Adiabatic elimination for multi-partite open quantum systems with non-trivial zero-order dynamics

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Abstract

We provide model reduction formulas for open quantum systems consisting of a target component which weakly interacts with a strongly dissipative environment. The time-scale separation between the uncoupled dynamics and the interaction allows to employ tools from center manifold theory and geometric singular perturbation theory to eliminate the variables associated to the environment (adiabatic elimination) with high-order accuracy. An important specificity is to preserve the quantum structure: reduced dynamics in (positive) Lindblad form and coordinate mappings in Kraus form. We provide formulas of the reduced dynamics. The main contributions of this paper are (i) to show how the decomposition of the environment into K components enables its efficient treatment, avoiding the quantum curse of dimension; and (ii) to extend the results to the case where the target component is subject to Hamiltonian evolution at the fast time-scale. We apply our theory to a microwave superconducting quantum resonator subject to material losses, and we show that our reduced-order model can explain the transmission spectrum observed in a recent pump probe experiment.

I. INTRODUCTION

The evolution of a quantum system interacting with an environment is rigorously described by a Schrödinger equation on the joint Hilbert space. However, the complexity of the environment hampers the study of the system as a whole and one often resorts to the Born-Markov approximation to obtain a Lindblad master equation [7] describing the target system alone, and the environment's effect summarized by dissipation or "decoherence" operators. Similarly, when a quantum system consists of several interacting components, e.g. a main computing subsystem coupled to an ancillary subsystem expressing a measurement device, one often seeks to analyze a dynamical equation for the main subsystem alone, approximately including the effect of the ancillary subsystem. In this perspective, model reduction methods come to aid to the physicists interested in gaining better physical insights, in running simplified numerical simulations, and in designing the dynamics of a target subsystem by smartly engineering its interaction with other subsystems, as in the case of reservoir engineering [18].

A classical approach to model reduction for quantum systems makes use of the time-scale separation between a slow subsystem of interest and the fast auxiliary subsystems coupled to it, and eliminates the fast variables in a procedure denominated as *adiabatic elimination*. In closed quantum systems – where the evolution stays unitary under Hamiltonian dynamics – adiabatic elimination is performed by means of standard perturbation theory techniques [22]. In contrast, the treatment of open quantum systems – including decoherence under Lindbladian dynamics – is more involved. In the literature, adiabatic elimination in the latter case has been addressed for specific examples separately: lambda systems up to second-order [8], a specific atom-optics example [2], systems where excited states decay toward n ground states [17], [21], systems with Gaussian dynamics and subject to continuous measurement [14].

However, general approaches to adiabatic elimination of Lindblad systems – and maintaining the positivity-preserving quantum structure, beyond a standard linear systems treatment via singular perturbation theory – have attracted much less attention. In [15], Kessler has developed a generalization of the Schrieffer-Wolff formalism; in [13], [6], the authors address quantum stochastic differential equations in the limit where the speed of the fast system goes to infinity. A geometric approach to adiabatic elimination has been introduced by [4], [3], where the authors explore an asymptotic expansion of the reduced dynamics by a careful application of center manifold techniques [9] and geometric singular perturbation theory [11]. In order to succesfully retain the physical interpretation, the reduced dynamics is expressed by Lindblad equations and is mapped to the original dynamics via a trace-preserving completely-positive (CPTP) map, also called Krauss map.

The present work builds upon the geometric approach of [4], [3] and brings forward two novel features. First, unlike in [4], [3] where the target system was assumed to be static in the ideal case, we here develop formulas for the case where the target system undergoes non-trivial fast Hamiltonian dynamics, when uncoupled from the environment. This appears in all practical situations where the target system is detuned from the reference frame, e.g. by a field to be measured in quantum metrology, or when it undergoes (in this paper constant) drives to implement quantum operations. Second, we consider environments that consist not of a single bulk system, but which can be decomposed into a not-necessarily-finite number of fast dissipative subsystems. Such situations often appear in practice when the target quantum system is corrupted by various imperfection

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sources [16]. We show how to take advantage of this decomposition towards more efficient model reduction computations. Indeed, the first-order approximation amounts to the sum of the contributions of each fast dissipative subsystem, and the same result holds for the second-order approximation under specific commutation properties of the operators involved in the computation. The proposed theory is applied to a model of a microwave superconducting resonator subject to dielectric losses due to a bath of many two-level-systems. We show how a reduced model resulting from our theory allows to explain the non-trivial transmission spectrum observed in a pump probe experiment.

The outline of the paper is as follows. Setting and main assumptions are introduced in Section II. Section III provides our main results with the formulas of our adiabatic elimination for the case of many fast dissipative subsystem weakly coupled to the target one. Section IV contains the application and comparison to experimental data. We conclude the paper with few final remarks. Proof and computation details are given in appendix.

