Blocking sets of tangent and external lines to a hyperbolic quadric in $PG(3, q)$

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Abstract

Let $\mathcal{H}$ be a hyperbolic quadric in $PG(3, q)$, where q is a prime power. Let $\mathbb{E}$ (respectively, $\mathbb{T}$) denote the set of all lines of $PG(3, q)$ which are external (respectively, tangent) to $\mathcal{H}$. We characterize the minimum size blocking sets in $PG(3, q)$, $q \neq 2$, with respect to the line set $\mathbb{E} \cup \mathbb{T}$. We also give an alternate proof characterizing the minimum size blocking sets in $PG(3, q)$ with respect to the line set $\mathbb{E}$ for all odd q.

Keywords: Projective space, Blocking set, Irreducible conic, Hyperbolic quadric

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1 Introduction

1.1 Hyperbolic quadrics in $PG(3, q)$

Throughout, q is a prime power. Let $PG(3, q)$ be the three-dimensional Desarguesian projective space defined over a finite field of order q. Let $\mathcal{H}$ be a hyperbolic quadric in $PG(3, q)$, that is, a non-degenerate quadric of Witt index two. One can refer to [9] for the basic properties of the points, lines and planes of $PG(3, q)$ with respect to the quadric $\mathcal{H}$. Every line of $PG(3, q)$ meets $\mathcal{H}$ in 0, 1, 2 or $q + 1$ points. We denote by $\mathbb{E}$ (respectively, $\mathbb{T}_1$, $\mathbb{S}$, $\mathbb{T}_0$) the set of lines of $PG(3, q)$ that intersect $\mathcal{H}$ in 0 (respectively, 1, 2, $q + 1$) points. The elements of $\mathbb{E}$ are called *external lines*, those of $\mathbb{S}$ *secant lines* and those of $\mathbb{T} := \mathbb{T}_0 \cup \mathbb{T}_1$ *tangent lines*. If $l \in \mathbb{T}_i$ with $i \in \{0, 1\}$, then l is also called a $\mathbb{T}_i$*-line*. The $\mathbb{T}_0$-lines are precisely the lines contained in $\mathcal{H}$.

The quadric $\mathcal{H}$ consists of $(q + 1)^2$ points and $2(q + 1)$ $\mathbb{T}_0$-lines. Every point of $\mathcal{H}$ lies on two $\mathbb{T}_0$-lines. So, $|\mathbb{T}_1| = (q + 1)^2(q - 1)$ and hence $|\mathbb{T}| = (q + 1)(q^2 + 1)$. Every point x of $PG(3, q)$ lies on $q^2 + q + 1$ lines of $PG(3, q)$, and $q + 1$ of them are tangent to $\mathcal{H}$. If x is a point of $\mathcal{H}$, then the remaining q^2 lines through x are secant to $\mathcal{H}$. If x is a point of $PG(3, q) \setminus \mathcal{H}$, then x lies on $q(q + 1)/2$ secant lines and $q(q - 1)/2$ external lines. We have $|\mathbb{S}| = \frac{1}{2}q^2(q + 1)^2$ and $|\mathbb{E}| = \frac{1}{2}q^2(q - 1)^2$.

With the quadric $\mathcal{H}$, there is naturally associated a polarity ζ which is symplectic if q is even and orthogonal if q is odd. For a point x of $PG(3,q)$, the plane x^ζ is called a *tangent plane* or a *secant plane* according as x is a point of $\mathcal{H}$ or not. For every point x of $\mathcal{H}$, the tangent plane x^ζ intersects $\mathcal{H}$ in the union of two $\mathbb{T}_0$-lines through x. The $q+1$ tangent lines through x are precisely the lines through x contained in x^ζ. Now let y be a point of $PG(3,q) \setminus \mathcal{H}$. Then the secant plane y^ζ intersects $\mathcal{H}$ in an irreducible conic $\mathcal{C}_y$. If q is even, then y is a point of y^ζ and is the nucleus of $\mathcal{C}_y$ in y^ζ. In this case, the $q+1$ tangent lines through y are precisely the lines through y contained in y^ζ. If q is odd, then y is not a point of y^ζ. In this case, the tangent lines through y are precisely the lines through y meeting $\mathcal{C}_y$.

If x is a point of $\mathcal{H}$, then we shall denote by π_x the tangent plane x^ζ. If π is a secant plane with $\pi = y^\zeta$ for some point y in $PG(3,q) \setminus \mathcal{H}$, then we also denote by $\mathcal{C}_\pi$ the conic $\mathcal{C}_y$ in $\pi = y^\zeta$ and thus $\mathcal{C}_\pi = \pi \cap \mathcal{H}$.

There are $(q+1)^2$ tangent planes and $q^3 - q$ secant planes. Every external line is contained in $q+1$ secant planes, every secant line is contained in two tangent planes and $q-1$ secant planes, every $\mathbb{T}_0$-line is contained in $q+1$ tangent planes, and every $\mathbb{T}_1$-line is contained in one tangent plane and q secant planes. Every point of $\mathcal{H}$ is in $2q+1$ tangent planes and $q^2 - q$ secant planes. Every point of $PG(3,q) \setminus \mathcal{H}$ is in $q+1$ tangent planes and q^2 secant planes.

