

# GLOBAL WELL-POSEDNESS OF THE KIRCHHOFF EQUATION AND KIRCHHOFF SYSTEMS

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ABSTRACT. This article is devoted to review the known results on global well-posedness for the Cauchy problem to the Kirchhoff equation and Kirchhoff systems with small data. Similar results will be obtained for the initial-boundary value problems in exterior domains with compact boundary. Also, the known results on large data problems will be reviewed together with open problems.

## 1. INTRODUCTION

G. Kirchhoff proposed the equation

$$\partial_t^2 u - \left( 1 + \int_{\Omega} |\nabla u(t, y)|^2 dy \right) \Delta u = 0 \quad (t \in \mathbf{R}, x \in \Omega)$$

in his book of mathematical physics in 1883, as a model equation for transversal motion of the elastic string, where  $\Omega$  is a domain in  $\mathbf{R}^n$  (see [13] and also [22]). Since then, it was the first that in 1940 Bernstein proved the existence of global in time analytic solutions on an interval of real line in his celebrated paper [2]. As is easily seen, this equation has a Hamilton structure, nevertheless it involves a mysterious problem *whether or not, one can prove the existence of time global solutions with large data in Gevrey class,  $C^\infty$  class or standard Sobolev spaces*. The results on global well-posedness in the Sobolev spaces  $H^{3/2}$ , or  $H^2$  with small data are well established in [3, 4, 5, 6, 8, 9, 11, 16, 17, 25, 26, 28, 29]. There, the classes of small data consist of compactly supported functions, or more generally, they are characterised by some weight conditions (see [3, 4, 5, 6, 8]) or oscillatory integrals (see [11, 16, 17, 26]). Recently, the present authors studied the global well-posedness for Kirchhoff systems with small data (see [19]), and extended all the previous results. Here, the class of data in [19] consists of Sobolev space  $(H^1)^m$ ,  $m$  being the order of system, and is characterised by some oscillatory integrals.

The object in this article is to review the known results cited above, and open problems for the Kirchhoff equation or Kirchhoff systems. More precisely, let us consider the Cauchy problem for the Kirchhoff equation of the form

$$\partial_t^2 u - \left( 1 + \int_{\mathbf{R}^n} |\nabla u(t, y)|^2 dy \right) \Delta u = 0, \quad t \neq 0, \quad x \in \mathbf{R}^n, \quad (1.1)$$

$$u(0, x) = f_0(x), \quad \partial_t u(0, x) = f_1(x), \quad x \in \mathbf{R}^n. \quad (1.2)$$

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Kirchhoff systems are of also interest. For  $m$  column vector-valued function  $U = U(t, x)$  on  $\mathbf{R} \times \mathbf{R}^n$ , consider

$$D_t U = A(s(t), D_x)U, \quad t \neq 0, \quad x \in \mathbf{R}^n, \quad (1.3)$$

$$U(0, x) = {}^t(f_0(x), f_1(x), \dots, f_{m-1}(x)), \quad x \in \mathbf{R}^n, \quad (1.4)$$

where

$$D = i^{-1}\partial \quad (i = \sqrt{-1}),$$

$A(s, D_x)$  is an  $m \times m$  pseudo-differential system with a suitably smooth element for  $s \in \mathbf{R}$  in a neighbourhood of 0;  $s(t)$  is a quadratic form

$$s(t) = \langle SU, U \rangle_{\mathbf{L}^2(\mathbf{R}^n)} = \int_{\mathbf{R}^n} {}^t(SU(t, x))\overline{U(t, x)} dx,$$

where  $S$  is an  $m \times m$  Hermitian matrix, and we put

$$\mathbf{L}^2(\mathbf{R}^n) = (L^2(\mathbf{R}^n))^m.$$

Assume that system (1.3) is strictly hyperbolic: Characteristic polynomial of  $D_t - A(s, D_x)$  has real and distinct roots  $\varphi_1(s, \xi), \dots, \varphi_m(s, \xi)$  for any  $s$  in the domain of definition of the  $A(s, \xi)$  and for any  $\xi \in \mathbf{R}^n \setminus 0$ , i.e.,

$$\det(\tau I - A(s, \xi)) = (\tau - \varphi_1(s, \xi)) \cdots (\tau - \varphi_m(s, \xi)). \quad (1.5)$$

Hence, for  $\delta > 0$  sufficiently small, we may assume that

$$\inf_{|\xi|=1, s \in [0, \delta]} |\varphi_j(s, \xi) - \varphi_k(s, \xi)| > 0 \quad (1.6)$$

for  $j \neq k$ . Needless to say, Kirchhoff systems (1.3) cover the Kirchhoff equation (1.1).

The approach of this paper yields new results already in the scalar case of the classical Kirchhoff equation but, in fact, we are able to make advances for coupled equations as well or, more generally, for Kirchhoff systems (see §2). To this end, Kirchhoff systems are of interest but present several major complications compared to the scalar case. First of all, even for the linearised system, it is much more difficult to find a suitable representation of solutions which would, on one hand, work with the low regularity ( $C^1$ ) of coefficients while, on the other hand, allow one to obtain sufficiently good estimates for solutions. Moreover, in the case of systems of higher order, it is impossible to find its characteristics explicitly, and the geometry of the system or rather of the level sets of the characteristics enters the picture.

We will present the three kinds of well-posedness considered in the literature, namely:

- (1) Kirchhoff equation with large data
- (2) Kirchhoff equation with small data
- (3) Kirchhoff systems with small data

The main idea behind items 2–3 is to approach the problem by developing the “asymptotic integration” method for the linearised equation to be able to control its solutions to the extent of being able to prove a priori estimates necessary for the handling of the fully nonlinear problem.

We recall the definition of Sobolev spaces. We denote by  $H^\sigma(\mathbf{R}^n) = (1-\Delta)^{-\sigma/2}L^2(\mathbf{R}^n)$  for  $\sigma \in \mathbf{R}$  the standard Sobolev spaces, and their homogeneous version is  $\dot{H}^\sigma(\mathbf{R}^n) = (-\Delta)^{-\sigma/2}L^2(\mathbf{R}^n)$ . We put  $H^\infty(\mathbf{R}^n) = \bigcap_{j=0}^{\infty} H^j(\mathbf{R}^n)$ .

