

FOURIER MULTIPLIERS IN HILBERT SPACES

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ABSTRACT. This is a survey on a notion of invariant operators, or Fourier multipliers on Hilbert spaces. This concept is defined with respect to a fixed partition of the space into a direct sum of finite dimensional subspaces. In particular this notion can be applied to the important case of $L^2(M)$ where M is a compact manifold M endowed with a positive measure. The partition in this case comes from the spectral properties of a fixed elliptic operator E .

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1. INTRODUCTION

These notes are based on our paper [DR16b] and have been prepared for the instructional volume associated to the Summer School on Fourier Integral Operators held in Ouagadougou, Burkina Faso, where the authors took part during 14–26 September 2015.

In this note we discuss invariant operators, or Fourier multipliers in a general Hilbert space \mathcal{H} . This notion is based on a partition of \mathcal{H} into a direct sum of finite dimensional subspaces, so that a densely defined operator on \mathcal{H} can be decomposed as acting in these subspaces. In the present exposition we follow our detailed description in [DR16b], with which there are intersections and to which we also refer for further details.

There are two main examples of this construction: operators on $\mathcal{H} = L^2(M)$ for a compact manifold M as well as operators on $\mathcal{H} = L^2(G)$ for a compact Lie group G . The difference in approaches to these settings is in the choice of partitions of \mathcal{H} into direct sums of subspaces: in the former case they are chosen as eigenspaces of a fixed elliptic pseudo-differential operator on M while in the latter case they are chosen as

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linear spans of matrix coefficients of inequivalent irreducible unitary representations of G .

Let M be a closed manifold (i.e. a compact smooth manifold without boundary) of dimension n endowed with a positive measure dx . Given an elliptic positive pseudo-differential operator E of order ν on M , by considering an orthonormal basis consisting of eigenfunctions of E we can associate a discrete Fourier analysis to the operator E in the sense introduced by Seeley ([See65], [See69]).

These notions can be applied to the derivation of conditions characterising those invariant operators on $L^2(M)$ that belong to Schatten classes. Furthermore, sufficient conditions for the r -nuclearity on L^p -spaces can also be obtained as well as the corresponding trace formulas relating operator traces to expressions involving their symbols. More details on these applications can be found in Section 8 of [DR16b].

A characteristic feature that appears is that no regularity is assumed neither on the symbol nor on the kernel. In the case of compact Lie groups, our results extend results on Schatten classes and on r -nuclear operators on L^p spaces that have been obtained in [DR16c] and [DRv2]. This can be shown by relating the symbols introduced in this paper to matrix-valued symbols on compact Lie groups developed in [RT61] and in [RT10].

To formulate the notions more precisely, let \mathcal{H} be a complex Hilbert space and let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a linear compact operator. If we denote by $T^* : \mathcal{H} \rightarrow \mathcal{H}$ the adjoint of T , then the linear operator $(T^*T)^{\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$ is positive and compact. Let $(\psi_k)_k$ be an orthonormal basis for \mathcal{H} consisting of eigenvectors of $|T| = (T^*T)^{\frac{1}{2}}$, and let $s_k(T)$ be the eigenvalue corresponding to the eigenvector ψ_k , $k = 1, 2, \dots$. The non-negative numbers $s_k(T)$, $k = 1, 2, \dots$, are called the singular values of $T : \mathcal{H} \rightarrow \mathcal{H}$. If $0 < p < \infty$ and the sequence of singular values is p -summable, then T is said to belong to the Schatten class $S_p(\mathcal{H})$, and it is well known that each $S_p(\mathcal{H})$ is an ideal in $\mathcal{L}(\mathcal{H})$. If $1 \leq p < \infty$, a norm is associated to $S_p(\mathcal{H})$ by

$$\|T\|_{S_p} = \left(\sum_{k=1}^{\infty} (s_k(T))^p \right)^{\frac{1}{p}}.$$

If $1 \leq p < \infty$ the class $S_p(\mathcal{H})$ becomes a Banach space endowed by the norm $\|T\|_{S_p}$. If $p = \infty$ we define $S_\infty(\mathcal{H})$ as the class of bounded linear operators on \mathcal{H} , with $\|T\|_{S_\infty} := \|T\|_{op}$, the operator norm. For the Schatten class S_2 we will sometimes write $\|T\|_{\text{HS}}$ instead of $\|T\|_{S_2}$. In the case $0 < p < 1$ the quantity $\|T\|_{S_p}$ only defines a quasi-norm, and $S_p(\mathcal{H})$ is also complete. The space $S_1(\mathcal{H})$ is known as the *trace class* and an element of $S_2(\mathcal{H})$ is usually said to be a *Hilbert-Schmidt* operator. For the basic theory of Schatten classes we refer the reader to [GK69], [RS75], [Sim79], [Sch70].

It is well known that the class $S_2(L^2)$ is characterised by the square integrability of the corresponding integral kernels, however, kernel estimates of this type are not effective for classes $S_p(L^2)$ with $p < 2$. This is explained by a classical Carleman's example [Car16] on the summability of Fourier coefficients of continuous functions (see [DRv2] for a complete explanation of this fact). This obstruction explains the relevance of symbolic Schatten criteria and here we will clarify the advantage of the

symbol approach with respect to this obstruction. With this approach, no regularity of the kernel needs to be assumed.

We introduce ℓ^p -style norms on the space of symbols Σ , yielding discrete spaces $\ell^p(\Sigma)$ for $0 < p \leq \infty$, normed for $p \geq 1$. Denoting by σ_T the matrix symbol of an invariant operator T provided by Theorem 4.1, Schatten classes of invariant operators on $L^2(M)$ can be characterised in terms of symbols. Here, the condition that T is invariant will mean that T is strongly commuting with E (see Theorem 4.1). On the level of the Fourier transform this means that

$$\widehat{Tf}(\ell) = \sigma(\ell)\widehat{f}(\ell)$$

for a family of matrices $\sigma(\ell)$, i.e. T assumes the familiar form of a Fourier multiplier.

