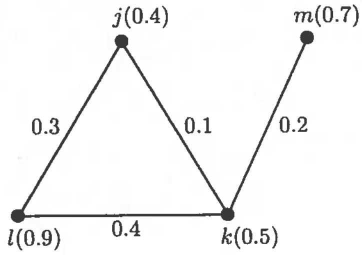


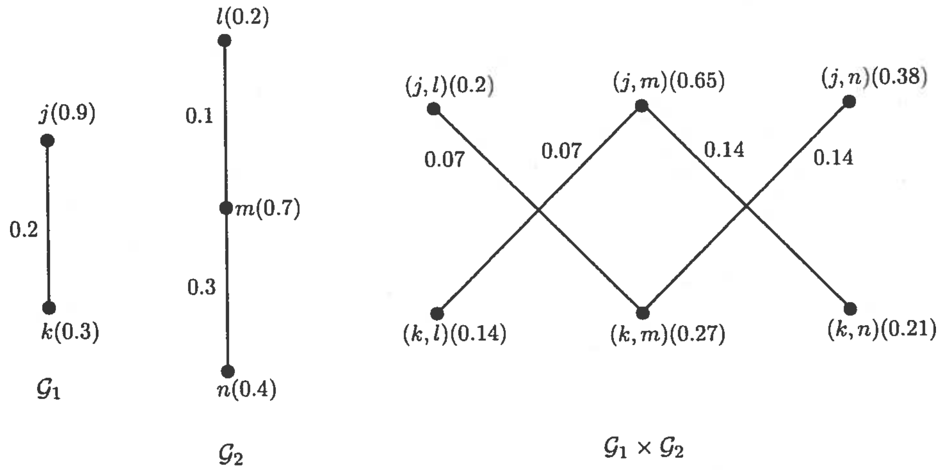
Figure 1: Dombi fuzzy graph

Definition 3.2. Let η_i be ~~the~~ a fuzzy subset of V_i and let ζ_i be ~~the~~ a fuzzy subset of E_i, $i = 1, 2$. Define the direct product $\mathcal{G}_1 \times \mathcal{G}_2 = (\eta_1 \times \eta_2, \zeta_1 \times \zeta_2)$ of the Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, as follows:

$$
\begin{aligned}
(\eta_1 \times \eta_2)(x_1, x_2) &= \frac{\eta_1(x_1)\eta_2(x_2)}{\eta_1(x_1) + \eta_2(x_2) - \eta_1(x_1)\eta_2(x_2)} \quad \text{for all } (x_1, x_2) \in V_1 \times V_2, \\
(\zeta_1 \times \zeta_2)((x_1, x_2)(y_1, y_2)) &= \frac{\zeta_1(x_1y_1)\zeta_2(x_2y_2)}{\zeta_1(x_1y_1) + \zeta_2(x_2y_2) - \zeta_1(x_1y_1)\zeta_2(x_2y_2)} \quad \text{for all } x_1y_1 \in E_1, \text{ for all } x_2y_2 \in E_2.
\end{aligned}
$$

Example 3.2. Consider two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\eta_1 = \{\frac{j}{0.9}, \frac{k}{0.3}\}$, $\zeta_1 = \{\frac{jk}{0.2}\}$, $\eta_2 = \{\frac{l}{0.2}, \frac{m}{0.7}, \frac{n}{0.4}\}$ and $\zeta_2 = \{\frac{lm}{0.1}, \frac{mn}{0.3}\}$. Then we have

$$
\begin{aligned}
(\zeta_1 \times \zeta_2)((j, l)(k, m)) &= 0.07, \quad (\zeta_1 \times \zeta_2)((j, m)(k, l)) = 0.07, \\
(\zeta_1 \times \zeta_2)((j, m)(k, n)) &= 0.14, \quad (\zeta_1 \times \zeta_2)((j, n)(k, m)) = 0.14.
\end{aligned}
$$



It is easy to check that $\mathcal{G}_1 \times \mathcal{G}_2$ is the Dombi fuzzy graph of $G_1 \times G_2$.

Proposition 3.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the Dombi fuzzy graphs of the graphs G_1 and G_2, respectively. The direct product $\mathcal{G}_1 \times \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the Dombi fuzzy graph of $G_1 \times G_2$.*

Proof. Consider $x_1y_1 \in E_1, x_2y_2 \in E_2$. Then

$$\begin{aligned}(\zeta_1 \times \zeta_2)((x_1,x_2)(y_1,y_2)) &= T(\zeta_1(x_1y_1), \zeta_2(x_2y_2)) \\ &\leq T\left(\frac{\eta_1(x_1)\eta_1(y_1)}{\eta_1(x_1)+\eta_1(y_1)-\eta_1(x_1)\eta_1(y_1)}, \frac{\eta_2(x_2)\eta_2(y_2)}{\eta_2(x_2)+\eta_2(y_2)-\eta_2(x_2)\eta_2(y_2)}\right)\end{aligned}$$

Putting $a = \eta_1(x_1), b = \eta_1(y_1), c = \eta_2(x_2), d = \eta_2(y_2)$.

$$(\zeta_1 \times \zeta_2)((x_1,x_2)(y_1,y_2)) \leq T\left(\frac{ab}{a+b-ab}, \frac{cd}{c+d-cd}\right)$$

$$= \frac{\frac{abcd}{(a+b-ab)(c+d-cd)}}{\frac{ab}{a+b-ab}+\frac{cd}{c+d-cd}-\frac{abcd}{(a+b-ab)(c+d-cd)}}$$

$$= \frac{\frac{abcd}{(a+c-ac)(b+d-bd)}}{\frac{ac}{a+c-ac}+\frac{bd}{b+d-bd}-\frac{abcd}{(a+c-ac)(b+d-bd)}}$$

$$= \frac{(\eta_1 \times \eta_2)((x_1,x_2))(\eta_1 \times \eta_2)((y_1,y_2))}{(\eta_1 \times \eta_2)((x_1,x_2)) + (\eta_1 \times \eta_2)((y_1,y_2)) - (\eta_1 \times \eta_2)((x_1,x_2))(\eta_1 \times \eta_2)((y_1,y_2))}.$$

Hence proved. $\square$

Definition 3.3. Let η_i be ~~the~~ a fuzzy subset of V_i and let ζ_i be ~~the~~ a fuzzy subset of E_i, $i = 1, 2$. Define the Cartesian product $\mathcal{G}_1 \square \mathcal{G}_2 = (\eta_1 \square \eta_2, \zeta_1 \square \zeta_2)$ of the Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, as follows:

$$\begin{aligned}(\eta_1 \square \eta_2)(x_1,x_2) &= \frac{\eta_1(x_1)\eta_2(x_2)}{\eta_1(x_1)+\eta_2(x_2)-\eta_1(x_1)\eta_2(x_2)} \quad \text{for all } (x_1,x_2) \in V_1 \times V_2, \\ (\zeta_1 \square \zeta_2)((x,x_2)(x,y_2)) &= \frac{\eta_1(x)\zeta_2(x_2y_2)}{\eta_1(x)+\zeta_2(x_2y_2)-\eta_1(x)\zeta_2(x_2y_2)} \quad \text{for all } x \in V_1, \text{ for all } x_2y_2 \in E_2, \\ (\zeta_1 \square \zeta_2)((x_1,z)(y_1,z)) &= \frac{\eta_2(z)\zeta_1(x_1y_1)}{\eta_2(z)+\zeta_1(x_1y_1)-\eta_2(z)\zeta_1(x_1y_1)} \quad \text{for all } z \in V_2, \text{ for all } x_1y_1 \in E_1.\end{aligned}$$

Remark 3.1. The Cartesian product of two Dombi fuzzy graphs is not necessarily a Dombi fuzzy graph.

