Towards Impossibility and Possibility Results for String Stability of Platoon of Vehicles

Over de mogelijkheid en onmogelijkheid om een voertuigpeloton naar ketenstabiliteit te regelen

Arash Farnam

Promotor: prof. dr. ir. A. Sarlette
Proefschrift ingediend tot het behalen van de graad van
Doctor in de ingenieurswetenschappen

Vakgroep Elektronica en Informatiesystemen
Voorzitter: prof. dr. ir. K. De Bosschere
Faculteit Ingenieurswetenschappen en Architectuur
Academiejaar 2018 - 2019

UNIVERSITEIT
GENT

ISBN 978-94-6355-164-9
NUR 950, 959
Wettelijk depot: D/2018/10.500/82

Members of the examination committee

Chair

Prof. dr. ir. Luc Taerwe, Department of Structural Engineering, Gent University, Gent, Belgium

Reading Committee

Prof. dr. ir. Guillaume Crevecoeur, EELAB, Gent University, Gent, Belgium

Prof. dr. ir. Mia Loccufier, EELAB, Gent University, Gent, Belgium

Prof. dr. ir. Oliver Mason, Maynooth University, Maynooth, Ireland

Prof. dr. ir. Jeroen Ploeg, Technische Universiteit Eindhoven, Eindhoven, Netherlands

Other members

Prof. dr. ir. Alain Sarlette, ELIS, Gent University, Gent, Belgium & INRIA, Paris, France (promotor)

Dankwoord

In science, one should use all available resources to solve difficult problems. One of our most powerful resources is the insight of our colleagues.

Peter Agre

I would like to take this opportunity to thank all those who have been involved with this thesis and my publications directly or indirectly and who helped me to successfully complete this work. First and foremost, I would like to really thank my supervisor Prof. Alain Sarlette for his strong support, patience, and encouragement during my research period at Ghent University. My sincere thanks and appreciation to Professors. Gert de Cooman, Dirk Aeyels and Mia Loccufier in Gent University, for the opportunity to work in their research group and for their outstanding help, support, guidance, and valuable discussions during the last four years.

Special thanks are reserved for all examination committee members, particularly for Prof. Luc Taerwe Prof. Guillaume Crevecoeur, Prof. Mia Loccufier, Prof. Jeroen Ploeg and Prof. Oliver Masonn for their useful comments and considerations.

I would like to express my gratitude several colleagues in Ghent University who in one way or another contributed in my research. I would like to express my sincere appreciation to:

Kevin Dekemele, Jasper Juchem, Simon Apers, Arthur Van Camp, and Michiel Burgelman from Gent University.

I would also thank to my friends Ashkan Hodaee, Ali shamseddinloo, Sharyar Abed for all the enjoyed moments, trips and weekends.

Most of all, I would like to give my deepest gratitude to my family, my dear parents Roghayeh and Majid, my sister Aysan.

Arash Farnam
Gent, 11 oktober 2018

Contents

@

List of figures and tables

Notations

k and N indicate the index of each vehicle and number of the vehicles in the vehicle chain, respectively.

$\dot{x}(t)$ denotes the time-derivative of the corresponding signal $x(t)$. $\dot{x}(t)$ is defined as follows

$$\dot{x}(t) = \frac{dx(t)}{dt}.$$

The imaginary unit is denoted by j, in which $j^2 = -1$. s and z are frequency parameters in continuous-time and discrete-time settings, respectively. Where $s = j\omega$. And ω indicates frequency.

$\mid x(j\omega_0) \mid$ indicates the magnitude of signal x at specific frequency ω_0.

The H_∞ norm of transfer function $C(s)$ is given by $\|C\|_\infty = sup_{\omega \geq 0}|C(j\omega)|$. *Re* and *Im* respectively denote the real and imaginary parts.

The L_2 norm of a time-dependent scalar signal in continuous-time setting is denoted by $\|x_k(\,\cdot\,)\|$. In which $\|x_k(\,\cdot\,)\|$ is defined as follows

$$\|x_k(\,\cdot\,)\| = \sqrt{\int_{-\infty}^{+\infty} |x_k(t)|^2 dt}.$$

For a time-dependent *vector* e.g. $x(t)$, a lower index will indicate the discrete norm used on the *vehicle index* dimension: the (L_2, l_2) norm in continuous-time setting is

$$\|x(\,\cdot\,)\|_2 = \sqrt{\sum_{k=0}^{N} \int_{-\infty}^{+\infty} |x_k(t)|^2 dt}$$

and the (L_2, l_∞) norm is $\|x(\,\cdot\,)\|_\infty = \max_k (\|x_k(\,\cdot\,)\|)$.

The l_2 norm of a scalar signal in discrete-time setting is denoted by $\mid\mid x_k(.) \mid\mid$ defined as follows

$$\| x_k(\cdot) \| = \sqrt{\sum_{t=0}^{n} | x_k(t) |^2}.$$

Where n is the number of time-steps.

The (L_2, l_2) norm of a vector of scalar signals $x_k(t)$ is denoted by $\| x(\cdot) \|_2$ defined as follows

$$\| x(\cdot) \|_2 = \sqrt{\sum_{k=0}^{N} \sum_{t=0}^{n} | x_k(t) |^2}.$$

Finally, the symbol □ is used to show the end of proofs.

Furthermore, some abbreviations used in this thesis are denoted as follows

LTI	:	linear time-invariant
CACC	:	cooperative adaptive cruise control
ITS	:	intelligent transportation systems
PD	:	proportional-derivative
PID	:	proportional-integral-derivative
AV	:	all vehicles
LV	:	leading vehicle

Samenvatting

Autonome voertuigen zijn vandaag de dag een beloftevolle technologie geworden, en zouden op korte termijn toelaten om intelligente netwerken op te bouwen die tot vloeiender verkeer leiden. Eerst is er echter nood aan algoritmes die een dergelijk grootschalig, gedistribueerd systeem efficiënt en veilig zouden regelen.

Eén fundamenteel doel zou zijn om autonome voertuigen op eenvoudige wegen (bv. snelwegen) dicht opeen te kunnen laten rijden, dankzij een cooperatieve regeling van hun besturingsalgoritmes. Bij een dergelijke coöperatieve sturing meten wagens hun onderlinge afstanden, en wisselen ze mogelijks onderling informatie uit. Op deze manier kunnen ze op gezamenlijke, gedistribueerde wijze een lager brandstofverbruik, dichter gepakt verkeer bij hoge snelheden, of verhoogd comfort en veiligheid voor de passagiers bereiken. In essentie komt het erop neer om een groot peloton, of een lange keten voertuigen zich te laten voortbewegen als één enkel lichaam, waarbij ieder deeltje (voertuig) de bewegingen van het leidend voertuig trouw volgt, en tegelijkertijd de onderlinge afstand tussen buren heel dicht bij een doelwaarde gestabiliseerd blijft. Een regelsysteem ontwerpen dat tegelijkertijd aan specificaties van zowel prestatie, robuustheid als stabiliteit voldoet is duidelijk een hele evenwichtsoefening; zeker voor grootschalige systemen zoals een heel peloton voertuigen. Wellicht een grotere verrassing is het feit dat enkel en alleen het garanderen van de begrensdheid van de afwijkingen op de gewenste afstand tussen voertuigen, onafhankelijk van de lengte van de keten, op zich al een heus probleem vormt. Deze centrale observatie werd in de literatuur geformaliseerd als “string (in)stability”, door ons vrij vertaald als “keten-(in)stabiliteit”, en heeft sindsdien de aandacht genoten van theoretische onderzoekers in de regeltechniek voor gedistribueerde systemen

Het hoofddoel van dit proefschrift is om onze kennis van het fenomeen keten-(in)stabiliteit te vervolledigen: welke aspecten van de probleemstelling beïnvloeden precies de mogelijkheid om keten-stabiliteit te garanderen met een degelijke regelaar, en welke variaties kunnen optreden in de academische

probleemstelling, die eigenlijk als abstractie wordt gebruikt van praktische situaties? De bijdragen van deze thesis kunnen worden opgesomd in drie delen.

In deel één verduidelijken we een aantal essentiële elementen gerelateerd tot de onmogelijkheid tot het ontwerpen van een regelaar die keten-stabiliteit garandeert. De regelaar moet werken louter op basis van relatieve metingen. Zo laten we het bv. niet toe om een meting van de absolute snelheid expliciet te gebruiken in de gedistribueerde regelaar. We beschouwen het systeem zowel in continue tijd als in discrete tijd. Een continue-tijd benadering is de meest gangbare bij het theoretisch ontwerpen van regelaars, en keten-stabiliteit vormt hierop geen uitzondering; deze resultaten knopen dus het dichtst aan bij de bestaande literatuur. Een discrete-tijd model ligt echter dichter bij een praktische implementatie via digitale regelaars, en bevat op een natuurlijke manier de geassocieerde beperkingen. Voor een veilig ontwerp zouden de twee benaderingen overeenstemmende resultaten moeten geven.

Voor het model in continue tijd geven we een aantal verduidelijkingen en uitbreidingen van bestaande resultaten. We beschouwen een algemener model van communicatie tussen voertuigen, waaronder ook de meest bestudeerde "Cooperative Adaptive Cruise Control (CACC)" valt, en verduidelijken dat communicatie op zich ook in de algemenere situatie niet volstaat om keten-stabiliteit te bekomen op basis van enkel relatieve metingen. Hierna onderzoeken we of een gewijzigde dynamica van de sensoren soelaas zou kunnen brengen. Deze laatste zou immers de symmetrie van relatieve metingen kunnen doorbreken, en dus eventueel met uitgebreidere informatie een oplossing kunnen bieden. De conclusie is echter terug negatief. In al deze gevallen beschouwen we verschillende definities die werden voorgesteld voor keten-stabiliteit, en vullen we zo de gaten op wanneer de bestaande literatuur zich beperkte tot éé of andere bepaalde definitie. Onze resultaten beperken zich tot een koppeling tussen directe buren, net zoals in het overgrote deel van de bestaande literatuur.

Voor het model in discrete tijd vestigen we veruit het meest algemene onmogelijkheidsresultaat tot nog toe voor keten-stabiliteit, met een oogwenk naar het feit dat dit verschijnsel veel fundamenteler zou kunnen zijn dan gesuggereerd werd door zijn studie in de lineaire systeemtheorie. We tonen namelijk aan dat, bij het toelaten van niet-lineaire regelaars, arbitraire koppelingen met een constant aantal voorliggers en achterliggers in de keten, én arbitraire (niet-lineaire, digitale, zeg maar) lokale communicatie, het nog steeds onmogelijk blijft om een regelaar te ontwerpen die keten-stabiliteit zou garanderen. We bewijzen dit voor de verschillende definities van keten-stabiliteit, met enkel de volgende beperkingen: (i) de regelaar is homogeen, i.e. elk voertuig reageert op dezelfde manier op zijn buren; (ii) de tijdstap dt van de digitale regelaar blijft eindig, ondanks een groeiende ketenlengte; en (iii) over een tijdstap dt kan elk voertuig enkel een beperkte aantal buren bereiken, onafhankelijk van de ketenlengte. Dit lijkt het eerste resultaat te zijn dat keten-instabiliteit vaststelt in een niet-lineaire context, en het zou dus volledig nieuwe perspectieven kunnen bieden gerelateerd met dit verschijnsel.

In deel twee beschouwen we keten-stabiliteit onder de voorwaarde dat storingen enkel op het leidend voertuig voorkomen. Dit sluit een groot aantal "gevaarlijke" storingen vanuit het eerste deel uit, met name de storing die tot het algemene onmogelijkheidsresultaat leidt in discrete tijd. Het beperken van storingen tot een paar leidende voertuigen wordt om verschillende redenen als relevant beschouwd in de literatuur: dit modelleert bijvoorbeeld de reactie van het leidend voertuig op obstakels, terwijl de andere voertuigen gewoon zouden volgen; of, wanneer subsystemen verdiepingen van een gebouw voorstellen in plaats van voertuigen, kan men hiermee de respons van een groot gebouw op aardbevingen bestuderen. De literatuur heeft inderdaad de twee gevallen beschouwd – met storingen enkel op het leidend voertuig én met storingen mogelijks op elk voertuig – maar de rol van dit verschil werd niet altijd verduidelijkt. In dit opzicht verduidelijken wij hoe sommige verschillen tussen de respectievelijke conclusies direct kunnen worden toegewijd aan het feit dat storingen beperkt zouden zijn tot het leidend voertuig. We bewijzen inderdaad dat, met storingen enkel op het leidend voertuig, een eenvoudige PD-regelaar die asymmetrisch reageert op zijn voorganger en zijn opvolger voldoet om keten-stabiliteit te bereiken. Dit wordt bewezen zowel in continue tijd alsook in discrete tijd.

In deel drie beschouwen we een aanpak met "time-headway", waar niet langer uitsluitend relatieve waarden worden gebruikt in de regelaar, maar ook de absolute snelheid, die namelijk de gewenste afstand tussen voertuigen bepaalt. Het was al geweten dat deze wijziging toelaat om te voorkomen dat een enkele storing langs de voertuigketen versterkt wordt, wanneer voertuigen enkel naar hun voorgangers kijken. Dit alles gebeurt ten (lichte) koste van het overlaten van de uiteindelijke afstand tussen voertuigen aan de individuele snelheidsregelaars. Het gaat ook uit van een bijkomende sensor, die betrouwbaar de snelheid zou opmeten ten opzichte van een gezamenlijke absolute referentie voor alle voertuigen.

Onze bijdrage in dit derde deel bestaat erin de eigenschappen van "time headway" uit te breiden naar de sterkere varianten van keten-stabiliteit. We geven eerst een negatief resultaat, waarbij een regelaar met eindige DC respons - dus bijvoorbeeld zonder integrerende actie - ondanks time-headway niet kan voldoen aan een sterkere definitie van keten-stabiliteit. Daarna geven we echter een reeks positieve resultaten, waarbij onder meer met een PID-regelaar aan alle definities wordt voldaan. We passen ook een expliciete formule aan voor de minimaal nodige time headway, voor het geval waarbij er ook communicatie is tussen de voertuigen (typisch CACC). Daarmee kunnen we meteen aantonen dat, zoals eerder al in simulaties gezien, het toevoegen van communicatie toelaat om de afstand tussen voertuigen minder afhankelijk te maken van hun snelheid (zonder deze afhankelijkheid volledig te kunnen verwijderen).

Abstract

Nothing is as simple as it seems at first.
Or as hopeless as it seems in the middle.
Or as finished as it seems in the end.

Nowadays, autonomous cars are an important and promising technology for intelligent transportation systems towards improving the traffic flow and decreasing network congestions. However, different advanced control systems methods must still be investigated to take full advantage of these possibilities and ensure safe operation of this inherently distributed system.

One fundamental objective is to allow autonomous vehicles to drive on simple road segments (e.g. highways) with tight inter-vehicle distances, thanks to cooperative driving systems. In such cooperative driving, vehicles measure their relative distances and possibly share information with neighboring vehicles; a proper distributed control system should then enable to achieve some important practical targets, such as optimizing fuel consumption, reducing air pollution, allowing more dense traffic at high speeds, and increasing comfort and safety for the passengers. In essence, the main objective of such platooning or chaining of vehicles is to allow a large set of vehicles to travel as a single rigid body, faithfully following the movements of the leading vehicle while precisely maintaining desired inter-vehicle distances.

It is clear that simultaneously satisfying different performance, robustness and stability objectives on such large scale system would not be a trivial task for control design. What might be more surprising is: just guaranteeing that deviations of inter-vehicle distances from their target value remain bounded, independently of the length of the vehicle chain, appears to be a very challenging task. This observation has been formalized in the literature as "string (in)stability", and it has since attracted specific attention from researchers in control theory for distributed systems.

The main objective of this thesis is to complete the state of knowledge about satisfying string stability: which ingredients of the setting are truly essential to make this possible or impossible, which variations can appear in the academic setting used as a proxy for practical applications. The thesis contributions can be subdivided into three parts.

In a first part, we clarify the essential elements that make it impossible to achieve string stability. We work only with relative state measurements between neighboring subsystems, e.g. we do not allow additional absolute velocity sensors. We consider discrete-time and continuous time settings. Continuous time is the most common theoretical framework for controller design, in the string stability literature as well, and thus relates most directly to existing results. The discrete-time setting is closer to digital controller implementations and incorporating related constraints in a direct way. For a robust practical model, the conclusions should be similar in the continuous-time and discrete-time settings.

For the continuous-time setting, we complete the existing results in several ways. We generalize the model of communication between vehicles, while previous papers have focused on a particular model of Cooperative Adaptive Cruise control (CACC) and mostly in conjunction with time-headway spacing instead of constant spacing policy. We investigate the impact of sensor dynamics which might break the symmetry of only relative-position information, and conclude that amplifying such effects to possibly build controllers on the basis of richer information would still not help. In all these cases, we explicitly address all the definitions of string stability, filling the occasional holes in the existing literature. Our study is restricted to nearest-neighbor coupling, as in most of the literature.

For the discrete-time setting, we establish the most general impossibility result so far, pointing to the fact that string instability is a much more fundamental issue than suggested so far by linear systems studies. Namely, we show that enabling nonlinear controllers, any couplings to a few vehicles in front and behind, any (nonlinear, quantized,...) local communication, and controller dependence on the chain length, all together do not allow to design a controller which would achieve string stability. We prove this for the different versions of the string stability definition, and with as only main constraints: (i) the controller is homogeneous and discrete-time, i.e. each vehicle in the chain reacts in the same way to its neighbors; (ii) the controller discretization step dt remains bounded away from zero despite increasing chain length; and (iii) over a time step dt, each vehicle can only reach a finite number of neighbors that is independent of the chain length. To our knowledge, this is the first result about string instability in a nonlinear

context, thus allowing more general controllers or e.g. quantized or event-based communication. This could open completely new perspectives related to this problem.

In a second part, we consider string stability with respect to a disturbance acting on the leading vehicle only. This excludes some of the constructions of "bad disturbances" in the first part, in particular the ones used for the very general impossibility result in discrete-time. Restricting disturbances to the leader has been considered relevant for various reasons in the literature: modeling the reaction of a leading vehicle to obstacles, while the others just follow; or modeling the behavior of buildings to earthquakes, with subsystems denoting the levels of the building. Anyways, both the case with disturbance on all vehicles, and with disturbance on the leading vehicle only, have been considered in the string stability literature. In this respect, we clarify that some differences in their conclusions can be traced back just to this difference in disturbance location, which may have been somewhat overlooked. We show indeed that with disturbance acting only on the leader, it is possible to guarantee the different definitions of string stability using just a PD controller with properly tuned gains, reacting in an asymmetric way to the directly preceding vehicle and to the directly following vehicle. We prove these results in both the discrete-time and the continuous-time settings.

In a third and last part, we consider the time-headway spacing policy, where the target value for the relative positions depends on the vehicles' absolute speed. This modification in the setting has been known to enable recovering the most basic version of string stability, i.e. avoiding to unboundedly amplify a disturbance that would act on the leading vehicle of a unidirectionally coupled chain. This happens at the (moderate) expense of delegating the true control on inter-vehicular distances to some exogenous system which would fix the absolute speed. It also introduces in the setting a new sensor element, i.e. sensing the absolute speed with respect to some absolute and thus global reference, in addition to relative values between vehicles.

The main aim of this last part is to extend this capability of the time-headway setting to stronger versions of string stability. We first obtain a negative result, showing that with controllers having bounded DC gain – thus excluding for instance integral action – the time-headway setting does not allow to satisfy a stronger version of string stability. We then obtain several positive results, showing that with a PID controller and time-headway setting, all the definitions of string stability can be satisfied. We also extend an explicit tuning rule for the value of the time-headway to the case when adding communication capabilities (typically CACC). This allows us to explicitly quantify a

point which was previously observed in simulations only, namely that adding communication allows to decrease the minimal value of the time-headway constant, i.e. the dependence of inter-vehicular distances on velocity.

Chapter 1

Introduction

Nowadays, autonomous cars are an important and promising technology for intelligent transportation systems towards improving the traffic flow and decreasing network congestions. From a longstanding dream, such intelligent transportation has started to move truly closer to reality since the 1990's, [1].

Improved transportation systems have also become an urgent need during the last decades. As reported in [2], according to the European Union Road Federation 2010, the tonne-kilometer on the EU-27 road network has grown by 45.5% over the period 1995-2008, at a rate of 2 percent per year. It is clear that such increase leads to traffic flow problems with all the related problems of wasted time, increased fuel consumption and pollution, and safety issues.

The development of intelligent transportation systems (ITS) relies on three technological elements. More efficient sensors or whole sensing infrastructures (like GPS and smartphone networks) provide a more complete and more reliable picture of the situation to automated systems. Vehicles can communicate information with each other and with the infrastructure in ever cheaper and more versatile ways. And the decision-taking computers are becoming more and more powerful, both at hardware and at software levels. This allows to address a whole range of objectives, from the most local concerns of passenger safety and comfort in individual vehicles, to network-wide optimization of real-time traffic routes.

1.1 Platooning of Vehicles

Cooperative driving systems are one of the important topics in Intelligent Transportation Systems. In cooperative driving systems vehicles cooperate, organize, and adapt their behavior to neighboring vehicles. This can be on the basis of sensor measurements, and of information communicated between the vehicles via local networks. A study made by TNO [4] estimates that within the next 10 to 15 years, cooperative driving systems can reduce vehicle hour loss by 50%, traffic death rate by 25%, CO2 emission by 10% and air pollution by 20%.

Especially for traffic congestion, platoons of cooperatively driving automated vehicles are a promising solution. The vision is indeed that efficient controllers should allow them to travel with minimum inter-vehicular distances at high speeds, at least on simple roads like highways. The resulting benefits would include not only increased traffic capacity, but also reduced fuel consumption, air pollution, comfort, and to some extent safety as everything runs more smoothly. See [2] for more details and numbers about these points. The idea of platooning of vehicles has thus been developed concretely by different researchers during the recent years, [17, 18, 21–25]. A particularly interesting case is the vehicle chain, in which vehicles are aligned one after the other on a path. A scheme of platooning of vehicles is shown in Figure 1.2, [3].

Figure 1.1: Platooning of vehicles.

1.2 The challenge of controlling platoons of vehicles

Successful platoons of vehicles will have to rely on accurate and powerful distributed control strategies to stabilize inter-vehicle distances close to their target value. Various design options can be followed at this stage. The control system may be homogeneous or heterogeneous (i.e. each vehicle reacting in the same way to its neighbors or not). It may be essentially linear or nonlinear. Interaction between the vehicles may involve communication, or measurements only. Furthermore, the interconnection between the vehicles may follow a unidirectional chain (only reacting to predecessors) or it can be bidirectional (i.e. each vehicle also reacting to the behavior of its followers).

The controller should be able to guarantee safety and comfort of the passengers, and other important objectives while complying to the actual system limitations like actuator limits, unavoidable noises, uncertainties and disturbances. When controlling a large interconnected distributed system, like the vehicle chain, a major issue is that subsystems influence each other throughout the chain, and approaches that guarantee stability for each individual vehicle are not applicable to design controllers achieving satisfactory performance of the interconnected system. For instance, while sudden acceleration or braking might be well under control at the single vehicle level, the effect of this sudden event on the other vehicles throughout the chain may cause serious problems. In fact, it has been observed that under certain general circumstances, such disturbances unavoidably get *amplified* throughout the chain such that inter-vehicle distances deviate more and more from their nominal values in longer chains, eventually leading to unavoidable collisions between vehicles, [17, 18]. This undesired issue in platoon of vehicles is called *string instability* in literature, [17, 18].

Improving the understanding of this issue of *string stability/instability*, from a control systems theory point of view, is the main focus of the present thesis. In this sense, in agreement with other authors working on this subject, we will consider a simplified abstract model of vehicles as double-integrators of the acceleration input, and we will neglect issues like measurement noise, or uncertainties in dynamical (actuation) models. We then just focus on the effect of disturbance inputs on the distributed system as its size keeps growing, with the aim to keep inter-vehicle distance within bounded deviations from their nominal values. From a practical viewpoint, our motivation for this is that the challenges appearing in this setting must of course be solved convincingly, before more accurate models can be claimed to be robustly and confidently under control. From an academic viewpoint, this situation appears like a particularly clean setting to investigate fundamental limitations on the capabilities of distributed control systems.

1.3 Thesis contributions

The main objective of this thesis is to provide new, definite answers about the question of satisfying string stability. The starting point is the early observation [17, 18] that under certain settings, there exists no linear controller which would allow to satisfy string stability. Follow-up work, as detailed later, has considered specific variants of this problem mostly on a case-by-case basis.

A first general objective of this thesis is to give a more comprehensive picture about the impossibility to satisfy string stability. *In particular, this is the first place identifying that linearity is not a main ingredient in such impossibility, as we provide an impossibility result when allowing nonlinear controllers.*

A second general objective is to provide stronger positive results, where string stability can be achieved. For these contributions, we give a proof by example, providing simple PD or PID controllers which do satisfy the academic objective.

The thesis is organized as follows.

- Chapter 2: Problem definition

  In this chapter, the interconnected vehicle chain model that will be used in the rest of the thesis is introduced. We explain the main restrictions on controller design and distinguish several important features in the problem setting. A major distinction is the use of constant inter-vehicle distance as a target reference, or of a time-headway space policy in which the target distance between the vehicles depends on their absolute velocity. In the latter case, the controller is thus allowed to depend on absolute velocity; in contrast, in the former more constraining settings, only the *relative* behavior of the vehicles is allowed to have an impact on their dynamics. Another distinction is unidirectional coupling, where vehicles only react to their predecessors, or bidirectional coupling in which each vehicle reacts to both following and preceding vehicles. A model for possible communication between the vehicles is given. We then introduce the different definitions of string stability which will be the focus of this work. Finally, we translate all these elements to the less standard discrete-time setting, for which we were able to obtain the most telling results.

- Chapter 3: Comprehensive impossibility results for string stability

  The aim of this chapter is to obtain a more precise understanding of string stability and clarifying the related controller design options, by narrowing down the essential elements that make it impossible to achieve string stability. We work only with *relative* state measurements between neighboring subsystems, e.g. we do not allow additional absolute velocity sensors. This also means, we suppose constant spacing policy in the controllers. We consider discrete-time and continuous-time settings. Continuous-time is the most common theoretical framework for controller design, in the string stability literature as well, and thus relates most directly to existing results. The discrete-time setting is closer to digital controller implementations and of incorporating related "natural" constraints in a direct way. Maybe surprisingly, the discrete-time framework is where we can get the most extensive impossibility results, pushing much further than the existing literature; the conclusions should be similar in practice to the continuous-time setting as long as numerical discretization schemes allow a faithful modeling of the system.

  For the continuous-time setting, we complete the existing results in several ways. We generalize the model of communication between vehicles, while previous papers have focused on a particular model of Cooperative Adaptive Cruise control (CACC) and mostly in conjunction with time-headway spacing instead of constant spacing policy. We investigate the impact of sensor dynamics which might break the symmetry of only relative-position information, and conclude that amplifying such effects to possibly build controllers on the basis of richer information would still not help. In all these cases, we explicitly address all the definitions of string stability, filling the occasional holes in the existing literature. Our study is restricted to nearest-neighbor coupling, as in most of the literature.

  For the discrete-time setting, we establish the most general impossibility result so far, pointing to the fact that string instability is a much more fundamental issue than suggested so far by linear systems studies. Namely, we show that enabling nonlinear controllers, any couplings to a few vehicles in front and behind, any (nonlinear, quantized,...) local communication, and controller dependence on the chain length, all together do not allow to design a controller which would achieve string stability. We prove this for the different versions of the string stability definition, and with as only main constraints: (i) the controller is homogeneous and discrete-time, i.e. each vehicle in the chain reacts in the same way to its neighbors; (ii) the controller discretization step dt remains bounded away from zero despite increasing chain length; and (iii) over a time step dt, each vehicle can only reach a finite number of neighbors that is independent of the chain length. To our knowledge, this is the first result about string instability in a nonlinear context, thus allowing more general controllers or e.g. quantized or event-based communication. This could open completely new perspectives related to this problem.

- Chapter 4: Possibility results for string stability, using constant space policy between the vehicles but with disturbance restricted to the leading vehicle

  In this chapter, we consider string stability with respect to a disturbance acting on the leading vehicle only. This excludes some of the constructions of "bad disturbances" in Chapter 3, in particular the ones used for the very general impossibility result in discrete-time. Restricting disturbances to the leader has been considered relevant for various reasons in the literature: modeling the reaction of a leading vehicle to obstacles, while the others just follow; or modeling the behavior of buildings to earthquakes, with subsystems denoting the levels of the building (see e.g. [19] and related papers). Anyways, both the case with disturbance on all vehicles (AV), and with disturbance on the leading vehicle only (LV), have been considered in the string stability literature, [19, 20]. In this respect, we clarify that some differences in their conclusions can be traced back just to this difference in disturbance location. For instance, in [19] an advanced linear controller with inerter has been proposed to satisfy some definition of string stability with symmetric bidirectional coupling. We highlight that compared to the results of Chapter 3, the main ingredient for this possibility are not some special controllers or variants on the string stability definition. Instead, a central but maybe somewhat overlooked point is just that disturbance in [19] is restricted to the leading subsystem. We show that in this case indeed, it is possible to guarantee the different definitions of string stability using just a PD controller with properly tuned gains, reacting in an asymmetric way to the directly preceding vehicle and to the directly following vehicle. We prove these results in both the discrete-time and the continuous-time settings.

- Chapter 5: About string stability with unidirectional controller using time-headway space policy

  In this chapter, we consider the time-headway spacing policy, where the target value for the relative positions depends on the vehicles' absolute speed. This modification in the setting has been known to enable recovering the most basic version of string stability [24], i.e. avoiding to unboundedly amplify a disturbance that would act on the leading vehicle of a unidirectionally coupled chain. This happens at the (moderate) expense of delegating the true control on inter-vehicular distances to some exogenous system which would fix the absolute speed. It also introduces in the setting a new sensor element, i.e. sensing the absolute speed *with respect to some absolute and thus global reference*, in addition to relative values between vehicles.

The main aim of this chapter is to extend this capability of the time-headway setting to stronger versions of string stability. We first obtain a negative result, showing that with controllers having bounded DC gain – thus excluding for instance integral action – the time-headway setting does not allow to satisfy a stronger version of string stability. We then obtain several positive results, showing that with a PID controller and time-headway setting, all the definitions of string stability given in Chapter 2 can be satisfied. We also extend an explicit tuning rule for the value of the time-headway [43] to the case when adding communication capabilities (typically CACC). This allows us to explicitly quantify a point which was previously observed in simulations only, namely that adding communication allows to decrease the minimal value of the time-headway constant, i.e. the dependence of inter-vehicular distances on velocity.

- Chapter 6: Conclusions

  In this chapter, we conclude the thesis with a recap of the results and a brief outlook on perspectives. Chapter 3 in particular raises several fundamental questions. A first point is how to address the control of long vehicle chains with other assumptions than the ones which allow to circumvent string instability in Chapters 4 and 5. A second point is about the possibly wider impact of the unprecedentedly general string instability result in discrete-time.

A more precise summary of the thesis contributions will be given after describing the problem setting in the following chapter. Tables summarizing our results are also provided in the Conclusion (Chapter 6).

Chapter 2

Problem definition

In this chapter we present the state of the art in the mathematical study of platooning of vehicles. We start with its origins and then we focus on some important aspects and problems that arise in this kind of interconnected vehicles. Specifically, we present the main problem definitions that will be investigated in detail in the other chapters of this thesis.

2.1 Origins and basic model for platooning of vehicles

As we mentioned in Chapter 1, there exist several works in the literature in theory and applications using different methods and techniques to optimize the usage of roads, towards intelligent transportation systems to improve more efficiently the capacity and safety of roads and the comfort of passengers.

The control of platoons of vehicles is not a new topic. In early works like [5] and [6], optimal controllers have been developed for this task which guarantee the satisfaction of constraints like following the leader, maintaining inter-vehicle distances within specified bounds, while optimizing for instance the fuel consumption. However, these first approaches were based on *centralized controllers*, where each agent/vehicle must have access to all the states of other vehicles. In particular, they must all have an accurate and direct measure of their behavior with respect to the leader, irrespective of how far they may be down the chain. Such approach appears to be poorly scalable.

Therefore, researchers have started looking at decentralized or distributed controllers, where each vehicle only has access to information from a few nearest neighbors. This turned out to lead to surprising challenges. Indeed, the authors in [17, 18] have observed that with distributed controllers, it already appears unavoidable that a disturbance acting on a given vehicle, gets *amplified* along the chain, such that deviations of inter-vehicle distances from their nominal value would grow unbounded mathematically – in practice, leading to unavoidable collision or breaking of the chain. More precisely, this observation involves a simplified model of vehicles, just keeping the dominating dynamics of a second-order integrator subjected to acceleration inputs, and a distributed system where each vehicle reacts in real-time to measurements of its distance to a few preceding vehicles. This observation has been formalized under the concept of "string stability". It has henceforth been the subject of intense research (see more references later in this thesis), and will be the main subject of the present thesis.

