



Smoothing estimates for non-dispersive equations

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Abstract This paper describes an approach to global smoothing problems for non-dispersive equations based on ideas of comparison principle and canonical transformation established in authors' previous paper (Ruzhansky and Sugimoto, Proc Lond Math Soc, 105:393–423, 2012), where dispersive equations were treated. For operators $a(D_x)$ of order m satisfying the dispersiveness condition $\nabla a(\xi) \neq 0$ for $\xi \neq 0$, the global smoothing estimate

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2)$$

is well-known, while it is also known to fail for non-dispersive operators. For the case when the dispersiveness breaks, we suggest the estimate in the form

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2)$$

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which is equivalent to the usual estimate in the dispersive case and is also invariant under canonical transformations for the operator $a(D_x)$. We show that this estimate and its variants do continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set. Moreover, other types of such estimates, and the case of time-dependent equations are also discussed.

1 Introduction

Various kinds of smoothing estimates for the solutions $u(t, x) = e^{ita(D_x)}\varphi(x)$ to equations of general form

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = \varphi(x) & \text{in } \mathbb{R}_x^n, \end{cases} \tag{1.1}$$

where $a(D_x)$ is the corresponding Fourier multiplier to a real-valued function $a(\xi)$, have been extensively studied under the ellipticity ($a(\xi) \neq 0$) or the dispersiveness ($\nabla a(\xi) \neq 0$) conditions, exclusive of the origin $\xi = 0$ (i.e. for $\xi \neq 0$) when $a(\xi)$ is a homogeneous function. Such conditions include Schrödinger equation as a special case ($a(\xi) = |\xi|^2$). Since Kato’s local gain of one derivative for the linearised KdV in [12], and the independent works by Ben-Artzi and Devinatz [1], Constantin and Saut [7], Sjölin [34] and Vega [38], the local, and then global smoothing estimates, together with their application to non-linear problem have been intensively investigated in a series of papers such as [2, 4, 5, 10, 11, 13–21, 27–30, 33, 35, 36, 39–41], to mention a few.

Among them, a comprehensive analysis is presented in our previous paper [30] for global smoothing estimates by using two useful methods, that is, *canonical transformations* and the *comparison principle*. Canonical transformations are a tool to transform one equation to another at the estimate level, and the comparison principle is a tool to relate differential estimates for solutions to different equations. These two methods work very effectively under the dispersiveness conditions to induce a number of new or refined global smoothing estimates, as well as many equivalences between them. Using these methods, the proofs of smoothing estimates are also considerably simplified.

The objective of this paper is to continue the investigation by the same approach in the case when the dispersiveness breaks down. We will conjecture what we may call an ‘invariant estimate’ extending the smoothing estimates to the non-dispersive case. Such an estimate yields the known smoothing estimates in dispersive cases, it is invariant under canonical transforms of the problem, and we will show its validity for a number of non-dispersive evolution equations of several different types.

The most typical example of a global smoothing estimate is of the form

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2), \tag{1.2}$$

where m denotes the order of the operator $a(D_x)$, and this estimate, together with other similar kind of global smoothing estimates, has been already justified under appropriate dispersiveness assumptions (see Sect. 2.1). Throughout this paper we use the standard notation

$$\langle x \rangle = (1 + |x|^2)^{1/2} \quad \text{and} \quad \langle D_x \rangle = (1 - \Delta_x)^{1/2}.$$

Note that the L^2 -norm of the solution is always the same as that of the Cauchy data φ for any fixed time t , but estimate (1.2) means that the extra gain of regularity of order $(m - 1)/2$ in the spacial variable x can be observed if we integrate the solution $e^{ita(D_x)}\varphi(x)$ to Eq. (1.1) in the time variable t . One interesting conclusion in [30] is that our method allowed us to carry out a global microlocal reduction of estimate (1.2) to the translation invariance of the Lebesgue measure.

On the other hand, despite their natural appearance in many problems, quite limited results are available for non-dispersive equations while the dispersiveness condition was shown to be necessary for most common types of global smoothing estimates (see [11]). To give an example, coupled dispersive equations are of high importance in applications while only limited analysis is available. Let $v(t, x)$ and $w(t, x)$ solve the following coupled system of Schrödinger equations:

$$\begin{cases} i\partial_t v = \Delta_x v + b(D_x)w, \\ i\partial_t w = \Delta_x w + c(D_x)v, \\ v(0, x) = v_0(x), w(0, x) = w_0(x). \end{cases} \quad (1.3)$$

This is the simplest example of Schrödinger equations coupled through linearised operators $b(D_x)$, $c(D_x)$. Such equations appear in many areas in physics. For example, this is a model of wave packets with two modes (in the presence of resonances), see Tan and Boyd [37]. In fibre optics they appear to describe certain types of a pair of coupled modulated wave-trains (see e.g. Manganaro and Parker [22]). They also describe the field of optical solitons in fibres (see Zen and Elim [42]) as well as Kerr dispersion and stimulated Raman scattering for ultrashort pulses transmitted through fibres. In these cases the linearised operators b and c would be of zero order. In models of optical pulse propagation of birefringent fibres and in wavelength-division-multiplexed systems they are of the first order (see Pelinovsky and Yang [25]). They may be of higher orders as well, for example in models of optical solitons with higher order effects (see Nakkeeran [24]).

Suppose now that we are in the simplest situation when system (1.3) can be diagonalised. Its eigenvalues are

$$a_{\pm}(\xi) = -|\xi|^2 \pm \sqrt{b(\xi)c(\xi)}$$

and the system uncouples into scalar equations of type (1.1) with operators $a(D_x) = a_{\pm}(D_x)$. Since the structure of operators $b(D_x)$, $c(D_x)$ may be quite involved, this motivates the study of scalar equations (1.1) with operators $a(D_x)$ of rather general form. Not only the presence of lower order terms is important in time global problems, the principal part may be rather general since we may have $\nabla a_{\pm} = 0$ at some points.

Let us also briefly mention another concrete example. Equations of the third order often appear in applications to water wave equations. For example, the Shrira equation [32] describing the propagation of a three-dimensional packet of weakly nonlinear internal gravity waves leads to third order polynomials in two dimensions. The same types of third order polynomials in two variables also appear in the Dysthe equation as

well as in the Hogan equation, both describing the behaviour of deep water waves in 2-dimensions. Strichartz estimates for the corresponding solutions have been analysed by e.g. Ghidaglia and Saut [9] and by Ben-Artzi et al. [3] by reducing the equations to pointwise estimates for operators in normal forms. In general, by linear changes of variables, polynomials $a(\xi_1, \xi_2)$ of order 3 are reduced to one of the following normal forms:

$$\begin{aligned} &\xi_1^3, \quad \xi_1^3 + \xi_2^3, \quad \xi_1^3 - \xi_1\xi_2^2, \\ &\xi_1^3 + \xi_2^2, \quad \xi_1\xi_2^2, \quad \xi_1\xi_2^2 + \xi_1^2, \\ &\xi_1^3 + \xi_1\xi_2, \quad \xi_1^3 + \xi_2^3 + \xi_1\xi_2, \quad \xi_1^3 - 3\xi_1\xi_2^2 + \xi_1^2 + \xi_2^2, \end{aligned} \tag{1.4}$$

modulo polynomials of order one. Strichartz estimates have been obtained for operators having their symbols in this list except for the cases $a(\xi_1, \xi_2) = \xi_1^3$ and $\xi_1\xi_2^2$, see Ben-Artzi et al. [3]. Some of normal forms listed here satisfy dispersive assumptions and can be discussed by existing results listed in Sect. 2.1. Indeed, $a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3$ and $\xi_1^3 - \xi_1\xi_2^2$ are homogeneous and satisfy the dispersiveness $\nabla a(\xi_1, \xi_2) \neq 0$ except for the origin [which is assumption (H) in Sect. 2.1]. However the dispersiveness assumption is sensitive to the perturbation by polynomials of order one. For example,

$$a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3 + \xi_1$$

still satisfies the dispersiveness $\nabla a(\xi_1, \xi_2) \neq 0$ everywhere [see assumption (L) in Sect. 2.1], but it breaks for

$$a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3 - \xi_1.$$

Furthermore the other normal forms do not satisfy neither of these assumptions, with the corresponding equation losing dispersiveness at some points (see Examples 3.2 and 3.5).

We now turn to describing an estimate which holds in such cases even when the dispersiveness fails at some points in the phase space. In this paper, based on the methods of comparison principle and canonical transforms, we develop several approaches to getting smoothing estimates in such non-dispersive cases. Since standard global smoothing estimates are known to fail in non-dispersive cases by [11], we will suggest an invariant form of global smoothing estimates instead, for the analysis, and call them *invariant estimates*, which we expect to continue to hold even in non-dispersive cases. As an example of it, estimate (1.2) may be rewritten in the form

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2). \tag{1.5}$$

Indeed the normal form $a(\xi_1, \xi_2) = \xi_1^3$ listed in (1.4) is known to satisfy this estimate (see estimate (5.6) in Corollary 5.7). Other types of invariant estimates are also suggested in Sect. 2.2. Such estimate has a number of advantages:

- in the dispersive case it is equivalent to the usual estimate (1.2);
- it does continue to hold for a variety of non-dispersive equations, where $\nabla a(\xi)$ may become zero on some set and when (1.2) fails;
- it does take into account possible zeros of the gradient $\nabla a(\xi)$ in the non-dispersive case, which is also responsible for the interface between dispersive and non-dispersive zone (e.g. how quickly the gradient vanishes);
- it is invariant under canonical transformations of the equation.