In Section 2 in Theorem 2.1 we discuss the abstract notion of symbol for operators densely defined in a general Hilbert space \mathcal{H} , and give several alternative formulations for invariant operators, or for Fourier multipliers, relative to a fixed partition of \mathcal{H} into a direct sum of finite dimensional subspaces,

$$\mathcal{H} = \bigoplus_j H_j.$$

Consequently, in Theorem 2.3 we give the necessary and sufficient condition for the bounded extendability of an invariant operator to $\mathcal{L}(\mathcal{H})$ in terms of its symbol, and in Theorem 2.5 the necessary and sufficient condition for the operator to be in Schatten classes $S_r(\mathcal{H})$ for $0 < r < \infty$, as well as the trace formula for operators in the trace class $S_1(\mathcal{H})$ in terms of their symbols. As our subsequent analysis relies to a large extent on properties of elliptic pseudo-differential operators on M , in Sections 3 and 4 we specify this abstract analysis to the setting of operators densely defined on $L^2(M)$. The main difference is that we now adopt the Fourier analysis to a fixed elliptic positive pseudo-differential operator E on M , contrary to the case of an operator $E_o \in \mathcal{L}(\mathcal{H})$ in Theorem 2.2.

The notion of invariance depends on the choice of the spaces H_j . Thus, in the analysis of operators on M we take H_j 's to be the eigenspaces of E . However, other choices are possible. For example, for $\mathcal{H} = L^2(G)$ for a compact Lie group G , choosing H_j 's as linear spans of representation coefficients for inequivalent irreducible unitary representations of G , we make a link to the quantization of pseudo-differential operators on compact Lie groups as in [RT10]. These two partitions coincide when inequivalent representations of G produce distinct eigenvalues of the Laplacian; for example, this is the case for $G = \text{SO}(3)$. However, the partitions are different when inequivalent representations produce equal eigenvalues, which is the case, for example, for $G = \text{SO}(4)$. For the more explicit example on $\mathcal{H} = L^2(\mathbb{T}^n)$ on the torus see Remark 2.6. A similar choice could be made in other settings producing a discrete spectrum and finite dimensional eigenspaces, for example for operators in Shubin classes on \mathbb{R}^n , see Chodosh [Cho11] for the case $n = 1$.

As an illustration we give an application to the spectral theory. The analogous concept to Schatten classes in the setting of Banach spaces is the notion of r -nuclearity introduced by Grothendieck [Gro55]. It has applications to questions of the distribution of eigenvalues of operators in Banach spaces. In the setting of compact Lie groups these applications have been discussed in [DRv2] and they include conclusions on the distribution or summability of eigenvalues of operators acting on L^p -spaces.

Another application is the Grothendieck-Lidskii formula which is the formula for the trace of operators on $L^p(M)$. Once we have r -nuclearity, most of further arguments are then purely functional analytic, so they apply equally well in the present setting of closed manifolds.

The paper is organised as follows. In Section 2 we discuss Fourier multipliers and their symbols in general Hilbert spaces. In Section 3 we associate a global Fourier analysis to an elliptic positive pseudo-differential operator E on a closed manifold M . In Section 4 we introduce the class of operators invariant relative to E as well as their matrix-valued symbols, and apply this to characterise invariant operators in Schatten classes in Section 5.

Throughout the paper, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also $\delta_{j\ell}$ will denote the Kronecker delta, i.e. $\delta_{j\ell} = 1$ for $j = \ell$, and $\delta_{j\ell} = 0$ for $j \neq \ell$.

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2. FOURIER MULTIPLIERS IN HILBERT SPACES

In this section we present an abstract set up to describe what we will call invariant operators, or Fourier multipliers, acting on a general Hilbert space \mathcal{H} . We will give several characterisations of such operators and their symbols. Consequently, we will apply these notions to describe several properties of the operators, in particular, their boundedness on \mathcal{H} as well as the Schatten properties.

We note that direct integrals (sums in our case) of Hilbert spaces have been investigated in a much greater generality, see e.g. Bruhat [Bru68], Dixmier [Dix96, Ch 2., §2], [Dix77, Appendix]. The setting required for our analysis is much simpler, so we prefer to adapt it specifically for consequent applications.

The main application of the constructions below will be in the setting when M is a compact manifold without boundary, $\mathcal{H} = L^2(M)$ and $\mathcal{H}^\infty = C^\infty(M)$, which will be described in detail in Section 3. However, several facts can be more clearly interpreted in the setting of abstract Hilbert spaces, which will be our set up in this section. With this particular example in mind, in the following theorem, we can think of $\{e_j^k\}$ being an orthonormal basis given by eigenfunctions of an elliptic operator on M , and d_j the corresponding multiplicities. However, we allow flexibility in grouping the eigenfunctions in order to be able to also cover the case of operators on compact Lie groups.