Example 3.3. Consider two Dombi fuzzy graphs as in Example 3.2. Then we have

$$\begin{aligned}(\zeta_1 \square \zeta_2)((j,l)(j,m)) &= 0.1, \quad (\zeta_1 \square \zeta_2)((j,m)(j,n)) = 0.29, \\ (\zeta_1 \square \zeta_2)((k,l)(k,m)) &= 0.08, \quad (\zeta_1 \square \zeta_2)((k,m)(k,n)) = 0.18, \\ (\zeta_1 \square \zeta_2)((j,l)(k,l)) &= 0.11, \quad (\zeta_1 \square \zeta_2)((j,m)(k,m)) = 0.18, \\ (\zeta_1 \square \zeta_2)((j,n)(k,n)) &= 0.15.\end{aligned}$$

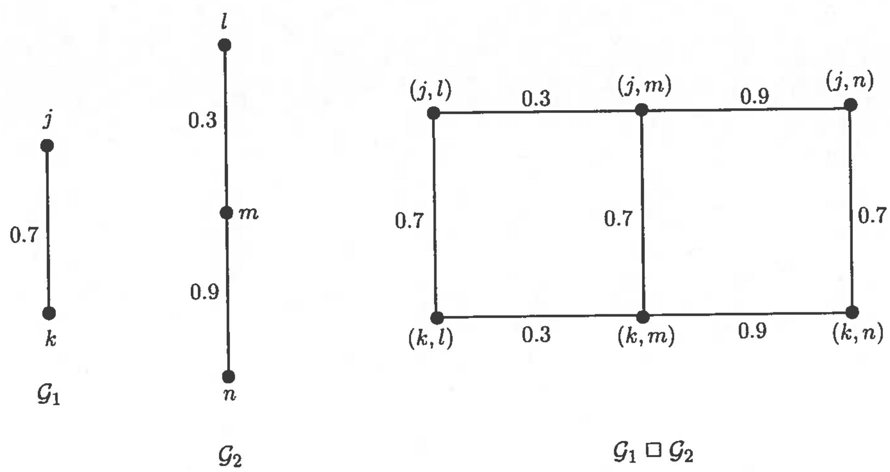
Consider $z \in V_2, x_1y_1 \in E_1$. Then

$$\begin{aligned}
&(\zeta_1 \square \zeta_2)((x_1, z)(y_1, z)) = T(\zeta_1(x_1y_1), \eta_2(z)) = T(\zeta_1(x_1y_1), 1) \\
&= \zeta_1(x_1y_1) \leq \frac{\eta_1(x_1)\eta_1(y_1)}{\eta_1(x_1) + \eta_1(y_1) - \eta_1(x_1)\eta_1(y_1)} \\
&= \frac{(\eta_1 \square \eta_2)((x_1, z))(\eta_1 \square \eta_2)((y_1, z))}{(\eta_1 \square \eta_2)((x_1, z)) + (\eta_1 \square \eta_2)((y_1, z)) - (\eta_1 \square \eta_2)((x_1, z))(\eta_1 \square \eta_2)((y_1, z))}.
\end{aligned}$$

Hence proved. $\square$

Example 3.4. Consider two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\eta_1(x) = 1$ for all $x \in V_1$, $\zeta_1 = \{\frac{jk}{0.7}\}$, $\eta_2(y) = 1$ for all $y \in V_2$, and $\zeta_2 = \{\frac{lm}{0.3}, \frac{mn}{0.9}\}$. Then we have

$$\begin{aligned}
(\zeta_1 \square \zeta_2)((j,l)(j,m)) &= 0.3, \quad (\zeta_1 \square \zeta_2)((j,m)(j,n)) = 0.9, \\
(\zeta_1 \square \zeta_2)((k,l)(k,m)) &= 0.3, \quad (\zeta_1 \square \zeta_2)((k,m)(k,n)) = 0.9, \\
(\zeta_1 \square \zeta_2)((j,l)(k,l)) &= 0.7, \quad (\zeta_1 \square \zeta_2)((j,m)(k,m)) = 0.7, \\
(\zeta_1 \square \zeta_2)((j,n)(k,n)) &= 0.7.
\end{aligned}$$



It is easy to check that $\mathcal{G}_1 \square \mathcal{G}_2$ is the Dombi fuzzy edge graph of $G_1 \square G_2$.

Definition 3.5. Let η_i be ~~the~~ a fuzzy subset of V_i and let ζ_i be ~~the~~ a fuzzy subset of E_i, $i = 1, 2$. Define the semi-strong product $\mathcal{G}_1 \bullet \mathcal{G}_2 = (\eta_1 \bullet \eta_2, \zeta_1 \bullet \zeta_2)$ of Dombi fuzzy graphs the
$\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, as follows:

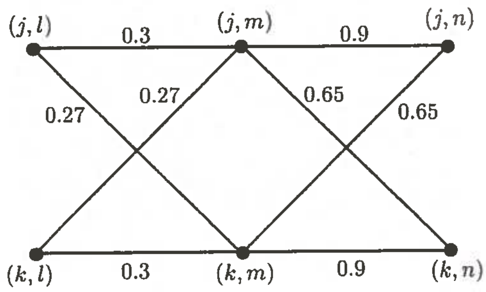
$$\begin{aligned}
(\eta_1 \bullet \eta_2)(x_1, x_2) &= \frac{\eta_1(x_1)\eta_2(x_2)}{\eta_1(x_1) + \eta_2(x_2) - \eta_1(x_1)\eta_2(x_2)} \quad \text{for all } (x_1, x_2) \in V_1 \times V_2, \\
(\zeta_1 \bullet \zeta_2)((x, x_2)(x, y_2)) &= \frac{\eta_1(x)\zeta_2(x_2y_2)}{\eta_1(x) + \zeta_2(x_2y_2) - \eta_1(x)\zeta_2(x_2y_2)} \quad \text{for all } x \in V_1, \text{ for all } x_2y_2 \in E_2, \\
(\zeta_1 \bullet \zeta_2)((x_1, x_2)(y_1, y_2)) &= \frac{\zeta_1(x_1y_1)\zeta_2(x_2y_2)}{\zeta_1(x_1y_1) + \zeta_2(x_2y_2) - \zeta_1(x_1y_1)\zeta_2(x_2y_2)} \quad \text{for all } x_1y_1 \in E_1, \text{ for all } x_2y_2 \in E_2.
\end{aligned}$$

Proposition 3.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the Dombi fuzzy edge graphs of the graphs G_1 and G_2, respectively. The semi-strong product $\mathcal{G}_1 \bullet \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the Dombi fuzzy edge graph of $G_1 \bullet G_2$.*