## 2. MODEL OF KIRCHHOFF SYSTEMS

In this section, for the convenience of the considerations, we shall give some examples of Kirchhoff systems (1.3). These examples are applicable for our result. The first example is:

**Example 2.1.** Let us consider the Cauchy problem to the Kirchhoff equations of higher order:

$$L(D_t, D_x, s(t))u \equiv D_t^m u + \sum_{j \leq m-1, |\nu|+j=m} b_{\nu,j}(s(t)) D_x^\nu D_t^j u = 0,$$

$$D_t^k u(0, x) = f_k(x), \quad k = 0, 1, \dots, m-1.$$

Here the quadratic form  $s(t)$  is given by

$$s(t) = \int_{\mathbf{R}^n} \sum_{|\beta|=|\gamma|=m-1} s_{\beta\gamma} D^\beta u(t, x) \overline{D^\gamma u(t, x)} dx,$$

where  $\beta = (\beta_t, \beta_x)$ ,  $\gamma = (\gamma_t, \gamma_x)$ ,  $D^\beta = D_t^{\beta_t} D_x^{\beta_x}$  and  $s_{\beta\gamma} = \overline{s_{\gamma\beta}}$ . We assume that the symbol  $L(\tau, \xi, s)$  of the differential operator  $L(D_t, D_x, s)$  has real and distinct roots  $\varphi_1(s, \xi), \dots, \varphi_m(s, \xi)$  for  $\xi \neq 0$  and  $0 \leq s \leq \delta$  with  $\delta > 0$ , i.e.,

$$L(\tau, \xi, s) = (\tau - \varphi_1(s, \xi)) \cdots (\tau - \varphi_m(s, \xi)),$$

$$\inf_{s \in [0, \delta], |\xi|=1} |\varphi_j(s, \xi) - \varphi_k(s, \xi)| > 0$$

for  $j \neq k$ . By taking the Fourier transform in the space variables and introducing the vector

$$V(t, \xi) = {}^T(|\xi|^{m-1} \hat{u}(t, \xi), |\xi|^{m-2} D_t \hat{u}(t, \xi), \dots, D_t^{m-1} \hat{u}(t, \xi)),$$

we reduce the problem to the system

$$D_t V = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -H_m(s(t), \xi) & -H_{m-1}(s(t), \xi) & \dots & -H_1(s(t), \xi) \end{pmatrix} |\xi| V = A(s(t), \xi) V,$$

where we put

$$H_j(s(t), \xi) = \sum_{|\nu|=j} b_{\nu, m-j}(s(t)) (\xi/|\xi|)^\nu, \quad (j = 1, \dots, m).$$

As a new example of (1.3), we can treat the completely coupled Kirchhoff equations of the following type.

**Example 2.2.** Let us consider the Cauchy problem to the coupled Kirchhoff equations of the following type:

$$\begin{aligned} \partial_t^2 u - a_1 (1 + \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2) \Delta u + P_1(t, D_x)v &= 0, \\ \partial_t^2 v - a_2 (1 + \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2) \Delta v + P_2(t, D_x)u &= 0, \\ \partial_t^j u(0, x) = u_j(x), \quad \partial_t^j v(0, x) = v_j(x), \quad j = 0, 1, \end{aligned} \quad (2.7)$$

for some second order homogeneous polynomials  $P_1(t, D_x), P_2(t, D_x)$ , and for some constants  $a_1, a_2 > 0$  with  $a_1 \neq a_2$ . The quadratic form is given here by

$$s(t) = \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2.$$

We assume that

$$|\xi|^{-2} P_k(t, \xi) \in \text{Lip}_{\text{loc}}(\mathbf{R}; L^\infty(\mathbf{R}^n \setminus \{0\})), \quad (2.8)$$

$$|\xi|^{-2} \partial_t P_k(t, \xi) \in L^1(\mathbf{R}; L^\infty(\mathbf{R}^n \setminus \{0\})), \quad (2.9)$$

for  $k = 1, 2$ , and that

$$\inf_{t \in \mathbf{R}, |\xi|=1} \{(a_1 - a_2)^2 + 4P_1(t, \xi)P_2(t, \xi)\} > 0, \quad (2.10)$$

$$\inf_{t \in \mathbf{R}, |\xi|=1} \{a_1^2 a_2^2 - P_1(t, \xi)P_2(t, \xi)\} > 0. \quad (2.11)$$

By taking the Fourier transform in the space variables and introducing the vector

$$V(t, \xi) = {}^t(|\xi|\widehat{u}(t, \xi), \widehat{u}'(t, \xi), |\xi|\widehat{v}(t, \xi), \widehat{v}'(t, \xi)),$$

we rewrite (2.7) as a system

$$D_t V = \begin{pmatrix} 0 & -i|\xi| & 0 & 0 \\ ic_1(t)^2|\xi| & 0 & iP_1(t, \xi)|\xi|^{-1} & 0 \\ 0 & 0 & 0 & -i|\xi| \\ iP_2(t, \xi)|\xi|^{-1} & 0 & ic_2(t)^2|\xi| & 0 \end{pmatrix} V =: A(s(t), \xi)V,$$

where

$$c_k(t) = \sqrt{a_k(1 + s(t))}, \quad k = 1, 2.$$

The four characteristic roots of the equation

$$\det(\tau I - A(s(t), \xi)) = 0$$

in  $\tau$  are given by

$$\varphi_{1,2,3,4}(s(t), \xi) = \pm \frac{|\xi|}{\sqrt{2}} \sqrt{c_1(t)^2 + c_2(t)^2 \pm \sqrt{\{c_1(t)^2 - c_2(t)^2\}^2 + 4P_1(t, \xi)P_2(t, \xi)|\xi|^{-4}}}.$$

Then it follows from (2.8)–(2.11) that

$$\inf_{s \in [0, \delta], |\xi|=1} |\varphi_j(s, \xi) - \varphi_k(s, \xi)| > 0$$

for  $j \neq k$ .