In agreement with the literature, we consider an idealized model of a vehicle as a second-order integrator, representing position as a state, controlled by acceleration (in fact force, with units such that the mass equals one) as inputs. An important comment in this respect is that if we would include a drag force, i.e. a damping term proportional to velocity, then this would solve the string stability problem. However, in a context where we search for more and more fuel-efficient means of transportation, it would be contradictory to rely solely on this dissipation in order to avoid collisions. Already with vacuum tube proposals, or chains of spaceship, the presence of drag force at all would have to be questioned. We would thus argue that string stability should be proved, even in the harder setting where drag force tends to zero. Moreover, the mathematical concept of string stability could be applied with mechanical subsystems being other than road vehicles, where again drag forces may be minimal. Thus, the absence of drag force is an important modeling assumption, meant to guarantee robustness of the "string stability" property. Other details of the dynamics, like actuation details, are not supposed to change the game, as long as they all rely on variables derived from *relative* position measurements between vehicles. Indeed, it really appears that the combination of double-pole at zero, and feedback relying on *relative* measurements only, are the key ingredients for string instability. Since we are allowing "any controller" for the rest, some of the actuation dynamics can in fact be delegated into the controller transfer function.

We thus consider $N+1$ acceleration-controlled vehicles, whose position along the road at time t we denote by $x(t) = (x_0(t), x_1(t), x_2(t), ..., x_N(t)) \in \mathbb{R}^{N+1}$. Their dynamics is described by

$$\ddot{x}_k(t) = u_k(t) + d_k(t)\ ,\ k = 0, 1, 2, ..., N\ , \tag{2.1}$$

or in Laplace domain

$$s^2 x_k = u_k + d_k \ , \ k = 0, 1, 2, ..., N \, . \tag{2.2}$$

Here x_k is the absolute position of vehicle k, while u_k and d_k are acceleration control input and disturbance input, respectively. These disturbances d_k can be used to model unavoidable external contingencies like road conditions, wind gusts, altitude changes, small inaccuracies in the motor/engine output; commands exogenous to the control system like human driver inputs, higher-level control loops, necessary driving commands to the leader $k = 0$, or safety braking in front of obstacles; and, by equivalent reformulation to some extent, deviations of the initial conditions from the target inter-vehicle distance.

In the following section, before giving more details about the objective of this thesis and in particular a mathematical definition of string stability, we distinguish different control methods existing in the related literature, and we clarify the positioning of the present thesis with respect to these aspects.

2.2 Different control systems for platoon of vehicles

In this section, we mention different control design methods and two different spacing methods between the vehicles in a vehicle chain. We follow the structure established in [43], where more discussion about these aspects can be found.

2.2.1 Homogenous and heterogenous control systems

It is a standard and rather valid abstraction to consider a chain in which different vehicles in the platoon have identical dynamics, as we do in the model (2.2). However, in addition, one might consider that the controllers of the different vehicles all follow the same rules with the same parameter values – this is called a *homogeneous* controller – or that they can be chosen differently – this is called a *heterogeneous* controller. The latter has been considered in [9] for instance, with additional weak coupling of each vehicle to the leader. Also optimal controllers (see below), for a finite chain, tend to lead to heterogeneous controllers (although usually weakly so, except at the boundaries, when the chain length is large). In [10], the authors have proved using heterogenous controllers between the vehicles where the control gains increase through the vehicle, it is possible to avoid string instability. This however implies that the control gain grows to infinity as the number of vehicles N increases, which is not an acceptable solution.

Besides these approaches, a large part of the literature has studied string stability for *homogeneous* controllers. These are arguably more scalable to design and deploy, since one just specifies a single controller that everybody uses. In line with this literature, the work presented in this thesis will use the assumption of a homogeneous platoon almost exclusively. Towards designs which do enable string stability, this is not a big restriction as soon as some minimal robustness can be guaranteed, which we believe is true. Towards impossibility results, i.e. showing that in certain situations no controller can guarantee string stability, it is true that homogeneous control is a simplifying assumption. Strictly speaking, in these settings, it may be that heterogeneous control can solve the issue. However, all indications so far rather point to the fact that this is not the case; for instance heterogeneity that would be periodic along the chain changes nothing to our conclusions, and the opposite approach with gains increasing monotonically along the chain is not acceptable. If very specific heterogeneous structures can help, it is certainly worth knowing for curiosity, although it may have debatable practical use.

2.2.2 Linear or nonlinear control systems

In the systems theory literature, the distinction between nonlinear and linear control is a major point regarding available tools for analysis and design.

According to [43], a few papers have considered vehicle chains with nonlinear dynamics or controllers. However, the vast majority of theoretical work has been done for linear time invariant systems (LTI systems), which are more straightforward to analyze with tools like the frequency response, using Bode diagrams, and the superposition principle for different disturbance inputs [8, 11, 17, 18, 22].

In a big part of the present thesis, we will propose several contributions to the literature about linear systems. As for the distinction heterogeneous/homogeneous, regarding positive results, this is a controller choice more than an actual restriction. For impossibility results, it does not exclude that nonlinear controllers would do better.

This last point has bothered us quite significantly. Indeed, nonlinear controllers are no curiosity anymore and routinely applied in some specific applications (e.g. nonlinear damping, mechanical elements with asymmetries like ropes). Moreover, nonlinear dynamics typically appear by themselves in systems, via actuator saturation and/or when disturbances are too large to justify a linear approximation; string instability is precisely about a situation where errors become unboundedly large. So, excluding nonlinear systems appears like a big limitation, and in some sense (see also below when we define it more comprehensively) string stability has from its origins been presented as an unavoidable shortcoming of *linear time-invariant* systems. Therefore, we really wanted to say something about nonlinear systems. In Section 3.3, we give a quite general impossibility result, in discrete-time though, showing that string instability is unavoidable in a quite large sense also with nonlinear homogeneous controllers.

2.2.3 Distributed parameter systems

Several authors have tried to gain insight on the behavior of large chains of vehicles by considering directly the infinite-length limit. A first approach [12] keeps the chain discrete but draws its conclusions explicitly from *spatially invariant systems*, which can does have no finite-length boundary. Other authors [30] derive insight from the continuous-chain limit, in the framework of partial differential equations (PDE). In particular, their insight that asymmetric bidirectional coupling can have advantages over symmetric bidirectional coupling has a direct interpretation in terms of dominating orders of spatial derivatives in the PDE context.

In the present thesis, the techniques used to prove string stability or string stability, always start from a finite chain length N whose limit is taken for increasing N. In this way, we are never confronted with the question of how much exactly inherently infinite-dimensions results are telling about finite chains; there are indeed some traps (or call it discontinuities) about this issue. An insight that we do take from the existing studies on infinitely long chains, is the possible advantage of *asymmetric* bidirectional controllers, specifically in Chapter 4.

2.2.4 Optimal control systems

We have already mentioned a few papers which have applied optimal control methods to the vehicle chain. More can be found in [43]. We are not pursuing this approach explicitly in the present thesis, in order to first focus on the issue of satisfying string stability, at all. One point that we can recall here is, in the context of time-headway spacing policy (see below) where it is known that some definition of string stability can be achieved, there have been studies including string stability in optimal control criteria, like [13] with receding horizon control. A minimum value for the time headway is found by iteration. We will present a criterion for the minimum time headway, which does not require any iteration as soon as some control transfer function is fixed.

2.2.5 Different spacing policies

In order to design control systems for platoon of vehicles, at first we should choose the spacing policy between the vehicles. We let

$$e_k(t) = x_{k-1}(t) - x_k(t) - c_k(t)$$

denote the error in vehicle spacing, where $c_k(t)$ is thus a desired inter-vehicle spacing between the vehicles with positions $x_k(t)$ and $x_{k-1}(t)$; we take the convention that the vehicle with the index k is the follower of the vehicle with the index $k-1$. We will consider two approaches that have been used in the literature about string stability.

In the first one, $c_k(t) = c$ for all k in the vehicle chain, with c a real positive constant. This spacing policy is called *constant spacing policy*. This is a very common selection and highly desirable policy since it implies that if the control architecture brings the errors $e_k(t)$ to zero, we know that the vehicles are all spaced by exactly c. Making c very small ensures a tight packing of the vehicles on the road — provided proper stabilization close to this value indeed allows to avoid collision. Furthermore, implementing this spacing policy a priori requires no extra measurements besides the relative distance between vehicles.

In [18], motivated in part by the difficulty to actually design controllers achieving the first goal, the authors have discussed another control method, called *time-headway spacing policy*. A headway time, measured in seconds, is the difference between the times at which two consecutive vehicles cross the same reference point in space. By factoring the headway time into the target distance between vehicles, the latter becomes velocity-dependent, in a way that reminds typical safety regulations. Explicitly, the spacing policy writes $c_k(t) = c + h\dot{x}_k(t)$, with $h > 0$ being the time headway parameter. A possible issue with the time-headway spacing policy is that we have to add a sensor, measuring accurately the absolute velocity of each vehicle with respect to a common global reference system, to implement the controller. For cars this might not be a big deal, but for ships, planes or space vehicles this might not always be innocent. Another issue is that this somewhat contradicts our goal of obtaining an advantageous packing of vehicles by making them move like a single rigid body, independently of their speed. Indeed, as shown in [18, 24], obtaining string stability in such context often involves a minimum value for the time headway, and this is a constraint on how much the spacing between vehicles will depend on their speed. Finally, from a more general viewpoint, in physically interconnected systems, Galilean invariance says that interaction forces would not depend on the global speed of the system. In any context where we would want to keep this property, time headway would not be a viable option.

For these reasons, we believe that it is worth studying string stability without velocity measurement, i.e. with the constant spacing policy, as we do in Chapters 3 and 4 of the present thesis. In Chapter 5, we give some results about systems with time-headway spacing policy.

2.2.6 Different interconnection methods

The interconnection specifies to which other vehicles each vehicle in the chain will react. This is an essential ingredient of the distributed control system.

Indeed, if the relative distance between each vehicle and the leading vehicle is known accurately, then it is sufficient that each vehicle *individually*, stabilizes its distance to the leader within accurate bounds, and the platoon would be safe. Such assumption is however not realistic in a scalable control system, where measurements will most likely be made locally, between neighboring vehicles. Even if the information about the relative distance from a vehicle k to the leader can be obtained indirectly from such measurements and assuming some communication, necessarily this "indirect knowledge" will be less and less accurate further down the chain; relying solely on such control with respect to the leader to stabilize vehicles k and $k+1$, would become less and less reliable to avoid collisions between them. Such setting is in fact better modeled as controllers which allow communication between vehicles, but introduce a robustness condition on the communication channel. We are thus back to local interconnections, but possibly with communication (see next subsection).

In this thesis, we will consider different settings for the local interconnection, as follows. The first three cases are the most common in the literature; the last one is also occasionally studied, and provides the most freedom, and thus the strongest impossibility result.

I. Unidirectional controller

In this case, each vehicle only reacts to preceding vehicles. Most often, researchers have considered the unidirectional control structure using the relative position of one predecessor, as this can be most directly measured. This is for instance the interconnection used in combination with the time-headway policy, typically in [18, 24] and also in our Chapter 5. The corresponding controller writes:

$$u_k = K(s)(x_{k-1} - x_k - c - hsx_k) \ , \ k = 1, 2, ..., N \ , \tag{2.3}$$

where $K(s)$ is the (here, homogeneous) control transfer function. The leading vehicle is supposed to be controlled by making it follow a virtual one, tracking the reference position x_0 and velocity v_0, on which we will here assume zero control input. In more practical settings, this could become a time-dependent signal but we should first be able to achieve string stability at least for the steady state situation.

II. Symmetric bidirectional controller

In this case, each vehicle reacts identically to its predecessors as to its followers. This can be viewed as a particular choice for cars on the road, but it becomes even more natural when the subsystems are connected through physical means, where a force applied between vehicles affects both of them symmetrically. Again the most common model is where each vehicle is connected symmetrically to one predecessor and one follower, e.g. as in [22, 23] using a constant spacing policy. The controller is then

$$\begin{aligned} u_k &= K(s)(x_{k-1} - x_k - c) + \\ &\quad K(s)(x_{k+1} - x_k + c), k = 1, 2, ..., N-1 \\ u_N &= K(s)(x_{N-1} - x_N - c) \ . \end{aligned} \tag{2.4}$$

For x_0 one can assume to have no control, like in the unidirectional case, or the symmetric situation of x_N.

III. Asymmetric bidirectional controller

The asymmetric bidirectional controller assumes reactions of each vehicle to both predecessors and followers, but possibly in a different way. Asymmetric bidirectional coupling to one predecessor and one follower has been proposed in [23] as a way to improve the scaling with N of the least stable closed loop eigenvalue of a chain of interconnected systems, in comparison with the symmetric coupling. The corresponding controller writes, with constant spacing policy:

$$\begin{aligned} u_0 &= K_2(s)(x_1 - x_0 + c) \\ u_k &= K_1(s)(x_{k-1} - x_k - c) + \\ &\quad K_2(s)(x_{k+1} - x_k + c), k = 1, 2, ..., N-1 \\ u_N &= K_1(s)(x_{N-1} - x_N - c) \ . \end{aligned} \tag{2.5}$$

In our model indeed, mostly considered in Chapter 4, the leading vehicle of the asymmetric bidirectional controller will be reacting to its follower.

The three schemes can in fact be viewed as particular variants of an asymmetric bidirectional controller. Schematically, this can be represented as on Figure 2.1.

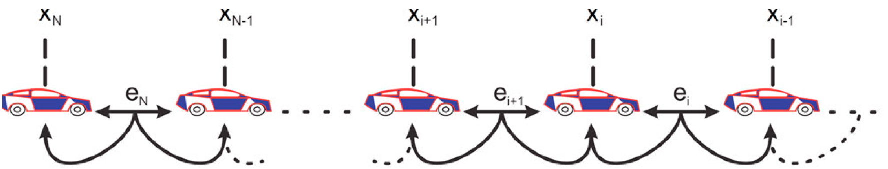

Figure 2.1: Scheme of asymmetric bidirectional controller of $N+1$ vehicles. The unidirectional controller would be the special case where the arrows from k to $k+1$ have zero gain. The symmetric bidirectional controller would impose the same transfer function on forward and backward arrows.

IV. Coupling to several forward and backward neighbors

A less stringent requirement that we can impose, while imposing some locality of feedback information, is that vehicle k can react to at most m_1 preceding vehicles and m_2 following vehicles, with m_1 and m_2 two constants fixed independently of the chain length N. In Chapter 3.3 we will consider this case with a very general controller, whose form we do not further specify. We would thus have:

$$u_k \quad = \quad f(e_k, e_{k-1}, ..., e_{k-m_1}, e_{k+1}, ..., e_{k+m_2}, \, [...]) \; . \qquad (2.6)$$

Here f is some arbitrary function and [...] denotes some other control variables; we will specify this model more concretely later in this Chapter, in the specific context of discrete-time control in which we will analyze it later.

In this context, we can briefly discuss that the present thesis is restricted to a single vehicle chain, with vehicles aligned after each other. This is indeed somewhat restrictive. A very similar behavior can however be expected if e.g. there is a junction somewhere with another vehicle chain; at the junction, the corresponding vehicle would just have two followers, and this does not seem to significantly change the game. Another structure that could be directly related to the present study – although probably not really with vehicles as applications – would be a lattice with N subsystems along several dimensions. Some of the techniques can be carried over to this case; e.g. when proving impossibility of string stability, if we assume the same disturbance is applied on each vehicle that has index k along one dimension, then the effective behavior of the lattice is equivalent to the case of a chain along the dimension of k.

2.2.7 Communication between the vehicles

In the above expressions, the controller at vehicle k only depends on local measurements of relative positions. Given the efficiency of modern communication infrastructure, it would seem natural to also consider possibly communication among neighboring cars in a vehicle chain. Controllers using communication between the vehicles have indeed been considered previously in the literature, see e.g. [13, 14, 22, 27, 28, 36–40]. In this model, some signals such as control signal or measurements of relative distance at each vehicle, are transmitted to neighboring vehicles; and possibly retransmitted further down the chain; along a communication channel with often a linear analog model (communication channel transfer function).

In this thesis, with the notable exception of Section 3.3, we will analyze a setting with this linear analog model. In Section 3.3 though, in discrete-time, we will allow a more general communication structure with enough freedom to include quantization, digital encoding and packing, and other elements closer to a true modern communication system. We detail the following communication settings.

I. Communication from leading vehicle

Assuming perfect communication between the vehicles allows to circumvent the issue of string stability [22]. Indeed, with perfect communication, vehicle k can get very fast knowledge of $e_1 + e_2 + ... + e_k = x_k - x_0 - kc$, and thus a controller could be designed on the basis of $x_k - x_0 - kc$ only, i.e. controlling each vehicle independently. This does guarantee a well-controlled chain, but it is not realistic. If we add a transfer function into the communication channel which is not identically one everywhere, e.g. expressing a bandwidth limitation, then this ideal picture does not hold anymore and the whole system has to be analyzed again. In [15], it has been proved if there exists time-delay in the communication channels it is not possible to guarantee string stability with the constant space policy and time-headway would be a requirement in the control structure proposed in [22].

In other words, the strategy proposed above would only work if we had communication "from leader to each vehicle" with error independent of chain length N; it is not clear how this would be done in practice. We will not consider this case in the present thesis.

II. Cooperative Adaptive Cruise Control (CACC)

This structure has been most popular in the literature, see e.g. variations on this theme in [27, 28, 36–40]. It assumes that the message sent by vehicle k to its follower $k+1$ is a filtered version of the input command u_k. Such communication has mostly been considered together with the time-headway policy – one goal of the present thesis was precisely to disentangle such things – and the corresponding model writes:

$$\begin{aligned} s^2 x_k &= K(s)(x_{k-1} - x_k - c - hsx_k) + H(s) r_k + d_k \\ u_k^{'} &= \tfrac{1}{B(s)}\big(K(s)(x_{k-1} - x_k - c - hsx_k) + H(s) r_k\big) \end{aligned} \quad (2.7)$$

$$r_{k+1}(s) = W(s) u_k^{'}(s) \; . \quad (2.8)$$

Here $u_k^{'}$ and r_{k+1} are scalar signals, respectively the signal sent by vehicle k and the filtered signal received by vehicle $k+1$. $W(s)$ is the transfer function of communication channel, including e.g. time delay and/or a low pass filter. The "encoding" filter $\frac{1}{B(s)}$ and "decoding" filter $H(s)$ express how the controller relates its internal logic to those signals on the communication line. One could for instance amplify the frequencies that will be attenuated by the communication line $W(s)$, but there is some limitation to this: $\frac{1}{B(s)}$ and $H(s)$ must both be bounded to avoid amplification of communication noise. In the analysis, obviously, only the product HW/B between signals used in the internal logic will play a role.

In this thesis, we will consider the CACC setting in Chapter 3 (with $h = 0$ i.e. constant spacing policy) and in Chapter 5 (with time-headway spacing policy).

III. More general linear communication between the vehicles

In Chapter 3, we prove it is not possible to guarantee string stability using CACC without time-headway. That is why we consider a slightly more general model of communication between the vehicles. We keep the assumption that each vehicle is connected just with one vehicle in front. However, we do not impose that the message u_k' sent by vehicle k corresponds to its control input: it can be a different combination of the information available to vehicle k, namely of the measured inter-vehicle distance error $e_k = x_{k-1} - x_k - c$, and of the signal r_k received from its predecessor. We moreover allow the communicated signals $u_k^{'}$ and r_k to be vectors of several components. This could result for instance from stacking e_1 and r_1 into $u_1^{'}$, then both $u_1^{'}$ and r_2 will both be of dimension two, by stacking e_2 and r_2 we get $u_2^{'}$ of dimension three, and so on. In this way one could effectively send $e_1, e_2, ...$ as a vector towards vehicle k; however, each of them would be passed through a (somewhat) realistic transfer function of the communication model. The control structure with general communication and constant spacing policy then writes:

$$\begin{aligned} s^2 x_k(s) &= K(s)(x_{k-1} - x_k - c) + H(s) r_k + d_k \\ r_k &= W(s)\, u_{k-1}^{'} \\ u_k^{'} &= F(s)(x_{k-1} - x_k - c) + G(s) r_k \end{aligned} \tag{2.9}$$

for $k = 1, 2, ..., N$, where now both $F(s)$, $G(s)$ are bounded transfer functions, possibly chosen independently of $H(s)$ and $K(s)$. This control structure will be discussed in Chapter 3 and Chapter 5 of the present thesis.

IV. Digital communication between the vehicles

In the present thesis, we will depart only in Section 3.3 from the linear model discussed in the previous points. In that case, we will barely specify any particular model for the communication. Indeed, the key point of our argument there is just that, up to a few vehicles close to the boundaries of the chain, each vehicle should behave in the same way (homogeneous controller).

For simplicity of the discussion, we will keep a deterministic model of communication in this case. For an impossibility result, this is not really restrictive compared to realistic communication models as in [**?**]. Indeed, we do allow a nonlinear controller which would include protection mechanisms against packet losses etc, but then we would just not include the fact that packets are actually lost with a finite probability.

We defer the presentation of the corresponding controller to the model of the discrete-time setting, later in the present Chapter. Possible generalizations should be clear from the (rather simple) corresponding proof in Chapter 3.3.

2.3 String stability: general idea

In a nutshell, string *instability* is a situation where the spacing error between consecutive vehicles in a vehicle chain grows unbounded when the number of vehicles increases to infinity, and string *stability* is the situation where this is avoided. We intentionally keep the definition of spacing error loose at this point as there are several versions, to be detailed below. This concept has spurred a lot of discussion and research since its definition in [17, 18]. Basically, it is known since [17, 18] that string stability cannot be achieved in a homogeneous string of interconnected second-order integrators (e.g. acceleration- controlled vehicles), with any controller that is linear and whose local control actions are determined from the relative distance to a few directly preceding vehicles, using constant space policy. This has attracted attention as a prototypical, unavoidable shortcoming of linear systems [17, 18]. When each vehicle only reacts to its immediate predecessor, a straightforward proof of string instability follows from the Bode integral theorem [16]. Indeed, the transfer function from error on vehicle $k-1$ to error on vehicle k takes the form of a complementary sensitivity function built on a double pole at the origin; this transfer function unavoidably amplifies some frequencies of the disturbance [22].

Let us recall this point in detail. Using the control structure (2.3) without time-headway policy $h = 0$, and assuming $d_k = 0$ for $k > 0$ i.e. only a disturbance on the leading vehicle, the error on the relative distance as vehicle k is easily computed to take the form:

$$e_k = T(s)^{k-1} \frac{1}{s^2 + K(s)} d_0 \;,\; k = 2, ..., N \,, \tag{2.10}$$

$$\text{with} \quad T(s) = \frac{K(s)}{s^2 + K(s)} = \frac{R(s)}{1 + R(s)}$$

where $R(s) = K(s)/s^2$. Thus $T(s)$ takes the form of a complementary sensitivity function built on $R(s)$. The latter has a double pole at the origin, under the reasonable assumption that $K(s)$ can have no zero at the origin — i.e. in practice, assuming that $K(s)$ cannot be made proportional to the derivative of $e(t)$, without any contribution from the value of $e(t)$ itself; this is a common assumption in realistic filters. To guarantee that e_k remains bounded, for any frequencies in the disturbance signal d_0 and with k arbitrarily large, it is then necessary in particular that $|T(j\omega)| \leq 1$ at all frequencies ω. One concludes that this is impossible for a stable system, from the statement of Bode's Complementary Sensitivity integral [16] which we recall below.

Proposition 2.1: *Assume that the loop transfer function $R(s)$ of a system has (at least) a double pole at $s = 0$. If the associated feedback system is stable, then the complementary sensitivity function $T(s) = \frac{R(s)}{1+R(s)}$ must satisfy:*

$$\int_0^\infty ln \mid T(j\omega) \mid d\omega/\omega^2 = \pi \sum_k \frac{1}{q_i^{(T)}} \geq 0 \,,$$

where $\{q_i^{(T)}\}$ are the zeros of $R(s)$ in the open right half plane. In particular, if $|T(j\omega)| < 1$ at some frequencies, then necessarily $|T(j\omega)| > 1$ at other frequencies.

So, no linear controller of the type $u_k(s) = K(s)e_k(s)$ can achieve string stability (under the reasonable assumption of no pole-cancellation, namely that $K(s)$ has no zero at $s = 0$).

As an illustrative example, we choose $K(s) = 2s + 1$ a PD controller and we plot the Bode diagram of the transfer function $T(s)$ on Figure 2.3. Clearly, the H_∞ norm of the transfer function $T(s)$ has a value larger than one. That means if we affect the system with the disturbance input d_0 in the frequencies where $\mid T(j\omega) \mid > 1$, then the error function e_k will grow unbounded as k increases. This can lead to collision between the vehicles or too large distances to still speak of an efficient platoon. I.e., we certainly want to avoid this situation in practice, and this is the purpose of imposing *string stability*.

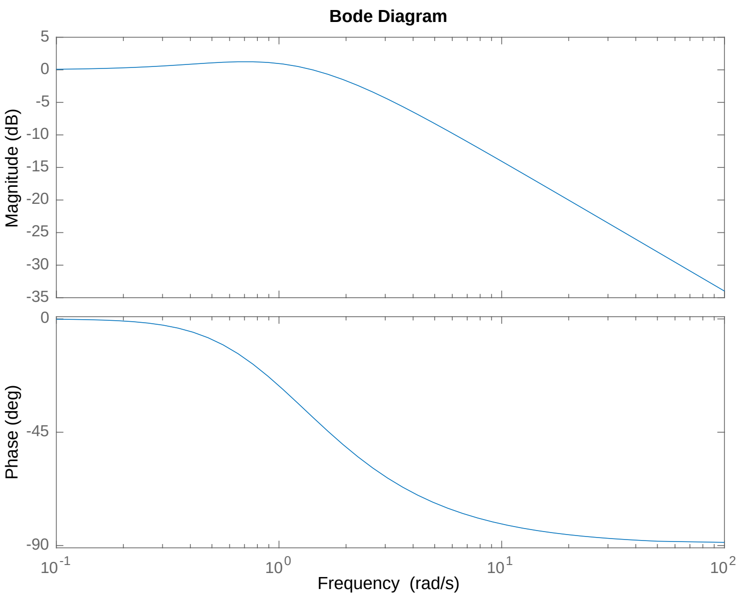

Figure 2.2: Bode diagram of the transfer function $T(s)$ illustrating Proposition 2.1.

To investigate in more detail this problem and options to solve it, a distinction among several string stability notions has been made. This distinction has sometimes been somewhat implicit in the literature, or using one as a proxy for others. We thus deem it worthwhile to distinguish them with some care in the following section.

2.4 Different definitions of string stability

2.4.1 Common features

In the vast majority if not all of the existing literature, researchers have concentrated on L_2 norms over time of input disturbance signals d_k and output error signals e_k. This may not be exactly the goal for the application, where a BIBO type property looks like the most natural specification; this would correspond to L_∞ norms over time. However, the L_2 approach has the big advantage to have a direct corresponding frequency-domain equivalent, via the Parseval equality. This is not different from many other control applications. For the same reason of analysis tools, we will stick to this dominating choice of the literature in the present thesis, whenever considering linear systems.

Another common point between all our definitions is the disturbances considered. Two types of approaches have been most popular in the literature: either considering disturbances on initial conditions, or considering disturbance input signals. These two versions are to some point equivalent, as bad initial conditions can be viewed as resulting from disturbance inputs. What is usually left out of string stability studies, is the possibility of having measurement noise; communication noise when there is communication; and model uncertainties. These are important points in practice. For impossibility results, clearly if we cannot satisfy string stability in absence of such disturbances, we cannot satisfy it with them either. For working solutions, this is another story.

In this thesis, like in the literature, we do not include explicitly disturbances, *besides on the input signals*. However, we will impose typical robustness constraints on our designs, like avoiding pole cancellations or too large transfer functions on the communication channel, such that one can expect other disturbances to be kept in check if the system can counter input disturbances. Of course, when we assume that disturbances affect the leader only (see Chapter 4 below), it also means that we effectively discard these other types of uncertainties on all vehicles, except the leader.

2.4.2 Distinct treatments of the vector of vehicles

A first version of string stability is to avoid that a single, local L_2-bounded disturbance signal would have unbounded effects far off in the chain; the Bode integral argument indeed shows that particular local disturbances would grow unbounded, unavoidably, with the simpler controllers (unidirectional coupling to one vehicle in front, no time-headway, no communication). We will call this version L_2 string stability. Many researchers have concentrated on this problem first [18, 22, 24–27], which is a prerequisite for stronger versions of string stability.

Stronger versions require that an l_p-bounded vector of L_2-bounded disturbance signals induces a bounded vector of inter-vehicle distance errors in the same norm [28]. For instance, (L_2, l_∞) string stability would request that if each vehicle is subject to an L_2-bounded input disturbance, then each inter-vehicle distance error should remain L_2-bounded. This appears to be the most practical formulation, but it seems it has only been recently examined in [29], and mentioned in [28]. Indeed, for the benefit of analysis tools, the (L_2, l_2) version has been its most popular proxy in standard work [22, 23].

In this thesis, we aim to address these three different definitions of string stability, using different control methods: so called L_2, (L_2, l_2), and (L_2, l_∞) string stability. The L_2 norm of a time-dependent scalar signal is denoted $\|x_k(t)\| = \sqrt{\int_{-\infty}^{+\infty} |x_k(t)|^2 dt}$. For a time-dependent *vector* e.g. $x(t)$, a lower index will indicate the discrete norm used on the *vehicle index* dimension. The (L_2, l_p) norm is thus

$$|| \ x(.) \ ||_{(L_2,l_p)} = (\sum_{k=0}^{N} (\int_{-\infty}^{+\infty} \ | \ x_k(t) \ |^2 \ dt)^{p/2})^{1/p} \ .$$

In particular, the (L_2, l_2) norm is given by $\|x(\cdot)\|_2 = \sqrt{\sum_{k=0}^{N} \int_{-\infty}^{+\infty} |x_k(t)|^2 dt}$ and the (L_2, l_∞) norm is $\|x(\cdot)\|_\infty = \max_k (\|x_k(\cdot)\|)$.

Definition 1 (L_2 String Stability): *The vehicle chain is L_2 string stable if, with the closed-loop dynamics, for every $\epsilon > 0$ there exists $\delta > 0$ such that: $\|d(\cdot)\|_2 < \delta$ implies $\|e_k(t)\| < \epsilon$, uniformly for all $N = 1, 2, ...$ and for all $k \in \{1, 2, ..., N\}$.*

Definition 2 ((L_2, l_2) String Stability): *The vehicle chain is (L_2, l_2) string stable if, with the closed-loop dynamics, for every $\epsilon > 0$ there exists $\delta > 0$ such that: $\|d(\cdot)\|_2 < \delta$ implies $\|e(\cdot)\|_2 < \epsilon$, uniformly for all $N = 1, 2, ...$.*

Definition 3 ((L_2, l_∞) String Stability): *The vehicle chain is* (L_2, l_∞) *string stable if, with the closed-loop dynamics, for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that:* $\|d_j(\cdot)\| < \delta$ *for all* $j \in \{0, 1, ..., N\}$ *(equivalently* $\|d(\cdot)\|_\infty < \delta$*) implies* $\|e_k(t)\| < \epsilon$ *for all* k *(equivalently* $\|e(\cdot)\|_\infty < \epsilon$*), uniformly for all* $N = 1, 2, \ldots$ *.*

In a nutshell, the focus of string stability is that the configuration error must be bounded *uniformly in* N. The weaker notion is L_2 string stability, as it bounds the sum of disturbance inputs but requests a bounded effect just independently for each e_k. This may look somewhat surprisingly asymmetric, but it has been considered very much in existing research as a proxy for stronger versions of string stability. Indeed, L_2 string stability is a necessary condition but not sufficient to guarantee the stronger versions: (L_2, l_2) string stability, where the sum-of-squares of the e_k must be bounded too; and (L_2, l_∞) string stability, where the e_k are considered individually but also the input disturbances d_k need to be bounded only individually. A priori, the (L_2, l_2) and (L_2, l_∞) string stability are not in a definite relation with respect to each other. Indeed, when N becomes infinite these norms are not in a finite ratio of each other; one may seek to bound one norm by another, but since this happens on both sides (disturbance input and error output), it says nothing as such about the related property. For curious readers, we can mention that the mirror of L_2 string stability is obviously impossible: i.e. allowing any bounded disturbance input on each vehicle, it appears readily too demanding to request that the resulting vector of output disturbances should be bounded in the (L_2, l_2) sense.

For practical purposes, it seems definitely realistic that disturbances could act on each vehicle, and that in this case we want each inter-vehicle distance to remain bounded. This makes (L_2, l_∞) the probably most relevant definition. The (L_2, l_2) version may be related more to energy-type interpretations, and appears to be somewhat harder to achieve as we will see. It can serve as a proxy for (L_2, l_∞) as has been done in the literature, and is more constraining than the necessary L_2 version.

Remark 2.1 (admissible disturbances): These variants of string stability sometimes restrict the structure of the disturbance vector, e.g. assuming $d_j = 0$ for all $j > 0$ to model disturbance on the leader only [19, 20], or the opposite. Disturbances on the leader are indeed special, both practically as this is the "active" boundary of the chain, and for analysis as the controller on the leading vehicle is different; we will see that some results can indeed differ.

