left(\nabla\delta u_i,\nabla\varphi\right)=\delta h_i\left(f,\varphi\right).
$$

Notice that for $i = 1$ the above difference is the equation itself as $\left(g_{2-\beta} * \delta^2 u\right)_0 = 0$, $u_0 = 0$ and $h_0 = 0$. We set $\varphi = \delta^2 u_i \tau$ and sum up equations for $1 \le i \le j$. By using Lemma 1.7.3, the Abel summation and Cauchy inequality, we gain the estimate from the lemma for $j \in \{1, ..., n\}$. □

**Lemma 5.3.7.** *Under the assumptions of Lemma 5.3.1 there exists a positive constant $C$ (independent of $n$) such that*

$$\left(g_{2-\beta} * \left\|\nabla \delta^2 u\right\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i) \left\|\nabla \delta^2 u_i\right\|^2 \tau + \sum_{i=1}^{j} \left\|\nabla \delta^2 u_i\right\|^2 \tau$$
$$+ \left\|\Delta \delta u_j\right\|^2 + \sum_{i=1}^{j} \left\|\Delta \delta u_i - \Delta \delta u_{i-1}\right\|^2 \le C \sum_{i=1}^{j} |\delta h_i| \left\|\nabla \delta^2 u_i\right\| \tau$$

*for every $j \in 1, ..., n$, $n \in \mathbb{N}$.*

*Proof.* Similarly as in the previous lemma, we make an difference, now, for $\Delta u_i$, using (5.8) we get that

$$\left(\delta \left(g_{2-\beta} * \delta^2 u\right)_i, \varphi\right) - \left(\Delta \delta u_i, \varphi\right) = \delta h_i \left(f, \varphi\right).$$

Setting $\varphi = -\Delta \delta^2 u_i \tau$, using the Green theorem and summing up for $1 \le i \le j$, we obtain

$$\sum_{i=1}^{j} \left(\delta \left(g_{2-\beta} * \nabla \delta^2 u\right)_i, \nabla \delta^2 u_i\right) \tau + \sum_{i=1}^{j} \left(\Delta \delta u_i, \Delta \delta^2 u_i\right) \tau = \sum_{i=1}^{j} \delta h_i \left(\nabla f, \nabla \delta^2 u_i\right) \tau.$$

Using Lemma 1.7.3, the Abel summation and Cauchy inequality leads us to the estimate in the lemma.

□

**Lemma 5.3.8.** *Under the assumptions of Lemma 5.3.4, there exists a positive constant $C$ (independent of $n$) such that*

$$\left(g_{2-\beta} * \left\|\Delta \delta^2 u\right\|^2\right)_j + \sum_{i=1}^{j} g_{2-\beta}(t_i) \left\|\Delta \delta^2 u_i\right\|^2 \tau + \sum_{i=1}^{j} \left\|\Delta \delta^2 u_i\right\|^2 \tau$$
$$+ \left\|\nabla \Delta \delta u_j\right\|^2 + \sum_{i=1}^{j} \left\|\nabla \Delta \delta u_i - \nabla \Delta \delta u_{i-1}\right\|^2 \le C \sum_{i=1}^{j} |\delta h_i| \left\|\Delta \delta^2 u_i\right\| \tau,$$

*for every $j \in 1, ..., n$, $n \in \mathbb{N}$.*

*Proof.* First, we make an difference from (5.9) to get

$$\delta\left(g_{2-\beta} * \nabla\delta^2 u\right)_i - \nabla\Delta\delta u_i = \delta h_i \nabla f,$$

then we multiply by $-\nabla\Delta\delta^2 u_i \tau$ and integrate over $\Omega$ to obtain

$$\left(\delta\left(g_{2-\beta} * \nabla\delta^2 u\right)_i, \nabla\Delta\delta^2 u_i\right)\tau + \left(\nabla\Delta\delta u_i, \nabla\Delta\delta^2 u_i\right)\tau = \delta h_i\left(\nabla f, \nabla\Delta\delta^2 u_i\right)\tau.$$

Using the Green theorem for the first term on the l.h.s. and for the term on the r.h.s, and then summing up for $1 \le i \le j$ we get

$$\sum_{i=1}^{j}\left(\delta\left(g_{2-\beta} * \Delta\delta^2 u\right)_i, \Delta\delta^2 u_i\right)\tau + \sum_{i=1}^{j}\left(\nabla\Delta\delta u_i, \nabla\Delta\delta^2 u_i\right)\tau = \sum_{i=1}^{j}\delta h_i\left(\Delta f, \Delta\delta^2 u_i\right)\tau.$$

We acquire the estimate from the lemma by the same manner as we did in the last step of the proof of Lemma 5.3.7. □

**Lemma 5.3.9.** *Under the assumptions of Lemma 5.3.4, if moreover it holds that $m \in C^3([0,T])$, then there exists a positive constant $C$ (independent of $n$) such that*