## 3. KNOWN RESULTS AND OPEN PROBLEMS WITH LARGE DATA

In this section we shall review some known results on global well-posedness of the Kirchhoff equation with large data. Let us introduce some classes  $\mathcal{N}_\varphi$  which ensure the global well-posedness for (1.1)–(1.2):

$$\mathcal{N}_\varphi = \bigcup_{\eta>0} \left\{ (f_0, f_1) \in H^1 \times L^2 : \int_{\mathbf{R}^n} \left\{ |\xi|^2 |\hat{f}_0(\xi)|^2 + |\hat{f}_1(\xi)|^2 \right\} e^{\eta\varphi(|\xi|)} d\xi < \infty \right\}$$

for some function  $\varphi : [0, \infty) \rightarrow [0, \infty)$ . Here  $\hat{f}(\xi)$  stands for the Fourier transform of  $f(x)$ .

When  $\varphi_0(|\xi|) = |\xi|$ ,  $\mathcal{N}_{\varphi_0}$  coincides with analytic class  $\mathcal{A}_{L^2}$  of  $L^2$  type. Bernstein proved that  $\mathcal{A}_{L^2}$  well-posedness for  $n = 1$  (see [2]). After him, Arosio and Spagnolo discussed analytic solutions in higher spatial dimensions (see [1]). Also, we refer to the result of Kajitani and Yamaguti who proved analytic well-posedness for the degenerate Kirchhoff equation (see [12]). There are also quasi-analytic classes which ensure the global well-posedness. Here quasi-analytic classes are intermediate ones between the analytic class and  $C^\infty$ -class. Nishihara found a weight  $\varphi_1(|\xi|) = |\xi|/\log(e + |\xi|)$ , and treated more general weight functions  $\varphi_1$  with convexity condition (see [23]). For the sake of simplicity, we treat only  $\varphi_1(|\xi|) = |\xi|/\log(e + |\xi|)$ . Then he proved that if the data belong to  $\mathcal{N}_{\varphi_1}$ ,  $H^\infty$ -solutions exist globally. Ghisi and Gobino generalised Nishihara class in the sense that the convexity condition is removed (see [7]). Summarising the above known results, we have the inclusions:

$$\mathcal{A}_{L^2} \subset \mathcal{N}_{\varphi_1} \subset \mathcal{G}^s \subset H^\infty \times H^\infty \quad (s > 1),$$

where inclusions are all strict. Here  $\mathcal{G}^s$  is the Gevrey class of  $L^2$  type and order  $s$ ;

$$\mathcal{G}^s = \mathcal{N}_{|\xi|^{1/s}}, \quad (s \geq 1).$$

When  $s = 1$ ,  $\mathcal{G}^1$  coincides with  $\mathcal{A}_{L^2}$ . It should be noted that  $\mathcal{G}^s$  well-posedness is still open.

We have other special classes which ensure the global well-posedness. The first class is of Pohožhaev. He introduced the following class (see [24]):

$$\mathcal{P} = \left\{ (f_0, f_1) \in H^\infty \times H^\infty : \limsup_{j \rightarrow \infty} 2^j \left( \|\Delta^{2j+(1/2)} f_0\|_{L^2}^2 + \|\Delta^{2j} f_1\|_{L^2}^2 \right)^{-\frac{1}{2j+2}} > 0 \right\}.$$

The second one is of Manfrin, the class  $\mathcal{B}_\Delta$  (see [15], and also [10]): we say that  $(f_0, f_1) \in \mathcal{B}_\Delta$  if  $(f_0, f_1) \in H^2 \times H^1$  and there exists a positive sequence  $\{\rho_j\}_{j \in \mathbf{N}}$ ,  $\rho_j \rightarrow \infty$ , and a constant  $\eta > 0$  such that

$$\sup_{j \in \mathbf{N}} \int_{|\xi| \geq \rho_j} \left\{ |\xi|^4 |\hat{f}_0(\xi)|^2 + |\xi|^2 |\hat{f}_1(\xi)|^2 \right\} \frac{e^{\eta \rho_j^2 / |\xi|}}{\rho_j^2} d\xi < \infty.$$

The inclusion among these classes are:

$$\mathcal{A}_{L^2} \subset \mathcal{P} \subset \mathcal{B}_\Delta \subset H^2 \times H^1,$$

$$\mathcal{P} \not\subset \mathcal{N}_{\varphi_1}, \quad \mathcal{G}^s \not\subset \mathcal{B}_\Delta \quad (s > 1).$$

In conclusion,  $\mathcal{G}^s$ -well-posedness,  $C^\infty$ -well-posedness and  $H^s$ -well-posedness are still open. These problems are quite fascinating.

## 4. KIRCHHOFF EQUATION WITH SMALL DATA

In this section we will review the recent results on global well-posedness for initial-boundary value problem to the Kirchhoff equation with small data. For an open set  $\Omega$  in  $\mathbf{R}^n$  with a smooth boundary  $\partial\Omega$ , we consider the initial-boundary value problem:

$$\partial_t^2 u - \left(1 + \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = 0, \quad t \in \mathbf{R}, \quad x \in \Omega, \quad (4.12)$$

$$u(0, x) = f_0(x), \quad \partial_t u(0, x) = f_1(x), \quad x \in \Omega, \quad (4.13)$$

$$u(t, x) = 0, \quad t \in \mathbf{R}, \quad x \in \partial\Omega. \quad (4.14)$$

Then we have:

**Theorem 4.1** (Matsuyama ([17])). *Let  $n \geq 1$ . If  $(f_0, f_1) \in Y(\Omega)$  and*

$$\|\nabla f_0\|_{L^2(\Omega)}^2 + \|f_1\|_{L^2(\Omega)}^2 + |(f_0, f_1)|_{Y(\Omega)} \ll 1,$$