II. Setting

A. K-partite systems with non-trivial zero-order dynamics

Open quantum systems are typically described by differential equations evolving on the manifold $\mathcal{M}$ of density operators ρ, namely the set of all linear Hermitian nonnegative operators from a Hilbert space $\mathcal{H}$ to itself, whose trace equals one. The evolution of an open quantum system is then described by the Lindblad master equation [7]:

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i\,[\boldsymbol{H}, \rho] + \sum_\mu \mathcal{D}_{\boldsymbol{L}_\mu}(\rho),$$

where each $\boldsymbol{L}_\mu$ is a "decoherence" operator on $\mathcal{H}$, $\boldsymbol{H}$ is a Hermitian "Hamiltonian" operator on $\mathcal{H}$, and $\mathcal{D}$ is a superoperator defined by:

$$\mathcal{D}_{\boldsymbol{L}_\mu}(\rho) := \boldsymbol{L}_\mu \rho \boldsymbol{L}_\mu^\dagger - \frac{1}{2}\boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu \rho - \frac{1}{2}\rho \boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu.$$

In this paper, we consider the composite Hilbert space $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ of a target quantum system on $\mathcal{H}_B$ and its environment on $\mathcal{H}_A$. The dynamics on $\mathcal{H}$ satisfies a time scale separation:

$$\frac{d\rho}{dt} = \mathcal{L}_A(\rho) + \varepsilon\mathcal{L}_{int}(\rho) + \varepsilon\mathcal{L}_B(\rho) + (-i)[\tilde{\boldsymbol{H}}_B, \rho], \tag{1}$$

where ε is a small positive parameter; $\mathcal{L}_A$ and $\mathcal{L}_B$ are Lindbladian super-operators acting exclusively on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively; $\mathcal{L}_{int}$ is a Lindbladian superoperator which captures the interaction between $\mathcal{H}_A$ and $\mathcal{H}_B$. Here we assume that this interaction is Hamiltonian and expressed as:

$$\mathcal{L}_{int}(\rho) := -i\left[\boldsymbol{A} \otimes \boldsymbol{B}^\dagger + \boldsymbol{A}^\dagger \otimes \boldsymbol{B},\ \rho\right],$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively are non-necessarily-Hermitian operators acting on $\mathcal{H}_A$ and $\mathcal{H}_B$ only. Finally, $\tilde{\boldsymbol{H}}_B$ is a Hamiltonian operator on $\mathcal{H}_B$, thus expressing fast unitary dynamics on the target system; its presence is the first novelty in our paper. For a set of interesting situations, the dynamics of typical quantum systems can be expressed in a rotating frame where the term $\tilde{\boldsymbol{H}}_B$ would vanish. However, several reasons can justify to keep this term. For instance, in many significant situations the vanishing of $\tilde{\boldsymbol{H}}_B$ is not rigorous and involves an additional treatment of appearing fast time-varying parameters in the equation via averaging theory; or, $\tilde{\boldsymbol{H}}_B$ can be a term of particular interest like a field to be measured with the quantum device or an actuation towards applying some operation on the target system.

As a second novelty, we consider a generalized setting where $\mathcal{H}_A = \bigotimes_k \mathcal{H}_A^{(k)}$ is composed of a non-necessarily-finite number of Hilbert spaces $\mathcal{H}_A^{(k)}$. Each subsystem on $\mathcal{H}_A^{(k)}$ is strictly dissipative and interacts with $\mathcal{H}_B$ only. Then, system (1) reads as:

$$\frac{d\rho}{dt} = \sum_k \left(\mathcal{L}_A^{(k)}(\rho) + \varepsilon\mathcal{L}_{int}^{(k)}(\rho)\right) + \varepsilon\mathcal{L}_B(\rho) + (-i)\left[\tilde{\boldsymbol{H}}_B, \rho\right] \tag{2}$$

where $\mathcal{L}_A^{(k)}$ acts on $\mathcal{H}_A^{(k)}$ only and where

$$\mathcal{L}_{int}^{(k)}(\rho) := -i\left[\boldsymbol{A}^{(k)} \otimes \boldsymbol{B}^\dagger + \boldsymbol{A}^{(k)\,\dagger} \otimes \boldsymbol{B},\ \rho\right],$$

captures the Hamiltonian interaction between $\mathcal{H}_A^{(k)}$ and $\mathcal{H}_B$, with $\boldsymbol{A}^{(k)}$ non-necessarily-Hermitian operators acting on $\mathcal{H}_A^{(k)}$ only.

For $\varepsilon = 0$, the system is uncoupled and the solution trajectories stay separable for all times, namely for $\rho(0) = \bigotimes_k \rho_A^{(k)}(0) \otimes \rho_B(0)$ we have $\rho(t) = \bigotimes_k \rho_A^{(k)}(t) \otimes \rho_B(t)$ for all times, with each factor in the product following its independent dynamics. To apply adiabatic elimination, we assume that each part of the environment is highly dissipative and relaxes fast to a unique steady state, i.e.: for any initial state ρ_0 on $\mathcal{H}_A \otimes \mathcal{H}_B$, the solution of the uncoupled system $\varepsilon = 0$ converges to $\bigotimes_k \bar{\rho}_A^{(k)} \otimes \rho_B(t)$ where, for each k, $\bar{\rho}_A^{(k)}$ is the unique solution of $\mathcal{L}_A^{(k)}\left(\bar{\rho}_A^{(k)}\right) = 0$; and $\rho_B(t)$ satisfies $\dot{\rho}_B = -i[\tilde{\boldsymbol{H}}_B, \rho_B]$ with $\rho_B(0) = \mathrm{Tr}_A(\rho_0)$. For ease of presentation, we will also denote $\bar{\rho}_A := \bigotimes_k \bar{\rho}_A^{(k)}$.

