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Stochastic exclusion processes versus coherent transport

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Stochastic exclusion processes play an integral role in the physics of non-equilibrium statistical mechanics. These models are Markovian processes, described by a classical master equation. In this paper a quantum mechanical version of a stochastic hopping process in one dimension is formulated in terms of a quantum master equation. This allows the investigation of coherent and stochastic evolution in the same formal framework. The focus lies on the non-equilibrium steady state. Two stochastic model systems are considered, the totally asymmetric exclusion process and the fully symmetric exclusion process. The steady state transport properties of these models is compared to the case with additional coherent evolution, generated by the XX-Hamiltonian.

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I. INTRODUCTION

Stochastic exclusion processes have been studied in statistical mechanics for a long time [1, 2]. These are simplified one-dimensional hopping models, that allow the study of nonequilibrium phenomena in many-particle systems. The exclusion processes are modeled by a classical master equation that determines the time evolution of the probability distribution. The steady state of the master equation exhibits interesting non-equilibrium behavior, such as the presence of a current, non-equilibrium phase transitions and entire phases with a diverging correlation length[3, 4]. The presence of currents, such as the current of particles, energy or momentum, is a common feature of non-equilibrium steady states and can have profound effects on the correlations present in the system [6, 7]. Non-equilibrium systems can develop long-range correlations in the presence of a high current.

The asymmetric exclusion process (ASEP) as well as the symmetric exclusion process (SEP) are prime examples for such model systems[3, 5]. These processes describe the hopping of hard-core particles in a one-dimensional chain, only driven by the inflow and the outflow of particles at the boundaries of the chain. Here one considers open boundary conditions, where particles are injected at the first site and are removed at the last site N of the chain. The steady state properties are entirely determined by the inflow and outflow. The dynamics of the particles in the bulk are given by translationally invariant hopping rates, that either constrain the hopping of particles to take place in only one direction (ASEP) or allow for a hopping in both directions (SEP).

Transport properties of open quantum mechanical systems, on the other hand, are subject to recent research activities. A general interest is placed on how external noise, generated by the environment, affects the coherent transport in the system. It has been found, that the presence of noise in the quantum mechanical systems can actually aid the transport process of excitations through heterogeneous environments [8, 9], such In this article we want to investigate the interplay between stochastic transport processes and coherent transport present in the same system. Here we consider only the steady-state properties of the system. To treat both processes on equal footing, we incorporate the classical hopping terms into the quantum master equation. The stochastic hopping is modeled by appropriately chosen quantum jump operators. Such a construction has also been used to find quantum master equations that describe a quantization of kinetic Ising models [11]. These models obey detailed balance and allow for an exact solution. Considering hopping models in this more general quantum framework allows now for additional quantum transport, so to speak, on top of the classical hopping evolution. We can choose an arbitary, particle number conserving Hamiltonian to mediate the coherent transport and investigate the effect this quantum pertubation has on the classical hopping process.

Our article is organized as follows: First, in section II, we introduce the hopping model and formulate the problem as a quantum master equation. We choose in different instances either the ASEP or the SEP as the underlying stochastic process. In the following section III, the quantum analog of the ASEP is treated numerically in the framework of matrix product density operators (MPDO). The master equation for a chain of N = 40 sites is evolved in time, until the steady state is reached. The current, the particle density, as well as the particle density-density correlations are computed. In section IV the SEP is considererd. The two-point correlation functions of the SEP can be calculated exactly in the steady state, and the scaling of the current for larger lattice sizes is investigated. The conclusion are then drawn in the final section V.

as bio-molecules. An optimal ratio between coherent transport and dephasing noise can be found. The dynamics of open quantum systems are generally formulated in terms of a Markovian Lindblad master equation that describes the time evolution of the density matrix [10].