$$\begin{aligned}
(i)\quad & \max_{0\le i\le n}\left(g_{2-\beta} * \left\|\delta^2 u\right\|^2\right)_i + \sum_{i=1}^{n} g_{2-\beta}(t_i)\left\|\delta^2 u_i\right\|^2\tau + \sum_{i=1}^{n}\left\|\delta^2 u_i\right\|^2\tau \\
& + \max_{0\le i\le n}\|\delta u_i\|^2_{H^1(\Omega)} + \sum_{i=1}^{n}\|\nabla\delta u_i - \nabla\delta u_{i-1}\|^2 + \max_{0\le i\le n}\left(g_{2-\beta} * \left\|\nabla\delta^2 u\right\|^2\right)_i \\
& + \sum_{i=1}^{n} g_{2-\beta}(t_i)\left\|\nabla\delta^2 u_i\right\|^2\tau + \sum_{i=1}^{n}\left\|\nabla\delta^2 u_i\right\|^2\tau + \max_{0\le i\le n}\|\Delta\delta u_i\|^2_{H^1(\Omega)} \\
& + \max_{0\le i\le n}\left(g_{2-\beta} * \left\|\Delta\delta^2 u\right\|^2\right)_i + \sum_{i=1}^{n} g_{2-\beta}(t_i)\left\|\Delta\delta^2 u_i\right\|^2\tau + \sum_{i=1}^{n}\left\|\Delta\delta^2 u_i\right\|^2\tau \\
& + \sum_{i=1}^{n}\|\Delta\delta u_i - \Delta\delta u_{i-1}\|^2_{H^1(\Omega)} \le C \\
(ii)\quad & |\delta h_i| \le C\left(1 + g_{2-\beta}(t_i)\right).
\end{aligned} \tag{5.16}$$

*Proof.* First, we estimate the difference of $h_i$. We get from (5.15) and (DMP$i$)

$$\delta\left(g_{2-\beta} * m''\right)_i - \left(\Delta\delta u_{i-1}, \omega\right)_\Gamma = \delta h_i\left(f, \omega\right)_\Gamma,$$

eliminating $\delta h_i$ from it and estimating the absolute value of it using the trace theorem and (2.10) gives us

$$\begin{aligned}
|\delta h_i| &\le C\left(g_{2-\beta}(t_i)m_0'' + \left(g_{2-\beta} * |\delta m''|\right)_i + \|\Delta\delta u_{i-1}\|_\Gamma\right) \\
&\le C\left(1 + g_{2-\beta}(t_i) + \|\Delta\delta u_{i-1}\| + \|\nabla\Delta\delta u_{i-1}\|\right).
\end{aligned}$$

Next, we sum all the results from Lemma 5.3.6 to Lemma 5.3.8 up, and on the r.h.s we can use the above estimate and the Young inequality to obtain

$$
\begin{aligned}
&\sum_{i=1}^{j} |\delta h_i| \left(\left\|\delta^2 u_i\right\| + \left\|\nabla\delta^2 u\right\| + \left\|\Delta\delta^2 u_i\right\|\right)\tau \\
&\le \sum_{i=1}^{j} \left(1 + g_{2-\beta}(t_i) + \|\Delta\delta u_{i-1}\| + \|\nabla\Delta\delta u_{i-1}\|\right) \Big( \left\|\delta^2 u_i\right\| + \left\|\nabla\delta^2 u\right\| \\
&\quad + \left\|\Delta\delta^2 u_i\right\| \Big)\tau \\
&\le \sum_{i=1}^{j} g_{2-\beta}(t_i) \left(\left\|\delta^2 u_i\right\| + \left\|\nabla\delta^2 u\right\| + \left\|\Delta\delta^2 u_i\right\|\right)\tau \\
&\quad + \sum_{i=1}^{j} \left(1 + \|\Delta\delta u_{i-1}\| + \|\nabla\Delta\delta u_{i-1}\|\right) \left(\left\|\delta^2 u_i\right\| + \left\|\nabla\delta^2 u\right\| + \left\|\Delta\delta^2 u_i\right\|\right)\tau \\
&\le C_\varepsilon \sum_{i=1}^{j} g_{2-\beta}(t_i)\tau + \varepsilon \sum_{i=1}^{j} g_{2-\beta}(t_i) \left(\left\|\delta^2 u_i\right\|^2 + \left\|\nabla\delta^2 u\right\|^2 + \left\|\Delta\delta^2 u_i\right\|^2\right)\tau \\
&\quad + C_\varepsilon \sum_{i=1}^{j} \left(1 + \|\Delta\delta u_{i-1}\|^2 + \|\nabla\Delta\delta u_{i-1}\|^2\right)\tau \\
&\quad + \varepsilon \sum_{i=1}^{j} \left(\left\|\delta^2 u_i\right\|^2 + \left\|\nabla\delta^2 u\right\|^2 + \left\|\Delta\delta^2 u_i\right\|^2\right)\tau.
\end{aligned}
$$

Now, we choose the appropriate $\varepsilon > 0$ on the r.h.s and move the terms to the l.h.s, then we are prepared to use the discrete Grönwall lemma to get $(i)$, consequently, we obtain also $(ii)$. $\square$

Next step is to introduce functions which helps us to define the approximate solution on the whole time frame. We define them as $u_n, \overline{u}_n, \tilde{u}_n : [0,T] \to \boldsymbol{V}$ and $v_n, \overline{v}_n : [0,T] \to \boldsymbol{V}$ with the prescription defined in the previous chapters, assuming $u_0 = 0$, $v_0 = 0$. In the similar way we also define functions $h_n, \overline{h}_n, \overline{g_{2-\beta}}_n, \overline{m''}_n$. As we told before, with those definitions we can extend the discretized solution to the whole interval $[0,T]$, so we rewrite (DP$i$) and (DMP$i$) to

$$
\left(\left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t_i), \varphi\right) + \left(\nabla\overline{u}_n(t), \nabla\varphi\right) = \overline{h}_n(t)\left(f, \varphi\right), \tag{DP}
$$

and

$$
\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) + \left(\Delta\tilde{u}_n(t), \omega\right)_\Gamma = \overline{h}_n(t)\left(f, \omega\right)_\Gamma, \tag{DMP}
$$

for $t \in (t_{i-1}, t_i]$. With the above definition and all the estimates we have, we may proceed to the existence theorem. We will prove that the subsequences of Rothe functions converge to a functions $u$ and $h$, and that (DP) and (DMP) converge to

(P) and (MP), respectively, so the functions $u$ and $h$ are then a solution of our problem.