An *ovoid* of $\mathcal{H}$ is a set of points intersecting each $\mathbb{T}_0$-line in a unique point. Every ovoid of $\mathcal{H}$ has exactly $q+1$ points. For every point y of $PG(3,q) \setminus \mathcal{H}$, the conic $\mathcal{C}_y$ in the secant plane y^ζ is an ovoid of $\mathcal{H}$.

1.2 Blocking sets in $PG(3,q)$

Let $\mathcal{X}$ be a point-line geometry and L a given nonempty subset of the line set of $\mathcal{X}$. A set B of points of $\mathcal{X}$ is called an *L-blocking set* if each line of L contains at least one point of B. Blocking sets in various geometries with respect to varying sets of lines have been studied by several authors. The first step in this regard has been to determine the smallest cardinality of a blocking set and, if possible, to describe all blocking sets of that cardinality.

Now consider $\mathcal{X} = PG(3,q)$. If L is the set of all lines of $PG(3,q)$ and B is an L-blocking set, then $|B| \geq q^2 + q + 1$ and equality holds if and only if B is a plane of $PG(3,q)$. This follows from a more general result by Bose and Burton [6, Theorem 1]. Biondi et al. characterized the minimum size $\mathbb{E}$-blocking sets in [4, Theorem 1.1] for $q \geq 8$ even (also see [11, Section 3] for a different proof which works for all even q) and in [5, Theorem 2.4] for $q \geq 9$ odd. When $q > 2$ is even, the minimum size $(\mathbb{E} \cup \mathbb{S})$-blocking sets were determined in [13, Theorem 1.3] using the properties of generalized quadrangles. For $L \in \{\mathbb{S}, \mathbb{T} \cup \mathbb{S}, \mathbb{E} \cup \mathbb{S}\}$, the minimum size L-blocking sets are described in [12] for all q. We shall prove the following in this paper.

Theorem 1.1. *Let B be an $(\mathbb{E} \cup \mathbb{T})$-blocking set in $PG(3,q)$, where $q \geq 3$. Then $|B| \geq q^2 + q$ with equality if and only if $B = \pi_x \setminus \{x\}$ for some point x of $\mathcal{H}$.*

We note that Theorem 1.1 was proved in [11, Proposition 1.5] for q even, however, the arguments used in [11] can not be applied for odd q. When $q = 2$, a similar result holds with one more class of examples of $(\mathbb{E} \cup \mathbb{T})$-blocking sets of minimum size six, see [11, Proposition 1.5(i)(b)]. Here our proof of Theorem 1.1 will work for all $q \geq 3$ irrespective of q odd or even. Along the way, we give an alternate proof to characterize the $\mathbb{E}$-blocking sets of minimum size $q^2 - q$ for q odd, which also includes the three smallest values of q, namely, $q = 3, 5, 7$.

2 Revisiting $\mathbb{E}$-blocking sets, q odd

Let $\mathcal{C}$ be an irreducible conic in $PG(2, q)$. One can refer to [10] for the following basic properties. Every line of $PG(2, q)$ meets $\mathcal{C}$ in 0, 1 or 2 points. A line of $PG(2, q)$ is called external (respectively, tangent, secant) with respect to $\mathcal{C}$ if it meets $\mathcal{C}$ in 0 (respectively, 1, 2) points. Suppose that q is odd. Then every point of $PG(2, q) \setminus \mathcal{C}$ lies on 0 or 2 tangent lines. Such a point is called *interior* to $\mathcal{C}$ in the first case and *exterior* to $\mathcal{C}$ in the latter. There are $q(q-1)/2$ interior points and $q(q+1)/2$ exterior points in $PG(2, q)$ with respect to $\mathcal{C}$. Every interior point lies on $(q+1)/2$ external lines and $(q+1)/2$ secant lines. Every exterior point lies on $(q - 1)/2$ external lines and $(q - 1)/2$ secant lines. Every external line contains $(q + 1)/2$ interior points and $(q + 1)/2$ exterior points. Every secant line contains $(q - 1)/2$ interior points and $(q - 1)/2$ exterior points.

For q odd, Aguglia and Korchmáros characterized in [3, Theorem 1.1] the minimum size blocking sets of the external lines in $PG(2, q)$ with respect to the conic $\mathcal{C}$.

Theorem 2.1 ([3]). *Let A be a blocking set of the external lines in $PG(2, q)$ with respect to $\mathcal{C}$, where q is odd. Then $|A| \geq q - 1$ and the following hold in case of equality:*

(i) If $q \geq 9$, then $|A| = q - 1$ if and only if $A = l \setminus \mathcal{C}$ for some line l of $PG(2, q)$ secant to $\mathcal{C}$.