Theorem 2.1. *Let \mathcal{H} be a complex Hilbert space and let $\mathcal{H}^\infty \subset \mathcal{H}$ be a dense linear subspace of \mathcal{H} . Let $\{d_j\}_{j \in \mathbb{N}_0} \subset \mathbb{N}$ and let $\{e_j^k\}_{j \in \mathbb{N}_0, 1 \leq k \leq d_j}$ be an orthonormal basis of \mathcal{H} such that $e_j^k \in \mathcal{H}^\infty$ for all j and k . Let $H_j := \text{span}\{e_j^k\}_{k=1}^{d_j}$, and let $P_j : \mathcal{H} \rightarrow H_j$ be the orthogonal projection. For $f \in \mathcal{H}$, we denote*

$$\widehat{f}(j, k) := (f, e_j^k)_{\mathcal{H}}$$

and let $\widehat{f}(j) \in \mathbb{C}^{d_j}$ denote the column of $\widehat{f}(j, k)$, $1 \leq k \leq d_j$. Let $T : \mathcal{H}^\infty \rightarrow \mathcal{H}$ be a linear operator. Then the following conditions are equivalent:

- (A) *For each $j \in \mathbb{N}_0$, we have $T(H_j) \subset H_j$.*

(B) For each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that for all e_j^k

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk} \delta_{j\ell}.$$

(C) If in addition, e_j^k are in the domain of T^* for all j and k , then for each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that

$$\widehat{Tf}(\ell) = \sigma(\ell) \widehat{f}(\ell)$$

for all $f \in \mathcal{H}^\infty$.

The matrices $\sigma(\ell)$ in (B) and (C) coincide.

The equivalent properties (A)–(C) follow from the condition

(D) For each $j \in \mathbb{N}_0$, we have $TP_j = P_jT$ on \mathcal{H}^∞ .

If, in addition, T extends to a bounded operator $T \in \mathcal{L}(\mathcal{H})$ then (D) is equivalent to (A)–(C).

Under the assumptions of Theorem 2.1, we have the direct sum decomposition

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} H_j, \quad H_j = \text{span}\{e_j^k\}_{k=1}^{d_j}, \quad (2.1)$$

and we have $d_j = \dim H_j$. The two applications that we will consider will be with $\mathcal{H} = L^2(M)$ for a compact manifold M with H_j being the eigenspaces of an elliptic pseudo-differential operator E , or with $\mathcal{H} = L^2(G)$ for a compact Lie group G with

$$H_j = \text{span}\{\xi_{km}\}_{1 \leq k, m \leq d_\xi}$$

for a unitary irreducible representation $\xi \in [\xi_j] \in \widehat{G}$. The difference is that in the first case we will have that the eigenvalues of E corresponding to H_j 's are all distinct, while in the second case the eigenvalues of the Laplacian on G for which H_j 's are the eigenspaces, may coincide. In Remark 2.6 we give an example of this difference for operators on the torus \mathbb{T}^n .

In view of properties (A) and (C), respectively, an operator T satisfying any of the equivalent properties (A)–(C) in Theorem 2.1, will be called an *invariant operator*, or a *Fourier multiplier relative to the decomposition* $\{H_j\}_{j \in \mathbb{N}_0}$ in (2.1). If the collection $\{H_j\}_{j \in \mathbb{N}_0}$ is fixed once and for all, we can just say that T is *invariant* or a *Fourier multiplier*.

The family of matrices σ will be called the *matrix symbol of T relative to the partition $\{H_j\}$ and to the basis $\{e_j^k\}$* . It is an element of the space Σ defined by

$$\Sigma = \{\sigma : \mathbb{N}_0 \ni \ell \mapsto \sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}\}. \quad (2.2)$$

A criterion for the extendability of T to $\mathcal{L}(\mathcal{H})$ in terms of its symbol will be given in Theorem 2.3.

For $f \in \mathcal{H}$, in the notation of Theorem 2.1, by definition we have

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) e_j^k \quad (2.3)$$

with the convergence of the series in \mathcal{H} . Since $\{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j}$ is a complete orthonormal system on \mathcal{H} , for all $f \in \mathcal{H}$ we have the Plancherel formula

$$\|f\|_{\mathcal{H}}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |(f, e_j^k)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j, k)|^2 = \|\widehat{f}\|_{\ell^2(\mathbb{N}_0, \Sigma)}^2, \quad (2.4)$$

where we interpret $\widehat{f} \in \Sigma$ as an element of the space

$$\ell^2(\mathbb{N}_0, \Sigma) = \left\{ h : \mathbb{N}_0 \rightarrow \prod_d \mathbb{C}^{d_j} : h(j) \in \mathbb{C}^{d_j} \text{ and } \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j, k)|^2 < \infty \right\}, \quad (2.5)$$

and where we have written $h(j, k) = h(j)_k$. In other words, $\ell^2(\mathbb{N}_0, \Sigma)$ is the space of all $h \in \Sigma$ such that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j, k)|^2 < \infty.$$

We endow $\ell^2(\mathbb{N}_0, \Sigma)$ with the norm

$$\|h\|_{\ell^2(\mathbb{N}_0, \Sigma)} := \left(\sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j, k)|^2 \right)^{\frac{1}{2}}. \quad (2.6)$$

We note that the matrix symbol $\sigma(\ell)$ depends not only on the partition (2.1) but also on the choice of the orthonormal basis. Whenever necessary, we will indicate the dependence of σ on the orthonormal basis by writing $(\sigma, \{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j})$ and we also will refer to $(\sigma, \{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j})$ as the *symbol* of T . Throughout this section the orthonormal basis will be fixed and unless there is some risk of confusion the symbols will be denoted simply by σ . In the invariant language, we have that the transpose of the symbol, $\sigma(j)^\top = T|_{H_j}$ is just the restriction of T to H_j , which is well defined in view of the property (A).