*then the initial-boundary value problem (4.12)–(4.14) admits a unique solution  $u \in C(\mathbf{R}; H^{3/2}(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbf{R}; H^{1/2}(\Omega))$ , where*

$$Y(\Omega) := \{(f, g) \in (H^{3/2}(\Omega) \cap H_0^1(\Omega)) \times H^{1/2}(\Omega) : |(f, g)|_{Y(\Omega)} < +\infty\},$$

*with*

$$\begin{aligned} |(f, g)|_{Y(\Omega)} &= \int_{-\infty}^{+\infty} \left\{ \left| (e^{i\tau H} H^{3/2} f, H^{3/2} f)_{L^2(\Omega)} \right| + \left| (e^{i\tau H} H^{3/2} f, H^{1/2} g)_{L^2(\Omega)} \right| \right. \\ &\quad \left. + \left| (e^{i\tau H} H^{1/2} g, H^{1/2} g)_{L^2(\Omega)} \right| \right\} d\tau. \end{aligned}$$

*Here  $(f, g)_{L^2(\Omega)}$  denotes the  $L^2(\Omega)$ -inner product of  $f$  and  $g$ .*

When  $\Omega = \mathbf{R}^n$ , a similar space is considered by Kajitani and Rzymowski (see [11, 26]). As to the previous known results on the global well-posedness for small data, we refer to the results of Greenberg and Hu, D’Ancona and Spagnolo, and of Yamazaki (see [8, 4, 5, 6, 27]). Greenberg and Hu proved that if the data belong to  $H^2(\mathbf{R}) \times H^1(\mathbf{R})$  and have compact supports,  $H^2(\mathbf{R})$ -solution exists globally. After them, D’Ancona and Spagnolo obtained the similar results in higher spatial dimensions if the data belong to the weighted Sobolev space  $H_{\kappa}^2(\mathbf{R}^n) \times H_{\kappa}^1(\mathbf{R}^n)$  for  $\kappa \in (1, n+1]$ , where we define

$$H_{\kappa}^s(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) : \langle x \rangle^{\kappa} f \in H^s(\mathbf{R}^n)\}, \quad s \in \mathbf{R}.$$

More general result was discussed by Yamazaki. She introduced a class  $\mathcal{Y}_{\kappa}(\mathbf{R}^n)$  which ensures the global well-posedness. More precisely,  $(f_0, f_1) \in \mathcal{Y}_{\kappa}(\mathbf{R}^n)$  is said to be  $(f_0, f_1) \in H^{3/2}(\mathbf{R}^n) \times H^{1/2}(\mathbf{R}^n)$  such that

$$\sum_{j,k=0}^1 \sup_{\tau \in \mathbf{R}} \langle \tau \rangle^{\kappa} \left| \int_{\mathbf{R}^n} e^{i\tau|\xi|} \widehat{f}_j(\xi) \overline{\widehat{f}_k(\xi)} |\xi|^{3-j-k} d\xi \right| < \infty.$$

We have the inclusion among these classes as follows:

$$H_{\kappa}^2(\mathbf{R}^n) \times H_{\kappa}^1(\mathbf{R}^n) \subset \mathcal{Y}_{\kappa}(\mathbf{R}^n) \subset Y(\mathbf{R}^n).$$

We notice that the first inclusion holds provided  $\kappa \in (1, n+1]$ , and the second one is valid for any  $\kappa > 1$ . In conclusion, our class  $Y(\mathbf{R}^n)$  is the most general which ensures

global well-posedness for the (scalar) Kirchhoff equation.

As to the exterior problems, we have

**Theorem 4.2** (Matsuyama ([16])). *Let  $\Omega$  be a domain of  $\mathbf{R}^n$  such that  $\mathbf{R}^n \setminus \Omega$  is compact and its boundary  $\partial\Omega$  is  $C^\infty$ . For  $\sigma \geq 0$  and  $\kappa \in \mathbf{R}$ , let  $H_{\kappa,0}^\sigma(\Omega)$  be the completion of  $C_0^\infty(\Omega)$  in the norm  $\|\cdot\|_{H_\kappa^\sigma(\Omega)}$ . If  $n \geq 3$  and  $\mathbf{R}^n \setminus \Omega$  is star-shaped with respect to the origin, then the inclusion*

$$H_{s(k),0}^{s_0+1}(\Omega) \times H_{s(k),0}^{s_0}(\Omega) \subset Y_k(\Omega) \subset Y(\Omega)$$

holds for any  $s_0 > (n+1)/2$ ,  $s(k) > \max(n+1/2, k+n/2)$  and  $k > 1$ .

The proof of Theorem 4.2 is based on the generalized Fourier transformation method. The crucial tool in our argument is the asymptotic expansion of resolvent of  $-\Delta$  around the origin in the complex plane.

Based on Theorems 4.1–4.2, we have:

**Theorem 4.3** (Matsuyama ([16])). *Let  $\Omega, n, s_0, s(k)$  be as in Theorem 4.2. If*

$$f_0(x) \in H_{s(k),0}^{s_0+1}(\Omega), \quad f_1(x) \in H_{s(k),0}^{s_0}(\Omega),$$

and

$$\|f_0\|_{H_{s(k),0}^{s_0+1}(\Omega)} + \|f_1\|_{H_{s(k),0}^{s_0}(\Omega)} \ll 1,$$

then the initial-boundary value problem (4.12) admits a unique solution

$$u \in \bigcap_{j=0}^2 C^j(\mathbf{R}; H^{s_0+1-j}(\Omega)).$$

Let us now make only a few short remarks to compare the result in Theorem 4.3 with what is known. In the results of Heiming and Racke the data are imposed to be small in the weighted Sobolev spaces, and the supports of generalised Fourier transform of data are away from the origin (see [9, 25]). However, our result covers the low frequencies of the generalised Fourier transform of data. Therefore, the statement of Theorem 4.3 goes beyond [9, 25].