(ii) If $q \in \{5, 7\}$, then $|A| = q - 1$ if and only if one of the following two cases occurs:

- *(a) $A = l \setminus \mathcal{C}$ for some line l of $PG(2, q)$ secant to $\mathcal{C}$;*
- *(b) A is a suitable set of $q - 1$ interior points with respect to $\mathcal{C}$.*

(iii) If $q = 3$, then $|A| = 2$ if and only if one of the following two cases occurs:

- *(a) $A = l \setminus \mathcal{C}$ for some line l of $PG(2, 3)$ secant to $\mathcal{C}$;*
- *(b) A consists of any two interior points with respect to $\mathcal{C}$.*

When $q = 3$, the possibility stated in Theorem 2.1(iii)(b) was not included in the statement of [3, Theorem 1.1]. We give a proof of Theorem 2.1(iii) below.

Proof of Theorem 2.1(iii). There are three interior points and three external lines in $PG(2, 3)$ with respect to $\mathcal{C}$. Every external line in $PG(2, 3)$ contains two interior points.

So any two interior points will block all the three external lines in $PG(2,3)$, justifying the statement in Theorem 2.1(iii)(b).

Conversely, let $A = \{x, y\}$ be a blocking set of minimum size 2 of the external lines in $PG(2,3)$ with respect to $\mathcal{C}$. Then the minimality of $|A|$ implies $A \cap \mathcal{C} = \emptyset$. Let $l := xy$ be the line of $PG(2,q)$ through x and y. We may assume that l is not secant to $\mathcal{C}$. Suppose that l is tangent to $\mathcal{C}$. Let $z \in l$ be the unique point such that $l \setminus \mathcal{C} = \{x, y, z\}$. Then the unique external line through z would not meet A, a contradiction. So l is external to $\mathcal{C}$. If at least one of x and y is not interior to $\mathcal{C}$, then there exists a point $b \in l \setminus \{x, y\}$ which is interior to $\mathcal{C}$. Then the external line through b, different from l, would be disjoint from A, again a contradiction. Thus both x and y are interior with respect to $\mathcal{C}$. $\square$

We have the following result for the $\mathbb{E}$-blocking sets in $PG(3,q)$ for all q.

Theorem 2.2. *Let B be an $\mathbb{E}$-blocking set in $PG(3,q)$. Then $|B| \geq q^2 - q$, and equality holds if and only if $B = \pi \setminus \mathcal{H}$ for some tangent plane π of $PG(3,q)$.*

We note that Theorem 2.2 was proved by Biondi et al. in [4, Theorem 1.1] and [5, Theorem 2.4], with exception of the equality case for some small values of q, namely $q \in \{2, 3, 4, 5, 7\}$. In [11, Section 3], a different proof was given to prove the equality case in Theorem 2.2 for all even q which includes the two smallest values of q ($q = 2, 4$). In the rest of this section, our aim is to give an alternate proof of the equality case in Theorem 2.2 which works for all odd q, in particular, for $q = 3, 5, 7$.

Let B be an $\mathbb{E}$-blocking set in $PG(3,q)$, q odd. As in the proof of [5, Proposition 2.1], by counting in two ways the cardinality of the set $\{(x, l) : x \in B, l \in \mathbb{E}, x \in l\}$, it follows that $|B| \geq q^2 - q$ with equality if and only if $B \cap \mathcal{H} = \emptyset$ and each external line contains exactly one point of B. Suppose now that B has minimum possible size $q^2 - q$.

Lemma 2.3. *For any external line l, exactly one of the planes through l contains q points of B and each of the remaining planes contains $q - 1$ points of B.*

Proof. Let $\pi_0, \pi_1, \cdots, \pi_q$ be the $q+1$ planes through l. Then each π_i is a secant plane and $\pi_i \cap B$ is a blocking set of the external lines in π_i with respect to the conic $\mathcal{C}_{\pi_i} = \pi_i \cap \mathcal{H}$. By Theorem 2.1, $|\pi_i \cap B| \geq q - 1$ for each i. Now the lemma follows from the three facts that $B = \bigcup_{i=0}^{q} (\pi_i \cap B)$, $|l \cap B| = 1$ and $|B| = q^2 - q$. $\square$

Lemma 2.4. *Let π be a secant plane of $PG(3,q)$. Then the following hold:*

(i) $|\pi \cap B| \in \{q - 1, q\}$.

(ii) *If $|\pi \cap B| = q - 1$, then $\pi \cap B = l \setminus \mathcal{C}_\pi$ for some secant line l contained in π.*

(iii) *If $|\pi \cap B| = q$, then each point of $\pi \cap B$ is exterior in π with respect to $\mathcal{C}_\pi$.*

Proof. Considering an external line l contained in π, (i) follows from Lemma 2.3. We prove (ii) and (iii).

