Vector valued Hardy spaces

Richard D. Carmichael
Department of Mathematics
Wake Forest University
Winston-Salem, N.C. 27109-7388
U.S.A.
E-mail: carmicha@wfu.edu

Stevan Pilipović
Department of Mathematics and Informatics
Faculty of Sciences
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad
Serbia
E-mail: Stevan.Pilipovic@dmi.uns.ac.rs

Jasson Vindas
Department of Mathematics
Ghent University
Krijgslaan 281 Building S25
B 9000 Gent
Belgium
E-mail: Jasson.Vindas@ugent.be

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Abstract. The Hardy space H^p of vector valued analytic functions in tube domains in $\mathbb{C}^n$ and with values in Banach space are defined. Vector valued analytic functions in tube domains in $\mathbb{C}^n$ with values in Hilbert space and which have vector valued tempered distributions as boundary value are proved to be in H^p corresponding to Hilbert space if the boundary value is in L^p with values in Hilbert space. A Poisson integral representation for such vector valued analytic

functions is obtained.

1 Introduction

The representation of tempered distributions as boundary values of analytic functions was first accomplished by H-G Tillmann [22] whose analysis was for functions analytic in half planes and tubes defined by n-rants. Meise ([15], [16]) extended results of this type to vector valued tempered distributions. Carmichael and Walker [2] have obtained a boundary value result for vector valued functions analytic in tubes defined by cones with the boundary value being in the vector valued tempered distribution space. The tempered distributions have also been extended to tempered ultradistributions; see Pilipović [17] and the book [7]. Recently in [10] and [11] a large class of distribution spaces has been introduced and studied whose definition is based on translation-invariant Banach spaces and which generalize the Schwartz $\mathcal{D}'_{L^p}$ spaces; a complete theory of boundary value results and analytic representations of these new distributions is obtained. Further, new results associated with the analysis of this paper are contained in [8] and [9]. Boundary value results concerning quasianalytic ultradistributions are obtained in [8], and classical results on boundary values in distribution spaces are important tools in the study of complex Tauberian theorems for Laplace transforms in [9]. The reader should especially note Schwartz ([19], [20]) for a general background of vector valued distributions.

In [18] Raina showed that if the distributional boundary value of an analytic function f in the upper half plane obtained in the dual space of a space of type $\mathcal{S}$ was in fact a L^p, $1 \leq p \leq \infty$, function then the analytic function f was in the Hardy space H^p in the upper half plane. This and associated results found applications in particle physics. In [5] Carmichael and Richters generalized the result of Raina to functions analytic in tube domains and for the tempered and other distribution topologies; associated results and representations were obtained. The main proof of [5] was obtained through representing the assumed analytic function as a Poisson integral in tubes.

As noted above Carmichael and Walker [2, Theorem 8] have obtained a vector valued distributional boundary value result concerning vector valued analytic functions in tubes in $\mathbb{C}^n$ which obtain tempered vector valued distributions as boundary value. In this paper we desire to extend the results of [5] to the vector valued case by showing that if the boundary value in [2, Theorem 8] is in fact a L^p function with values in a Hilbert space $\mathcal{H}$, then the analytic function, which has values in $\mathcal{H}$, must be in the vector valued Hardy space H^p which we define in this paper.

2 Notation and definitions

Throughout $\mathcal{B}$ will denote a Banach space, $\mathcal{H}$ will denote a Hilbert space, $\mathcal{N}$ will denote the norm of the specified Banach or Hilbert space, and Θ will denote the zero vector of the specified Banach or Hilbert space. We reference Dunford and Schwartz [12] for integration of vector valued functions and for vector valued analytic functions. For foundational information concerning vector valued distributions we refer to Schwartz ([19], [20]).

The n-dimensional notation used in this paper will be the same as that in [4] and [7]. Basic information concerning cones in $\mathbb{R}^n$ can be found in Vladimirov [23] and in the books [6] and [7]. We recall some needed concepts of cones that are important for this paper. $C \subset \mathbb{R}^n$ is a cone (with vertex at $\overline{0} = (0, 0, ..., 0)$ in $\mathbb{R}^n$) if $y \in C$ implies $\lambda y \in C$ for all positive scalars λ. The intersection of C with the unit sphere $|y| = 1$ is called the projection of C and is denoted $pr(C)$. A cone C' such that $pr(\overline{C'}) \subset pr(C)$ is a compact subcone of C. The function

$$u_C(t) = \sup_{y \in pr(C)} (- < t, y >), t \in \mathbb{R}^n, \tag{1}$$

is the indicatrix of C. The dual cone C^* of C is $C^* = \{t \in \mathbb{R}^n : < t, y > \geq 0 \ for\ all\ y \in C\} = \{t \in \mathbb{R}^n : u_C(t) \leq 0\}$. A convex cone which does not contain any entire straight line is called a regular cone. Let $v = (v_1, v_2, ..., v_n)$ be any of the 2^n n-tuples whose entries are 0 or 1; the 2^n n-rants $C_v = \{y \in \mathbb{R}^n : (-1)^{v_j} y_j > 0, j = 1, 2, ..., n\}$ are examples of regular cones in $\mathbb{R}^n$ as are the future and past light cones [23, p. 219]. The n-rants and the light cones are also examples of self dual cones in which the closure of the cone is equal to the dual cone of the cone.

Vector valued functions will be denoted by bold letters; and the space of strongly measurable functions $\mathbf{f}$ with the property $\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(t)))^p dt < \infty, p \in [1, \infty)$, is denoted by $L^p(\mathbb{R}^n, \mathcal{B})$. The symbol $|\mathbf{h}|_p$ will denote the norm of $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{B})$. In the case $p = \infty$ we make the obvious change of the definition. Schwartz spaces $\mathcal{S}$ and $\mathcal{S}^{(m)}, m \in \mathbb{N}$, over $\mathbb{R}^n$ are the test spaces; and $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ and $\mathcal{S}^{(m)'}(\mathbb{R}^n, \mathcal{B})$ are the spaces of tempered vector valued distributions with values in $\mathcal{B}$ with the dual pairing

$$\langle \mathbf{f}(t), \phi(t) \rangle \in \mathcal{B}, \phi \in \mathcal{S}.$$

Elements of $L^p(\mathbb{R}^n, \mathcal{B})$ define the regular elements of $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ by

$$\langle \mathbf{f}(t), \phi(t) \rangle = \int_{\mathbb{R}^n} \mathbf{f}(t) \phi(t) dt, \phi \in \mathcal{S} \text{ (or } \phi \in \mathcal{S}^{(m)}).$$

We define the Fourier transform in $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ as a transpose mapping

$$\langle \widehat{\mathbf{f}}(t), \phi(t) \rangle = \langle \mathbf{f}(t), \widehat{\phi}(t) \rangle, \phi \in \mathcal{S},$$

where the Fourier transform of $\phi \in \mathcal{S}$ (or for any $L^1(\mathbb{R}^n)$ function) is

$$\widehat{\phi}(x) = \mathcal{F}(\phi)(x) = \mathcal{F}[\phi(t); x] = \int_{\mathbb{R}^n} e^{2\pi i < x, t >} \phi(t) dt, \ x \in \mathbb{R}^n.$$

Appropriate references for vector valued weighted functions in this paper are [1] and [14]. If $\mathbf{f} \in L^1(\mathbb{R}^n, \mathcal{B})$, then its Fourier transform is defined by the

Bochner integral

$$\mathcal{F}(\mathbf{f})(x) = \mathcal{F}[\mathbf{f}(t); x] = \int_{\mathbb{R}^n} e^{2\pi i <x,t>} \mathbf{f}(t) dt, x \in \mathbb{R}^n;$$

while $\mathcal{F}^{-1}(\mathbf{f})(x) = \mathcal{F}(\mathbf{f})(-x)$. Then ([1], [14]) $\mathcal{F}(\mathbf{f}) \in L^\infty(\mathbb{R}^n, \mathcal{B})$ and the Hausdorff-Young inequality holds. Moreover, $\widehat{\mathbf{f}} \in C_0(\mathbb{R}^n, \mathcal{B})$ which means that $\widehat{\mathbf{f}}$ is continuous and $\mathcal{N}(\widehat{\mathbf{f}}(x)) \to 0$ as $|x| \to \infty$.

For the case $p = 2$ in order to have an isomorphism of $\mathcal{F}$ of $L^2(\mathbb{R}^n, \mathcal{B})$ onto itself (with the Parseval identity $||\mathcal{N}(\widehat{\mathbf{f}})||_{L^2(\mathbb{R}^n)} = ||\mathcal{N}(\mathbf{f})||_{L^2(\mathbb{R}^n)}$) it is necessary and sufficient that $\mathcal{B} = \mathcal{H}$ is a Hilbert space. Recall that in this case $\mathcal{F}$ is defined first on a dense set and then extended on $L^2(\mathbb{R}^n, \mathcal{H})$ [1].

By the Fubini theorem ([1]), the Fourier transform of $\mathbf{f}$ in the cases of $L^1(\mathbb{R}^n, \mathcal{B})$ and $L^2(\mathbb{R}^n, \mathcal{H})$ defined in the sense of Bochner integral imbedded into the $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ coincide with the $\widehat{\mathbf{f}}$ defined as the distributional Fourier transform. We will use this fact in the sequel.