We will also try to justify invariant estimates for non-dispersive equations. The combination of the proposed two methods (canonical transformations and the comparison principles) has a good power again on the occasion of this analysis. Besides the simplification of the proofs of global smoothing estimates for standard dispersive equations, we have the following advantage in treating non-dispersive equations where the dispersiveness condition $\nabla a(\xi) \neq 0$ breaks:

- in radially symmetric cases, we can use the comparison principle of radially symmetric type (Theorem 3.1);
- in polynomial cases we can use the comparison principle of one dimensional type (Theorem 3.2);
- in the homogeneous case with some information on the Hessian, we can use canonical transformation to reduce the general case to some well-known model situations (Theorem 3.3);
- around non-dispersive points where the Hessian is non-degenerate, we can microlocalise and apply the canonical transformation based on the Morse lemma (Theorem 3.4);

In particular, in the radially symmetric cases, we will see that estimate (1.5) is valid in a generic situation (Theorem 3.1). And as another remarkable result, it is also valid for any differential operators with real constant coefficients, including operators corresponding to normal forms listed in (1.4) with perturbation by polynomials of order one (Theorem 3.2). Some normal forms are also covered by Theorem 3.4.

In addition, we will derive estimates for equations with time dependent coefficients. In general, the dispersive estimates for equations with time dependent coefficients may be a delicate problem, with decay rates heavily depending on the oscillation in coefficients (for a survey of different results for the wave equation with lower order terms see, e.g. Reissig [26]; for more general equations and systems and the geometric analysis of the time-decay rate of their solution see [31] or [8]). However, we will show in Sect. 4 that the smoothing estimates still remain valid if we introduce an appropriate factor into the estimate. Such estimates become a natural extension of the invariant estimates to the time dependent setting.

We will explain the organisation of this paper. In Sect. 2, we list typical global smoothing estimates for dispersive equations, and then discuss their invariant form which we expect to remain true also in non-dispersive situations. In Sect. 3, we establish invariant estimates for several types of non-dispersive equations. The case of time-dependent coefficients will be treated in Sect. 4. In Appendix A, we review our fundamental tools, that is, the canonical transformation and the comparison principle, which is used for the analysis in Sect. 3.

Finally we comment on the notation used in this paper. When we need to specify the entries of the vectors $x, \xi \in \mathbb{R}^n$, we write $x = (x_1, x_2, \dots, x_n), \xi = (\xi_1, \xi_2, \dots, \xi_n)$ without any notification. As usual we will denote $\nabla = (\partial_1, \dots, \partial_n)$ where $\partial_j = \partial_{x_j}$, $D_x = (D_1, D_2, \dots, D_n)$ where $D_j = -\sqrt{-1} \partial_j (j = 1, 2, \dots, n)$, and view operators $a(D_x)$ as Fourier multipliers. We denote the set of the positive real numbers $(0, \infty)$ by \mathbb{R}_+ . Constants denoted by letter C in estimates are always positive and may differ on different occasions, but will still be denoted by the same letter.

2 Invariant smoothing estimates for dispersive equations

In this section we collect known smoothing estimates for dispersive equations under several different assumptions. Then we show that the invariant estimate (1.5) holds in these cases and is, in fact, equivalent to the known estimates. Thus, let us consider the solution

$$u(t, x) = e^{ita(D_x)}\varphi(x)$$

to the equation

$$\begin{cases} (i\partial_t + a(D_x))u(t, x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = \varphi(x) & \text{in } \mathbb{R}_x^n, \end{cases}$$

where we always assume that function $a(\xi)$ is real-valued. We sometimes decompose the initial data φ into the sum of the *low frequency* part φ_l and the *high frequency* part φ_h , where

$$\text{supp } \widehat{\varphi}_l \subset \{\xi : |\xi| < 2R\}$$

and

$$\text{supp } \widehat{\varphi}_h \subset \{\xi : |\xi| > R\}$$

with sufficiently large $R > 0$. First we review a selection of known results on global smoothing estimates established in [30] when the dispersiveness assumption $\nabla a(\xi) \neq 0$ for $\xi \neq 0$ is satisfied, and then rewrite them in a form which is expected to hold even in non-dispersive situations.

2.1 Smoothing estimates for dispersive equations

First we collect known results for dispersive equations. Let

$$a_m = a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\}),$$

the *principal* part of $a(\xi)$, be a positively homogeneous function of order m , that is, satisfy

$$a_m(\lambda\xi) = \lambda^m a_m(\xi) \quad \text{for all } \lambda > 0 \quad \text{and} \quad \xi \neq 0.$$

First we consider the case that $a(\xi)$ has no lower order terms, and assume that $a(\xi)$ is *dispersive*:

$$a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0). \tag{H}$$

Theorem 2.1 *Assume (H). Then there exists a constant $C > 0$ such that the following estimates hold true.*

- *Suppose $n \geq 1, m > 0$, and $s > 1/2$. Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{2.1}$$

- *Suppose $m > 0$ and $(m - n + 1)/2 < \alpha < (m - 1)/2$. Or, in the elliptic case $a(\xi) \neq 0$ ($\xi \neq 0$), suppose $m > 0$ and $(m - n)/2 < \alpha < (m - 1)/2$. Then we have*

$$\left\| |x|^{\alpha-m/2} |D_x|^\alpha e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{2.2}$$

- *Suppose $n - 1 > m > 1$, but in the elliptic case $a(\xi) \neq 0$ ($\xi \neq 0$) suppose $n > m > 1$. Then we have*

$$\left\| \langle x \rangle^{-m/2} \langle D_x \rangle^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{2.3}$$

In the Schrödinger equation case $a(\xi) = |\xi|^2$ and for $n \geq 3$, estimate (2.1) was obtained by Ben-Artzi and Klainerman [2]. It follows also from a sharp local smoothing estimate by Kenig et al. [15, Theorem 4.1]), and also from the one by Chihara [5] who treated the case $m > 1$. For the range $m > 0$ and any $n \geq 1$ it was obtained in [30, Theorem 5.1].

Compared to (2.1), the estimate (2.2) is scaling invariant with the homogeneous weights $|x|^{-s}$ instead of non-homogenous ones $\langle x \rangle^{-s}$. The estimate (2.2) was obtained in [30, Theorem 5.2], and it is a generalisation of the result by Kato and Yajima [14] who treated the case $a(\xi) = |\xi|^2$ with $n \geq 3$ and $0 \leq \alpha < 1/2$, or with $n = 2$ and $0 < \alpha < 1/2$, and also of the one by Sugimoto [35] who treated elliptic $a(\xi)$ of order $m = 2$ with $n \geq 2$ and $1 - n/2 < \alpha < 1/2$.

The smoothing estimate (2.3) is of yet another type replacing $|D_x|^{(m-1)/2}$ by its non-homogeneous version $\langle D_x \rangle^{(m-1)/2}$, obtained in [30, Corollary 5.3]. It is a direct consequence of (2.1) with $s = m/2$ and (2.2) with $\alpha = 0$ (note also the L^2 -boundedness of $\langle D_x \rangle^{(m-1)/2} (1 + |D_x|^{(m-1)/2})^{-1}$), and it also extends the result by Kato and Yajima [14] who treated the case $a(\xi) = |\xi|^2$ and $n \geq 3$, the one by Walther [40] who treated the case $a(\xi) = |\xi|^m$, and the one by the authors [27] who treated the elliptic case with $m = 2$.

We can also consider the case that $a(\xi)$ has lower order terms, and assume that $a(\xi)$ is dispersive in the following sense:

$$a(\xi) \in C^\infty(\mathbb{R}^n), \quad \nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \\ |\partial^\alpha(a(\xi) - a_m(\xi))| \leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \geq 1. \quad (\mathbf{L})$$

Condition (L) may be formulated equivalently in the following way

$$a(\xi) \in C^\infty(\mathbb{R}^n), \quad |\nabla a(\xi)| \geq C \langle \xi \rangle^{m-1} \quad (\xi \in \mathbb{R}^n) \quad \text{for some } C > 0, \\ |\partial^\alpha(a(\xi) - a_m(\xi))| \leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \geq 1. \quad (\mathbf{L})$$

The last line of these assumptions simply amount to saying that the principal part a_m of a is positively homogeneous of order m for $|\xi| \geq 1$. Then we have the following estimates:

Theorem 2.2 *Assume (L). Then there exists a constant $C > 0$ such that the following estimates hold true.*

- Suppose $n \geq 1, m > 0$, and $s > 1/2$. Then we have

$$\left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \quad (2.4)$$

- Suppose $n \geq 1, m \geq 1$ and $s > 1/2$. Then we have

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \quad (2.5)$$

The estimate (2.4) was established in [30, Theorem 5.4]. Consequently, (2.5) is a straightforward consequence of (2.4) and the L^2 -boundedness of the Fourier multiplier $|D_x|^{(m-1)/2} \langle D_x \rangle^{-(m-1)/2}$ with $m \geq 1$. It is an analogue of (2.1) for operators $a(D_x)$ with lower order terms, and also a generalisation of [15, Theorem 4.1] who treated essentially polynomial symbols $a(\xi)$. For (2.5) in its full generality we refer to [30, Corollary 5.5].