Let α (respectively, β) denote the number of points in $\pi \cap B$ which are interior (respectively, exterior) with respect to $\mathcal{C}_\pi$. Since $B \cap \mathcal{C}_\pi = \emptyset$, we have $\alpha + \beta = |\pi \cap B|$. Consider the following set of point-line pairs:

$$X = \{(x, l) : x \in \pi \cap B, l \in \mathbb{E}, l \subseteq \pi, x \in l\}.$$

Counting $|X|$ in two ways, we get

$$\alpha \left(\frac{q+1}{2}\right) + \beta \left(\frac{q-1}{2}\right) = |X| = \frac{q(q-1)}{2}.$$

This gives

$$(\alpha + \beta)q + \alpha - \beta = q(q-1). \tag{1}$$

If $|\pi \cap B| = q - 1$, then putting $\alpha + \beta = q - 1$ in (1), we get $\alpha = \beta$. Thus, half of the points of $\pi \cap B$ are interior and the other half are exterior with respect to $\mathcal{C}_\pi$. Then (ii) follows from Theorem 2.1, since $\pi \cap B$ is a blocking set of minimum size $q - 1$ of the external lines in π with respect to $\mathcal{C}_\pi$.

If $|\pi \cap B| = q$, then we have $\alpha + \beta = q$. Then (1) implies that $\alpha - \beta = -q$. It follows that $\alpha = 0$ and $\beta = q$. Thus, all the points of $\pi \cap B$ are exterior with respect to $\mathcal{C}_\pi$, implying (iii). □

As a consequence of Lemmas 2.3 and 2.4(ii), we have the following.

Corollary 2.5. *There exists a secant line l such that $l \setminus \mathcal{H}$ is contained in B.*

We now prove Theorem 2.2 for all odd q.

Proof of Theorem 2.2. Consider a secant line l such that $l \setminus \mathcal{H}$ is contained in B and the planes through it. Let π be any secant plane through l. By Lemma 2.4(i), we have $|\pi \cap B| \in \{q-1, q\}$. Since half of the points of $l \setminus \mathcal{H} = l \setminus \mathcal{C}_\pi$ are interior in π with respect to $\mathcal{C}_\pi$, Lemma 2.4(iii) implies that $|\pi \cap B| \neq q$. So $|\pi \cap B| = q-1$ and hence $\pi \cap B = l \setminus \mathcal{H}$. It follows that the points of $B \setminus l$ are contained in the two tangent planes through l. Since $|B \setminus l| = q^2 - q - (q-1) = (q-1)^2$, one of the tangent planes through l, say π_0, contains at least $(q-1)^2/2$ points of $B \setminus l$. Then π_0 contains at least $q - 1 + (q-1)^2/2 = (q^2-1)/2$ points of B and so $|B \setminus \pi_0| \leq (q-1)^2/2$.

We prove that $B = \pi_0 \setminus \mathcal{H}$. It is enough to show that each point of $\pi_0 \setminus \mathcal{H}$ is in B. On the contrary, suppose that there exists a point $x \in \pi_0 \setminus \mathcal{H}$ which is not in B. There are $q(q-1)/2$ external lines through x and each of them meets B at a unique point outside π_0. This defines an injective map from the set of external lines through x to the set $B \setminus \pi_0$. But such a map is not possible, since $q(q-1)/2 > (q-1)^2/2 \geq |B \setminus \pi_0|$, a contradiction. □

3 Proof of Theorem 1.1

We need two results related to blocking sets in $PG(2,q)$. The first of these results was proved by Aguglia et al. in [1, Theorem 1.2] for q even and in [2, Theorem 1.1] for q odd. The second one was proved by Giulietti in [8, Theorems 1 and 2].

Proposition 3.1 ([1, 2]). *Let $\mathcal{C}$ be an irreducible conic in $PG(2,q)$. If A is a blocking set of the external and tangent lines in $PG(2,q)$ with respect to $\mathcal{C}$, then $|A| \geq q$. Further, equality holds if and only if one of the following three cases occurs:*

(i) $A = l \setminus \mathcal{C}$ for some tangent line l;

(ii) $A = (l \setminus \mathcal{C}) \cup \{z\}$ for some secant line l, where z is the pole of l if q is odd and the nucleus of $\mathcal{C}$ if q is even;

(iii) q is a square and $A = \Pi \setminus (\Pi \cap \mathcal{C})$, where Π is a Baer subplane of $PG(2,q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in Π.

Proposition 3.2 ([8]). *Let $\mathcal{C}$ be an irreducible conic in $PG(2,q)$, q even, and let n be its nucleus. If A is a blocking set of the external lines in $PG(2,q)$ with respect to $\mathcal{C}$, then $|A| \geq q-1$. Further, equality holds if and only if one of the following three cases occurs:*

(i) $A = l \setminus \mathcal{C}$ for some secant line l;

(ii) $A = l \setminus (\mathcal{C} \cup \{n\})$ for some tangent line l;

(iii) q is a square and $A = \Pi \setminus \Big((\Pi \cap \mathcal{C}) \cup \{n\}\Big)$, where Π is a Baer subplane of $PG(2,q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in Π.