We will also sometimes write T_σ to indicate that T_σ is an operator corresponding to the symbol σ . It is clear from the definition that invariant operators are uniquely determined by their symbols. Indeed, if $T = 0$ we obtain $\sigma = 0$ for any choice of an orthonormal basis. Moreover, we note that by taking $j = \ell$ in (B) of Theorem 2.1 we obtain the formula for the symbol:

$$\sigma(j)_{mk} = \widehat{T e_j^k}(j, m), \quad (2.7)$$

for all $1 \leq k, m \leq d_j$. The formula (2.7) furnishes an explicit formula for the symbol in terms of the operator and the orthonormal basis. The definition of Fourier coefficients tells us that for invariant operators we have

$$\sigma(j)_{mk} = (T e_j^k, e_j^m)_H. \quad (2.8)$$

In particular, for the identity operator $T = I$ we have $\sigma_I(j) = I_{d_j}$, where $I_{d_j} \in \mathbb{C}^{d_j \times d_j}$ is the identity matrix.

Let us now indicate a formula relating symbols with respect to different orthonormal basis. If $\{e_\alpha\}$ and $\{f_\alpha\}$ are orthonormal bases of \mathcal{H} , we consider the unitary

operator U determined by $U(e_\alpha) = f_\alpha$. Then we have

$$(Te_\alpha, e_\beta)_\mathcal{H} = (UTE_\alpha, Ue_\beta)_\mathcal{H} = (UTU^*Ue_\alpha, Ue_\beta)_\mathcal{H} = (UTU^*f_\alpha, f_\beta)_\mathcal{H}.$$

Thus, if $(\sigma_T, \{e_\alpha\})$ denotes the symbol of T with respect to the orthonormal basis $\{e_\alpha\}$ and $(\sigma_{UTU^*}, \{f_\alpha\})$ denotes the symbol of UTU^* with respect to the orthonormal basis $\{f_\alpha\}$ we have obtained the relation

$$(\sigma_T, \{e_\alpha\}) = (\sigma_{UTU^*}, \{f_\alpha\}). \quad (2.9)$$

Thus, the equivalence relation of basis $\{e_\alpha\} \sim \{f_\alpha\}$ given by a unitary operator U induces the equivalence relation on the set Σ of symbols given by (2.9). In view of this, we can also think of the symbol being independent of a choice of basis, as an element of the space Σ / \sim with the equivalence relation given by (2.9).

We make another remark concerning part (C) of Theorem 2.1. We use the condition that e_j^k are in the domain $\text{Dom}(T^*)$ of T^* in showing the implication (B) \implies (C). Since e_j^k 's give a basis in \mathcal{H} , and are all contained in $\text{Dom}(T^*)$, it follows that $\text{Dom}(T^*)$ is dense in \mathcal{H} . In particular, by [RS80, Theorem VIII.1], T must be closable (in part (C)). These conditions are not restrictive for the further analysis since they are satisfied in several natural applications.

The principal application of the notions above will be as follows, except for in the sequel we sometimes need more general operators E unbounded on \mathcal{H} . In order to distinguish from this general case, in the following theorem we use the notation E_o .

Theorem 2.2. *Continuing with the notation of Theorem 2.1, let $E_o \in \mathcal{L}(\mathcal{H})$ be a linear continuous operator such that H_j are its eigenspaces:*

$$E_o e_j^k = \lambda_j e_j^k$$

for each $j \in \mathbb{N}_0$ and all $1 \leq k \leq d_j$. Then equivalent conditions (A)–(C) imply the property

$$(E) \text{ For each } j \in \mathbb{N}_0 \text{ and } 1 \leq k \leq j, \text{ we have } TE_o e_j^k = E_o T e_j^k,$$

and if $\lambda_j \neq \lambda_\ell$ for $j \neq \ell$, then (E) is equivalent to properties (A)–(C).

Moreover, if T extends to a bounded operator $T \in \mathcal{L}(\mathcal{H})$ then equivalent properties (A)–(D) imply the condition

$$(F) TE_o = E_o T \text{ on } \mathcal{H},$$

and if also $\lambda_j \neq \lambda_\ell$ for $j \neq \ell$, then (F) is equivalent to (A)–(E).

For an operator $T = F(E_o)$, when it is well-defined by the spectral calculus, we have

$$\sigma_{F(E_o)}(j) = F(\lambda_j)I_{d_j}. \quad (2.10)$$

In fact, this is also well-defined then for a function F defined on λ_j , with finite values which are e.g. j -uniformly bounded (also for non self-adjoint E_o).

We have the following criterion for the extendability of a densely defined invariant operator $T : \mathcal{H}^\infty \rightarrow \mathcal{H}$ to $\mathcal{L}(\mathcal{H})$, which was an additional hypothesis for properties (D) and (F). In the statements below we fix a partition into H_j 's as in (2.1) and the invariance refers to it.

Theorem 2.3. *An invariant linear operator $T : \mathcal{H}^\infty \rightarrow \mathcal{H}$ extends to a bounded operator from \mathcal{H} to \mathcal{H} if and only if its symbol σ satisfies $\sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)} < \infty$.*

Moreover, denoting this extension also by T , we have

$$\|T\|_{\mathcal{L}(\mathcal{H})} = \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}.$$

We also record the formula for the symbol of the composition of two invariant operators:

Proposition 2.4. *If $S, T : \mathcal{H}^\infty \rightarrow \mathcal{H}$ are invariant operators with respect to the same orthonormal partition, and such that the domain of $S \circ T$ contains \mathcal{H}^∞ , then $S \circ T : \mathcal{H}^\infty \rightarrow \mathcal{H}$ is also invariant with respect to the same partition. Moreover, if σ_S denotes the symbol of S and σ_T denotes the symbols of T with respect to the same orthonormal basis then*

$$\sigma_{S \circ T} = \sigma_S \sigma_T,$$

i.e. $\sigma_{S \circ T}(j) = \sigma_S(j) \sigma_T(j)$ for all $j \in \mathbb{N}_0$.