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FIG. 1: spin-1/2 chain with jump-operators and coherent evolution. The inflow is given by rate α , the outflow with rate β , stochastic hopping to the left with rate φ_L and to the right φ_R . The strength of the coherent evolution is given by λ

II. FORMULATION OF THE QUANTUM MASTER EQUATION

We study a system of hard-core particles in a one dimensional chain of length N, where each site $(1 \le k \le N)$ can be either occupied or empty. In the spin chain picture this corresponds to either spin up (occupied), or spin down (empty). At the boundary k = 1, we allow for an inflow with a rate α and at k = N for an outflow of particles, given by a rate β . The particles at each site are allowed to hop stochastically to the left with a rate φ_L and to the right with a rate φ_R . In this article we consider only two cases. First, the fully asymmetric case (ASEP), where the particles only hop to the right, $\varphi_R = 1$ and $\varphi_L = 0$. Second, the fully symmetric case (SEP) where both hopping rates are equal $\varphi_L = \varphi_R = 1$.

These stochastic particle jumps correspond to the classical stochastic exclusion process and are formulated in terms of quantum-jump operators L_{μ} . These operators govern the incoherent evolution of the quantum master equation. These jumps can be formulated in terms of spin-flip operations:

$$L_{1} = \sqrt{\alpha} \, \sigma^{-} \qquad (1)$$

$$L_{N} = \sqrt{\beta} \, \sigma^{+}$$

$$L_{k,k+1}^{R} = \sqrt{\varphi_{R}} \, \sigma_{k}^{-} \otimes \sigma_{k+1}^{+}$$

$$L_{k-1,k}^{L} = \sqrt{\varphi_{L}} \, \sigma_{k-1}^{+} \otimes \sigma_{k}^{-}.$$

Here, the σ^{\pm} correspond to the Pauli raising and lowering operators. The master equation written with only these operators reproduces exactly the classical stochastic behavior with the appropriate jump rates. This can be seen, when one restricts oneself to density matrices diagonal in the computational basis. With the chosen definition of the Lindblad operators, the classical master equation for the probabilities is reproduced. In this generalized framework, we can now also allow for an additional coherent evolution of the system, when choosing an appropriate Hamiltonian. The XX-Hamiltonian

$$H = \lambda \sum_{k=1}^{N-1} \sigma_k^x \otimes \sigma_{k+1}^x + \sigma_k^y \otimes \sigma_{k+1}^y.$$
(2)

gives rise to the free coherent evolution of the hardcore particles. Furthermore it has the property that it conserves the number of particles. The full quantum master equation for the density operator ρ can be written as:

$$\partial_t \rho = -i \left[\rho, H \right] + \sum_{\mu} L_{\mu} \rho L_{\mu}^{\dagger} - \frac{1}{2} \left\{ L_{\mu}^{\dagger} L_{\mu}; \rho \right\} \quad (3)$$
$$\equiv \mathcal{L} \left[\rho \right].$$

The central observables are the particle density $n_k = \sigma_k^+ \sigma_k^$ and the current j_k of particles. To find the right expression for the particle current, we consider the continuity equation for the density n_k . The continuity equation is obtained in the Heisenberg picture when the adjoined of \mathcal{L} is acting on n_k .

$$\partial_t n_k = \mathcal{L}^{\dagger} [n_k]$$

$$= -i [H, n_k] + \sum_{\mu} L^{\dagger}_{\mu} n_k L_{\mu} - \frac{1}{2} \{ L^{\dagger}_{\mu} L_{\mu}; n_k \}$$
(4)

Let us first consider only the (ASEP) with a single Lindblad operator in the bulk, since $\varphi_L = 0$. We find, that n_k obeys a continuity equation of the form:

$$\partial_t n_k = \left(j_{k,k+1}^{co} + j_{k,k+1}^{st}\right) - \left(j_{k-1,k}^{co} + j_{k-1,k}^{st}\right).$$
(5)

We may now interpret the sum of the terms

as the total current-density $j_{k,k+1}$ of the system. Note, that there are two different contributions to the current, the coherent part j^{co} due to the dynamics generated by the Hamiltonian and the stochastic contribution j^{st} originating from the quantum-jump operator induced hopping. We observe, that the stochastic contribution to the current corresponds exactly to the current present in the classical model [3, 12].