2.4.3 Precisions on previous work

Several papers have considered the impossibility of L_2 string stability under relative information only, and proved how alternative settings using e.g. absolute velocity feedback (as in the time-headway policy, see [5, 9, 10, 13, 15, 24–26, 28, 36–39]) do allow to achieve L_2 string stability with appropriate tuning. While this absolute velocity solution has gathered serious attention as solving L_2 string stability [24–27, 36–40], sometimes in conjunction with inter-vehicle communication and in particular with simple PD controllers, it appears that no result so far has established its power for the stronger yet practically important (L_2, l_p) versions. Those have only been investigated with even more information, e.g. controllers relying on absolute position and/or on non-deteriorated knowledge of the leader's velocity profile [29]. We precisely set to answer these missing points in this thesis.

It has also been proved, using *symmetric bidirectional* controller (2.4) it is possible to guarantee L_2 string stability with constant spacing policy between the vehicles, if only there exists disturbance input on the leading vehicle [19, 20]. However, regarding (L_2, l_2) string stability, as well as the less-studied (L_2, l_∞) string stability, the situation is more negative. In the symmetric bidirectional control setting of [19, 20], it has been proved that (L_2, l_2) norm string stability cannot be achieved using any linear symmetric bidirectional controllers, see [22, 23].

2.5 Discrete-time setting

In this section, we translate the discussions given in the previous sections from continuous-time setting to discrete-time setting. We give the model of the system, and the three different definitions of string stability in discrete-time setting.

2.5.1 Model description

The dynamics of a chain of undamped second-order integrators, with discrete-time controller, writes:

$$\begin{aligned} x_k(t+dt) &= x_k(t) + v_k(t)\,dt + u_{k,1}(t) + d_{k,1}(t) \ , \\ v_k(t+dt) &= v_k(t) + u_{k,2}(t) + d_{k,2}(t) \ , \end{aligned} \tag{2.11}$$

where $x_k, v_k \in \mathbb{R}$ for $k = 0, 1, 2, ..., N$ denote position and velocity respectively, $u_{k,1}, u_{k,2}$ come from feedback control inputs, $d_{k,1}, d_{k,2}$ come from perturbation forces, and dt is the time increment of a typically digital controller.

The model (2.11) is obtained by integrating the corresponding continuous-time model over a finite time interval $[t, t+dt)$. When accelerations $u_k(t)$ and $d_k(t)$ are applied to the system in continuous-time, the associated discrete-time inputs and disturbances are obtained via single integration of the signals over $[t, t+dt)$ for $u_{k,2}$ and $d_{k,2}$, and double integration of the same signals for $u_{k,1}$ and $d_{k,1}$. Regarding control inputs, we may mention that by choosing different profiles for $u(t)$ during the interval $[t, t+dt)$, the values of $u_{k,1}$ and $u_{k,2}$ can be specified independently.

We will consider the discrete-time model only under the more constraining constant spacing policy, where the feedback signals $u_{k,1}, u_{k,2}$ must be based on only *relative* displacement measurements $e_k = x_{k-1} - x_k - c$ between consecutive vehicles in the chain, including possibly relative speeds $\dot{e}_k = v_{k-1} - v_k$, internal variables and memory; but not using any information about absolute position nor absolute velocity. However, we will consider a very general controller structure which can include nonlinear and digital elements. Indeed, denoting by $e_{k:\ell}$ for $\ell > k$ the set of values $e_k, e_{k+1}, ..., e_\ell$, we will consider:

$$\begin{aligned}
u_{k,1} &= f_1(e_{(k-m_1):(k+m_2)}, \dot{e}_{(k-m_1):(k+m_2)}, c_{k,+}, c_{k,-}, \xi_k, N, t)\ , \\
u_{k,2} &= f_2(e_{(k-m_1):(k+m_2)}, \dot{e}_{(k-m_1):(k+m_2)}, c_{k,+}, c_{k,-}, \xi_k, N, t)\ , \\
c_{k+1,+} &= g_1(e_{(k-m_1):(k+m_2)}, \dot{e}_{(k-m_1):(k+m_2)}, c_{k,+}, c_{k,-}, \xi_k, N, t)\ , \\
c_{k-1,-} &= g_2(e_{(k-m_1):(k+m_2)}, \dot{e}_{(k-m_1):(k+m_2)}, c_{k,+}, c_{k,-}, \xi_k, N, t)\ , \\
\xi_k(t+dt) &= h(\xi_k(t), \xi_k(t-dt)..., \xi_k(t-Mdt), \\
&\qquad e_{(k-m_1):(k+m_2)}, \dot{e}_{(k-m_1):(k+m_2)}, c_{k,+}, c_{k,-}, N, t)
\end{aligned} \tag{2.12}$$

In this model, m_1 and m_2 are the number of agents ahead and behind to which a vehicle can react, respectively; $c_{k,+}, c_{k,-} \in \mathbb{R}^{n_c}$ with n_c some bounded integer are communication signals from $k-1$ to k and from $k+1$ to k respectively; the $\xi_k \in \mathbb{R}^{n_\xi}$ for some finite integers n_ξ and M, allow for a dynamical controller with finite memory of length M; and f_1, f_2, g_1, g_2, h are arbitrary functions, with some minimal regularity just to ensure that the solution to the dynamical system is well-defined at all times. The controller (2.12) is applied on all vehicles $k \in (m_1, N - m_2)$, whereas adapted versions are applied on the m_1 leading and m_2 last vehicles. The exact form of the latter will play no role in our analysis. We recall that, by tailoring the continuous-time acceleration $u(\tau)$ that is applied to the physical system during the time interval $\tau \in [t, t + dt)$ between two updates of the digital controller, it is possible to *independently assign* the values of $u_{k,1} = f_1 = \int\int_t^{t+dt} u(\tau)d\tau^2$ and $u_{k,2} = f_2 = \int_t^{t+dt} u(\tau)d\tau$.

All the controller functions can be nonlinear and time-dependent (e.g. modulated at specific frequencies), thereby vastly extending the traditional LTI setting. They contain unidirectional coupling and symmetric bidirectional coupling as special cases. They also allow arbitrary communication functions, for instance with digital encoding and decoding. Like in the continuous-time case and in line with the mainstream literature, we do not model the communication noise (e.g. packet drops) explicitly. Once a general communication scheme is allowed this is not really a restriction, as we will see that even assuming that the deployed communication infrastructure works perfectly (say without any packet loss), one can establish impossibility results. The way that communication noise intervenes is at the design stage, where one would not allow to assume e.g. communication to all vehicles during the time span dt.

The main constraints which will remain on our analysis of the discrete-time setting are:

- no dependence on absolute position nor absolute velocity (i.e. constant spacing policy, no time-headway)
- homogeneous controller, i.e. the functions f_1, f_2, g_1, g_2, h do not depend on vehicle index k and the internal variables are initialized with the same default values for each k. This can obviously be relaxed to approximately homogeneous, or to periodically homogeneous. In fact we believe that heterogeneous control could not essentially solve the issue of string instability, but we do not cover this in full generality in our proof.
- finite discretization time: the time step dt will be kept constant, although possibly very small, as the chain length N grows.

2.5.2 Different definitions of string stability

We now translate the different definitions of string stability to the discrete-time setting.

Definition 4: *The ℓ_2 string stability requires that there exist some $C_2, C_1 > 0$ independent of N such that: for any disturbances satisfying*

$$\sum_{k=0}^{N} \sum_{n} |d_{k,1}(n\,dt)/dt^2|^2 dt + \sum_{k=0}^{N} \sum_{n} |d_{k,2}(n\,dt)/dt|^2 dt < C_1$$

, it is ensured that $\sum_n |e_j(n\,dt)|^2 dt < C_2$ for each j.

Definition 5: *The (ℓ_2, ℓ_2) string stability requires that there exist some $C_1, C_2 > 0$ independent of N such that: for any disturbances satisfying*

$$\sum_{k=0}^{N} \sum_{n} |d_{k,1}(n\,dt)/dt^2|^2 dt + \sum_{k=0}^{N} \sum_{n} |d_{k,2}(n\,dt)/dt|^2 dt < C_1$$

, it is ensured that $\sum_{k=0}^{N} \sum_{n} |e_k(n\,dt)|^2 dt < C_2$.

Definition 6: *The (ℓ_2, ℓ_∞) string stability requires that there exist $C_1, C_2 > 0$ independent of N such that: for any disturbances satisfying $\sum_n |d_{k,1}(n\,dt)/dt^2|^2$ $dt < C_1$ and $\sum_n |d_{k,2}(n\,dt)/dt|^2 dt < C_1$ for all k, it is ensured that $\sum_n$ $|e_j(n\,dt)|^2 dt < C_2$ for all j.*

The dt factors are introduced as $d_{k,1}$ and $d_{k,2}$ are supposed to result from integrating (twice or once) a continuous-time acceleration perturbation signal $d_k(t)$ over the time interval $[t, t + dt)$. In principle we could just choose units such that $dt = 1$, but we choose to keep it for the benefit of later discussion. Similarly to the continuous-time case, Definition 5 is stronger than Definition 4, as it requires the influence of a same disturbance signal to not only be bounded on each e_j, but also on their sum-of-squares; this essentially supposes exponential decay of the influence of $d_{k,1}$ on e_j as j gets farther away from k. Definition 6 is similarly stronger than Definition 4, as it requires the same effect but allows stronger disturbances. Its relation to Definition 5 appears undetermined in general.

Remark 2.2: Those definitions were initially stated in the linear context, where the constants C_1 and C_2 can be rescaled, such that in this context it makes no difference in which order they are chosen (e.g. variants like “for each C_1, there exists a C_2” become equivalent to our statement). In the nonlinear context this might differ, and we have chosen the weaker constraint: this ensures that our impossibility result in Chapter 3 will also hold for stronger variants of the definitions.

We are now ready to state the aim of this thesis more precisely.

2.6 The aim of this thesis

The problem setting for the thesis is the following.

- We consider the dynamics of a vehicle chain where every vehicle follows 2nd order dynamics, according to (2.1) in continuous-time or (2.11) in discrete-time.

- The aim of the feedback controllers u_k of each vehicle is to stabilize configuration errors e_k of the vehicle chain, that is the distances between consecutive vehicles

- Moreover, when subject to local force noise d_k, the controllers must keep the errors e_k bounded independently of the chain length N, in the sense of satisfying string stability according to some selected Definitions 1-6.

- As illustrated in Section 2.4, the feedbacks u_k will have to be designed under various information constraints, among which the most important ones are: dependence on local variables only and, except for time-headway where absolute velocity can play a role, relying only on *relative* position measurements.

The particular theoretical interest in this problem comes from the observation in [17, 18] that, at least for some design constraints and string stability definition, there exists no linear controller at all which would solve this task. Subsequent investigation, e.g. proposing time-headway or bidirectional coupling, have provided working designs or further impossibility results for particular scenarios. The aim of this thesis is to complete the picture by providing comprehensive results about which scenarios (combination of design constraints and string stability definition) are possible or impossible to solve.

In order to facilitate navigation among the various results of this thesis, we anticipate the conclusions in the following tables. The meaning of each item should be clarified by reading the corresponding section of the thesis. Conversely, the context of a particular result later in the thesis can be checked by referring to the general tables below. The previously existing results are indicated with lowercase letters, our contributions in this thesis are highlighted with uppercase letters.

Table 2.1: Satisfying string stability with **disturbances on all vehicles** and **no time-headway (**$h = 0$**)**. We only list our generalizations with respect to existing results, which are worked out in Chapter 3.

Generalized CACC communication (2.7), unidirectional coupling; any definitions	IMPOSSIBLE
General linear scalar communication (2.9), unidirectional coupling; any definitions	IMPOSSIBLE
General vector communication , (2.9), finite $K(0)$ unidirectional coupling; (L_2, l_2)	IMPOSSIBLE
Sensor dynamics breaking, relative position symmetry unidirectional coupling; any definitions	IMPOSSIBLE
Any Digital Controller (2.12), possibly nonlinear, communicating, bidirectional; any definitions	IMPOSSIBLE

Table 2.2: Satisfying string stability with **disturbance on the leader only** and **no time-headway (**$h = 0$**)**. Our new contributions are in capital letters and worked out in Chapter 4.

	Symmetric Bidirectional Coupling Bidirectional Coupling	Asymmetric Bidirectional Coupling
$L_2 \equiv (L_2, l_\infty)$	possible with advanced linear controller	POSSIBLE WITH PD
(L_2, l_2)	impossible	POSSIBLE WITH PD

Table 2.3: Satisfying string stability **with time-headway** and **unidirectional coupling** (reacting only to the preceding vehicle). Our new contributions are in capital letters and worked out in Chapter 5.

	finite $K(0)$ e.g. PD control	infinite $K(0)$ e.g. PID/Integral Controllers
L_2	possible	possible
(L_2, l_2), $\lVert d_0 \rVert_2 \neq 0$	IMPOSSIBLE	POSSIBLE
(L_2, l_2), $\lVert d_0 \rVert_2 = 0$	POSSIBLE	POSSIBLE
(L_2, l_∞)	POSSIBLE	POSSIBLE

Chapter 3

Comprehensive impossibility results for string stability

The objective of the present chapter is to narrow down the conditions of *impossibility results* towards a more precise understanding of string stability, and consequently hopefully related issues. As a small variation, we consider both discrete-time and continuous-time settings for the impossibility, in which discrete-time setting has the advantage of being closer to digital controller implementations and of incorporating related "natural" constraints in a direct way; the conclusions should carry over in practice to the continuous-time setting as long as numerical discretization schemes allow a faithful modeling of the system. We work only with *relative* state measurements between neighboring subsystems, e.g. we do not allow additional absolute velocity sensors as in the time-headway spacing strategies. In accordance with the existing literature, we do not model measurement errors nor communication noise explicitly. For impossibility results, this is not a restriction. Our main approach is to relax some of the conditions found in the literature, and show that still under these relaxed conditions, achieving string stability, according to any of the three definitions provided above, is impossible. Specifically:

1. In Section 3.1, we consider a more general model of linear communication in continuous-time, and study it with constant spacing policy, while most of the literature about communicating vehicles has also combined it with time headway. We prove how such general communication alone, i.e. without resorting to a time-headway strategy, still makes it impossible to guarantee string stability. This extension is done under the common assumption of unidirectional coupling with one vehicle in front.

2. In Section 3.2, we consider that the realistic dynamics of relative distance sensors are in fact *not* exactly dependent on relative positions between the vehicles. Everything still depends on relative positions though, of sensor parts with respect to the vehicles and so on. It therefore comes at no surprise that however strongly one would amplify these effects, it does not solve the string instability issue. This is meant as an illustration confirming that string stability is robust to dynamic model details, as long as one does not explicitly resort to an external reference frame (like for measuring absolute velocity).

3. In Section 3.3, we provide a most general string instability result in discrete-time. We indeed establish that enabling nonlinear controllers, any couplings to a few vehicles in front and behind, any (nonlinear, quantized,...) local communication, and controller dependence on the chain length and on time explicitly, all together do not allow to design a controller which would achieve string stability with respect to any disturbances acting on the subsystems of the chain. We prove this for the various string stability definitions, and with as only key constraints: (i) the controller is homogeneous, i.e. each vehicle in the chain reacts in the same way to its neighbors; (ii) the controller discretization step dt remains bounded away from zero despite increasing chain length.
Once the setting has been identified, the proof comes down to working out a counterexample with rather basic mathematical concepts, and we believe that the contribution mainly rests on the unprecedented generality of the conclusions. Essentially: *string instability in a chain of second-order integrators is an unavoidable property of distributed sensing, for a (much) larger class of controllers than LTI systems.* The only realistic opening left by this result is, we would say, the assumption of a *homogeneous chain*. We would rather conjecture though, that designing heterogeneous chains would not allow to solve this issue either — although of course we may be wrong on this point.

The insight from this chapter could allow to seriously narrow down researchers' attempts at designing string stable controllers, in particular by resorting to nonlinear control means. Indeed, as a conclusion of Section 3.3, it would rather seem that string stability in this general definition would be too strong a goal for any system based on relative measurements only.

3.1 Communication between the vehicles

In this section we consider the problem of impossibility of string stability using communication between the vehicles in which each vehicle is only interconnected with one preceding vehicle, and in the case where we use constant space policy between the vehicles. The presence of communication has indeed been considered mostly in combination with a time-headway spacing policy, which thus combines two facilitating elements: communication, and measurement of absolute velocity. We here want to disentangle them, and consider a more general linear communication model than in the existing literature.

3.1.1 Cooperative Adaptive Cruise Control (CACC)

We first consider the impossibility of string stability using Cooperative Adaptive Cruise Control (CACC). With respect to the existing literature, this will clarify in particular whether we need a time-headway policy to guarantee L_2 string stability or not.

Computing the dynamics of $e_k = x_{k-1} - x_k - c$ from the one of x_k in the control structure (2.7) without time-headway, and defining $z_k = [e_k \; ; \; u^{'}_{k-1} - u^{'}_k]$, we get the closed-loop dynamics described by:

$$z_{k+1} \;=\; \mathbf{T}(s)\, z_k \;+\; \begin{bmatrix} \frac{1}{s^2+K} \\ \frac{1}{B} \cdot \frac{-K}{s^2+K} \end{bmatrix} (d_k - d_{k+1}) \tag{3.1}$$

in which

$$\mathbf{T}(s) = \begin{bmatrix} \frac{K}{s^2+K} & \frac{HW}{s^2+K} \\ \frac{K}{B} \cdot \frac{s^2}{s^2+K} & \frac{HW}{B} \cdot \frac{s^2}{s^2+K} \end{bmatrix}.$$

This takes the form of an iteration for the propagation of disturbances from vehicle k to vehicle $k+1$, so for each frequency $s = j\omega$ we must investigate the stability of the matrix $\mathbf{T}(j\omega)$.

In the following Theorem, we prove it is not possible to guarantee any of the definitions of string stability mentioned in Chapter 2 for a vehicle chain using such CACC and when there is no time-headway. Before giving the results, let us summarize the constraints again.

- Transfer function W is imposed. $W(j\omega)$ is bounded at all frequencies and decreases towards zero at high frequencies ω. This is a model transfer function, which might have some uncertainty; this must be taken into account in the sense that we cannot rely on its perfect knowledge to ensure some cancellations.
- For stability, $s^2 + K$ must have all zeros with negative real part.

- For stability, we need a negative real part for all eventual poles of $1/B$, H, W.
- For a realistic communication, we must give some minimal conditions that avoid to amplify communication noise unboundedly. This has not been included into (2.7) because we will replace it by a simple constraint; indeed, by adding communication noise, the above matrix equations include terms in $\frac{H}{K}$ and $\frac{H}{KB}$ multiplying noise terms. Therefore we will just require that at least $H(s)/(B(s)K(s))$ remains bounded for all $s = j\omega$.

Theorem 3.1: *It is not possible to guarantee any of the Definitions 1-3 of string stability, in a vehicle chain model with Cooperative Adaptive Cruise Control given by (2.7), with $H(s)W(s)/(B(s)K(s))$ bounded for all $s = j\omega$ and without time-headway ($h = 0$).*

Proof: We must thus prove that L_2 string stability cannot be achieved. The key step is to notice that $\mathbf{T}(s)$ is a singular matrix for all s, since the right column equals HW/K times the left column. Thus the single nonzero eigenvalue of $T(s)$ equals its trace,

$$\text{trace}(\mathbf{T}(s)) = \frac{K + \frac{HW}{B}s^2}{K + s^2} \; .$$

We can rewrite $\text{trace}(\mathbf{T}(s)) = \frac{R}{1+R}$ with $R = \frac{K+\frac{HW}{B}s^2}{s^2(1-HW/B)} = \frac{1+\frac{HW}{BK}s^2}{s^2(1/K-HW/BK)}$. Since HW/BK is bounded, the numerator of R tends to 1 as s tends to zero, while $1/K - HW/BK$ cannot be tuned to have a pole precisely at $s = 0$. Therefore, R has at least a double-pole at $s = 0$ and we are in the conditions to apply *Proposition 2.1* mentioned in Chapter 2 (Bode complementary sensitivity integral). This implies that there will be a range of frequencies ω where $\mathbf{T}(j\omega)$ has an eigenvalue with norm $|\text{trace}(\mathbf{T}(j\omega))|$ larger than 1. The corresponding eigenvectors,

$$[e_k \; ; \; v_{k-1} - v_k] \propto [K(j\omega) \; ; \; -\omega^2/B(j\omega)] \; ,$$

unavoidably make the system string unstable.

Since L_2 string stability is a necessary condition for (L_2, l_2) and (L_2, l_∞) string stabilities, we can conclude the proof. □

3.1.2 General communication scheme

We now want to investigate whether a more general (linear) communication scheme could allow to drop the requirement for time-headway.

Similarly to the case of CACC using the control structure (2.9), by defining $z_k = [e_k \; ; \; v_{k-1}]$ we can reformulate the dynamics as:

$$z_{k+1} \;\; = \;\; \mathbf{T}(s) z_k \; + \left[\begin{array}{c} \frac{1}{s^2+K} \\ 0 \end{array} \right] (d_k - d_{k+1}) \tag{3.2}$$

in which now

$$\mathbf{T}(s) = \left[\begin{array}{cc} \frac{K-HWF}{s^2+K} & \frac{HW(I-GW)}{s^2+K} \\ F & GW \end{array} \right] .$$

Here I is the identity matrix, emphasizing that v_k might be a vector and F, G, H, W matrices/vectors of appropriate size.

Before giving the results, let us summarize the constraints again in this setting.

- Transfer function W is imposed. $W(j\omega)$ decreases towards zero at high frequencies ω. This is a model transfer function which might have some uncertainty, i.e. we cannot rely on perfect tuning/cancellation with respect to W.
- For stability, $s^2 + K$ must have all zeros with negative real part; K can have at most one more zero than pole (e.g. term proportional to velocity, but not to acceleration).
- For stability, we need a negative real part for all eventual poles of F, G H, W.
- $G(j\omega)$ and $F(j\omega)$ must be bounded for all ω to avoid amplification of communication noise, while H may be of a similar form as K.

We do not have the complete picture for this case, but we can give two complementary results: one with bounded controllers, and one when communication signals $u_k^{'}$ are scalar.

Theorem 3.2a: *The vehicle chain with communication-based control (2.9) cannot achieve (L_2, l_2) string stability with a bounded control gain $K(0)$.*

Proof: A problem can be identified in the neighborhood of $s = 0$. The proof starts by showing that $\mathbf{T(0)}$ has an eigenvalue 1. This is easily seen by noting that

$$\mathbf{T}(0) - I = \left[\begin{array}{cc} \frac{-HW}{K} F & \frac{-HW}{K}(GW - I) \\ F & (GW - I) \end{array} \right]$$

is obviously singular. A signal d_0 concentrated on arbitrarily low frequencies, can then propagate along the whole chain without significant damping, preventing the satisfaction of (L_2, l_2) string stability; for a more explicit argument on this fact, see the proof of *Theorem 5.2* in Chapter 5. The only way to avoid that this leads to unbounded $\|e\|_2$, is to avoid that such d_0 gets into the chain in the first place, i.e. making $\frac{1}{s^2+K(s)} \to 0$ as s approaches 0.
□

So, using any controller $K(s)$ with a bounded DC gain, it is impossible to guarantee (L_2, l_2) string stability. We do not mention the other definitions, because in fact we believe that the following stronger result should also hold for communication signals of arbitrary dimension. Indeed, several (moderately handwaving) arguments would suggest that nothing significant can be won by sending a vector signal of information to a follower, instead of sending a particular scalar signal that would be of interest for the control decisions. We were not able however to explicitly analyze the system in its full generality, beyond the case where the $u_k^{'}$ are scalar (i.e. one-dimensional) signals.

Theorem 3.2b: *The vehicle chain with communication-based control (2.9) cannot achieve L_2 string stability, when each $u_k^{'}$ is a scalar signal and K, G, W are rational transfer functions satisfying the conditions summarized before Theorem 3.2a.*

Proof: We use a Routh-Hurwitz type criterion for discrete-time systems ([41]). For a two-dimensional state matrix A, it states that the eigenvalues belong to the open unit circle provided

$$\begin{aligned}&|\mathrm{det}(A)| < 1 \quad \text{and} \\ &|\mathrm{det}(A)^{*}\mathrm{trace}(A) - \mathrm{trace}(A)^{*}| < 1 - |\mathrm{det}(A)|^2 \;.\end{aligned}$$

The determinant of $\mathbf{T}(s)$ imposes

$$|\mathrm{det}| = \left|\frac{GWK - HWF}{s^2+K}\right| =: \frac{|A|}{|s^2+K|} \leq 1 \;,$$

where we have defined $A = (GK - HF)W$. Next, we need

$$\begin{aligned} 1 \;&\geq\; \frac{|\mathrm{trace} - \mathrm{trace}^{*}\mathrm{det}|}{1-|\mathrm{det}|^2} \\ &=\; \left|1 + \frac{s^2}{1-|\mathrm{det}|^2}\left(\frac{GW-1}{s^2+K} - \frac{(GW-1)^{*}}{(s^2+K)^{*}}\frac{A}{s^2+K}\right)\right| \;. \end{aligned}$$

Since $\frac{s^2}{1-|\mathrm{det}|^2}$ is real negative for $s = j\omega$ and $|\mathrm{det}| < 1$, the above equation cannot be satisfied if $(GW-1)/(s^2+K)$ takes a real negative value for some $s = j\omega$. Indeed, for any c_1, c_2 real negative and c_3 complex but of norm smaller than one, we have that $1 + c_1c_2(1 - c_3)$ lies outside the unit disk. Thus to conclude the proof, there remains to show that $(GW - 1)/(s^2 + K)$ will always take a real negative value for some $s = j\omega$.

Since $s^2 + K$ has two more zeros than poles, and all zeros must satisfy stability, we have that the phase Bode plot of $1/(s^2 + K)$ goes down at least by 180 degrees, to end at -180 degrees for ω tending to infinity. In contrast, $GW - 1$ has as many zeros as poles; all poles are stable, implying 90 degrees down in the phase Bode plot, such that overall with $GW - 1$ we either go down or stay, and again we end at -180 degrees for ω tending to infinity. Now assume as a first possibility, that $GW - 1$ starts at another value than -180 degrees. In this case, it must go down nontrivially, i.e. we must go sown by strictly more than 180 degrees to end up at -360 degrees: somewhere in between, there will be a 180 degree phase, proving impossibility. (Note indeed that we forbid any perfect cancellation with $GW = 1$ at a target value of ω.) So the only choice left is that GW starts at -180 degrees. Then for $K(0)$ finite we would have a negative real phase at $s = 0$, thus impossible. There remains the case with K having a pole of order $m > 0$ at $s = 0$. In this case, $1/(s^2 + K)$ has m of its zeros at $s = 0$, and $\frac{1}{s^m}\frac{1}{s^2+K}$ has a phase Bode plot going down by $(180 + m90)$ degrees overall. This means, $\frac{GW-1}{s^2+K}$ would start with a phase of $-180 + m90$ degrees at $s = 0$, then go down by $180 + m90$ degrees to end up at -360 degrees for ω tending to infinity, with $m > 0$. Again, this implies a phase of -180 degrees for some intermediate ω. There are no possibilities left, so the proof is concluded. □

The achievement of string stability with communicated signal *vectors* thus remains a possibility, in theory.

3.2 Modeling dynamic sensor parts

The main difficulty in achieving string stability is to rely on measurements of only *relative* positions e_k. We thus wondered whether a positive result could be obtained in some way if we can use slightly more than the relative information between the vehicles.

In particular, one can take into account that the sensors are usually composed of two parts, mounted on the rear of car k and the front of car $k + 1$, and whose relative distance is actually measured. Those mounts are not exactly static, and one might wonder if it could be beneficial to tune their dynamics in some way. We thus consider the rear and front parts to be linked to the vehicle via transfer functions

$$x_k^{(r)} = M^{(r)}(s)x_k \quad , \quad x_k^{(f)} = M^{(f)}(s)x_k \ ,$$

while the actual measurement replaces e_k by

$$e'_k = x_{k-1}^{(r)} - x_k^{(f)} \ .$$

We consider the system (2.1). Writing the dynamics with e_k replaced by e'_k, without time-headway nor communication, yields

$$\begin{aligned} e_k = & \frac{M^{(r)}(s)K(s)}{s^2 + M^{(f)}(s)K(s)}\, e_{k-1} + \\ & \frac{1}{s^2 + M^{(f)}(s)K(s)}\, (d_{k-1} - d_k) \, , \end{aligned} \tag{3.3}$$

for $k = 1, 2, ..., N$. There remains to clarify the mount models $M^{(r)}$ and $M^{(f)}$. The sensor part dynamics, is itself sensitive to its relative position with respect to the vehicle on which it is mounted, e.g. $x_k^{(r)} - x_k$. Thus, consistently with the rest of this section, we should have

$$s^2 x_k^{(r)} = K^{(r)}(s)\,(x_k - x_k^{(r)})$$

and the same on the front sensor part, which yields

$$\begin{aligned} M^{(r)} &= \frac{K^{(r)}}{s^2 + K^{(r)}} \ , \\ M^{(f)} &= \frac{K^{(f)}}{s^2 + K^{(f)}} \ . \end{aligned} \tag{3.4}$$

It is not hard to see that this implies: tailoring $K^{(r)}$ and $K^{(f)}$ cannot solve the string stability issue.

Theorem 3.3: *The vehicle chain with dynamics* (3.3),(3.4) *cannot achieve L_2 string stability, namely for any choice of stabilizing $K(s)$, $K^{(r)}$ and $K^{(f)}$ there will always be a frequency $s = j\omega$ at which a perturbation is amplified exponentially along the vehicle chain.*

Proof: We rewrite (3.3) as

$$e_k = M^{(r)} \cdot \frac{1}{M^{(f)}} \cdot \frac{K'(s)}{s^2 + K'(s)}\, e_{k-1} =: A(s)\, e_{k-1} \tag{3.5}$$

where $K'(s) = M^{(f)}(s) \cdot K(s)$. In this product of 3 transfer functions, $M^{(r)}$ takes the form of a complementary sensitivity function, with loop transfer function $K^{(r)}(s)/s^2$ satisfying the requirements of *Proposition 2.1*; $M^{(f)}(s)$ does as well, although here its inverse appears; and the last factor $T'(s) = \frac{K'(s)}{s^2+K'(s)}$ again is a complementary sensitivity function, with loop transfer function $K'/s^2 = M^{(f)}K/s^2$. The latter must of course still be stabilizing and since $M^{(f)}(0) = 1$, K'/s^2 has the same double pole at $s = 0$ as K/s^2, i.e. again *Proposition 2.1* applies. Overall, we thus have

$$\begin{aligned}\int_0^\infty \ln|A(j\omega)|.d\omega/\omega^2 &= \int_0^\infty \ln|M^{(r)}(j\omega)|.d\omega/\omega^2 \\ &\quad - \int_0^\infty \ln|M^{(f)}(j\omega)|.d\omega/\omega^2 \\ &\quad + \int_0^\infty \ln|T'(j\omega)|.d\omega/\omega^2 \\ &= \sum_k \frac{1}{q_k^{(M^{(r)})}} - \frac{1}{q_k^{(M^{(f)})}} + \frac{1}{q_k^{(T')}} \,,\end{aligned}$$

where the q_k denote the zeros associated to the transfer functions; for notational simplicity we use a single sum over the zeros of the different loop transfer functions, assuming appropriate padding if their number of zeros differ. Like for the basic case mentioned before (see *Proposition 2.1*), achieving string stability requires that the last line is negative. The only way to obtain this is if $K^{(f)}(s)/s^2$ has zeros in the open right half plane, without having the same zeros in the other terms. However, the latter would mean that $M^{(f)}(s)$ has zeros in the right half plane, unmatched by the other transfer functions, and by (3.5) this would imply that the vehicle chain has a pole in the right half plane i.e. it is unstable. □

Another case of sensor dynamics that is even easier to check is delay. Considering that distances are deduced from a time-of-flight measurement, one could say that the actual measurement is $x_{k-1}(t-\tau) - x_k(t)$ rather than $x_{k-1}(t) - x_k(t)$. In Laplace domain, this amounts to the previous situation with

$$M^{(r)} = \exp(-s\tau) \quad , \qquad M^{(f)} = 1 \,.$$

The dynamics thus comes down to $e_k = \exp(-s\tau)\, T(s)\, e_{k-1}$ with $T(s)$ the transfer function in absence of delay. This dynamics trivially has the same problems for $s = j\omega$, as $|T(j\omega)|$ in absence of delay.

So, we can conclude, by using a realistic class of sensor dynamics, it is not possible to circumvent the problem of L_2, (L_2, l_2) nor (L_2, l_∞) string instabilities. As mentioned earlier, we were motivated to consider this particular variation in order to illustrate that the impossibility results for string stability are robust to nontrivial modifications in the dynamical model. Based on Theorem 3.3, we would dare to conjecture that adding any (reasonable) model detail to the vehicle dynamics, as long as it involves *relative distances only*, would maintain the string instability problem.

3.3 String stability is impossible with any homogeneous controllers that can be nonlinear, time-varying, and locally communicating

In this section, we significantly extend the impossibility results of string stability, from the LTI setting to any homogeneous controllers that can be nonlinear, unidirectional, bidirectional, time-varying, and locally communicating. We should mention this important point that we consider discrete-time setting model called digital controllers in this section.