Assumption (L) requires the condition $\nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n)$ for the full symbol, besides the same one $\nabla a_m(\xi) \neq 0 \quad (\xi \neq 0)$ for the principal term. To discuss what happens if we do not have the condition $\nabla a(\xi) \neq 0$, we can introduce an intermediate assumption between (H) and (L):

$$a(\xi) = a_m(\xi) + r(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \quad r(\xi) \in C^\infty(\mathbb{R}^n) \\ |\partial^\alpha r(\xi)| \leq C \langle \xi \rangle^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha. \quad (\mathbf{HL})$$

Theorem 2.2 remain valid if we replace assumption (H) by (HL) and functions $\varphi(x)$ in the estimates by their (sufficiently large) high frequency parts $\varphi_h(x)$. However we cannot control the low frequency parts $\varphi_l(x)$, and so have only the time local estimates on the whole, which we now state for future use:

Theorem 2.3 [30, Theorem 5.6] *Assume (HL). Suppose $n \geq 1, m > 0, s > 1/2$, and $T > 0$. Then we have*

$$\int_0^T \left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} e^{ia(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_x^n)}^2 dt \leq C \|\varphi\|_{L^2(\mathbb{R}^n)}^2,$$

where $C > 0$ is a constant depending on $T > 0$.

2.2 Invariant estimates

Let us now suggest an invariant form of time global smoothing estimates which might remain valid also in some areas without dispersion $\nabla a(\xi) \neq 0$, where standard smoothing estimates are known to fail (see Hoshiro [11]). We can equivalently rewrite estimates above under the dispersiveness assumption (H) or (L) in the form

$$\left\| w(x) \zeta(|\nabla a(D_x)|) e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2), \tag{2.6}$$

where w is a weight function of the form $w(x) = |x|^\delta$ or $\langle x \rangle^\delta$, and the smoothing is given by the function ζ on \mathbb{R}_+ of the form $\zeta(\rho) = \rho^\eta$ or $(1 + \rho^2)^{\eta/2}$, with some $\delta, \eta \in \mathbb{R}$. For example, we can rewrite estimate (2.1) of Theorem 2.1 as well as estimate (2.5) of Theorem 2.2 for the dispersive equations in the form

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{2.7}$$

Similarly we can rewrite estimate (2.2) of Theorem 2.1 in the form

$$\left\| |x|^{\alpha-m/2} |\nabla a(D_x)|^{\alpha/(m-1)} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (m \neq 1), \tag{2.8}$$

and estimate (2.3) of Theorem 2.1 (with $s = -m/2$) as well as estimate (2.4) of Theorem 2.2 (with $s > 1/2$) in the form

$$\left\| \langle x \rangle^{-s} \langle \nabla a(D_x) \rangle^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{2.9}$$

Indeed, under assumption (H) we clearly have $|\nabla a(\xi)| \geq c|\xi|^{m-1}$, so the equivalence between estimate (2.7) and estimate (2.1) in Theorem 2.1 follows from the fact that the Fourier multipliers $|\nabla a(D_x)|^{1/2} |D_x|^{-(m-1)/2}$ and $|\nabla a(D_x)|^{-1/2} |D_x|^{(m-1)/2}$ are bounded on $L^2(\mathbb{R}^n)$. Under assumption (L) the same argument works for large $|\xi|$, while for small $|\xi|$ both $\langle \xi \rangle^{(m-1)/2}$ and $|\nabla a(\xi)|^{1/2}$ are bounded away from zero. Thus we have the equivalence between estimate (2.7) and estimate (2.5) in Theorem 2.2. The same is true for the other equivalences. As we will see later (Theorem 5.2), estimate (2.6), and hence estimates (2.7)–(2.9) are invariant under canonical transformations.

On account of it, we will call estimate (2.6) an *invariant estimate*, and indeed we expect invariant estimates (2.7), (2.8), and (2.9) to hold without dispersiveness assumption (H) or (L), for $s > 1/2$, $(m - n)/2 < \alpha < (m - 1)/2$, and $s = -m/2$ ($n > m > 1$), respectively in ordinary settings (elliptic case for example), where $m > 0$ is the order of $a(D_x)$. We will discuss and establish them in Sect. 3 in a variety of situations.

Here is an intuitive understanding of the invariant estimate (2.7) with $s > 1/2$ by spectral argument. Let $E(\lambda)$ be the spectral family of the self-adjoint realisation of $a(D_x)$ on $L^2(\mathbb{R}^n)$, that is

$$(E(\lambda)f, g) = \int_{a(\xi) \leq \lambda} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi.$$

Then its spectral density

$$A(\lambda) = \frac{d}{d\lambda} E(\lambda) = \frac{1}{2\pi i} (R(\lambda + i0) - R(\lambda - i0)),$$

where $R(\lambda \pm i0)$ denotes the “limit” of the resolvent $R(\lambda \pm i\varepsilon)$ as $\varepsilon \searrow 0$, is given by

$$(A(\lambda)f, g) = \frac{d}{d\lambda} (E(\lambda)f, g) = \int_{a(\xi)=\lambda} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \frac{d\xi}{|\nabla a(\xi)|}$$

whenever $\nabla a(\xi) \neq 0$. Hence we have the identity

$$(A(\lambda)|\nabla a(D_x)|^{1/2}f, |\nabla a(D_x)|^{1/2}g) = \int_{a(\xi)=\lambda} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi$$

which continues to have a meaning even if $\nabla a(\xi)$ may vanishes (although this is just a formal observation). On the other hand, the right hand side of this identity with $f = g$ has the uniform estimate

$$\int_{a(\xi)=\lambda} |\widehat{f}(\xi)|^2 d\xi \leq C \|\langle x \rangle^s f\|_{L^2}^2$$

in $\lambda \in \mathbb{R}$ by the one-dimensional Sobolev embedding (It can be justified at least when $\nabla a(\xi) \neq 0$. See [6, Lemma 1]). If once we justify them, we could have the estimate

$$\sup_{\lambda \in \mathbb{R}} |(A(\lambda)H^*f, H^*f)| \leq C \|f\|_{L^2}^2, \tag{2.10}$$

where H^* is the adjoint operator of $H = \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2}$. Estimate (2.10) can be regarded as the $a(D_x)$ -smooth property

$$\sup_{\mu \notin \mathbb{R}} |([R(\mu) - R(\bar{\mu})]H^*f, H^*f)| \leq C \|f\|_{L^2}^2$$

which is equivalent to invariant estimate (2.7) (see [12, Theorem 5.1]). Or we may proceed as in [1] or [2] to obtain the same conclusion. In fact, we have

$$\begin{aligned} \left(H e^{i t a(D_x)} \varphi(x), \psi(t, x) \right)_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i t \lambda} (A(\lambda) \varphi, H^* \psi(t, \cdot)) d \lambda d t \\ &= \int_{-\infty}^{\infty} (A(\lambda) \varphi, H^* \tilde{\psi}(\lambda, \cdot)) d \lambda, \end{aligned}$$

where $\tilde{\psi}(\lambda, x) = \int_{-\infty}^{\infty} e^{-i t \lambda} \psi(t, x) d t$, and

$$\begin{aligned} \left| (A(\lambda) \varphi, H^* \tilde{\psi}(\lambda, \cdot)) \right| &\leq (A(\lambda) \varphi, \varphi)^{1/2} (A(\lambda) H^* \tilde{\psi}(\lambda, \cdot), H^* \tilde{\psi}(\lambda, \cdot))^{1/2} \\ &\leq C (A(\lambda) \varphi, \varphi)^{1/2} \|\tilde{\psi}(\lambda, \cdot)\|_{L^2} \end{aligned}$$

by estimate (2.10). Hence by Schwartz inequality, Plancherel’s theorem, and the fact $\int_{-\infty}^{\infty} (A(\lambda) \varphi, \varphi) d \lambda = \|\varphi\|_{L^2(\mathbb{R}_x^n)}^2$, we have the estimate

$$\left| \left(H e^{i t a(D_x)} \varphi(x), \psi(t, x) \right)_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \right| \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \|\psi(t, x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)}$$

which is again equivalent to invariant estimate (2.7).

Let us now show that the invariant estimate (2.7) with $s > 1/2$ is also a refinement of another known estimate for non-dispersive equations, namely of the smoothing estimate obtained by Hoshiro [10]. If operator $a(D_x)$ has real-valued symbol $a = a(\xi) \in C^1(\mathbb{R}^n)$ which is positively homogeneous of order $m \geq 1$ and no dispersiveness assumption is made, Hoshiro [10] showed the estimate

$$\left\| \langle x \rangle^{-s} \langle D_x \rangle^{-s} |a(D)|^{1/2} e^{i t a(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2). \tag{2.11}$$

But once we prove (2.7) with $s > 1/2$, we can have a better estimate

$$\left\| \langle x \rangle^{-s} \langle D_x \rangle^{-1/2} |a(D)|^{1/2} e^{i t a(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2)$$

with respect to the number of derivatives. In fact, using the Euler’s identity

$$m a(\xi) = \xi \cdot \nabla a(\xi),$$

we see that this estimate trivially follows from

$$\begin{aligned} \left\| \langle x \rangle^{-s} \langle D_x \rangle^{-1/2} |D_x|^{1/2} |\nabla a(D)|^{1/2} e^{i t a(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \\ \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2), \end{aligned}$$

which in turn follows from (2.7) with $s > 1/2$ because the Fourier multiplier operator $\langle D_x \rangle^{-1/2} |D_x|^{1/2}$ is $L^2(\mathbb{R}^n)$ -bounded. In fact, estimate (2.11) holds only because of the homogeneity of a , since in this case by Euler’s identity zeros of a contain zeros of ∇a . In general, estimate (2.11) cuts off too much, and therefore does not reflect the nature of the problem for non-homogeneous symbols, as (2.7) still does.