We now show another application of the above notions to give a characterisation of Schatten classes of invariant operators in terms of their symbols.

Theorem 2.5. *Let $0 < r < \infty$. An invariant operator $T \in \mathcal{L}(\mathcal{H})$ with symbol σ is in the Schatten class $S_r(\mathcal{H})$ if and only if*

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r(H_\ell)}^r < \infty.$$

Moreover

$$\|T\|_{S_r(\mathcal{H})} = \left(\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r(H_\ell)}^r \right)^{1/r}. \quad (2.11)$$

In particular, if T is in the trace class $S_1(\mathcal{H})$, then we have the trace formula

$$\mathrm{Tr}(T) = \sum_{\ell=0}^{\infty} \mathrm{Tr}(\sigma(\ell)). \quad (2.12)$$

Remark 2.6. We note that the membership in $\mathcal{L}(\mathcal{H})$ and in the Schatten classes $S_r(\mathcal{H})$ does not depend on the decomposition of \mathcal{H} into subspaces H_j as in (2.1). However, the notion of invariance does depend on it. For example, let $\mathcal{H} = L^2(\mathbb{T}^n)$ for the n -torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Choosing

$$H_j = \mathrm{span}\{e^{2\pi i j \cdot x}\}, \quad j \in \mathbb{Z}^n,$$

we recover the classical notion of invariance on compact Lie groups and moreover, invariant operators with respect to $\{H_j\}_{j \in \mathbb{Z}^n}$ are the translation invariant operators on the torus \mathbb{T}^n . However, to recover the construction of Section 4 on manifolds, we take \widetilde{H}_ℓ to be the eigenspaces of the Laplacian E on \mathbb{T}^n , so that

$$\widetilde{H}_\ell = \bigoplus_{|j|^2=\ell} H_j = \mathrm{span}\{e^{2\pi i j \cdot x} : j \in \mathbb{Z}^n \text{ and } |j|^2 = \ell\}, \quad \ell \in \mathbb{N}_0.$$

Then translation invariant operators on \mathbb{T}^n , i.e. operators invariant relative to the partition $\{H_j\}_{j \in \mathbb{Z}^n}$, are also invariant relative to the partition $\{\widetilde{H}_\ell\}_{\ell \in \mathbb{N}_0}$ (or relative to the Laplacian, in terminology of Section 4).

If we have information on the eigenvalues of E , like we do on the torus, we may sometimes also recover invariant operators relative to the partition $\{\widetilde{H}_\ell\}_{\ell \in \mathbb{N}_0}$ as linear combinations of translation invariant operators composed with phase shifts and complex conjugation.

3. FOURIER ANALYSIS ASSOCIATED TO AN ELLIPTIC OPERATOR

One of the main applications of the described setting is to study operators on compact manifolds, so we start this section by describing the discrete Fourier analysis associated to an elliptic positive pseudo-differential operator as an adaptation of the construction in Section 2. In order to fix the further notation we give some explicit expressions for notions of Section 2 in this setting.

Let M be a compact smooth manifold of dimension n without boundary, endowed with a fixed volume dx . We denote by $\Psi^\nu(M)$ the Hörmander class of pseudo-differential operators of order $\nu \in \mathbb{R}$, i.e. operators which, in every coordinate chart, are operators in Hörmander classes on \mathbb{R}^n with symbols in $S_{1,0}^\nu$, see e.g. [Shu01] or [RT10]. For simplicity we may be using the class $\Psi_{cl}^\nu(M)$ of classical operators, i.e. operators with symbols having (in all local coordinates) an asymptotic expansion of the symbol in positively homogeneous components (see e.g. [Dui11]). Furthermore, we denote by $\Psi_+^\nu(M)$ the class of positive definite operators in $\Psi_{cl}^\nu(M)$, and by $\Psi_e^\nu(M)$ the class of elliptic operators in $\Psi_{cl}^\nu(M)$. Finally,

$$\Psi_{+e}^\nu(M) := \Psi_+^\nu(M) \cap \Psi_e^\nu(M)$$

will denote the class of classical positive elliptic pseudo-differential operators of order ν . We note that complex powers of such operators are well-defined, see e.g. Seeley [See67]. In fact, all pseudo-differential operators considered from now on will be classical, so we may omit explicitly mentioning it every time, but we note that we could equally work with general operators in $\Psi^\nu(M)$ since their powers have similar properties, see e.g. [Str72].

We now associate a discrete Fourier analysis to the operator $E \in \Psi_{+e}^\nu(M)$ inspired by those constructions considered by Seeley ([See65], [See69]), see also Greenfield and Wallach [GW73]. However, we adapt it to our purposes and in the sequel also indicate several auxiliary statements concerning the eigenvalues of E and their multiplicities, useful to us in the subsequent analysis. In general, the construction below is exactly the one appearing in Theorem 2.1 with a particular choice of a partition.

The eigenvalues of E (counted without multiplicities) form a sequence $\{\lambda_j\}$ which we order so that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots \quad (3.1)$$

For each eigenvalue λ_j , there is the corresponding finite dimensional eigenspace H_j of functions on M , which are smooth due to the ellipticity of E . We set

$$d_j := \dim H_j, \text{ and } H_0 := \ker E, \lambda_0 := 0.$$

We also set $d_0 := \dim H_0$. Since the operator E is elliptic, it is Fredholm, hence also $d_0 < \infty$ (we can refer to [Ati68], [Hör85] for various properties of H_0 and d_0).