⊥ the *Proof.* The proof follows at once from ⊥ proof of Propositions 3.1 and 3.2. □

Example 3.5. Consider two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ as in Example 3.4. Then we have

$$\begin{aligned}
(\zeta_1 \bullet \zeta_2)((j,l)(j,m)) &= 0.3, \quad (\zeta_1 \bullet \zeta_2)((j,m)(j,n)) = 0.9,\\
(\zeta_1 \bullet \zeta_2)((k,l)(k,m)) &= 0.3, \quad (\zeta_1 \bullet \zeta_2)((k,m)(k,n)) = 0.9,\\
(\zeta_1 \bullet \zeta_2)((j,l)(k,m)) &= 0.27, \quad (\zeta_1 \bullet \zeta_2)((j,m)(k,l)) = 0.27,\\
(\zeta_1 \bullet \zeta_2)((j,m)(k,n)) &= 0.65, \quad (\zeta_1 \bullet \zeta_2)((j,n)(k,m)) = 0.65.
\end{aligned}$$



$\mathcal{G}_1 \bullet \mathcal{G}_2$

It is easy to check that $\mathcal{G}_1 \bullet \mathcal{G}_2$ is the Dombi fuzzy edge graph of $G_1 \bullet G_2$.

Definition 3.6. Let η_i be ~~the~~ a fuzzy subset of V_i and let ζ_i be ~~the~~ a fuzzy subset of E_i, $i = 1, 2$. Define the strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2 = (\eta_1 \boxtimes \eta_2, \zeta_1 \boxtimes \zeta_2)$ of ⊥ Dombi fuzzy graphs ⊥ the $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, as follows:

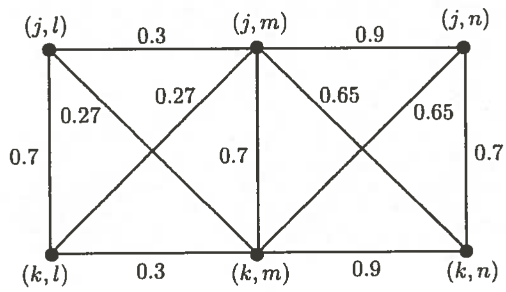
$$\begin{aligned}
(\eta_1 \boxtimes \eta_2)(x_1, x_2) &= \frac{\eta_1(x_1)\eta_2(x_2)}{\eta_1(x_1) + \eta_2(x_2) - \eta_1(x_1)\eta_2(x_2)} \quad \text{for all } (x_1, x_2) \in V_1 \times V_2,\\
(\zeta_1 \boxtimes \zeta_2)((x, x_2)(x, y_2)) &= \frac{\eta_1(x)\zeta_2(x_2y_2)}{\eta_1(x) + \zeta_2(x_2y_2) - \eta_1(x)\zeta_2(x_2y_2)} \quad \text{for all } x \in V_1, \text{ for all } x_2y_2 \in E_2,\\
(\zeta_1 \boxtimes \zeta_2)((x_1, z)(y_1, z)) &= \frac{\eta_2(z)\zeta_1(x_1y_1)}{\eta_2(z) + \zeta_1(x_1y_1) - \eta_2(z)\zeta_1(x_1y_1)} \quad \text{for all } z \in V_2, \text{ for all } x_1y_1 \in E_1,\\
(\zeta_1 \boxtimes \zeta_2)((x_1, x_2)(y_1, y_2)) &= \frac{\zeta_1(x_1y_1)\zeta_2(x_2y_2)}{\zeta_1(x_1y_1) + \zeta_2(x_2y_2) - \zeta_1(x_1y_1)\zeta_2(x_2y_2)} \quad \text{for all } x_1y_1 \in E_1, \text{ for all } x_2y_2 \in E_2.
\end{aligned}$$

Proposition 3.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the Dombi fuzzy edge graphs of the graphs G_1 and G_2, respectively. The strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the Dombi fuzzy edge graph of $G_1 \boxtimes G_2$.*

Proof. The proof follows at once from the proof of Propositions 3.1 and 3.2. $\square$

Example 3.6. Consider two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ as in Example 3.4. Then we have

$$\begin{aligned}
(\zeta_1 \boxtimes \zeta_2)((j,l)(j,m)) &= 0.3, \quad (\zeta_1 \boxtimes \zeta_2)((j,m)(j,n)) = 0.9,\\
(\zeta_1 \boxtimes \zeta_2)((k,l)(k,m)) &= 0.3, \quad (\zeta_1 \boxtimes \zeta_2)((k,m)(k,n)) = 0.9,\\
(\zeta_1 \boxtimes \zeta_2)((j,l)(k,l)) &= 0.7, \quad (\zeta_1 \boxtimes \zeta_2)((j,m)(k,m)) = 0.7,\\
(\zeta_1 \boxtimes \zeta_2)((j,n)(k,n)) &= 0.7, \quad (\zeta_1 \boxtimes \zeta_2)((j,l)(k,m)) = 0.27,\\
(\zeta_1 \boxtimes \zeta_2)((j,m)(k,l)) &= 0.27, \quad (\zeta_1 \boxtimes \zeta_2)((j,m)(k,n)) = 0.65,\\
(\zeta_1 \boxtimes \zeta_2)((j,n)(k,m)) &= 0.65.
\end{aligned}$$



$\mathcal{G}_1 \boxtimes \mathcal{G}_2$

It is easy to check that $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is the Dombi fuzzy edge graph of $G_1 \boxtimes G_2$.

Definition 3.7. Let η_i be ~~the~~ a fuzzy subset of V_i and let ζ_i be ~~the~~ a fuzzy subset of E_i, $i = 1, 2$. Define the lexicographic product $\mathcal{G}_1[\mathcal{G}_2] = (\eta_1 \circ \eta_2, \zeta_1 \circ \zeta_2)$ of the Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, as follows:

$$(\eta_1 \circ \eta_2)(x_1, x_2) = \frac{\eta_1(x_1)\eta_2(x_2)}{\eta_1(x_1) + \eta_2(x_2) - \eta_1(x_1)\eta_2(x_2)} \quad \text{for all } (x_1, x_2) \in V_1 \times V_2,$$

$$(\zeta_1 \circ \zeta_2)((x, x_2)(x, y_2)) = \frac{\eta_1(x)\zeta_2(x_2y_2)}{\eta_1(x) + \zeta_2(x_2y_2) - \eta_1(x)\zeta_2(x_2y_2)} \quad \text{for all } x \in V_1, \text{ for all } x_2y_2 \in E_2,$$

$$(\zeta_1 \circ \zeta_2)((x_1, z)(y_1, z)) = \frac{\eta_2(z)\zeta_1(x_1y_1)}{\eta_2(z) + \zeta_1(x_1y_1) - \eta_2(z)\zeta_1(x_1y_1)} \quad \text{for all } z \in V_2, \text{ for all } x_1y_1 \in E_1,$$

$$(\zeta_1 \circ \zeta_2)((x_1, x_2)(y_1, y_2)) = \frac{\eta_2(x_2)\eta_2(y_2)\zeta_1(x_1y_1)}{\eta_2(x_2)\eta_2(y_2) + \eta_2(y_2)\zeta_1(x_1y_1) + \eta_2(x_2)\zeta_1(x_1y_1) - 2\eta_2(x_2)\eta_2(y_2)\zeta_1(x_1y_1)}$$

for all $x_1y_1 \in E_1$, $x_2 \neq y_2$.