The Hardy space $H^p(T^C, \mathcal{B})$, $1 \le p < \infty$, consists of those analytic functions $\mathbf{f}(z)$ in the tube $T^C = \mathbb{R}^n + iC$ with values in $\mathcal{B}$ such that

$$\textstyle\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x + iy)))^p dx \ \le \ A, \, y \in C,$$

where the constant A is independent of $y \in C$. The Hardy space $H^\infty(T^C, \mathcal{B})$ is defined with the usual changes.

Let C be an open convex cone in $\mathbb{R}^n$. The following function $d_y(t), t \in \mathbb{R}^n, y \in C$, will be used. Let $s(u) \in \mathcal{E}(\mathbb{R}), u \in \mathbb{R}$, such that $s(u) = 1, u \ge 0, s(u) = 0, u \le -\epsilon, \epsilon > 0$ and fixed, and $0 \le s(u) \le 1$. Put

$$d_y(t) = s(< t, y >), \ t \in \mathbb{R}^n, \ y \in C. \tag{2}$$

We have $d_y(t) \in \mathcal{E}(\mathbb{R}^n)$, for any $y \in C$.

3 Preliminary results

We indicate results in this section which we need to prove the main results of this paper contained in section 4. Recall that $C \subset \mathbb{R}^n$ is a regular cone if it is an open convex cone which does not contain any entire straight line.

For C being a regular cone, the Cauchy kernel corresponding to the tube $T^C = \mathbb{R}^n + iC$ is

$$K(z - t) = \textstyle\int_{C^*} e^{2\pi i <z-t,\eta>} d\eta, \, t \in \mathbb{R}^n, \, z \in T^C,$$

where C^* is the dual cone of C. The Poisson kernel corresponding to T^C is

$$Q(z; t) = \tfrac{K(z-t)\overline{K(z-t)}}{K(2iy)} = \tfrac{|K(z-t)|^2}{K(2iy)}, \, t \in \mathbb{R}^n, \, z \in T^C.$$

Referring to [7] for details we know that $K(z-\cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}, 1 < p \leq \infty$, and $Q(z;\cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}, (z \in T^C) 1 \leq p \leq \infty$, where $*$ is Beurling (M_p) or Roumieu $\{M_p\}$. These ultradifferentiable functions are contained in the Schwartz space $\mathcal{D}(L^p, \mathbb{R}^n)$. We will use the results of [5, Lemmas 3.1 and 3.2] here as well as the calculation [5, (4.8)] and calculations in [5, section 2], and they will not be restated here.

Because of these properties of the Cauchy and Poisson kernels, we know that the Cauchy and Poisson integrals

$$\int_{\mathbb{R}^n} \mathbf{h}(t) K(z-t)dt \text{ and } \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t)dt,\ z \in T^C,$$

are well defined for $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{B}), 1 \leq p < \infty$, and $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{B}), 1 \leq p \leq \infty$, respectively.

Several lemmas are proved now which are needed for section 4.

LEMMA 3.1. *Let C be an open connected cone and C' be an arbitrary compact subcone of C. Let $r > 0$ be arbitrary. Let $\mathbf{g}(t), t \in \mathbb{R}^n$, be a continuous function with support in C^* and with values in a Banach space $\mathcal{B}$ which satisfies*

$$\mathcal{N}(\mathbf{g}(t)) \leq M(C', r) exp(2\pi(< y', t > + \sigma|y'|)),\ t \in \mathbb{R}^n, \tag{3}$$

for all $\sigma > 0$, where $M(C', r)$ is a constant which depends on C' and on $r > 0$ and (3) is independent of $y' \in (C' \setminus (C' \cap N(\overline{0}, r)))$ (that is, (3) holds for all $y' \in (C' \setminus (C' \cap N(\overline{0}, r))))$. Let y be any point of C. We have

$$(e^{-2\pi <y,\cdot>} \mathbf{g}) \in L^p(\mathbb{R}^n, \mathcal{B})$$

for all $p, 1 \leq p < \infty$.

Proof. Let $y \in C$ be arbitrary but fixed. There exists $C' \subset C$ and $r > 0$ such that $y \in (C' \setminus (C' \cap N(\overline{0}, r)))$. Choose μ such that $1 > \mu > (r/|y|) > 0$ and put $y' = \mu y$. We have $y' \in C'$ since C' is a cone and $|\mu y| = \mu |y| > r > 0$; thus $y' \in C' \setminus (C' \cap N(\overline{0}, r))$. By [6, Lemma 4.3.2, p. 155] there is a $\delta > 0$ such that

$$< y, t > \geq \delta |y||t|,\ t \in C^*,\ y \in C', \tag{4}$$

and δ depends only on C' and not on $y \in C'$. Now taking $y' = \mu y$ in (3) we have for $t \in C^*$ and $y \in C'$

$$\begin{aligned} &\mathcal{N}(e^{-2\pi <y,t>} \mathbf{g}(t)) = e^{-2\pi <y,t>} \mathcal{N}(\mathbf{g}(t)) \\ &\leq M(C', r) exp(2\pi(\mu < y, t > + \sigma\mu|y| - < y, t >)) \\ &= M(C', r) e^{2\pi\sigma\mu|y|} exp(2\pi(1-\mu)(- < y, t >)) \\ &\leq M(C', r) e^{2\pi\sigma\mu|y|} exp(-2\pi\delta(1-\mu)|y||t|) \end{aligned}$$

with $\delta > 0$ and $(1-\mu) > 0$. Since $supp(\mathbf{g}) \subseteq C^*$, we have for $1 \leq p < \infty$ and the arbitrary but fixed $y \in C$ at the beginning of the proof

$$
(5) \qquad \int_{\mathbb{R}^n} (\mathcal{N}(e^{-2\pi<y,t>}\mathbf{g}(t)))^p dt = \int_{C^*} (\mathcal{N}(e^{-2\pi<y,t>}\mathbf{g}(t)))^p dt
$$
$$
\leq (M(C',r))^p e^{2\pi\sigma\mu p|y|} \int_{C^*} exp(-2\pi\delta(1-\mu)p|y||t|)dt
$$
$$
\leq Z_n (M(C',r))^p e^{2\pi\sigma\mu p|y|} \int_0^\infty u^{n-1} exp(-2\pi\delta(1-\mu)p|y|u)du
$$
$$
= Z_n (M(C',r))^p e^{2\pi\sigma\mu p|y|}(n-1)!(2\pi\delta(1-\mu)p|y|)^{-n}
$$

where Z_n is the surface area of the unit sphere in $\mathbb{R}^n$, which proves $e^{-2\pi<y,t>}\mathbf{g}(t) \in L^p(\mathbb{R}^n, \mathcal{B}), 1 \leq p < \infty$, for any $y \in C$.

In the following Lemmas 3.2, 3.3, and 3.4 C will be a regular cone. The vector valued functions in Lemma 3.2 have values in a Hilbert space as opposed to a general Banach space because we use the Fourier transform in the proof; recall that if $\mathcal{B} = \mathcal{H}$, a Hilbert space, the Fourier transform is a bijection on $L^2(\mathbb{R}^n, \mathcal{H})$. For Lemma 3.4 $\mathcal{B}$ is an arbitrary Banach space.

LEMMA 3.2. *Let* $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ *and* $\mathbf{G}(\eta) = \mathcal{F}^{-1}[\mathbf{g}(t);\eta], \eta \in \mathbb{R}^n$, *in* $L^2(\mathbb{R}^n, \mathcal{H})$. *Assume* $\mathbf{G}exp(2\pi i < z, \cdot >) \in L^1(\mathbb{R}^n, \mathcal{H})$ *for* $z \in T^C$ *and that* $supp(\mathbf{G}) \subseteq C^*$ *almost everywhere. We have*

$$
\int_{\mathbb{R}^n} \mathbf{G}(\eta) e^{2\pi i<z,\eta>} d\eta = \int_{\mathbb{R}^n} \mathbf{g}(t) K(z-t) dt, \ z \in T^C.
$$

Proof. Let $I_{C^*}(\eta)$ denote the characteristic function of C^*. By [5, Lemma 2.1] $I_{C^*}(\eta)e^{2\pi i<z,\eta>} \in L^p, 1 \leq p \leq \infty$, as a function of $\eta \in \mathbb{R}^n$ for $z \in T^C$. Recalling the Cauchy kernel $K(z-t), t \in \mathbb{R}^n, z \in T^C$, we have

$$
\int_{\mathbb{R}^n} \mathbf{g}(t) K(z-t) dt = \int_{\mathbb{R}^n} \mathbf{g}(t) \int_{C^*} e^{2\pi i<z-t,\eta>} d\eta dt
$$
$$
= \int_{\mathbb{R}^n} \mathbf{g}(t) \mathcal{F}^{-1}[I_{C^*}(\eta)e^{2\pi i<z,\eta>};t] dt = \int_{C^*} \mathcal{F}^{-1}[\mathbf{g}(t);\eta] e^{2\pi i<z,\eta>} d\eta
$$
$$
= \int_{C^*} \mathbf{G}(\eta) e^{2\pi i<z,\eta>} d\eta = \int_{\mathbb{R}^n} \mathbf{G}(\eta) e^{2\pi i<z,\eta>} d\eta.
$$

LEMMA 3.3. *Let* z_o *be an arbitrary but fixed point in* T^C. *Let* $1 \leq p \leq \infty$. *There exists a closed neighborhood* $N(z_o,\rho) = \{z : |z - z_o| \leq \rho, \ \rho > 0\}$ *of* z_o *which is contained in* T^C *and a constant* $B(z_o)$ *depending only on* z_o *such that*

$$
||Q(z;t)||_{L^p} \ \leq \ B(z_o) \ < \ \infty, \ z \in N(z_o,\rho),
$$

where the L^p *norm is with respect to* $t \in \mathbb{R}^n$ *and* $Q(z;t), t \in \mathbb{R}^n$, $z \in T^C$, *is the Poisson kernel.*

Proof. See [5, Lemma 3.4].