We should refer to the results of Yamazaki (see [28, 29]), who gave some sufficient conditions without any weight condition on data. That is, she assumed that the data belong to  $W^{s,q}(\Omega) \times W^{s-1,q}(\Omega)$  for some  $s > 2$  and  $q \in (1, 2)$  depending on  $n(\geq 3)$ , where  $\Omega$  is a non-trapping domain in  $\mathbf{R}^n$ . We have an advantage of considering the classes of Theorem 4.3; it is more useful in the scattering problem rather than the ones in [9, 25, 28, 29].

As a final remark in this section, we have a new result on the Cauchy problem. Indeed, we can remove the additional class  $\mathcal{Y}$  of data. This result will appear elsewhere.

## 5. KIRCHHOFF SYSTEMS WITH SMALL DATA

Recall Kirchhoff system (1.3):

$$\begin{aligned} D_t U &= A(s(t), D_x)U, & t \neq 0, \quad x \in \mathbf{R}^n, \\ U(0, x) &= {}^t(f_0(x), f_1(x), \dots, f_{m-1}(x)), & x \in \mathbf{R}^n. \end{aligned}$$

Denote

$$\mathbf{H}^\sigma(\mathbf{R}^n) = (H^\sigma(\mathbf{R}^n))^m$$

for  $\sigma \in \mathbf{R}$ .

Then we have:

**Theorem 5.1** (Matsuyama and Ruzhansky ([19])). *Let  $n \geq 1$ . Suppose that  $A(s, \xi) = (a_{jk}(s, \xi))_{j,k=1}^m$  is an  $m \times m$  matrix positively homogeneous order one in  $\xi$ , whose entries  $a_{jk}(s, \xi/|\xi|)$  are in  $\text{Lip}([0, \delta]; L^\infty(\mathbf{R}^n \setminus 0))$  for some  $0 < \delta \ll 1$ , and satisfies the strictly hyperbolic condition (1.5)–(1.6). If  $U_0(x) \in \mathbf{L}^2(\mathbf{R}^n) \cap \mathcal{Y}(\mathbf{R}^n)$  satisfy*

$$\|U_0\|_{\mathbf{L}^2(\mathbf{R}^n)}^2 + |U_0|_{\mathcal{Y}(\mathbf{R}^n)} \ll 1, \quad (5.15)$$

then system (1.3)–(1.4) has a unique solution  $U(t, x) \in C(\mathbf{R}; \mathbf{L}^2(\mathbf{R}^n))$ . In addition to (5.15), if  $U_0 \in \mathbf{H}^1(\mathbf{R}^n)$ , then the solution  $U(t, x)$  exists uniquely in the class  $C(\mathbf{R}; \mathbf{H}^1(\mathbf{R}^n)) \cap C^1(\mathbf{R}; \mathbf{L}^2(\mathbf{R}^n))$ .

Precise definition of  $\mathcal{Y}(\mathbf{R}^n)$  is as follows:  $U_0 = {}^t(f_0, f_1, \dots, f_{m-1}) \in \mathcal{Y}(\mathbf{R}^n)$  if and only if  $U_0 \in (\mathcal{S}'(\mathbf{R}^n))^m$  satisfies

$$\begin{aligned} |U_0|_{\mathcal{Y}(\mathbf{R}^n)} &:= \sum_{j,k=0}^{m-1} \int_{-\infty}^{\infty} \left( \int_{\mathbf{S}^{n-1}} \left| \int_0^{\infty} e^{i\tau\rho} \widehat{f}_j(\rho\omega) \overline{\widehat{f}_k(\rho\omega)} \rho^n d\rho \right| d\sigma(\omega) \right) d\tau \\ &< +\infty, \end{aligned}$$

where  $d\sigma(\omega)$  is the  $(n-1)$ -dimensional Hausdorff measure.

Let us compare Theorem 5.1 with what is known. Callegari and Manfrin introduced the following class (see [3], and also [14]):

$$\mathcal{M}(\mathbf{R}^n) = \{U_0(x) = {}^t(f_0(x), f_1(x), \dots, f_{m-1}(x)) : |U_0|_{\mathcal{M}(\mathbf{R}^n)} < \infty\},$$

where

$$|U_0|_{\mathcal{M}(\mathbf{R}^n)} = \sum_{k=0}^2 \sum_{j=0}^{m-1} \sup_{\omega \in \mathbf{S}^{n-1}} \int_0^{\infty} \left| \partial_\rho^k \widehat{f}_j(\rho\omega) \right|^2 (1 + \rho^{\max\{n, 2\}}) d\rho.$$

The inclusion among this class and ours is:

$$(C_0^\infty(\mathbf{R}^n))^m \subset \mathbf{L}_2^1(\mathbf{R}^n) \cap \mathbf{H}_2^1(\mathbf{R}^n) \subset \mathcal{M}(\mathbf{R}^n) \subset \mathcal{Y}(\mathbf{R}^n),$$

where denoting  $\mathbf{L}_2^1(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) : \langle x \rangle^2 f \in L^1(\mathbf{R}^n)\}$  and  $\mathbf{H}_2^1(\mathbf{R}^n) = \{f \in H^1(\mathbf{R}^n) : \langle x \rangle^2 f \in L^2(\mathbf{R}^n)\}$ , we put

$$\mathbf{L}_2^1(\mathbf{R}^n) = (L_2^1(\mathbf{R}^n))^m, \quad \mathbf{H}_\kappa^1(\mathbf{R}^n) = (H_\kappa^1(\mathbf{R}^n))^m.$$

Therefore, we can understand that Theorem 5.1 is the most general result in the framework of small data problem.



