Every 4-connected graph with crossing number 2 is Hamiltonian

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Abstract. A seminal theorem of Tutte states that planar 4-connected graphs are Hamiltonian. Applying a result of Thomas and Yu, one can show that every 4-connected graph with crossing number 1 is Hamiltonian. In this paper, we continue along this path and prove the titular statement. We also discuss the traceability and Hamiltonicity of 3-connected graphs with small crossing number and few 3-cuts, and present applications of our results.

Key Words. Hamiltonian cycle, crossing number, 3-cuts
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1 Introduction

In a graph G, a cycle or a path is *Hamiltonian* if it contains all vertices of G. A graph is called *Hamiltonian* if it has a Hamiltonian cycle, *traceable* if it contains a Hamiltonian path, and *Hamiltonian-connected* if there is a Hamiltonian path between any two vertices in the graph. When we speak of a *cut*, we refer to a set of vertices whose removal disconnects the graph (when the cut is a single vertex, we will emphasise this and write *cut-vertex*), while *edge-cuts* will always be explicitly mentioned as such. A cut is *trivial* when it is the neighbourhood of a vertex.

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Whitney proved in 1931 that 4-connected planar triangulations are Hamiltonian [16], and a quarter century later Tutte extended this to all planar 4-connected graphs [15]. These theorems have had an enormous impact on graph theory as we know it, and have seen an abundance of extensions. Recently, it was investigated what conclusions can be drawn concerning the Hamiltonicity of 3-connected planar graphs if we restrict the number of 3-cuts. We refer the reader to the survey [6]. The strongest available result in this direction is by Brinkmann and the second author [1] and states that every 3-connected planar graph with at most three 3-cuts is Hamiltonian. For plane triangulations with at most three separating triangles, this had been shown by Jackson and Yu [3].

Two important notions to classify non-planar graphs are the crossing number and the genus. For a graph G, its *crossing number* $\text{cr}(G)$ is the minimum number of edge crossings over all plane drawings of G. For a rigorous definition and a survey, we refer to work of Richter and Salazar [7]. The *genus* of a graph is the smallest k such that the graph can be embedded (i.e. drawn without edge crossings) on a sphere with k handles. In the seventies, Grünbaum [2] and Nash-Williams [5] independently conjectured that every 4-connected graph of genus 1 is Hamiltonian. Forty years later, this problem remains unsolved—we *do* know, by a result of Thomas, Yu, and Zang, that 4-connected toroidal graphs are traceable [11]. $K_{3,5}$ shows that there exist graphs of genus 1 with exactly one 3-cut that are non-traceable. We see this abrupt end of the story as further motivation to investigate the Hamiltonian properties of 4-connected graphs with few crossings.

Kawarabayashi and the first author [4] showed that every 4-connected projective-planar graph is Hamiltonian-connected—this strengthens two classic results due to Thomassen [14], and Thomas and Yu [10]. Since any graph with crossing number 1 can be embedded into the projective plane, it follows that every 4-connected graph with crossing number at most 1 is Hamiltonian-connected. Brinkmann (see [17]) showed that if e and f are the crossing edges in a 4-connected graph G with crossing number 1, then G contains a Hamiltonian cycle avoiding e and f.

The principal contribution of this article is the following result.

Theorem 1 *Every* 4*-connected graph with crossing number at most* 2 *is Hamiltonian.*

This paper is organised as follows. In the next section we present all ingredients necessary for the proof of Theorem 1, which is then given in Section 3. Thereafter, in Section 4, we give applications of Theorem 1—in particular, we (i) discuss how our results relate to the traceability of 3-connected graphs with few 3-cuts and few crossings, (ii) provide, via a toughness argument, 3-connected non-Hamiltonian and non-traceable graphs with small crossing number and few 3-cuts, (iii) show that every 3-connected graph with at most one crossing and containing at most one 3-cut is Hamiltonian (this extends a result of Thomassen [13] which extends the aforementioned result of Tutte [15]), and (iv) comment on hypohamiltonian graphs. The latter includes an extension of a theorem of the second author [17] which extends a theorem of Thomassen [13]. In the last section, we give tabular overviews of certain Hamiltonian properties of 3-connected graphs with few crossings and a small number of 3-cuts.

2 Preliminaries

We shall require the following two results on the Hamiltonicity of 4-connected graphs. (The proof of the latter uses the former.)

Theorem 2 (Thomas and Yu [10]) *For every 4-connected planar graph G and any vertices u and v, the graph $G - \{u, v\}$ is Hamiltonian.*

Theorem 3 (Brinkmann, see [17]) *Let G be a 4-connected graph with one edge crossing formed by the edges e and f. Then $G - \{e, f\}$ contains a Hamiltonian cycle.*

Let $\mathfrak{h}$ be a subgraph of a graph G. An $\mathfrak{h}$-*bridge* of G is either (i) an edge of $G - E(\mathfrak{h})$ with both ends on $\mathfrak{h}$ or (ii) a subgraph of G induced by the edges in a component of $G - V(\mathfrak{h})$ together with all edges between that component and $\mathfrak{h}$. An $\mathfrak{h}$-bridge satisfying (i) is *trivial*, while an $\mathfrak{h}$-bridge satisfying (ii) is *non-trivial*. For an $\mathfrak{h}$-bridge P of G, the vertices in $V(P) \cap V(\mathfrak{h})$ are the *attachments* of P (on $\mathfrak{h}$), and those vertices of P that are not attachments are *non-attachments*. We say that $\mathfrak{h}$ is a *Tutte subgraph in G* if every $\mathfrak{h}$-bridge of G has at most three attachments on $\mathfrak{h}$. For a subgraph D of G, $\mathfrak{h}$ is a *D-Tutte subgraph in G* if $\mathfrak{h}$ is a Tutte subgraph in G and every $\mathfrak{h}$-bridge of G containing an edge of D has at most two attachments on $\mathfrak{h}$. A *Tutte path* (respectively, a *Tutte cycle*) is a path (respectively, a cycle) that is a Tutte subgraph. Similarly, we define a *D-Tutte path* and a *D-Tutte cycle*. By the definition, we see that a Tutte cycle $\mathfrak{h}$ in a 4-connected graph is nothing but a Hamiltonian cycle if $|V(\mathfrak{h})| \geq 4$.