We fix an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of E :

$$\{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j}, \quad (3.2)$$

where $\{e_j^k\}^{1 \leq k \leq d_j}$ is an orthonormal basis of H_j . Let $P_j : L^2(M) \rightarrow H_j$ be the corresponding projection. We shall denote by (\cdot, \cdot) the inner product of $L^2(M)$. We observe that we have

$$P_j f = \sum_{k=1}^{d_j} (f, e_j^k) e_j^k,$$

for $f \in L^2(M)$. The ‘Fourier’ series takes the form

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (f, e_j^k) e_j^k,$$

for each $f \in L^2(M)$. The Fourier coefficients of $f \in L^2(M)$ with respect to the orthonormal basis $\{e_j^k\}$ will be denoted by

$$(\mathcal{F}f)(j, k) := \widehat{f}(j, k) := (f, e_j^k). \quad (3.3)$$

We will call the collection of $\widehat{f}(j, k)$ the *Fourier coefficients of f relative to E* , or simply the *Fourier coefficients of f* .

Since $\{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j}$ forms a complete orthonormal system in $L^2(M)$, for all $f \in L^2(M)$ we have the Plancherel formula (2.4), namely,

$$\|f\|_{L^2(M)}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |(f, e_j^k)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j, k)|^2 = \|\widehat{f}\|_{\ell^2(\mathbb{N}_0, \Sigma)}^2, \quad (3.4)$$

where the space $\ell^2(\mathbb{N}_0, \Sigma)$ and its norm are as in (2.5) and (2.6).

We can think of $\mathcal{F} = \mathcal{F}_M$ as of the Fourier transform being an isometry from $L^2(M)$ into $\ell^2(\mathbb{N}_0, \Sigma)$. The inverse of this Fourier transform can be then expressed by

$$(\mathcal{F}^{-1}h)(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} h(j, k) e_j^k(x). \quad (3.5)$$

If $f \in L^2(M)$, we also write

$$\widehat{f}(j) = \begin{pmatrix} \widehat{f}(j, 1) \\ \vdots \\ \widehat{f}(j, d_j) \end{pmatrix} \in \mathbb{C}^{d_j},$$

thus thinking of the Fourier transform always as a column vector. In particular, we think of

$$\widehat{e}_j^k(\ell) = \left(\widehat{e}_j^k(\ell, m) \right)_{m=1}^{d_\ell}$$

as of a column, and we notice that

$$\widehat{e}_j^k(\ell, m) = \delta_{j\ell} \delta_{km}. \quad (3.6)$$

Smooth functions on M can be characterised by

$$\begin{aligned} f \in C^\infty(M) &\iff \forall N \exists C_N : |\widehat{f}(j, k)| \leq C_N(1 + \lambda_j)^{-N} \text{ for all } j, k \\ &\iff \forall N \exists C_N : |\widehat{f}(j)| \leq C_N(1 + \lambda_j)^{-N} \text{ for all } j, \end{aligned} \quad (3.7)$$

where $|\widehat{f}(j)|$ is the norm of the vector $\widehat{f}(j) \in \mathbb{C}^{d_j}$. The implication ‘ \iff ’ here is immediate, while ‘ \implies ’ follows from the Plancherel formula (2.4) and the fact that for $f \in C^\infty(M)$ we have $(I + E)^N f \in L^2(M)$ for any N .

For $u \in \mathcal{D}'(M)$, we denote its Fourier coefficient

$$\widehat{u}(j, k) := u(\overline{e_j^k}),$$

and by duality, the space of distributions can be characterised by

$$f \in \mathcal{D}'(M) \iff \exists M \exists C : |\widehat{u}(j, k)| \leq C(1 + \lambda_j)^M \text{ for all } j, k.$$

We will denote by $H^s(M)$ the usual Sobolev space over L^2 on M . This space can be defined in local coordinates or, by the fact that $E \in \Psi_{+e}^\nu(M)$ is positive and elliptic with $\nu > 0$, it can be characterised by

$$\begin{aligned} f \in H^s(M) &\iff (I + E)^{s/\nu} f \in L^2(M) \iff \{(1 + \lambda_j)^{s/\nu} \widehat{f}(j)\}_j \in \ell^2(\mathbb{N}_0, \Sigma) \\ &\iff \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (1 + \lambda_j)^{2s/\nu} |\widehat{f}(j, k)|^2 < \infty, \end{aligned} \quad (3.8)$$

the last equivalence following from the Plancherel formula (2.4). For the characterisation of analytic functions (on compact manifolds M) we refer to Seeley [See69] and [DR16a].

4. INVARIANT OPERATORS AND SYMBOLS ON COMPACT MANIFOLDS

We now discuss an application of a notion of an invariant operator and of its symbol from Theorem 2.1 in the case of $\mathcal{H} = L^2(M)$ and $\mathcal{H}^\infty = C^\infty(M)$ and describe its basic properties. We will consider operators T densely defined on $L^2(M)$, and we will be making a natural assumption that their domain contains $C^\infty(M)$. We also note that while in Theorem 2.2 it was assumed that the operator E_o is bounded on \mathcal{H} , this is no longer the case for the operator E here. Indeed, an elliptic pseudo-differential operator $E \in \Psi_{+e}^\nu(M)$ of order $\nu > 0$ is not bounded on $L^2(M)$.

Moreover, we do not want to assume that T extends to a bounded operator on $L^2(M)$ to obtain analogues of properties (D) and (F) in Section 2, because this is too restrictive from the point of view of differential operators. Instead, we show that in the present setting it is enough to assume that T extends to a continuous operator on $\mathcal{D}'(M)$ to reach the same conclusions.

So, we combine the statement of Theorem 2.1 and the necessary modification of Theorem 2.2 to the setting of Section 3 as follows.

We also remark that Part (iv) of the following theorem provides a correct formulation for a missing assumption in [DRv1, Theorem 3.1, (iv)].