The main idea of the impossibility proof is to construct a disturbance input that is badly countered by any distributed controller. While exactly solvable situations may appear hard to find, we take advantage of a simple construction that focuses on the central part of the chain only, in order to give a lower bound on the induced error. As a result, the counterexample does not need to rely on linearity, and in fact it is not hard to consider with the same approach possibly other variants of string stability than Definitions 4-6, for instance an (l_∞, l_∞) type which would correspond to the maybe most practical BIBO criterion.

3.3.1 A badly countered disturbance situation

Consider disturbances of the following form:

$$\begin{aligned}
&d_{k,1}(t) = d_{k,2}(t) = 0 && \text{for all } t < 0,\quad k = 0, 1, ..., N\ ; \\
&\left.\begin{aligned} d_{k,1}(t) &= \tfrac{\alpha k dt^2}{N}, \\ d_{k,2}(t) &= \tfrac{\alpha k dt}{N} \end{aligned}\right\} && \text{for all } t = 0, dt, ..., T, \ \text{ and } \ k = 0, 1, ..., N\ ; \\
&d_{k,1}(t) = d_{k,2}(t) = 0 && \text{for all } t > T,\quad k = 0, 1, ..., N\ ,
\end{aligned} \tag{3.6}$$

with constants $\alpha > 0$ and $T > 0$ to be specified later.

We note that this disturbance takes a very particular form, maybe at odds with the intuitive picture of a disturbance propagating along a chain. However, it fits with the definitions of string stability in Chapter 2, and it is a possibility – although maybe not too probable – for a disturbance acting on a distributed system. To show impossibility, a single counterexample is sufficient. Maybe this indicates that the general definition of string stability is too demanding for vehicle chains. However, in this sense, we must mention that the next Chapter 4 shows how if we restrict disturbances to the leader only, instead of allowing such distributed diturbances, then string stability can be recovered, counter to common consensus about the problem setting. For other applications than vehicles, the proposed perturbation may suggest a way to disrupt large chains with arbitrary couplings. Anyways, the following mathematical analysis stands as a fact.

To compute the evolution of the system under these disturbances, the trick is to exploit the finite propagation speed of signals along the chain — namely at most $m = \max(m_1, m_2)$ vehicles per time step — in order to restrict our attention to a central subset of vehicles, for which the computations are easy.

• We thus consider the discrete-time system (2.11) with general controller (2.12) presented in Chapter 2. Consider the evolution of $e_k = x_{k-1} - x_k$ and $\dot{e}_k = v_{k-1} - v_k$ over one time step, when the $N+1$ vehicles all start with the same state $x_k(0) = v_k(0) = 0$ for all k, and with controllers initialized at $\xi_k = c_{k,+} = c_{k,-} = 0$ for all k. We get

$$\begin{aligned} e_k(dt) &= e_k(0) + dt\,\dot{e}_k(0) + u_{k-1,1}(0) - u_{k,1}(0) + d_{k-1,1}(0) - d_{k,1}(0) \\ &= u_{k-1,1}(0) - u_{k,1}(0) + \alpha dt^2/N \;; \\ \dot{e}_k(dt) &= \dot{e}_k(0) + u_{k-1,2}(0) - u_{k,2}(0) + d_{k-1,2}(0) - d_{k,2}(0) \\ &= u_{k-1,2}(0) - u_{k,2}(0) + \alpha dt/N \;. \end{aligned}$$

Here we have dropped all the arguments of u_1 and u_2 except time, to avoid heavy notation. It is clear however that, since the internal variables are all equal at time $t = 0$, the controller values are too, i.e. $u_{k-1,1} = u_{k,1}$ and $u_{k-1,2} = u_{k,2}$, for all vehicles that are not in the "boundary layer" with specific controllers, i.e. for all vehicles with $m_1 < k < N - m_2$. For those central vehicles, *completely irrespectively of the controller chosen*, we have

$$e_k(dt) = \alpha dt^2/N \text{ and } \dot{e}_k(dt) = \alpha dt/N \quad , \text{ for all } m_1 < k < N - m_2 \,.$$

For the same reason, the $c_{k,+}(dt), c_{k,-}(dt)$ and $\xi_k(dt)$ of all these vehicles will be equal.

• Now consider a time $t = ndt$ for some integer $n > 0$ and assume that all the state variables $e_k(t) = e_j(t)$, $\dot{e}_k(t) = \dot{e}_j(t)$, $c_{k,+}(t) = c_{j,+}(t)$, $c_{k,-}(t) = c_{j,-}(t)$ and $\xi_k(t) = \xi_j(t)$ for all $j, k \in [N_{\text{lead}},\; N - N_{\text{tail}}]$ for some integers

$N_{\text{lead}}, N_{\text{tail}} > 0$. Slightly extending the above example, we get:

$$\begin{aligned}
e_k(t+dt) &= e_k(t) + dt\,\dot{e}_k(t) + u_{k-1,1}(t) - u_{k,1}(t) + d_{k-1,1}(t) - d_{k,1}(t) \\
&= e_k(t) + dt\,\dot{e}_k(t) + \alpha dt^2/N \;=\; e_j(t+1) \\
&\quad \text{for all } j,k \in [N_{\text{lead}} + m_1,\; N - (N_{\text{tail}} + m_2)]\;; \\
\dot{e}_k(t+dt) &= \dot{e}_k(t) + \alpha dt/N \;=\; \dot{e}_j(t+1) \\
&\quad \text{for all } j,k \in [N_{\text{lead}} + m_1,\; N - (N_{\text{tail}} + m_2)]\;, \qquad (3.7)
\end{aligned}$$

and similarly we maintain $c_{k,+}(t) = c_{j,+}(t)$, $c_{k,-}(t) = c_{j,-}(t)$ and $\xi_k(t) = \xi_j(t)$ for $j,k \in [N_{\text{lead}} + m_1,\; N - (N_{\text{tail}} + m_2)]$.

We can thus iterate the above argument and get the following property.

Lemma 3.1: *Consider the system (2.11),(2.12) subject to the particular disturbance (3.6) and zero initial conditions. Then for any (well-defined) controller choice, the solution satisfies:*

$$\begin{aligned}
e_k(t) &= t(t+dt)\;\alpha\;/\;(2N) \qquad (3.8)\\
\dot{e}_k(t) &= t\,\alpha/N\,,
\end{aligned}$$

for all $t \in [0,T]$ and all $k \in (\frac{t}{dt}m_1,\; N - \frac{t}{dt}m_2)$.

<u>Proof:</u> The main argument is provided by the explanations preceding the statement. From (3.7), it is first clear that $\dot{e}_k$ is obtained as a sum of t/dt times the bias $\alpha dt/N$. Then replacing this into the expression of e_k in (3.7), one observes that the increment of e_k at time $n = t/dt$ is linear in n, so the standard formula for a linearly progressing series gives the result. □

To be useful at time t, the solution (3.8) of *Lemma 3.1* should cover at least 1 vehicle, i.e. $N - \frac{t}{dt}(m_1 + m_2) \geq 1$. For fixed m_1, m_2 and dt, we can ensure to have a valid solution for at least $N/2$ vehicles over the interval $[0,T]$, when taking $T = \frac{N\,dt}{2(m_1+m_2)}$. As m_1 and m_2 are constants independent of N, we just mean that we would select the duration T of the "bad" disturbance applied at the inputs, to be of order $N\,dt$.

3.3.2 Consequences for string stability

We now investigate what the above construction implies for string stability. First take Definition 4 defined in Chapter 2. To satisfy the condition on $d_{k,1},\; d_{k,2}$, we must thus have, by using the formula for a quadratically progressing series,

$$\sum_{k=0}^{N}\sum_{n} \qquad |d_{k,1}(n\,dt)/dt^2|^2 dt + \sum_{k=0}^{N}\sum_{n} |d_{k,2}(n\,dt)/dt|^2 dt$$
$$= \; 2T\,(\tfrac{\alpha}{2N})^2 \sum_{k=0}^{N} k^2 = 2T\,(\tfrac{\alpha}{N})^2\,(\tfrac{N^3}{3}+\tfrac{N^2}{2}+\tfrac{N}{6}) \; < C_1$$

or in other words, α of order $1/\sqrt{NT}$. This fixes the maximum amplitude that we can give to the input disturbance, as a function of N and T. Looking at the output, for the vehicles covered by *Lemma 3.1*, we then have

$$\sum_{n} |e_j(n\,dt)|^2 dt \;\; \gtrsim \;\; (\tfrac{\alpha}{2N})^2 \sum_{n=0}^{T/dt} [n(n+1)]^2 dt^5 \sim (\tfrac{\alpha}{N})^2 (\tfrac{T}{dt})^5 dt^5 \sim N\,dt^4\,.$$

Here we have replaced α by $1/\sqrt{NT}$ as just discussed and T by $N\,dt$ as suggested a few lines earlier, and we use the fact that a series progressing as $\{t^k\}_{t=0,1,\ldots,M}$ will have a leading term of order M^{k+1} – just to get an order of magnitude, as we denote by $\gtrsim$ and $\sim$. The $>$ comes from the fact that on the right we compute the sum for $t=0,1,...,T/dt$, whereas the actual perturbation on e_j will of course remain nonzero, in general, for many $t>T$ as well.

The basic conclusion is: the relevant norm of e_j grows at least linearly in N (for large values of N). Thus despite allowing a very general choice of *homogeneous* controllers, ℓ_2 string stability in the sense of Definition 4 cannot be satisfied — unless dt converges to zero with increasing N. We will comment about dt in the next subsection.

A similar argument can be repeated for the other definitions of Chapter 2, only the relative scalings of α, N, T, dt must be adapted. Altogether, this yields the following results.

Theorem 3.4: *For the system (2.11),(2.12), there exist disturbances $d_{k,1}$ and $d_{k,2}$ satisfying the required respective bounds according to the definitions of Chapter 2 and such that, irrespectively of any (well-defined) controller choice, for large N:*

- ***[ℓ_2, Definition 4]:*** *$\sum_n |e_j(n\,dt)|^2\,dt$ grows as $N dt^4$;*
- ***[(ℓ_2,ℓ_2), Definition 5]:*** *$\sum_{k=0}^{N}\sum_n |e_k(n\,dt)|^2\,dt$ grows as $N^2 dt^4$;*
- ***[(ℓ_2,ℓ_∞), Definition 6]:*** *$\sum_n |e_j(n\,dt)|^2\,dt$ grows as $N^2 dt^4$.*

Proof: The computation for Definition 4 is given above. For Definition 5 it is the same, but summing the disturbance over the number of vehicles for which *Lemma 3.1* is valid – this can be of order N as mentioned in the last sentence of Section 3.3.1. For Definition 6, the disturbance can be larger i.e. α of order $1/\sqrt{T}$, and with respect to the Definition 4 computation this adds a factor N to $\sum_n |e_j(n\,dt)|^2\,dt$. □

For dt fixed, this result establishes impossibility to satisfy any of the definitions of string stability given in Chapter 2, in a very general setting, allowing unidirectional or bidirectional symmetric or asymmetric coupling, looking a number of vehicles ahead and behind (as long as that number is independent of N), communicating with neighbors with general protocols, and processing all this in an arbitrary nonlinear control system with memory.

Remark 3.1: We may consider the generalization of this impossibility result to other graph structures than a chain. In [32], the behavior of a lattice of simple *linearly coupled* systems under stochastic noise is examined as a function of lattice dimension, and higher-dimensional lattices turn out to perform better. The investigation of [32] is motivated by more profound implications for the stability of physical matter, which after all appears to be governed by forces depending on *relative* states. Implications are also expected for the numerical simulation of related PDEs. It is not hard to extend the analysis towards our *Theorem 3.4* to lattices of possibly *nonlinear* systems:

- The number of time steps over which we can compute before the boundary effects appear everywhere in the lattice is $T = N^{1/D}$, with D the lattice dimension and N the total number of subsystems. We can then keep our counterexample with d_k increasing along one dimension of the lattice from 0 to α with steps $\frac{\alpha}{N^{1/D}}$, and constant along the other dimensions.

- Computing the acceptable α for each case, we get the relevant error to grow like $N^{2/D-1}$ for Definition 1, and like $N^{2/D}$ for Definitions 2-3.

- Thus there remains a doubt, as our counterexample fails to feature string instability, for Definition 1 as soon as $D \geq 2$. Note that this is *only a hint* at a dimensionality effect, to be confirmed with more general disturbance inputs and with an upper-bound rather than a lower-bound analysis.

- For Definitions 2-3, the internal dissipation is insufficient and we witness string instability with our counterexample, for any lattice dimension.

Compared to [32], we thus generalize the setting by allowing any nonlinear, time-varying local interactions towards improving the situation, but we obtain a more negative result. A major difference however is that we consider *the worst* disturbance distribution, over time and over subsystem indices. These bad disturbances might in fact become negligibly probable with increasing N and D. This suggests that maybe, not only the system setting and objective, but also the disturbance assumptions may have to be revised towards new, positive string stability results.

3.3.3 How telling is the discrete-time controller setting?

Besides the worth of this discrete-time result on its own right, we must briefly comment on the relevance of this result, keeping dt bounded away from zero, in comparison with the continuous-time literature about string stability.

As a common point with the existing literature, we can mention that low-frequency disturbances indeed appear to cause most of the problem in continuous time string instability proofs, with e.g. transfer functions along the chain unavoidably larger than 1 at some frequency. Thus, even though our precise computations towards *Theorem 3.4* say nothing rigorously about how this disturbance fares with a continuous-time system, it seems that adding the contributions of all the neglected vehicles and time-steps to the norm of e would finally yield string instability in continuous-time too – at least in simple cases, like any unidirectional PD controller, in perfect agreement with existing results. Also in accordance with the existing literature, the result of *Theorem 3.4* models no measurement errors nor communication noise explicitly. In their presence, the resulting behavior can only be worse and possibly counterbalance the possible benefits of controllers with smaller dt.

Regarding differences to the continuous-time setting, one should not forget that our result only follows a sufficient construction, where the discrete-time dt assumption serves as a tradeoff for the generality of the class of controllers covered. In other words, *Theorem 3.4* proves that it is *necessary* – yet possibly not even sufficient – to let dt go to zero with increasing N in order to satisfy string stability. This strongly suggests that controllers which would do well on string stability, should be those that cannot be modeled well with finite dt. Therefore, let us try to list and discuss which controller features would typically necessitate a very small dt:

- One obvious effect of smaller dt is the possibility of faster communication across the vehicle chain. Intuitively, finite communication speed appears realistic. If conversely one could communicate arbitrarily fast, with arbitrarily small dt and without communication noise, then vehicle k could get very fast knowledge of $e_1 + e_2 + ... + e_k = x_k - x_0$. We know that a controller based on the latter quantity can work to achieve string stability: just control each $x_k - x_0$ independently to stabilize each vehicle with respect to the leader. The "distributed system" setting and chain size N play no role anymore. Of course this idealized situation is unrealistic. In presence of communication noise, precision of the message is in a clear relation to communication time, at least for some traditional noise models like white Gaussian channels. In such context, requiring smaller dt appears to mean requiring increased communication bandwidth per signal when N increases. This conclusion is in agreement with the impossibility results for continuous-time controllers that assume a fixed bandwidth, as in the models of Chapter 3.

- Setting the communication aside, in practice, the controller's discretization step dt is chosen as the desired control dwell-time or delay needed, before vehicle k reacts to a measurement of vehicle $k - m_1$ or $k + m_2$; thus in practice dt converging to zero would mean, controller bandwidth tending to infinity. This is typically linked to situations where the feedback signals e_k would possibly move a lot over a short time span, and thus points towards controllers with high gain, e.g. increasing gain as a function of N.

  It is known indeed that this can work: in continuous-time, without communication, a LTI controller whose gain increases fast enough with N, can ensure string stability. However, this causes other problems related to measurement noise and commanded acceleration inputs. It indeed becomes questionable whether measurement noise can still be neglected when measurement outputs must be given at increasingly high speed (see the discussion about communication); and having control feedback gains tending to infinity is also know to pose a number of robustness problems.

- *Theorem 3.4* thus shows that with the general setting (2.11),(2.12), string stability is not robust to time-discretization. This is important to know towards typical minimal robustness tests in system simulations, where situations that work only for infinitesimal dt are quickly considered singular for all practical purposes.

In a sense, this robustness to finite dt can even be mathematically compared to the traditional requirement of "no poles cancellation" in the continuous-time setting. Indeed, allowing a decreasingly small dt without any measurement noises can be compared to allowing precise computation of $\lim_{dt\to 0} \frac{s(t+dt)-s(t)}{dt}$ for a signal s, i.e. evaluating pure derivatives. For a double-integrator system, this implies the possibility of pole cancellation at zero frequency, which is often excluded from the allowed settings.

Finally, a closer look shows that the dependence on dt is rooted in the fact that we analyze the system before the signals from the edges of the chain reach all the vehicles and make a detailed analysis harder. This does not mean of course that the vehicle chain would automatically be stabilized as soon as the signals from the edges have crossed the chain. Thus even when dt is allowed to decrease with N, it seems that efforts to circumvent string instability will have to take into account the apparently important role played by the boundary controllers.

While these arguments are of course no match for a full mathematical proof, they give strong indications to conjecture that string stability would be impossible with any "reasonable" homogeneous, possibly nonlinear and communicating controllers in continuous-time too. At this point of detail, we might argue as well that the digital-controller model is in fact closer to many typically cited applications, than the traditional continuous-time one.

The present chapter has thus completed some gaps in the literature about string instability with constant spacing policy in continuous-time, before providing a much generalized impossibility result in discrete-time. To summarize its contributions, we here repeat the table that was anticipated at the end of Chapter 2.

Table 3.1: Impossibility results established in Chapter 3 about satisfying string stability with **disturbances on all vehicles** and **no time-headway (**$h = 0$**)**.

Generalized CACC communication (2.7), unidirectional coupling; any definitions	IMPOSSIBLE
General linear scalar communication (2.9), unidirectional coupling; any definitions	IMPOSSIBLE
General vector communication (2.9), finite $K(0)$, unidirectional coupling; (L_2, l_2)	IMPOSSIBLE
Sensor dynamics breaking relative position symmetry, unidirectional coupling; any definitions	IMPOSSIBLE
Any Digital Controller (2.12), possibly nonlinear, communicating, bidirectional; any definitions	IMPOSSIBLE

Chapter 4

About the possibility to satisfy string stability, using constant spacing policy, with respect to a disturbance on the leading vehicle only

In this chapter, we consider string stability with respect to a disturbance acting on the leading vehicle only. This excludes the construction of the previous chapter, and it has been considered relevant in applications where a dominant disturbance is expected on the first subsystem only: modeling the reaction of a leading vehicle to obstacles, while the others just follow; or of buildings to earthquakes, with subsystems denoting the levels of the building (see e.g. [19, 20] and related papers). Anyways, both the case with disturbance on all vehicles (AV), and with disturbance on the leading vehicle only (LV), have been considered in the string stability literature. We have provided a very general impossibility result for the Definitions 4-6 in the AV case in Chapter 3, in the discrete time setting, and extended some more basic impossibility results in the continuous-time setting.

In [19, 20], it has been shown that a linear system using symmetric coupling can solve Definition 1 (see Chapter 2) of string stability in the LV sense. The focus there was on passivity-based analysis, and further study of this setting has been left open — maybe not realizing the importance of this assumption, having disturbance on the leading vehicle only, in a bidirectional case. Indeed, for unidirectional coupling, a disturbance on the leader would be a good representative of the general situation (see also our next Chapter 5), as an intermediate vehicle may be viewed like the leader of the remainder of the queue. For bidirectional coupling however the situation changes more drastically, and it is possible to achieve a string stable vehicle chain when input disturbances are known to act only on a fixed number of leading vehicle. Indeed, to complete the observation of [19, 20], we now prove that a simple linear PD controller can satisfy Definitions 1-6 (see Chapter 2) of string stability in the LV sense, both in discrete-time and in continuos-time settings. For the discrete-time setting, this also somehow confirms that our derived model with discretization step dt is reasonable, in the sense that it does allow to retrieve the same feasibility result as in continuous-time.

4.1 Controller structure and disturbance constraint

We take the simplest possible controller, namely:

- no communication;
- no dependence on N of the controllers;
- each vehicle uses just the relative information of the directly preceding vehicle and the directly following vehicle with constant spacing policy;
- the vehicles apply a homogeneous PD controller to this information, but asymmetrically to the information coming from upfront and the one coming from behind (see also [30, 33, 34]).

The bidirectional coupling is essential. Asymmetric coupling allows to achieve the stronger versions of string stability, while symmetric coupling is restricted to the L_2 type in [19, 20]. The specific choice of a PD controller is just for more concreteness in stability and string stability analysis. It should be clear though, how the proofs and associated results can be repeated for other types of controllers that one might prefer in practice.

The main point in this chapter is thus that disturbances only act on the beginning of the chain. More specifically, from a mathematical viewpoint we must assume that there exists a constant m, independent of the chain length N, such that disturbance inputs $d_k = 0$ for all $k > m$. Thus, while the chain length increases unboundedly, the number of vehicles on which disturbances can act remains fixed. This is the precise setting that we will consider in the continuous-time case. In practice, the most realistic reason for satisfying such property is probably when disturbances are restricted to the leading vehicle (or subsystem for more general applications); see e.g. [19, 20]. In order to provide a simpler proof, the latter assumption is made directly in our study of the discrete-time setting; we do believe though that the result still holds with disturbances restricted to a fixed number of leading vehicles, like in continuous-time.

4.2 Discrete-time setting

In this section, we consider an asymmetric bidirectional controller in discrete-time setting to guarantee the Definitions 4-6 for a disturbance input only on the leading vehicle.

4.2.1 Particularizing the Model

Consider the model defined in (2.11). We assume $d_{k,1} = d_{k,2} = 0$ for all k except $k = 0$ in order to consider the special case where only there exists disturbance on the leading vehicle, and the controls are defined as:

$$\begin{aligned} u_0(t) &= -a_2 e_1(t) - b_2 \dot{e}_1(t) && (4.1)\\ u_k(t) &= a_1 e_k(t) + b_1 \dot{e}_k(t) - a_2 e_{k+1}(t) - b_2 \dot{e}_{k+1}(t) \quad \text{for } 1 \leq k \leq N-1 \\ u_N(t) &= a_1 e_N(t) + b_1 \dot{e}_N(t) \ , \end{aligned}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ are positive constants, representing respectively proportional and derivative gains for the information coming from upfront and from behind.

For the remainder of this section, we make the following simplifications to avoid cluttering notation; the results however do hold in a more general setting. (i) assume $dt = 1$, without loss of generality; (ii) choose the controller tuning $b_1 = a_1$, $b_2 = a_2$ (i.e. proportional gain equals derivative gain, in each direction); and (iii) denote the only active disturbances $d_1 := d_{0,1}$ and

$d_2 := d_{0,2}$. This yields the error dynamics:

$$\begin{aligned}
e_1(t+1) &= e_1(t) + \dot{e}_1(t) + \frac{a_2}{2}(e_2(t) - e_1(t) + \dot{e}_2(t) - \dot{e}_1(t)) \\
&\quad -\frac{a_1}{2}(e_1(t) + \dot{e}_1) + d_1(t) \\
\dot{e}_1(t+1) &= \dot{e}_1(t) + a_2(e_2(t) - e_1(t) + \dot{e}_2(t) - \dot{e}_1(t)) \\
&\quad -a_1(e_1(t) + \dot{e}_1(t)) + d_2(t) \\
e_k(t+1) &= e_k(t) + \dot{e}_k(t) + \frac{a_1}{2}(e_{k-1}(t) - e_k(t) + \dot{e}_{k-1}(t) - \dot{e}_k(t)) \\
&\quad +\frac{a_2}{2}(e_{k+1}(t) - e_k(t) + \dot{e}_{k+1}(t) - \dot{e}_k(t)) \\
&\quad \text{for } 2 \leq k \leq N-1 \\
\dot{e}_k(t+1) &= \dot{e}_k(t) + a_1(e_{k-1}(t) - e_k(t) + \dot{e}_{k-1}(t) - \dot{e}_k(t)) \\
&\quad +a_2(e_{k+1}(t) - e_k(t) + \dot{e}_{k+1}(t) - \dot{e}_k(t)) \\
&\quad \text{for } 2 \leq k \leq N-1 \\
e_N(t+1) &= e_N(t) + \dot{e}_N(t) + \frac{a_1}{2}(e_{N-1}(t) - e_N(t) + \\
&\quad \dot{e}_{N-1}(t) - \dot{e}_N(t)) - \frac{a_2}{2}(e_N(t) + \dot{e}_N(t)) \\
\dot{e}_N(t+1) &= \dot{e}_N(t) + a_1(e_{N-1}(t) - e_N(t) + \dot{e}_{N-1}(t) - \dot{e}_N(t)) \\
&\quad -a_2(e_N(t + \dot{e}_N(t)) \; .
\end{aligned} \tag{4.2}$$

In matrix notation and frequency domain, with $z = e^{j\omega}$ the frequencies associated to a discrete-time system, (4.2) becomes:

$$S\,E = D \;, \tag{4.3}$$

where the matrix S and vectors E and D are defined as:

$$E = col(E_1, E_2, \dots, E_N),\; E_k = col(e_k, \dot{e}_k)\;;\; D = col(D_1, 0, \dots, 0),$$

$$D_1 = col(d_1, d_2)$$

$$\begin{aligned}
S &= \begin{bmatrix} p & -a_2\,p_0 & 0 & \dots & 0 \\ -a_1\,p_0 & p & -a_2\,p_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -a_1\,p_0 & p & -a_2\,p_0 \\ 0 & \dots & 0 & -a_1\,p_0 & p \end{bmatrix} \quad \text{with} \\
p_0 &= \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \;, \quad p = (a_1 + a_2)\,p_0 + \begin{bmatrix} (z-1) & -1 \\ 0 & (z-1) \end{bmatrix} .
\end{aligned}$$

4.2.2 Explicit expression of the induced errors

The model (4.3) gives D as a function of E, we must invert this relation to compute E as a function of D. We carry out this inversion in two steps, loosely inspired by a procedure based on directional flows from [35], to which we add an explicit analysis of the boundary effects.

Lemma 4.1: *The system of equations (4.3) with associated definitions is equivalent to the following system :*

$$\begin{bmatrix} a_1^{N-1}(p-\beta p_0) & \beta^N\, p_0 \\ \beta^N\, p_0 & a_2^{N-1}(p-\beta p_0) \end{bmatrix} \begin{bmatrix} E_1 \\ E_N \end{bmatrix} = \begin{bmatrix} a_1^{N-1}\, D_1 \\ \beta^{N-1}\, p_z\, D_1 \end{bmatrix} \tag{4.4}$$

and

$$(p-2\beta p_0)\, E_k \;=\; \frac{\beta^{k-1}}{a_2^{k-1}} p_z D_1 - \frac{\beta^k}{a_2^{k-1}}\, p_0\, E_1 - \frac{\beta^{N+1-k}}{a_1^{N-k}}\, p_0\, E_N \text{for} 2 \leq k \leq N-1$$

$$\textit{with} \qquad p_z = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \begin{bmatrix} (z-1) & z \end{bmatrix} \;/\; \tfrac{3z-1}{2}\,,$$

$$\beta = \gamma - \sqrt{\gamma^2 - a_1 a_2}\,, \quad \gamma = \frac{a_1+a_2}{2} + \frac{(z-1)^2}{3z-1}\;.$$

<u>Proof:</u> Consider the matrix M with structure:

$$M = \begin{bmatrix} I & C_2 & C_2^2 & \dots & C_2^{N-1} \\ C_1 & I & C_2 & \dots & C_2^{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C_1^{N-2} & \dots & C_1 & I & C_2 \\ C_1^{N-1} & \dots & C_1^2 & C_1 & I \end{bmatrix},$$

where I is the identity matrix and C_1, C_2 are 2×2 transfer functions to be computed. This idea comes from [35], with C_1 denoting the gain for flows from subsystem k to $k+1$ and C_2 the gain for flows from $k+1$ to k. Multiplying both sides of (4.3) by M, we have

$$M\,S\,E = Q\,E = M\,D \tag{4.5}$$

where we have defined $Q = M\,S$, and we want it to have the structure:

$$Q = \begin{bmatrix} q_{1,1} & 0 & 0 & \dots & q_{1,N} \\ q_{2,1} & q & 0 & \dots & q_{2,N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ q_{N-1,1} & \dots & 0 & q & q_{N-1,N} \\ q_{N,1} & \dots & 0 & 0 & q_{N,N} \end{bmatrix}. \tag{4.6}$$

This would indeed yield a system driven by the boundary conditions E_1 and E_N.

Satisfying the zero components of (4.6) imposes the following relations:

$$\begin{array}{rcl} -C_2^{k+1} a_1 p_0 + C_2^k p - C_2^{k-1} a_2 p_0 & = & 0 \,, \\ -C_1^{k+1} a_2 p_0 + C_1^k p - C_1^{k-1} a_1 p_0 & = & 0 \,, \quad \text{for } k = 1, 2, ..., N-2. \end{array}$$

The relations with C_1 are all satisfied once the quadratic matrix equation corresponding to $k = 1$ holds, that is:

$$-a_1 p_0 + C_1 p - C_1^2 a_2 p_0 = 0 \,. \tag{4.7}$$

Multiplying this equation on the right by $col(1, -1)$, we get the condition

$$C_1 \begin{bmatrix} -z \\ (z-1) \end{bmatrix} = 0$$

so the matrix C_1 must be singular. Writing $C_1 = V_1\, W_1^T$, with $V_1 = col(v_1, v_1')$ and $W_1 = col(w_1, w_1')$ both two-dimensional column vectors, we thus know that $w_1 = \frac{z-1}{z} w_1'$. Multiplying both sides of (4.7) from the left by the row vector $V_1^{\perp}$ orthogonal to V_1, we get

$$V_1^{\perp}(-a_1 p_0 + C_1 p - C_1^2 a_2 p_0) = V_1^{\perp} a_1 p_0 = a_1 \begin{bmatrix} V_1^{\perp} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} & V_1^{\perp} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \end{bmatrix} = 0$$

so we can identify $V_1 = col(1/2, 1)$. There remains to identify the normalization of W_1. By plugging all our knowledge into (4.7), we get

$$w_1 \;=\; \frac{z-1}{z} w_1' = \frac{(z-1)\beta}{(3z-1)/2}$$

with β defined as in the statement, and where in principle we can (consistently) choose the meaning to give to the square root. A strictly similar procedure gives $C_2 = \frac{a_2}{a_1} C_1$. In particular, C_1 and C_2 commute as they are equal up to a scalar factor, and we can write $C_1 = C_0/a_2$, $C_2 = C_0/a_1$ with $C_0 = \beta\, p_z$.

The remaining equations from (4.5),(4.6) just define the nonzero terms of Q. By taking into account the just computed result from (4.7) we get:

$$\begin{array}{rcl} q & = & p - 2C_0 p_0 \\ q_{1,1} & = & q_{N,N} = p - C_0 p_0 \,; \\ q_{k,1} & = & \dfrac{C_0^k}{a_2^{k-1}}\, p_0 \quad \text{for } k = 2, 3, ..., N; \\ q_{k,N} & = & \dfrac{C_0^{N+1-k}}{a_1^{N-k}}\, p_0 \quad \text{for } k = 1, 2, ..., N-1. \end{array}$$

Writing down the related terms, carrying out the algebra with the particular property $C_0^k\, p_0 = \beta^k p_0$ and computing $M\, D$ yields the announced result. $\square$

Lemma 4.2: *The explicit solution of the system (4.3) is given by:*

$$E_1 = \begin{bmatrix}(z+1)s_1 + (2d_1 - d_2)\\ 2(z-1)s_1 - (2d_1 - d_2)\end{bmatrix} \;/\; (3z-1) \quad ,$$

$$E_k = \begin{bmatrix}(z+1)\\ 2(z-1)\end{bmatrix} \; s_k \;/\; (3z-1) \;\; \textit{for}\; k = 2, 3, ..., N \;, \; \textit{with}$$

$$s_k = \frac{(a_1^{N-k+1} a_2^{N-k+1} \text{-} \beta^{2N-2k+2}) a_1^{k-1} \beta^{k\text{-}1}}{a_1^{N+1} a_2^{N+1} \text{-} \beta^{2N+2}} \; \frac{2\beta}{(3z\text{-}1)} \; \begin{bmatrix}(z\text{-}1) & z\end{bmatrix} D_1$$

$$\textit{for}\; k = 1, 2, ..., N \;.$$

<u>Proof:</u> We first consider the second row of the matrix equation in (4.4) and we notice that all terms except $a_2^{N-1} p\, E_N$ have a vector $col(1/2 \;,\; 1)$ on the left. Hence the term $a_2^{N-1} p\, E_N$ must also take this form. This already gives the stated expression for E_N, where the value of the parameter $s_N = row(1 \;,\; 1)\, E_N$ still has to be identified.