As another application of Theorem 5.1, we consider the Cauchy problem for the second order equation of the form

$$\partial_t^2 u - \left(1 + \int_{\mathbf{R}^n} |u(t, y)|^2 dy\right) \Delta u = 0, \quad t \neq 0, \quad x \in \mathbf{R}^n, \quad (5.16)$$

with data

$$u(0, x) = f_0(x), \quad \partial_t u(0, x) = f_1(x). \quad (5.17)$$

In this particular case, the nonlocal term  $s(t)$  is defined by

$$s(t) = \|u(t)\|_{L^2(\mathbf{R}^n)}^2.$$

Introducing another class of data

$$\tilde{\mathcal{Y}}(\mathbf{R}^n) = \left\{ (f_0, f_1) \in \mathcal{S}'(\mathbf{R}^n) \times \mathcal{S}'(\mathbf{R}^n) : |(f_0, f_1)|_{\tilde{\mathcal{Y}}(\mathbf{R}^n)} < \infty \right\},$$

where we put

$$\begin{aligned} |(f_0, f_1)|_{\tilde{\mathcal{Y}}(\mathbf{R}^n)} = & \\ & \sum_{j,k=0}^1 \int_{-\infty}^{\infty} \left( \int_{\mathbf{S}^{n-1}} \left| \int_0^{\infty} e^{i\tau\rho} \widehat{f}_j(\rho\omega) \overline{\widehat{f}_k(\rho\omega)} \rho^{n-j-k} d\rho \right| d\sigma(\omega) \right) d\tau, \end{aligned}$$

we have:

**Theorem 5.2** (Matsuyama and Ruzhansky ([19])). *Let  $n \geq 1$ . For any  $(f_0, f_1) \in (H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)) \cap \tilde{\mathcal{Y}}(\mathbf{R}^n)$ , the Cauchy problem (5.16)–(5.17) has a unique solution  $u \in \bigcap_{k=0,1} C^k(\mathbf{R}; H^{1-k}(\mathbf{R}^n))$ , provided that*

$$\|f_0\|_{L^2(\mathbf{R}^n)}^2 + \|f_1\|_{H^{-1}(\mathbf{R}^n)}^2 + |(f_0, f_1)|_{\tilde{\mathcal{Y}}(\mathbf{R}^n)} \ll 1.$$

When  $n \geq 3$ , a similar result was obtained in Callegari and Manfrin and D’Ancona and Spagnolo (see [3, 5]). However, the regularity of data in Theorem 5.2 is lower than that in the previous results. It should be noted that Theorem 5.2 also covers low dimensions  $n = 1, 2$ , the case that remained open since Callegari and Manfrin, and D’Ancona and Spagnolo.

## 6. OUTLINE OF PROOF OF THEOREM 5.1

The strategy of the proof of Theorem 5.1 is to employ the Schauder-Tychonoff fixed point theorem via asymptotic integrations method. Consider the *linear* Cauchy problem:

$$D_t U = A(t, D_x) U, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n, \quad (6.18)$$

$$U(0, x) = U_0(x) = {}^t(f_0(x), f_1(x), \dots, f_{m-1}(x)) \quad (6.19)$$

where  $A(t, D_x)$  is a first order  $m \times m$  pseudo-differential system, with symbol  $A(t, \xi)$  of the form  $A(t, \xi) = (a_{jk}(t, \xi))_{j,k=1}^m$ . We assume that  $\det(\tau I - A(t, \xi)) = 0$  has real and distinct roots  $\varphi_1(t, \xi), \dots, \varphi_m(t, \xi)$ ,

$$a_{jk}(t, \xi/|\xi|) \in \text{Lip}_{\text{loc}}(\mathbf{R}; L^\infty(\mathbf{R}^n \setminus 0)) \quad \text{and} \quad \partial_t a_{jk}(t, \xi/|\xi|) \in L^1(\mathbf{R}; L^\infty(\mathbf{R}^n \setminus 0)). \quad (6.20)$$

We prepare the next lemma.

**Lemma 6.1** (c.f. Proposition 6.4 from [21]). *Let  $A(t, \xi)$  be a symbol of differential operator  $A(t, D_x)$  satisfying (6.20). Then there exists a matrix  $\mathcal{N} = \mathcal{N}(t, \xi)$  of homogeneous order 0 in  $\xi$  satisfying the following properties:*

(i)  $\mathcal{N}(t, \xi)A(t, \xi/|\xi|) = \mathcal{D}(t, \xi)\mathcal{N}(t, \xi)$ , where

$$\mathcal{D}(t, \xi) = \text{diag}(\varphi_1(t, \xi/|\xi|), \dots, \varphi_m(t, \xi/|\xi|));$$

(ii)  $\inf_{\xi \in \mathbf{R}^n \setminus 0, t \in \mathbf{R}} |\det \mathcal{N}(t, \xi)| > 0$ ;

(iii)  $\mathcal{N}(t, \xi) \in \text{Lip}_{\text{loc}}(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2})$  and  $\partial_t \mathcal{N}(t, \xi) \in L^1(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2})$ .

We have asymptotic integrations for (6.18)–(6.19).

**Proposition 6.2** (Matsuyama and Ruzhansky ([18])). *Let  $A(t, \xi)$  be a symbol of regularly hyperbolic operator  $A(t, D_x)$  satisfying (6.20), and  $\mathcal{N}(t, \xi)$  the diagonaliser of  $A(t, \xi/|\xi|)$ . Then there exist vector-valued functions  $\mathbf{a}^j(t, \xi)$ ,  $j = 0, 1, \dots, m-1$ , determined by the initial value problem*

$$D_t \mathbf{a}^j(t, \xi) = C(t, \xi) \mathbf{a}^j(t, \xi), \quad (\mathbf{a}^1(0, \xi), \dots, \mathbf{a}^m(0, \xi)) = \mathcal{N}(0, \xi),$$

with

$$C(t, \xi) = \Phi(t, \xi)^{-1} (D_t \mathcal{N}(t, \xi)) \mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \in L^1(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2}),$$

such that the solution  $U(t, x)$  of (6.18)–(6.19) is represented by

$$U(t, x) = \sum_{j=0}^{m-1} \mathcal{F}^{-1} \left[ \mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \mathbf{a}^j(t, \xi) \widehat{f}_j(\xi) \right] (x), \quad (6.21)$$

where we put

$$\Phi(t, \xi) = \text{diag} \left( e^{i \int_0^t \varphi_1(s, \xi) ds}, \dots, e^{i \int_0^t \varphi_m(s, \xi) ds} \right).$$

Let us introduce a class of symbols of differential operators, which is convenient for the fixed point argument.