We use the following result by Sanders. (Note that an earlier but weaker result by Thomassen [14] is sufficient for our proof, but we introduce a more general statement.)

Theorem 4 (Sanders [9]) *Let G be a 2-connected plane graph with facial cycle D, let $x, y \in V(G)$ and $e \in E(D)$. Then G contains a D-Tutte path from x to y through e.*

The following theorems appear in [10, (2.8)], in [10, (3.2)] and in [10, (4.1)], respectively—see also [8, Lemma 3] for the first one.

Theorem 5 (Thomas and Yu [10]) *Let G be a 2-connected plane graph with a facial cycle D and let $e_1, e_2, e_3 \in E(D)$. Then G contains a D-Tutte cycle passing through e_1, e_2, and e_3.*

Theorem 6 (Thomas and Yu [10]) *Let G be a 2-connected plane graph with two facial cycles D_1 and D_2, and let $e \in E(D_1)$. Then G contains a $(D_1 \cup D_2)$-Tutte cycle $\mathfrak{h}$ passing through e such that no $\mathfrak{h}$-bridge contains edges of both D_1 and D_2.*

Theorem 7 (Thomas and Yu [10]) *Let G be a 2-connected graph on the projective plane with a facial cycle D, and let $e \in E(D)$. Then G contains a D-Tutte cycle $\mathfrak{h}$ passing through e such that every $\mathfrak{h}$-bridge that contains a non-nullhomotopic cycle is edge-disjoint from D.*

We also need the following lemma. We denote the subgraph of G induced by a vertex set $S \subseteq V(G)$ with $G[S]$.

Lemma 8 *Let G be a 4-connected graph, let $S \subseteq V(G)$ be a cut of G, let A be a component of $G - S$, and let $G_A = G[A \cup S]$. Then for any Tutte subgraph $\mathfrak{h}$ in G_A, if $|V(\mathfrak{h})| \geq 4$ and $\mathfrak{h}$ contains all vertices in S, then $\mathfrak{h}$ contains all vertices in G_A.*

Proof. Suppose to the contrary that for a Tutte subgraph $\mathfrak{h}$ in G_A, there is a non-trivial $\mathfrak{h}$-bridge P of G_A. Since $\mathfrak{h}$ is a Tutte subgraph, P has at most three attachments on $\mathfrak{h}$. As G is 4-connected and $|V(\mathfrak{h})| \geq 4$, the three attachments of P on $\mathfrak{h}$ do not form a cut of G, and hence P contains a vertex in S as a non-attachment. However, this contradicts that $\mathfrak{h}$ contains all vertices in S. □

3 Proof of Theorem 1

By Theorem 3, we may assume that G has a drawing in the plane with exactly two edge crossings. Let e_1, f_1, e_2, and f_2 be edges in G such that e_i and f_i form an edge crossing for $i \in \{1, 2\}$.

Claim 1 *We may assume that all of e_1, f_1, e_2, and f_2 are distinct edges.*

Proof. Suppose that some of them are not distinct. Since trivially $e_1 \neq f_1$ and $e_2 \neq f_2$, we may assume that $e_1 = e_2$. This implies that $G' = G - \{e_1\}$ is a plane graph. Let $e_1 = e_2 = x_A x_B$. Specifying an appropriate edge as e, by Theorem 4 the graph G' contains a Tutte path $\mathfrak{h}$ from x_A to x_B with $|V(\mathfrak{h})| \geq 4$. (We do not need to consider the facial cycle D.) Since $x_A x_B = e_1 \in E(G)$, to obtain a Hamiltonian cycle in G, it suffices to show that $\mathfrak{h}$ is a Hamiltonian path. Suppose it is not. Then there is a non-trivial $\mathfrak{h}$-bridge P of G'. Since $\mathfrak{h}$ is a Tutte path, P has at most three attachments. As G is 4-connected and $|V(\mathfrak{h})| \geq 4$, the attachments of P on $\mathfrak{h}$ do not form a cut of G of order at most three, and hence P must contain x_A or x_B as a non-attachment. This however contradicts the fact that x_A and x_B are end-vertices of $\mathfrak{h}$. □

We use Claim 1 implicitly in the remaining proof.

Consider the graph G' obtained from G by replacing the crossing between e_i and f_i with a vertex of degree 4, for $i \in \{1, 2\}$. In other words, the plane graph G' is obtained from $G - \{e_1, f_1, e_2, f_2\}$ by adding two new vertices u_1 and u_2 together with four edges between each of them and the corresponding end-vertices of e_1, f_1, e_2, f_2. If G' is 4-connected, then it follows from Theorem 2 that $G' - \{u_1, u_2\} = G - \{e_1, f_1, e_2, f_2\}$ is Hamiltonian, and hence we are done. So we may assume that G' contains a cut S' of order at most three. Since G is 4-connected, S' must contain at least one of u_1 and u_2. Let A' and B' be the components of $G' - S'$. The strategy used in the upcoming proof is as follows. First, by showing four claims, the terrain is prepared to get the graph $G'' - \{e_2, f_2\}$ (where G'' is obtained from G by replacing the crossing edges e_1 and f_1 as shown in Figure 2), which by Theorem 3 has a Hamiltonian cycle $\mathfrak{q}''$. Second, by means of a case analysis it is shown that from $\mathfrak{q}''$ one can obtain a Hamiltonian cycle of G.