III. THE ASYMETRIC EXCLUSION PROCESS

The steady state of the ASEP, without any additional quantum evolution, i.e. $\lambda = 0$, can be calculated exactly [12] and its solution can be written in terms of a matrix product density operator (MPDO) [13]. The general form of a MPDO is given by,

$$\rho = \sum_{i_1,\dots,j_N=0}^{d-1} \langle V | M_{i_1,j_1},\dots,M_{i_N,j_N} | W \rangle \bigotimes_{k=1}^N | i_k \rangle \langle j_k |,$$
(7)

where d denotes the dimension of the local Hilbert space and M_{i_k,j_k+1} are $D_k \times D_{k+1}$ dimensional matrices. In the case of the ASEP, the steady state solution of the master equation is given by the choice, see Appendix A:

$$\langle W | = \sum_{k=0}^{N} \frac{1}{\alpha^{k}} \langle k |$$

$$M_{1,0} = M_{0,1} = 0$$

$$M_{0,0} = \sum_{k=0}^{N} |k\rangle \langle k-1 |$$

$$M_{1,1} = \sum_{n=0}^{N} \frac{1}{\beta} |0\rangle \langle n | + \sum_{n=1}^{N} \sum_{m=1}^{n} |m\rangle \langle n |$$

$$|V\rangle = |0\rangle ,$$

$$(8)$$



FIG. 2: Classical phase diagram of the asymmetric exclusion process. The ASEP processes three phases, LD (low-density), HD (highdensity) and the maximal current phase MC.

which is a MPDO with a matrix dimension of D = N + 1. In [12] the classical phase diagram with respect to α and β has been found FIG. 2. In general there are three phases, the low-density phase LD, the high density phase HD, and the maximal current phase MC. The three phases exhibit different behavior in the current and density. The statistical current and the corresponding density in the large N limit are given by,

as has been calculated in [12]. We want to understand the system's response to a quantum mechanical perturbation at distinct points in the phase diagram.

Note, there is a special line in the classical phase diagram marked by $\alpha + \beta = 1$. Along this line the density operator for the classical probabilities factorizes, and mean-field theory becomes exact, Appendix A. If we now add the quantum perturbation, we see, that the state,

$$\rho = \bigotimes_{k=1}^{N} \begin{pmatrix} \beta & 0\\ 0 & \alpha \end{pmatrix}, \tag{9}$$

is still the steady state of the system for arbitrary values of λ . That is, when the system is classically uncorrelated, the quantum perturbation has no effect on the system.

To see that in the regime, where the stochastic steady state is correlated, the coherent evolution alters the steady state, we need to calculate the steady state of the system numerically by time-evolving the density matrix, until we reach the steady state. The numerical simulations of the real time evolution is performed by a modification of the TEBD algorithm for mixed states [13, 14], see Appendix B for details. As initial state for the evolution we have chosen the classical steady state (7). We then changed the value of $\lambda = 0$ to, $\lambda = 1/2$ and $\lambda = 1$, and evolved the MPDO, until the steady state was reached, i.e. until all considered observables did not change any more. Negative values of λ were also simulated and led to the same results. We conclude from this, that the systems response only depends on the absolute value of λ . The simulations were done for a lattice with N = 40 sites. The matrix bond dimension of the MPDO was chosen as D = 60 and we chose a Trotter step $\Delta t = 10^{-4}$.