**Theorem 5.3.1** (existence of a solution). *Let $\omega \in L^2(\Gamma)$, $f \in H^2(\Omega)$ with $\nabla f \cdot \boldsymbol{\nu} = 0$ on $\Gamma$, $(f,\omega)_\Gamma \neq 0$ and $m \in C^3([0,T])$.*

*Then there exists a solution $(u,h)$ to the (P), (MP) obeying $u \in C\left([0,T], \boldsymbol{V}\right)$ with $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), \boldsymbol{V}\right)$, $\partial_{tt} u \in L^2\left((0,T), H^1(\Omega)\right)$, $\partial_{tt}\Delta u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* Based on the estimate $(ii)$ of Lemma 5.3.9, we get

$$|h_n'(t)| = |\delta h_i| \leq C t_i^{1-\beta} + C \leq C t^{1-\beta} + C,$$

for $t \in (t_{i-1}, t_i]$. Then for $t, s \in [0,T]$, such that $|t-s| \leq \varepsilon$, for $\varepsilon > 0$, it holds

$$\begin{aligned}|h_n(t) - h_n(s)| \leq \left| \int_s^t |h_n'(r)| \,\mathrm{d}r \right| &\leq C \left| \int_s^t \left(r^{1-\beta} + 1\right) \mathrm{d}r \right| \\ &= C \frac{\left|t^{2-\beta} - s^{2-\beta}\right|}{2-\beta} + \varepsilon C = C(\varepsilon^{2-\beta} + \varepsilon),\end{aligned}$$

which means that sequence $\{h_n\}$ is uniform equi-continuous. The equi-boundedness of the sequence is obtained from the estimate $(ii)$ of Lemma 5.3.5. The Arzelà-Ascoli theorem 1.2.5 gives us the existence of $h \in C([0,T])$ to which the subsequence $\{h_{n_k}\}$ (from now on denoted as $\{h_n\}$) converges in $C([0,T])$.

From Lemma 5.3.5$(i)$, we obtain the estimate of the Rothe functions $u_n, \overline{u}_n$

$$\max_{0 \leq t \leq T} \|\overline{u}_n(t)\|_{\boldsymbol{V}}^2 + \max_{0 \leq t \leq T} \|\partial_t u_n(t)\|^2 \leq C,$$

since $\boldsymbol{V} \Subset L^2(\Omega)$, we can use Lemma 1.5.1, which says that there exist $u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), \boldsymbol{V}\right)$ such that $\partial_t u \in L^2\left((0,T), L^2(\Omega)\right)$ and subsequences $\{u_{n_k}\}_{k\in\mathbb{N}}$, $\{\overline{u}_{n_k}\}_{k\in\mathbb{N}}$ (from now on indexed by $n$, for the sake of simplicity) for which it holds

$$\begin{cases} u_n \to u, & \text{in} \quad C\left([0,T], L^2(\Omega)\right) & (5.17\text{a}) \\ u_n(t) \rightharpoonup u(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t \in [0,T] & (5.17\text{b}) \\ \overline{u}_n(t) \rightharpoonup u(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t \in [0,T] & (5.17\text{c}) \\ \partial_t u_n \rightharpoonup \partial_t u, & \text{in} \quad L^2\left((0,T), L^2(\Omega)\right). & (5.17\text{d}) \end{cases}$$

Moreover, for $\partial_t u$ we have the estimate $\max_{t\in[0,T]} \|\partial_t u(t)\|_{\boldsymbol{V}} \leq C$ also from Lemma 5.3.9$(i)$. The estimate gives us the boundedness of $\partial_t u$ in the reflexive space $L^2\left((0,T), \boldsymbol{V}\right)$. Therefore, for a subsequence of $\{\partial_t u_n\}$, we get that

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{in} \quad L^2\left((0,T), \boldsymbol{V}\right),$$

and, consequently,

$$u(t) - u(s) = \int_s^t \partial_t u(r)\, \mathrm{d}r$$
$$\Rightarrow \|u(t) - u(s)\|_{\boldsymbol{V}} \le |t-s|^{\frac{1}{2}} \left( \int_0^T \|\partial_t u(r)\|_{\boldsymbol{V}}^2 \, \mathrm{d}r \right)^{\frac{1}{2}} \le C\,|t-s|^{\frac{1}{2}},$$

so we have $u \in C\left([0,T], \boldsymbol{V}\right)$.