The following claim provides useful properties of S'.

Claim 2 *We may assume that* $|S'| = 3$ *and either* $u_1 \notin S'$ *or* $u_2 \notin S'$.

Proof. Suppose first $|S'| = 2$. Since G is 4-connected, we have $S' = \{u_1, u_2\}$ and each edge of e_1, f_1, e_2, f_2 connects A' and B'. Let $e_1 = x_A x_B$, $f_1 = y_A y_B$, $e_2 = z_A z_B$ and $f_2 = w_A w_B$ with $x_A, y_A, z_A, w_A \in V(A')$ and $x_B, y_B, z_B, w_B \in V(B')$. By symmetry, we may assume that x_A, y_A, z_A, w_A appear in the boundary of A' in counterclockwise order. Since e_i and f_i form an edge crossing for $i \in \{1, 2\}$ but no other pairs form an edge crossing, the vertices y_B, x_B, w_B, z_B appear in the boundary of B' in clockwise order (see the left-hand side of Figure 1, ignoring Q_A, Q_B, s). However, if we flip B' vertically, then we obtain a planar embedding of G (see the right-hand side of Figure 1, ignoring Q_A, Q_B, D_1, D_2, u, v), and hence Tutte's result implies that G is Hamiltonian. Therefore, we may assume $|S'| = 3$.

Suppose next that $u_1, u_2 \in S'$. Since G is 4-connected, each edge of e_1, f_1, e_2, f_2 connects A' and B'. As above, let $e_1 = x_A x_B$, $f_1 = y_A y_B$, $e_2 = z_A z_B$, and $f_2 = w_A w_B$ with $x_A, y_A, z_A, w_A \in V(A')$ and $x_B, y_B, z_B, w_B \in V(B')$, and assume that x_A, y_A, z_A, w_A appear in the boundary of A' in counterclockwise order. Then the vertices y_B, x_B, w_B, z_B appear in the boundary of B' in clockwise order. We may also assume that all neighbours of s, which is the vertex in S' with $s \neq u_1, u_2$, in A' (respectively in B') appear in Q_A (respectively in Q_B), where Q_A is the subpath of the boundary of A' from w_A to x_A in counterclockwise order (respectively Q_B is the subpath of the boundary of B' from z_B to y_B in clockwise order). See the left-hand side of Figure 1.

Note that $G - s$ behaves as described in the first paragraph of this proof. Therefore, by flipping B' vertically, we obtain a plane embedding of $G - s$. Since G is 4-connected, the vertex s has at least two neighbours in Q_A or in Q_B. By symmetry, we may assume that the former occurs. Take two neighbours u and v of s so that they are as close on Q_A as possible, and add an edge between them if they are not

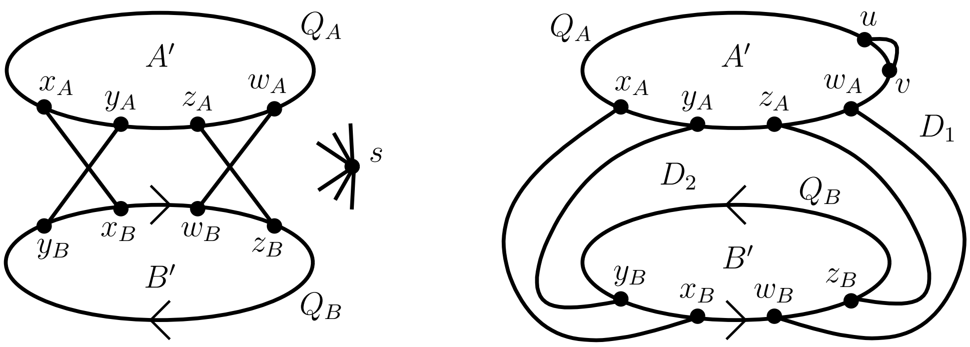

Figure 1: Vertical flipping of B' in $G - s$ to obtain the plane graph.

consecutive in Q_A. Let H be the resulting (plane) graph, let D_1 be the facial cycle of H containing the edge uv and the vertices w_A, x_A, x_B, w_B, and let D_2 be the facial cycle of H containing the path Q_B and the vertices y_A, z_A, z_B, y_B. See the right-hand side of Figure 1.

By Theorem 6, H contains a $(D_1 \cup D_2)$-Tutte cycle $\mathfrak{h}$ with $uv \in E(\mathfrak{h})$. It is easy to see that $|V(\mathfrak{h})| \geq 4$. Suppose that there is a non-trivial $\mathfrak{h}$-bridge P of H. Since $\mathfrak{h}$ is a Tutte subgraph, P has at most three attachments on $\mathfrak{h}$. As G is 4-connected, the three attachments of P on $\mathfrak{h}$ do not form a cut of G, and hence P contains a neighbour of s in G as a non-attachment. By the choice of Q_A, Q_B, u, v, all neighbours of s are contained in D_1 or D_2. Thus, P contains an edge in $D_1 \cup D_2$. Since $\mathfrak{h}$ is a $(D_1 \cup D_2)$-Tutte subgraph, P has at most two attachments on $\mathfrak{h}$, but these two attachments, together with s, form a cut of order at most three in G, a contradiction.