Proposition 3.5. *The lexicographic product $\mathcal{G}_1[\mathcal{G}_2]$ of two Dombi fuzzy edge graphs of G_1 and G_2 is the Dombi fuzzy edge graph of $G_1[G_2]$.*

For page 13

ii/ Suppose;

$x \in V_1 \setminus V_2$, $y \in V_1 \cap V_2$

Then we obtain:

$$(\zeta_1 \cup \zeta_2)(xy) = \zeta_1(xy) \quad \text{(by definition 3.8)}$$

$$\le T(\eta_1(x), \eta_1(y)) \quad \text{(by def. of Dombi fuzzy graph)}$$

$$= T(\eta_1 \cup \eta_2(x), \eta_1(y)) \quad (*)$$

? $\eta_1(y) \le \dfrac{\eta_1(y)+\eta_2(y) - 2\eta_1(y)\eta_2(y)}{1-\eta_1(y)\eta_2(y)}$

putting $\eta_1(y) = f$ $\eta_2(y) = h$.

? $f \le \dfrac{f+h-2fh}{1-fh}$.

? $\cancel{f} - f^2h \le \cancel{f} + h - 2fh$.

$-f^2 \le 1-2f$.

$0 \le f^2 - 2f + 1$.

$0 \le (f-1)^2$ OK.

So from (*) we get:

$$(\zeta_1 \cup \zeta_2)(xy) \le T(\eta_1 \cup \eta_2(x), \eta_1 \cup \eta_2(y))$$

since T is increasing.

iii/

iii) Suppose: $x \in V_1 \cap V_2$, $y \in V_1 \cap V_2$.
Then we get:

$$(\zeta_1 \cup \zeta_2)(xy) = \zeta_1(xy)$$
$$\leq T(\eta_1(x), \eta_1(y))$$
$$\leq T(\eta_1 \cup \eta_2(x), \eta_1 \cup \eta_2(y))$$

since as in ii):

$$\eta(x) \leq \eta_1 \cup \eta_2(x) \text{ and } \eta_1(y) \leq \eta_1(y) \cup \eta_2(y).$$

Similarly, because of symmetry, we find for $xy \in E_2 \setminus E_1$ in the three possible cases:

$$(\zeta_1 \cup \zeta_2)(xy) \leq \frac{(\eta_1 \cup \eta_2)(x), (\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}.$$

Finally suppose $xy \in E_1 \cap E_2$. Then:

provide a counterexample for for ex. semi-strong product!

Remark 3.2. In general the semi-strong product, strong product and lexicographic product of two Dombi fuzzy graphs are not Dombi fuzzy graphs.

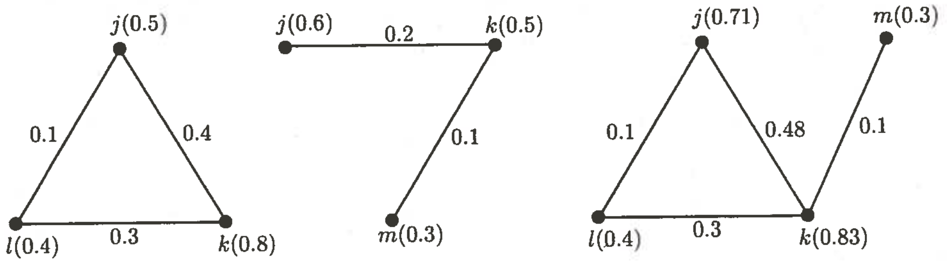
Definition 3.8. Let η_i be ~~the~~ a fuzzy subset of V_i and let ζ_i be ~~the~~ a fuzzy subset of E_i, $i = 1, 2$. Define the union $\mathcal{G}_1 \cup \mathcal{G}_2 = (\eta_1 \cup \eta_2, \zeta_1 \cup \zeta_2)$ of the Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ as follows:

$$(\eta_1 \cup \eta_2)(x) = \begin{cases} \eta_1(x) & \text{if } x \in V_1 - V_2, \\ \eta_2(x) & \text{if } x \in V_2 - V_1, \\ \frac{\eta_1(x)+\eta_2(x)-2\eta_1(x)\eta_2(x)}{1-\eta_1(x)\eta_2(x)} & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$(\zeta_1 \cup \zeta_2)(xy) = \begin{cases} \zeta_1(xy) & \text{if } xy \in E_1 - E_2, \\ \zeta_2(xy) & \text{if } xy \in E_2 - E_1, \\ \frac{\zeta_1(xy)+\zeta_2(xy)-2\zeta_1(xy)\zeta_2(xy)}{1-\zeta_1(xy)\zeta_2(xy)} & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Example 3.8. We consider two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\eta_1 = \{\frac{j}{0.5}, \frac{k}{0.8}, \frac{l}{0.4}\}$, $\zeta_1 = \{\frac{jk}{0.4}, \frac{kl}{0.3}, \frac{jl}{0.1}\}$, $\eta_2 = \{\frac{j}{0.6}, \frac{k}{0.5}, \frac{m}{0.3}\}$ and $\zeta_2 = \{\frac{jk}{0.2}, \frac{km}{0.1}\}$. Then we have

$$\eta_1 \cup \eta_2 = \left\{\frac{j}{0.71}, \frac{k}{0.83}, \frac{l}{0.4}, \frac{m}{0.3}\right\}, \quad \zeta_1 \cup \zeta_2 = \left\{\frac{jk}{0.48}, \frac{kl}{0.3}, \frac{jl}{0.18}, \frac{km}{0.1}\right\}.$$



$\mathcal{G}_1$ $\quad$ $\mathcal{G}_2$ $\quad$ $\mathcal{G}_1 \cup \mathcal{G}_2$

Clearly, $\mathcal{G}_1 \cup \mathcal{G}_2$ is a Dombi fuzzy graph of $G_1 \cup G_2$.

Proposition 3.6. *The union $\mathcal{G}_1 \cup \mathcal{G}_2$ of two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of G_1 and G_2 is the Dombi fuzzy graph of $G_1 \cup G_2$.*

Proof. Consider $xy \in E_1 - E_2$. Then there are three posibilities, (i) $x, y \in V_1 - V_2$, (ii) $x \in V_1 - V_2, y \in V_1 \cap V_2$ and (iii) $x, y \in V_1 \cap V_2$. $\perp$ s

(i) Suppose ~~If~~ $x, y \in V_1 - V_2$

$$(\zeta_1 \cup \zeta_2)(xy) = \zeta_1(xy) \leq T(\eta_1(x), \eta_1(y)) = \frac{\eta_1(x)\eta_1(y)}{\eta_1(x) + \eta_1(y) - \eta_1(x)\eta_1(y)}$$
$$= \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}.$$ $\perp$,