In terms of invariant estimates, we can also give another explanation to the reason why we do not have time-global estimate in Theorem 2.3. The problem is that the symbol of the smoothing operator $\langle D_x \rangle^{(m-1)/2}$ does not vanish where the symbol of $\nabla a(D_x)$ vanishes, as should be anticipated by the invariant estimate (2.7). If zeros of $\nabla a(D_x)$ are not taken into account, the weight should change to the one as in estimate (2.3).

3 Smoothing estimates for non-dispersive equations

Canonical transformations and the comparison principle recalled for convenience in Appendix A are still important tools when we discuss the smoothing estimates for *non-dispersive equations*

$$\begin{cases} (i \partial_t + a(D_x)) u(t, x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = \varphi(x) & \text{in } \mathbb{R}_x^n, \end{cases}$$

where real-valued functions $a(\xi)$ fail to satisfy dispersive assumption (H) or (L) in Sect. 2.1. However, the secondary comparison tools stated in Appendix A.3 work very effectively in such situations. In Corollary 5.8 for example, even if we lose the dispersiveness assumption at zeros of f' , the estimate is still valid because σ must vanish at the same points with the order determined by condition $|\sigma(\rho)| \leq A|f'(\rho)|^{1/2}$. The same is true in other comparison results in Corollary 5.9. In this section, we will treat the smoothing estimates of non-dispersive equations based on these observations.

3.1 Radially symmetric case

The following result states that we still have estimate (2.7) of invariant form suggested in Sect. 2.2 even for non-dispersive equations in a general setting of the radially symmetric case. Remarkably enough, it is a straight forward consequence of the second comparison method of Corollary 5.8, and in this sense, it is just an equivalent expression of the translation invariance of the Lebesgue measure (see Appendix A.3):

Theorem 3.1 *Suppose $n \geq 1$ and $s > 1/2$. Let $a(\xi) = f(|\xi|)$, where $f \in C^1(\mathbb{R}_+)$ is real-valued. Assume that f' has only finitely many zeros. Then we have*

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{3.1}$$

Proof Noticing $|\nabla a(\xi)| = |f'(|\xi|)|$, use Corollary 5.8 for $\sigma(\rho) = |f'(\rho)|^{1/2}$ in each interval where f is strictly monotone. □

Example 3.1 As a consequence of Theorem 3.1, we have the estimate of invariant form (2.7) if $a(\xi)$ is a real polynomial of $|\xi|$. This fact is not a consequence of Theorems 2.1 or 2.2. For example, let

$$a(\xi) := f(|\xi|^2)^2,$$

with $f(\rho)$ being a non-constant real polynomial on \mathbb{R} . The principal part $a_m(\xi)$ of $a(\xi)$ is a power of $|\xi|^2$ multiplied by a constant, hence it satisfies $\nabla a_m(\xi) \neq 0$ ($\xi \neq 0$). If $f(\rho)$ is a homogeneous polynomial, then $a(\xi)$ satisfies assumption (H) and we have estimate (2.7) by Theorem 2.1. In the case when $f(\rho)$ is not homogeneous, trivially $a(\xi)$ does not satisfy (H). Furthermore $a(\xi)$ does not always satisfy assumption (L) either since

$$\nabla a(\xi) = 4f(|\xi|^2)f'(|\xi|^2)\xi$$

vanishes on the set $|\xi|^2 = c$ such that $f(c) = 0$ or $f'(c) = 0$ as well as at the origin $\xi = 0$. Hence Theorem 2.2 does not assure the estimate (2.7) for $a(\xi) = (|\xi|^2 - 1)^2$ (we take $f(\rho) = \rho - 1$), for example, but even in this case, we have the invariant smoothing estimate (3.1) by Theorem 3.1.

3.2 Polynomial case

Another remarkable fact is that we can obtain invariant estimate (2.7) for all differential equations with real constant coefficients if we use second comparison method of Corollary 5.9 (hence again it is just an equivalent expression of the translation invariance of the Lebesgue measure):

Theorem 3.2 *Suppose $n \geq 1$ and $s > 1/2$. Let $a(\xi)$ be a real polynomial. Then we have*

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{3.2}$$

Proof We can assume that the polynomial a is not a constant since otherwise the estimate is trivial. Thus, let $m \geq 1$ be the degree of polynomial $a(\xi)$, and we write it in the form

$$a(\xi) = p_0 \xi_1^m + p_1(\xi') \xi_1^{m-1} + \dots + p_{m-1}(\xi') \xi_1 + p_m(\xi'),$$

where $p_k(\xi')$ is a real polynomial in $\xi' = (\xi_2, \dots, \xi_n)$ of degree k ($k = 0, 1, \dots, m$). The polynomial equation

$$\partial_1 a(\xi) = m p_0 \xi_1^{m-1} + (m - 1) p_1(\xi') \xi_1^{m-2} + \dots + p_{m-1}(\xi') = 0$$

in ξ_1 has at most $m - 1$ real roots. For $k = 0, 1, \dots, m - 1$, let U_k be the set of $\xi' \in \mathbb{R}^{n-1}$ for which it has k distinct real simple roots $\lambda_{k,1}(\xi') < \lambda_{k,2}(\xi') \dots < \lambda_{k,k}(\xi')$, and let

$$\Omega_{k,l} := \{(\xi_1, \xi') \in \mathbb{R}^n : \xi' \in U_k, \lambda_{k,l}(\xi') < \xi_1 < \lambda_{k,l+1}(\xi')\}$$

for $l = 0, 1, \dots, k$, regarding $\lambda_{k,0}$ and $\lambda_{k,k+1}$ as $-\infty$ and ∞ respectively. Then we have the decomposition

$$\Omega = \{\xi \in \mathbb{R}^n : \partial_1 a(\xi) \neq 0\} = \bigcup_{\substack{k=0,1,\dots,m-1, \\ l=0,1,\dots,k-1}} \Omega_{k,l},$$

and $a(\xi)$ is strictly monotone in ξ_1 on each $\Omega_{k,l}$. Hence by taking χ in Corollary 5.9 to be the characteristic functions $\chi_{k,l}$ of the set $\Omega_{k,l}$, we have the estimate

$$\left\| \langle x_1 \rangle^{-s} \chi_{k,l}(D_x) |\partial_1 a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Taking the sum in k, l and noting the fact that the Lebesgue measure of the complement of the set Ω is zero, we have

$$\left\| \langle x_1 \rangle^{-s} |\partial_1 a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Regarding other variables ξ_j ($j = 2, \dots, n$) as the first one, we also then have

$$\left\| \langle x_j \rangle^{-s} |\partial_j a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

and combining them all we have

$$\left\| \langle x \rangle^{-s} \left(|\partial_1 a(D_x)|^{1/2} + \dots + |\partial_n a(D_x)|^{1/2} \right) e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

by trivial inequalities $\langle x \rangle^{-s} \leq \langle x_j \rangle^{-s}$. Substituting $\eta(D_x)\varphi$ for φ in the estimate, where

$$\eta(\xi) = |\nabla a(\xi)|^{1/2} \left(|\partial_1 a(\xi)|^{1/2} + \dots + |\partial_n a(\xi)|^{1/2} \right)^{-1},$$

we have estimate (3.2) if we note the boundedness of $\eta(\xi)$ and use Plancherel’s theorem. □

Example 3.2 Some of normal forms listed in (1.4) in Introduction satisfy dispersiveness assumptions. Indeed, $a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3$ and $\xi_1^3 - \xi_1 \xi_2^2$ are homogeneous and satisfy the assumption (H). The other normal forms however satisfy neither (H) nor (L). For example, $a(\xi_1, \xi_2) = \xi_1^3$ and $\xi_1 \xi_2^2$ are homogeneous but $\nabla a(\xi_1, \xi_2) = 0$ when $\xi_1 = 0$ and $\xi_2 = 0$ respectively. On the other hand $a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^2$ and $\xi_1 \xi_2^2 + \xi_1^2$ are not homogeneous and satisfy $\nabla a(\xi_1, \xi_2) = 0$ at the origin. See Example 3.5 for others. But even for them we still have invariant smoothing estimate (2.7) with $s > 1/2$ by Theorem 3.2.

3.3 Hessian at non-dispersive points

Now we will present another approach to treat non-dispersive equations. Recall that, in [30, Section 5], the method of canonical transformation effectively works to reduce smoothing estimates for dispersive equations listed in Sect. 2.1 to some model estimates. For example, as mentioned in the beginning of Appendix A.3, estimate (2.1) in Theorem 2.1 is reduced to model estimates (5.6) and (5.7) in Corollary 5.7. We explain here that this strategy works for non-dispersive cases as well.