Therefore, there is no non-trivial $\mathfrak{h}$-bridge of H, and hence $\mathfrak{h}$ is a Hamiltonian cycle of H. Then by adding the path usv to $\mathfrak{h}$ instead of the edge uv, we obtain a Hamiltonian cycle of G. This completes the proof of Claim 2. $\square$

By Claim 2 and symmetry, we may assume that there exists a cut S' of G' with $|S'| = 3$ and $u_2 \notin S'$. Since G is 4-connected, we have $u_1 \in S'$ and both e_1 and f_1 connect A' and B'. Let $e_1 = x_A x_B$, and $f_1 = y_A y_B$ with $x_A, y_A \in V(A')$ and $x_B, y_B \in V(B')$. In this case $S = S' - \{u_1\}$ is a 2-cut in $G - \{e_1, f_1\}$, and let A and B be the components of $G - \{e_1, f_1\} - S$, corresponding to A' and B', respectively. Let $\{s_1, s_2\} = S$, $G_A = G[A \cup S]$ and $G_B = G[B \cup S]$. For G_A and G_B, we show the following claim.

Claim 3 *At least one of G_A and G_B does not contain an edge crossing.*

Proof. If neither e_2 nor f_2 connects s_1 and s_2, then clearly at least one of G_A and G_B does not contain the edge crossing by e_2 and f_2. Suppose that either e_2 or f_2, say e_2 by symmetry, connects s_1 and s_2. Since S is a 2-cut in $G - \{e_1, f_1\}$, f_2 does not connect A and B. Since the end-vertices of f_2 are distinct from those of e_2,

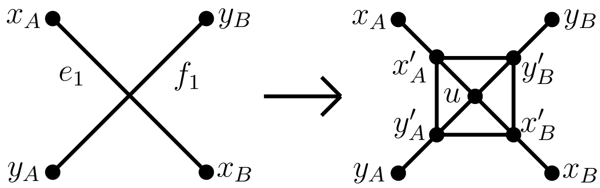

Figure 2: Replacement of the edge crossing with five new vertices and twelve new edges.

either $f_2 \in E(A)$ or $f_2 \in E(B)$ holds. Then G_B or G_A contains no edge crossing, respectively. $\Box$

By Claim 3 and symmetry, we may assume that G_A does not contain an edge crossing, i.e. G_A is a plane graph. Let D_A be the boundary of G_A. By symmetry, we may further assume that the vertices s_1, s_2, x_A, and y_A appear in D_A in this order. Now, we take a cut S' as in Claim 2 so that $|A|$ is as small as possible. This choice gives the following:

Claim 4 *G_A is 2-connected.*

Proof. Suppose that G_A contains a cut-vertex t. Then there is a component C of $G_A - t$ that contains at most two vertices in $\{s_1, s_2, x_A, y_A\}$. Furthermore, if C contains two vertices in the set, then we may assume that one of them is x_A or y_A.

If C contains x_A or y_A, say x_A by symmetry, then t and x_B possibly together with y_A or s_1 or s_2 form a cut of order at most three in G, a contradiction. Therefore, we may assume that C contains neither x_A nor y_A. By the choice of C, C contains exactly one of s_1 and s_2, say s_1 by symmetry. If C contains a vertex other than s_1, then $\{s_1, t\}$ is a 2-cut of G, a contradiction again. Thus, we may assume that C consists of only s_1. However $\{t, s_2, u_1\}$ is also a cut of G', contradicting the choice of S'. $\Box$

Let G'' be the graph obtained from $G - \{e_1, f_1\}$ by adding five new vertices x'_A, y'_A, x'_B, y'_B, and u together with the twelve edges $x_A x'_A$, $y_A y'_A$, $x_B x'_B$, $y_B y'_B$, $x'_A y'_A$, $y'_A x'_B$, $x'_B y'_B$, $y'_B x'_A$, $x'_A u$, $y'_A u$, $x'_B u$, and $y'_B u$; see Figure 2. The graph G'' has a drawing in the plane with only one edge crossing, which is formed by e_2 and f_2. We regard $G - \{e_1, f_1\}$ as a subgraph of G''. Let R be the subgraph of G'' induced by the added twelve edges. We now show the following.

Claim 5 *The graph G'' is 4-connected.*

Proof. To show this claim, we use Menger's theorem and prove that for every pair of vertices a and b of G'' there exist four pairwise internally disjoint paths connecting them, depending where a and b lie.

We first prove the case $a, b \in V(G)$. Since G is 4-connected, there exist four pairwise internally disjoint paths in G connecting a and b. If they use neither e_1 nor f_1, then they are also paths in G''. If they use exactly one of e_1 and f_1, then we obtain the desired paths by replacing e_1 or f_1 with an appropriate path in R. Thus, we may assume that they use both e_1 and f_1. In this case, replacing the two edges e_1 and f_1 with either the two paths $x_A x'_A y'_A y_A$ and $x_B x'_B y'_B y_B$ or the two paths $x_A x'_A y'_B y_B$ and $y_A y'_A x'_B x_B$, we obtain the paths we are looking for.

Next, consider the case $a, b \in V(G'') - V(G)$. Now we only prove the case $a = u$ and $b = x'_A$, but the other cases can be shown in the same way. For the vertices u and x'_A, we have the three paths ux'_A, $uy'_A x'_A$, and $uy'_B x'_A$ in R. Furthermore, since $G - \{e_1, f_1\}$ is 2-connected, we can find a path Q in $G - \{e_1, f_1\}$ connecting x_A and x_B. Then the extension of Q using the paths $x'_A x_A$ and $x_B x'_B u$ gives a fourth path, and we are done.

So, by symmetry, it only remains to treat the case $a \in V(G)$ and $b \in V(G'') - V(G)$. Since G is 4-connected, there exist four paths in G from a to the set $\{x_A, y_A, x_B, y_B\}$ that share only a. Extending these paths to b in R, we obtain the desired paths.