Hence proved. $\square$

The converse of above result holds if $V_1 \cap V_2 = \emptyset$. the

⊥ the **Theorem 3.1.** *The union $\mathcal{G}_1 \cup \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is ~~a~~ Dombi fuzzy graph of $G_1 \cup G_2$ if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are Dombi fuzzy graphs of G_1 ~~and~~ G_2, respectively, where η_1, η_2, ζ_1 and ζ_2 are ~~the~~ fuzzy subsets of V_1, V_2, E_1 and E_2, respectively, and $V_1 \cap V_2 = \emptyset$.*

Proof. Assume that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a Dombi fuzzy graph. Let $xy \in E_1$, then $xy \notin E_2$ and $x, y \in V_1 - V_2$. Thus

$$\begin{aligned}
\zeta_1(xy) = (\zeta_1 \cup \zeta_2)(xy) &\leq \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)} \\
&= \frac{\eta_1(x)\eta_1(y)}{\eta_1(x) + \eta_1(y) - \eta_1(x)\eta_1(y)}.
\end{aligned}$$

Thus $\mathcal{G}_1$ is a Dombi fuzzy graph of G_1. Similarly, it is easy to show that $\mathcal{G}_2$ is a Dombi fuzzy graph of G_2. $\square$

a a

Definition 3.9. Let η_i be ~~the~~ fuzzy subset of V_i and let ζ_i be ~~the~~ fuzzy subset of E_i, ⊥ the
$i = 1, 2$. Define the ring-sum $\mathcal{G}_1 \oplus \mathcal{G}_2 = (\eta_1 \oplus \eta_2, \zeta_1 \oplus \zeta_2)$ of Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ as follows:

$$\begin{aligned}
(\eta_1 \oplus \eta_2)(x) &= (\eta_1 \cup \eta_2)(x) \quad \text{if } x \in V_1 \cup V_2, \\
(\zeta_1 \oplus \zeta_2)(xy) &= \begin{cases} \zeta_1(xy) & \text{if } xy \in E_1 - E_2, \\ \zeta_2(xy) & \text{if } xy \in E_2 - E_1, \\ 0 & \text{if } xy \in E_1 \cap E_2. \end{cases}
\end{aligned}$$

Proposition 3.7. *The ring-sum $\mathcal{G}_1 \oplus \mathcal{G}_2$ of two Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of G_1 and G_2 is the Dombi fuzzy graph of $G_1 \oplus G_2$.*

Definition 3.10. Consider the join $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where E' is the set of all edges joining the vertices of V_1 a
a and V_2, $V_1 \cap V_2 = \emptyset$. Let η_i be ~~the~~ fuzzy subset of V_i and let ζ_i be ~~the~~ fuzzy subset of E_i, $i = 1, 2$. Define the join $\mathcal{G}_1 + \mathcal{G}_2 = (\eta_1 + \eta_2, \zeta_1 + \zeta_2)$ of Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ ⊥ the and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ as follows:

$$\begin{aligned}
(\eta_1 + \eta_2)(x) &= (\eta_1 \cup \eta_2)(x) \quad \text{if } x \in V_1 \cup V_2, \\
(\zeta_1 + \zeta_2)(xy) &= (\zeta_1 \cup \zeta_2)(xy) \quad \text{if } xy \in E_1 \cup E_2, \\
(\zeta_1 + \zeta_2)(xy) &= \frac{\eta_1(x)\eta_2(y)}{\eta_1(x) + \eta_2(y) - \eta_1(x)\eta_2(y)} \quad \text{if } xy \in E'.
\end{aligned}$$

Theorem 3.2. *The join $\mathcal{G}_1 + \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is a Dombi fuzzy graph of $G_1 + G_2$ if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are Dombi fuzzy graphs of G_1 and G_2, respectively, where η_1, η_2, ζ_1 and ζ_2 are ~~the~~ fuzzy subsets of V_1, V_2, E_1 and E_2, respectively, and $V_1 \cap V_2 = \emptyset$.*

Proof. Suppose that $\mathcal{G}_1 + \mathcal{G}_2$ is a Dombi fuzzy graph. Then from the proof of Theorem 3.1, $\mathcal{G}_1$ and $\mathcal{G}_2$ are Dombi fuzzy graphs.

Conversely, suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are Dombi fuzzy graphs of G_1 and G_2, respectively. Consider $xy \in E_1 \cup E_2$. Then the required result follows from Proposition 3.6. Let $xy \in E'$. Then

$$\begin{aligned}
(\zeta_1 + \zeta_2)(xy) &= \frac{\eta_1(x)\eta_2(y)}{\eta_1(x) + \eta_2(y) - \eta_1(x)\eta_2(y)} \\
&= \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)} \\
&= \frac{(\eta_1 + \eta_2)(x)(\eta_1 + \eta_2)(y)}{(\eta_1 + \eta_2)(x) + (\eta_1 + \eta_2)(y) - (\eta_1 + \eta_2)(x)(\eta_1 + \eta_2)(y)}.
\end{aligned}$$

? why

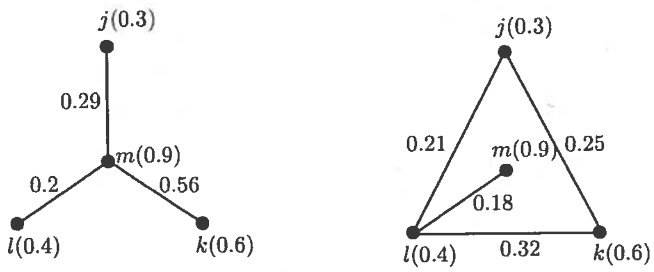
Hence proved. □

Definition 3.11. The complement of a Dombi fuzzy graph $\mathcal{G} = (\eta, \zeta)$ of $G = (V, E)$ is a Dombi fuzzy graph $\overline{\mathcal{G}} = (\overline{\eta}, \overline{\zeta})$, where $\overline{\eta}(x) = \eta(x)$ for all $x \in V$ and $\overline{\zeta}$ is defined as follows:

$$\overline{\zeta}(xy) = \begin{cases} \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)} & \text{if } \zeta(xy) = 0, \\ \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)} - \zeta(xy) & \text{if } 0 < \zeta(xy) \leq 1. \end{cases}$$

Example 3.9. Consider a Dombi fuzzy graph over $V = \{j, k, l, m\}$ defined by

$$\eta = \left\{ \frac{j}{0.3}, \frac{k}{0.6}, \frac{l}{0.4}, \frac{m}{0.9} \right\}, \quad \zeta = \left\{ \frac{jm}{0.29}, \frac{km}{0.56}, \frac{lm}{0.2} \right\}.$$



$\mathcal{G}$ $\overline{\mathcal{G}}$

We have

$$\overline{\overline{\eta(x)}} = \overline{\eta(x)} = \eta(x) \quad \text{for all } x \in V,$$

and

$$\begin{aligned}
\overline{\overline{\zeta(xy)}} &= \frac{\overline{\eta(x)\eta(y)}}{\overline{\eta(x)} + \overline{\eta(y)} - \overline{\eta(x)\eta(y)}} - \overline{\zeta(xy)} \\
&= \frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)} - \left(\frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)} - \zeta(xy) \right) \\
&= \zeta(xy) \quad \text{for all } x, y \in V.
\end{aligned}$$

Hence $\overline{\overline{\mathcal{G}}} = \mathcal{G}$.