From Theorem 2.1 and Proposition 3.2, we are able to prove the following.

Lemma 3.3. *Let $\mathcal{C}$ be an irreducible conic in $PG(2,q)$, and let k be a secant line to $\mathcal{C}$. If A is a blocking set of the external and tangent lines to $\mathcal{C}$ that is disjoint from k, then A contains at least q points of $\overline{\mathcal{C}} := PG(2,q) \setminus \mathcal{C}$.*

Proof. Suppose to the contrary that A contains at most $q-1$ points of $\overline{\mathcal{C}}$. Since $A \cap \overline{\mathcal{C}}$ is a blocking set with respect to the external lines, Theorem 2.1 and Proposition 3.2 then imply that $|A \cap \overline{\mathcal{C}}| = q-1$. Moreover, $A \cap \overline{\mathcal{C}}$ is one of the following.

(1) $A \cap \overline{\mathcal{C}} = l \setminus \mathcal{C}$ for some secant line l. If we put $l \cap \mathcal{C} = \{x_1, x_2\}$, then the fact that $k \cap A = \emptyset$ implies that $k \cap l = \{x_i\}$ for some $i \in \{1,2\}$. The tangent line through the point $x_i \notin A$ would then be disjoint from A, a contradiction.

(2) $q \in \{3,5,7\}$ and $A \cap \overline{\mathcal{C}}$ is a suitable set of $q-1$ interior points. But then the tangent line through a point of $k \cap \mathcal{C}$ (which cannot contain interior points) would be disjoint from A, a contradiction.

(3) q is even and there exists a tangent line l such that $A \cap \overline{\mathcal{C}} = l \setminus (\mathcal{C} \cup \{n\})$, where n is the nucleus of $\mathcal{C}$. As k is a secant line, the nucleus n cannot belong to it. As the line k is

disjoint from A, the unique point x of $\mathcal{C}$ on l belongs to k. If y denotes the other point of $\mathcal{C}$ on the line k, then the tangent line through y would be disjoint from A, a contradiction.

(4) q is an even square and $A \cap \overline{\mathcal{C}} = \Pi \setminus \Big((\Pi \cap \mathcal{C}) \cup \{n\}\Big)$, where Π is a Baer subplane of $PG(2,q)$ such that $\Pi \cap \mathcal{C}$ is an irreducible conic in Π. Here, n is again the nucleus of $\mathcal{C}$. Since k is a secant line, we have $n \notin k$. Every line of Π contains a point of A, implying that k cannot intersect Π in a Baer subline. So, k intersects Π in a unique point, say x. Since k is disjoint from A, we have that $x \in \Pi \cap \mathcal{C}$. If y denotes the other point of $\mathcal{C}$ on the line k, then the tangent line through y would intersect Π in the point n and hence be disjoint from A, which is again impossible. $\square$

We now proceed to prove Theorem 1.1. Let B be an $(\mathbb{E} \cup \mathbb{T})$-blocking set of $PG(3,q)$ of minimum possible size, where $q \geq 3$. Observe that, for every point x of $\mathcal{H}$, the set $\pi_x \setminus \{x\}$ is an $(\mathbb{E} \cup \mathbb{T})$-blocking set of size $q^2 + q$. So

$$|B| \leq q^2 + q. \tag{2}$$

Since $B \cap \mathcal{H}$ blocks every $\mathbb{T}_0$-line, we have $|B \cap \mathcal{H}| \geq q + 1$. Every external line meets B outside $\mathcal{H}$. So $B \setminus \mathcal{H}$ is an $\mathbb{E}$-blocking set of $PG(3,q)$ and hence $|B \setminus \mathcal{H}| \geq q^2 - q$ by Theorem 2.2. Thus, we have

$$q^2 - q \leq |B \setminus \mathcal{H}| \leq q^2 - 1 \tag{3}$$

and

$$q + 1 \leq |B \cap \mathcal{H}| \leq 2q. \tag{4}$$

Lemma 3.4. *Let π be any plane of $PG(3,q)$. Then the following hold:*

- *(i) If π is a secant plane, then $|\pi \cap B| \geq q$.*
- *(ii) If $\pi = \pi_x$ is a tangent plane for some point x in $\mathcal{H} \setminus B$, then $|\pi \cap B| \geq q + 1$.*

Proof. (i) The set $\pi \cap B$ is a blocking set of the external and tangent lines in π with respect to the conic $\mathcal{C}_\pi$. So $|\pi \cap B| \geq q$ by Proposition 3.1.

(ii) This follows from the facts that $x \notin B$ and that each of the $q + 1$ tangent lines through x in π_x meets B. $\square$

For a given $\mathbb{T}_1$-line l, we denote by x_l the tangency point of l in $\mathcal{H}$, that is, the unique point of l contained in $\mathcal{H}$. Let $\overline{\mathbb{T}}$ denote the set of all $\mathbb{T}_1$-lines l such that $|l \cap B| = 1$ and $x_l \notin B$.