This completes the proof of Claim 5. $\square$

By Claim 5 and Theorem 3, $G'' - \{e_2, f_2\}$ contains a Hamiltonian cycle $\mathfrak{q}''$. Note that $\mathfrak{q}''$ uses exactly two or four edges in $F = \{x_A x'_A, y_A y'_A, x_B x'_B, y_B y'_B\}$, which is a 4-edge-cut in G''. Let $\mathfrak{q}_G = \mathfrak{q}'' - \{u, x'_A, y'_A, x'_B, y'_B\}$ and $\mathfrak{q}_B = \mathfrak{q}_G - V(A)$. Observe that both $\mathfrak{q}_G$ and $\mathfrak{q}_B$ are subgraphs of G, consisting of one path or two disjoint paths.

We divide the remaining proof into a brief discussion of the situations which can be dealt with fairly straightforwardly, followed by three more involved cases, depending on how $\mathfrak{q}''$ passes through F. In the former cases, adding e_1 or f_1 to $\mathfrak{q}_G$ immediately gives a Hamiltonian cycle in G, while in the latter three cases we can only use $\mathfrak{q}_B$ and need to modify $\mathfrak{q}''$ inside of G_A. Such modifications are allowed by Theorems 4 and 5 by specifying appropriate edges.

If $E(\mathfrak{q}'') \cap F = \{x_A x'_A, x_B x'_B\}$, we obtain a Hamiltonian cycle of G by adding the edge e_1 to $\mathfrak{q}_G$. Symmetrically, if $E(\mathfrak{q}'') \cap F = \{y_A y'_A, y_B y'_B\}$, we obtain a Hamiltonian cycle in G by adding the edge f_1 to $\mathfrak{q}_G$. Furthermore, assume all of the edges in F are used in $\mathfrak{q}''$. In this situation, $\mathfrak{q}_G$ consists of two disjoint paths with end-vertices x_A, y_A, x_B and y_B. In particular, since $G'' - \{e_2, f_2\}$ is a plane graph, there are only the following two possibilities:

- One path connects x_A and y_A, and the other connects x_B and y_B.
- One path connects x_A and y_B, and the other connects y_A and x_B.

In either case, a Hamiltonian cycle in G can be obtained by adding the two edges e_1 and f_1. Now we deal with each of the other cases.

Case 1. $E(\mathfrak{q}'') \cap F = \{x_A x'_A, y_A y'_A\}$.

In this case $\mathfrak{q}_B$ is a Hamiltonian path in G_B connecting s_1 and s_2. Let $G_A^{(1)}$ be the graph obtained from G_A by adding the edge s_1s_2 on the outer facial cycle D_A of G_A. Note that the added edge s_1s_2 and the vertices x_A and y_A are contained in a new facial cycle $D_A^{(1)}$ of $G_A^{(1)}$. Take any edges e_x and e_y in $D_A^{(1)}$ that are incident with x_A and y_A, respectively. By Theorem 5, $G_A^{(1)}$ contains a $D_A^{(1)}$-Tutte cycle $\mathfrak{h}_A^{(1)}$ with $s_1s_2, e_x, e_y \in E(\mathfrak{h}_A^{(1)})$. Then $\mathfrak{h}_A^{(1)} - \{s_1s_2\}$ is a Tutte path in G_A connecting s_1 and s_2 and containing x_A and y_A. Thus, by Lemma 8, it contains all vertices of G_A. Combining $\mathfrak{q}_B$ and $\mathfrak{h}_A^{(1)} - \{s_1s_2\}$ with the vertices s_1 and s_2, we obtain a Hamiltonian cycle in G. $\square$

Case 2. $E(\mathfrak{q}'') \cap F = \{x_Ax'_A, y_By'_B\}$.

Here $\mathfrak{q}_B$ is a path in G_B connecting y_B and s_i for some $i \in \{1,2\}$. We need to find a suitable path $\mathfrak{h}_A^{(2)}$ in G_A, considering two subcases depending on whether $\mathfrak{q}_B$ passes through s_{3-i}.

Suppose first that $\mathfrak{q}_B$ passes through s_{3-i}. Let $G_A^{(2)}$ be the graph obtained from G_A by adding the edge s_1s_2 on D_A. Note that the added edge s_1s_2 and the vertices x_A and y_A are contained in a new facial cycle $D_A^{(2)}$ of $G_A^{(2)}$. By Theorem 4, $G_A^{(2)}$ contains a $D_A^{(2)}$-Tutte path $\mathfrak{h}_A^{(2)}$ from y_A to s_{3-i} passing through the edge s_1s_2.

Suppose next that $\mathfrak{q}_B$ does not pass through s_{3-i}. Let $G_A^{(2)} = G_A$ and $D_A^{(2)} = D_A$, and let e_s be an edge of D_A that is incident with s_{3-i}. By Theorem 4, $G_A^{(2)}$ contains a $D_A^{(2)}$-Tutte path $\mathfrak{h}_A^{(2)}$ from y_A to s_i passing through the edge e_s.

We remark that in either case, $\mathfrak{h}_A^{(2)}$ contains y_A, s_1 and s_2, and we now show that it also contains x_A. Suppose to the contrary that $\mathfrak{h}_A^{(2)}$ does not contain x_A. Then there exists a non-trivial $\mathfrak{h}_A^{(2)}$-bridge P of $G_A^{(2)}$ containing x_A as non-attachment. Since x_A is contained in $D_A^{(2)}$ and $\mathfrak{h}_A^{(2)}$ is a $D_A^{(2)}$-Tutte path in $G_A^{(2)}$, P has at most two attachments. However, those attachments together with x_B form a cut of G of order at most three, a contradiction. Therefore $\mathfrak{h}_A^{(2)}$ also contains x_A.

By Lemma 8, $\mathfrak{h}_A^{(2)}$ contains all vertices of G_A. Then combining $\mathfrak{q}_B$ and $\mathfrak{h}_A^{(2)} - s_{3-i}$ (when $\mathfrak{q}_B$ passes through s_{3-i}) or $\mathfrak{h}_A^{(2)}$ (otherwise) with the vertex s_i and the edge f_1, we obtain a Hamiltonian cycle in G. $\square$

If $E(\mathfrak{q}'') \cap F = \{y_Ay'_A, x_Bx'_B\}$, we may proceed as in Case 2—simply switch the names of x's and y's.