LEMMA 3.4. *Let* $\mathbf{f}$ *be analytic in* T^C *with values in a Banach space* $\mathcal{B}$ ($\mathbf{f} \in \mathcal{A}(T^C, \mathcal{B})$) *and have the Poisson integral representation*

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t) dt, \ z \in T^C,$$

for $\boldsymbol{h} \in L^p(\mathbb{R}^n, \mathcal{B}), 1 \leq p \leq \infty$. *We have* $\mathbf{f} \in H^p(T^C, \mathcal{B}), 1 \leq p \leq \infty$. *For* $p = \infty$, $\mathbf{f}(x+iy) \to \mathbf{h}(x)$ *in the weak-star topology of* $L^\infty(\mathbb{R}^n, \mathcal{B})$ *as* $y \to \overline{0}, y \in C$; *for* $1 \leq p < \infty$, $\mathbf{f}(x+iy) \to \mathbf{h}(x), x \in \mathbb{R}^n$, *as* $y \to \overline{0}, y \in C$, *in* $L^p(\mathbb{R}^n, \mathcal{B})$; *for* $1 < p \leq 2$

$$\mathcal{N}(\mathbf{f}(x+iy)) \leq M(C') |\mathbf{h}|_p |y|^{-n/p}, \ z = x+iy \in T^{C'}, \tag{6}$$

for all compact subcones $C' \subset C$, $M(C')$ *being a constant depending on* $C' \subset C$ *and not on* $y \in C'$, *while*

$$\mathcal{N}(\mathbf{f}(x+iy)) \leq M_y |\mathbf{h}|_p |y|^{-n/p}, \ z = x+iy \in T^C,$$

where M_y *is a constant depending on* $y \in C$; *and for* $2 < p < \infty$

$$\begin{aligned}&\mathcal{N}(\mathbf{f}(x+iy)) \leq M(C', r)|\mathbf{h}|_p, \\ &z = x+iy \in T(C', r) \ = \ \{z = x+iy : x \in \mathbb{R}^n, \ y \in (C' \setminus (C' \cap N(\overline{0}, r)))\},\end{aligned}$$

for all compact subcones $C' \subset C$ *and all* $r > 0, M(C', r)$ *being a constant depending on* $C' \subset C$ *and on* $r > 0$ *but not on* $y \in (C' \setminus (C' \cap N(\overline{0}, r)))$ *while*

$$\mathcal{N}(\mathbf{f}(x+iy)) \leq M_y |\mathbf{h}|_p, \ z = x+iy \in T^C,$$

where M_y *is a constant depending on* $y \in C$.

Proof. For $p = \infty$ and $z \in T^C$ we use [5, (3.3) and (3.4)] to obtain

$$\mathcal{N}(\mathbf{f}(z)) \leq \int_{\mathbb{R}^n} \mathcal{N}(\mathbf{h}(t)) Q(z;t) dt \ \leq \ A$$

where A is the bound on $\mathbf{h}$; and $\mathbf{f} \in H^\infty(T^C, \mathcal{B})$. Also for $p = \infty$ the weak-star convergence of $\mathbf{f}(x+iy)$ to $\mathbf{h}(x)$ as $y \to \overline{0}, \ y \in C$, is proved as in the scalar valued case using the approximate identity properties of the Poisson kernel; see [5, Lemma 3.5]. For $1 \leq p < \infty$ we use Jensen's inequality [13, 2.4.19, p. 91], which holds for Banach spaces $\mathcal{B}$, and [5, Lemma 3.1] to obtain for $y \in C$

$$\begin{aligned}&\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x+iy)))^p dx \leq \int_{\mathcal{R}^n} \int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(t)))^p Q(z;t) dt dx \\ &= \int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(t)))^p \int_{\mathbb{R}^n} Q(z;t) dx dt = \int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(t)))^p dt \ < \ \infty,\end{aligned}$$

and $\mathbf{f} \in H^p(T^C, \mathcal{B})$.

We use the properties of $Q(z;t)$ in [5, Lemma 3.1] and the associated properties of $Q(u;y), u \in \mathbb{R}^n, y \in C$, stated in [21, p. 105]. For $z = x + iy \in T^C$ and $\rho > 0$

$$
\begin{aligned}
&|\mathbf{f}(x+iy) - \mathbf{h}(x)|_p = \left(\int_{\mathbb{R}^n} (\mathcal{N}(\int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t) dt - \int_{\mathbb{R}^n} \mathbf{h}(x) Q(z;t) dt))^p dx \right)^{1/p} \\
&\leq (\int_{\mathbb{R}^n} 2^p ((\mathcal{N}(\int_{|u| \leq \rho} (\mathbf{h}(x-u) - \mathbf{h}(x)) Q(u;y) du))^p \\
&+ (\mathcal{N}(\int_{|u| > \rho} (\mathbf{h}(x-u) - \mathbf{h}(x)) Q(u;y) du))^p) dx)^{1/p} \\
&\leq 2^{(p+1)/p} (\int_{|u| \leq \rho} \left(\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(x-u) - \mathbf{h}(x)))^p dx \right)^{1/p} Q(u;y) du \\
&+ \int_{|u| > \rho} \left(\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(x-u) - \mathbf{h}(x)))^p dx \right)^{1/p} Q(u;y) du) \\
&\leq 2^{(p+1)/p} \left(\sup_{|u| \leq \rho} \left(\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(x-u) - \mathbf{h}(x)))^p dx \right)^{1/p} \right) \int_{\mathbb{R}^n} Q(u;y) du \\
&+ 2^{(3p+2)/p} |\mathbf{h}|_p \int_{|u| > \rho} Q(u;y) du \, .
\end{aligned}
$$

The sup term $\to 0$ as $|u| \to 0$; thus we can choose $\rho > 0$ such that

$$2^{(p+1)/p} (\sup_{|u| \leq \rho} (\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(x-u) - \mathbf{h}(x)))^p dx)^{1/p}) \ < \ \epsilon$$

for $\epsilon > 0$. For this chosen ρ we have

$$\int_{|u| > \rho} Q(u;y) du \ \to \ 0$$

as $y \to \overline{0}, y \in C$, from [21, p. 105]. Combining we obtain $\mathbf{f}(x+iy) \to \mathbf{h}(x)$ in $L^p(\mathbb{R}^n, \mathcal{B})$ as $y \to \overline{0}, y \in C$.

For the remainder of the conclusions recall from section 3 that for $z \in T^C, K(z-t) \in \mathcal{D}(*, L^p) \subset \mathcal{D}(L^p, \mathbb{R}^n), 1 < p \leq \infty$, and $Q(z;t) \in \mathcal{D}(*, L^p) \subset \mathcal{D}(L^p, \mathbb{R}^n), 1 \leq p \leq \infty$, where $*$ is either Beurling or Roumieu. Now let $1 < p \leq 2$. By Hölder's inequality

$$\mathcal{N}(\mathbf{f}(x+iy)) \ \leq \ |\mathbf{h}|_p ||Q(x+iy;t)||_{L^q}, \ z = x + iy \in T^C, \tag{7}$$

$1/p + 1/q = 1$. From the definition of the Poisson kernel for $z \in T^C$

$$||Q(x+iy;t)||_{L^q} = (K(2iy))^{-1} (\int_{\mathbb{R}^n} |K(x+iy-t)|^{2q} dt)^{1/q}, \ z = x + iy \ \in \ T^C. \tag{8}$$

By [5, Lemma 2.1] $I_{C^*}(\eta) e^{2\pi i <z,\eta>} \ \in \ L^1 \cap L^p, \ 1 < p \ \leq 2$, for $z \in T^C$. Thus

$$K(z-t) \;=\; \mathcal{F}^{-1}[I_{C^*}(\eta)e^{2\pi i<z,\eta>};t],\; z \;\in\; T^C,$$

and by the Parseval inequality

$$||K(z-t)||_{L^q} \;\leq\; ||I_{C^*}(\eta)e^{2\pi i<z,\eta>}||_{L^p}\,,\; z \;\in\; T^C.$$

By this Parseval inequality and analysis as in (5) we have

$$(9) \qquad ||K(x+iy-t||_{L^q} \;\leq\; \left(\frac{Z_n(n-1)!}{(2\pi p\delta)^n}\right)^{1/p} \; |y|^{-n/p}$$

with δ depending on $y \in C$ if (9) is taken to hold for all $y \in C$ while δ depends on $C' \subset C$, and not on $y \in C' \subset C$, if (9) is taken to hold for $y \in C' \subset C$ for C' being any compact subcone of C. From [4, Lemma 2],

$$(10) \qquad K(2iy) \;\geq\; B(C)|y|^{-n},\; y \;\in\; C,$$

where the constant $B(C)$ depends only on C and not on $y \in C$. Additionally from [3, Lemma 3],

$$(11) \qquad |K(z-t)| \;\leq\; Z_n(n-l)!\delta^{-n}|y|^{-n}\,,\; t \;\in\; \mathbb{R}^n,$$

with $\delta > 0$ depending on $y \in C$ and (11) holding for $z = x + iy \in T^C$ while (11) holds for all $z = x+iy \in T^{C'}, C'$ being an arbitrary compact subcone of C, with δ depending only on $C' \subset C$ and not on $y \in C' \subset C$. Now the desired norm growth on $\mathcal{N}(\mathbf{f}(\cdot + iy))$ for $1 < p \leq 2$ follows by combining (7), (8), (9), (10), and (11).