Next, we multiply the whole matrix equation in (4.4) by $\beta\, p_z \otimes I_2$ on the left. Using $p_z^2 = p_z$ and the property $\beta p_z (p - \beta p_0) = a_1 a_2 p_0$ deduced from (4.7) (we recall that $C_1 = \beta p_z / a_2$), this yields two scalar equations in two unknowns $s_N = row(1 \;,\; 1)\, E_N$ and $s_1 = row(1 \;,\; 1)\, E_1$, namely:

$$\begin{bmatrix} a_1^N a_2 & \beta^{N+1} \\ \beta^{N+1} & a_2^N a_1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_N \end{bmatrix} = \begin{bmatrix} \frac{2 a_1^{N-1} \beta}{(3z-1)} \begin{bmatrix}(z-1) & z\end{bmatrix} D_1 \\ \frac{2\beta^N}{(3z-1)} \begin{bmatrix}(z-1) & z\end{bmatrix} D_1 \end{bmatrix} .$$

Analytically inverting this system yields the expressions for s_1 and s_N.

Then we multiply the first row of the matrix equation in (4.4) by $row(2 \;,\; -1)$ on the left, such that only the term with p (and thus E_1) remains on the left hand side. We rewrite it as a function of $(e_1,\, s_1 = e_1 + \dot{e}_1)$ instead of $E_1 = (e_1, \dot{e}_1)$. From there we can readily deduce the expression for e_1, and thus via s_1 for all of E_1.

We are now left with a 2×2 equation to invert for each k with $2 \leq k \leq N-1$. To solve this, we first plug the obtained expressions of E_1 and E_N into the result of *Lemma 4.1*. Again, we notice that all terms except $p\, E_k$ have a vector $col(1/2 \;,\; 1)$ on the left, and this allows us to deduce the expression for E_k, where only s_k remains to be identified. The latter easily follows as the system now reduces to a scalar equation for each k. $\square$

4.2.3 Proving string stability

Having the explicit expression of *Lemma 4.2*, there remains to check that a controller tuning exists for which (i) the transfer functions are stable and (ii) their frequency response guarantees the satisfaction of the string stability definitions.

Proposition 4.1: *The system (4.3) is stable provided* $\frac{a_1+a_2}{2} + \sqrt{a_1 a_2} < 1$ *and* $a_1 \neq a_2$.

Proof: This analysis concerns all $z \in \mathbb{C}$. We must thus show that all the poles of the transfer functions in the solution from D_1 to E, whose expression is explicitly written in *Lemma 4.2*, belong to the unit circle centered on $z = 0$. The factors $1/(3z-1)$ and β always satisfy this requirement. So there remains to analyze the denominator of s_k, thus checking when $(a_1 a_2)^{N+1} - (\beta^2)^{N+1} = 0$, or equivalently when $(\beta/\sqrt{a_1 a_2})^{N+1} = 1$.
A necessary condition for this is to have $|\beta/\sqrt{a_1 a_2}| = 1$. We can rewrite

$$\left|\frac{\beta}{\sqrt{a_1 a_2}}\right| = \left|\frac{1-\sqrt{1-x}}{\sqrt{x}}\right| =: |f(x)| \quad \text{with} \quad x = a_1 a_2/\gamma^2 \in \mathbb{C}\ .$$

Complex analysis shows that $|f(x)| = 1$ if an only if $x \in [1, +\infty]$, while $|f(x)| < 1$ for all other $x \in \mathbb{C}$. Thus the poles are defined by

$$\frac{\gamma^2}{a_1 a_2} \in [0,1] \quad \text{or equivalently} \quad \frac{\frac{a_1+a_2}{2} + \frac{(z-1)^2}{3z-1}}{\sqrt{a_1 a_2}} \in [-1,1]\ .$$

In particular this requires the imaginary part of this last expression to vanish, which a quick analysis reveals to happen in the following cases:

(a) $z = 1$: In this case we have $\frac{\gamma}{\sqrt{a_1 a_2}} = (\sqrt{a_1/a_2} + \sqrt{a_2/a_1})/2 \geq 1$ with equality holding if and only if $a_1 = a_2$. Thus provided $a_1 \neq a_2$, this will give no pole.

(b) $z \in \mathbb{R}$ in general: For $z > 1$ we have $\frac{\gamma}{\sqrt{a_1 a_2}} > (\sqrt{a_1/a_2} + \sqrt{a_2/a_1})/2 \geq 1$ so no pole can be found. For $z < -1$, the value of $\frac{\gamma}{\sqrt{a_1 a_2}}$ decreases monotonically as z decreases towards $-\infty$; indeed, its derivative with respect to real z is positive for all $z < -1/3$. A sufficient condition to have no pole for $z \in [-\infty, -1]$ is thus, to have $\frac{\gamma}{\sqrt{a_1 a_2}} < -1$ for $z = -1$, which is equivalent to $\frac{a_1+a_2}{2} + \sqrt{a_1 a_2} < 1$.

(c) $z = \frac{1}{3} - \frac{2}{3}e^{j\theta}$ for all $\theta \in [-\pi, \pi)$. This implies $|z| \leq 1$ with equality holding only at $z = 1$. The latter has already been excluded in (a), so case (c) can only yield poles inside the unit circle.

So under the stated conditions, the property $|\beta/\sqrt{a_1 a_2}| = 1$ necessary for a pole is excluded for all z satisfying $|z| \geq 1$, and the system is stable. □

Note that the conditions of *Proposition 4.1* are also "kind of necessary" for stability. Indeed, once $|\beta/\sqrt{a_1 a_2}| = 1$, a pole is obtained when the related phase provides a $(2N+2)^{\text{th}}$ root of unity. As N can grow to infinity, this phase condition boils down to the rational multiples of 2π.

Towards string stability, there remains to examine how the frequency response of this stable system depends on N.

Proposition 4.2: *For any $\lambda \in (0,1)$, there exists a tuning of a_1, a_2 such that the stability conditions of Proposition 4.1 are satisfied and simultaneously the transfer function from $D_1(z)$ to each $E_k(z)$ is bounded by $C\,\lambda^k$, for all k, N and all $z = e^{j\omega}$, $\omega \in [-\pi, \pi)$, and with $C > 0$ independent of all those parameters.*

Proof: This analysis only concerns z of the form $z = e^{j\omega}$, with $\omega \in [-\pi, \pi)$. Since the vector components $2(z-1)$ and $z+1$, as well as the explicit factor $1/|3z-1| = 1/|3e^{j\omega} - 1| = 1/\sqrt{10 - 6\cos\omega} \leq 1/2$, provide bounded nonzero contributions independently of N, a_1, a_2, there only remains to examine the dependence on N, k, a_1, a_2 of the fraction

$$\frac{(a_1^{N-k+1} a_2^{N-k+1} - \beta^{2N-2k+2}) a_1^{k-1} \beta^k}{a_1^{N+1} a_2^{N+1} - \beta^{2N+2}} = \tag{4.8}$$

$$\frac{1 - \left(\frac{\beta}{\sqrt{a_1 a_2}}\right)^{2N+2-2k}}{1 - \left(\frac{\beta}{\sqrt{a_1 a_2}}\right)^{2N+2}} \left(\frac{\beta}{\sqrt{a_1 a_2}}\right)^k \frac{a_1^{k/2-1}}{a_2^{k/2}}$$

appearing in s_k. Let us write $a_2/a_1 = \alpha$ and $a_1 = \kappa/(1+\alpha)$, for some positive real κ and α. Then we have $a_1 + a_2 = \kappa$ and $\sqrt{a_1 a_2} = \frac{\kappa}{1+\alpha}\sqrt{\alpha}$. The conditions for stability of *Proposition 3* become $\alpha \neq 1$ and $\kappa(1/2 + \sqrt{\alpha}/(1+\alpha)) < 1$, for which a sufficient condition independent of α is $\kappa < 1$. We will henceforth consider $\kappa < 1$ fixed and adjust α to satisfy the stated property.

We first consider γ. Given the locus of $\frac{(z-1)^2}{3z-1}$ for $z = e^{j\omega}$, $\omega \in [-\pi, \pi)$, see Figure 4.1, for any $\kappa \in (0,1)$ there exists a value $g \in (0, \kappa)$ such that $|\gamma| = |\frac{(z-1)^2}{3z-1} + \kappa/2| > g$ for all $z = e^{j\omega}$. From this, we can take a sufficiently small $\alpha/(1+\alpha)^2 \ll g^2/\kappa^2$ to apply in β the Taylor expansion of $f(x) = \sqrt{1+x}$ with small approximation error at first order, and we get

$$\beta \;=\; \gamma\left(1 - \sqrt{1 - \frac{\kappa^2}{\gamma^2}\frac{\alpha}{1+\alpha^2}}\right) \;=\; \frac{\gamma}{2}\frac{\kappa^2}{\gamma^2}\frac{\alpha}{(1+\alpha)^2} + \epsilon$$

with $|\epsilon| < \frac{\kappa^4}{g^3}\frac{\alpha^2}{(1+\alpha)^4}$. We will here take a *sufficiently large* $\alpha \gg 1$ to ensure this condition. From there we can ensure

$$\left|\frac{\beta}{\sqrt{a_1 a_2}}\right| = \left|\frac{\kappa}{2\gamma}\frac{\sqrt{\alpha}}{1+\alpha} + \epsilon'\right| < \frac{\kappa}{g\sqrt{\alpha}} \ ,$$

with α chosen large enough such that $|\epsilon'| < (\frac{\kappa\sqrt{\alpha}}{g(1+\alpha)})^3 < \frac{\kappa}{2g\sqrt{\alpha}}$. With these developments and taking α large enough such that $\frac{\kappa}{g\sqrt{\alpha}} < 1$, the fraction (4.8) becomes

$$\left|\frac{1-\left(\frac{\beta}{\sqrt{a_1a_2}}\right)^{2N+2-2k}}{1-\left(\frac{\beta}{\sqrt{a_1a_2}}\right)^{2N+2}}\left(\frac{\beta}{\sqrt{a_1a_2}}\right)^k \frac{a_1^{k/2-1}}{a_2^{k/2}}\right| \leq$$
$$\frac{1+\frac{\kappa^2}{g^2\alpha}}{1-\frac{\kappa^2}{g^2\alpha}}\left(\frac{\kappa}{g\sqrt{\alpha}}\right)^k \frac{1}{\sqrt{\alpha}^{k-2}}\frac{1+\alpha}{\kappa\alpha} \ .$$

Recall that here $g < \kappa$. For fixed κ, taking all things together, we thus have:

$$\text{for } k=1: \qquad \max(|e_1(z)|, |\dot{e}_1(z)|) \leq (C_1\frac{\kappa}{g}+C_2)\ \max(|d_1(z)|, |d_2(z)|)$$
$$\text{for } k\geq 2: \qquad \max(|e_k(z)|, |\dot{e}_k(z)|) \leq C_1\left(\frac{\kappa^2}{g^2\alpha}\right)^{k-1} \max(|d_1(z)|, |d_2(z)|) \ ,$$

for all $z = e^{j\omega}, \omega \in [-\pi, \pi)$ and all $\alpha > \overline{\alpha}$; with $C_1, C_2 > 0$ some constants independent of α, ω; and $\overline{\alpha} > 1$ a sufficiently large lower bound on α. The

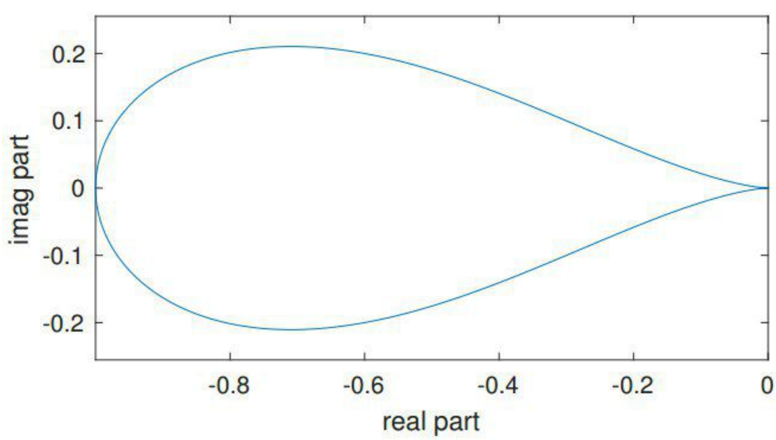

Figure 4.1: Locus of $\frac{(z-1)^2}{3z-1}$ for $z = e^{j\omega}, \omega \in [-\pi, \pi)$.

exponential rate $\lambda = \frac{\kappa^2}{g^2 \alpha}$ can be decreased at will by taking large enough α, without deteriorating the constants C_1, C_2. □

One might wonder what is the tradeoff for lower λ, i.e. for faster rejection along the chain of a perturbation on the initial vehicle. A quick analysis shows that taking a higher asymmetry α between forward and backward coupling a_1 and a_2, implies that disturbances acting on the last vehicles would be amplified more when travelling from the back to the front of the chain. Thus in practice, at some point, the neglected disturbances $d_{k,1}, d_{k,2}$ for $k > 0$ will become relevant for too low λ. The setting of this section however assumes, as in previous work like [19], that disturbances strictly act on the first vehicle only. We then have the following result.

Theorem 4.1: *There exists a tuning of a_1, a_2, with $a_1 + a_2 < 1$ and $a_2/a_1 \gg 1$, such that the system (4.3) is string stable towards disturbances D_1 acting on the leading vehicle only, according to all of Definitions 4-6 in the LV sense.*

Proof: By *Propositions 4.1-4.2* there exists a tuning as stated such that the system is stable and the transfer functions from D_1 to E_k are bounded by $C\,\lambda^k$, with $\lambda < 1$ and C both independent of N, for any frequency $z = e^{j\omega}$. We can then use Parseval's equality to rewrite the ℓ_2 norm over time of the signals as $\frac{1}{2\pi}\int_{-\pi}^{\pi} |d_1(e^{j\omega})|^2 d\omega$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |d_2(e^{j\omega})|^2 d\omega$ and respectively

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |e_i(e^{j\omega})|^2 d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{1,i} d_1(e^{j\omega}) + H_{2,i} d_2(e^{j\omega})|^2 d\omega \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |H_{1,i}(e^{j\omega})|^2 |d_1(e^{j\omega})|^2 + \\
&\quad |H_{2,i}(e^{j\omega})|^2 |d_2(e^{j\omega})|^2 \, d\omega \\
&\leq \frac{1}{\pi} C^2 \lambda^{2i} \left(\int_{-\pi}^{\pi} |d_1(e^{j\omega})|^2 d\omega + \int_{-\pi}^{\pi} |d_2(e^{j\omega})|^2 d\omega \right) \\
&\leq 2C^2\, C_1\, \lambda^{2i} \;.
\end{aligned}$$

where $H_{1,j}$ and $H_{2,j}$ are the transfer functions from disturbances d_1 and d_2 respectively to e_j, and the last line follows from the hypothesis in Definitions 4-6. Since $\lambda < 1$ and C, C_1 are independent of N, this result readily proves the satisfaction of Definition 4 = Definition 6 of string stability in the LV sense. Moreover, taking the sum over j we have a geometric series, yielding

$$\sum_{i=1}^{N} \sum_{t} |e_i(t)|^2 \leq \frac{2C^2\, C_1}{1 - \lambda^2}$$

independently of N, i.e. satisfying Definition 5 in the LV sense. □

While *Theorem 4.1* states the result for inputs acting on the leading vehicle only, a similar result could be obtained for disturbances acting on the $M \geq 1$ first vehicles only, with M fixed *independently of* N, and strictly zero disturbance input on vehicles $k > M$. It is however harder to see which application would motivate such setting.

4.3 Continuos-time setting

In this section, we establish the same positive results derived in the previous section not for the continuous-time model. We still assume that each vehicle is connected with one preceding vehicle and one following vehicle asymmetrically with PD control gains.

4.3.1 Particularizing the Model

As in discrete-time, we thus consider asymmetric bidirectional coupling (??) with a PD controller, i.e.:

$$\begin{aligned} u_0 &= a_2(x_1 - x_0) + b_2(\dot{x}_1 - \dot{x}_0) \\ u_k &= a_2(x_{k+1} - x_k) + b_2(\dot{x}_{k+1} - \dot{x}_k) + a_1(x_{k-1} - x_k) + \\ &\quad b_1(\dot{x}_{k-1} - \dot{x}_k) \quad \text{for } 1 \leq k \leq N-1 \\ u_N &= a_1(x_{N-1} - x_N) + b_1(\dot{x}_{N-1} - \dot{x}_N)\,, \end{aligned} \tag{4.9}$$

where a_1, a_2, b_1 and b_2 are constant parameters. Plugging (4.9) into (2.2), we write the dynamics of the configuration error $e_k = x_{k-1} - x_k$ in Laplace domain:

$$\begin{aligned} s^2 e_1 &= (a_2 + b_2 s)(e_2 - e_1) - (a_1 + b_1 s)e_1 + d'_1 \\ s^2 e_k &= (a_2 + b_2 s)(e_{k+1} - e_k) + (a_1 + b_1 s)(e_{k-1} - e_k) + d'_k \\ &\qquad \text{for } 1 \leq k \leq N-1 \\ s^2 e_N &= (a_1 + b_1 s)(e_{N-1} - e_N) - (a_2 + b_2 s)e_N + d'_N \end{aligned} \tag{4.10}$$

Here $d'_k = d_{k-1} - d_k$ and by the triangle inequality, $\|d\|_2 < \delta/2$ implies $\|d'\|_2 < \delta$. Thus investigating reactions to bounded D' instead of bounded D, is a sufficient condition (yet not fully necessary) for string stability.

4.3.2 Proof of string stability with respect to the leading vehicle

I. Partial inversion of the dynamics

The error dynamics (4.10) can be written compactly as

$$SE = D^{'} \tag{4.11}$$

with matrix S and column vectors E, D' given by

$$\begin{aligned}
E &= (e_1, e_2, \dots, e_N) \\
D' &= (d'_1, d'_2, \dots, d'_N) \\
S &= \begin{bmatrix} s^2+q & -p_2 & 0 & \dots & 0 \\ -p_1 & s^2+q & -p_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -p_1 & s^2+q & -p_2 \\ 0 & \dots & 0 & -p_1 & s^2+q \end{bmatrix}
\end{aligned}$$

where we have defined the elementary transfer functions $p_1 = a_1 + b_1 s$, $p_2 = a_2 + b_2 s$ and $q = p_1 + p_2$.

To analyze in detail the effect of D' on E, we essentially want to invert equation (4.11). We will do this in two steps, as for the discrete-time analysis. Namely, first we apply a transformation that makes (4.11) *almost* diagonal – i.e. after transformation each component follows a diagonal dynamics, plus a drive by the boundary vehicles e_1 and e_N. We are then able to analyze the resulting system by hand. For the first step (transformation), we define the matrix

$$\begin{aligned}
M &= \frac{1}{m} \begin{bmatrix} C & C_2 & C_2^2 & \dots & C_2^{N-1} \\ C_1 & C & C_2 & \dots & C_2^{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C_1^{N-2} & \dots & C_1 & C & C_2 \\ C_1^{N-1} & \dots & C_1^2 & C_1 & C \end{bmatrix} \\
\text{with } m &= \sqrt{(s^2+q)^2 - 4p_1p_2}\,,
\end{aligned}$$

and C, C_1, C_2 to be found. Multiplying both sides of (4.11) by the proposed matrix M, we want to obtain

$$MSE = QE = MD^{'} \tag{4.12}$$

with a matrix Q easy to invert. In particular, we impose the structure:

$$Q = MS = \begin{bmatrix} q_{1,1} & 0 & 0 & \dots & q_{1,N} \\ q_{2,1} & 1 & 0 & \dots & q_{2,N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ q_{N-1,1} & \dots & 0 & 1 & q_{N-1,N} \\ q_{N,1} & \dots & 0 & 0 & q_{N,N} \end{bmatrix} .$$

By working out the matrix multiplication, this imposes the following relations:

$$\begin{aligned} 0 &= -C_2^k p_2 + C_2^{k+1}(s^2+q) - C_2^{k+2} p_1 \\ 0 &= -C_1^{k+2} p_2 + C_1^{k+1}(s^2+q) - C_1^k p_1 \\ & \quad \text{for } k = 1, 2, ..., N-3; \\ 0 &= -C p_2 + C_2(s^2+q) - C_2^2 p_1 \\ 0 &= -C_1^2 p_2 + C_1(s^2+q) - C p_1 \\ m &= -C_1 p_2 + C(s^2+q) - C_2 p_1 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} m q_{k,N} &= -C_2^{N-(k+1)} p_2 + C_2^{N-k}(s^2+q) \\ m q_{k+2,1} &= C_1^{k+1}(s^2+q) - C_1^k p_1 \\ & \quad \text{for } k = 1, 2, ..., N-2; \\ m q_{1,1} &= C(s^2+q) - C_2 p_1 \\ m q_{2,1} &= C_1(s^2+q) - C p_1 \\ m q_{N-1,N} &= -C p_2 + (s^2+q) C_2 \\ m q_{N,N} &= -C_1 p_2 + (s^2+q) C \, . \end{aligned} \tag{4.14}$$

The second set of equations (4.14) just defines the $q_{k,1}$ and q_k, N, to which we will come back later. The first set of equations (4.13) define C, C_1, C_2; one checks that they are satisfied if and only if we take

$$\begin{aligned} C &= 1 \, , \quad C_1 = \frac{(s^2+q)-m}{2p_2} \, , \quad C_2 = \frac{(s^2+q)-m}{2p_1} \\ & \text{with} \quad m = \sqrt{(s^2+q)^2 - 4p_1p_2} \, . \end{aligned} \tag{4.15}$$

In particular, the last line imposes the sign in front of m in the expressions of C_1 and C_2. To obtain proper transfer functions ([35]), the complex square root of m should be interpreted along the branch for which the dominant s^2 terms cancel at high frequencies.

Using (4.15) the error dynamics of the vehicles rewrites:

$$
\begin{aligned}
e_1 &= \frac{1}{q_{1,1}}\left(-q_{1,N}e_N + d'_1/m + \sum_{k=1}^{N-1} C_2^k\, d'_{1+k}/m\right) \qquad (4.16)\\
e_k &= -q_{k,1}e_1 - q_{k,N}e_N + d'_k/m \\
&\quad + \sum_{\ell=1}^{k-1} C_1^\ell\, d'_{k-\ell}/m + \sum_{\ell=1}^{N-k} C_2^\ell\, d'_{k+\ell}/m \\
&\quad \text{for } k = 2, 3, ..., N-1 \\
e_N &= \frac{1}{q_{N,N}}\left(-q_{N,1}e_1 + d'_N/m + \sum_{k=1}^{N-1} C_1^k\, d'_{N-k}/m\right) \; .
\end{aligned}
$$

We see that the pair e_1, e_N now forms a system of its own, which drives the other vehicles inside the chain. The latter are in addition driven by their local disturbance d'_k and by two flows: a flow of disturbances coming from the front, which we denote

$$f_k = \sum_{\ell=1}^{k-1} C_1^\ell\, d'_{k-\ell} \;=\; C_1(f_{k-1} + d'_{k-1}) \; ,$$

and a flow coming from the rear,

$$g_k = \sum_{\ell=1}^{N-k} C_2^\ell\, d'_{k+\ell} \;=\; C_2(g_{k+1} + d'_{k+1}) \, .$$

In the next subsection, we analyze separately the parts of e_k related to the disturbance flows and to the e_1, e_N pair.

II. Bounding the flow transfer functions

We first consider the flows f_k and g_k. In order to ensure (L_2, l_2) boundedness of those signals, the H_∞ norm of both C_1 and C_2 would have to be lower than one. We next show that we can tune the controller such that one of those two constraints is satisfied, but not both. We typically choose to have $\|C_1(j\omega)\|_\infty < 1$. This leaves the hope of achieving string stability with respect to disturbance inputs e.g. $d'_1 \neq 0$ on the leading vehicle only. We will then conclude by showing that indeed, assuming $d'_k = 0$ for all $k > 1$, the e_1, e_N part of the dynamics has an (L_2, l_2) bounded influence on the dynamics as well, and thus the asymmetric system can be string stable in that sense.

Note that if we take a symmetric controller, with $a_1 = a_2$, then we have $C_1(\omega = 0) = C_2(\omega = 0) = 1$, so the series defining f_k, g_k does not converge. In particular, for a very low frequency disturbance $d_1' \neq 0$ on the leader only, we have $f_k = d_1'$ for all k; so the l_2-norm of the vector $(f_2, f_3, ..., f_N)$ would be proportional to $N\,d_1'$, i.e. not bounded independently of N. The asymmetry thus appears necessary to obtain string stability. We then obtain the following.

Lemma 4.3: *Consider the controller (4.9) with $a_1 \neq 0 \neq a_2$ (no poles cancellation). It is impossible to have both $\|C_1(j\omega)\|_\infty \leq 1$ and $\|C_2(j\omega)\|_\infty \leq 1$. (Allowing $a_1 = a_2$ might at best weaken the $\leq$ into $<$, as just discussed.)*

Proof: Let us assume $a_2 > a_1$; the converse case is similar. We have

$$\begin{aligned}
C_2 &= \tfrac{(s^2+p_1+p_2)-\sqrt{(s^2+p_1+p_2)^2-4p_1p_2}}{2p_1} \\
&= \tfrac{1}{2p_1}\left(p_1 + s^2 + p_2 - \sqrt{(s^2+p_2-p_1)^2 + 4s^2p_1}\right) \\
&= \tfrac{1}{2}\left(1 + \tfrac{s^2+p_2}{p_1} - \left(\tfrac{s^2+p_2}{p_1} - 1\right)\sqrt{1 + \tfrac{4s^2p_1}{(s^2+p_2-p_1)^2}}\right) \\
&\simeq \tfrac{1}{2}\left(1 + \tfrac{s^2+p_2}{p_1} - \left(\tfrac{s^2+p_2}{p_1} - 1\right) - \tfrac{2s^2}{s^2+p_2-p_1}\right) + O(|s|^4) \\
&= 1 + \tfrac{s^2}{p_1-p_2-s^2}\,.
\end{aligned}$$

The second line is obtained by square completion. The third line is valid for $|s| \ll 1$, taking into account that $a_2 > a_1$ for the phase of the factor taken out of the square root. The next line is Taylor approximation for the square root for $|s| \ll 1$; the higher order terms are of order $|s|^4$ provided $a_1 \neq a_2$ and $a_1 \neq 0 \neq a_2$, which is the condition to avoid pole cancellation. Replacing $s = j\omega$ in the last line we obtain

$$|C_2(j\omega)| \simeq \left|\tfrac{(a_2-a_1)+(b_2-b_1)j\omega}{(a_2-a_1)+(b_2-b_1)j\omega-\omega^2}\right| > 1$$

for low frequencies. □

On the positive side, we have the following results.

Lemma 4.4: *Consider the controller (4.9) with $a_1 \neq 0 \neq a_2$ (no poles cancellation).*
(a) For any choice of the control parameters we have $\|\, C_1(j\omega)C_2(j\omega)\,\|_\infty \leq 1$.
(b) Taking $p_2 = \alpha p_1$, for any $1 \neq \alpha > 0$ and any $a_1, b_1 > 0$, we have $\|C_1(j\omega)C_2(j\omega)\|_\infty < 1$.
(c) Taking case (b) and writing $p_1 = \frac{\kappa}{1+\alpha}p$, with any $\kappa > 0$ and $p = a + bs$

for some fixed $a, b, \kappa > 0$, there exists $\bar{\alpha}$ such that for $\alpha > \bar{\alpha}$, we have $|| C_1(j\omega) ||_\infty < 1$.

Proof: (a), (b) We have $C_1 C_2 = (1 - \sqrt{1-x})/x =: f(x)$ with $x = \frac{4p_1p_2}{(s^2+p_1+p_2)^2}$. The property follows from the fact that $f(1) = 1$ and $f(x) < 1$ for all $x \in \mathbb{C}$. For the particular choice of (b), we have $x = \frac{4\alpha p_1^2}{(s^2+(1+\alpha)p_1)^2}$. Since $x(j\omega)$ can be real positive, only if the phases of numerator and denominator match, this can happen only for $(j\omega)^2$ parallel to $(1+\alpha)p_1$, i.e. p_1 real. With $b_1 \neq 0$ this happens only at $\omega = 0$, for which we have $x = 4\alpha/(1+\alpha)^2 < 1$. Thus with (b) we never have $x \in [1, +\infty)$, so $f(x) < 1$.

(c) We here propose a particular design with $p_1 = \beta p$ and $p_2 = \alpha\beta p$, for some $\alpha, \beta > 0$ to be tuned and a more or less arbitrarily fixed transfer function $p(s) = a + bs$. We have

$$C_1 = \frac{s^2 + \kappa p - \sqrt{(s^2+\kappa p)^2 - 4\alpha\beta^2 p^2}}{2\alpha\beta p},$$

where $\kappa = (1+\alpha)\beta$. Denote by g the minimum norm of $h(s) = (s^2+\kappa p)^2/p^2$ over all $s = j\omega$; recall from standard Bode diagram approximations that $g > 0$ as long as perfect undamped resonance is avoided. We can now decrease the value of $\alpha\beta^2$ to make it arbitrarily smaller than g, while keeping κ and hence $h(s)$ constant, by decreasing β and increasing α at the same time. This allows to apply the Taylor expansion of $\sqrt{1+x}$ to the square root in C_1, uniformly for all ω:

$$C_1 = \frac{\frac{4\alpha\beta^2p^2}{(s^2+\kappa p)} + O(\,(\alpha\beta^2/g)^2\,)}{2\alpha\beta p} = \frac{2\beta}{h(s)} + O(\alpha\beta^3/g^2).$$

It is clear that the norm of this last expression can be made arbitrarily small by decreasing β while maintaining $\kappa = (1+\alpha)\beta$ constant, such that we can make $\|C_1(j\omega)\|_\infty$ smaller than 1 or in fact any other value. □

Those results indicate that one cannot expect L_2 string stability with this controller, however tuned, when all the vehicles are subject to disturbances d'_k. However, thanks to *Lemma 4.4(c)*, string stability might hold when the disturbance is only concentrated on the first vehicle(s) not on other vehicles. We now further analyze this situation.

III. Analysis III: the e_1,e_N subsystem and conclusion

Let us rewrite the first and last line of (4.16):

$$\begin{aligned} q_{1,1}e_1 &= -q_{1,N}e_N + d'_1/m + g_1/m \\ q_{N,N}e_N &= -q_{N,1}e_1 + d'_N/m + f_N/m\,. \end{aligned}$$

Multiplying the first one by $q_{N,N}$ and substituting the second one into it (respectively conversely), we obtain

$$\begin{aligned}\frac{e_1}{d_1'} &= \frac{mq_{N,N} - mq_{1,N}C_1^{N-1}}{m^2 q_{1,1}q_{N,N} - m^2 q_{1,N}q_{N,1}} \\ \frac{e_N}{d_1'} &= \frac{-mq_{N,1} + mq_{1,1}C_1^{N-1}}{m^2 q_{1,1}q_{N,N} - m^2 q_{1,N}q_{N,1}} \,.\end{aligned}$$

With this expression we can state the following result.

Theorem 4.2: *With appropriate tuning (see Lemma 4.4), the vehicle chain described by the controller (4.9) is string stable with respect to disturbances d'* ***restricted to the first $\bar{k}$ vehicles only, for some integer $\bar{k}$ independent of*** *N; in other words, it is string stable provided we impose $d_k' = 0$ for all $k > \bar{k}$, according to all of Definitions 1-3.*

We will use the following facts later in the proof.

(a) By choosing $\alpha > \bar{\alpha}$ large enough in the conditions of Lemma 4.4(c), it is possible to ensure that $m(s) \neq 0$ in the RHP, and thus in particular $m(j\omega)$ bounded away from 0. Indeed, in this setting we have $m = (s^2+p)\sqrt{1 - \frac{4\alpha}{(1+\alpha)^2}\,(\frac{p}{s^2+p})^2}$. The second-order polynomial s^2+p has all roots in LHP for positive coefficients. Since moreover $\frac{p}{s^2+p}$ goes to 0 for $|s|$ going to infinity, we can upper bound $|\frac{p}{s^2+p}|^2 < \bar{\eta}$ in the RHP. Then by taking α large enough, we can make $\frac{4\alpha}{(1+\alpha)^2}$ small enough, in particular such that $\sqrt{1 - \frac{4\alpha}{(1+\alpha)^2}\bar{\eta}} > 0$, thus implying the property.

(b) For a tuning as in *Lemma 4.4(c)*, we can give a lower bound $\eta_2 > 0$ for the norm of $(\frac{s^2+q+m}{2})^2$ in the RHP. Indeed, note that $s^2+q+m = (s^2+p) + (s^2+p)\sqrt{1 - \frac{4\alpha}{(1+\alpha)^2}\,(\frac{p}{s^2+p})^2}$. The factor s^2+p has roots in LHP, like for point (a). We have also explained in point (a) that by choosing α large enough, we can make the term $\frac{4\alpha}{(1+\alpha)^2}\,(\frac{p}{s^2+p})^2$ arbitrarily small in the RHP. It is then clear that we can ensure $1+\sqrt{1 - \frac{4\alpha}{(1+\alpha)^2}\,(\frac{p}{s^2+p})^2} \neq 0$ in the RHP.

Proof: The basic case is of course when $\bar{k} = 1$ i.e. only the leader is subject to a disturbance. We here provide the proof for this case; the general case is similar.