Definition 3.12. A homomorphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ of two Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ is a mapping $\psi : V_1 \to V_2$ satisfying the following conditions:

(a) $\eta_1(x) \leq \eta_2(\psi(x))$ for all $x \in V_1$,

(b) $\zeta_1(xy) \leq \zeta_2(\psi(x)\psi(y))$ for all $x, y \in V_1$.

Definition 3.13. An isomorphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ of two Dombi fuzzy graphs $\mathcal{G}_1 = (\eta_1, \zeta_1)$ and $\mathcal{G}_2 = (\eta_2, \zeta_2)$ is a bijective mapping $\psi : V_1 \to V_2$ satisfying the following conditions:

(denoted as $\mathcal{G}_1 \cong \mathcal{G}_2$)

(c) $\eta_1(x) = \eta_2(\psi(x))$ for all $x \in V_1$,

(d) $\zeta_1(xy) = \zeta_2(\psi(x)\psi(y))$ for all $x, y \in V_1$.

A weak isomorphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ is a bijective homomorphism with the condition (c) above and a co-weak isomorphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ is a bijective homomorphism with the condition (d) above.

Definition 3.14. A Dombi fuzzy graph $\mathcal{G} = (\eta, \zeta)$ is said to be self-complementary if $\mathcal{G} = (\eta, \zeta) \cong \overline{\mathcal{G}} = (\overline{\eta}, \overline{\zeta})$.

Proposition 3.8. *Let $\mathcal{G} = (\eta, \zeta)$ be a self-complementary Dombi fuzzy graph, then*

$$\sum_{x \neq y} \zeta(xy) = \frac{1}{2} \sum_{x \neq y} \frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)}.$$

Proof. Let $\mathcal{G}$ be a self-complementary Dombi fuzzy graph. Then there exists an isomorphism $\psi : V \to V$ such that $\overline{\eta}(\psi(x)) = \eta(x)$ for all $x \in V$ and $\overline{\zeta}(\psi(x)\psi(y)) = \zeta(xy)$ for all $xy \in E$.
By definition of $\overline{G}$, we have

$$\begin{aligned}
\overline{\zeta}(\psi(x)\psi(y)) &= \frac{\overline{\eta}(\psi(x))\overline{\eta}(\psi(y))}{\overline{\eta}(\psi(x)) + \overline{\eta}(\psi(y)) - \overline{\eta}(\psi(x))\overline{\eta}(\psi(y))} - \zeta(\psi(x)\psi(y)) \\
\zeta(xy) &= \frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)} - \zeta(\psi(x)\psi(y)) \\
\sum_{x \neq y} \zeta(xy) + \sum_{x \neq y} \zeta(\psi(x)\psi(y)) &= \sum_{x \neq y} \frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)} \\
2\sum_{x \neq y} \zeta(xy) &= \sum_{x \neq y} \frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)} \\
\sum_{x \neq y} \zeta(xy) &= \frac{1}{2} \sum_{x \neq y} \frac{\eta(x)\eta(y)}{\eta(x) + \eta(y) - \eta(x)\eta(y)}.
\end{aligned}$$

Hence proved. √ OK ∎

29/04/16

Proposition 3.9. *Let $\mathcal{G} = (\eta, \zeta)$ be ~~a~~ the Dombi fuzzy graph of G. If $\zeta(xy) = \frac{1}{2}\left(\frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}\right)$ for all $x, y \in V$, then $\mathcal{G}$ is self-complementary.*

Proof. ~~If~~ Let $\mathcal{G}$ ~~is~~ be a Dombi fuzzy graph satisfying $\zeta(xy) = \frac{1}{2}\left(\frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}\right)$ for all $x, y \in V$. Then the identity mapping $I : V \to V$ is an isomorphism from $\mathcal{G}$ to $\overline{\mathcal{G}}$. Clearly, I satisfies the condition (c) of Definition 3.13. Since $\zeta(xy) = \frac{1}{2}\left(\frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}\right)$ for all $x, y \in V$, we have

$$\begin{aligned}
\overline{\zeta}(I(x)I(y)) &= \overline{\zeta}(xy) \\
&= \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)} - \zeta(xy) \\
&= \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)} - \frac{1}{2}\left(\frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}\right) \\
&= \frac{1}{2}\left(\frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}\right) \\
&= \zeta(xy).
\end{aligned}$$

Thus the condition (d) of Definition 3.13 is also satisfied by I. Therefore $\mathcal{G}$ is self-complementary. □

Proposition 3.10. *The complements of two isomorphic Dombi fuzzy graphs are isomorphic and conversely.*

Proof. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two isomorphic Dombi fuzzy graphs. Then there exists a bijective mapping $\psi : V_1 \to V_2$ satisfying

$$\begin{aligned}
\eta_1(x) &= \eta_2(\psi(x)) \quad \text{for all } x \in V_1, \\
\zeta_1(xy) &= \zeta_2(\psi(x)\psi(y)) \quad \text{for all } xy \in E_1.
\end{aligned}$$

Using the definition of complement, we have

$$\begin{aligned}
\overline{\zeta_1}(xy) &= \frac{\eta_1(x)\eta_1(y)}{\eta_1(x)+\eta_1(y)-\eta_1(x)\eta_1(y)} - \zeta_1(xy) \\
&= \frac{\eta_2(\psi(x))\eta_2(\psi(y))}{\eta_2(\psi(x))+\eta_2(\psi(y))-\eta_2(\psi(x))\eta_2(\psi(y))} - \zeta_2(\psi(x)\psi(y)) = \overline{\zeta_2}(\psi(x)\psi(y)).
\end{aligned}$$

Hence $\overline{\mathcal{G}_1} \cong \overline{\mathcal{G}_2}$. Similarly, we can prove the converse part. □

Proposition 3.11. *The complements of two weak isomorphic Dombi fuzzy graphs are weak isomorphic.*

pag 18, proof of prop. 3.11.

$$\xi_1(xy) \le \xi_2(\psi(x)\psi(y))$$

$$\Longrightarrow \quad -\xi_1(xy) \ge -\xi_2(\psi(x)\psi(y))$$

$$\Longrightarrow T(\eta_1(x),\eta_1(y)) - \xi_1(xy)$$

$$\ge T(\eta_1(x),\eta_1(y)) - \xi_2(\psi(x)\psi(y))$$

$$\Longrightarrow \quad T(\eta_1(x),\eta_1(y)) - \xi_1(xy)$$

$\eta_1(x)=\eta_2(\psi(x))$

$\eta_1(y)=\eta_2(\psi(y))$

$$\ge T(\eta_2(\psi(x)),\eta_2(\psi(y))) - \xi_2(\psi(x)\psi(y))$$

$$\Longrightarrow \quad \bar{\xi}_1(xy) \ge \bar{\xi}_2(\psi(x)\psi(y)).$$

Proof. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two weak-isomorphic Dombi fuzzy graphs. Then there exists a bijective mapping $\psi : V_1 \to V_2$ satisfying

$$\begin{aligned} \eta_1(x) &= \eta_2(\psi(x)) \quad \text{for all } x \in V_1, \\ \zeta_1(xy) &\leq \zeta_2(\psi(x)\psi(y)) \quad \text{for all } xy \in E_1. \end{aligned}$$

Using the definition of complement, for all $xy \in E_1$, we have

$$\begin{aligned} \overline{\zeta_1}(xy) &= \frac{\eta_1(x)\eta_1(y)}{\eta_1(x)+\eta_1(y)-\eta_1(x)\eta_1(y)} - \zeta_1(xy) \\ &\leq \frac{\eta_2(\psi(x))\eta_2(\psi(y))}{\eta_2(\psi(x))+\eta_2(\psi(y))-\eta_2(\psi(x))\eta_2(\psi(y))} - \zeta_2(\psi(x)\psi(y)) = \overline{\zeta_2}(\psi(x)\psi(y)). \end{aligned}$$

elementary math!