Now let $2 < p < \infty$. Again by Hölder's inequality as in (7), we combine (8), (10), and (11) to obtain

$$\mathcal{N}(\mathbf{f}(\cdot + iy)) \leq |\mathbf{h}|_p(|y|^n/B(C))(Z_n(n-1)!/\delta^n|y|^n)||K(\cdot + iy - t)||_{L^q}$$

with the $\delta > 0$ depending on $C' \subset C$ for $y \in C' \subset C$ and not on y while $\delta > 0$ depends on y if $y \in C$. From the proof of [7, Theorem 4.1.1] or by using (9), for $z = x+iy \in T(C',r) = \{z = x+iy : x \in \mathbb{R}^n,\; y \in (C' \setminus (C' \cap N(\overline{0},r)))\}$ where C' is any compact subcone of C and $r > 0$ is arbitrary we have

$$||K(x+iy-t)||_{L^q} \leq P(C',r),\; z = x+iy \in T(C',r),$$

while

$$||K(x+iy-t)||_{L^q} \leq P_y,\; z = x+iy \in T^C,$$

with the constant P_y depending on $y \in C$. Combining the inequalities we have the desired estimates for $\mathcal{N}(\mathbf{f}(x+iy))$ when $2 < p < \infty$.

The following lemma is proved by the same proof as [5, Lemma 3.6].

LEMMA 3.5. *Let* $\mathbf{h} \in L^\infty(\mathbb{R}^n, \mathcal{B})$. *Let* C *be a regular cone. Put*

$$X_\epsilon(t) = \prod_{j=1}^n (1 - i\epsilon(-1)^{v_j} t_j)^{R+n+2},\ \epsilon > 0,\ t \in \mathbb{R}^n,$$

where $R \geq 0$ is a fixed real number, n is the dimension, and $v = (v_1, v_2, ..., v_n)$ is any of the 2^n n-tuples whose entries are 0 or 1 that defines the quadrant C_v. We have

$$\lim_{\epsilon\to 0+} \mathcal{N}\left(\int_{\mathbb{R}^n} (\mathbf{h}(t) - \frac{\mathbf{h}(t)}{X_\epsilon(t)}) Q(z;t)dt\right) \ = \ 0$$

uniformly in z on compact subsets of T^C.

4 $H^p(T^C, \mathcal{H})$ functions, $2 \leq p \leq \infty$

We begin by restating [2, Theorem 8, p. 327] in a more general form which is proved by the same analysis of [2, Theorem 8, p. 327]. In Theorem 4.1 $\mathcal{B}$ can be any space assumed in [2, Theorem 8]. However $\mathcal{B}$ being a Banach space is of primary interest in this paper.

THEOREM 4.1. *Let C be an open convex cone. Let $\mathbf{f} \in \mathcal{A}(T^C, \mathcal{B})$. For every compact subcone $C' \subset C$ and every $r > 0$ let*

$$(12)\qquad \begin{aligned} &\mathcal{N}(\mathbf{f}(x+iy)) \leq M(C', r)(1+|x|)^R |y|^{-k}, \\ &z = x+iy \ \in \ T(C', r) \ = \ \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0}, r))), \end{aligned}$$

where $M(C', r)$ is a constant depending on $C' \subset C$ and on r, R is a nonnegative integer, k is an integer greater than 1, and neither R nor k depend on C' or r. There exists a positive integer m and a unique element $\mathbf{U} \in \mathcal{S}^{(m)'}(\mathbb{R}^n, \mathcal{B}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ such that

$$(13)\qquad \lim_{y\to\overline{0}, y\in C} \mathcal{N}(\langle \mathbf{f}(x+iy), \phi(x)\rangle \ - \ \langle \mathbf{U}, \phi\rangle) \ = \ 0,\ \phi \in \mathcal{S}^{(m)}.$$

$\mathbf{U}$ here will be called the $\mathcal{S}^{(m)'}(\mathbb{R}^n, \mathcal{B}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ boundary value of $\mathbf{f}(\cdot + iy)$. In Theorem 4.1 and in the other theorems in this section, by $y \to \overline{0},\ y \in C$, we mean that $y \to \overline{0},\ y \in C' \subset C$ for every compact subcone C' of C.

The element $\mathbf{U}$ in the conclusion of Theorem 4.1 could be a function $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{B})$. We now prove that if this is the case for $\mathcal{B}$ being a Hilbert space $\mathcal{H}$ and $2 \leq p \leq \infty$, the analytic function $\mathbf{f}$ in Theorem 4.1 is in fact an element of $H^p(T^C, \mathcal{H})$. We prove this in two steps. First we consider the cone C to be contained in or be any of the 2^n n-rants C_v; we then use this case to prove our result for C being any regular cone.

Before proceeding to the statement and proof of our desired result for the case that the cone C satisfies $C \subseteq C_v$, we first give an outline of the proof. First note that the functions and distributions are assumed to have values in a Hilbert space $\mathcal{H}$ now as opposed to a general Banach space; the reason for this is that we use the function Fourier transform in the proof. We first consider the case $p = 2$. Starting with the assumed analytic function $\mathbf{f}(z),\ z \in T^C$, which has boundary value $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, we multiply $\mathbf{f}(z)$ by an analytic function depending on $\epsilon > 0$ such that the product $\mathbf{g}_\epsilon(z)$ has a stronger growth than $\mathbf{f}(z)$ and has a boundary value $\mathbf{U}_\epsilon \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0},\ y \in C,\ \epsilon > 0$. A function of $t \in \mathbb{R}^n$ depending on $\epsilon > 0$, $\mathbf{G}_\epsilon(t)$, is constructed as the Fourier-Laplace transform of $\mathbf{g}_\epsilon(z)$ and needed properties of $\mathbf{G}_\epsilon(t)$ are obtained. We show $\mathbf{U}_\epsilon = \mathcal{F}[\mathbf{G}_\epsilon]$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ and proceed to construct a function $\mathbf{H}_\epsilon \in L^2(\mathbb{R}^n, \mathcal{H})$, which equals $\mathbf{G}_\epsilon$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, from which we show that the Poisson integral of $\mathcal{F}[\mathbf{H}_\epsilon(x); t]$ in L^2 equals $\mathbf{g}_\epsilon(z),\ z \in T^C$. From this Poisson integral representation of $\mathbf{g}_\epsilon(z)$ and its analyticity in T^C we prove that the Poisson integral of the assumed boundary value $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$ in the theorem is analytic in T^C using a limit argument and that this limit Poisson integral is in $H^2(T^C, \mathcal{H})$. To conclude the proof we show that the original assumed $\mathbf{f}(z)$ equals this limit Poisson integral of $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$ for $z \in T^C$ and is thus in $H^2(T^C, \mathcal{H})$. The proof of our result for $2 < p \leq \infty$ follows by the same procedure as for the case $p = 2$.