We will choose p_1, p_2 according to *Lemma 4.4(c)* such that $\|C_1(j\omega)C_2(j\omega)\|_\infty < 1$, and with the controller parameterized via α and p.

We thus assume $d'_k = 0$ for all $k > 1$, which implies $g_k = 0$ for all k and $f_k = C_1^{k-1}\, d'_1$.

We first analyze e_1. A few computations lead to

$$\frac{e_1}{d'_1} = \frac{1-(C_1C_2)^N}{\frac{s^2+q+m}{2}[1-(C_1C_2)^{N+1}]}. \quad =: \quad H_1(s) =: G_1(s)\,.$$

A first point is to prove stability of $H_1(s)$. By the property (b) above, this comes down to proving that $(C_1C_2)^{N+1} \neq 1$ in the RHP. Since $C_1C_2 = (\frac{s^2+q-m}{2})\,/\,(\frac{s^2+q+m}{2})$, we have to prove that

$$(\frac{s^2+q+m}{2})^N \neq (\frac{s^2+q-m}{2})^N\,.$$

As N can take arbitrary integer values, we will show that $|s^2+q+m| \neq |s^2+q+m|$. To have equality, we would need m perpendicular to s^2+q in the complex plane. But analyzing m as in property (a) above, we can choose α such that $m = (s^2+p)\sqrt{1+\eta_3}$ with $|\eta_3| \ll 1$, such that perpendicularity cannot be achieved. Thus, H_1 is stable.

We next check string stability. For large ω we have $\|C_1(j\omega)C_2(j\omega)\| = O(1/\omega^2)$, so $H_1(s)$ behaves like $O(\omega^2)\,/\,O(\omega^4)$ for large ω, N, with leading coefficients independent of N. For any $\xi > 0$, we can thus define $\bar{\omega}$ such that $|H_1(j\omega)| < \xi$ for all $\omega > \bar{\omega}$ and for all $N > 3$. For the compact domain $\omega < \bar{\omega}$, thanks to property (b) above and to $\|C_1(j\omega)C_2(j\omega)\| < 1$, we have a bound on $\|H_1(j\omega)\|_\infty$ which is independent of N.

We next turn to e_N.
Similarly we have

$$\begin{aligned}\frac{e_N}{d'_1} &= \frac{(\frac{s^2+q+m}{2})C_1^{N-1} + [C_1^{N-2}p_1 - C_1^{N-1}(s^2+q)]}{(\frac{s2+q+m}{2})^2 - (C_1C_2)N - 2p_1p_2[1-\frac{(s^2+q-m)(s^2+q)}{2p_1p_2}]}\\ &= \frac{(\frac{s^2+q+m}{2})C_1^{N-1} + p_1C_1^{N-2}[1-\frac{(s^2+q-m)(s^2+q)}{2p_1p_2}]}{(\frac{s^2+q+m}{2})^2 - (C_1C_2)^{N-2}p_1p_2[1-\frac{(s^2+q-m)(s^2+q)}{2p_1p_2}]}\\ &= \frac{mC_1C_2}{p_2(\frac{s^2+q+m}{2})} \cdot \frac{C_1^{N-2}}{1-(C_1C_2)^{N+1}} =: H_N(s)\,.\end{aligned}$$

By the same arguments the transfer function H_N is stable and the transfer function $G_N := H_N/C_1^{N-2}$ from $C_1^{N-2}d'_1$ to e_N is bounded independently of N.

For the other vehicles, we then have

$$\begin{aligned} \frac{e_k}{d_1'} &= -q_{k,1}\frac{e_1}{d_1'} - q_{k,N}\frac{e_N}{d_1'} + C_1^{k-1}/m \\ &= C_1^{k-2}\left(\frac{C_1}{m} - \frac{[1-(C_1C_2)^N]C_1C_2(\frac{s^2+q-m}{2}) + m(C_1C_2)^{N-k+2}}{mp_2[1-(C_1C_2)^{N+1}]}\right) \\ &=: H_k(s) =: G_k(s)C_1^{k-2}\ . \end{aligned}$$

Proving stability involves the same elements as for vehicle 1, plus requiring $m \neq 0$ in the RHP; the latter property is proved in item (a) above. Towards proving string stability, one can also apply the same arguments as for $G_1(s)$ to the different terms of $G_k(s)$: they are bounded for $\omega \gg 1$, and for finite ω we can bound $\|G_k(j\omega)\|_\infty$, independently of N and of k, provided we have a lower bound on $\|m(j\omega)\|_\infty$. The latter is also ensured by property (a) above.

This proves L_2 and (L_2, l_∞) string stability. Indeed, here Definition 1 is equivalent to Definition 3 because we consider the case where there is only disturbance input on the leading vehicle: thus having a constraint on the sum over vehicles of disturbance inputs, or on their maximum, amounts to the same.

For (L_2, l_2) string stability, taking all things together, we have

$$\begin{aligned} \|e(\,\cdot\,)\|_2^2 &\leq \sum_{k=1}^{N} \|H_k(j\omega)\|_\infty^2\ \|d_1'(\,\cdot\,)\|_2^2 \\ &= \sum_{k=2}^{N} \|G_k(j\omega)\|_\infty^2\ \|C_1(j\omega)^{k-2}\|_\infty^2\ \|d_1'(\,\cdot\,)\|_2^2 \\ &\quad +\|G_1(j\omega)\|_\infty^2\ \|d_1'(\,\cdot\,)\|_2^2 \\ &\leq \|d_1'(\,\cdot\,)\|_2^2\ \|G_{\max}(j\omega)\|_\infty^2\ r_N\ . \end{aligned}$$

Here $G_{\max}$ is the transfer function, among the G_k, with the largest H_∞ norm; we have just shown that this norm is bounded independently of N. And

$$r_N := 1 + \sum_{k=2}^{N} r^{2(k-2)} = 1 + \frac{1-r^{2(N-1)}}{1-r^2}$$

with $r := \|C_1(j\omega)\|_\infty$ is bounded independently of N when $r < 1$; the latter condition can be satisfied by *Lemma 4.4(c)*. So, the (L_2, l_2) string stability according to Definition 2 is proved. This concludes the proof. □

To conclude these theoretical results, we should repeat that the PD controllers considered here are a only a particular, simple example of a controller that achieves string stability. We have imposed them to be able to give concrete proofs where all elements are worked out. However, the steps of those proofs can be repeated in very similar manner for other controllers which one may prefer in practice.

4.4 Simulation results

We now illustrate in simulation the possibility of string stability for disturbance acting on the leading vehicle only. We will thus investigate the criteria of Definitions 1-6 using asymmetric bidirectional controllers, both in discrete-time and in continuos-time. Furthermore, for completeness we illustrate the problematic behavior of (l_2, l_2) string stability using a *symmetric* bidirectional controller. Indeed, it is known from the literature [23, 30] that *symmetric* bidirectional controllers do not allow to attain (l_2, l_2). The same authors have also proposed to use slight asymmetry, in order to obtain a better scaling of the spectral gap as a function of N. We here propose a more radical approach, where achieving string stability really relies on a strong enough asymmetry between forward and backward coupling. So, we wanted to illustrate how indeed the importance of asymmetry readily shows up in a practical example.

We recall that when assuming disturbance on the leader only, L_2 and $(L_2, l\infty)$ string stability become equivalent: having a constraint on the sum over vehicles of disturbance inputs, or on their maximum, amounts to the same for string stability, as soon as the number of vehicles on which disturbances differ from zero, remains bounded independently of N. So Definition 1 = Definition 3 and likewise Definition 4 = Definition 6, under the special circumstance that one restricts disturbance inputs to the leader or to a fixed number of leading vehicles that is independent of N.

4.4.1 Discrete-time setting

We can illustrate the effectiveness of the proposed asymmetric bidirectional controller with a concrete example. We take the values $a_1 = 0.01$ and $a_2 = 0.1$ for the controller parameters. We apply a pulse disturbance input on the leading vehicle. This contains all the frequencies which could affect the vehicle chain and for a linear system it should be a good indicator of its general behavior. We then simulate the behavior of the vehicle chain for various chain lengths, from 1 to 50 vehicles. Figure 4.2 shows the most stringent criterion, that is the (l_2, l_2) norm of the error vector $E = col(\, e_1, \dot{e}_1, e_2, \dot{e}_2, \ldots, e_N, \dot{e}_N\,)$. It must remain bounded independently of N

to ensure string stability according to Definition 5, which then automatically implies Definition 4 = Definition 6. Figure 4.3 shows the ℓ_2 norm of E_N, i.e. the error between the last two vehicles always, even if this is not the largest one. Indeed the largest $||E_k||$, which would determine the L_2 string stability, always lies at the beginning of the chain while disturbances further down exponentially decrease, as suggested by Figure 4.3. Finally, Figure 4.4 shows the evolution over time of the errors E_k induced at different vehicles by the disturbance acting on the leader. While the signal shape stabilizes to the least damped frequencies, the scale of the vertical axis contracts rather dramatically, confirming the good damping of any signal while propagating along the chain.

Intuitively, this exponential decrease of the leader's influence comes from the asymmetric design of our controller, in which each subsystem gives a larger weight to the rear subsystem than to the preceding subsystem in the chain. Other simulations confirm that conversely, when a stronger weight is given to the front subsystem — or equivalently, when a disturbance acts on the last subsystem with our tuning — the induced error on the first subsystems of the chain grows as the chain gets longer (not shown, as this is just a growing exponential). This is in agreement with the theoretical result that when disturbances can act everywhere on the chain, it is impossible in a very wide sense to satisfy string stability.

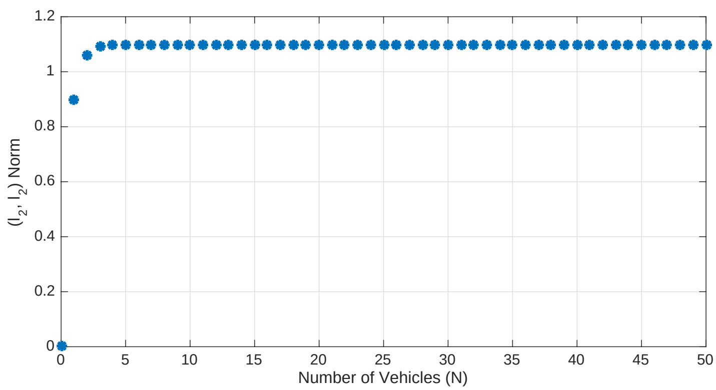

Figure 4.2: Discrete-time; (l_2, l_2) norm of the vector $E(t)$ of error functions as a function of N, for a pulse disturbance acting on the leader, using asymmetric bidirectional PD controller (4.1).

Next we illustrate what happens, in contrast, when taking a *symmetric* bidirectional controller, taking $a_1 = a_2 = 0.1$. We again apply a pulse as disturbance input on the leading vehicle. We simulate the behavior of the system for a varying number of vehicles, from 1 up to 51. Figure 4.5 shows the

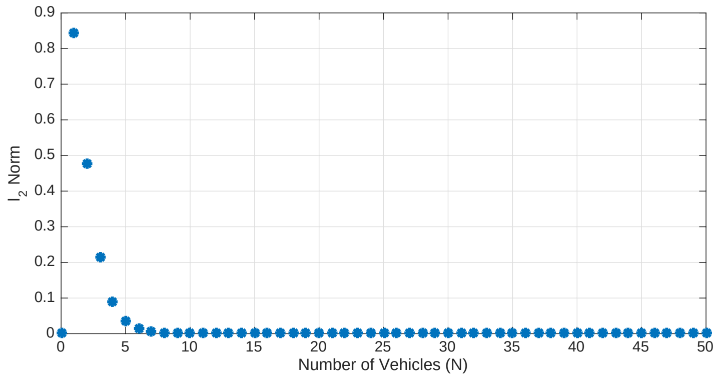

Figure 4.3: Discrete-time; l_2 norm of error function $E_N(t)$ between the last two subsystems as a function of N, for a pulse disturbance acting on the leader, using asymmetric bidirectional PD controller (4.1).

(l_2, l_2) norm of the error vector, thus the criterion for (l_2, l_2) string stability. This norm keeps increasing with N, unlike with the asymmetric controller. On Figure 4.6, we can see that the individual components of the error do not decrease anymore with increasing values of k, as was the case with a properly tuned asymmetric controller. This is in agreement with results from the literature, showing that a symmetric bidirectional controller would not work; it is also in agreement with our results, where we require a large enough level of asymmetry to make the proofs work.

4.4.2 Continuous-time setting

We now go over to the continuos-time setting. The vehicle chain is controlled via asymmetric bidirectional coupling between vehicles and PD parameters $a_1 = 0.01$, $b_1 = 0.01$, $a_2 = 0.1$ and $b_2 = 0.1$. This is not exactly the "practical" tuning $p_2 = \alpha\, p_1$ exploited in the proof, but it appears to work as well, showing some (expected) robustness with respect to the tuning parameters.

We first illustrate the transfer function $C_1(s)$ which plays a major role in the discussion of the previous section. Since it involves a square-root, we use the continued-fraction expansion

$$\sqrt{z^2 + y} = z + \cfrac{y}{2z + \cfrac{y}{2z + \cfrac{y}{2z + \cfrac{y}{\ddots}}}}$$

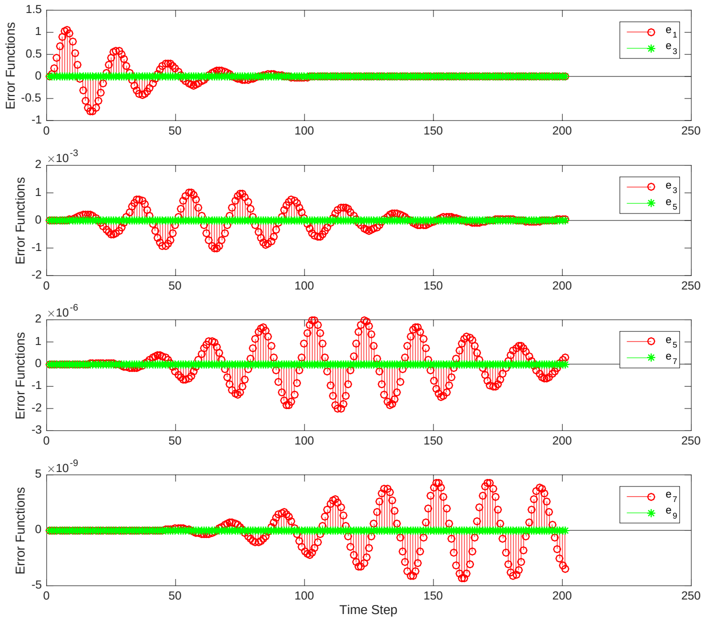

Figure 4.4: Discrete-time; Error functions e_k for various vehicles $k = 1, 3, 5, 7, 9$, for a pulse disturbance acting on the leader, using asymmetric bidirectional PD controller (4.1), and $N = 10$.

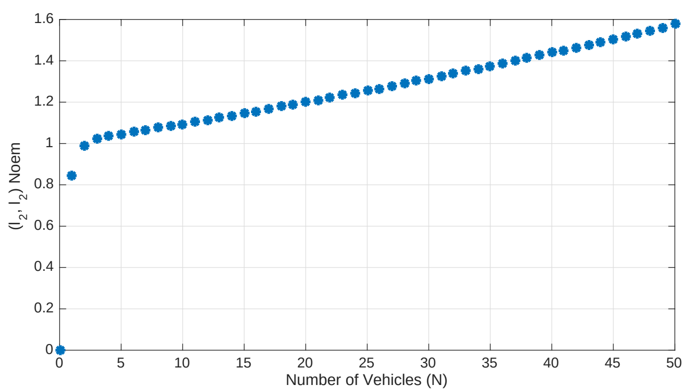

Figure 4.5: Discrete-time; (l_2, l_2) norm of the vector $E(t)$ of error functions as a function of N, for a pulse disturbance acting on the leader, under a symmetric bidirectional PD controller.

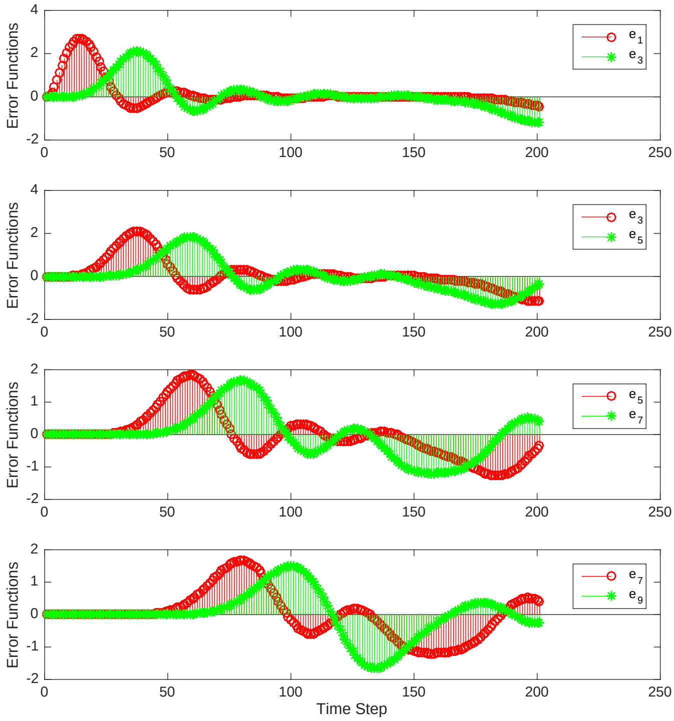

Figure 4.6: Discrete-time; Error functions e_k for various vehicles $k = 1, 3, 5, 7, 9$, for a pulse disturbance acting on the leader, under a symmetric bidirectional PD controller, and for $N = 10$.

in order to write

$$C_1 = 1/2(\frac{s^2+q}{p_2} - \sqrt{(\frac{s^2+q}{p_2})^2 - 4p_1/p_2}) = \frac{p_1/p_2}{\frac{s^2+q}{p_2} - \frac{p_1/p_2}{\frac{s^2+q}{p_2} - \frac{p_1/p_2}{\frac{s^2+q}{p_2} - \frac{p_1/p_2}{\ddots}}}} \tag{4.17}$$

. Using (4.17), the Bode diagram of the transfer function C_1 can be plot as on Figure 4.7. In particular, with the chosen tuning values, C_1 has a H_∞ norm strictly lower than 1.

Next we show the simulations of this asymmetric PD controller coupling on a chain of vehicles, where again we apply a short pulse disturbance on the

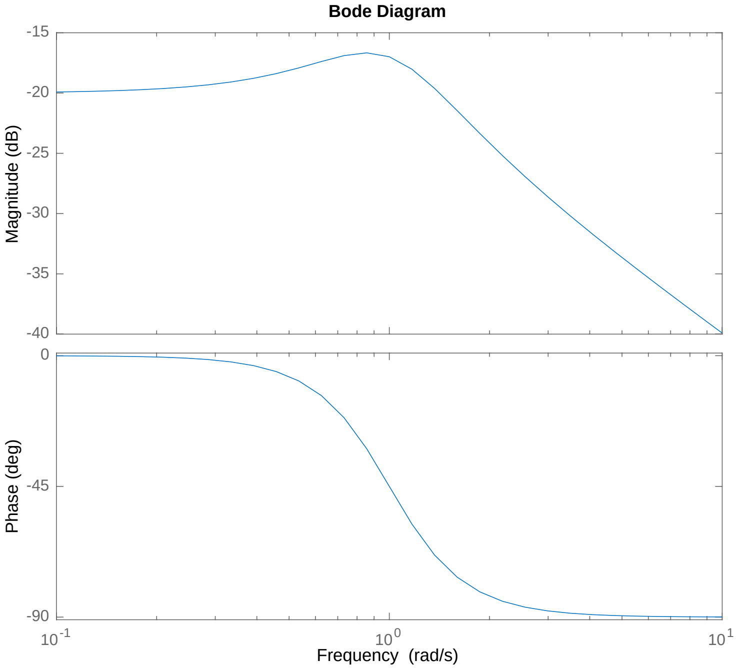

Figure 4.7: Bode diagram of transfer function C_1, (4.17).

leading vehicle of the platoon. Figure 4.8 shows the (L_2, l_2) norm of the vector of error functions as a function of the number of vehicles in the vehicle chain. It appears to converge to a constant value independently of number of vehicles, confirming the results obtained in *Theorem 4.2* for the possibility of (L_2, l_2) string stability. Like for the discrete-time case, this is the most stringent of the three definitions of string stability when disturbance inputs are restricted to the leader, so Figure 4.8 also implies that the vehicle chain satisfies the L_2 and (L_2, l_∞) definitions of string stability. Also like for the discrete-time case, the input disturbance that affects the leader appears to have a smaller and smaller impact down the chain, as illustrated on Figure 4.9. For more detail, Figure 4.10 shows the evolution in time of the spacing errors $e_k(t)$, for a network of 12 vehicles. It is apparent that the error decreases not only in time but also along the vehicle chain – after 3 vehicles essentially, it becomes barely visible on the plot.

We recall that these simulations must be viewed mainly as conceptual confirmations of the theoretical results. To draw detailed practical conclusions, one would first have to include more dynamics, measurement noise, parameter uncertainties,... into the model of the vehicle chain; all these things are

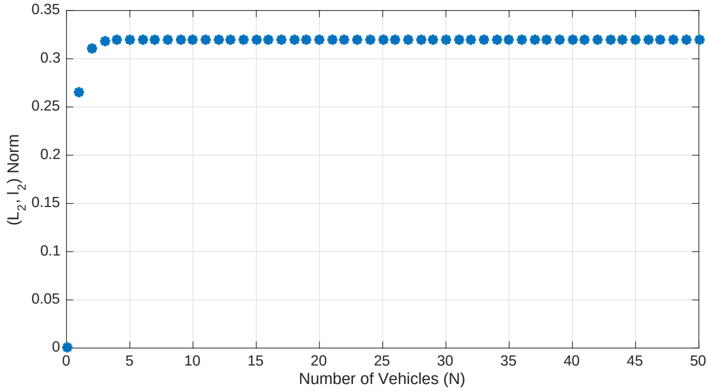

Figure 4.8: Continuous-time; (L_2, l_2) norm of the vector $e(t)$ of error functions as a function of N, for a pulse disturbance acting on the leader, using asymmetric bidirectional PD controller (4.9).

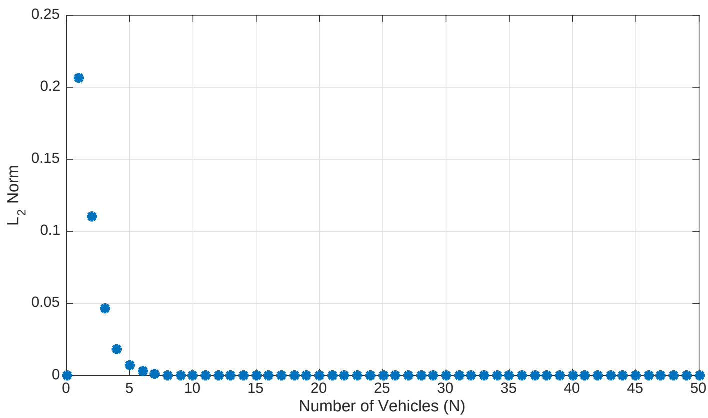

Figure 4.9: Continuous-time; L_2 norm of error function e_N as a function of number of vehicles N, for a pulse disturbance acting on the leader, using asymmetric bidirectional PD controller (4.9).

discarded in the academic study of string stability. Nevertheless, the conceptual idea remains the same that we are providing results to avoid errors e_k growing unbounded as the length of the chain increases.

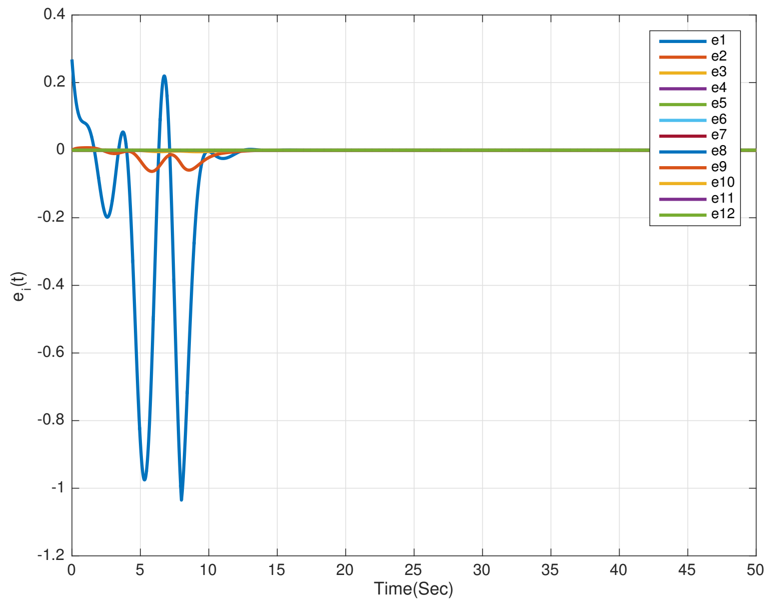

Figure 4.10: Continuous-time; Spacing errors $e_k(t)$ of a platoon with 12 following vehicles, for a pulse disturbance acting on the leader, using asymmetric bidirectional PD controller (4.9).

Chapter 5

About string stability with unidirectional controller using time-headway space policy

In the previously existing literatures, it has been shown that L_2 string stability can be solved by adding a sufficiently strong feedback term proportional to *absolute* velocity – thus slightly enlarging the setting compared to purely relative information. This absolute velocity can be viewed in several ways. It can be viewed as a natural drag force on the vehicles [25, 26], although this would be less in line with developing ever more fuel efficient transportation means. In a somewhat subtler way, the absolute velocity term can also be obtained from a so-called *time-headway policy*, where the desired distance from a vehicle to its predecessor increases with the vehicle's velocity [24, 28]. While this absolute velocity solution has gathered serious attention as solving L_2 string stability [28, 36–40], sometimes in conjunction with inter-vehicle communication and in particular with simple PD controllers, it appears that no result so far has established its power for the stronger yet practically important (L_2, l_2) and (L_2, l_∞) versions. (L_2, l_∞) only has been investigated with even more information, e.g. controllers relying on absolute position and/or on non-deteriorated knowledge of the leader's velocity profile [29].

Therefore, the possibilities for time-headway to satisfy (L_2, l_2) and (L_2, l_∞) string stability have remained, somewhat surprisingly, open to date. Establishing these results is precisely the purpose of the present chapter. We have both a positive result – characterizing a PD controller which satisfies the "practical" (L_2, l_∞) string stability as requested in [29]; and a negative result – suggesting why these results were missing, namely because the more standard (L_2, l_2) string stability notion cannot be satisfied by any controller that has bounded DC gain. Although bounded DC gain discards standards like PID controllers. Furthermore, in the symmetric bidirectional controllers in which each vehicle is connected with one vehicle in front and one vehicle in behind with the same gains, controllers with integrator terms were proven to be unstable [23]. The contribution of this chapter is thus to establish a simple way to satisfy the strong yet practical (L_2, l_∞) string stability; and to clarify that if one truly wants the (L_2, l_2) version, then a PID controller will be necessary, with possibly the need to more carefully analyze in the future the other effects related to the associated infinite DC gain.

5.1 Unidirectional controller without communication

In this section, we consider the unidirectional controller (2.3) presented in Chapter 2, and without any explicit communication between the vehicles. The benefits of adding communication will be addressed in the next section. We consider the three different definitions of string stability mentioned as Definitions 1-3 in Chapter 2.

5.1.1 Previous results: time headway model and L_2 string stability

We thus consider the unidirectional controller eqrefunidirectional with constant time-headway $h > 0$. We recall that the latter expresses that the desired distance between x_k and x_{k-1} is proportional to velocity $\dot{x}_k$. With the configuration error $e_k = x_{k-1} - x_k - hsx_i$, this controller implies the closed-loop equation

$$\begin{aligned} e_k \;&=\; \frac{K(s)}{s^2 + (1+hs)K(s)} e_{k-1} \\ &\quad + \frac{1}{s^2 + (1+hs)K(s)} (d_{k-1} - (1+hs)d_k) \;, \end{aligned} \tag{5.1}$$

for $k = 2, 3, ..., N$, and $e_1 = \frac{1}{s^2+(1+hs)K(s)}(d_0 - (1+hs)d_1)$. The controller transfer function $K(s)$ should satisfy $K(0) \neq 0$, but for the rest it is left open for future tuning. This control strategy was motivated by the following result [24, 43]. The proof is simple enough to be repeated here.

Proposition 5.1: *The norm at $s = j\omega$ of transfer function $T(s) = \frac{K(s)}{s^2+(1+hs)K(s)}$ in (5.1) is < 1 at all frequencies $\omega \neq 0$, and its H_∞ norm equals $T(0) = 1$, if and only if one of the following equivalent conditions hold:*
(a), see [43]: If one chooses $\bar{K}(s) = K(s)(1 + hs)$ first and then derives $K(s)$ from h, then we should ensure that h satisfies

$$h > \sqrt{\max_\omega \frac{\left|\frac{\bar{R}(j\omega)}{1+\bar{R}(j\omega)}\right|^2 - 1}{\omega^2}}$$

in which $\bar{R}(s) = \bar{K}(s)/s^2$.
(b) If one chooses $K(s)$ first, then the criterion becomes

$$h > \max_\omega \sqrt{K_R(j\omega)\left(2 - \omega^2 K_R(j\omega)\right)} + \omega K_J(j\omega)$$

where $K_R(j\omega) = \frac{1}{2}(\frac{1}{K(j\omega)} + \frac{1}{K(j\omega)^*})$, $K_J(j\omega) = \frac{1}{2j}(\frac{1}{K(j\omega)} - \frac{1}{K(j\omega)^*})$, and the maximization runs over all ω for which the argument of the square root is positive.

Proof: For case (a), we reformulate $T(s) = \frac{1}{1+hs} \frac{\bar{K}(s)/s^2}{1+\bar{K}(s)/s^2}$. Then writing

$$|T(j\omega)|^2 = \frac{1}{1+\omega^2 h^2} \left|\frac{R(j\omega)}{1+R(j\omega)}\right|^2 < 1 \quad \text{for all } \omega \neq 0$$

directly yields the expression, where the Bode integral (see Chapter 2) ensures that the max inside the square root will be non-negative.

For case (b), we just write $1/|T(j\omega)|^2 = |-\omega^2/K(j\omega) + (1 + hj\omega)|^2 > 1$ and we group real and imaginary parts to finally isolate h. □

The first criterion may appear less natural from a design perspective, but easier to check; the second one is a reformulation of ours towards perhaps more natural control tuning. For particular controllers one can get easy criteria, e.g. for a PD controller $K(s) = bs + a$, it is not hard to see that if $a > 2b^2$ the right hand side in case (b) is decreasing with ω, and one gets the simple condition $h > \sqrt{2/a}$.

Note that the system with time headway is not subject to the Bode Integral, because we have $T(s) = \frac{R(s)}{1+R(s)}$ with $R(s) = K(s)/(s^2 + hsK(s))$ having a *single* pole at $s = 0$. The Bode Integral requires a double-pole at the origin to impose its severe limitations.

The result of Proposition 5.1 ensures that one can avoid amplification of disturbances when a disturbance propagates with transfer function $T(s)^k$ throughout the chain; i.e. , the fact that controllers like the above PD example can satisfy the conditions of Proposition 5.1, proves that L_2 string stability can be satisfied. The objective of the present section is to provide a rigorous study for the stronger definitions of string stability defined in Chapter 2.

5.1.2 Impossibility of (L_2, l_2) string stability using bounded linear controllers with time-headway

We first prove the impossibility of achieving (L_2, l_2) string stability using any *bounded* stabilizing controller $K(s)$, in particular any controller satisfying $|K(0)| < \infty$, even in presence of time headway. (As we just recalled, without time headway i.e. for $h = 0$, it is already impossible to just achieve L_2 string stability.) We do this in two steps to identify that the main culprit is the disturbance on the leading vehicle: in essence, we can avoid that it gets amplified, but we cannot damp its effect significantly enough along the chain with any bounded linear controller. In contrast, if disturbances on the leader reduce to zero, then (L_2, l_2) string stability can be satisfied. In practice, this distinction by itself seems to be of little importance, as the leader is the one most likely subject to disturbances; see also the literature, where disturbances are expected either everywhere, or exclusively on the leader like in [20]. However, from an academic research point of view, this informs us that the focus should be on this "boundary effect" at the leader. In any case, the general conclusion may explain why a result about more than L_2 string stability was still missing regarding controllers with time-headway policy. Indeed, authors have had a tendency to focus on the (L_2, l_2) string stability definition whenever more than L_2 was considered; assuming no disturbance at the leader was never a realistic consideration; and most controller design attempts have focused on bounded controllers like variations on PD controllers. Our result proves that in this context indeed, it is impossible to achieve string stability.

The reader may note that this is the opposite observation compared to Chapter 4. This comes from the fact that we are considering a *unidirectional* control strategy here, with vehicles not reacting to their followers. The results in the present Chapter may be viewed as closer to the ones of the existing literature, in the sense that we justify their focus on disturbances that affect the leader — here indeed these are shown to be the worst ones, in contrast to the context of Chapter 4.