$\geq$!

Hence $\overline{\mathcal{G}_1}$ and $\overline{\mathcal{G}_2}$ are weak isomorphic. □

We state the following proposition without proof.

Proposition 3.12. *The complements of two co-weak isomorphic Dombi fuzzy graphs are homomorphic.*

4 Strong Dombi fuzzy graphs

Definition 4.1. A Dombi fuzzy graph $\mathcal{G} = (\eta, \zeta)$ is called a strong Dombi fuzzy graph of $G = (V, E)$ if $\zeta(xy) = \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}$ for all $xy \in E$.

Example 4.1. Consider a Dombi fuzzy graph over $V = \{j, k, l, m\}$ defined by

$$\eta = \left\{\frac{j}{0.5}, \frac{k}{0.7}, \frac{l}{0.8}, \frac{m}{0.3}\right\}, \quad \zeta = \left\{\frac{jk}{0.41}, \frac{jm}{0.23}, \frac{ml}{0.28}\right\}.$$

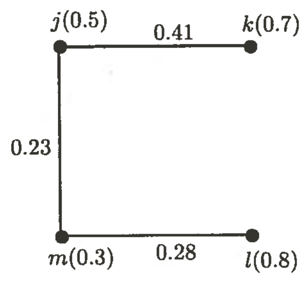


Figure 2: Strong Dombi fuzzy graph.

Proposition 4.1. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are strong Dombi fuzzy graphs, then $\mathcal{G}_1 \times \mathcal{G}_2$ and $\mathcal{G}_1 + \mathcal{G}_2$ are also strong Dombi fuzzy graphs.*

$(\zeta_1 \cup \zeta_2)(xy) = \zeta_1(xy) \le T(\eta_1(x), \eta_1(y))$ ----?

ii) Suppose ~~If~~ $x \in V_1 - V_2, y \in V_1 \cap V_2$ why

$$(\zeta_1 \cup \zeta_2)(xy) \le T\left(\eta_1(x), \frac{\eta_1(y) + \eta_2(y) - 2\eta_1(y)\eta_2(y)}{1 - \eta_1(y)\eta_2(y)}\right)$$

see page (1) en (2)

Putting $e = \eta_1(x), f = \eta_1(y), h = \eta_2(y)$, we get

$$\begin{aligned}
(\zeta_1 \cup \zeta_2)(xy) &\le T\left(e, \frac{f+h-2fh}{1-fh}\right) \\
&= \frac{e(\frac{f+h-2fh}{1-fh})}{e + \frac{f+h-2fh}{1-fh} - e(\frac{f+h-2fh}{1-fh})} \\
&= \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}.
\end{aligned}$$

If $x, y \in V_1 \cap V_2$

$$(\zeta_1 \cup \zeta_2)(xy) \le T\left(\frac{\eta_1(x) + \eta_2(x) - 2\eta_1(x)\eta_2(x)}{1 - \eta_1(x)\eta_2(x)}, \frac{\eta_1(y) + \eta_2(y) - 2\eta_1(y)\eta_2(y)}{1 - \eta_1(y)\eta_2(y)}\right)$$

Putting $e = \eta_1(x), f = \eta_1(y), g = \eta_2(x), h = \eta_2(y)$, we get

$$\begin{aligned}
(\zeta_1 \cup \zeta_2)(xy) &\le T\left(\frac{e+g-2eg}{1-eg}, \frac{f+h-2fh}{1-fh}\right) \\
&= \frac{\frac{e+g-2eg}{1-eg}\frac{f+h-2fh}{1-fh}}{\frac{e+g-2eg}{1-eg} + \frac{f+h-2fh}{1-fh} - \frac{e+g-2eg}{1-eg}, \frac{f+h-2fh}{1-fh}} \\
&= \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}.
\end{aligned}$$

Similarly, it is easy to show that if $xy \in E_2 - E_1$, then

$$(\zeta_1 \cup \zeta_2)(xy) \le \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}.$$

Let $xy \in E_1 \cap E_2$. Then

$$\begin{aligned}
&(\zeta_1 \cup \zeta_2)(xy) = S(\zeta_1(xy), \zeta_2(xy)) \\
&\le S\left(\frac{\eta_1(x)\eta_1(y)}{\eta_1(x) + \eta_1(y) - \eta_1(x)\eta_1(y)}, \frac{\eta_2(x)\eta_2(y)}{\eta_2(x) + \eta_2(y) - \eta_2(x)\eta_2(y)}\right) \\
&= S\left(\frac{ef}{e+f-ef}, \frac{gh}{g+h-gh}\right) = S(T(e,f), T(g,h)) \overset{?}{\le} T(S(e,g), S(f,h)) \\
&= \frac{\frac{ef}{e+f-ef} + \frac{gh}{g+h-gh} - 2\frac{ef}{e+f-ef}\frac{gh}{g+h-gh}}{1 - \frac{ef}{e+f-ef}\frac{gh}{g+h-gh}} \\
&\le \frac{\frac{e+g-2eg}{1-eg}\frac{f+h-2fh}{1-fh}}{\frac{e+g-2eg}{1-eg} + \frac{f+h-2fh}{1-fh} - \frac{e+g-2eg}{1-eg}\frac{f+h-2fh}{1-fh}} \\
&= \frac{(\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}{(\eta_1 \cup \eta_2)(x) + (\eta_1 \cup \eta_2)(y) - (\eta_1 \cup \eta_2)(x)(\eta_1 \cup \eta_2)(y)}.
\end{aligned}$$

This inequality is true if the t-norm T is idempotent ($T(a,a) = a$) which is not the case for the Dombi t-norm. Can you prove it for Dombi?

02/05/2016

Proof. The proof follows at once from the proof of Proposition 3.1 and Theorem 3.2. ☐

Proposition 4.2. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are strong Dombi fuzzy edge graphs, then $\mathcal{G}_1 \Box \mathcal{G}_2$, $\mathcal{G}_1 \bullet \mathcal{G}_2$, $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ and $\mathcal{G}_1[\mathcal{G}_2]$ are also strong Dombi fuzzy edge graphs.*

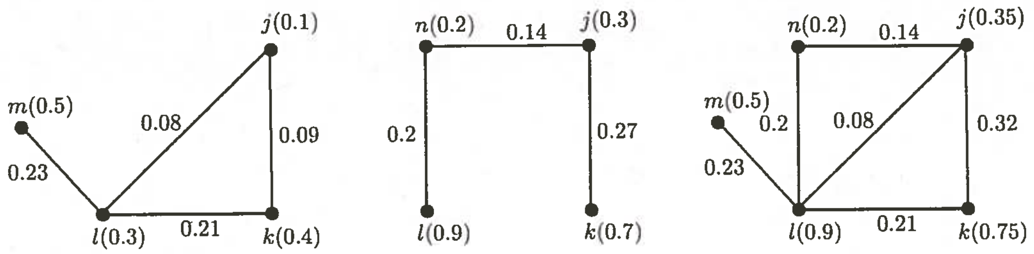
Proof. The proof follows at once from the proof of Propositions 3.2, 3.3, 3.4 and 3.5. ☐

Remark 4.1. In general the Cartesian product, semi-strong product, strong product and lexicographic product of two strong Dombi fuzzy graphs are not Dombi fuzzy graphs.