THEOREM 4.2. *Let C be an open convex cone which is contained in or is any of the 2^n n-rants C_v in $\mathbb{R}^n$. Let $f \in \mathcal{A}(T^C, \mathcal{H})$ which satisfies (12). Let the unique boundary value $\mathbf{U}$ of Theorem 4.1 be $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$. We have $\mathbf{f} \in H^2(T^C, \mathcal{H})$ and*

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t)\, dt,\ z \in T^C. \tag{14}$$

Proof. Put $\mathbf{g}_\epsilon(z) = \mathbf{f}(z)/X_\epsilon(z)$, $z \in T^C$, $\epsilon > 0$, where

$$X_\epsilon(z) = \prod_{j=1}^n (1 - i\epsilon(-1)^{v_j} z_j)^{R+n+2},\ \epsilon > 0.$$

Clearly, $\mathbf{g}_\epsilon$ satisfies (12). By Theorem 4.1 there is a unique $\mathbf{U}_\epsilon \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ such that $\mathbf{g}_\epsilon(x+iy) \to \mathbf{U}_\epsilon(x), x \in \mathbb{R}^n$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0}$, $y \in C$; that is, (13) holds for $\mathbf{g}_\epsilon$ and $\mathbf{U}_\epsilon$. From (12) and the calculations of [5, (4.8)] there is a constant $M'(C', r, \epsilon)$ such that

$$\begin{aligned} &\mathcal{N}(\mathbf{g}_\epsilon(z)) \leq M'(C', r, \epsilon)(1 + |z|)^{-n-2}, \\ &z \in T(C', r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0}, r))), \end{aligned} \tag{15}$$

for all compact subcones $C' \subset C$ and all $r > 0$. Put

$$\mathbf{G}_\epsilon(t) = \int_{\mathbb{R}^n} \mathbf{g}_\epsilon(x+iy) e^{-2\pi i <x+iy,t>} dx,\ y \in C,\ t \in \mathbb{R}^n. \tag{16}$$

For any $y \in C$, $y \in C' \subset C$ and $|y| > r$ for some compact subcone $C' \subset C$ and some $r > 0$; thus $\mathbf{G}_\epsilon(t)$ is a well defined function of $t \in \mathbb{R}^n$ for any $y \in C$ and any $\epsilon > 0$ and is a continuous function of $t \in \mathbb{R}^n$ for $y \in C$ and $\epsilon > 0$. Let C'' be an arbitrary compact subdomain of C. From (15),

$$\int_{C''} \mathcal{N}(\mathbf{g}_\epsilon(x+iy)e^{-2\pi i<x+iy,t>})dy \ \to \ 0$$

as $|x| \to \infty$; hence an application of the Cauchy-Poincare theorem yields that $\mathbf{G}_\epsilon$ is independent of $y \in C''$ and is thus independent of $y \in C$ since C'' is an arbitrary compact subdomain of C.

We now show that $supp(\mathbf{G}_\epsilon) \subseteq C^* = \{t \in \mathbb{R}^n :< t, y > \geq 0 \ for\ all\ y \in C\} = \{t \in \mathbb{R}^n : u_C(t) \leq 0\}$. Let $t_o \in \mathbb{R}^n \setminus C^* = C_*$; thus $u_C(t_o) > 0$. By the proof of [23, Lemma, p. 241] there is a point $y' \in pr(C)$ and a number $\rho = \rho(t_o) > 0$ which can be chosen small enough in order that

$$- < t_o, y' > \geq \ u_C(t_o) - \rho \ > \ 0.$$

Letting $\lambda > 0$ be arbitrary, set $C' = \{y : y = \lambda y'\} \subset C$ and recall that $\mathbf{G}_\epsilon$ is independent of $y \in C$. Using (15) we have for $\lambda > r > 0$

$$\begin{aligned}(17)\qquad \mathcal{N}(\mathbf{G}_\epsilon(t_o)) &\leq \int_{\mathbb{R}^n} \mathcal{N}(\mathbf{g}_\epsilon(x+i\lambda y')e^{-2\pi i<x+i\lambda y',t_o>})dx \\ &\leq M'(C',r,\epsilon)e^{2\pi\lambda<y',t_o>}\int_{\mathbb{R}^n}(1+|x|)^{-n-2}dx \\ &\leq M''(C',r,\epsilon)e^{2\pi\lambda(\rho-u_C(t_o))}\end{aligned}$$

with $(\rho - u_C(t_o)) < 0$. Letting $\lambda \to \infty$ (17) yields $\mathcal{N}(\mathbf{G}_\epsilon(t_o)) = 0$ and $\mathbf{G}_\epsilon(t_o) = \Theta$, the zero element of $\mathcal{H}$. Thus $supp(\mathbf{G}_\epsilon) \subseteq C^*$ since t_o was any point of $\mathbb{R}^n \setminus C^* = C_*$.

For any compact subcone $C' \subset C$ and any $r > 0$ (15) yields

$$(18)\qquad \mathcal{N}(\mathbf{G}_\epsilon(t)) \leq M''(C',r,\epsilon)e^{2\pi<y,t>},\ t \in \mathbb{R}^n,\ y \in (C' \setminus (C' \cap N(\overline{0},r))),$$

as in (17). Also from (15) $\mathbf{g}_\epsilon(x+iy) \in L^1(\mathbb{R}^n,\mathcal{H}) \cap L^2(\mathbb{R}^n,\mathcal{H})$ as a function of $x \in \mathbb{R}^n$ for $y \in C$. Thus from (16) $e^{-2\pi<y,\cdot>}\mathbf{G}_\epsilon = \mathcal{F}^{-1}[\mathbf{g}_\epsilon(x+iy);\cdot],\ y \in C$; $e^{-2\pi<y,\cdot>}\mathbf{G}_\epsilon \in L^2(\mathbb{R}^n,\mathcal{H}),\ y \in C$; and in $L^2(\mathbb{R}^n,\mathcal{H})$

$$(19)\qquad \mathbf{g}_\epsilon(x+iy) = \mathcal{F}[e^{-2\pi<y,t>}\mathbf{G}_\epsilon(t);x],\ z = x+iy \ \in \ T^C.$$

Now $\mathbf{G}_\epsilon$ is continuous, $supp(\mathbf{G}_\epsilon) \subseteq C^*$, and (18) holds; thus by Lemma 3.1 $e^{-2\pi<y,\cdot>}\mathbf{G}_\epsilon \in L^p(\mathbb{R}^n,\mathcal{H})$ for all $p, 1 \leq p < \infty$, and for $y \in C$. Thus the Fourier transform in (19) can be interpreted in the $L^1(\mathbb{R}^n,\mathcal{H})$ sense as well as in the $L^2(\mathbb{R}^n,\mathcal{H})$ sense, and (19) becomes

$$(20)\qquad \mathbf{g}_\epsilon(x+iy) = \int_{\mathbb{R}^n}\mathbf{G}_\epsilon(t)e^{2\pi i<x+iy,t>}dt,\ z = x+iy \in T^C.$$

Both $\mathbf{G}_\epsilon$ and $e^{-2\pi<y,\cdot>}\mathbf{G}_\epsilon,\ y \in C$, are elements of $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$. Also $\mathbf{g}_\epsilon(\cdot + iy) \in \mathcal{S}'(\mathbb{R}^n,\mathcal{H}),\ y \in C$. Let $\phi \in \mathcal{S}$ and $\psi = \mathcal{F}[\phi(t);\cdot]$. We have

$$(21)\qquad \begin{aligned}&\langle \mathbf{g}_\epsilon(x+iy),\psi(x)\rangle = \langle e^{-2\pi<y,t>}\mathbf{G}_\epsilon(t),\phi(t)\rangle\\ &\to \langle \mathbf{G}_\epsilon(t),\phi(t)\rangle = \langle \mathcal{F}[\mathbf{G}_\epsilon],\psi\rangle\end{aligned}$$

as $y \to \overline{0},\ y \in C$. As noted previously in this proof, by Theorem 4.1 there is a unique $\mathbf{U}_\epsilon \in \mathcal{S}'(\mathbb{R}^n,\mathcal{H})$ such that $\mathbf{g}_\epsilon(\cdot+iy) \to \mathbf{U}_\epsilon$ in $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$ as $y \to \overline{0},\ y \in C$; hence $\mathcal{F}^{-1}[\mathbf{U}_\epsilon] \in \mathcal{S}'(\mathbb{R}^n,\mathcal{H})$. Thus, $\mathbf{G}_\epsilon = \mathcal{F}^{-1}[\mathbf{U}_\epsilon] \in \mathcal{S}'(\mathbb{R}^n,\mathcal{H})$. Moreover, in the sense of the convergence in that space,

$$(22)\qquad \lim_{y\to\overline{0},y\in C} \mathbf{g}_\epsilon(x+iy) = \mathbf{U}_\epsilon = \mathcal{F}[\mathbf{G}_\epsilon] \in \mathcal{S}'(\mathbb{R}^n,\mathcal{H}).$$