Lemma 3.5. *$\overline{\mathbb{T}}$ is nonempty.*

Proof. Let R be the set of all $\mathbb{T}_1$-lines l for which $x_l \notin B$. Using the upper bound for $|B \cap \mathcal{H}|$ given in (4), we get $|R| = [(q+1)^2 - |B \cap \mathcal{H}|]\,(q-1) \geq (q^2+1)(q-1)$.

If each line of R meets B in at least two points, then counting the cardinality of the set $Z = \{(x,l) : x \in B,\ l \in \mathbb{T},\ x \in l\}$ in two ways, we get $|B|(q+1) = |Z| \geq 2|R| + |\mathbb{T} \setminus R| = |R| + |\mathbb{T}|$. Since $|B| \leq q^2 + q$ and $|\mathbb{T}| = (q+1)(q^2+1)$, it follows that $|R| \leq q^2 - 1$, a contradiction. $\square$

Lemma 3.6. *For every $l \in \overline{\mathbb{T}}$, there exists a secant plane through l containing exactly q points of B.*

Proof. Suppose that this is not the case. By Lemma 3.4, we then know that each of the q secant planes through l contains at least $q+1$ points of B. Since $x_l \notin B$, the tangent plane through l contains at least $q+1$ points of B. Using the fact that $|l \cap B| = 1$, it follows that all the planes through l together contain at least $(q+1)q + 1 = q^2 + q + 1$ points of B, a contradiction to (2). □

As a consequence of Lemmas 3.5 and 3.6, we have the following.

Corollary 3.7. *There exists a secant plane containing exactly q points of B.*

Note that there are lines secant to $\mathcal{H}$ which are disjoint from B. Otherwise, $|B| \geq q^2 + q + 1$, as B would be a blocking set with respect to all the lines of $PG(3, q)$.

Lemma 3.8. *Let l be a secant line disjoint from B. If π_a and π_b, for points a, b of $\mathcal{H}$, are the two tangent planes through l, then at least one of a and b is in B.*

Proof. Suppose that none of a and b is in B. By Lemma 3.4, each of π_a and π_b contains at least $q+1$ points of B and each of the $q-1$ secant planes through l contains at least q points of B. Since $l \cap B$ is empty, we get $|B| \geq 2(q+1) + q(q-1) = q^2 + q + 2$, a contradiction to (2). □

Now, let π be any secant plane of $PG(3, q)$ containing exactly q points of B. Since $\pi \cap B$ is a minimum size blocking of the external and tangent lines in π with respect to $\mathcal{C}_\pi$, there are three possibilities for $\pi \cap B$ by Proposition 3.1:

(I) $\pi \cap B = l \setminus \{x_l\}$ for some tangent line l contained in π;

(II) $\pi \cap B = (l \setminus \mathcal{C}_\pi) \cup \{\alpha\}$ for some secant line l in π, where α is the pole of l if q is odd and the nucleus of $\mathcal{C}_\pi$ if q is even;

(III) q is a square and $\pi \cap B = \Pi \setminus (\Pi \cap \mathcal{C}_\pi)$, where Π is a Baer subplane of π such that $\Pi \cap \mathcal{C}_\pi$ is an irreducible conic in Π.

Lemma 3.9. *Possibility (I) occurs for every secant plane that contains q points of B.*

Proof. Suppose that π is a secant plane containing q points of B for which possibility (I) does not occur. The number of secant lines in π that are disjoint from $\pi \cap B$ is then equal to $2(q-1)$ or $(\sqrt{q}+1)(q-\sqrt{q}) = \sqrt{q}(q-1)$ depending on whether possibility (II) or (III) occurs. Each of these secant lines is contained in exactly two tangent planes, implying that the number of points $a \in \mathcal{H} \setminus \mathcal{C}_\pi$ for which $\pi_a \cap \pi$ is a secant line disjoint from $\pi \cap B$ is equal to $4(q-1)$ or $2\sqrt{q}(q-1)$. Since $|B \cap \mathcal{H}| \leq 2q$ by equation (4) and $2q < \min\{4(q-1), 2\sqrt{q}(q-1)\}$ for $q \geq 3$, there exists a point $a^* \in \mathcal{H} \setminus (B \cup \mathcal{C}_\pi)$ such that $l^* := \pi_{a^*} \cap \pi$ is a secant line disjoint from $\pi \cap B$.

There are $q-1$ secant planes through l^*. For each such plane π', the set $\pi' \cap B$ (disjoint from l^*) is a blocking set in π' of the external and tangent lines to $\mathcal{C}_{\pi'}$ and so $\pi' \cap B$ contains at least q points of $B \setminus \mathcal{H}$ by Lemma 3.3. As $a^* \notin B$, the tangent plane π_{a^*} through l^* contains at least $q-1$ points of $B \setminus \mathcal{H}$. Hence, $|B \setminus \mathcal{H}| \geq (q-1)q + q - 1 = q^2 - 1$. As $|B \setminus \mathcal{H}| \leq q^2 - 1$ by equation (3), we thus have that $|B \setminus \mathcal{H}| = q^2 - 1$. As $|B| \leq q^2 + q$ and $q + 1 \leq |B \cap \mathcal{H}|$ by equations (2) and (4), we then know that $|B| = q^2 + q$ and $|B \cap \mathcal{H}| = q + 1$.