Remark 4.2. The union of two strong Dombi fuzzy graphs need not be a strong Dombi fuzzy graph. Here is a counterexample.

Example 4.2. We consider two strong Dombi fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\eta_1 = \{\frac{j}{0.1}, \frac{k}{0.4}, \frac{l}{0.3}, \frac{m}{0.5}\}$, $\zeta_1 = \{\frac{jk}{0.09}, \frac{kl}{0.21}, \frac{jl}{0.08}, \frac{lm}{0.23}\}$, $\eta_2 = \{\frac{j}{0.3}, \frac{k}{0.7}, \frac{l}{0.9}, \frac{n}{0.2}\}$ and $\zeta_2 = \{\frac{jk}{0.27}, \frac{nl}{0.2}, \frac{nj}{0.14}\}$. Then we have

$$\eta_1 \cup \eta_2 = \left\{\frac{j}{0.35}, \frac{k}{0.75}, \frac{l}{0.9}, \frac{n}{0.2}, \frac{m}{0.5}\right\}, \quad \zeta_1 \cup \zeta_2 = \left\{\frac{jk}{0.32}, \frac{kl}{0.21}, \frac{ln}{0.2}, \frac{nj}{0.14}, \frac{jl}{0.08}, \frac{ml}{0.23}\right\}.$$



$\mathcal{G}_1$ $\mathcal{G}_2$ $\mathcal{G}_1 \cup \mathcal{G}_2$

It is easy to check that $\mathcal{G}_1 \cup \mathcal{G}_2$ is not ~~a~~ the strong Dombi fuzzy graph of $\mathcal{G}_1 \cup \mathcal{G}_2$.

Proposition 4.3. *If $\mathcal{G}_1 \times \mathcal{G}_2$ is strong Dombi fuzzy graph, then at least $\mathcal{G}_1$ or $\mathcal{G}_2$ must be strong.*

Proof. Assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are not strong Dombi fuzzy graphs. Then there exists $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$ such that

$$\begin{aligned} \zeta_1(x_1y_1) &< \frac{\eta_1(x_1)\eta_1(y_1)}{\eta_1(x_1) + \eta_1(y_1) - \eta_1(x_1)\eta_1(y_1)} = \frac{ab}{a+b-ab}, \\ \zeta_2(x_2y_2) &< \frac{\eta_2(x_2)\eta_2(y_2)}{\eta_2(x_2) + \eta_2(y_2) - \eta_2(x_2)\eta_2(y_2)} = \frac{cd}{c+d-cd}. \end{aligned}$$

Suppose that

$$\zeta_2(x_2y_2) \leq \zeta_1(x_1y_1) < \frac{ab}{a+b-ab} \leq a.$$

$E(\mathcal{G}_1 \times \mathcal{G}_2)$

Let $(x_1, x_2)(y_1, y_2) \in \cancel{E}$, then we have

$$
\begin{aligned}
&(\zeta_1 \times \zeta_2)((x_1, x_2)(y_1, y_2)) = T\,(\zeta_1(x_1y_1), \zeta_2(x_2y_2)) \\
&< T\left(\frac{ab}{a+b-ab}, \frac{cd}{c+d-cd}\right) \\
&= \frac{\frac{abcd}{(a+b-ab)(c+d-cd)}}{\frac{ab}{a+b-ab} + \frac{cd}{c+d-cd} - \frac{abcd}{(a+b-ab)(c+d-cd)}} \\
&= \frac{\frac{abcd}{(a+c-ac)(b+d-bd)}}{\frac{ac}{a+c-ac} + \frac{bd}{b+d-bd} - \frac{abcd}{(a+c-ac)(b+d-bd)}} \\
&= \frac{(\eta_1 \times \eta_2)((x_1, x_2))(\eta_1 \times \eta_2)((y_1, y_2))}{(\eta_1 \times \eta_2)((x_1, x_2)) + (\eta_1 \times \eta_2)((y_1, y_2)) - (\eta_1 \times \eta_2)((x_1, x_2))(\eta_1 \times \eta_2)((y_1, y_2))}.
\end{aligned}
$$

⊥a That is, $\mathcal{G}_1 \times \mathcal{G}_2$ is not strong Dombi fuzzy graph, a contradiction. ~~Hence if $\mathcal{G}_1 \times \mathcal{G}_2$ is strong Dombi fuzzy graph, then at least $\mathcal{G}_1$ or $\mathcal{G}_2$ must be strong.~~ □

⊥a **Proposition 4.4.** *If $\mathcal{G}_1 + \mathcal{G}_2$ is strong Dombi fuzzy graph, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are both strong.*

Proof. Obvious. □

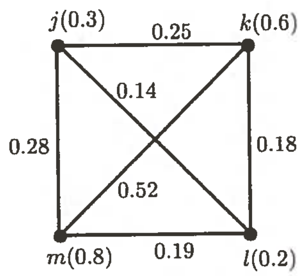
Definition 4.2. The complement of a strong Dombi fuzzy graph $\mathcal{G} = (\eta, \zeta)$ of $G = (V, E)$ is a strong Dombi fuzzy graph $\overline{\mathcal{G}} = (\overline{\eta}, \overline{\zeta})$ of $\overline{G} = (\overline{V}, \overline{E})$, where $\overline{\eta}(x) = \eta(x)$ for all $x \in V$ and $\overline{\zeta}$ is defined as follows:

$$
\overline{\zeta}(xy) = \begin{cases} \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)} & \text{if } \zeta(xy) = 0, \\ 0 & \text{if } 0 < \zeta(xy) \leq 1. \end{cases}
$$

Definition 4.3. A Dombi fuzzy graph $\mathcal{G} = (\eta, \zeta)$ is said to be complete if $\zeta(xy) = \frac{\eta(x)\eta(y)}{\eta(x)+\eta(y)-\eta(x)\eta(y)}$ for all $x, y \in V$.

Example 4.3. Consider a Dombi fuzzy graph over $V = \{j, k, l, m\}$ defined by

$$
\eta = \{\frac{j}{0.3}, \frac{k}{0.6}, \frac{l}{0.2}, \frac{m}{0.8}\}, \ \zeta = \{\frac{jk}{0.25}, \frac{kl}{0.18}, \frac{lm}{0.19}, \frac{mj}{0.28}, \frac{jl}{0.14}, \frac{km}{0.52}\}.
$$



Remark 4.3. Every complete Dombi fuzzy graph is strong.