Recalling the definition of $\mathbf{g}_\epsilon(z),\ z \in T^C$, we have $\mathbf{f}(z) = \mathbf{g}_\epsilon(z)X_\epsilon(z),\ z \in T^C$, $\epsilon > 0$, and $\mathbf{f}(\cdot+iy)$ has boundary value $\mathbf{h} \in L^2(\mathbb{R}^n,\mathcal{H})$ in $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$ as $y \to \overline{0},\ y \in C$, and $X_\epsilon(\cdot+iy)\mathbf{g}_\epsilon(\cdot+iy) \to X_\epsilon\mathcal{F}[\mathbf{G}_\epsilon] = X_\epsilon\mathbf{U}_\epsilon$ in $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$ as $y \to \overline{0},\ y \in C$. Thus $X_\epsilon\mathbf{U}_\epsilon = \mathbf{h}$ in $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$, $\epsilon > 0$. Now for $\phi \in \mathcal{S}$

$$\langle \tfrac{\mathbf{h}(x)}{X_\epsilon(x)},\phi(x)\rangle = \langle \mathbf{h}(x),\tfrac{\phi(x)}{X_\epsilon(x)}\rangle = \langle \mathbf{U}_\epsilon,\phi\rangle$$

and $\mathbf{U}_\epsilon = \frac{\mathbf{h}(x)}{X_\epsilon(x)} \in \mathcal{S}'(\mathbb{R}^n,\mathcal{H})$. Clearly, $\mathbf{H}_\epsilon = \mathcal{F}^{-1}[\mathbf{h}(x)/X_\epsilon(x);\cdot] \in L^2(\mathbb{R}^n,\mathcal{H})$. Since $supp(\mathbf{G}_\epsilon) \subseteq C^*$ then $supp(\mathbf{H}_\epsilon) \subseteq C^*$ almost everywhere. For the function $d_y(t)$ defined in (2) we have $d_y(t)e^{2\pi i<z,t>} \in \mathcal{S},\ z \in T^C$. Thus

$$(23)\qquad \begin{aligned}&\int_{C^*}\mathbf{G}_\epsilon(t)e^{2\pi i<z,t>}dt = \langle \mathbf{G}_\epsilon(t),d_y(t)e^{2\pi i<z,t>}\rangle\\ &= \langle \mathbf{U}_\epsilon,\mathcal{F}^{-1}[d_y(t)e^{2\pi i<z,t>};\eta]\rangle\\ &= \langle \frac{\mathbf{h}(\eta)}{X_\epsilon(\eta)},\mathcal{F}^{-1}[d_y(t)e^{2\pi i<z,t>};\eta]\rangle\\ &= \langle \mathbf{H}_\epsilon(t),d_y(t)e^{2\pi i<z,t>}\rangle = \int_{C^*}\mathbf{H}_\epsilon(t)e^{2\pi i<z,t>}dt\end{aligned}$$

with $\mathbf{H}_\epsilon \in L^2(\mathbb{R}^n,\mathcal{H})$ and $z \in T^C$. From [5, Lemma 2.1] $I_{C^*}(t)e^{2\pi i<z,t>} \in L^p$ for all $p,\ 1 \le p \le \infty,\ z \in T^C$, where $I_{C^*}(t)$ is the characteristic function of C^*. Recalling (20) and (23) we have for $z \in T^C$

$$(24)\qquad \begin{aligned}&\mathbf{g}_\epsilon(x+iy) = \int_{C^*}\mathbf{G}_\epsilon(t)e^{2\pi i<z,t>}dt\\ &= \int_{C^*}\mathbf{H}_\epsilon(t)e^{2\pi i<z,t>}dt = \langle \mathbf{H}_\epsilon(t),I_{C^*}(t)e^{2\pi i<z,t>}\rangle\\ &= \langle \mathcal{F}^{-1}[\mathbf{h}(x)/X_\epsilon(x);t],I_{C^*}(t)e^{2\pi i<z,t>}\rangle\\ &= \langle \mathbf{h}(\eta)/X_\epsilon(\eta),\mathcal{F}^{-1}[I_{C^*}(t)e^{2\pi i<z,t>};\eta]\rangle\\ &= \langle \mathbf{h}(\eta)/X_\epsilon(\eta),\int_{C^*}e^{2\pi i<z-\eta,t>}dt\rangle = \int_{\mathbb{R}^n}\frac{\mathbf{h}(t)}{X_\epsilon(t)}K(z-t)dt.\end{aligned}$$

Now let w be an arbitrary point of T^C and for this arbitrary w consider the function $K(z+w)\mathbf{g}_\epsilon(z),\ z \in T^C$. Using [5, Lemma 3.2] we have $K(z+w)\mathbf{g}_\epsilon(z)$ is analytic in T^C, $|K(z+w)| \leq M_{Im(w)} < \infty,\ z \in T^C$, where $M_{Im(w)}$ is a constant which depends only on $Im(w)$. Thus $K(z+w)\mathbf{g}_\epsilon(z) = K(z+w)\mathbf{f}(z)/X_\epsilon(z)$ satisfies the growth of $\mathbf{f}(z),\ z \in T^C$, and

$$\lim_{y\to\overline{0},y\in C} K(x+iy+w)\mathbf{g}_\epsilon(x+iy) = K(x+w)\mathbf{U}_\epsilon = \tfrac{K(x+w)\mathbf{h}(x)}{X_\epsilon(x)}$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ with $K(x+w)\mathbf{h}(x)/X_\epsilon(x) \in L^2(\mathbb{R}^n, \mathcal{H})$ since both $K(x+w)$ and $1/X_\epsilon(x)$ are bounded for $x \in \mathbb{R}^n$. Combining the facts in this paragraph, the same proof leading to (24) applied to $K(z+w)\mathbf{g}_\epsilon(z),\ z \in T^C$, yields

$$K(z+w)\mathbf{g}_\epsilon(z) = \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} K(t+w)K(z-t)dt,\ z \ \in \ T^C. \tag{25}$$

Recalling that w above is an arbitrary point of T^C we choose $w = -x+iy \in T^C$ for $z = x+iy \in T^C$. Then (25) combined with (24) becomes

$$\begin{aligned} \mathbf{g}_\epsilon(z) &= \int_{C^*} \mathbf{G}_\epsilon(t) e^{2\pi i<z,t>} dt \\ &= \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} K(z-t)dt = \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} Q(z;t)dt,\ z \in T^C. \end{aligned} \tag{26}$$

We now need to construct the function from which the conclusion of this theorem will follow. First we need to show that $\mathbf{h}/X_\epsilon \ \to \ \mathbf{h}$ in $L^2(\mathbb{R}^n, \mathcal{H})$ as $\epsilon \ \to \ 0$. Since $|1/X_\epsilon(x)| \leq 1,\ x \in \mathbb{R}^n,\ \epsilon > 0$, note that

$$\begin{aligned} &(\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x) \ - \ \mathbf{h}(x)))^2 \ \leq \ (\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x)) \ + \ \mathcal{N}(\mathbf{h}(x)))^2 \\ &\leq 2^2((\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x)))^2 \ + \ (\mathcal{N}(\mathbf{h}(x)))^2) \ \leq \ 4((\mathcal{N}(\mathbf{h}(x)))^2 \ + \ (\mathcal{N}(\mathbf{h}(x)))^2) \\ &= 8(\mathcal{N}(\mathbf{h}(x)))^2 \end{aligned}$$

and the right side is independent of $\epsilon > 0$. Also

$$\lim_{\epsilon\to 0+} \mathcal{N}(\tfrac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x)) = \lim_{\epsilon\to 0+} |(X_\epsilon(x))^{-1} - 1|\mathcal{N}(\mathbf{h}(x)) \ = \ 0,\ x \in \mathbb{R}^n.$$

By the Lebesgue dominated convergence theorem

$$\lim_{\epsilon\to 0+} |\frac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x)|_2 = \lim_{\epsilon\to 0+} (\int_{\mathbb{R}^n} (\mathcal{N}(\frac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x)))^2 dx)^{1/2} \ = \ 0. \tag{27}$$

Now put

$$\mathbf{G}(z) \ = \ \int_{\mathbb{R}^n} \mathbf{h}(t)Q(z;t)dt,\ z \in T^C. \tag{28}$$

Let z_o be an arbitrary but fixed point of T^C. Choose the closed neighborhood $N(z_o,\rho) = \{z : |z - z_o| \leq \rho,\ \rho > 0\} \subset T^C$ of Lemma 3.3. Using (26), (28), Hölder's inequality, and Lemma 3.3

$$\begin{aligned} \mathcal{N}(\mathbf{g}_\epsilon(z) - \mathbf{G}(z)) &\leq (\int_{\mathbb{R}^n} (\mathcal{N}(\frac{\mathbf{h}(t)}{X_\epsilon(t)} - \mathbf{h}(t)))^2 dt)^{1/2} ||Q(z;t)||_{L^2} \\ &\leq B(z_o)(\int_{\mathbb{R}^n} (\mathcal{N}(\frac{\mathbf{h}(t)}{X_\epsilon(t)} - \mathbf{h}(t)))^2 dt)^{1/2} \end{aligned} \tag{29}$$

for $z \in N(z_o,\rho) \subset T^C$. (27) and (29) now yield

$$\lim_{\epsilon\to 0+}\ \mathbf{g}_\epsilon(z)\ =\ \mathbf{G}(z)$$

uniformly in $z \in N(z_o,\rho)$. Since $\mathbf{g}_\epsilon(z)$ is analytic in T^C for each $\epsilon > 0$, we have that $\mathbf{G}(z)$ is analytic at $z_o \in T^C$ and hence in T^C since z_o is an arbitrary point of T^C. Applying Lemma 3.4 we have $\mathbf{G}(z)$ of (28) is an element of $H^2(T^C,\mathcal{H})$.

For $\phi \in \mathcal{S}$ we use Hölder's inequality to obtain

$$\mathcal{N}(\langle \mathbf{G}(x+iy), \phi(x)\rangle - \langle \mathbf{h}(x), \phi(x)\rangle) = \mathcal{N}(\int_{\mathbb{R}^n} (\mathbf{G}(x+iy) - \mathbf{h}(x))\phi(x)dx)$$

$$\leq |\mathbf{G}(x+iy) - \mathbf{h}(x)|_2 ||\phi||_{L^2}. \tag{30}$$

By Lemma 3.4 $\mathbf{G}(x+iy) \to \mathbf{h}(x)$ in $L^2(\mathbb{R}^n,\mathcal{H})$ as $y \to \overline{0},\ y \in C$; hence $\mathbf{G}(x+iy) \to \mathbf{h}(x)$ in $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$ as $y \to \overline{0},\ y \in C$.