We will however look at the rank of the Hessian $\nabla^2 a(\xi)$, instead of the principal type assumption $\nabla a(\xi) \neq 0$. Assume now that $a = a(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ is real-valued and positively homogeneous of order two. It can be noted that from Euler’s identity we obtain

$$\nabla a(\xi) = {}^t \xi \nabla^2 a(\xi) \tag{3.3}$$

since $\nabla a(\xi)$ is homogeneous of order one (here ξ is viewed as a row). Then the condition $\text{rank } \nabla^2 a(\xi) = n$ implies $\nabla a(\xi) \neq 0$ ($\xi \neq 0$), and as we have already explained, we have invariant estimates (2.7), (2.8) and (2.9) by Theorem 2.1 in this favourable case. We will show that in the non-dispersive situation the rank of $\nabla^2 a(\xi)$ still has a responsibility for smoothing properties. We assume

$$\text{rank } \nabla^2 a(\xi) \geq k \quad \text{whenever } \nabla a(\xi) = 0 \quad (\xi \neq 0) \tag{3.4}$$

with some $1 \leq k \leq n - 1$. We note that condition (3.4) is invariant under the canonical transformation in the following sense:

Lemma 3.1 *Let $a = \sigma \circ \psi$, with $\psi : U \rightarrow \mathbb{R}^n$ satisfying $\det D\psi(\xi) \neq 0$ on an open set $U \subset \mathbb{R}^n$. Then, for each $\xi \in U$, $\nabla a(\xi) = 0$ if and only if $\nabla \sigma(\psi(\xi)) = 0$. Furthermore the ranks of $\nabla^2 a(\xi)$ and $\nabla^2 \sigma(\psi(\xi))$ are equal on Γ whenever $\xi \in U$ and $\nabla a(\xi) = 0$.*

Proof Differentiation gives

$$\nabla a(\xi) = \nabla \sigma(\psi(\xi)) D\psi(\xi)$$

and we have the first assertion. Another differentiation gives

$$\nabla^2 a(\xi) = \nabla^2 \sigma(\psi(\xi)) D\psi D\psi$$

when $\nabla a(\xi) = 0$. This implies the second assertion. □

To fix the notation, we assume

$$\nabla a(e_n) = 0 \quad \text{and} \quad \text{rank } \nabla^2 a(e_n) = k \quad (1 \leq k \leq n - 1), \tag{3.5}$$

where $e_n = (0, \dots, 0, 1)$. Then we have $a(e_n) = 0$ by Euler’s identity $2a(\xi) = \xi \cdot \nabla a(\xi)$. We claim that there exists a conic neighbourhood $\Gamma \subset \mathbb{R}^n \setminus 0$ of e_n and a homogeneous C^∞ -diffeomorphism $\psi : \Gamma \rightarrow \tilde{\Gamma}$ (satisfying $\psi(\lambda\xi) = \lambda\psi(\xi)$) for all $\lambda > 0$ and $\xi \in \Gamma$) as appeared in Appendix A.1 such that we have the form

$$a(\xi) = (\sigma \circ \psi)(\xi), \quad \sigma(\eta) = c_1\eta_1^2 + \dots + c_k\eta_k^2 + r(\eta_{k+1}, \dots, \eta_n), \quad (3.6)$$

where $\eta = (\eta_1, \dots, \eta_n)$ and $c_j = \pm 1$ ($j = 1, 2, \dots, k$). We remark that r must be real-valued and positively homogeneous of order two. We will prove the existence of such ψ that will satisfy (3.6). By (3.3), (3.5), and the symmetricity, all the entries of the matrix $\nabla^2 a(e_n)$ are zero except for the (perhaps) non-zero upper left $(n - 1) \times (n - 1)$ corner matrix. Moreover, by a linear transformation involving only the first $(n - 1)$ variables of $\xi = (\xi_1, \dots, \xi_{n-1}, \xi_n)$, we may assume $\partial^2 a / \partial \xi_1^2(e_n) \neq 0$. We remark that (3.5) still holds under this transformation. Then, by the Malgrange preparation theorem, we can write

$$a(\xi) = \pm c(\xi)^2(\xi_1^2 + a_1(\xi')\xi_1 + a_2(\xi')), \quad \xi' = (\xi_2, \dots, \xi_n). \quad (3.7)$$

locally in a neighbourhood of e_n , where $c(\xi) > 0$ is some strictly positive function, while function a_1 and a_2 are smooth and real-valued. Restricting this expression to the hyperplane $\xi_n = 1$, and using the homogeneity

$$a(\xi) = \pm \xi_n^2 a(\xi_1/\xi_n, \dots, \xi_{n-1}/\xi_n, 1),$$

we can extend the expression (3.7) to a conic neighbourhood Γ of e_n , so that functions $c(\xi)$, $a_1(\xi')$ and $a_2(\xi')$ are positively homogeneous of orders zero, one, and two, respectively. Let us define $\psi_0(\xi) = c(\xi)\xi$ and

$$\tau(\eta) = \pm(\eta_1^2 + a_1(\eta')\eta_1 + a_2(\eta')),$$

so that

$$a(\xi) = (\tau \circ \psi_0)(\xi),$$

where we write $\eta = (\eta_1, \eta')$, $\eta' = (\eta_2, \dots, \eta_n)$. Furthermore, let us define $\psi_1(\xi) = (\xi_1 + \frac{1}{2}a_1(\xi'), \xi')$, so that $\tau(\xi) = (\sigma \circ \psi_1)(\xi)$ with $\sigma(\eta) = \eta_1^2 + r(\eta')$, where $r(\eta') = a_2(\eta') - \frac{1}{4}a_1(\eta')^2$ is positively homogeneous of degree two. Then we have $a = \sigma \circ \psi$, where $\psi = \psi_1 \circ \psi_0$, and thus we have the expression (3.6) with $k = 1$.

We note that, by the construction, we have

$$\psi(e_n) = c(e_n) \left(\frac{1}{2}a_1(e'_n), e'_n \right),$$

where $c(e_n) > 0$ and $e'_n = (0, \dots, 0, 1) \in \mathbb{R}^{n-1}$. Then we can see that the function $r(\eta')$ of $(n - 1)$ -variables is defined on a conic neighbourhood of e'_n in \mathbb{R}^{n-1} . On account of this fact and Lemma 3.1, we can apply the same argument above to $r(\eta')$, and repeating the process k -times, we have the expression (3.6).

To complete the argument, we check that $\det D\psi_0(\xi) = c(\xi)^n$, which clearly implies $\det D\psi(\xi) = c(\xi)^n$, and assures that it does not vanish on a sufficiently narrow Γ . We observe first that

$$D\psi_0(\xi) = c(\xi)I_n + {}^t\xi \nabla c(\xi),$$

where I_n is the identity n by n matrix. We note that if we consider the matrix $A = (\alpha_i \beta_j)_{i,j=1}^n = {}^t \alpha \cdot \beta$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, then A has rank one, so its eigenvalues are $n - 1$ zeros and some λ . But $\text{Tr } A$ is also the sum of the eigenvalues, hence $\lambda = \text{Tr } A$. Now, let $\alpha = \xi$, $\beta = \nabla c(\xi)$, and $A = {}^t \alpha \cdot \beta$. Since $c(\xi)$ is homogeneous of order zero, by Euler's identity we have

$$\text{Tr } A = \xi \cdot \nabla c(\xi) = 0,$$

hence all eigenvalues of A are zero. It follows now that there is a non-degenerate matrix S such that $S^{-1}AS$ is strictly upper triangular. But then $\det D\psi_0(\xi) = \det(c(\xi)I_n + S^{-1}AS)$, where matrix $c(\xi)I_n + S^{-1}AS$ is upper triangular with n copies of $c(\xi)$ at the diagonal. Hence $\det D\psi_0(\xi) = c(\xi)^n$.

On account of the above observations, we have the following result which states that invariant estimates (2.7), (2.8) and (2.9) with $m = 2$ still hold for a class of non-dispersive equations:

Theorem 3.3 *Let $a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be real-valued and satisfy $a(\lambda\xi) = \lambda^2 a(\xi)$ for all $\lambda > 0$ and $\xi \neq 0$. Assume that $\text{rank } \nabla^2 a(\xi) \geq n - 1$ whenever $\nabla a(\xi) = 0$ and $\xi \neq 0$.*

- *Suppose $n \geq 2$ and $s > 1/2$. Then we have*

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

- *Suppose $(4 - n)/2 < \alpha < 1/2$, or $(3 - n)/2 < \alpha < 1/2$ in the elliptic case $a(\xi) \neq 0$ ($\xi \neq 0$). Then we have*

$$\left\| |x|^{\alpha-1} |\nabla a(D_x)|^\alpha e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

- *Suppose $n > 4$, or $n > 3$ in the elliptic case $a(\xi) \neq 0$ ($\xi \neq 0$). Then we have*

$$\left\| \langle x \rangle^{-1} \langle \nabla a(D_x) \rangle^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Proof By microlocalisation and an appropriate rotation, we may assume $\text{supp } \widehat{\varphi} \subset \Gamma$, where $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ is a sufficiently narrow conic neighbourhood of the direction $e_n = (0, \dots, 0, 1)$. Since everything is alright in the dispersive case $\nabla a(e_n) \neq 0$ by Theorem 2.1, we may assume $\nabla a(e_n) = 0$. We may also assume $n \geq 2$ since $\nabla a(e_n) = 0$ implies $\nabla a(\xi) = 0$ for all $\xi \neq 0$ in the case $n = 1$. Then we have $\text{rank } \nabla^2 a(e_n) \neq n$ by the relation (3.3), hence $\text{rank } \nabla^2 a(e_n) = n - 1$ by the assumption $\text{rank } \nabla^2 a(\xi) \geq n - 1$. In the setting (3.5) and (3.6) above, we have

$$\text{rank } \nabla^2 \widetilde{r}(\psi(e_n)) = 0 \tag{3.8}$$

by Lemma 3.1, where $\widetilde{r}(\eta) = r(\eta_{k+1}, \dots, \eta_n)$. Since $k = n - 1$ in our case, we can see that r is a function of one variable and r'' vanishes identically by (3.8) and the homogeneity of r . Then r is a polynomial of order one, but is also positively

homogeneous of order two. Hence we can conclude that $r = 0$ identically, and have the relation

$$a(\xi) = (\sigma \circ \psi)(\xi), \quad \sigma(\eta) = c_1 \eta_1^2 + \dots + c_{n-1} \eta_{n-1}^2.$$