Case 3. $E(\mathfrak{q}'') \cap F = \{y_By'_B, x_Bx'_B\}$.

In this case, $\mathfrak{q}_B$ consists of two disjoint paths connecting $\{x_B, y_B\}$ and $\{s_1, s_2\}$. Let $G_A^{(3)}$ be the graph obtained from G_A by adding the edge y_As_1 on D_A. Note that the added edge and the two vertices x_A and s_2 are contained in a new facial cycle $D_A^{(3)}$ of $G_A^{(3)}$. By Theorem 5, $G_A^{(3)}$ contains a $D_A^{(3)}$-Tutte path $\mathfrak{h}_A^{(3)}$ from x_A to s_2 passing through the edge y_As_1. By Lemma 8, $\mathfrak{h}_A^{(3)}$ contains all vertices of G_A. By planarity of G_A, $\mathfrak{h}_A^{(3)} - \{y_As_1\}$ consists of two disjoint paths in G_A such that one path connects x_A and y_A and the other connects s_1 and s_2. Therefore, combining $\mathfrak{q}_B$

and $\mathfrak{h}_A^{(3)} - \{y_A s_1\}$ with the vertices s_1 and s_2 and the edges e_1 and f_1, we obtain a Hamiltonian cycle in G. $\square$

4 Applications

4.1 Traceability

In this section we give an application of Theorem 1 concerning traceability by using the following proposition. For a 3-connected graph G, let $\phi(G)$ denote the number of 3-cuts present in G.

Proposition 9 *Let $k, \ell \geq 0$. If every 3-connected graph with crossing number at most k and at most ℓ 3-cuts is Hamiltonian, then every 3-connected graph with crossing number at most $k+1$ and at most ℓ 3-cuts is traceable.*

Proof. Let G be a 3-connected graph with $\mathrm{cr}(G) \leq k+1$ and $\phi(G) \leq \ell$. We replace one of the edge crossings as depicted in Figure 2. Note that other edges of G may cross e_1 or f_1; we perform the transformation such that these edge crossings remain unchanged. Thus we obtain a graph G' with $\mathrm{cr}(G') \leq k$ and $\phi(G') \leq \ell$; the latter follows by using the same argument as in Claim 5 from the proof of Theorem 1. Therefore, by assumption, G' contains a Hamiltonian cycle $\mathfrak{q}$. Let $F = \{x_A x'_A, y_A y'_A, x_B x'_B, y_B y'_B\}$, which is a 4-edge-cut in G'.

Suppose that exactly two of the edges in F are used in $\mathfrak{q}$. If $E(\mathfrak{q}) \cap F = \{x_A x'_A, y_A y'_A\}$, then we remove from $\mathfrak{q}$ the vertices $\{u, x'_A, x'_B, y'_A, y'_B\}$ and obtain a Hamiltonian path in G from x_A to y_A. Similarly we find a Hamiltonian path in G for all other cases.

Assume next that all of the edges in F are used in $\mathfrak{q}$. Then removing the vertices $\{u, x'_A, x'_B, y'_A, y'_B\}$ from $\mathfrak{q}$ and adding the edges e_1 and f_1, we obtain either a Hamiltonian cycle in G or two disjoint cycles containing all vertices in G. In the former case, we are done, while in the latter case, by adding an edge between the two cycles and deleting a suitable edge from each of the two cycles, we obtain a Hamiltonian path in G. $\square$

We shall use this proposition to give a tabular overview in the final section, concerning the Hamiltonicity of 3-connected graphs with few 3-cuts and small crossing number.

4.2 Crossing number and number of 3-cuts

In this section, we discuss the Hamiltonicity of 3-connected graphs featuring both a small crossing number and few 3-cuts. We begin with a negative result on their Hamiltonicity and traceability.

Proposition 10 *There exist infinitely many 3-connected graphs G which are*

- *non-Hamiltonian and satisfy* $\mathrm{cr}(G) + \phi(G) = 6$, *and*
- *non-traceable and satisfy* $\mathrm{cr}(G) + \phi(G) = 8$.

Proof. Consider a quadrilateral $Q = abcd$. Inserting into Q a vertex w and adding the edges aw, bw, cw, and dw, we say that we 1-*augment* Q. If we insert into Q two vertices u and v and eight edges $au, bu, cu, du, av, bv, cv$, and dv, then we say that we 2-*augment* Q. Note that if Q is a quadrangular face, then by 1-augmenting Q, the new vertex w can be added without creating any edge crossings, while by 2-augmenting Q, the new vertices u and v can be added with two edge crossings.

Let G be a 3-connected quadrangulation with three pairwise disjoint quadrilaterals Q_1, Q_2, and Q_3. Let G' be the graph obtained by 2-augmenting Q_1, Q_2, and Q_3 and 1-augmenting all other quadrilaterals. Then G' is a 4-connected graph with $\mathrm{cr}(G') \leq 6$. We now show that in fact $\mathrm{cr}(G') = 6$. Let F be a set of edges of G' with $|F| \leq 5$, and for $i \in \{1, 2, 3\}$, let $Q_i = a_i b_i c_i d_i$, and u_i and v_i be the two new vertices obtained by 2-augmenting Q_i. Since Q_1, Q_2, and Q_3 are pairwise disjoint, one of them, say Q_1 by symmetry, satisfies that the subgraph induced by $\{a_1, b_1, c_1, d_1, u_1, v_1\}$ contains at most one edge in F. By symmetry, we may assume that all of the edges $au_1, bu_1, cu_1, av_1, bv_1$, and cv_1 exist in $G' - F$. Then we can find a minor of $K_{3,3}$ such that a_1, b_1, and c_1 form the vertices in one partite set and u_1, v_1, and the remaining vertices form those in the other partite set. Therefore, $G' - F$ is not planar. Since such an edge-set F can be chosen arbitrarily, we see that $\mathrm{cr}(G') = 6$.