We now consider $\mathbf{f}(z) - \mathbf{G}(z),\ z \in T^C$, which is analytic in T^C; f satisfies the growth (12) and $\mathbf{G}$ satisfies the growth (6). Thus

$$\begin{aligned} &\mathcal{N}(\mathbf{f}(z) - \mathbf{G}(z)) \leq P(C',r)(1+|z|)^R, \\ &z = x+iy \in T(C',r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0},r))), \end{aligned} \tag{31}$$

for any compact subcone $C' \subset C$ and any $r > 0$, where $P(C',r)$ is a constant depending on $C' \subset C$ and on $r > 0$, and

$$\lim_{y\to\overline{0}} (\mathbf{f}(x+iy) - \mathbf{G}(x+iy))\ =\ \mathbf{h}(x) - \mathbf{h}(x)\ =\ \Theta \tag{32}$$

in $\mathcal{S}'(\mathbb{R}^n,\mathcal{H})$). Now put $\mathbf{F}(z) = \mathbf{f}(z) - \mathbf{G}(z),\ z \in T^C$, and $\mathbf{F}(z)$ satisfies (31) and (32). Letting $\epsilon = 1$ in the function $X_\epsilon(z)$ at the beginning of this proof, consider $\mathbf{g}(z) = \mathbf{F}(z)/X_1(z), z \in T^C$. As in (15), for any compact subcone $C' \subset C$ and any $r > 0$

$$\begin{aligned} &\mathcal{N}(\mathbf{g}(z) \leq P'(C',r)(1+|z|)^{-n-2}, \\ &z = x+iy \in T(C',r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0},r))). \end{aligned}$$

Now putting as in (16)

$$\mathbf{A}(t) \ = \ \int_{\mathbb{R}^n} \mathbf{g}(x+iy)e^{-2\pi i<x+iy,t>}dx, \ y \in C, \ t \in \mathbb{R}^n,$$

and proceeding with the form of the proof from (16) to (20) we have that $\mathbf{A}$ is continuous, is independent of $y \in C$, has support in C^*, satisfies a growth as in (18), $e^{-2\pi<y,t>}\mathbf{A}(t) \ = \ \mathcal{F}^{-1}[\mathbf{g}(x+iy);t], \ t \in \mathbb{R}^n, \ y \in C$, with $e^{-2\pi<y,\cdot>}\mathbf{A} \in L^2(\mathbb{R}^n, \mathcal{H}), \ \mathbf{g}(x+iy) = \mathcal{F}[e^{-2\pi<y,t>}\mathbf{A}(t);x], \ x \in \mathbb{R}^n, \ y \in C$, and

$$(33) \qquad \mathbf{g}(x+iy) = \int_{\mathbb{R}^n} \mathbf{A}(t)e^{2\pi i<x+iy,t>}dt, \ z = x+iy \in T^C.$$

For $\phi \in \mathcal{S}$ and $y \in C$

$$\begin{aligned}&\langle \mathbf{F}(z)/X_1(z), \phi(x)\rangle = \langle \mathbf{g}(z), \phi(x)\rangle \\ &= \langle \mathcal{F}[e^{-2\pi<y,t>}\mathbf{A}(t);x], \phi(x)\rangle = \langle e^{-2\pi<y,t>}\mathbf{A}(t), \hat{\phi}(t)\rangle \\ &\to \langle \mathbf{A}(t), \hat{\phi}(t)\rangle = \langle \mathcal{F}[\mathbf{A}], \phi\rangle\end{aligned}$$

as $y \to \overline{0}, \ y \in C$. Thus

$$\begin{aligned}&\langle \mathbf{F}(x+iy), \phi(x)\rangle = \langle \mathbf{g}(x+iy), X_1(x+iy)\phi(x)\rangle \\ &\to \langle \mathcal{F}[\mathbf{A}], X_1(x)\phi(x)\rangle = \langle X_1(x)\mathcal{F}[\mathbf{A}], \phi(x)\rangle\end{aligned}$$

as $y \to \overline{0}, \ y \in C$. Combining this fact with (32) we have $X_1(x)\mathcal{F}[\mathbf{A}] = \Theta$ which yields $\mathbf{A} = \Theta$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$.

Now put

$$\Delta \ = \ \prod_{j=1}^n \left(1 - i(-1)^{v_j}\left(\frac{-1}{2\pi i}\frac{\partial}{\partial t_j}\right)\right)^{R+n+2}.$$

From (33) and for $z \in T^C$

$$\begin{aligned}&\mathbf{F}(z) = X_1(z)\int_{\mathbb{R}^n} \mathbf{A}(t)e^{2\pi i<z,t>}dt \\ &= \langle \Delta\mathbf{A}(t), d_y(t)e^{2\pi i<z,t>}\rangle = \mathbf{\Theta}\end{aligned}$$

which yields

$$\mathbf{f}(z) = \mathbf{G}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t)Q(z;t)dt, \ z \in T^C,$$

and $\mathbf{f}(z) \in H^2(T^C, \mathcal{H})$. The proof of Theorem 4.2 is complete.

Extending Theorem 4.2 to the cases $2 < p \leq \infty$ we have the following result.

THEOREM 4.3. *Let C be an open convex cone which is contained in or is any of the 2^n n-rants C_v in $\mathbb{R}^n$. Let $\mathbf{f}(z) \in \mathcal{A}(T^C, \mathcal{H})$ which satisfies (12). Let the unique boundary value $\mathbf{U}$ of Theorem 4.1 be $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H})$, $2 < p \leq \infty$. We have $\mathbf{f} \in H^p(T^C, \mathcal{H})$, $2 < p \leq \infty$, and*

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t) dt,\ z \in T^C.$$

Proof. Note that the analysis from the beginning of the proof of Theorem 4.2 through the fact that $\mathbf{U}_\epsilon = \frac{\mathbf{h}(x)}{X_\epsilon(x)} \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ above (23) is independent of the value of p, $2 \le p \le \infty$. From the definition of $X_\epsilon(z)$ at the beginning of the proof of Theorem 4.2

$$|1/X_\epsilon(x)| = \epsilon^{-n(R+n+2)} \prod_{j=1}^n (\epsilon^{-2} + x_j^2)^{-1-R/2-n/2},\ x \in \mathbb{R}^n,$$

and $1/X_\epsilon \in L^q,\ 1 \le q \le \infty$. Thus $\mathbf{h}/X_\epsilon \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^p(\mathbb{R}^n, \mathcal{H}),\ 2 < p \le \infty$. If $p = \infty, \mathbf{h}(x)/X_\epsilon(x) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H}) \cap L^\infty(\mathbb{R}^n, \mathcal{H})$. Further, if $2 < p < \infty$

$$\begin{aligned}&\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x)))^2 dx = \int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{h}(x))^2 |1/X_\epsilon(x)|^2 dx\\ &\le ||(\mathcal{N}(\mathbf{h}(x)))^2||_{L^{p/2}} ||\ |1/X_\epsilon(x)|^2\ ||_{L^{p/(p-2)}} \ <\ \infty.\end{aligned}$$

Thus $\mathbf{h}/X_\epsilon \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H}) \cap L^p(\mathbb{R}^n, \mathcal{H})$ for the value of $p, 2 < p \le \infty$. As noted before in this proof the analysis from the beginning of the proof of Theorem 4.2 through the fact that $\mathbf{U}_\epsilon = \frac{\mathbf{h}(x)}{X_\epsilon(x)} \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ above (23) is independent of the value of p in the hypotheses of Theorem 4.2 and also now of the hypotheses of Theorem 4.3. Since $\mathbf{h}/X_\epsilon \in L^2(\mathbb{R}^n, \mathcal{H})$ for $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H}),\ 2 < p \le \infty$, here, the analysis from this fact that $\mathbf{U}_\epsilon = \frac{\mathbf{h}(x)}{X_\epsilon(x)} \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ to equation (23) to equation (26) holds exactly as in the proof of Theorem 4.2 for the present case. For $2 < p < \infty$

$$\begin{aligned}&(\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x) - \mathbf{h}(x)))^p \le (\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x)) + \mathcal{N}(\mathbf{h}(x)))^p\\ &\le 2^p((\mathcal{N}(\mathbf{h}(x)/X_\epsilon(x)))^p + (\mathcal{N}(\mathbf{h}(x)))^p) \le 2^{p+1}(\mathcal{N}(\mathbf{h}(x)))^p.\end{aligned}$$

The use of the Lebesgue dominated convergence theorem as in (27) now shows

$$\lim_{\epsilon \to 0+} |\mathbf{h}(x)/X_\epsilon(x) - \mathbf{h}(x)|_p \ =\ 0. \tag{34}$$