Now, we have the estimates

$$\begin{aligned} \left\| \langle x \rangle^{-s} |\nabla \sigma(D_x)|^{1/2} e^{it\sigma(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}, \\ \left\| |x|^{\alpha-1} |\nabla \sigma(D_x)|^\alpha e^{it\sigma(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}, \\ \left\| \langle x \rangle^{-1} \langle \nabla \sigma(D_x) \rangle^{1/2} e^{it\sigma(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}, \end{aligned}$$

if we use the trivial inequalities $\langle x \rangle^{-s} \leq \langle x' \rangle^{-s}$, $|x|^{\alpha-1} \leq |x'|^{\alpha-1}$ and $\langle x \rangle^{-1} \leq \langle x' \rangle^{-1}$, Theorem 2.1 with respect to x' , and the Plancherel theorem in x_n , where $x = (x', x_n)$ and $x' = (x_1, \dots, x_{n-1})$. On account of Theorem 5.2 and the L^2_{-s} , $\dot{L}^2_{\alpha-1}$, L^2_{-1} -boundedness of the operators $I_{\psi, \gamma}$ and $I_{\psi, \gamma}^{-1}$ for $(1/2 <)s < n/2$, $-n/2 < \alpha - 1 (< -1/2)$ (see Theorem 5.4), respectively, we have the conclusion. \square

Example 3.3 The function

$$a(\xi) = b(\xi)^2$$

satisfies the condition in Theorem 3.3, where $b(\xi)$ is a positively homogeneous function of order one such that $\nabla b(\xi) \neq 0$ ($\xi \neq 0$). Indeed, if $b(\xi)$ is elliptic, then $\nabla a(\xi) = 2b(\xi)\nabla b(\xi) \neq 0$ ($\xi \neq 0$). If $b(\xi_0) = 0$ at a point $\xi_0 \neq 0$, then $\nabla a(\xi_0) = 0$ and further differentiation immediately yields $\nabla^2 a(\xi_0) = 2t' \nabla b(\xi_0) \nabla b(\xi_0)$, and clearly we have $\text{rank } \nabla^2 a(\xi_0) \geq 1$. Especially in the case $n = 2$, $a(\xi)$ meets the condition in Theorem 3.3. As an example, we consider

$$a(\xi) = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2}.$$

Setting $b(\xi) = \xi_1 \xi_2 / |\xi|$, we clearly have $a(\xi) = b(\xi)^2$ and $\nabla b(\xi) = \left(\frac{\xi_2^3}{|\xi|^3}, \frac{\xi_1^3}{|\xi|^3} \right)$, hence $\nabla b(\xi) \neq 0$ ($\xi \neq 0$). Although $\nabla a(\xi) = 0$ on the lines $\xi_1 = 0$ and $\xi_2 = 0$, we have invariant estimates (2.7), (2.8) and (2.9) in virtue of Theorem 3.3. This is an illustration of a smoothing estimate for the Cauchy problem for an equation like

$$i \partial_t \Delta u + D_1^2 D_2^2 u = 0,$$

which can be reduced to the second order non-dispersive pseudo-differential equation with symbol $a(\xi)$ above. Similarly, we have these estimates for more general case

$$a(\xi) = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} + \xi_3^2 + \dots + \xi_n^2$$

since we obtain $\text{rank } \nabla^2 a(\xi) \geq n - 1$ from the observation above.

3.4 Isolated critical points

Next we consider more general operators $a(\xi)$ of order m which may have some lower order terms. Then even the most favourable case $\det \nabla^2 a(\xi) \neq 0$ does not imply the dispersive assumption $\nabla a(\xi) \neq 0$. The method of canonical transformation, however, can also allow us to treat this problem by obtaining localised estimates near points ξ where $\nabla a(\xi) = 0$.

Assume that ξ_0 is a non-degenerate critical point of $a(\xi)$, that is, that we have $\nabla a(\xi_0) = 0$ and $\det \nabla^2 a(\xi_0) \neq 0$. Let us microlocalise around ξ_0 , so that we only look at what happens around ξ_0 . In this case, the order of the symbol $a(\xi)$ does not play any role and we do not distinguish between the main part and lower order terms. Let Γ denote a sufficiently small open bounded neighbourhood of ξ_0 so that ξ_0 is the only critical point of $a(\xi)$ in Γ . Since $\nabla^2 a(\xi_0)$ is symmetric and non-degenerate, we may assume

$$\nabla^2 a(\xi_0) = \text{diag}\{\pm 1, \dots, \pm 1\}$$

by a linear transformation. By Morse lemma for $a(\xi)$, there exists a diffeomorphism $\psi : \Gamma \rightarrow \tilde{\Gamma} \subset \mathbb{R}^n$ with an open bounded neighbourhood of the origin such that

$$a(\xi) = (\sigma \circ \psi)(\xi), \quad \sigma(\eta) = c_1 \eta_1^2 + \dots + c_n \eta_n^2,$$

where $\eta = (\eta_1, \dots, \eta_n)$ and $c_j = \pm 1$ ($j = 1, 2, \dots, n$). From Theorem 2.1 applied to operator $\sigma(D_x)$, we obtain the estimates

$$\begin{aligned} \left\| \langle x \rangle^{-s} |\nabla \sigma(D_x)|^{1/2} e^{i\sigma(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (s > 1/2), \\ \left\| \langle x \rangle^{-1} |\nabla \sigma(D_x)|^{1/2} e^{i\sigma(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (n > 2). \end{aligned}$$

Hence by Theorem 5.2, together with the L^2_{-s}, L^2_{-1} -boundedness of the operators $I_{\psi, \gamma}$ and $I_{\psi, \gamma}^{-1}$ (which is assured by Theorem 5.3), we have these estimates with $\sigma(D_x)$ replaced by $a(D_x)$ assuming $\text{supp } \widehat{\varphi} \subset \Gamma$. On the other hand, we have the same estimates for general φ by Theorem 2.2 if we assume condition (L). The above argument, however, assures that the following weak assumption is also sufficient if $a(\xi)$ has finitely many critical points and they are non-degenerate:

$$\begin{aligned} a(\xi) &\in C^\infty(\mathbb{R}^n), \quad |\nabla a(\xi)| \geq C \langle \xi \rangle^{m-1} \quad (\text{for large } \xi \in \mathbb{R}^n) \quad \text{for some } C > 0, \\ |\partial^\alpha (a(\xi) - a_m(\xi))| &\leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \gg 1. \quad (\mathbf{L}') \end{aligned}$$

Thus, summarising the above argument, we have established the following result of invariant estimates (2.7) and (2.9):

Theorem 3.4 *Let $a \in C^\infty(\mathbb{R}^n)$ be real-valued and assume that it has finitely many critical points, all of which are non-degenerate. Assume also (L').*

- *Suppose $n \geq 1, m \geq 1$, and $s > 1/2$. Then we have*

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

- *Suppose $n > 2$ and $m \geq 1$. Then we have*

$$\left\| \langle x \rangle^{-1} \langle \nabla a(D_x) \rangle^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Example 3.4 It is easy to see that

$$a(\xi) = \xi_1^4 + \dots + \xi_n^4 + |\xi|^2$$

satisfies the assumption of Theorem 3.4. We remark that Morii [23] also established the first estimate in Theorem 3.4 under a more restrictive condition but which also allows this example.

Example 3.5 Some normal forms listed in (1.4) are also covered by Theorem 3.4 although they are covered by Theorem 3.2. Indeed

$$a(\xi_1, \xi_2) = \xi_1^3 + \xi_1 \xi_2$$

has a unique critical point at the origin and it is non-degenerate. It is also easy to see that

$$a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3 + \xi_1 \xi_2$$

and

$$\xi_1^3 - 3\xi_1 \xi_2^2 + \xi_1^2 + \xi_2^2$$

have their critical points at $(\xi_1, \xi_2) = (0, 0), (-1/3, -1/3)$ and $(\xi_1, \xi_2) = (0, 0), (1/3, \pm 1/\sqrt{3}), (-2/3, 0)$ respectively, and all of them are non-degenerate.

4 Equations with time-dependent coefficients

Finally we discuss smoothing estimates for equations with time-dependent coefficients:

$$\begin{cases} (i\partial_t + b(t, D_x)) u(t, x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x^n, \\ u(0, x) = \varphi(x) & \text{in } \mathbb{R}_x^n. \end{cases} \tag{4.1}$$

If the symbol $b(t, \xi)$ is independent of t , invariants estimates (2.7), (2.8) and (2.9) say that $\nabla_{\xi} b(t, D_x)$ is responsible for the smoothing property. The natural question here is what quantity replaces it if $b(t, \xi)$ depends on t .

We can give an answer to this question if $b(t, \xi)$ is of the product type

$$b(t, \xi) = c(t)a(\xi),$$

where we only assume that $c(t) > 0$ is a continuous function. In the case of dispersive and Strichartz estimates for higher order (in ∂_t) equations the situation may be very delicate and in general depends on the rates of oscillations of $c(t)$ (see e.g. Reissig [26] for the case of the time-dependent wave equation, or [8, 31] for more general equations).