We next show that G' is non-Hamiltonian. By Euler's formula, the number of faces in G is $n - 2$, where n is the number of vertices of G. In G' we colour blue the vertices originally belonging to G, and red the vertices added when augmenting. Then no two red vertices are adjacent and we have in G' exactly n blue vertices and $3 \cdot 2 + (n - 2 - 3) = n + 1$ red vertices. Thus, removing all n blue vertices, we obtain $n + 1$ components, so G' is not 1-tough. Thus G' is non-Hamiltonian.

Since we can choose arbitrary 3-connected quadrangulations with three pairwise disjoint quadrilaterals as G, this shows that there exist infinitely many 3-connected graphs G which are non-Hamiltonian and satisfy $\mathrm{cr}(G) = 6$ and $\phi(G) = 0$. It is easy to check that when removing an edge which decreases the crossing number (by one), one new (trivial) 3-cut appears, and hence the first statement holds for any $0 \leq \phi(G) \leq 6$.

For the non-traceable case we proceed in the same manner, but we 2-augment four pairwise disjoint quadrilaterals and 1-augment all other quadrilaterals of a given quadrangulation. □

It may come as a surprise that there exists a graph with $\mathrm{cr}(G) + \phi(G) = 3$ which is, as the graphs described above, also not 1-tough, namely $K_{3,4}$. In contrast to

Proposition 10, Theorem 1 guarantees the Hamiltonicity for 3-connected graphs G with $\text{cr}(G) \leq 2$ and $\phi(G) = 0$. Let us now give another positive result.

Theorem 11 *Every 3-connected graph with crossing number at most 1 and containing at most one 3-cut is Hamiltonian.*

Proof. If G is a 3-connected graph satisfying $\text{cr}(G) + \phi(G) \leq 1$, then G is Hamiltonian by theorems of Thomas and Yu [10] or Thomassen [13]. Henceforth, let G be a 3-connected graph with $\text{cr}(G) = \phi(G) = 1$. Denote the unique 3-cut of G with S.

Observe that G can be embedded into the projective plane without edge crossings. If there are two vertices u and v in S with $uv \notin E(G)$, then there exists a facial cycle D with $u, v \in V(D)$, and let $G' = G$. On the other hand, if S forms a triangle, then let $G' = G - \{uv\}$ for some $u, v \in S$ and let D be the new facial cycle. In either case G' is 3-connected and any 3-cut S' satisfies either

(1) $S' = S$ or

(2) each component of $G' - S'$ contains one of u and v.

In either case, for any 3-cut S' of G', the choice of D implies that D contains a vertex in both components of $G' - S'$. We choose an arbitrary edge e in D that is incident with u. By Theorem 7, G' contains a D-Tutte cycle $\mathfrak{h}$ passing through e. We show that $\mathfrak{h}$ is a Hamiltonian cycle in G' and hence in G.

Suppose to the contrary that there is a non-trivial $\mathfrak{h}$-bridge P of G'. Since $\mathfrak{h}$ is a Tutte subgraph, P has at most three attachments on $\mathfrak{h}$. Since G' is 3-connected, P has exactly three attachments and they form a 3-cut in G'. So, either (1) or (2) above is satisfied, and P contains an edge in D. Since $\mathfrak{h}$ is a D-Tutte cycle, P has at most two attachments, a contradiction. $\square$

4.3 Hypohamiltonian Graphs

A graph G is *hypohamiltonian* if G is non-Hamiltonian yet $G - v$ is Hamiltonian for every vertex v in G. Using a result of Tutte [15], Thomassen [13] showed that planar hypohamiltonian graphs always contain a cubic vertex—put in a different, perhaps more appealing way, Thomassen's result states that if in a planar graph with minimum degree at least 4 every vertex-deleted subgraph is Hamiltonian, then the graph itself must be Hamiltonian. (We cannot replace the "4" with a "3", since by another result of Thomassen planar cubic hypohamiltonian graphs exist [12].) The second author strengthened this result in several directions [17], one of which states that every hypohamiltonian graph with crossing number 1 contains at least one cubic vertex. We here present an extension of this theorem, but first need two more definitions.

Every non-complete graph G of connectivity exactly k has non-empty induced subgraphs G_1, G_2 such that $G = G_1 \cup G_2$, $|V(G_1)| \geq k+1$, $|V(G_2)| \geq k+1$, and $V(G_1) \cap V(G_2) = S$, where $|S| = k$. Then G_1 (and also G_2) is called a *k-fragment* of G and S is the set of *ends* of G_1.

Corollary 12 *Let G be a graph with crossing number 1 in which every vertex-deleted subgraph is Hamiltonian. If G contains at most one cubic vertex, then G is Hamiltonian.*

Proof. Since every vertex-deleted subgraph of G is Hamiltonian, we have that G is 3-connected, and thus has minimum degree at least 3. Assume G is non-Hamiltonian. Since every vertex-deleted subgraph of G is Hamiltonian, G is hypohamiltonian. By Theorem 11, G contains at least two 3-cuts. Since G has at most one cubic vertex, one of these 3-cuts must be non-trivial. We denote this 3-cut with S. Let H and H' be the 3-fragments with ends S. By a lemma of the second author [17], no vertex in S is cubic, so any cubic vertex present in G lies either in $V(H) \setminus S$ or $V(H') \setminus S$—say the former. Clearly we have $\mathrm{cr}(H) \leq 1$. If H is planar, by using a result of Thomassen [13] we identify the ends of H and a copy thereof. We obtain a planar hypohamiltonian graph with at most two cubic vertices. This, however, contradicts [17, Theorem 3]. If $\mathrm{cr}(H) = 1$, then we glue two copies of H' and are led, in the same way, to a contradiction. $\square$

We would very much like to extend above result to the case of crossing number 2. Unfortunately, we did not succeed. We were however able to prove the following.