Theorem 4.1. *Let M be a closed manifold and let $T : C^\infty(M) \rightarrow L^2(M)$ be a linear operator. Then the following conditions are equivalent:*

- (i) *For each $j \in \mathbb{N}_0$, we have $T(H_j) \subset H_j$.*

- (ii) For each $j \in \mathbb{N}_0$ and $1 \leq k \leq j$, we have $TEe_j^k = ETe_j^k$.
 (iii) For each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that for all e_j^k

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk} \delta_{j\ell}. \quad (4.1)$$

- (iv) If, in addition, the domain of T^* contains $C^\infty(M)$, then for each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that

$$\widehat{Tf}(\ell) = \sigma(\ell) \widehat{f}(\ell)$$

for all $f \in C^\infty(M)$.

The matrices $\sigma(\ell)$ in (iii) and (iv) coincide.

If T extends to a linear continuous operator $T : \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ then the above properties are also equivalent to the following ones:

- (v) For each $j \in \mathbb{N}_0$, we have $TP_j = P_jT$ on $C^\infty(M)$.
 (vi) $TE = ET$ on $L^2(M)$.

If any of the equivalent conditions (i)–(iv) of Theorem 4.1 are satisfied, we say that the operator $T : C^\infty(M) \rightarrow L^2(M)$ is *invariant* (or is a *Fourier multiplier*) *relative to E* . We can also say that T is *E -invariant* or is an *E -multiplier*. This recovers the notion of invariant operators given by Theorem 2.1, with respect to the partitions H_j 's in (2.1) which are fixed being the eigenspaces of E . When there is no risk of confusion we will just refer to such kind of operators as invariant operators or as Fourier multipliers. It is clear from (i) that the operator E itself or functions of E defined by the functional calculus are invariant relative to E .

We note that the boundedness of T on $L^2(M)$ needed for conditions (D) and (F) in Theorem 2.1 and in Theorem 2.2 is now replaced by the condition that T is continuous on $\mathcal{D}'(M)$ which explored the additional structure of $L^2(M)$ and allows application to differential operators.

We call σ in (iii) and (iv) the *matrix symbol of T* or simply the *symbol*. It is an element of the space $\Sigma = \Sigma_M$ defined by

$$\Sigma_M := \{\sigma : \mathbb{N}_0 \ni \ell \mapsto \sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}\}. \quad (4.2)$$

Since the expression for the symbol depends only on the basis e_j^k and not on the operator E itself, this notion coincides with the symbol defined in Theorem 2.1.

Let us comment on several conditions in Theorem 4.1 in this setting. Assumptions (v) and (vi) are stronger than those in (i)–(iv). On one hand, clearly (vi) contains (ii). On the other hand, it can be shown that assumption (v) implies (i) without the additional hypothesis that T is continuous on $\mathcal{D}'(M)$.

In analogy to the strong commutativity in (v), if T is continuous on $\mathcal{D}'(M)$, so that all the assumptions (i)–(vi) are equivalent, we may say that T is *strongly invariant relative to E* in this case.

The expressions in (vi) make sense as both sides are defined (and even continuous) on $\mathcal{D}'(M)$.

We also note that without additional assumptions, it is known from the general theory of densely defined operators on Hilbert spaces that conditions (v) and (vi) are generally not equivalent, see e.g. Reed and Simon [RS80, Section VIII.5]. If T is a differential operator, the additional assumption of continuity on $\mathcal{D}'(M)$ for parts

(v) and (vi) is satisfied. In [GW73, Section 1, Definition 1] Greenfield and Wallach called a differential operator D to be an E -invariant operator if $ED = DE$, which is our condition (vi). However, Theorem 4.1 describes more general operators as well as reformulates them in the form of Fourier multipliers that will be explored in the sequel.

There will be several useful classes of symbols, in particular the moderate growth class

$$\mathcal{S}'(\Sigma) := \{\sigma \in \Sigma : \exists N, C \text{ such that } \|\sigma(\ell)\|_{op} \leq C(1 + \lambda_\ell)^N \forall \ell \in \mathbb{N}_0\}, \quad (4.3)$$

where

$$\|\sigma(\ell)\|_{op} = \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}$$

denotes the matrix multiplication operator norm with respect to $\ell^2(\mathbb{C}^{d_\ell})$.

In the case when M is a compact Lie group and E is a Laplacian on G , left-invariant operators on G , i.e. operators commuting with the left action of G , are also invariant relative to E in the sense of Theorem 4.1. However, we need an adaptation of the above construction since the natural decomposition into H_j 's in (2.1) may in general violate the condition (3.1).

As in Section 2 since the notion of the symbol depends only on the basis, for the identity operator $T = I$ we have

$$\sigma_I(j) = I_{d_j},$$

where $I_{d_j} \in \mathbb{C}^{I_{d_j} \times I_{d_j}}$ is the identity matrix, and for an operator $T = F(E)$, when it is well-defined by the spectral calculus, we have

$$\sigma_{F(E)}(j) = F(\lambda_j)I_{d_j}. \quad (4.4)$$

We now discuss how invariant operators can be expressed in terms of their symbols.