For smoothing estimates, we will be able to state a rather general result in Theorem 4.1 below. The final formulae show that a natural extension of the invariant estimates of the previous section still remain valid in this case. In this special case, the Eq. (4.1) can be transformed to the equation with time-independent coefficients. In fact, by the assumption for $c(t)$, the function

$$C(t) = \int_0^t c(s) ds$$

is strictly monotone and the inverse $C^{-1}(t)$ exists. Then the function

$$v(t, x) := u(C^{-1}(t), x)$$

satisfies

$$\partial_t v(t, x) = \frac{1}{c(C^{-1}(t))} (\partial_t u)(C^{-1}(t), x),$$

hence $v(t, x)$ solves the equation

$$\begin{cases} (i \partial_t + a(D_x)) v(t, x) = 0, \\ v(0, x) = \varphi(x), \end{cases}$$

if $u(t, x)$ is a solution to Eq. (4.1). By this argument, invariant estimates for $v(t, x) = e^{ita(D_x)} \varphi(x)$ should imply also estimates for the solution

$$u(t, x) = v(C(t), x) = e^{i \int_0^t b(s, D_x) ds} \varphi(x)$$

to Eq. (4.1). For example, if we notice the relations

$$\|v(\cdot, x)\|_{L^2} = \left\| |c(\cdot)|^{1/2} u(\cdot, x) \right\|_{L^2}$$

and

$$c(t)\nabla a(D_x) = \nabla_\xi b(t, D_x),$$

we obtain the estimate

$$\left\| \langle x \rangle^{-s} |\nabla_\xi b(t, D_x)|^{1/2} e^{i \int_0^t b(s, D_x) ds} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \tag{4.2}$$

from the invariant estimate (2.7). Estimate (4.2) is a natural extension of the invariant estimate (2.7) to the case of time-dependent coefficients, which says that $\nabla_\xi b(t, D_x)$ is still responsible for the smoothing property. From this point of view, we may call it an *invariant estimate* too. We can also note that estimate (4.2) may be also obtained directly, by formulating an obvious extension of the comparison principles to the time dependent setting. We also have similar estimates from the invariant estimates (2.8) and (2.9). The same method of the proof yields the following:

Theorem 4.1 *Let $[\alpha, \beta] \subset [-\infty, +\infty]$. Assume that function $c = c(t)$ is continuous on $[\alpha, \beta]$ and that $c \neq 0$ on (α, β) . Let $u = u(t, x)$ be the solution of Eq. (4.1) with $b(t, \xi) = c(t)a(\xi)$, where $a = a(\xi)$ satisfies assumptions of any part of Theorems 2.1, 2.2, 2.3, 3.1, 3.3 or 3.4. Then the smoothing estimate of the corresponding theorem holds provided we replace $L^2(\mathbb{R}_t, \mathbb{R}_x^n)$ by $L^2([\alpha, \beta], \mathbb{R}_x^n)$, and insert $|c(t)|^{1/2}$ in the left hand side norms.*

We note that it is possible that $\alpha = -\infty$ and that $\beta = +\infty$, in which case by continuity of c at such points we simply mean that the limits of $c(t)$ exist as $t \rightarrow \alpha+$ and as $t \rightarrow \beta-$.

To give an example of an estimate from Theorem 4.1, let us look at the case of the first statement of Theorem 2.1. In that theorem, we suppose that $a(\xi)$ satisfies assumption (H), and we assume $n \geq 1, m > 0$, and $s > 1/2$. Theorem 2.1 assures that in this case we have the smoothing estimate (2.1), which is

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Theorem 4.1 states that solution $u(t, x)$ of Eq. (4.1) satisfies this estimate provided we replace $L^2(\mathbb{R}_t, \mathbb{R}_x^n)$ by $L^2([\alpha, \beta], \mathbb{R}_x^n)$, and insert $|c(t)|^{1/2}$ in the left hand side norm. This means that u satisfies

$$\left\| \langle x \rangle^{-s} |c(t)|^{1/2} |D_x|^{(m-1)/2} u(t, x) \right\|_{L^2([\alpha, \beta] \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

The same is true with statements of any of Theorems 2.1, 2.2, 2.3, 3.1, 3.3 or 3.4.

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Appendix A: Canonical transformation and comparison principle

For convenience of the reader in this appendix we briefly recall two powerful tools introduced in [30] for getting smoothing estimates, that is, the canonical transformation and the comparison principle, which enable us to induce global smoothing estimates for dispersive equations rather easily, and formulate several corollaries of these methods to be used in the analysis of this paper. In particular, Theorem 5.2 explains the invariance of (1.5) and similar estimates under canonical transforms. Also, Corollary 5.9 is instrumental in treating equations with polynomial symbols (i.e. differential evolution equations) in Sect. 3.2.

We remark that all known smoothing estimates from Sect. 2.1 were proved in [30] by using these two methods.

A.1 Canonical transformation

The first tool is the canonical transformation which transforms the equation with the operator $a(D_x)$ and the Cauchy data $\varphi(x)$ to that with $\sigma(D_x)$ and $g(x)$ at the estimate level, where $a(D_x)$ and $\sigma(D_x)$ are related with each other as $a(\xi) = (\sigma \circ \psi)(\xi)$.

Let $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^n$ be open sets and $\psi : \Gamma \rightarrow \tilde{\Gamma}$ be a C^∞ -diffeomorphism (we do not assume them to be cones since we do not require homogeneity of phases). We always assume that

$$C^{-1} \leq |\det \partial\psi(\xi)| \leq C \quad (\xi \in \Gamma), \tag{5.1}$$

for some $C > 0$. Let $\gamma \in C^\infty(\Gamma)$ and $\tilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\tilde{\Gamma})$ be cut-off functions which satisfy $\text{supp } \gamma \subset \Gamma, \text{supp } \tilde{\gamma} \subset \tilde{\Gamma}$. Then we set

$$\begin{aligned} I_{\psi,\gamma}u(x) &= \mathcal{F}^{-1} \left[\gamma(\xi) \mathcal{F}u(\psi(\xi)) \right] (x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\Gamma} e^{i(x \cdot \xi - y \cdot \psi(\xi))} \gamma(\xi) u(y) dy d\xi, \\ I_{\psi,\gamma}^{-1}u(x) &= \mathcal{F}^{-1} \left[\tilde{\gamma}(\xi) \mathcal{F}u(\psi^{-1}(\xi)) \right] (x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\tilde{\Gamma}} e^{i(x \cdot \xi - y \cdot \psi^{-1}(\xi))} \tilde{\gamma}(\xi) u(y) dy d\xi. \end{aligned} \tag{5.2}$$

In the case that $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^n \setminus \{0\}$ are open cones, we may consider the homogeneous ψ and γ which satisfy $\text{supp } \gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1}$ and $\text{supp } \tilde{\gamma} \cap \mathbb{S}^{n-1} \subset \tilde{\Gamma} \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$. Then we have the expressions for compositions

$$I_{\psi,\gamma} \cdot \sigma(D_x) = (\sigma \circ \psi)(D_x) \cdot I_{\psi,\gamma}, \quad I_{\psi,\gamma}^{-1} \cdot (\sigma \circ \psi)(D_x) = \sigma(D_x) \cdot I_{\psi,\gamma}^{-1}$$

which enable us to relate $a(D_x)$ with $\sigma(D_x)$ when $a(\xi) = (\sigma \circ \psi)(\xi)$.

We also introduce the weighted L^2 -spaces. For a weight function $w(x)$, let $L^2_w(\mathbb{R}^n; w)$ be the set of measurable functions $f: \mathbb{R}^n \rightarrow \mathbb{C}$ such that the norm

$$\|f\|_{L^2(\mathbb{R}^n; w)} = \left(\int_{\mathbb{R}^n} |w(x)f(x)|^2 dx \right)^{1/2}$$

is finite. Then we have the following fundamental theorem:

Theorem 5.1 [30, Theorem 4.1] *Assume that the operator $I_{\psi, \gamma}$, defined by (5.2) is $L^2(\mathbb{R}^n; w)$ -bounded. Suppose that we have the estimate*

$$\left\| w(x)\rho(D_x)e^{i\sigma(D_x)}g(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|g\|_{L^2(\mathbb{R}_x^n)}$$

for all g such that $\text{supp } \widehat{g} \subset \text{supp } \widetilde{\gamma}$. Assume also that the function

$$q(\xi) = \frac{\gamma \cdot \zeta}{\rho \circ \psi}(\xi)$$

is bounded. Then we have

$$\left\| w(x)\zeta(D_x)e^{i\sigma(D_x)}\varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

for all φ such that $\text{supp } \widehat{\varphi} \subset \text{supp } \gamma$, where $a(\xi) = (\sigma \circ \psi)(\xi)$.

We remark that invariant estimate (2.6) introduced in Sect. 2.2, hence estimates (2.7)–(2.9) are invariant under canonical transformations by Theorem 5.1. More precisely, we have the following theorem:

Theorem 5.2 *Let ζ be a function on \mathbb{R}_+ of the form $\zeta(\rho) = \rho^\eta$ or $(1 + \rho^2)^{\eta/2}$ with some $\eta \in \mathbb{R}$. Assume that the operators $I_{\psi, \gamma}$ and $I_{\psi, \gamma}^{-1}$ defined by (5.2) are $L^2(\mathbb{R}^n; w)$ -bounded. Then the following two estimates*

$$\begin{aligned} \left\| w(x)\zeta(|\nabla a(D_x)|)e^{i\sigma(D_x)}\varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (\text{supp } \widehat{\varphi} \subset \text{supp } \gamma), \\ \left\| w(x)\zeta(|\nabla \sigma(D_x)|)e^{i\sigma(D_x)}\varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad (\text{supp } \widehat{\varphi} \subset \text{supp } \widetilde{\gamma}) \end{aligned}$$

are equivalent to each other, where $a = \sigma \circ \psi \in C^1$ on $\text{supp } \gamma$.