Furthermore, from Lemma 5.3.9$(i)$ we gain

$$\max_{0\le t\le T} \|\overline{v}_n(t)\|_{\boldsymbol{V}}^2 + \int_0^T \|\partial_t v_n(t)\|^2 \le C,$$

using the Lemma 1.5.1 we are obtaining $v \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), \boldsymbol{V}\right)$ with $\partial_t v \in L^2\left((0,T), L^2(\Omega)\right)$ and subsequences $\{v_{n_k}\}_{k\in\mathbb{N}}$, $\{\overline{v}_{n_k}\}_{k\in\mathbb{N}}$ (from now on indexed by $n$) such that

$$\begin{cases} v_n \to v, & \text{in} \quad C\left([0,T], L^2(\Omega)\right) & (5.18a) \\ v_n(t) \rightharpoonup v(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t \in [0,T] & (5.18b) \\ \overline{v}_n(t) \rightharpoonup v(t), & \text{in} \quad \boldsymbol{V}, \quad \forall t \in [0,T] & (5.18c) \\ \partial_t v_n \rightharpoonup \partial_t v, & \text{in} \quad L^2\left((0,T), L^2(\Omega)\right). & (5.18d) \end{cases}$$

To see the connection between $u$ and $v$, we start with the equality

$$(u_n(t) - u_0, \varphi) = \int_0^t (\overline{v}_n(s), \varphi)\, ds, \qquad \text{for} \;\; \varphi \in L^2(\Omega),$$

since $\partial_t u_n = \overline{v}_n$, by passing the limit for $n \to \infty$ it is obtained

$$(u(t) - u_0, \varphi) = \int_0^t (v(s), \varphi)\, ds, \qquad \text{for} \;\; \varphi \in L^2(\Omega).$$

From this we see that $v(t) = \partial_t u(t)$ in $L^2(\Omega)$ for a.a. $t \in [0,T]$.

Next, from Lemma 5.3.9 (i) we have the estimate $\sum_{i=1}^n \left\|\Delta\delta^2 u_i\right\|^2 \tau \le C$ that can be rewritten as

$$\int_0^T \|\Delta\partial_t v_n(t)\|^2 \, \mathrm{d}t \le C,$$

and that together with the reflexivity of $L^2\left((0,T), L^2(\Omega)\right)$ imply the weak convergence of a subsequence of $\{\Delta\partial_t v_n\}$ (indexed again by $n$) to $z \in L^2\left((0,T), L^2(\Omega)\right)$. Since it holds that

$$\int_0^T (\Delta\partial_t v_n(t), \varphi)\, \mathrm{d}t = \int_0^T (\partial_t v_n(t), \Delta\varphi)\, \mathrm{d}t,$$

for every $\varphi \in C_0^\infty(\Omega)$, by passing to the limit $n \to \infty$ we obtain

$$\int_0^T (z(t), \varphi) \, \mathrm{d}t = \int_0^T (\partial_t v(t), \Delta\varphi) \, \mathrm{d}t = \int_0^T (\Delta\partial_t v(t), \varphi) \, \mathrm{d}t,$$

for all $\varphi \in C_0^\infty(\Omega)$, so $\Delta\partial_{tt} u = z \in L^2\left((0,T), L^2(\Omega)\right)$. Analogously we get similar result for $\nabla\partial_{tt} u$.

The rest of the proof will consist of proving the convergence of (DMP) and (DP) to (MP) and (P), respectively. First,

$$\begin{aligned}
&\left|\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t)\right| \\
&\le \left|\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\overline{m''}_n(s) \, \mathrm{d}s\right| + \left|\int_0^t \left(\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right) \overline{m''}_n(s) \, \mathrm{d}s\right| \\
&\le C \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \, \mathrm{d}s + C \int_0^t \left|\overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t-s)\right| \, \mathrm{d}s.
\end{aligned}$$

As $\overline{g_{2-\beta}}_n \to g_{2-\beta}$ in $(0,T)$ pointwise, the Lebesgue dominated theorem gives

$$\left(\overline{g_{2-\beta}}_n * \overline{m''}_n\right)(t_i) \to \left(g_{2-\beta} * m''\right)(t).$$

Next, notice that since $\max_{0\le i\le n} \|\Delta\delta u_i\|^2_{H^1(\Omega)} \le C$, we get

$$\int_0^T \|\Delta\tilde{u}_n(t) - \Delta\overline{u}_n(t)\|_\Gamma \, \mathrm{d}t \le \int_0^T \|\Delta\tilde{u}_n(t) - \Delta\overline{u}_n(t)\|_{H^1(\Omega)} \, \mathrm{d}t = \mathcal{O}(\tau).$$

Thanks to Lemma 5.3.9 $(ii)$, it also holds that

$$\int_0^T \left|\overline{h}_n(t) - h_n(t)\right| \, \mathrm{d}t = \mathcal{O}(\tau).$$

We next integrate (DMP) for $\xi \in [0,T]$, and, thanks to the above facts and the convergences we got, we can pass to the limit $n \to \infty$. Then by the differentiation with respect to $\xi$ we obtain (MP).

The problematic term in (DP) is the first one on the l.h.s., several estimates need to be done to be able to pass the limit, we start with

$$\left|\int_0^\xi \left(\left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t_i) - \left(\overline{g_{2-\beta}}_n * \partial_t v_n\right)(t), \varphi\right) \, \mathrm{d}t\right|$$

$$
\begin{aligned}
&\leq \int_0^\xi \left| \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\, (\partial_t v_n(s), \varphi)\ \mathrm{d}s \right| dt \\
&\quad + \int_0^\xi \left| \int_0^t \left( \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right) (\partial_t v_n(s), \varphi)\ \mathrm{d}s \right| \mathrm{d}t \\
&\leq \int_0^\xi \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \left\| \partial_t v_n(s) \right\| \left\| \varphi \right\| \mathrm{d}s\, \mathrm{d}t \\
&\quad + \int_0^\xi \int_0^t \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \left\| \partial_t v_n(s) \right\| \left\| \varphi \right\| \mathrm{d}s\, \mathrm{d}t.
\end{aligned} \tag{5.19}
$$