As mentioned above, there are $2(q-1)$ or $\sqrt{q}(q-1)$ secant lines in π disjoint from $\pi \cap B$. For each such secant line l, we know from Lemma 3.8 that there exists a point $a \in B \cap \mathcal{H}$ for which $\pi_a \cap \pi = l$. In this way, we get a collection of $N \in \{2(q-1), \sqrt{q}(q-1)\}$ points of $B \cap \mathcal{H}$. Since $N \leq |B \cap \mathcal{H}| = q + 1$ and $q \geq 3$, we find that $q = 3$ and that possibility (II) occurs for the secant plane π.

We thus have that $q = 3$, $|B| = q^2 + q = 12$, $|B \setminus \mathcal{H}| = q^2 - 1 = 8$ and $|B \cap \mathcal{H}| = q + 1 = 4$ (so $B \cap \mathcal{H}$ is an ovoid of $\mathcal{H}$). Moreover, for each of the four secant lines l contained in π and disjoint from $\pi \cap B$, there exists a unique point $a \in B \cap \mathcal{H}$ for which $\pi_a \cap \pi = l$ and a unique point $b \in \mathcal{H} \setminus (B \cup \mathcal{C}_\pi)$ for which $\pi_b \cap \pi = l$. Among the eight points of $B \setminus \mathcal{H}$, there are two contained in π_b and six contained in the two secant planes through l (recall Lemma 3.3). So, the tangent plane π_a through l cannot contain further points of $B \setminus \mathcal{H}$.

As l ranges over all four secant lines of π disjoint from $\pi \cap B$, the point a will range over all four points of $B \cap \mathcal{H}$. As none of the four tangent planes π_a, $a \in B \cap \mathcal{H}$, contains points of $B \setminus \mathcal{H}$, we thus have:

($*$) any $\mathbb{T}_1$-line through a point of $B \cap \mathcal{H}$ does not contain points of $B \setminus \mathcal{H}$.

For every point x of $PG(3,3) \setminus \mathcal{H}$, the conic $\mathcal{C}_x$ in x^ζ is an ovoid of $\mathcal{H}$. The map $x \mapsto \mathcal{C}_x$ from $PG(3,3) \setminus \mathcal{H}$ to the set of ovoids of $\mathcal{H}$ is a bijection (see e.g. [7]). Any two distinct ovoids of $\mathcal{H}$ intersect in at most two points. If x_1 and x_2 are two distinct points of $PG(3,3) \setminus \mathcal{H}$, then x_1x_2 is a tangent, secant or external line whenever $|\mathcal{C}_{x_1} \cap \mathcal{C}_{x_2}|$ is equal to 1, 2 or 0, respectively. By ($*$), we have

For every $x \in B \setminus \mathcal{H}$, the ovoid $\mathcal{C}_x$ is disjoint from $B \cap \mathcal{H}$.

We can now label the points of $\mathcal{H}$ by x_{ij}, where $i, j \in \{1, 2, 3, 4\}$, such that two distinct points x_{ij} and $x_{i'j'}$ of $\mathcal{H}$ are incident with a $\mathbb{T}_0$-line if either $i = i'$ or $j = j'$. Without loss of generality, we may suppose that $B \cap \mathcal{H} = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. Then the ovoids of $\mathcal{H}$ disjoint from $B \cap \mathcal{H}$ are the following:

$$
\begin{aligned}
&O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\},\ O_2 = \{x_{13}, x_{31}, x_{24}, x_{42}\},\ O_3 = \{x_{14}, x_{41}, x_{23}, x_{32}\},\\
&O_4 = \{x_{12}, x_{24}, x_{31}, x_{43}\},\ O_5 = \{x_{12}, x_{23}, x_{34}, x_{41}\},\ O_6 = \{x_{13}, x_{24}, x_{32}, x_{41}\},\\
&O_7 = \{x_{13}, x_{21}, x_{34}, x_{42}\},\ O_8 = \{x_{14}, x_{21}, x_{32}, x_{43}\},\ O_9 = \{x_{14}, x_{23}, x_{31}, x_{42}\}.
\end{aligned}
$$

The collection $\{\mathcal{C}_x \mid x \in B \setminus \mathcal{H}\}$ consists of eight of these nine ovoids. So, one of the above ovoids is missing in this collection.

Suppose one of the ovoids O_1, O_2 and O_3 is missing in the above collection. Without loss of generality, we may suppose that O_1 is the ovoid that is missing. Since $O_4 \cap O_5 =$

$\{x_{12}\}$ is a singleton, the two points of $PG(3,3) \setminus \mathcal{H}$ corresponding to O_4 and O_5 lie on the same $\mathbb{T}_1$-line through x_{12}. Then the other $\mathbb{T}_1$-line through x_{12} would not contain any point of B, a contradiction.