Proof Note that $\nabla a(\xi) = \nabla \sigma(\psi(\xi))D\psi(\xi)$ and $C|\nabla a(\xi)| \leq |\nabla \sigma(\psi(\xi))| \leq C'|\nabla a(\xi)|$ on $\text{supp } \gamma$ with some $C, C' > 0$, which is assured by the assumption (5.1). Then the result is obtained from Theorem 5.1. □

As for the $L^2(\mathbb{R}^n; w)$ -boundedness of the operator $I_{\psi, \gamma}$, we have criteria for some special weight functions. For $\kappa \in \mathbb{R}$, let $L^2_\kappa(\mathbb{R}^n)$, $\dot{L}^2_\kappa(\mathbb{R}^n)$ be the set of measurable functions f such that the norm

$$\|f\|_{L^2_\kappa(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\langle x \rangle^\kappa f(x)|^2 dx \right)^{1/2}, \quad \|f\|_{\dot{L}^2_\kappa(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |x|^\kappa |f(x)|^2 dx \right)^{1/2}$$

is finite, respectively. Then we have the following:

Theorem 5.3 [30, Theorem 4.2] *Suppose $\kappa \in \mathbb{R}$. Assume that all the derivatives of entries of the $n \times n$ matrix $\partial\psi$ and those of γ are bounded. Then the operators $I_{\psi,\gamma}$ and $I_{\psi,\gamma}^{-1}$ defined by (5.2) are $L^2_\kappa(\mathbb{R}^n)$ -bounded.*

Theorem 5.4 [30, Theorem 4.3] *Let $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^n \setminus \{0\}$ be open cones. Suppose $|\kappa| < n/2$. Assume $\psi(\lambda\xi) = \lambda\psi(\xi)$, $\gamma(\lambda\xi) = \gamma(\xi)$ for all $\lambda > 0$ and $\xi \in \Gamma$. Then the operators $I_{\psi,\gamma}$ and $I_{\psi,\gamma}^{-1}$ defined by (5.2) are $L^2_\kappa(\mathbb{R}^n)$ -bounded and $\dot{L}^2_\kappa(\mathbb{R}^n)$ -bounded.*

A.2 Comparison principle

The second tool is the comparison principle which relates the smoothing estimate for the solution $u(t, x) = e^{itf(D_x)}\varphi(x)$ with the operator $f(D_x)$ of the smoothing $\sigma(D_x)$ to that for $v(t, x) = e^{itg(D_x)}\varphi(x)$ with $g(D_x)$ of $\tau(D_x)$:

Theorem 5.5 [30, Theorem 2.5] *Let $f, g \in C^1(\mathbb{R}_+)$ be real-valued and strictly monotone on the support of a measurable function χ on \mathbb{R}_+ . Let $\sigma, \tau \in C^0(\mathbb{R}_+)$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq A \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

for all $\rho \in \text{supp } \chi$ satisfying $f'(\rho) \neq 0$ and $g'(\rho) \neq 0$. Then we have

$$\|\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A \|\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}^n$.

Theorem 5.6 [30, Corollary 2.2] *Let $f, g \in C^1(\mathbb{R}^n)$ be real-valued functions such that, for almost all $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$, $f(\xi)$ and $g(\xi)$ are strictly monotone in ξ_1 on the support of a measurable function χ on \mathbb{R}^n . Let $\sigma, \tau \in C^0(\mathbb{R}^n)$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\xi)|}{|\partial_1 f(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|\partial_1 g(\xi)|^{1/2}}$$

for all $\xi \in \text{supp } \chi$ satisfying $\partial_1 f(\xi) \neq 0$ and $\partial_1 g(\xi) \neq 0$. Then we have

$$\begin{aligned} & \left\| \chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x_1, x') \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})} \\ & \leq A \left\| \chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(\tilde{x}_1, x') \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})} \end{aligned}$$

for all $x_1, \tilde{x}_1 \in \mathbb{R}$, where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$.

Let us now repeat here important examples of the use of the comparison principle discussed in [30]. Applying Theorem 5.6 with $n = 1$ in two directions, we immediately obtain that for $l, m > 0$, we have

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{\frac{l}{m}} \left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \tag{5.3}$$

for every $x \in \mathbb{R}$, assuming that $\text{supp } \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. Here we neglect $x' = (x_2, \dots, x_n)$ in a natural way and just write $x = x_1$ and $D_x = D_1$. Applying Theorem 5.6 with $n = 2$, we similarly obtain that for $l, m > 0$, we have

$$\begin{aligned} & \left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \\ &= \left\| |D_y|^{(l-1)/2} e^{itD_x|D_y|^{l-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \end{aligned} \tag{5.4}$$

for every $x \in \mathbb{R}$. Here we have used the notation $(x, y) = (x_1, x_2)$, and $(D_x, D_y) = (D_1, D_2)$. On the other hand, in the case $n = 1$, we have easily

$$\left\| e^{itD_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \quad \text{for all } x \in \mathbb{R}, \tag{5.5}$$

which is a straightforward consequence of the fact $e^{itD_x} \varphi(x) = \varphi(x+t)$ and the translation invariance of the Lebesgue measure. By using equality (5.5), we can estimate the right hand sides of equalities (5.3) and (5.4) with $l = 1$, and as a result, we have the following low dimensional pointwise estimates

$$\begin{aligned} & \left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}, \\ & \left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \leq C \|\varphi\|_{L^2(\mathbb{R}_{x,y}^2)} \end{aligned}$$

for all $x \in \mathbb{R}$, from which we straightforwardly obtain the following result:

Corollary 5.7 [30, Corollary 3.3] *Suppose $n \geq 1, m > 0$, and $s > 1/2$. Then we have*

$$\left\| \langle x_1 \rangle^{-s} |D_1|^{(m-1)/2} e^{it|D_1|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{5.6}$$

Suppose $n \geq 2, m > 0$, and $s > 1/2$. Then we have

$$\left\| \langle x_1 \rangle^{-s} |D_n|^{(m-1)/2} e^{itD_1|D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{5.7}$$

We remark that estimate (5.6) has been already the invariant estimate (2.7) for the normal form $a(\xi) = \xi_1^m$ (if we replace the weight $\langle x_1 \rangle^{-s}$ by a smaller one $\langle x \rangle^{-s}$).

A.3 Secondary comparison

We remark that Corollary 5.7 is just a consequence of trivial equality (5.5), and the proof of Theorem 2.1 was carried out in [30] by reducing estimate (2.1) to estimate (5.6) (elliptic case) or estimate (5.7) (non-elliptic case) in Corollary 5.7 via canonical transformations discussed in Appendix A.1. Let us further compare estimates in Theorem 2.1 and Corollary 5.7 by using the comparison principle again to obtain *secondary comparison* results. In this sense, the results stated below are obtained from just the translation invariance of the Lebesgue measure via a combination use of the comparison principle and the canonical transformation.

Now, in notation of Theorem 5.5, setting $\tau(\rho) = \rho^{(m-1)/2}$ and $g(\rho) = \rho^m$, we have $|\tau(\rho)|/|g'(\rho)|^{1/2} = m^{-1/2}$. Hence, noticing that $\chi(D_x)$ is L^2 -bounded for $\chi \in L^\infty$, we obtain the following result from Theorem 2.1 with $a(\xi) = |\xi|^m$:

Corollary 5.8 [30, Corollary 7.3] *Suppose $n \geq 1, s > 1/2$. Let $\chi \in L^\infty(\mathbb{R}_+)$. Let $f \in C^1(\mathbb{R}_+)$ be real-valued and strictly monotone on $\text{supp } \chi$. Let $\sigma \in C^0(\mathbb{R}_+)$ be such that for some $A > 0$ we have*

$$|\sigma(\rho)| \leq A|f'(\rho)|^{1/2}$$

for all $\rho \in \text{supp } \chi$. Then we have

$$\left\| \langle x \rangle^{-s} \chi(|D_x|) \sigma(|D_x|) e^{itf(|D_x|)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

Similarly, in notation of Theorem 5.6, setting $\tau(\xi) = |\xi_1|^{(m-1)/2}$ and $g(\xi) = |\xi_1|^m$, we have $|\tau(\xi)|/|\partial g / \partial \xi_1(\xi)|^{1/2} = m^{-1/2}$. Then we obtain the following result from estimate (5.6) of Corollary 5.7:

Corollary 5.9 *Suppose $n \geq 1$ and $s > 1/2$. Let $\chi \in L^\infty(\mathbb{R}^n)$. Let $f \in C^1(\mathbb{R}^n)$ be a real-valued function such that, for almost all $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$, $f(\xi)$ is strictly monotone in ξ_1 on $\text{supp } \chi$. Let $\sigma \in C^0(\mathbb{R}^n)$ be such that for some $A > 0$ we have*

$$|\sigma(\xi)| \leq A |\partial_1 f(\xi)|^{1/2}$$

for all $\xi \in \text{supp } \chi$. Then we have

$$\left\| \langle x_1 \rangle^{-s} \chi(D_x) \sigma(D_x) e^{itf(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

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