Theorem 13 *Every hypohamiltonian graph G with crossing number 2 containing at least two 3-cuts must contain a cubic vertex.*

Proof. In addition to above properties, let G have minimum order. Now assume that G has minimum degree at least 4. Let S be one of the (necessarily non-trivial) 3-cuts present in G. Let H and H' be the two 3-fragments with ends S. Arguing as in [17, Lemma 9], we obtain that either one of the two 3-fragments is planar, or both have crossing number 1. In the former case, let H be the planar 3-fragment. Glue H and a copy of itself using [17, Lemma 2]. We obtain a planar hypohamiltonian graph with no cubic vertices, contradicting a theorem of Thomassen [13]. In the latter case, by [13, Lemma 3], w.l.o.g. H has order strictly less than $\frac{1}{2}(|V(G)|+3)$ since G contains at least two 3-cuts. Gluing two copies of H yields a hypohamiltonian graph with crossing number 2 with fewer vertices than G, a contradiction. (Would there be only one 3-cut in G, the fragment H could have exactly $\frac{1}{2}(|V(G)|+3)$ vertices and no contradiction would occur.) $\square$

We derive that if there exists a hypohamiltonian graph with crossing number 2 which has minimum degree at least 4, then it contains exactly one non-trivial 3-cut S such that each of the two 3-fragments with ends S has crossing number 1.

5 Conclusion

By Theorem 1, as well as Propositions 9 and 10 we obtain Table 1, which shows the known results on the Hamiltonicity and traceability of 4-connected graphs with given crossing number. Question marks indicate open problems, which are interesting but challenging.

cr	0	1	2	3	4	5	6	7	8
Hamiltonian	✓(A)	✓(B)	✓(C1)	?	?	?	✗✗	✗✗	✗✗
Traceable	✓(A)	✓(B)	✓(C1)	✓(C2)	?	?	?	?	✗✗

(A) Tutte [15]
(B) Thomas and Yu [10]
(C1) This paper, Theorem 1
(C2) This paper, Proposition 9

Table 1: On the Hamiltonicity and traceability of 4-connected graphs with small crossing number. Green cells (marked ✓) signify that the corresponding graphs are Hamiltonian and traceable, respectively, while red cells (marked ✗✗) represent that not all graphs in the respective family are Hamiltonian or traceable.

ϕ\cr	0	1	2	3	4	5	6
0	✓(A)	✓(C)	✓(F1)	?	?	?	✗✗
1	✓(B)	✓(F2)	✗ $K_{3,4}$	✗	✗ $K_{3,5}$	✗✗	
2	✓(C)	?	✗	✗	✗✗		
3	✓(D)	?	✗	✗✗			
4	?	✗(E)	✗✗				
5	?	✗✗					
6	✗✗						

(A) Tutte [15]
(B) Thomassen [14]
(C) Thomas and Yu [10]
(D) Brinkmann and Zamfirescu [1]
(E) Barish (personal communication)
(F1) This paper, Theorem 1
(F2) This paper, Theorem 11

Table 2: On the Hamiltonicity of 3-connected graphs with small crossing number and few 3-cuts. Green cells (marked ✓) signify that the corresponding graphs are Hamiltonian, while orange (marked ✗) and red cells (marked ✗✗) represent that not all graphs in the respective family are Hamiltonian, because of graphs related to $K_{3,4}$, $K_{3,5}$ or $K_{4,5}$, and Proposition 10, respectively.

Considering Theorem 11 and Proposition 10, one may think that Table 1 can be extended to 3-connected graphs with few 3-cuts, but the situation is more complicated, see Table 2. The complete bipartite graph $K_{3,4}$ is not Hamiltonian, while $\phi(K_{3,4}) = 1$ and it is easy to check that $\text{cr}(K_{3,4}) = 2$. However, we do not know of any other 3-connected non-Hamiltonian graphs G with $\phi(G) = 1$ and $\mathrm{cr}(G) = 2$. On the other hand, for any $(i, j) \in \{(1, 3), (1, 4), (2, 2), (2, 3), (3, 2)\}$, suitable combinations of $K_{3,4}$ with 4-connected plane graphs give infinitely many 3-connected non-Hamiltonian graphs G with $\phi(G) = i$ and $\mathrm{cr}(G) = j$. Furthermore, we can construct infinitely many 3-connected non-Hamiltonian graphs G with $\phi(G) = 4$ and $\mathrm{cr}(G) = 1$; this can be achieved by considering as base graph $K_{4,5}$ with a slightly modified edge-set, as pointed out by Robert Barish in personal correspondence. We leave the details to the reader.

Note that a similar situation occurs for traceability, as well. Proposition 10 shows that for any (i, j) with $i + j \leq 8$, there are infinitely many non-traceable graphs G with $\phi(G) = i$ and $\mathrm{cr}(G) = j$, but there exist 3-connected non-traceable graphs for which the sum of crossing number and number of 3-cuts is less than 8. The simplest example is $K_{3,5}$, which is not traceable and satisfies $\phi(K_{3,5}) = 1$ and $\mathrm{cr}(K_{3,5}) = 4$. We can modify $K_{3,5}$ with non-traceable graphs G with $\phi(G) = i$ and $\mathrm{cr}(G) = j$ for certain pairs (i, j). Since the situation seems much more complicated, we leave this as future work.

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