Suppose one of the ovoids $O_4, O_5, \ldots, O_9$ is missing in the above collection. Without loss of generality, we may suppose that O_4 is the ovoid that is missing. The ovoids $O_4 = \{x_{12}, x_{24}, x_{31}, x_{43}\}$ and $O' := \{x_{11}, x_{23}, x_{34}, x_{42}\}$ are disjoint and hence correspond to points y_4 and y' of $PG(3,3) \setminus \mathcal{H}$ such that the line y_4y' is external. Denote by y'' and y''' the other two points of the line y_4y', and by O'' and O''' the corresponding ovoids of $\mathcal{H}$. Then $\{O_4, O', O'', O'''\}$ is a partition of the point set of $\mathcal{H}$ in ovoids. So, these ovoids determine a partition of $B \cap \mathcal{H}$. Since $(B \cap \mathcal{H}) \cap O_4 = \emptyset$ and $|(B \cap \mathcal{H}) \cap O'| = 1$, each of the ovoids O'' and O''' intersects $B \cap \mathcal{H}$ in 1 or 2 points. It follows that none of the points y_4, y', y'', y''' belongs to B. This would imply that the external line y_4y' is disjoint from B, a contradiction. □

By recycling some of the arguments in the proof of Lemma 3.9, we show the following.

Lemma 3.10. *We have $|B| = q^2 + q$, $|B \cap \mathcal{H}| = 2q$ and $|B \setminus \mathcal{H}| = q^2 - q$. Moreover, there exist two intersecting $\mathbb{T}_0$-lines l_0 and l_1 such that $B \cap \mathcal{H} = (l_0 \cup l_1) \setminus (l_0 \cap l_1)$.*

Proof. By Corollary 3.7 and Lemma 3.9, there exists a secant plane π containing q points of B for which possibility (I) occurs. So, there exists a $\mathbb{T}_1$-line m contained in π such that $\pi \cap B = m \setminus \{x_m\}$. Let l_0 and l_1 denote the two $\mathbb{T}_0$-lines through x_m. In the plane π, there are exactly q secant lines disjoint from $\pi \cap B$, and each of these lines contains the point x_m. For each of these secant lines l, there exists by Lemma 3.8 a point $a \in B \cap \mathcal{H}$, necessarily belonging to $(l_0 \cup l_1) \setminus \{x_m\}$, for which $\pi_a \cap \pi = l$. In this way, we get a collection of $q \geq 3$ points belonging to $(l_0 \cup l_1) \cap B$. So, $B \cap \mathcal{H}$ cannot be an ovoid of $\mathcal{H}$ and hence $|B \cap \mathcal{H}| > q + 1$. As $|B| \leq q^2 + q$ by equation (2), this implies that $|B \setminus \mathcal{H}| < q^2 - 1$.

Suppose now there exists a point a of $(l_0 \cup l_1) \setminus \{x_m\}$ which is not in B. Then $\pi_a \cap \pi$ is a secant line l disjoint from $\pi \cap B$. Applying a similar argument as in the proof of Lemma 3.9, each of the $q-1$ secant planes through l contains at least q points of $B \setminus \mathcal{H}$ and the tangent plane π_a contains at least $q-1$ points of $B \setminus \mathcal{H}$. This would again lead to the inequality $|B \setminus \mathcal{H}| \geq q^2 - 1$, which is impossible.

Hence, all the $2q$ points of $(l_0 \cup l_1) \setminus \{x_m\}$ belong to B. As $|B| \leq q^2 + q$, $q^2 - q \leq |B \setminus \mathcal{H}|$ and $|B \cap \mathcal{H}| \leq 2q$ by equations (2), (3) and (4) respectively, this implies that $|B \cap \mathcal{H}| = 2q$, $|B \setminus \mathcal{H}| = q^2 - q$ and $|B| = q^2 + q$. Moreover, the $2q$ points of $B \cap \mathcal{H}$ are precisely the points of $(l_0 \cup l_1) \setminus \{x_m\}$. □

Invoking Lemma 3.10, we can now prove the following.

Proposition 3.11. *There exists a point x of $\mathcal{H}$ such that $B = \pi_x \setminus \{x\}$.*

Proof. Since $B \setminus \mathcal{H}$ is a set of size $q^2 - q$ blocking all external lines, Theorem 2.2 implies that there exists a point $x \in \mathcal{H}$ such that $B \setminus \mathcal{H} = \pi_x \setminus \mathcal{H}$. Every point y of $\pi_x \cap \mathcal{H}$ distinct from x is contained in a $\mathbb{T}_1$-line that is not contained in π_x. As this $\mathbb{T}_1$-line contains a point of B, we must have $y \in B$. So, $\pi_x \setminus \{x\}$ is contained in and hence equal to B (as both sets contain $q^2 + q$ points). □

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