**Class  $\mathcal{K}$ .** *Given two constants  $\Lambda > 0$  and  $K > 0$ , we say that a symbol  $A(t, \xi)$  of a pseudo-differential operator  $A(t, D_x)$  belongs to  $\mathcal{K} = \mathcal{K}(\Lambda, K)$  if  $A(t, \xi/|\xi|)$  belongs to  $\text{Lip}_{\text{loc}}(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2})$  and satisfies*

$$\begin{aligned} \|A(t, \xi/|\xi|)\|_{L^\infty(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2})} &\leq \Lambda, \\ \int_{-\infty}^{\infty} \|\partial_t A(t, \xi/|\xi|)\|_{(L^\infty(\mathbf{R}^n \setminus 0))^{m^2}} dt &\leq K. \end{aligned}$$

The next lemma is the heart of our argument. It will be applied with a sufficiently small constant  $K_0 > 0$  which will be fixed later.

**Lemma 6.3.** *Let  $n \geq 1$ . Assume that the symbol  $A(t, \xi)$  of a pseudo-differential operator  $A(t, D_x)$  satisfies strictly hyperbolic condition and (6.20), and belongs to  $\mathcal{K} = \mathcal{K}(\Lambda, K)$  for some  $\Lambda > 0$  and  $0 < K \leq K_0$  with a sufficiently small constant  $K_0 > 0$ . Let  $U \in C(\mathbf{R}; \mathbf{L}^2(\mathbf{R}^n))$  be a solution to the Cauchy problem*

$$D_t U = A(t, D_x) U, \quad U(0, x) = U_0(x) \in \mathbf{L}^2(\mathbf{R}^n) \cap \mathcal{Y}(\mathbf{R}^n),$$

and let  $s(t)$  be the function

$$s(t) = \langle SU(t, \cdot), U(t, \cdot) \rangle_{\mathbf{L}^2(\mathbf{R}^n)}.$$

Then there exist two constants  $M > 0$  and  $c > 0$  independent of  $U$  and  $K$  such that

$$\|A(s(t), \omega)\|_{(L^\infty(\mathbf{S}^{n-1}))^{m^2}} \quad (6.22)$$

$$\leq \|A(s(0), \omega)\|_{(L^\infty(\mathbf{S}^{n-1}))^{m^2}} + M \left( K \|U_0\|_{\mathbf{L}^2(\mathbf{R}^n)}^2 + \frac{1}{1 - cK} \|U_0\|_{\mathcal{Y}(\mathbf{R}^n)} \right),$$

$$\int_{-\infty}^{\infty} \|\partial_t [A(s(t), \omega)]\|_{(L^\infty(\mathbf{S}^{n-1}))^{m^2}} dt \quad (6.23)$$

$$\leq M \left( K \|U_0\|_{\mathbf{L}^2(\mathbf{R}^n)}^2 + \frac{1}{1 - cK} \|U_0\|_{\mathcal{Y}(\mathbf{R}^n)} \right).$$

*Outline of proof.* The proof is based on Lemma 6.2 via Fourier transform. More precisely, writing

$$s(t) = \langle S\widehat{U}(t, \xi), \widehat{U}(t, \xi) \rangle_{\mathbf{L}^2(\mathbf{R}^n)},$$

we calculate the derivative  $s'(t)$  and plugging (6.21) into  $s'(t)$ . Then we can write

$$s'(t) = 2\operatorname{Re} \left\langle S\widehat{U}'(t, \xi), \widehat{U}(t, \xi) \right\rangle_{\mathbf{L}^2(\mathbf{R}^n)} = 2\{I(t) + J(t)\}, \quad (6.24)$$

where

$$\begin{aligned} I(t) &= \\ \operatorname{Re} \sum_{j,k=0}^{m-1} &\left\langle S\mathcal{N}(t, \xi)^{-1} \partial_t \Phi(t, \xi) \mathbf{a}^j(t, \xi) \widehat{f}_j(\xi), \mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \mathbf{a}^k(t, \xi) \widehat{f}_k(\xi) \right\rangle_{\mathbf{L}^2(\mathbf{R}^n)}, \\ J(t) &= \\ \operatorname{Re} \sum_{j,k=0}^{m-1} &\left\langle S\partial_t \mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \mathbf{a}^j(t, \xi) \widehat{f}_j(\xi), \mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \mathbf{a}^k(t, \xi) \widehat{f}_k(\xi) \right\rangle_{\mathbf{L}^2(\mathbf{R}^n)} \\ &+ \left\langle S\mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \partial_t \mathbf{a}^j(t, \xi) \widehat{f}_j(\xi), \mathcal{N}(t, \xi)^{-1} \Phi(t, \xi) \mathbf{a}^k(t, \xi) \widehat{f}_k(\xi) \right\rangle_{\mathbf{L}^2(\mathbf{R}^n)}. \end{aligned}$$

We can proceed our analysis and conclude the proof.

*Outline of proof of Theorem 5.1.* We employ the Schauder-Tychonoff fixed point theorem. Let  $A(t, \xi) \in \mathcal{K}$ , and we fix the data  $U_0 \in \mathbf{L}^2(\mathbf{R}^n) \cap \mathcal{Y}(\mathbf{R}^n)$ . Then it follows from Lemma 6.3 that the mapping

$$\Theta : A(t, \xi) \mapsto A(s(t), \xi)$$

maps  $\mathcal{K} = \mathcal{K}(\Lambda, K)$  into itself provided that  $\|U_0\|_{\mathbf{L}^2(\mathbf{R}^n)}^2 + \|U_0\|_{\mathcal{Y}(\mathbf{R}^n)}$  is sufficiently small, with  $\Lambda > 2\|A(0, \xi/|\xi|)\|_{(L^\infty(\mathbf{R}^n \setminus 0))^{m^2}}$  and sufficiently small  $0 < K < K_0$ . Now  $\mathcal{K}$  may be regarded as the convex subset of the Fréchet space  $L_{\text{loc}}^\infty(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2})$ , and we endow  $\mathcal{K}$  with the induced topology. We can show that  $\mathcal{K}$  is compact in  $L_{\text{loc}}^\infty(\mathbf{R}; (L^\infty(\mathbf{R}^n \setminus 0))^{m^2})$  and the mapping  $\Theta$  is continuous on  $\mathcal{K}$ . Thus the Schauder-Tychonoff fixed point theorem allows us to conclude the proof.