We use Hölder's inequality and Lemma 5.3.9 for the first term on the r.h.s. to get

$$
\begin{aligned}
&\int_0^\xi \int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \left\| \partial_t v_n(s) \right\| \left\| \varphi \right\| \mathrm{d}s\, \mathrm{d}t \\
&\leq \left\| \varphi \right\| \int_0^\xi \sqrt{\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s)\, \mathrm{d}s} \sqrt{\int_t^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \left\| \partial_t v_n(s) \right\|^2 \mathrm{d}s\, \mathrm{d}t} \\
&\leq \left\| \varphi \right\| \sqrt{\tau^{2-\beta}} \int_0^\xi \sqrt{\int_0^{t_i} \overline{g_{2-\beta}}_n(t_i - s) \left\| \partial_t v_n(s) \right\|^2 \mathrm{d}s ds} \\
&\leq C \left\| \varphi \right\| \sqrt{\tau^{2-\beta}}.
\end{aligned}
$$

The second term in (5.19) is estimated after switching the order of integration and using Hölder's inequality, as follows

$$
\begin{aligned}
&\int_0^\xi \int_0^t \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \left\| \partial_t v_n(s) \right\| \left\| \varphi \right\| \mathrm{d}s\, \mathrm{d}t \\
&\leq \left\| \varphi \right\| \int_0^\xi \int_s^\xi \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \left\| \partial_t v_n(s) \right\| \mathrm{d}t\, \mathrm{d}s \\
&\leq \left\| \varphi \right\| \int_0^\xi \left\| \partial_t v_n(s) \right\| \int_s^\xi \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \mathrm{d}t\, \mathrm{d}s \\
&\leq \left\| \varphi \right\| \sqrt{\int_0^\xi \left\| \partial_t v_n(s) \right\|^2 \mathrm{d}s} \sqrt{\int_0^\xi \left( \int_s^\xi \left| \overline{g_{2-\beta}}_n(t_i - s) - \overline{g_{2-\beta}}_n(t - s) \right| \mathrm{d}t \right)^2 \mathrm{d}s} \\
&\leq C \left\| \varphi \right\|.
\end{aligned}
$$

The fact that $\overline{g_{2-\beta}}_n \to g_{2-\beta}$ in $(0, T)$ pointwise enables using of Lebesgue's convergence theorem and brings

$$
\lim_{n \to \infty} \left| \int_0^\xi \left( \left( \overline{g_{2-\beta}}_n * \partial_t v_n \right)(t_i) - \left( \overline{g_{2-\beta}}_n * \partial_t v_n \right)(t), \varphi \right) \mathrm{d}t \right| = 0.
$$

Next, we apply the Cauchy, Hölder and Young inequalities to get

$$\left|\int_0^\xi \left(\left[\overline{g_{2-\beta}}_n - g_{2-\beta}\right] * (\partial_t v_n, \varphi)\right)(t)\,\mathrm{d}t\right|$$
$$\leq \int_0^\xi \left|\overline{g_{2-\beta}}_n(t) - g_{2-\beta}(t)\right|\,\mathrm{d}t \sqrt{\int_0^\xi \|\partial_t v_n(t)\|^2\,\mathrm{d}t}\sqrt{\int_0^\xi \|\varphi\|^2\,\mathrm{d}t} \leq C\,\|\varphi\|\,, \qquad (5.20)$$

and by using Lebesgue's convergence theorem, we acquire

$$\lim_{n\to\infty}\left|\int_0^\xi \left(\left[\overline{g_{2-\beta}}_n - g_{2-\beta}\right] * (\partial_t v_n, \varphi)\right)(t)\,\mathrm{d}t\right| = 0.$$

Furthermore,

$$\left|\int_0^\xi (g_{2-\beta} * (\partial_t v_n, \varphi))(t)\,\mathrm{d}t\right| \leq \int_0^\xi g_{2-\beta}(t)\,\mathrm{d}t \sqrt{\int_0^\xi \|\partial_t v_n(t)\|^2\,\mathrm{d}t}\sqrt{\int_0^\xi \|\varphi\|^2\,\mathrm{d}t}$$
$$\leq C\,\|\partial_t v_n\|_{L^2((0,T),L^2(\Omega))}\,\|\varphi\|\,,$$

which means that the estimated term can be seen as the linear bounded functional on $L^2\left((0,T), L^2(\Omega)\right)$, and using (5.18d), we arrive to

$$\lim_{n\to\infty}\int_0^\xi (g_{2-\beta} * (\partial_t v_n, \varphi))(t)\,\mathrm{d}t = \int_0^\xi (g_{2-\beta} * (\partial_t v, \varphi))(t)\,\mathrm{d}t.$$

In the last step we integrate (DP) over $(0,\xi)$ and pass to the limit $n\to\infty$ to obtain

$$\int_0^\xi ((g_{2-\beta} * \partial_{tt} u)(t), \varphi)\,\mathrm{d}t + \int_0^\xi (\nabla u(t), \nabla\varphi)\,\mathrm{d}t = \int_0^\xi h(t)\,(f,\varphi)\,\mathrm{d}t, \qquad (5.21)$$

where we used the estimates, convergences and relations above. Differentiation of the equality (5.21) with respect to $\xi$ brings (P). □

## 5.4 Uniqueness

In this section we will prove the uniqueness of the solution in the appropriate spaces.