Proposition 4.2. *An invariant operator T_σ associated to the symbol σ can be written in the following way:*

$$\begin{aligned} T_\sigma f(x) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (\sigma(\ell) \widehat{f}(\ell))_m e_\ell^m(x) \\ &= \sum_{\ell=0}^{\infty} [\sigma(\ell) \widehat{f}(\ell)]^\top e_\ell(x), \end{aligned} \quad (4.5)$$

where $[\sigma(\ell) \widehat{f}(\ell)]$ denotes the column-vector, and $[\sigma(\ell) \widehat{f}(\ell)]^\top e_\ell(x)$ denotes the multiplication (the scalar product) of the column-vector $[\sigma(\ell) \widehat{f}(\ell)]$ with the column-vector $e_\ell(x) = (e_\ell^1(x), \dots, e_\ell^{d_\ell}(x))^\top$. In particular, we also have

$$(T_\sigma e_j^k)(x) = \sum_{m=1}^{d_j} \sigma(j)_{mk} e_j^m(x). \quad (4.6)$$

If $\sigma \in \mathcal{S}'(\Sigma)$ and $f \in C^\infty(M)$, the convergence in (4.5) is uniform.

Proof. Formula (4.5) follows from Part (iv) of Theorem 4.1, with uniform convergence for $f \in C^\infty(M)$ in view of (4.3). Then, using (4.5) and (3.6) we can calculate

$$\begin{aligned}
(T_\sigma e_j^k)(x) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (\sigma(\ell) \widehat{e}_j^k(\ell))_m e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \left(\sum_{i=1}^{d_\ell} (\sigma(\ell))_{mi} \widehat{e}_j^k(\ell, i) \right) e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{i=1}^{d_\ell} (\sigma(\ell))_{mi} \delta_{j\ell} \delta_{ki} e_\ell^m(x) \\
&= \sum_{m=1}^{d_j} (\sigma(j))_{mk} e_j^m(x),
\end{aligned}$$

yielding (4.6). \square

Theorem 2.3 characterising invariant operators bounded on $L^2(M)$ now becomes

Theorem 4.3. *An invariant linear operator $T : C^\infty(M) \rightarrow L^2(M)$ extends to a bounded operator from $L^2(M)$ to $L^2(M)$ if and only if its symbol σ satisfies*

$$\sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{op} < \infty,$$

where $\|\sigma(\ell)\|_{op} = \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}$ is the matrix multiplication operator norm with respect to $H_\ell \simeq \ell^2(\mathbb{C}^{d_\ell})$. Moreover, we have

$$\|T\|_{\mathcal{L}(L^2(M))} = \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{op}.$$

This can be extended to Sobolev spaces. We will use the multiplication property for Fourier multipliers which is a direct consequence of Proposition 2.4:

Proposition 4.4. *If $S, T : C^\infty(M) \rightarrow L^2(M)$ are invariant operators with respect to E such that the domain of $S \circ T$ contains $C^\infty(M)$, then $S \circ T : C^\infty(M) \rightarrow L^2(M)$ is also invariant with respect to E . Moreover, if σ_S denotes the symbol of S and σ_T denotes the symbols of T with respect to the same orthonormal basis then*

$$\sigma_{S \circ T} = \sigma_S \sigma_T,$$

i.e. $\sigma_{S \circ T}(j) = \sigma_S(j) \sigma_T(j)$ for all $j \in \mathbb{N}_0$.

Recalling Sobolev spaces $H^s(M)$ in (3.8) we have:

Corollary 4.5. *Let an invariant linear operator $T : C^\infty(M) \rightarrow C^\infty(M)$ have symbol σ_T for which there exists $C > 0$ and $m \in \mathbb{R}$ such that*

$$\|\sigma_T(\ell)\|_{op} \leq C(1 + \lambda_\ell)^{\frac{m}{\nu}}$$

holds for all $\ell \in \mathbb{N}_0$. Then T extends to a bounded operator from $H^s(M)$ to $H^{s-m}(M)$ for every $s \in \mathbb{R}$.

Proof. We note that by (3.8) the condition that $T : H^s(M) \rightarrow H^{s-m}(M)$ is bounded is equivalent to the condition that the operator

$$S := (I + E)^{\frac{s-m}{\nu}} \circ T \circ (I + E)^{-\frac{s}{\nu}}$$

is bounded on $L^2(M)$. By Proposition 4.4 and the fact that the powers of E are pseudo-differential operators with diagonal symbols, see (4.4), we have

$$\sigma_S(\ell) = (1 + \lambda_\ell)^{-\frac{m}{\nu}} \sigma_T(\ell).$$

But then $\|\sigma_S(\ell)\|_{op} \leq C$ for all ℓ in view of the assumption on σ_T , so that the statement follows from Theorem 4.3. \square

5. SCHATTEN CLASSES OF OPERATORS ON COMPACT MANIFOLDS

In this section we give an application of the constructions in the previous section to determine the membership of operators in Schatten classes and then apply it to a particular family of operators on $L^2(M)$.

As a consequence of Theorem 2.5, we can now characterise invariant operators in Schatten classes on compact manifolds. We note that this characterisation does not assume any regularity of the kernel nor of the symbol. Once we observe that the conditions for the membership in the Schatten classes depend only on the basis e_j^k and not on the operator E , we immediately obtain:

Theorem 5.1. *Let $0 < r < \infty$. An invariant operator $T : L^2(M) \rightarrow L^2(M)$ is in $S_r(L^2(M))$ if and only if $\sum_{\ell=0}^{\infty} \|\sigma_T(\ell)\|_{S_r}^r < \infty$. Moreover*

$$\|T\|_{S_r(L^2(M))}^r = \sum_{\ell=0}^{\infty} \|\sigma_T(\ell)\|_{S_r}^r.$$

If an invariant operator $T : L^2(M) \rightarrow L^2(M)$ is in the trace class $S_1(L^2(M))$, then

$$\mathrm{Tr}(T) = \sum_{\ell=0}^{\infty} \mathrm{Tr}(\sigma_T(\ell)).$$

An interested reader can find in [DRT17] further applications to Schatten classes and the so-called r -nuclearity of operators.

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