To get an understanding of how the system responds to the quantum perturbation in each phase respectively, we have chosen to compute the steady state at different values of α and β , which correspond to points chosen to lie in different phases. Four different points were chosen. Corresponding to the coexistence line (CL), where $\alpha = \beta$ and $\beta + \alpha \leq 1$, we chose the point $\alpha = \beta = 1/4$. In the maximum current phase (MC), we chose $\alpha = \beta = 3/4$. For the low-density (LD) and the high-density (HD) phase, we chose $\alpha = 1/4$ $\beta = 1/2$ and $\alpha = 1/2$ $\beta = 1/4$ respectively.

The observable we considered first was the density distribution $\langle n_k \rangle = \text{Tr} \{ n_k \rho \}$ as a function of the lattice site k, FIG 3.



FIG. 3: (color online) Density distribution $\langle n_k \rangle$ for the different points (α, β) , (a) HD (1/2, 1/4), (b) LD (1/4, 1/2) and (c) MC (3/4, 3/4)as well as (d) CL (3/4, 3/4). The black solid line corresponds to $\lambda = 0$, i.e. the classical solution. The red dashed line corresponds to a quantum perturbation with $\lambda = 1/2$, and the blue dashed-dotted line to a perturbation $\lambda = 1$.

Furthermore, we calculated the values of the two-point correlation functions, of the densities $n_k = \sigma_k^+ \sigma_k^-$ for all pairs (i, j) of sites,

$$\langle n_i n_j \rangle^c = \langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle.$$
 (10)

The expectation values are taken with respect to the system's steady state. In the figures FIG. 4,5,6,7, the correlation functions are compared to the different contributions to the current j^{tot} defined in (6), for different values of $\lambda = 0, 1/2, 1$.



FIG. 4: (color online) HD phase: The left column shows the densitydensity correlations (10) for $\alpha = 1/2$ and $\beta = 1/4$ (HD) and different values of coherent driving λ . In the right column the currentdensity is plotted as a function of the lattice site k. The black dashdotted line depicts the total current j^{tot} . The blue solid line corresponds to the stochastic contribution j^{st} and the red dashed line shows the coherent contribution j^{co} . The plots are ordered from top to bottom as (a) $\lambda = 0$ (b) $\lambda = 0.5$ and (c) $\lambda = 1$.

The first observation to be made is that the individual contributions to the total current are no longer constant throughout the lattice anymore, they show a dependence on the lattice site. The total current however, i.e. $j_{tot} = j_{co} + j_{st}$, is still constant at each site of the lattice, as is required, since the system is in a steady state.

a. The high- and low-density phases FIG. 4,5: These two phases are, just as in the classical set up, related by a particle-hole transformation and exchanging the numbering of the lattice sites from left to right. All the plots reflect this symmetry. One observes that the correlation functions in the classical regime, $\lambda = 0$, are already quite short ranged and decay rapidly. The quantum perturbation in both cases leads to a further decay of the correlations. The total current remains stable with respect to the quantum perturbation and does not change its value notably. The individual constituents to the current however change their behavior. At the boundaries the stochastic contributions are increased, whereas the coherent current gives rise to a flow in the opposite direction.



FIG. 5: (color online) LD phase: In the left column we plotted the density-density correlations (10) for $\alpha = 1/4$ and $\beta = 1/2$ (LD). The right column depicts the current-density as a function of the lattice site k. The black dash-dotted line amounts to the total current j^{tot} . The blue solid line corresponds to the stochastic contribution j^{st} and the red dashed line shows the coherent contribution j^{co} . The plots are ordered from top to bottom as (a) $\lambda = 0$ (b) $\lambda = 0.5$ and (c) $\lambda = 1$.

b. The maximal current phase FIG. 6: In the classical process, the MC phase corresponds to the maximum amount of current the system can carry. Allowing for a quantum perturbation, the system makes use of the additional transport capacity and increases its total current. For these boundary conditions the stochastic as well as the coherent contributions flow in the same direction. Note, however, that the system in the bulk prefers to use the stochastic transport, whereas at the boundaries coherent transport is preferred. The classical correlation function initially assumes negative values close to the boundaries. The onset of the quantum perturbation also reduces the magnitude of the correlations in this phase, even though the total amount of current is increased.