I. No disturbance on leader, $d_0 = 0$

While $d_0 = 0$ is certainly not a practical situation, we treat it first to show, by linearity, that all problems essentially arise from d_0. We will show indeed that for $d_0 = 0$, one can achieve (L_2, l_2) string stability using PD controllers with time-headway.

Theorem 5.1: *There exists a pair* $(K(s),\ h)$*, where* $h \geq 0$ *is a sufficiently large constant time-headway satisfying Proposition 5.1 and* $K(s) = bs + a$ *is a stabilizing PD controller, that achieves* (L_2, l_2) *string stability provided* $d_0 = 0$.

Proof: The use of a PD controller simplifies simplifies the checking of *stability*, since the denominators of the transfer functions are all based on second-order polynomials in s; the latter are stable for any positive coefficients.

Towards *string stability*, the key point is to recognize that two effects of d_k tend to compensate each other in e_m with $m > k$. Indeed, we rewrite (5.1) as

$$\begin{aligned} e_1 &= -L(s)d_1 \\ e_k &= -L(s)d_k + \sum_{m=2}^{k} T(s)^{k-m} P(s) d_{m-1} \ , \end{aligned}$$

with $P(s) = \frac{s^2}{(s^2+(1+hs)K(s))^2}$ and $L(s) = \frac{1+hs}{s^2+(1+hs)K(s)}$ and $T(s) = \frac{K(s)}{s^2+(1+hs)K(s)}$. In matrix form, this means

$$e(s) = \big(- L(s)\mathbf{A} + P(s)\mathbf{B}(s) \big)\, d(s)$$

with the $N \times (N+1)$ matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} ,$$

$$\mathbf{B}(s) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & T(s) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & T(s)^{N-2} & T(s)^{N-3} & \dots & 0 \end{bmatrix} .$$

We first use the triangle inequality to bound

$$\|e(s)\|_2 \quad \leq \quad \big(|L(s)|\,\|\mathbf{A}\|_2 + |P(s)|\,\|\mathbf{B}\|_2\big)\|d(s)\|_2$$

with the induced matrix norms, i.e.

$$\|\mathbf{D}\|_2 = \sqrt{\lambda_{\mathsf{max}}(\mathbf{D}^*\mathbf{D})}$$

with * the complex conjugate transpose. The proof now comes down to proving a bounded norm, independent of N and $s = j\omega$, for the coefficient in front of $\|d\|_2$.

For the first term, since $\mathbf{A}^*\mathbf{A} = \text{diag}(0, 1, 1, 1, ..., 1)$, we immediately have $|L(s)|\,\|\mathbf{A}\|_2 = |L(s)|$, and the latter can be bounded independently of $s = j\omega$ for a stable system.

For the second term, we obtain that the element (m, n) of the matrix $\mathbf{B}^*\mathbf{B}$ equals

$$T(s)^{m-n} \sum_{j=0}^{N-m} |T(s)|^{2j}$$

for $m, n \in \{2, 3, ..., N\}$, $m \geq n$, symmetrically for $n > m$, and zero for the remaining terms. The Gerschgorin circle theorem thus says that all the eigenvalues of $\mathbf{B}^*\mathbf{B}(j\omega)$ are comprised in the circles of respective center and radius

$$\begin{aligned} c^{(m)} &= \sum_{j=0}^{N-m} |T(j\omega)|^{2j} \ , \\ r^{(m)} &= \left(\sum_{j=0}^{N-m} |T(j\omega)|^{2j}\right) \left(\sum_{n=2, n\neq m}^{N} |T|^{|m-n|}\right) . \end{aligned}$$

With a PD controller satisfying *Proposition 5.1*, we have $|T(j\omega)| < 1$ for all $\omega > 0$ and we can bound each sum by the result of an infinite geometric series, provided we investigate the limit of this diverging sum at $\omega = 0$ when multiplied by $P(s)$. This yields

$$\begin{aligned} |P(j\omega)|^2\,\|\mathbf{B}(j\omega)\|_2^2 &\leq |P(j\omega)|^2 \max\nolimits_m (c^{(m)} + r^{(m)}) \\ &\leq \frac{1}{1-|T(j\omega)|^2} \cdot \frac{2}{1-|T(j\omega)|} \cdot |P(j\omega)|^2 \\ &= \frac{|L(j\omega)|^2}{1-|T(j\omega)|^2} \cdot \frac{2|R(j\omega)|^2}{1-|T(j\omega)|} \end{aligned}$$

where $R(s) = \frac{s^2}{(1+hs)(s^2+(1+hs)K(s))}$. Every factor in this expression is bounded at large frequencies, so there just remains to investigate the limit at $\omega = 0$. For the first factor we have

$$\frac{|L(j\omega)|^2}{1-|T(j\omega)|^2} = \frac{1}{|\text{-}\omega^2 + (1+jh\omega)K(j\omega)|^2 - |K(j\omega)|^2} \simeq \frac{1}{a^2}\cdot\frac{1}{h^2\omega^2}$$

for ω close to zero. For the second factor, we have $|R(s)| \simeq \frac{\omega^2}{a}$, while $\frac{1}{1-|T(j\omega)|}$ has a leading term of order $1/\omega$ at low frequencies. Thus in fact $|P(j\omega)|^2 \, \|\mathbf{B}(j\omega)\|_2^2$ is of order ω^4/ω^3 and converges to zero for low ω. This gives a uniform bound on $|P(j\omega)|^2 \, \|\mathbf{B}(j\omega)\|_2^2$ at all frequencies and thus concludes the proof. □

Like for the other positive results in this thesis, we must mention that the PD controller is only a particular, simple example showing that string stability can be achieved. The key in the analysis is the satisfaction of Proposition 5.1, in conjunction with the analysis for $s = j\omega$ close to zero. In this sense, the proof can be repeated and the same result should easily hold for other types of controllers which might be used in practice.

II. Disturbance concentrated on d_0

The chain's reaction to disturbances on the leader is slightly different, and we now show that this precludes the achievement of (L_2, l_2) string stability with any linear controller of bounded DC gain.

Theorem 5.2: *There exists no pair* $(K(s),\ h)$*, with* $h \geq 0$ *a constant time-headway and* $K(s)$ *a stabilizing controller with* $K(0)$ *finite, which would guarantee* (L_2, l_2) *norm string stability of system* (5.1) *when* $||\ d_0(t)\ ||_2 \neq 0$.

Proof: We consider only a disturbance input d_0 that affects the leading vehicle, which leads to

$$\begin{aligned} e_1 &= \frac{1}{s^2+(1+hs)K(s)} d_0 \\ e_k &= T(s)^{k-1}\frac{1}{s^2+(1+hs)K(s)} d_0 \quad , \quad k \in \{2,3,...,N\} \end{aligned} \tag{5.2}$$

with still $T(s) = \frac{K(s)}{s^2+(1+hs)K(s)}$. Then

$$\sum_{k=1}^{N} |e_k(s)|^2 = \sum_{k=0}^{N-1} |\ T(s)\ |^{2k} \cdot \frac{|d_0(s)|_2^2}{|s^2+(1+hs)K(s)|^2} \ .$$

Take some $\beta > 0$ and define $\alpha > 0$ such that $|s^2 + (1+hs)K(s)|^2|_{s=j\omega} < \alpha$ for all $\omega \in (-\beta, \beta)$. Now select any $\epsilon \in (0, \beta)$, and take an input disturbance concentrated at low frequencies such that

$$\int_{-\epsilon}^{\epsilon} |d_0(j\omega)|^2 d\omega \geq \frac{1}{2} \int_{-\infty}^{+\infty} |d_0(j\omega)|^2 d\omega \,.$$

Then

$$\begin{aligned} \|e(.)\|_2^2 &\geq \int_{-\epsilon}^{\epsilon} \sum_{k=0}^{N-1} |T(s)|^{2k} \frac{|d_0(s)|^2}{|s^2 + (1+hs)K(s)|^2}\Bigg|_{s=j\omega} d\omega \\ &\geq \frac{\|d_0(.)\|_2^2}{2} \frac{1}{\alpha} \quad \min_{\omega \in (-\epsilon, \epsilon)} \sum_{k=0}^{N-1} \mid T(j\omega) \mid^{2k} \quad . \end{aligned} \tag{5.3}$$

Since $T(0) = 1$, for any given $K(s)$ and h and any $\delta > 0$, there will always exist an ϵ such that $\min_{\omega \in (-\epsilon, \epsilon)} |T(j\omega)|^2 > 1 - \delta$. As δ can tend towards 0 and N towards infinity, the geometric sum in the second line of (5.3) then cannot be bounded independently of N. ☐

Theorem 5.2 implies the impossibility to achieve (L_2, l_2) string stability in the general case, and in all practical cases where disturbances are expected *at least* on the leading vehicle. As we mentioned earlier, this might explain why results in the literature are restricted to L_2 string stability, because the next-most popular setting would indeed be (L_2, l_2) with bounded controllers $K(s)$.

Luckily, there are two possible workarounds for this negative result. A first one is to allow $K(s)$ with unbounded DC gain, like a PID controller. This might require to investigate other effects more carefully though, as unmodeled measurement noises or saturation effects could seriously deteriorate the situation. Another solution is to recognize that (L_2, l_∞) string stability might be a satisfactory achievement in practice. Indeed, for the latter case, we have the positive result that we present next.

5.1.3 Satisfying (L_2, l_∞) String Stability with PD controller

We now turn to the positive part of the results, first showing how one does guarantee string stability in the (L_2, l_∞) sense using a PD controller with time headway, for all bounded disturbances d.

Theorem 5.3: *There exists a pair* $(K(s),\ h)$*, where* $h \geq 0$ *is a sufficiently large constant time-headway satisfying Proposition 5.1 and* $K(s) = bs + a$ *is a stabilising PD controller, that achieves* (L_2, l_∞) *string stability.*

Proof: Again the *stability* is easy to achieve with PD controllers as they lead to second-order polynomials in the denominators, also with time headway.

For *string stability*, consider the worst case where there are disturbance inputs satisfying $\|d_k\| = \delta$ on all the vehicles $k \in \{0, 1, ..., N\}$. From (5.1), we have

$$\begin{aligned} \|e_k\| &\leq \max_{s=j\omega} \left| (\sum_{m=2}^{k} T^{k-m} P + T^{k-1} L - L)(s) \right| \delta \\ &\leq \max_{s=j\omega} \left(\sum_{m=2}^{k} |T^{k-m} P| + |T^{k-1} L| + |L| \right) \delta \end{aligned} \tag{5.4}$$

where $L(s) = \frac{1+hs}{s^2+(1+hs)K(s)}$, while $T(s) = \frac{K(s)}{1+hs} L(s)$ and $P(s) = \frac{s^2}{(s^2+(1+hs)K)^2}$ as before. By satisfying *Proposition 5.1*, we know that $\mid T(j\omega) \mid < 1$ for all $\omega > 0$, and then $|L(s)| = |T(s)| \cdot |1 + hs| / |K(s)| < 1/a$ for all $s = j\omega$ with the PD controller. The last two terms in (5.4) are thus bounded independently of k and of N.

For the remaining term, we have

$$\sum_{m=2}^{k} |T^{k-m}(j\omega) P(j\omega)| \leq \frac{1}{1 - |T(j\omega)|} \cdot |P(j\omega)| \tag{5.5}$$

$$= \frac{1}{| -\omega^2 + (1 + hj\omega) K(j\omega) \mid - \mid K(j\omega) \mid} \cdot \frac{\mid -\omega^2 \mid}{\mid -\omega^2 + (1 + hj\omega) K(j\omega) \mid}.$$

We first check its behavior at low frequencies. By Taylor expansion we find $\frac{1}{|-\omega^2+(1+j\omega h)K(j\omega)|-|K(j\omega)|} \simeq \frac{1}{\omega a}$ and $\mid \frac{-\omega^2}{-\omega^2+(1+hj\omega)K(j\omega)} \mid \simeq \frac{\omega^2}{a}$. For $\omega = 0$ thus, (5.5) converges to 0. At low frequencies $\omega > 0$, the deviation from 0 in the right-hand side of (5.5) is independent of k and of N, and this provides a bound independent of k and N for the left-hand side. For any given controller satisfying *Proposition 5.1*, it is thus straightforward to identify some $\omega_0 > 0$ such that $\sum_{m=2}^{k} |T^{k-m}(j\omega) P(j\omega)| < 1/a$ for instance (this value is chosen comparable to the other term $|L(s)|$), for all $\omega \in (-\omega_0, \omega_0)$. There remains to prove that the same term remains bounded independently, of k and N, for all $\omega > \omega_0$. With the proposed PD controller, for any $\omega_0 > 0$, there exists $\alpha < 1$ such that $\mid T(j\omega) \mid \leq \alpha$ for all $\omega > \omega_0$; this is checked for instance by ensuring a monotone decreasing Bode amplitude diagram, as illustrated in simulations below. Then we have, for all $\omega > \omega_0$, a uniform bound on $\frac{1}{1-|T(j\omega)|} < \frac{1}{1-\alpha}$ and also on $|P(j\omega)| = |T(j\omega)|^2 \cdot |\omega / K(j\omega)|^2$. Together, all this provides a uniform bound on the first term of (5.4) and concludes the proof. □

Like for the other positive results in this thesis,the PD controller is only a particular simple example showing that string stability can be achieved. The proof can be easily repeated for other types of controllers that would satisfy Proposition 5.1. The main analysis for such other cases would require to check stability; ensure a similarly acceptable behavior for $s = j\omega$ close to zero; and check the Bode diagram to be bounded away from 1 for $s = j\omega$ away from zero.

5.1.4 Satisfying (L_2, l_2) string stability with a PID controller

In the proof of *Theorem 5.2*, the prior bound α plays a key role in establishing the existence of a "bad" disturbance profile. This points to a possible solution for string stability by using an unbounded $K(0)$, as appears in integral control. We here show that indeed, combining this with the time-headway policy, one can ensure (L_2, l_2) norm string stability. We in fact give a particular solution which will be illustrated by simulation in a later section.

We must now consider errors on all vehicles and, recognizing the special occurrences of d_0 and d_k for subsystem k, we rewrite (5.1) as

$$\begin{aligned}
e_1 &= L(s)(d_0/(1+hs) - d_1) \\
e_k &= L(s)(-d_k) + T(s)^{k-1}L(s)d_0/(1+hs) + \\
&\quad \sum_{m=2}^{k} T(s)^{k-m}P(s)d_{m-1}\,, \quad \text{with} \\
P(s) &= \frac{s^2}{(s^2+(1+hs)K(s))^2} \\
L(s) &= \frac{1+hs}{s^2+(1+hs)K(s)} \\
T(s) &= \frac{K(s)}{s^2+(1+hs)K(s)}\,.
\end{aligned}$$

Note that applying the filter $1/(1+hs)$ to the disturbance d_0 just comes down to an attenuation of its high-frequency components, so to simplify the following we can redefine $d(s) = col[d_0/(1+hs),\ d_1,\ d_2,\ ...,\ d_N]$; indeed this new $d(s)$ will automatically be bounded in (L_2, l_2) and (L_2, l_∞) norms when the original input disturbance is. Then in matrix form, we have

$$e(s) = \big(-L(s)\mathbf{A} + L(s)\mathbf{B}(s) + P(s)\mathbf{C}(s)\big)\,d(s)$$

with the $N \times (N+1)$ matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

$$\mathbf{B}(s) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ T(s) & 0 & 0 & \dots & 0 \\ T(s)^2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ T(s)^{N-1} & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$\mathbf{C}(s) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & T(s) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & T(s)^{N-2} & T(s)^{N-3} & \dots & 0 \end{bmatrix}.$$

Theorem 5.4: *Using a stabilizing PID controller $K(s) = c/s + bs + a$ with time-headway h satisfying Proposition 5.1, we can ensure (L_2, l_2) string stability for any disturbances d.*

Proof: In other words, we must prove that we can tune the gains such that the system is stable and we can guarantee $\| e(.) \|_2 < C_0 \, \| d(.) \|_2$ in which the constant C_0 is bounded independently of number of vehicles N.

In the further analysis, we will impose no particular tuning values to h and to the parameters of the PID controller. To satisfy stability, it is thus sufficient to find a PID controller and h which make the system stable while fulfilling the conditions of Proposition 5.1. Considering the first criterion in Proposition 5.1, we will thus fixe some tuning of the polynomial $\bar{K}(s)$ which makes the system stable (just checking always the same denominator $s^3 + s(1 + h\,s)\,K(s) = s^3 + s\,\bar{K}(s)$). Once $\bar{K}(s)$ has been selected, we would then choose h according to the related criterion, while adapting the other parameters in order to maintain $\bar{K}(s)$ fixed as selected. For this to be possible, the only essential element is to prove that the first criterion in Proposition 5.1 always remains bounded for a stable PID controller.

We thus consider $s^3 + s\,\bar{K}(s)$ to be any third-order polynomial with roots in the open left half plane. Then in the criterion,

$$\frac{\bar{R}}{1+\bar{R}} = \frac{s\,\bar{K}(s)}{s^3 + s\,\bar{K}(s)}$$

remains bounded for all $s = j\omega$ and we must only investigate the behavior for ω close to 0. From the inverse triangle inequality $|\frac{\bar{R}}{1+\bar{R}}|^2 - 1 \leq |(\frac{\bar{R}}{1+\bar{R}})^2 - 1|$ a sufficient criterion for Proposition 5.1(a) is

$$h > \sqrt{\left|\frac{(j\omega)^2\bar{K}^2(j\omega) - [(j\omega)^3 + (j\omega)\bar{K}(j\omega)]^2}{\omega^2[(j\omega)^3 + (j\omega)\bar{K}(j\omega)]^2}\right|}\,,$$

which just comes down to

$$h > \sqrt{\left|\frac{\omega^4 - 2\omega^2 K(j\omega)}{(j\omega\bar{K}(j\omega) - j\omega^3)^2}\right|}\,.$$

For ω close to 0 and $K(j\omega)$ a PID controller, the dominating term is $h > \sqrt{|2\omega\, k_I/k_I^2|}$, with k_I the integral gain. This imposes a bounded constraint on h and it is thus possible indeed to satisfy stability and the criterion of Proposition 5.1 simultaneously with a PID controller.

Towards proving string stability, the proof proceeds similarly to Theorem 5.1. We first use the triangle inequality to bound

$$\begin{aligned}\|e(s)\|_2 \;\; &\leq \;\; \big(|L(s)|\,\|\mathbf{A}\|_2 + |L(s)|\,\|\mathbf{B}(s)\|_2 \\ &\qquad +|P(s)|\,\|\mathbf{C}(s)\|_2\,\big)\;\|d(s)\|_2\,,\end{aligned}$$

with the induced matrix norms, i.e.

$$\|\mathbf{D}\|_2 = \sqrt{\lambda_{\max}(\mathbf{D}^*\mathbf{D})}$$

with * the complex conjugate transpose. The proof now comes down to proving a bounded norm, independent of N and $s = j\omega$, for each of the three terms in the matrix sum.

For the first term, since $\mathbf{A}^*\mathbf{A} = \text{diag}(0, 1, 1, 1, ..., 1)$, we immediately have $|L(s)|\,\|\mathbf{A}\|_2 = |L(s)|$, and the latter can be bounded independently of $s = j\omega$ for a stable system.

For the second term, we have

$$\begin{aligned}\mathbf{B}^*\mathbf{B} \;\; &= \;\; \text{diag}\big(\sum_{k=0}^{N-1} |T(s)|^{2k},\; 0,\; 0, ..., 0\big) \\ &= \;\; \text{diag}\big(\frac{1 - |T(s)|^{2N}}{1 - |T(s)^2|},\; 0,\; 0, ..., 0\big)\,.\end{aligned}$$

Under the conditions of *Proposition 5.1*, the numerator is lower than 1 and

$$|L(j\omega)|\,\|\mathbf{B}(j\omega)\|_2 \leq \sqrt{\frac{|L(j\omega)|^2}{1-|T(j\omega)|^2}}\,,$$

up to a possibly diverging result at $\omega = 0$ where $|T(j\omega)| = 1$. For this behavior at $\omega = 0$, the unbounded DC gain here plays an essential role by making $L(s)$ converge to zero as well. Indeed, writing

$$\begin{aligned}\frac{|L(j\omega)|^2}{1-|T(j\omega)|^2} &= \frac{1}{|\text{-}\omega^2 + (1+jh\omega)K(j\omega)|^2 - |K(j\omega)|^2} \\ &= \frac{1}{|K(j\omega)|^2}\,\frac{1}{\left|\frac{-\omega^2}{K(j\omega)} + (1+jh\omega)\right|^2 - 1}\,,\end{aligned}$$

we clearly recover that for $K(0)$ finite the right hand side cannot be bounded in a neighborhood of $\omega = 0$, while for $K(0)$ unbounded it may be. More precisely, with a PID controller where $K(j\omega) \simeq jc/\omega$ for ω close to zero, the term $\frac{-\omega^2}{K(j\omega)}$ becomes negligible and we have

$$\frac{|L(j\omega)|^2}{1-|T(j\omega)|^2} \simeq \frac{\omega^2}{c^2}\cdot\frac{1}{h^2\omega^2} = \frac{1}{c^2h^2}$$

i.e. the limit for $\omega \to 0$ of $|L(j\omega)|\,\|\mathbf{B}(j\omega)\|_2$ is bounded, independently of N. It is then easy to find a bound that is valid independently of N and at all frequencies ω, for a given PID controller and h satisfying *Proposition 5.1*.

For the third term, we obtain that the element (m,n) of the matrix $\mathbf{C}^*\mathbf{C}$ equals

$$T(s)^{m-n}\sum_{j=0}^{N-m}|T(s)|^{2j}$$

for $m,n \in \{2,3,...,N\}$, $m \geq n$, symmetrically for $n > m$, and zero for the remaining terms. The Gerschgorin circle theorem thus says that all the eigenvalues of $\mathbf{C}^*\mathbf{C}(j\omega)$ are comprised in the circles of respective center and radius

$$\begin{aligned}c^{(m)} &= \sum_{j=0}^{N-m}|T(j\omega)|^{2j}\,, \\ r^{(m)} &= \left(\sum_{j=0}^{N-m}|T(j\omega)|^{2j}\right)\left(\sum_{n=2,n\neq m}^{N}|T|^{|m-n|}\right).\end{aligned}$$

Again for $|T(j\omega)| \leq 1$ we can bound each sum by the result of an infinite geometric series, provided we investigate the limit at $\omega = 0$ when the factor $P(s)$ is included. This yields

$$\begin{aligned} |P(j\omega)|^2 \, \|\mathbf{C}(j\omega)\|_2^2 \leq |P(j\omega)|^2 \max\nolimits_m (c^{(m)} + r^{(m)}) \\ \leq \frac{1}{1-|T(j\omega)|^2} \cdot \frac{2}{1-|T(j\omega)|} \cdot |P(j\omega)|^2 \\ = \frac{|L(j\omega)|^2}{1-|T(j\omega)|^2} \cdot \frac{2|R(j\omega)|^2}{1-|T(j\omega)|} \end{aligned}$$

where $R(s) = \frac{s^2}{(1+hs)(s^2+(1+hs)K(s))}$. We have already shown in the previous paragraph that the first factor is bounded, independently of ω and N. The second factor must likewise be investigated for ω close to 0, as it is trivially bounded at other frequencies. For low frequencies, we get $|R(s)| \simeq \frac{hc}{c/\omega} = h\omega$, while $\frac{1}{1-|T(j\omega)|}$ has a leading term of order $1/\omega$. Thus in fact $|P(j\omega)|^2 \, \|\mathbf{C}(j\omega)\|_2^2$ converges to zero for $\omega \to 0$ and our proof is concluded. □

Note that from the computations in the proof, the disturbance d_0 (captured by $\mathbf{B}$) is the only dominant one at low frequencies. This is in agreement with the observations of the previous sections in this Chapter, and with the fact that the literature has focused attention on rejecting disturbances that act on the leader.

So, using stabilizing PID controllers with time-headway policy it is possible to guarantee (L_2, l_2) string stability with respect to disturbances acting on the leading vehicle and on other vehicles as well, thus in the general sense of Definition 2. Again, we wish to mention that the PID controller is chosen mainly to provide a concrete example that works; it should be clear how the analysis relies only on a few key points which can be repeated for other controllers.

5.1.5 Satisfying (L_2, l_∞) String Stability with PID controller

For (L_2, l_∞) string stability, we have already shown previously that a PD controller is sufficient. For completeness, and acknowledging at least once explicitly that further design considerations could motivate a controller other than a PD type, we next prove that also a PID controller allows to achieve (L_2, l_∞) string stability with time-headway, for all bounded disturbances d.

Theorem 5.5: *There exists a pair $(K(s),\ h)$, where $h \geq 0$ is a sufficiently large constant time-headway satisfying Proposition 5.1 and $K(s) = c/s+bs+a$ is a stabilizing PID controller, that achieves (L_2, l_∞) string stability.*

Proof: For the stability the proof is exactly equal to the (L_2, l_2) case of course.

For string stability, the proof follows similar lines as Theorem 5.3 (the one with the PD controller). Explicitly, consider the worst case where there are disturbance inputs satisfying $\|d_k\| = \delta$ on all the vehicles $k \in \{0, 1, ..., N\}$, leading to the expression (5.4), just with K now a PID controller. The last two terms in (5.4) are bounded independently of k and of N in the same way as for the PD controller. There remains to consider the term corresponding to (5.5), where now $\frac{1}{|-\omega^2+(1+j\omega h)K(j\omega)|-|K(j\omega)|} \simeq \frac{1}{hc}$ and $| \frac{-\omega^2}{-\omega^2+(1+hj\omega)K(j\omega)} | \simeq \frac{\omega^3}{c}$. For $\omega = 0$ thus, (5.5) converges to 0. From here we can follow the steps of Theorem 5.3 verbatim. At low frequencies $\omega > 0$, the deviation from 0 in the right-hand side of (5.5) is independent of k and of N, and this provides a bound independent of k and N for the left-hand side. For any given controller satisfying *Proposition 1*, it is thus straightforward to identify some $\omega_0 > 0$ such that $\sum_{m=2}^{k} |T^{k-m}(j\omega)P(j\omega)| < 1/a$ for instance (this value is chosen comparable to the other term $|L(s)|$), for all $\omega \in (-\omega_0, \omega_0)$. There remains to prove that the same term remains bounded independently, of k and N, for all $\omega > \omega_0$. With the proposed PID controller, for any $\omega_0 > 0$, there exists $\alpha < 1$ such that $| \ T(j\omega) \ | \leq \alpha$ for all $\omega > \omega_0$; this is checked for instance by ensuring a monotone decreasing Bode amplitude diagram. Then we have, for all $\omega > \omega_0$, a uniform bound on $\frac{1}{1-|T(j\omega)|} < \frac{1}{1-\alpha}$ and also on $|P(j\omega)| = |T(j\omega)|^2 \cdot |\omega/K(j\omega)|^2$. Together, all this provides a uniform bound on the first term of (5.4) and concludes the proof. □

5.2 Cooperative Adaptive Cruise Control (CACC)

In the structure of Cooperative Adaptive Cruise Control (CACC) [36–40], we assume that the message sent by vehicle k to its follower $k+1$ is a filtered version of the input command $u_k^{'}$ as mentioned in (2.7).

Computing the dynamics of e_k from the one of x_k that is defined as $e_k = x_{k-1} - x_k - hsx_k$, and defining $z_k = [e_k \ ; \ v_{k-1} - v_k]$, we get the closed-loop dynamics described by:

$$z_{k+1} \quad = \quad \mathbf{T}(s)\, z_k \ + \left[\begin{array}{c} \frac{1}{s^2+K(1+hs)} \\ \frac{1}{B} \cdot \frac{-K(1+hs)}{s^2+K(1+hs)} \end{array} \right] (d_k - (1+hs)d_{k+1}) \qquad (5.6)$$

in which

$$\mathbf{T}(s) = \left[\begin{array}{cc} \frac{K}{s^2+K(1+hs)} & \frac{HW}{s^2+K(1+hs)} \\ \frac{K}{B} \cdot \frac{s^2}{s^2+K(1+hs)} & \frac{HW}{B} \cdot \frac{s^2}{s^2+K(1+hs)} \end{array} \right] .$$

This takes the form of an iteration for the propagation of disturbances from vehicle k to vehicle $k+1$, so for each frequency $s = j\omega$ we must investigate the stability of the matrix $\mathbf{T}(j\omega)$.

We already proved in Chapter 3 that it is not possible to guarantee string stability using CACC without time-headway policy, $h = 0$.

In the following Proposition, we give a sufficient result in order to obtain the minimum time-headway requirement in CACC. This generalizes the result of [43] reported in Proposition 5.1, to the case with communication. It may seem rather direct after reading the other elements of our thesis, in particular the formulation as (5.6). However, at first sight this formulation may not have been evident. Indeed in the literature so far, the benefit of adding CACC to time-headway systems was only investigated in full-system simulations, by trial-and-error with various values of h. The below criterion only requires to plot a single Bode diagram in order to identify the minimal requirement for the time-headway. In particular, by comparing it to Proposition 5.1, one can explicitly identify the benefit of CACC in this criterion.

Proposition 5.2: *The norm at $s = j\omega$ of transfer function $\mathbf{T}(s)$ in (5.6) is < 1 at all frequencies $\omega \neq 0$, and its H_∞ norm equals $\mathbf{T}(0) = 1$, if and only if the following condition holds:*
Choosing $\bar{K}(s) = K(s)(1+hs)$ and $\frac{\bar{H}\bar{W}}{\bar{B}}(s) = \frac{HW}{B}(s)(1+hs)$ first and then deriving $K(s)$, $HW(s)$ from h, then we should ensure that h satisfies

$$h > \sqrt{\max_{\omega} \frac{\left|\frac{\bar{N}(j\omega)+\bar{M}(j\omega)}{1+\bar{N}(j\omega)}\right|^2 - 1}{\omega^2}}$$

in which $\bar{N}(s) = \bar{K}(s)/s^2$ and $\bar{M} = \frac{\bar{H}\bar{W}}{\bar{B}}(s)$.

Proof:

The key step is to notice that $\mathbf{T}(s)$ is a singular matrix for all s, since the right column equals HW/K times the left column. Thus $\mathbf{T}(s)$ has one zero eigenvalue, which would robustly ensure string stability; and the single nonzero eigenvalue of $\mathbf{T}(s)$ equals its trace,

$$\text{trace}(\mathbf{T}(s)) = \frac{K + \frac{HW}{B}s^2}{K(1+hs)+s^2} \; .$$

We reformulate $\text{trace}(\mathbf{T}(s)) = \frac{1}{1+hs} \frac{\bar{K}(s)/s^2 + \frac{\bar{H}\bar{W}}{\bar{B}}}{1+\bar{K}(s)/s^2}$. Then writing

$$|\text{trace}(\mathbf{T}(s))|^2 = \frac{1}{1+\omega^2 h^2} \left|\frac{\bar{M}(j\omega)+\bar{N}(j\omega)}{1+\bar{N}(j\omega)}\right|^2 < 1 \quad \text{for all } \omega \neq 0$$

directly yields the expression. The expression inside the square root will necessarily have a positive value, because we have proved in Chapter 3 that with CACC but no time-headway, a unidirectional controller will not be able to avoid amplifying some disturbance signals (Theorem 3.1). □

5.3 Simulation results

We now illustrate the various results of this chapter. To compute the minimal time headway requirement, we will use the form as in Proposition 5.2, where first we specify fixed target transfer functions with $(1+hs)$ implicitly included, and then we derive what the actual controller should be when removing the effect of $(1+hs)$. We thus fix $\bar{K}$ first and then derive K from it; or, we fix $\bar{H}\bar{W}/\bar{B}$ first and then derive HW/B from it. For the latter situation, in a theoretical framework, there is no unique way to specify the factors H, B, W, so we just arbitrarily split them according to transfer functions which appear reasonable for each element, e.g. incorporating a delay in the channel $W(s)$ and $\bar{W}(s)$.

We first consider a PD controller $\bar{K} = s + 1$ and compute the minimal $h = \sqrt{2.32}$ according to Proposition 5.1 without communication, and $h = \sqrt{1.9}$ according to Proposition 5.1 with CACC (same PD parameters but adding communication with $\bar{H}(s) = 1/\bar{B}(s) = \frac{1}{s+1}$ and $\bar{W}(s) = \frac{1}{s+1}e^{-0.01s}$, i.e. including a time delay of 0.01). So, CACC can lead to a smaller time-headway constant h compared to controller (2.3), using the same control gain $\bar{K}$. Figure 5.6 illustrates this point, by showing the full frequency response of the transfer functions associated to the computation of the minimal value of h in the cases with (red) and without (blue) communication.

According to our results, the PD controller should be able to satisfy L_2 and (L_2, l_∞) string stability, but not (L_2, l_2) string stability. In simulations, we can first illustrate that a PD type controller does not achieve (L_2, l_2) string stability; although of course this is only a particular example, not really an illustration of impossibility. In this sense, Figure 5.1 illustrates the (L_2, l_2) norm of the error functions grows unbounded when increasing the number of vehicles, with a disturbance acting on the leading vehicle. Using the same controllers with the disturbance inputs acting on the second and third vehicles only, Figure 5.2 illustrates (L_2, l_2) norms of the vector of the error functions converge to a constant value; this confirms the particular distinction that we have made about disturbances acting on the leader or elsewhere (Theorems 5.1 and 5.2). Figure 5.3 finally shows how individual errors remain bounded when bounded input errors are applied to all the vehicles, thus illustrating how a PD controller allows to satisfy (L_2, l_∞) string stability in a controller with time-headway and unidirectional coupling.