**Theorem 5.4.1** (uniqueness)**.** *Let $f \in H^2(\Omega)$ with $\nabla f \cdot \boldsymbol{\nu} = 0$ on $\Gamma$, $\omega \in L^2(\Gamma)$, $(f,\omega)_\Gamma \neq 0$, $m \in C^2([0,T])$. Then there exists at most one solution $(u,h)$ to (P), (MP) which obeys $u \in C([0,T], \boldsymbol{V})$, $\partial_t u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), H^1(\Omega)\right)$, $\partial_t \Delta u \in L^2((0,T), L^2(\Omega))$, $\partial_{tt} u \in L^2\left((0,T), H^1(\Omega)\right)$, $\partial_{tt}\Delta u \in L^2\left((0,T), L^2(\Omega)\right)$ and $h \in C([0,T])$.*

*Proof.* We prove this in the classical way by contradiction. Let there be two solutions $(u_1, h_1), (u_2, h_2)$ of the (P), (MP) belonging to the spaces written in the theorem. We define $u = u_1 - u_2$ and $h = h_1 - h_2$ which then obey

$$(\Delta u(t), \omega)_\Gamma = h(t)\,(f, \omega)_\Gamma\,, \tag{5.22}$$

and

$$((g_{2-\beta} * \partial_{tt} u)\,(t), \varphi) + (\nabla u(t), \nabla \varphi) = h(t)\,(f, \varphi)\,, \tag{5.23}$$

for every $\varphi \in H^1(\Omega)$, a.a. $t \in [0, T]$ and $u(0) = 0$, $\partial_t u(0) = 0$. We can eliminate $h$ from (5.22) and using the trace theorem estimate as

$$|h(t)| \leq \left| \frac{(\Delta u(t), \omega)_\Gamma}{(f, \omega)_\Gamma} \right| \leq C\,(\|\Delta u(t)\| + \|\nabla \Delta u(t)\|)\,.$$

We put $\varphi = \partial_t u(t)$ in (5.23), integrate over $(0, \xi)$ with $\xi \in (0, T]$, and for the first term on the l.h.s. we use the relationship $(g_{2-\beta} * \partial_{tt} u)\,(t) = \partial_t\,(g_{2-\beta} * \partial_t u)\,(t)$, which is true since $\partial_t u = 0$, to obtain that

$$\int_0^\xi (\partial_t\,(g_{2-\beta} * \partial_t u)\,(t), \partial_t u(t))\ \mathrm{d}t + \frac{1}{2}\,\|\nabla u(\xi)\|^2 = \int_0^\xi h(t)\,(f, \partial_t u(t))\ \mathrm{d}t.$$

Using Lemma 1.7.2, the Cauchy, the Young inequalities and choosing the appropriate $\varepsilon$ lead us to the estimate

$$\int_0^\xi \|\partial_t u(t)\|^2\ \mathrm{d}t + \|\nabla u(\xi)\|^2 \leq C \int_0^\xi |h(t)|^2\ \mathrm{d}t,$$

similar to the one in the Lemma 5.3.2. Thanks to the assumption from the theorem, we may use the Green identity $(\nabla u(t), \nabla \varphi) = (\Delta u(t), \varphi)$ in (5.23), then in a comparable manner as in Lemma 5.3.3 and 5.3.4, we derive that

$$\int_0^\xi \|\partial_t \nabla u(t)\|^2\ \mathrm{d}t + \|\Delta u(\xi)\|^2 \leq C \int_0^\xi |h(t)|^2\ \mathrm{d}t,$$

and

$$\int_0^\xi \|\partial_t \Delta u(t)\|^2\ \mathrm{d}t + \|\nabla \Delta u(\xi)\|^2 \leq C \int_0^\xi |h(t)|^2\ \mathrm{d}t.$$

Summing the last three estimates up and using estimate $\|u(\xi)\|^2 \leq \int_0^\xi \|\partial_t u(t)\|^2\ \mathrm{d}t$, we obtain

$$\|u(\xi)\|^2 + \|\nabla u(\xi)\|^2 + \|\Delta u(\xi)\|^2 + \|\nabla \Delta u(\xi)\|^2 \leq C \int_0^\xi \left( \|\Delta u(t)\|^2 + \|\nabla \Delta u(t)\|^2 \right)\ \mathrm{d}t.$$

The Grönwall's argument is applied to get

$$\|u(\xi)\|^2 + \|\nabla u(\xi)\|^2 + \|\Delta u(\xi)\|^2 + \|\nabla \Delta u(\xi)\|^2 \leq 0, \tag{5.24}$$

which is true for any $\xi \in [0, T]$. This imply that $u = 0$ a.e. in $\Omega \times [0, T]$, and then also $h = 0$ a.e. in $[0, T]$. $\square$

## 5.5 Numerical Experiments

In the section two numerical experiments are presented. The first one is a demonstration of the algorithm arising from the above time discretization. On the $i-$th time layer $h_i$ is calculated from (DMP$i$) and $u_i$ from (DP$i$), then we move to the next time level. In the second experiment we propose a way how to deal with the data containig some percentage of noise.