c. Coexistence line FIG. 7: For the chosen boundary conditions, the amount of correlations initially present in the steady-state are decreased, when switching on the quantum perturbation. The final steady-state, were $\lambda = 1$, however, still shows the presence of correlation to a higher degree than in the other phases. The total amount of current carried by the



FIG. 6: (color online) MC phase: The left column shows the densitydensity correlations (10) for $\alpha = 3/4$ and $\beta = 3/4$ (MC) and different values of coherent driving λ . In the right column the currentdensity is plotted as a function of the lattice site k. The black dashdotted line depicts the total current j^{tot} . The blue solid line corresponds to the stochastic contribution j^{st} and the red dashed line shows the coherent contribution j^{co} . We plotted three different values for (a) $\lambda = 0$ (b) $\lambda = 0.5$ and (c) $\lambda = 1$.

system, however, is decreased. The coherent contribution is negative throughout the system.

IV. THE SYMMETRIC EXCLUSION PROCESS

If we allow for stochastic hopping in both directions with an equal rate φ and turn off the coherent evolution, the model describes the classical symmetric exclusion process. The symmetric exclusion process is known to possess only a single phase classically with a vanishing current [5]. We now consider the quantum perturbation to the system. Note, that now the quantum-jump operators are related via $L_{k,k+1}^R = L_{k,k+1}^{L^{\dagger}} \equiv L_{k,k+1}$. This allows us to rewrite the full master



FIG. 7: (color online) Coexistence line: The left column shows the density-density correlations (10) for $\alpha = 1/4$ and $\beta = 1/4$ (CL). In the right column the current-density is plotted. The black dash-dotted line amounts to the total current j^{tot} . The blue solid line corresponds to the stochastic contribution j^{st} and the red dashed line shows the coherent contribution j^{co} . The values for the coherent driving are chosen as (a) $\lambda = 0$ (b) $\lambda = 0.5$ and (c) $\lambda = 1$.

equation as,

$$\partial_t \rho = -i[\rho; H]$$

$$+ \sum_{k=1}^{N-1} [[L_{k,k+1}; \rho]; L_{k,k+1}^{\dagger}] + [[L_{k,k+1}^{\dagger}; \rho]; L_{k,k+1}]$$

$$+ L_1 \rho L_1^{\dagger} - \frac{1}{2} \{L_1^{\dagger}, L_1; \rho\} + L_N \rho L_N^{\dagger} - \frac{1}{2} \{L_N^{\dagger}, L_N; \rho\}.$$
(11)

It is possible to calculate the nearest neighbor two-point correlation functions exactly. To see why this is possible, we first transform the Pauli raising and lowering operators, σ^+ and σ^- to fermionic modes by means of the Jordan-Wigner transformation. The fermionic modes read then,

$$a_{k}^{\dagger} = -\left(\bigotimes_{i=1}^{k-1} \sigma^{z}\right) \sigma^{+}$$

$$a_{k} = -\left(\bigotimes_{i=1}^{k-1} \sigma^{z}\right) \sigma^{-}$$

$$(12)$$

One can verify, that these modes now obey the fermionic anti-commutation relations, $\{a_k; a_l^{\dagger}\} = \delta_{k,l}$. It is possible to calculate the evolution of the fermionic two-point function $\langle a_k^{\dagger} a_m \rangle$ from the master equation (11) via,

$$\partial_t \langle a_k^{\dagger} a_m \rangle = \text{Tr} \left\{ \mathcal{L}^{\dagger}[a_k^{\dagger} a_m] \rho \right\} \equiv \langle \mathcal{L}^{\dagger}[a_k^{\dagger} a_m] \rangle.$$
(13)