## REFERENCES

- [1] Arosio, A., Spagnolo, S.: Global solutions to the Cauchy problem for a nonlinear hyperbolic equation, *Nonlinear partial differential equations and their applications*, Collège de France seminar, Vol. VI (Paris, 1982/1983), pp. 1–26, Res. Notes in Math., 109, Pitman, Boston, MA, 1984.
- [2] Bernstein, S.: Sur une classe d'équations fonctionnelles aux dérivées partielles, *Izv. Akad. Nauk SSSR Ser. Mat.* **4**, 17–27 (1940)
- [3] Callegari, E., Manfrin, R.: Global existence for nonlinear hyperbolic systems of Kirchhoff type, *J. Differential Equations* **132**, 239–274 (1996)
- [4] D'Ancona, P., Spagnolo, S.: A class of nonlinear hyperbolic problems with global solutions, *Arch. Rational Mech. Anal.*, **124**, 201–219 (1993)
- [5] D'Ancona, P., Spagnolo, S.: Nonlinear perturbations of the Kirchhoff equation, *Comm. Pure Appl. Math.*, **47**, 1005–1029 (1994)
- [6] D'Ancona, P., Spagnolo, S.: Kirchhoff type equations depending on a small parameter, *Chin. Ann. of Math.* **16B**, 413–430 (1995)
- [7] Ghisi, M., Gobino, M.: Kirchhoff equation from quasi-analytic to spectral-gap data, *Bull. London Math. Soc.* **43**, 374–385 (2011)
- [8] Greenberg, J.M., Hu, S.C.: The initial-value problem for a stretched string, *Quart. Appl. Math.* **38**, 289–311 (1980)
- [9] Heiming, C. (=Kerler, C.), *Mapping properties of generalized Fourier transforms and applications to Kirchhoff equations*, NoDEA Nonlinear Differential Equations Appl. **7**, 389–414 (2000)
- [10] Hirose, F.: Global solvability for Kirchhoff equation in special classes of non-analytic functions, *J. Differential Equations* **230**, 49–70 (2006)
- [11] Kajitani, K.: The global solutions to the Cauchy problem for multi-dimensional Kirchhoff equation, *Advance in Phase Space Analysis of Partial Differential Equations*, (A. Bove, D. Del Santo and M. K. V. Murthy eds.), Progress in Nonlinear Differential Equations and Their Applications, vol. 78, Birkhäuser, Boston 2009, pp. 141–153.
- [12] Kajitani, K., Yamaguti, K.: *On global analytic solutions of the degenerate Kirchhoff equation*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **21**, 279–297 (1994)
- [13] Kirchhoff, G.: *Vorlesungen über Mechanik*, Teubner, Leipzig (1883)
- [14] Manfrin, R.: On the global solvability of symmetric hyperbolic systems of Kirchhoff type, *Discrete Contin. Dynam. Systems* **3**, 91–106 (1997)
- [15] Manfrin, R.: On the global solvability of Kirchhoff equation for non-analytic initial data, *J. Differential Equations* **211**, 38–60 (2005)
- [16] Matsuyama, T.: Global well-posedness for the exterior initial-boundary value problem to the Kirchhoff equation, *J. Math. Soc. Japan* **64**, 1167–1204 (2010)
- [17] Matsuyama, T.: The Kirchhoff equation with global solutions in unbounded domains, *Rend. Istit. Mat. Univ. Trieste.* **42** Suppl., 125–141 (2010)
- [18] Matsuyama, T., Ruzhansky, M.: Scattering for strictly hyperbolic systems with time-dependent coefficients, *Math. Nachr.* **286**, 1191–1207 (2013)
- [19] Matsuyama, T., Ruzhansky, M.: *Global well-posedness of Kirchhoff systems*, *J. Math. Pures Appl.* **100**, 220–240 (2013)
- [20] Matsuyama, T., Ruzhansky, M.: Asymptotic integration and dispersion for hyperbolic equations, *Adv. Differential Equations* **15**, 721–756 (2010)
- [21] Mizohata, S.: “The Theory of Partial Differential Equations,” Cambridge Univ. Press, 1973.
- [22] Nishida, T.: A note on the nonlinear vibrations of the elastic string, *Mem. Fac. Eng. Kyoto Univ.* **33**, 329–341 (1971)
- [23] Nishihara, K.: On a global solution of some quasilinear hyperbolic equation, *Tokyo J. Math.* **7**, 437–459 (1984)
- [24] Pohožhaev, S.I.: On a class of quasilinear hyperbolic equations, *Math. USSR Sb.* **25**, 145–158 (1975)
- [25] Racke, R.: *Generalized Fourier transforms and global, small solutions to Kirchhoff equations*, *Asymptotic Analysis* **58**, 85–100 (1995)

- [26] Rzymowski, W.: One-dimensional Kirchhoff equation, *Nonlinear Analysis* **48**, 209–221 (2002)
- [27] Yamazaki, T.: Scattering for a quasilinear hyperbolic equation of Kirchhoff type, *J. Differential Equations* **143**, 1–59 (1998)
- [28] Yamazaki, T.: Global solvability for the Kirchhoff equations in exterior domains of dimension larger than three, *Math. Methods Appl. Sci.* **27**, 1893–1916 (2004)
- [29] Yamazaki, T.: Global solvability for the Kirchhoff equations in exterior domains of dimension three, *J. Differential Equations* **210**, 290–316 (2005)

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