Both experiments have the following setting. We assume $(x, y) \in \Omega = (0, \pi) \times (0, \pi)$, $T = 3$ and $\beta = 1.3$, next

$$f(x, y) = \cos x + \cos y,$$

and the equation is accompanied with the initial and boundary condition

$$\begin{aligned} u_0(x, y) &= 5(\cos x + \cos y), \\ v_0(x, y) &= 0, \\ -\nabla u(x, y, t) \cdot \boldsymbol{\nu} &= 0. \end{aligned}$$

The measurement function takes form

$$m(t) = \int_\Gamma u(x, y, t)\omega(x, y) dS = \left(\frac{\pi}{2} + 1\right) \left(t^3 - 2t^2 + 5\right),$$

where

$$\omega(x, y) = \begin{cases} 1, & \left|y - \frac{\pi}{4}\right| \le \frac{\pi}{4}, x = 0, \\ 0, & otherwise. \end{cases}$$

It can be easily showed that the functions

$$\begin{aligned} u(x, y, t) &= \left(t^3 - 2t^2 + 5\right)(\cos x + \cos y), \\ h(t) &= \frac{6}{(2 - \beta)(3 - \beta)\Gamma(2 - \beta)} t^{3-\beta} - \frac{4}{(2 - \beta)\Gamma(2 - \beta)} t^{2-\beta} + t^3 - 2t^2 + 5 \end{aligned}$$

are the solution of the inverse problem with the above settings. We implement the algorithm in Python using the finite element library DOLFIN from the FEniCS Project [73]. The domain is divided into 50 cells in each $x-$ and $y-$ direction. In each time step the Lagrange basis function are used which leads to the system with 10201 degrees of freedom.

### 5.5.1 Exact data

Using the above setting, we calculate the approximate solution for several values of time step $\tau$. On Fig. 5.1(a) the reconstruction of $h$ is depicted. The development of relative error of $h$ and $u$ in time can be seen on Fig. 5.1(b) and (c), respectively. The decay of maximal relative error of $h$ and $u$ for various values of $\tau$ is shown on Fig. 5.1(d) and (e), respectively. The graph of the solid line in (d) and (e) is given by $0.9899 \log_2 \tau + 0.1965$ for the error of $h$ and $1.0121 \log_2 \tau + 0.6473$ for the error of $u$, respectively.

### 5.5.2 Noisy data

In this experiment, we model a noisy measurement in the following way

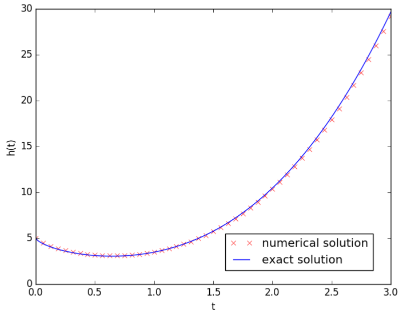
$$m_\epsilon(t) = m(t) + \varepsilon \delta m_{max},$$

where $\delta$ is the Gaussian distributed noise with mean and standard deviation equal to 0 and 1, respectively, $m_{max}$ is the maximum value of measurement $m$ and $\varepsilon$ is a scale representing the amount of the noise.

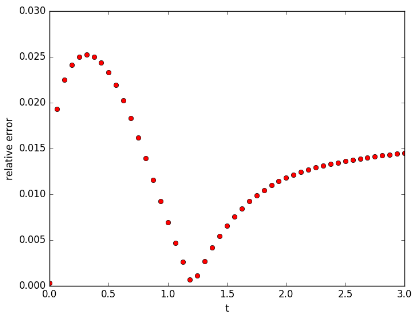
Since our algorithm requires the continuous second derivative of the measurement, we need to apply some kind of smoothing on the data. We use the least square method to obtain a function of the form

$$m_{app}(t) = at^3 + bt^2 + ct + d$$

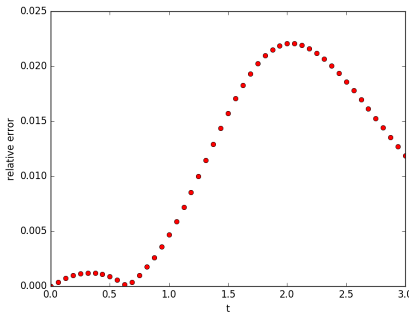
which is smooth enough. This function is then used instead of $m$ in the algorithm. We use the same setting as in the previous experiment. On the Fig. 5.2(b) we can see reconstruction of source term for the several various amount of noise and on the Fig. 5.2(c) and (d) the corresponding relative error in time can be seen for $h$ and $u$, respectively.



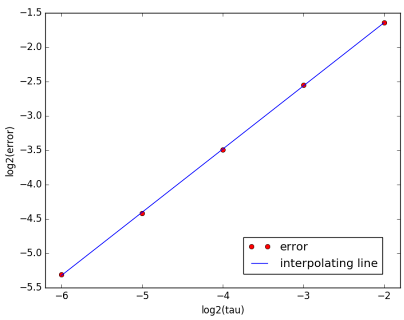

(a) Reconstruction of $h$ together with exact $h$.



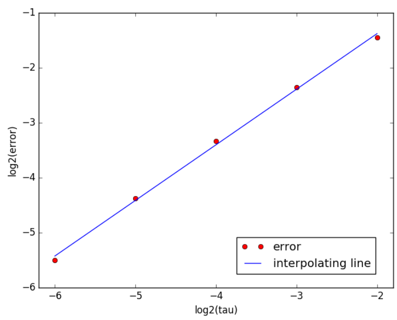

(b) Relative error $\frac{|h_{approx}(t)-h_{exact}(t)|}{|h_{exact}(t)|}$.




(c) Relative error $\frac{||u_{approx}(t)-u_{exact}(t)||}{||u_{exact}(t)||}$.

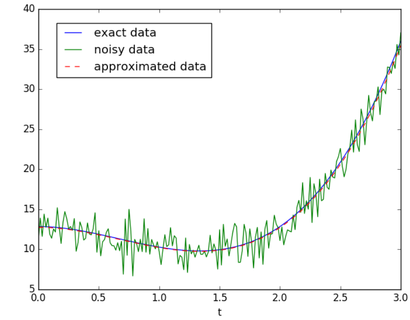



(d) Logarithm of maximal relative error in time of $h$ for different values of $\tau$. Slope of the line is 0.9899.