Since the commutator of two pairs of fermionic modes is again an operator made up from two fermionic modes, we see, that the time-evolution of the fermionic two-point functions again only depends on two-point functions. So the two-point correlation functions of the steady state can be computed exactly. The equations for the correlation functions read,

$$\begin{aligned} \partial_{t} \langle a_{m}^{\dagger} a_{l} \rangle &= \\ \frac{\alpha}{2} (\delta_{m,0} + \delta_{l,0}) \langle a_{l} a_{m}^{\dagger} \rangle - \frac{\beta}{2} (\delta_{m,N} + \delta_{l,N}) \langle a_{m}^{\dagger} a_{l} \rangle \\ - \frac{\varphi}{2} \left([\langle a_{m+1} a_{m+1}^{\dagger} \rangle - \langle a_{m+1}^{\dagger} a_{m+1} \rangle \\ + \langle a_{m-1} a_{m-1}^{\dagger} \rangle - \langle a_{m-1}^{\dagger} a_{m-1} \rangle] \delta_{l,m} \\ + 2 [\langle a_{m}^{\dagger} a_{l} \rangle - \langle a_{l} a_{m}^{\dagger} \rangle] \right) \\ - i\lambda \left([\langle a_{m}^{\dagger} a_{l+1} \rangle + \langle a_{m}^{\dagger} a_{l} - 1 \rangle] \\ - [\langle a_{m+1}^{\dagger} a_{l} \rangle + \langle a_{m-1}^{\dagger} a_{l} \rangle] \right), \end{aligned}$$
(14)

and,

$$\begin{aligned} \partial_t \langle a_m^{\dagger} a_l^{\dagger} \rangle &= \\ -\frac{\alpha}{2} \left(\delta_{0,m} + \delta_{0,l} \right) \langle a_m^{\dagger} a_l^{\dagger} \rangle - \frac{\beta}{2} \left(\delta_{l,N} + \delta_{m,N} \right) \langle a_m^{\dagger} a_l^{\dagger} \rangle \\ -\varphi \left(\left[\delta_{l+1,m} \langle a_l^{\dagger} a_{l+1}^{\dagger} \rangle - \delta_{m+1,l} \langle a_m^{\dagger} a_{m+1}^{\dagger} \rangle \right] + 2 \langle a_m^{\dagger} a_l^{\dagger} \rangle \right) \\ + i\lambda \left(\langle a_m^{\dagger} a_{l-1}^{\dagger} \rangle + \langle a_m^{\dagger} a_{l+1}^{\dagger} \rangle + \langle a_{m-1}^{\dagger} a_l^{\dagger} \rangle + \langle a_{m+1}^{\dagger} a_l^{\dagger} \rangle \right) \\ 15) \end{aligned}$$

The other correlation functions are related to these two by the identities imposed due to the anti-commutation relations of the fermionic modes, thus $\langle a_m^{\dagger} a_l^{\dagger} \rangle = \langle a_m a_l \rangle^*$ and $\langle a_m^{\dagger} a_l \rangle = \delta_{l,m} - \langle a_l a_m^{\dagger} \rangle^*$.

The steady-state correlations can be computed from these equations by requiring stationarity, i.e. $\partial_t \langle a_m^{\dagger} a_l \rangle = \partial_t \langle a_m^{\dagger} a_l^{\dagger} \rangle = 0$. One needs to solve the corresponding system of linear difference equations.

The current as well as the particle number density can be expressed in terms of these correlators. One finds for the particle number density $\langle n_k \rangle = \langle a_k^{\dagger} a_k \rangle$ and the two contributions to the current read,

$$j_{k,k+1}^{st} = \varphi\left(\langle n_k \rangle - \langle n_{k+1} \rangle\right)$$
(16)
$$j_{k,k+1}^{co} = \frac{2\lambda}{i} \left(\langle a_k^{\dagger} a_{k+1} \rangle - \langle a_k a_{k+1}^{\dagger} \rangle\right).$$

Note, that the stochastic current now only depends on the difference of the densities at adjacent sites. The equations (14) and (15) are solved numerically for different parameters α and β .