Define $\mathbf{G}$ as in (28) and recall that Lemma 3.3 holds for all $p,\ 1 \le p \le \infty$. For the present case of $2 < p < \infty$ we use an estimate as in (29) together with (34) to obtain that $\mathbf{G}$ is analytic in T^C. For the case $p = \infty$ choose a closed neighborhood contained in T^C about each fixed point $z_o \in T^C$, which can be done since C is open; and $\mathbf{h}/X_\epsilon \in L^2(\mathbb{R}^n, \mathcal{H})$ for $\mathbf{h} \in L^\infty(\mathbb{R}^n, \mathcal{H})$. From (26)

$$\mathbf{g}_\epsilon(z) = \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} Q(z;t) dt,\ z \in T^C,$$

is analytic in T^C, and from Lemma 3.5 $\mathbf{g}_\epsilon(z) \to \mathbf{G}(z)$ uniformly on the closed neighborhood about $z_o \in T^C$ and hence on the corresponding open neighborhood about $z_o \in T^C$ as $\epsilon \to 0+$. Thus for the case $\mathbf{h} \in L^\infty(\mathbb{R}^n, \mathcal{H})$, $\mathbf{G}$ is analytic on T^C since z_o was an arbitrary point of T^C. Since $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H}), 2 < p \le \infty$, here, Lemma 3.4 yields $\mathbf{G} \in H^p(T^C, \mathcal{H}), 2 < p \le \infty$. Arguing as in (30) we have for $2 < p < \infty, 1/p + 1/q = 1$,

$$\mathcal{N}(\langle \mathbf{G}(x+iy), \phi(x)\rangle - \langle \mathbf{h}(x), \phi(x)\rangle) \le |\mathbf{G}(x+iy) - \mathbf{h}(x)|_p ||\phi||_{L^q},\ \phi \in \mathcal{S},$$

and using Lemma 3.4 we get $\mathbf{G}(x+iy) \to \mathbf{h}(x)$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0},\ y \in C$; and this convergence holds also in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ for $p = \infty$.

We now have both $\mathbf{f}$ and $\mathbf{G}$ are analytic in T^C, and $\mathbf{G}(z)$ satisfies the stated growth in Lemma 3.4 for $z \in T(C', r)$ in the cases $2 < p < \infty$ and is bounded for all $z \in T^C$ independent of z if $p = \infty$. Further both $\mathbf{f}$ and $\mathbf{G}$ have $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H})$ as boundary value in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. Thus we may consider $(\mathbf{f}(z) - \mathbf{G}(z)),\ z \in T^C$, exactly as in the proof of Theorem 4.2 starting at (31) and continuing through the end of the proof for the case $p = 2$ to conclude that

$$\mathbf{f}(z) = \mathbf{G}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t) dt,\ z \in T^C,$$

and $\mathbf{f} \in H^p(T^C, \mathcal{H}),\ 2 < p \le \infty$, from Lemma 3.4. The proof of Theorem 4.3 is complete.

We now extend Theorems 4.2 and 4.3 to a tube T^C where C is an arbitrary regular cone.

THEOREM 4.4. *Let C be a regular cone. Let $\mathbf{f} \in \mathcal{A}(T^C, \mathcal{H})$ and satisfy (12). Let the unique $\mathcal{S}^{(m)'}(\mathbb{R}^n, \mathcal{H}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ boundary value of $\mathbf{f}$ from Theorem 4.1 be $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H}),\ 2 \le p \le \infty$. We have $\mathbf{f} \in H^p(T^C, \mathcal{H}),\ 2 \le p \le \infty$.*

Proof. For each of the 2^n n-rants C_v consider $C \cap C_v$. Let $S_j,\ j = 1, ..., k$, be an enumeration of the intersections $C \cap C_v$ which are non-empty; each S_j is an open regular cone which is contained in or is a n-rant C_v in $\mathbb{R}^n$. Put

$$\mathbf{f}_j(z) = \mathbf{f}(z),\ z \in T^{S_j} = \mathbb{R}^n + iS_j,\ j = 1, ..., k. \tag{35}$$

Each $\mathbf{f}_j$ satisfies the analyticity and growth hypotheses of Theorems 4.2 and 4.3, and each $\mathbf{f}_j$ obtains the unique $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H})$ as $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ boundary value. By Theorems 4.2 and 4.3 each $\mathbf{f}_j \in H^p(T^{S_j}, \mathcal{H}),\ 2 \le p \le \infty,\ j = 1, ..., k$; and

$$f(z) = \mathbf{f}_j(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t) dt,\ z \in T^{S_j},\ j = 1, ..., k. \tag{36}$$

For $2 \le p < \infty$ there are constants $A_j,\ j = 1, ..., k$, independent of $y = Im(z)$ such that

(37)
$$\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x+iy)))^p dx = \int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}_j(x+iy)))^p dx \le A_j^p\ ,\ y \in S_j,\ j = 1, ..., k.$$

Put

$$A \; = \; max\{A_1, A_2, ..., A_k\}. \tag{38}$$

Now let $y \in C$ such that $y \notin S_j, j = 1, ..., k$. Then y is on the topological boundary of some S_j. For this S_j choose a sequence of points $\{y_{j,l}\} \subset S_j$ such that $y_{j,l} \to y$ as $l \to \infty$. Since $\mathbf{f}(z)$ is analytic in T^C, by Fatou's lemma we have for $y \in C$ such that $y \notin S_j, \; j = 1, ..., k$,

$$\begin{aligned} &\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x+iy)))^p dx \leq \liminf_{l \to \infty} \int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x+iy_{j,l})))^p dx \\ &\leq A_j^p \; \leq \; A^p. \end{aligned} \tag{39}$$

Combining (35), (37), (38), and (39) we have for $2 \leq p < \infty$

$$\textstyle\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x+iy)))^p dx \; \leq \; A^p, \; y \in C,$$

where A is independent of $y \in C$. Thus $\mathbf{f} \in H^p(T^C, \mathcal{H}), 2 \leq p < \infty$.

For $p = \infty$ each $\mathbf{f}_j(z) \in H^\infty(T^{S_j}, \mathcal{H}), j = 1, ..., k$; and $\mathcal{N}(\mathbf{f}_j(z)) \leq B_j, z \in T^{S_j}, j = 1, ..., k$, for positive constants $B_j, j = 1, ..., k$, which are independent of $z \in T^{S_j}, j = 1, ..., k$. Put $B = max\{B_1, B_2, ..., B_k\}$. If $y \in C$ such that $y \notin S_j, j = 1, ..., k$, again choose a sequence $\{y_{j,l}\} \subset S_j$ for some j such that $y_{j,l} \to y$ as $l \to \infty$; since $\mathbf{f}(z)$ is analytic and hence continuous, a simple continuity argument yields $\mathcal{N}(\mathbf{f}(x+iy)) \; \leq \; 1 + B$ for any $y \in C$ such that $y \notin S_j, j = 1, ..., k$. Combining these facts we thus have $\mathcal{N}(\mathbf{f}(x+iy)) \; \leq \; 1 + B$ for all $z \in T^C$, and $\mathbf{f} \in H^\infty(T^C, \mathcal{H})$.

We conclude $\mathbf{f} \in H^p(T^C, \mathcal{H})$ for each choice of $p, \; 2 \leq p \leq \infty$, and the proof of Theorem 4.4 is complete.

5 Results for $1 \leq p < 2$

For the cases $1 \leq p < 2$ Theorem 4.3, and hence Theorem 4.4, will not follow from the $p = 2$ case of Theorem 4.2 by a proof like the one used to obtain the $2 < p \leq \infty$ cases from the $p = 2$ case in this paper. To obtain results like Theorems 4.2, 4.3, and 4.4 for the cases $1 \leq p < 2$ we must use separate proof techniques. We propose to do so by using the concept of Banach space $\mathcal{B}$ with Fourier type $p, 1 \leq p < 2$, as discussed in [14, section 6].

From [14, section 6] the Banach space $\mathcal{B}$ has Fourier type p with respect to $\mathbb{R}^n$ if there is a constant $K > 0$ such that for every compactly supported $\mathbf{f} \in L^p(\mathbb{R}^n, \mathcal{B}), 1 \leq p \leq 2$,

$$\left(\int_{\mathbb{R}^n} (\mathcal{N}(\widehat{\mathbf{f}}(t)))^q dt\right)^{1/q} \; \leq \; K\left(\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(t)))^p dt\right)^{1/p}, \; q \; = \; p/(p-1). \tag{40}$$

By completeness, for a Banach space with Fourier type $p, 1 \leq p \leq 2$, we have for every $\mathbf{f} \in L^p(\mathbb{R}^n, \mathcal{B})$ that $\widehat{\mathbf{f}} \in L^q(\mathbb{R}^n, \mathcal{B})$ and (40) holds. Again we refer to [14, section 6] for details.

If $\mathcal{B}$ is of Fourier type $p, 1 \leq p \leq 2$, the Fourier transform of $\mathbf{f}$ defined in the sense of the Bochner integral imbedded into $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ coincides with $\mathbf{g} = \mathcal{F}[\mathbf{f}]$ defined as the distributional Fourier transform, and $\mathbf{f} = \mathcal{F}^{-1}[\mathbf{g}]$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$.

In future research we use the concept of a Banach space $\mathcal{B}$ having Fourier type $p, 1 \leq p \leq 2$, to obtain the results Theorems 4.2, 4.3, and 4.4 for the cases $1 \leq p < 2$.

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