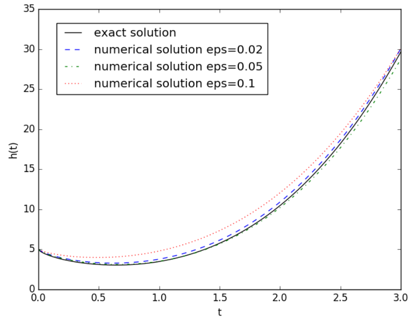



(e) Logarithm of maximal relative error in time of $u$ for different values of $\tau$. Slope of the line is 1.0121.

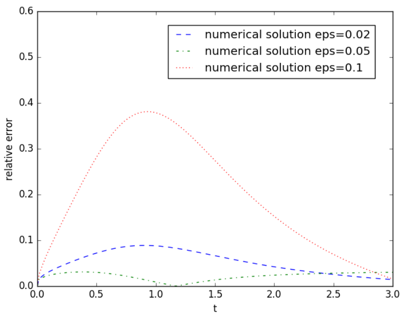
Figure 5.1: The results of the reconstruction algorithm $\tau = 0.015625$



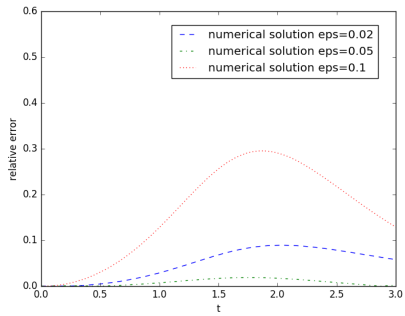

(a) Exact and noisy data for $\epsilon = 0.05$. Approximating curve has the form $m_{app}(t) = 2.529t^3 - 5.0772t^2 + 0.0143t + 12.7607$.




(b) Reconstruction of $h$ together with exact $h$ for $\tau = 0.015625$ and for the different amount of noise.




(c) Relative error $\frac{|h_{approx}(t) - h_{exact}(t)|}{|h_{exact}(t)|}$.




(d) Relative error $\frac{||u_{approx}(t) - u_{exact}(t)||}{||u_{exact}(t)||}$.

Figure 5.2: The results of the reconstruction algorithm for the noisy data and various amount of noise

# Chapter 6

# Conclusion

In this thesis, we were studying several inverse source problems for the time-fractional PDEs. The order of the fractional derivative ranged between zero and two, which corresponds to the fractional diffusion-wave equation. On a simple example, it has been illustrated that for the order between zero and one the solution displays the slow-diffusion behavior while for the order between one and two the behavior of the solution carries signs both of the diffusion and the wave transport. The equations themselves can be derived as a generalization of the Brownian motion.

In the first chapter, we provided the necessary mathematical background required for good understanding of the later chapters. At the end of the chapter, two important lemmas have been formulated and proved, enabling the convenient estimation of the integrals and sums containing the fractional derivative and its discretized version.

The second chapter addressed the inverse problem of determining a solely time-dependent source for a fractional diffusion equation with a nonlinear term on the right hand side. The well-posedness of the solution was studied; the uniqueness and the existence of the solution were established.

The next three chapters dealt with the recognition of the time-dependent part of the source term in the fractional wave equation. Firstly, the nonlinear term on the right hand side was considered. The measurement was in the form of the integral over the domain which can be restricted to the integral over a subdomain. Secondly, we assumed the dynamical boundary conditions; those conditions are often used to gain more physically corresponding models. The considered measurement is in the form of the integral over the subdomain. With the fractional derivative in the boundary condition, we needed to follow a different approach to the weak formulation where a test function was chosen in the form of the Laplacian of a function from the suitable space which contains functions properly defined on the boundary. Lastly, we assumed the noninvasive measurement in the form of a boundary integral. Assuming this kind of a measurement caused complications in a priori estimates since the controllability of the Laplacian on the boundary is required. In all cases the uniqueness of the solution was addressed and the existence of the solution was proved using the Rothe method. We performed simple numerical experiments to illustrate the algorithm and the convergence of the algorithm for the decreasing time-step in each case. The treatment of the noisy data was suggested and performed for all cases. The error estimate was calculated in the case of the dynamical boundary conditions. The interesting results is that the rate of the convergence was shown to be dependent on the order of the fractional derivative.

The unsolved problem, which might be of a future interest and research, is the identification of the time-dependent part of the source term in the fractional diffusion equation considering the boundary measurement. This problem was partly solved in [134] where the uniqueness of the solution was addressed and Tikhonov regularization was used for the calculation of the approximate solution. However, the existence of the solution was not proved, yet, and it remains an open question. The main issue in this case is the estimation of the Laplacian of a solution on the boundary of the domain that is not possible to handle in the same way as for the fractional wave equation case.

Other possibility of future research might be equations containing a so-called distributed order fractional derivative [111] where the term with the fractional derivative is multiplied by the weight function and integrated over the order of the fractional derivative. Such an equation is used to model decelerating anomalous diffusion and ultra slow diffusive processes.

Among another naturally rising questions are the reconstruction of the space-dependent part of a source and the reconstruction of a source in equations containing fractional derivatives in a space direction. In our thesis we also presented simple numerical experiments. An interesting question to ponder could be the quality of the reconstruction for different shapes of the domain and for different treatments of the noise.

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