The classical SEP without any further driving obeys the detailed balance condition, and thus does not support a steady state current. When one allows for an external driving of the particles at the boundaries, as we do in our example, a current is induced in the SEP steady state. This current however vanishes as $\sim 1/N$ in the system size N, see FIG. 8(a). If we turn to the coherent part, we also see this behavior, i.e. the current vanishes in the limit $N \rightarrow \infty$ FIG. 8(b). We deduce from this, that the quantum perturbation to the SEP is an irrelevant perturbation and does not lead to a qualitatively different behavior of the systems transport properties. This point is also affirmed by considering the particle density distribution as seen in FIG. 9(a). Here one sees that the density distribution in the presence of a quantum perturbation almost coincides with the classical distribution. The different contributions to the current, FIG. 9(b), are identical in the system's bulk, and add up to an increased current J_{tot} . Whether the quantum perturbation is completely irrelevant however can not be deduced from just considering the steady state density and the current alone. Here one would need to take higher order correlations into account, as for instance the current-current correlation function at unequal times.



FIG. 8: (color online) The plots demonstrate the scaling of the currents for large N. Plot (a) depicts the scaling of the total current $j^{tot} = j^{st}$, blue dashed line for $\lambda = 0$. The plot (b) corresponds to $\lambda = 1$. Here the total current j^{tot} is represented by the black dash-dotted line. The coherent contribution corresponds to the blue solid line. The boundary values were chosen as $\alpha = \beta = 3/4$

V. CONCLUSIONS AND OUTLOOK

We have investigated a quantum perturbation to the dynamics of the stochastic asymetric exclusion process as well as to the symmetric exclusion process. We find that we can rephrase the stochastic master equation as a quantum equation that fully reproduces the classical dynamics. The quantum perturbations modify the steady state behavior and allow for two different types of currents, which each on their own can vary as a function of the site. Numerical simulations of the full master equation indicate, that the underlying classical phase-diagram of the stochastic process is respected. The steady state responds to driving due to the boundary terms with a different behavior in current and density. A further step



FIG. 9: (color online) Plot (a) shows the density distributions for the values $\lambda = 0, 1$ and for $\alpha = \beta = 3/4$. Calculations for larger system sizes indicate, that this deviation vanishes in the thermodynamic limit. The second plot (b) shows the dependence of the current distribution on the lattice site for $\lambda = 1$. The stochastic contribution j_{st} as well as the coherent contribution j_{co} only marginally differ at the boundaries

would be to investigate the current-current correlation function of the SEP, to see whether the quantum perturbation has an effect to the current-fluctuations. Furthermore,other, more complex models with an interplay between stochastic and coherent dynamics can be investigated along these lines.

VI. ACKNOWLEDGMENTS

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Appendix A: Derivation of the steady state

The steady state can be calculated in matrix product form by applying the Derrida method of the quadratic algebra [12]. To better understand this, we shall rewrite the MPDO with translationally invariant matrices $M_{i,j}$ as,

$$\rho = \langle W | \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}^{\otimes N} | V \rangle$$
(A1)

This MPDO now serves as steady state ansatz for the master equation. To derive the quadratic algebra, consider the total master equation (3), where we now regrouped the terms in local two-body interactions and the single particle boundary terms. This can be written as,

$$\mathcal{L}[\rho] = \mathcal{L}_1[\rho] + \sum_{k=1}^{N-1} \mathcal{L}_{k,k+1}[\rho] + \mathcal{L}_1[\rho].$$
(A2)