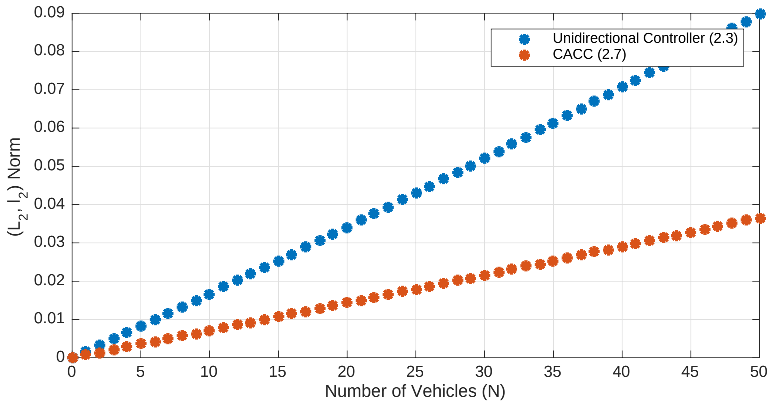

Figure 5.1: (L_2, l_2) norm of the vector of error functions for a disturbance acting on the leading vehicle, as a function of N.

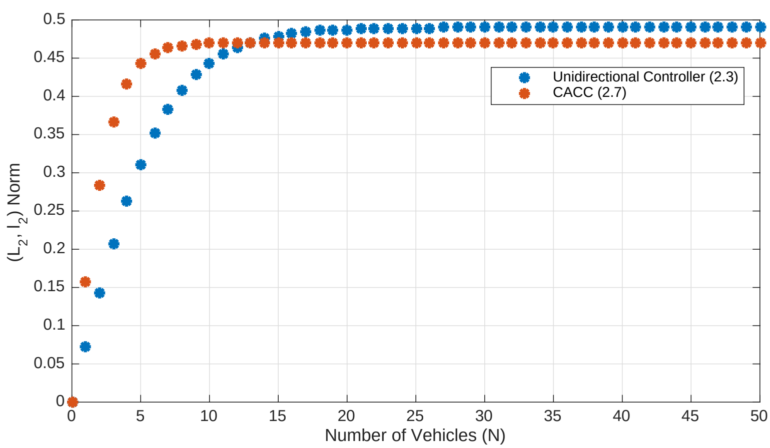

Figure 5.2: (L_2, l_2) norm of the vector of error functions for disturbances not acting on the leading vehicle, as a function of N.

We next turn to the PID controller. We choose $\bar{K}(s) = 1 + 2s + 1/s$ for the controllers (2.3) and (2.7), with the same communication model as before. Using *Proposition 5.1* and *Proposition 5.2* the time-headway requirement is equal to $\sqrt{2.1}$ and $\sqrt{1.78}$ for the controllers (2.3) and (2.7), respectively. Figure 5.4 shows the (L_2, l_2) norm of the vector of error functions, when disturbances are applied on the leading, second and third vehicles. This norm

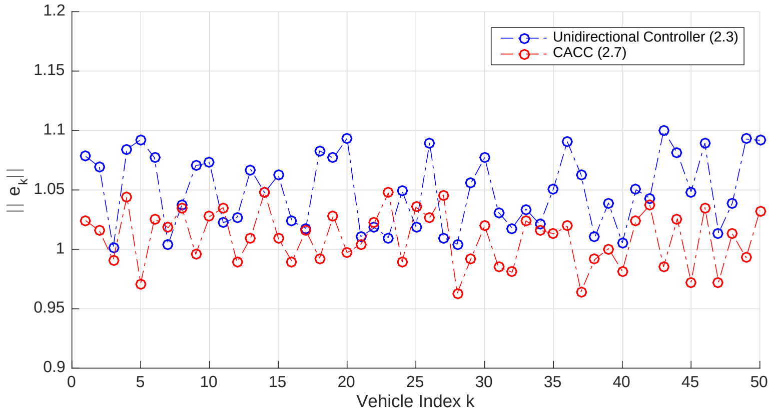

Figure 5.3: L_2 norm of error functions $\|\ e_k\ \|_2$ as a function of k, for disturbances acting on all the vehicles, as a function of N, using PD controller. The (L_2, l_∞) string stability criterion for chain length N is given by $\|\ e(.)\ \|_\infty = max_{k \leq N}(\|\ e_k\ \|_2)$, so this illustrates the possibility of (L_2, l_∞) string stability.

converges to a constant value, unlike on Figure 5.1 for the PD controller; this illustrates the possibility of (L_2, l_2) string stability using PID control. Figure 5.5 illustrates the (L_2, l_∞) criterion for the same PID controllers, with and without communication. Note that in presence of communication, we have chosen a *lower value of* h instead of trying to improve some performance criterion. So one should not attribute real importance to the slightly larger effect of disturbances that can be observed on the figures, for low number of vehicles with communication but thus also with smaller h.

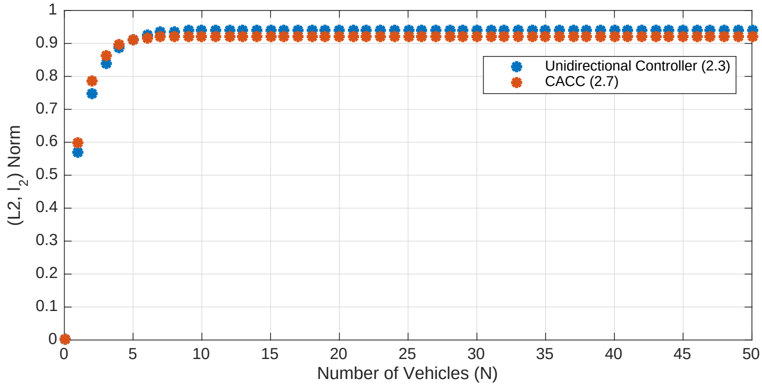

Figure 5.4: (L_2, l_2) norm of the vector of error functions for disturbances acting on the leading, second and third vehicles, as a function of N, using PID control.

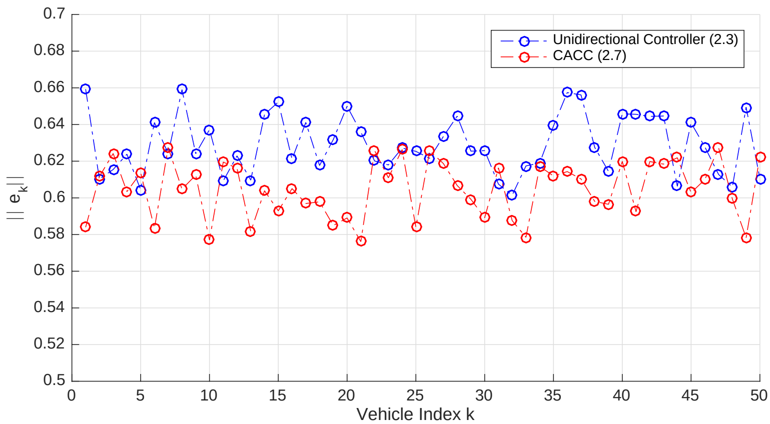

Figure 5.5: L_2 norm of error functions $\|\ e_k\ \|_2$ as a function of k, for disturbances acting on all the vehicles, as a function of N, using PID controller. The (L_2, l_∞) string stability criterion for chain length N is given by $\|\ e(.)\ \|_\infty = max_{k \leq N}(\|\ e_k\ \|_2)$, so this illustrates the possibility of (L_2, l_∞) string stability.

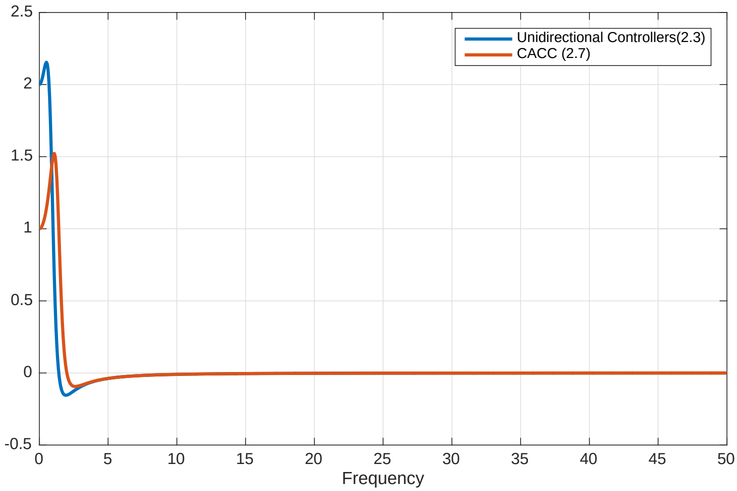

Figure 5.6: The red and blue lines show the frequency responses for the transfer functions $\frac{|\frac{\bar{N}(\omega)+\bar{M}(\omega)}{1+\bar{N}(\omega)}|^2-1}{\omega^2}$ and $\frac{|\frac{\bar{R}(\omega)}{1+\bar{R}(\omega)}|^2-1}{\omega^2}$, respectively, used in the computation of the minimal time-headway h, for a PD controller.

Chapter 6

Conclusions

The true delight is in the finding out rather than in the knowing.
Isaac Asimov

This thesis has investigated how to control the relative distances between automated/cooperative driving vehicles, in particular when such vehicles interact in very long chains. Such tightly packed chains would be a powerful way to mitigate congestion problems with clear benefits in traffic flow, fuel economy and air pollution. However, ensuring that safety, stability and control performance do not deteriorate unboundedly with increasing chain length, is a surprisingly challenging problem, which has been formalized as "string stability" a few decades ago. The focus of this thesis has been i) to identify more precisely situations and controllers which can guarantee string stability for vehicle platoons and ii) to narrow down the different control options thanks to a more precise understanding of impossibilities to satisfy string stability.

Towards more general design of intelligent transportation systems, the expected impact of such a theoretical study of string stability is likewise, to helps researchers in the area to narrow down the field of investigation when constructing control architectures which, among other requirements, may likely include string stability as a specification.

An advantage of the simplified academic model of our theoretical study, is that the general picture should also be valid for other applications and give useful insight there. For instance, string stability can be used to investigate the reaction of buildings to earthquakes [19, 20]. If not as a truly practical goal for this application, the insights provided in the context of string stability like in the present thesis, can at least serve as a guiding principle on particular design issues, or inspire new ideas for better rejecting disturbances. The results with disturbance restricted to the leader for instance, appear meaningful for stabilizing buildings more than for vehicle chains; in contrast, obtaining asymmetric coupling between different levels may require specific designs in a mechanical structure instead of a control algorithm. Another extension which should follow similar principles is towards subsystems interacting according to a lattice structure, instead of just a chain. Some of our conclusions, e.g. impossibility results, should not be too hard to extend to this setting. Lattices are an ubiquitous model for systems from microscopic to the most macroscopic scales. In this sense, the following conclusions will hopefully help inspire diverse working directions.

6.1 Chapter 1

In Chapter 1, we have briefly discussed how the academic concept of string stability is related to real-world problems and the benefits of more coordinated, smarter control in transport networks.

6.2 Chapter 2

In Chapter 2, we have motivated the basic setting of a second-order integrator model for an isolated vehicle, and presented different approaches to design controllers towards coordinating them in a vehicle chain. We distinguish linear/nonlinear settings, homogeneous or heterogeneous strings, allowing communication or not, and various coupling strategies – e.g. reacting only to preceding vehicles, or reacting both to preceding and to following vehicles. The most important aspect however, in the context of this thesis, is the type of sensing. The most fundamental sensors in long chains would measure relative properties between consecutive vehicles, e.g. distances between cars. On the basis of solely this information, i.e. without relying on measurements with respect to a common external reference, satisfying string stability appears to be particularly hard, as we show in the following chapters. Therefore, another line of work has investigated how to control the system when additional sensors reliably measure the absolute velocity of each vehicle with respect to a common reference frame, e.g. the road for cars. Such measurements allow to implement a time-headway spacing policy, in which the desired distances between the vehicles depend on absolute velocity; this somewhat simplifies the control task towards enabling string stability. Furthermore, we have highlighted that the existing literature about string stability has considered different measures of disturbances. We have hence presented three different definitions of string stability in continuous-time and discrete-time settings, which will be considered in the other chapters of thesis to derive the possibility and impossibility results. The basic target is always to keep disturbances on the relative distances between consecutive vehicles bounded, in the limit where the chain becomes infinitely long. The definitions investigated in this thesis all consider the standard 2-norm of signals over time, but differ in the way they consider vehicles, individually or as a sum. The main motivation for the 2-norm over time seems to have been theoretical, namely its straightforward reformulation in frequency domain, rather than resulting from deep practical considerations. This is also mentioned in [28] where other norms are considered for string stability, including the L_∞ norm over time. Such BIBO type criterion would appear to us as the practically most relevant measure, but has not been addressed directly in the literature. Now, given the various conclusions that we have found by clarifying the situation for the various definitions of string stability in this thesis, we do believe that one should be more careful about choosing one type of norm as a proxy for another one on this apparently sensitive problem. Hence, a targeted study of the BIBO criterion, even if requiring other tools than the popular frequency-domain energy spectrum, now seems worth carrying out specifically in future work. We must mention however that the strongest impossibility result in Chapter 3, also holds for the BIBO criterion.

With respect to differences in the behavior of infinite-length platoon models compared to the limit of a finite-length platoon as it becomes increasingly long [42], we have to mention that we have always been working in the latter situation. I.e., no conclusions were drawn explicitly starting from a model with infinite platoon length, so we are properly investigating the limit behavior of a finite chain that becomes increasingly long. In this sense, we believe to be properly capturing what happens for finite but long chains.

6.3 Chapter 3

In Chapter 3, we give the first results of the thesis, significantly extending the scope of an impossibility result regarding string stability towards disturbances acting on all subsystems. A main point was to show that communication between vehicles is not a sufficient ingredient to achieve string stability. Introducing time headway or a combination of communication and time headway is beneficial, as seen in Chapter 5 and observed to some extent in the existing literature; but communication on its own does not succeed, even when extending the communication scheme with respect to the existing impossibility result of [36]. The obtained results are summarized in the following table:

Table 6.1: Satisfying string stability with **disturbances on all vehicles** and **no time-headway ($h = 0$)**. We only list our generalizations with respect to existing results.

Generalized CACC communication (2.7), unidirectional coupling; any definitions	IMPOSSIBLE
General linear scalar communication (2.9), unidirectional coupling; any definitions	IMPOSSIBLE
General vector communication , (2.9), finite $K(0)$ unidirectional coupling; (L_2, l_2)	IMPOSSIBLE
Sensor dynamics breaking, relative position symmetry unidirectional coupling; any definitions	IMPOSSIBLE
Any Digital Controller (2.12), possibly nonlinear, communicating, bidirectional; any definitions	IMPOSSIBLE

We here want in particular to highlight our strongest result, Theorem 3.4, which has been obtained in a discrete-time setting but should give strong indications for all practical purposes about continuous-time design too. Indeed, while existing results have centered on LTI systems, we show that it is impossible to achieve string stability even if we allow controllers to be nonlinear, N-dependent, time-varying, thus possibly modulated, digitally quantized,... as well as communicating locally. The analysis has involved no complicated elements once the setting and example have been identified, but as the search for alternative controllers had remained open so far, it appears to give a definite answer clearly narrowing down the options towards achieving string stability. Essential features for the impossibility are:

- Second-order integrator model for individual subsystems: if the dynamics was first-order, our counterexample would not work, and indeed working controller designs are known in continuous-time;

- Relative measurements: variations that do solve string stability by adding an absolute velocity term are known, see e.g. time-headway policies. With respect to this criterion, academically, string instability appears more than ever as a property of distributed sensing. In practice, the presence of absolute velocity in the feedback controller or damping becomes a question of hardware and application tradeoff. We have to mention that the string instability issue is *not* directly linked to the low observability for long-range modes in distributed systems with relative measurements [31]. Indeed, here the target variables are not the absolute displacements x_k, for which indeed there would be an observability issue, but rather the relative displacements e_k, which are directly measured. Also see the previous point (second vs. first order integrator), which plays no role in observability.

- Homogeneous controller, i.e. same logic with same parameter values at all vehicles: technically, the possibility remains that heterogeneous controllers, i.e. letting the different vehicles react *differently* to the same signals, could solve the issue. However, we currently have no clue how to design this heterogeneity — unless one would allow parameters increasing unboundedly with chain length N, which however would pose other obvious problems. At least the simplest linear attempts, and controllers periodic in vehicle number, do not seem to work. We would thus rather conjecture that this assumption is not essential.

- Discrete-time controller: this should be representative in practice of a realistic digital controller. Rigorously, our counterexample analysis would break down when reducing the discretization step dt with increasing N. However, a property that only works with infinitely large bandwidth $1/dt$ for communication and/or control, is usually not robust in practice, and this suggests that any "reasonable" continuous-time controller would fail too. Note that the standard string stability model here includes no measurement nor communication noises, while with extreme continuous-time controllers those noises can be expected to become important.

Also note that we have only identified one particular, badly rejected disturbance input. In practice, for a generic disturbance, the situation might often be better, but also worse.

With this we believe to have given at least a much more comprehensive picture of what can be done on the standard academic property of string stability. If this string stability property appears critical in some key applications, those results should help guide a possible search for very particular controllers to achieve it, if it is feasible at all without relying on absolute velocities.

A different option for the future is to acknowledge that the academic definition of string stability is too strong to act as a useful proxy for applications, even in an extended framework with nonlinear controllers and so on. In that sense, distributed controller design should probably not right-away impose string stability, i.e. working fine for *infinite* N, but instead one could quantify the tradeoff in a more integrated picture: to have a given acceptable error, what are the best possible combinations of chain length N, absolute-velocity-dependence h, control+communication bandwidths $1/dt$, possibly nonlinear effects, and associated N-dependent gains in presence of other noises? This, knowing that the limit for infinite N will not work, but will also not be essential for most applications.

Meanwhile, as we are discussing practical applications, we should mention that it is not hard to extend the strong impossibility result of *Theorem 3.4* towards a BIBO version, namely: if we impose that disturbances must satisfy $|d_{k,1}(t)|/dt^2 < C_1$ and $|d_{k,2}(t)|/dt < C_1$ for all k, t, then we can choose α of order 1 in the “badly countered” disturbance, and this can result in $|e_k(t)|$ of order Ndt^2. Thus, if the setting of Theorem 3.4 is deemed representative of a practical situation, then this BIBO version is no direct solution for achieving string stability either.

A point that we did not study is whether one might be able to achieve string stability with respect to disturbances only on the initial state, instead of on input signals. For practical considerations, we would argue that both types of disturbances must be properly rejected, so our impossibility results for disturbances in input signals close the question in the negative sense. Moreover, it is well-known that a perturbed initial condition can be viewed as resulting from some disturbance input signal that has acted on the system in the past; in this sense, modulo adapting the norms used to characterize the disturbances, our results should also give indications for the vehicle chain's reaction to initial state disturbances only.

6.4 Chapter 4

In this chapter, we have shown that an asymmetric bidirectional controller using constant spacing policy among vehicles allows to solve the different definitions of string stability, provided disturbances act only on the leading vehicle. In the the controller proposed in this chapter we have supposed each vehicle is connected with one follower and one predecessor, but with different coupling gains. We have given the results in continuous-time and discrete-time settings. It is not hard to extend the result to the case where disturbances act on a few leading vehicles – the key is that the number of vehicles subject to disturbances remains fixed, while the length of the chain grows to infinity.

Table 6.2: Satisfying string stability with **disturbance on the leader only** and **no time-headway ($h = 0$)**. Our new contributions are in capital letters.

	Symmetric Bidirectional Coupling Bidirectional Coupling	Asymmetric Bidirectional Coupling
$L_2 \equiv (L_2, l_\infty)$	possible with advanced linear controller	POSSIBLE WITH PD
(L_2, l_2)	impossible	POSSIBLE WITH PD

The significance of such results are motivated by two points.

First, in several applications, it is reasonable to assume that the leading vehicle or subsystem will be subject to the strongest disturbances, as it is the "boundary" of the chain. For cars on roads for instance, typically obstacles are quasi-static and the leading vehicle will be the one encountering them by surprise. In other applications, the string could be physically attached at this boundary, e.g. when subsystems represent floors of a building. The "leading floor" would be the ground floor, and it is the only one directly subject to the action of earthquakes [19].

Second, in the literature it has not always been clearly distinguished whether one seeks to counter a disturbance acting on the leader or possibly on all vehicles, in addition to having different definitions of string stability in different papers. From there it is sometimes unclear whether successes really come from a new controller type, or just rely on a different definition of the objective. We here thus single out that when disturbances are confined to the head of the chain, nothing fancy has to be sought for: a simple asymmetric bidirectional PD coupling is sufficient to solve string stability.

6.5 Chapter 5

In this chapter, we turn to the model with time headway spacing policy, and assuming unidirectional coupling i.e. each vehicle reacting to its predecessor only. The time headway concept has indeed been proposed as a solution to satisfy L_2 string stability in this context; but it had never been proved rigorously, to our knowledge, that it also allows to satisfy stronger definitions of string stability. We have completed these proofs and obtained a somewhat nuanced picture, as summarized in the following table.

Table 6.3: Satisfying string stability **with time-headway** and **unidirectional coupling** (reacting only to the preceding vehicle). Our new contributions are in capital letters.

	finite $K(0)$ e.g. PD control	infinite $K(0)$ e.g. PID/Integral Controllers
L_2	possible	possible
(L_2, l_2), $\|\| d_0 \|\|_2 \neq 0$	IMPOSSIBLE	POSSIBLE
(L_2, l_2), $\|\| d_0 \|\|_2 = 0$	POSSIBLE	POSSIBLE
(L_2, l_∞)	POSSIBLE	POSSIBLE

We have to briefly give some perspective on the case of (L_2, l_2) string stability using any *bounded* controller and time-headway. We have shown that string stability cannot be achieved in this case, as soon as there is a disturbance on the leading vehicle. We have also shown that it is possible to guarantee (L_2, l_2) string stability using a PD controller, and with sufficiently large value of time-headway, if there exists no disturbance on the leading vehicle. This seems to identify the leading vehicle as the "only culprit", in apparent contradiction with the results of Chapter 4. The key point is that we here impose *unidirectional* coupling. The bidirectional coupling, with in fact stronger reaction to followers than to predecessors, is really essential in obtaining the string stability results of Chapter 4. It is thus not surprising that when the opposite is imposed — i.e. infinitely stronger reaction to predecessors than to followers, as the latter is zero — the conclusions become completely different.

At the end of this chapter, we have briefly considered the combination of inter-vehicle communication (CACC) and time-headway. This is in fact the context in which communication has been mostly considered in the literature. While simulations have confirmed the benefits in performance introduced by CACC, it has been observed but seemingly not explicitly proved that CACC also allows to obtain string stability with a lower value of the time-headway constant, and thus a weaker dependence of inter-vehicle distances on velocity. For practical purposes, this can be important to keep vehicles more tightly packed even at high velocities. We here generalize an explicit criterion from [43] for the time-headway constant, towards the case including CACC; this allows to readily show the benefits of communication on the criterion.

Our two concluding messages would be, in a nutshell:

- Be very careful about the assumptions on the setting in which string stability is investigated. Which elements make things different or not, is not always intuitive.
- The issue of string instability, on the basis of purely relative measurements, appears to be much more deeply rooted (see Theorem 3.4) than what one might suspect from its origins and standard treatment in the LTI context.

6.6 Contributions

The contributions of this thesis can be found in the following publications:

- A. Farnam, A. Sarlette, "About string stability of a vehicle chain with unidirectional controller," *Automatica*, under review.
- A. Farnam, A. Sarlette, "A most general impossibility result for string stability," *IEEE Trans. Automatic Control*, under review.
- A. Farnam, A. Sarlette, "About practical string stability of a vehicle chain,"*Proc. IEEE Conf. on Decision and Control*, to be presented in December 2018.
- A. Farnam, A. Sarlette, "String stability towards leader thanks to asymmetric bidirectional controller," *IFAC World Congress*, July 2017.

Bibliography

[1] https://www.scottautomation.com/assets/Uploads/AGV-Brochure.pdf.

[2] R. Kinafar "On Cooperative Control of Automated Driving Systems from a Stability and Safety Perspective", *PHD. Thesis, Chalmers University of Technology*, 2014.

[3] https://www.fhwa.dot.gov/publications/research/operations/its/

jpo98009/itssecanalysis.pdf.

[4] B. J. Bart van Arem and M. van Noort. "Smarter and better-the advantages of intelligent traffic", *Technical Report 2008-D-R0996/A-S, TNO, Van Mourik Broekmanweg 6 P.O. Box 49 2600 AA Dekft, The Netherlands*, 2008.

[5] W. Levine and M. Athans, "On the optimal error regulation of a string of moving vehicles," *IEEE Trans. Automatic Control*, Volume. 11, Pages. 355-36, 1966.

[6] S. Melzer and B. Kuo, "Optimal regulation of systems described by a countably infinite number of objects," *Automatica*, Volume. 7, Pages. 359-366, 1971.

[7] L. Peppard and V. Gourishankar, "An optimal automatic car-following system," *IEEE Trans. Vehicular Technology*, Volume. 21, Pages. 67-73, 1972.

[8] K. C. Chu, "Decentralized control of high-speed vehicular strings," *Transportation Science*, Volume. 8, Pages. 361-384, 1974.

[9] I. Lestas and G. Vinnicombe, "Scalability in heterogeneous vehicle platoons," *American Control Conference*, Pages. 4678-4683, 2007.

[10] M. E. Khatir and E. J. Davidson, "Bounded stability and eventual string stability of a large platoon of vehicles using non-identical controllers" *Proc. IEEE Conf. on Decision and Control*, 2004.

[11] R. Middleton, “Trade-offs in linear control system design," *Automatica*, Volume. 27, Pages. 281-292, 1991.

[12] B. Bamieh, F. Paganini, and M. Dahleh, “Distributed control of spatially invariant systems," *IEEE Trans. Automatic Control*, Volume. 47, Pages. 1091-1107, 2002.

[13] R. Kianfar, P. Falcone, and J. Fredriksson, “A receding horizon approach to string stable cooperative adaptive cruise control," *International IEEE Conference on Intelligent Transportation Systems*, Pages. 734-739, 2011.

[14] F. Lin, M. Fardad, and M. R. Jovanovic, “Optimal control of vehicular formations with nearest neighbor interactions," *IEEE Trans. Automatic Control*, Volume. 57, Pages. 2203-2218, 2012.

[15] A. A. Petersa, R. H. Middleton, and O. Masona,“Leader tracking in homogeneous vehicle platoons with broadcast delays," *Automatica*, Volume. 50, Pages. 64-74, 2014.

[16] D. Looze and J. Freudenberg, “Tradeoffs and limitations in feedback systems," *Ch.31 in The Control Handbook, W. Levine, Ed. CRC Press*, Pages. 537-549, 1996.

[17] D. Swaroop and J. Hedrick, “String stability of interconnected systems,” *IEEE Trans. Automatic Control*, Volume. 41, Pages 349-357, 1996.

[18] D. Swaroop, “String stability of interconnected systems: An application to platooning in automated highway systems,” *PhD thesis, University of California, Berkeley*, 1994.

[19] Y. Yamamoto, and M. C. Smith, “Bounded disturbance amplification for mass chains with passive interconnection,” *IEEE Trans. Automatic Control*, Volume. 47, Pages. 2534-2542, 2015.

[20] Y. Yamamoto, and M. C. Smith, “Design of passive interconnections in tall buildings subject to earthquake disturbances to suppress inter-storey drifts,” *Journal of Physics: Conference Series*, Volume. 744, Pages. 1-11, 2016.

[21] S. Sheikholeslam and C. Desoer, “Longitudinal control of a platoon of vehicles,” *Proc. American Control Conf.*, Pages. 291-297, 1990.

[22] P. Seiler, A. Pant, and K. Hedrick, “Disturbance propagation in vehicle strings,” *IEEE Trans. Automatic Control*, Volume. 37, Pages. 1835-1842, 1996.

[23] P. Barooah and J. P. Hespanha, "Error amplification and disturbance propagation in vehicle strings with decentralized linear control," *Proc. IEEE Conf. on Decision and Control*, Pages. 4964-4969, 2005.

[24] S. Klinge, and R. H. Middleton, "Time headway requirements for string stability of homogenous linear unidirectionally connected systems," *Proc. IEEE Conf. on Decision and Control*, Pages. 1992-1997, 2009.

[25] J. A. Rogge and D. Aeyels, "Vehicle Platoons Through Ring Coupling," *IEEE Trans. Automatic Control*, Volume. 53, Pages. 1370-1377, 2008.

[26] S. Knorn, A. Donaire, J. C. Aguero b and R. H. Middleton, "Passivity-based control for multi-vehicle systems subject to string constraints," *Automatica*, Volume. 50, Pages. 3224-3230, 2014.

[27] J. Ploeg, D.P. Shukla, N. van de Wouw and H. Nijmeijer, "Controller synthesis for string stability of vehicle platoons," *IEEE Trans. Intelligent Transportation Systems*, Volume. 15, Pages. 854-865, 2014.

[28] J. Ploeg, N. van de Wouw, and H. Nijmeijer, "L_p String Stability of Cascaded Systems: Application to Vehicle Platooning," *IEEE Trans. Control Systems Technology* Volume. 22, Pages 786-793, 2014.

[29] B. Besselink and K. H. Johansson, "String Stability and a Delay-Based Spacing Policy for Vehicle Platoons Subject to Disturbances," *IEEE Trans. Automatic Control*, Volume. 62, Pages. 4376-4391, 2017.

[30] P. Barooah, P. G. Mehta and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability of vehicular platoons," *IEEE Trans. Automatic Control*, Volume. 54, Pages. 2100-2113, 2009.

[31] A. Sarlette and R. Sepulchre, "Control limitations from distributed sensing: theory and Extremely Large Telescope application," *Automatica*, Volume. 50, Pages. 421-430, 2014.

[32] B. Bamieh, M. R. Jovanovic, P. Mitra, and S. Patterson, "Coherence in Large-Scale Networks: Dimension Dependent Limitations of Local Feedback," *IEEE Trans. Automatic Control*, Volume. 57, Pages. 2235-2249, 2012.

[33] I. Herman, S. Knorn and A. Ahlen, "Disturbance scaling in bidirectional vehicle platoons with different asymmetry in position and velocity coupling," *Automatica*, Volume. 82, Pages. 13-20, 2017.

[34] I. Herman, D. Martinec, Z. Hurak and M. Sebek, "Nonzero bound on Fiedler eigenvalue causes exponential growth of H-infinity norm of vehicular platoon," *IEEE Trans. Automatic Control*, Volume. 60, Pages. 2248-2253, 2015.

[35] W. J. O. Connor, "Wave-Based Analysis and Control of Lump-Modeled Flexible Robots," *IEEE Trans. Robotics*, Volume. 23, Pages. 342-352, 2007

[36] G. J. L. Naus, R. P. A. Vugts, J. Ploeg, M. J. G. v. Molengraft, and M. Steinbuch, "String-Stable CACC Design and Experimental Validation: A Frequency-Domain Approach," *IEEE Trans. Vehicular Technology*, Volume. 59, Pages. 4268-4279, 2010.

[37] S. Oncu, J. Ploeg, N. van. Wouw, and H. Nijmeijer, "Cooperative Adaptive Cruise Control: Network-Aware Analysis of String Stability," *IEEE Trans. Intelligent Transportation Systems*, Volume. 15, Pages. 1527-1537, 2014.

[38] S. Oncu, N. V. Wouw, W. P. M. H. Heemels, and H. Nijmeijer, "String Stability of Interconnected Vehicles Under Communication Constraints," *Proc. IEEE Conf. on Decision and Control*, 2012.

[39] J. Ploeg, E. S. Kazerooni, G. Lijster, N. v. Wouw, and H. Nijmeijer, "Graceful Degradation of Cooperative Adaptive Cruise Control," *IEEE Trans. Intelligent Transportation Systems*, Volume. 16, Pages. 488-497, 2015.

[40] B. v. Arem, C. J. G. V. Driel, and R. Visser, "The Impact of Cooperative Adaptive Cruise Control on Traffic-Flow Characteristics," *IEEE Trans. Intelligent Transportation Systems*, Volume. 7, Pages. 429-436, 2006.

[41] I. Serban and M. Najim, "Multidimensional Systems: BIBO Stability Test Based on Functional Schur Coefficients," *IEEE Trans. Signal Processing*, Volume. 55, Pages. 5277-5285, 2007.

[42] R. F. Curtain, et al., "System theoretic properties of a class of spatially invariant systems," *Automatica*, Volume. 45, Pages. 1619-1627, 2009.

[43] A. A. Peters, "Stability and String Stability Analysis of Formation Control Architectures for Platooning". *PhD thesis, National University of Ireland Maynooth.*