We now try to find relations for the matrices $M_{i,j}$ so that ρ will correspond to the steady state of the system. We introduce

additional ancilla matrices M_{ij} and require, that

$$\mathcal{L}_{k,k+1} \left[\begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \otimes \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \right] = \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \otimes \begin{pmatrix} \hat{M}_{0,0} & \hat{M}_{0,1} \\ \hat{M}_{1,0} & \hat{M}_{1,1} \end{pmatrix} - \begin{pmatrix} \hat{M}_{0,0} & \hat{M}_{0,1} \\ \hat{M}_{1,0} & \hat{M}_{1,1} \end{pmatrix} \otimes \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \right].$$
(A3)

Furthermore we require, that the single site operators at the boundaries have to satisfy,

$$\langle V | \mathcal{L}_{1} \left[\begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \right] = \langle V | \begin{pmatrix} \hat{M}_{0,0} & \hat{M}_{0,1} \\ \hat{M}_{1,0} & \hat{M}_{1,1} \end{pmatrix}$$
$$\mathcal{L}_{N} \left[\begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \right] | W \rangle = - \begin{pmatrix} \hat{M}_{0,0} & \hat{M}_{0,1} \\ \hat{M}_{1,0} & \hat{M}_{1,1} \end{pmatrix} | W \rangle (A4)$$

We see that the total sum (A2) telescopes to zero and ρ is the steady state solution of the equation. To find an explicit solution one needs to construct a representation of the matrices M_{ij} that obey the given algebraic constraints. In the simple case, where $\lambda = 0$ and $\varphi = 1$, we reproduce the known classical algebra for the ASEP steady state [12]. Here one chooses $M_{0,1} = M_{1,0} = 0$, as well as $\hat{M}_{0,1} = \hat{M}_{1,0} = 0$. The diagonal terms, upon stetting, $\hat{M}_{0,0} = -\hat{M}_{1,1} = 1$, have to satisfy the algebra $M_{1,1}M_{0,0} = M_{0,0} + M_{1,1}$. The boundary terms have to fulfill, $\langle V | M_{0,0} = 1/\alpha \langle V |$ and $M_{1,1} | W \rangle = 1/\beta | W \rangle$. One possible representation for this algebra is given by (8), when considering $N \to \infty$. Note, that when one wants to reproduce the steady state of a finite system of size N, it suffices to choose a matrix dimension of D = N + 1 for this specific representation. One can also verify, that at $\alpha + \beta = 1$ the representation of the algebra can be chosen one-dimensional [12].

Appendix B: Numerical Implementation

For the numerical evolution of the density-matrix according to (3), we apply a numerical scheme developed in [13]. Starting from the initial density-matrix ρ_0 given as an MPDO, we apply the CP-map $\mathcal{E}(t) = \exp(t\mathcal{L})$ for a small time step Δt and approximate the resulting density operator, that has now an increased bond dimension, with an MPDO that has a bonddimension D_k corresponding to that of the original MPDO. The approximation of the operator $\rho(t + \Delta t) = \mathcal{E}(\Delta t) \rho(t)$ is chosen, such that the Hilbert-Schmidt norm $\|\rho(t + \Delta t) - \rho_{new}\|_{HS}^2 = \operatorname{Tr} \left[(\rho(t + \Delta t) - \rho_{new})^2 \right]$ is minimized. This optimization can be performed efficiently by sweeping from left to right over the individual sites and optimizing the matrices locally. For the application of the CP-map to be computable, we perform a second-order Trotter expansion of the CP-map as follows,

$$\mathcal{E}(\mathcal{L}, \Delta t) \simeq \mathcal{E}(\mathcal{L}_o, \Delta t/2) \mathcal{E}(\mathcal{L}_e, \Delta t) \mathcal{E}(\mathcal{L}_o, \Delta t/2),$$
 (B1)

where $\mathcal{L} = \mathcal{L}_e + \mathcal{L}_o$ corresponds to splitting the Liouvillian into commuting terms which act on the sites (2k, 2k + 1) and

(2k-1,2k), respectively. The resulting MPDO ρ_{new} is then chosen as initial condition for the next step and the procedure

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