B. Asymptotic expansion

Both in the bi-partite and the K-partite case, for the uncoupled system $\varepsilon = 0$, there exists an asymptotically stable center manifold $\mathcal{M}_0$ of same dimension as $\mathcal{H}_B$, on which the dynamics have imaginary eigenvalues. It thus follows from Fenichel's Invariant Manifold Theorem [11] that, for small enough $\varepsilon > 0$, there exists an invariant and attractive manifold $\mathcal{M}_\varepsilon$ which has the same dimension as $\mathcal{M}_0$ and which is ε−close to it. Furthermore, by virtue of Carr's result [9], an approximation of $\mathcal{M}_\varepsilon$ can be computed up to arbitrary precision. The quantum particularity, as explained in [3], is that such approximation should retain a physical interpretation by preserving the quantum structure:

- the mapping from the reduced space to the complete space is a mapping between density operators, and $\mathcal{M}_\varepsilon$ can be parameterized by $\mathcal{M}_\varepsilon := \{\rho \in \mathcal{H} : \rho = \mathcal{K}(\rho_s),\ \rho_s \in \mathcal{H}_s\}$ for some Hilbert space $\mathcal{H}_s$ that has same dimension as $\mathcal{H}_B$, and where $\mathcal{K}(\cdot)$ is a Kraus map[1];
- the reduced dynamics on $\mathcal{M}_\varepsilon$ are Lindbladian, i.e. $\dot{\rho}_s = \mathcal{L}_s(\rho_s)$ for some Lindbladian superoperator $\mathcal{L}_s$.

In other words, we aim to find a Kraus map $\rho = \mathcal{K}(\rho_s)$ and a Lindbladian $\mathcal{L}_s$ such that the following invariance equation is satisfied for all ε small enough and for all ρ_s:

$$
\begin{aligned}
&\mathcal{L}_A(\mathcal{K}(\rho_s)) + \varepsilon\mathcal{L}_{int}(\mathcal{K}(\rho_s)) + \varepsilon\mathcal{L}_B(\mathcal{K}(\rho_s)) \\
&+ (-i)[\tilde{\boldsymbol{H}}_B, \mathcal{K}(\rho_s)] \quad = \mathcal{K}(\mathcal{L}_s(\rho_s)).
\end{aligned}
\tag{3}
$$

We impose that both the Kraus map and the Lindbladian are parameterized as an infinite series:

$$
\mathcal{K}(\rho_s) := \sum_{h=0}^{+\infty} \varepsilon^h \, \mathcal{K}_h(\rho_s), \quad \mathcal{L}_s(\rho_s) := \sum_{h=0}^{+\infty} \varepsilon^h \, \mathcal{L}_{s,h}(\rho_s).
$$

Then, by identifying the terms of the same order of ε in the invariance equation (3), we obtain an invariance relation at all orders h. At zero-order, we have:

$$
\mathcal{L}_A\left(\mathcal{K}_0(\rho_s)\right) + (-i)\left[\tilde{\boldsymbol{H}}_B, \mathcal{K}_0(\rho_s)\right] = \mathcal{K}_0\left(\mathcal{L}_{s,0}(\rho_s)\right).
\tag{4}
$$

Similarly, the first-order invariance condition reads as:

$$
\begin{aligned}
&\mathcal{L}_A\left(\mathcal{K}_1(\rho_s)\right) + \mathcal{L}_{int}\left(\mathcal{K}_0(\rho_s)\right) + \mathcal{L}_B\left(\mathcal{K}_0(\rho_s)\right) \\
&- i\left[\tilde{\boldsymbol{H}}_B, \mathcal{K}_1(\rho_s)\right] = \mathcal{K}_0\left(\mathcal{L}_{s,1}(\rho_s)\right) + \mathcal{K}_1\left(\mathcal{L}_{s,0}(\rho_s)\right),
\end{aligned}
\tag{5}
$$

whereas the second-order invariance condition reads as:

$$
\begin{aligned}
&\mathcal{L}_A\left(\mathcal{K}_2(\rho_s)\right) + \mathcal{L}_{int}\left(\mathcal{K}_1(\rho_s)\right) \\
&+ \mathcal{L}_B\left(\mathcal{K}_0(\rho_s)\right) - i\left[\tilde{\boldsymbol{H}}_B, \mathcal{K}_2(\rho_s)\right] \\
&= \mathcal{K}_0\left(\mathcal{L}_{s,2}(\rho_s)\right) + \mathcal{K}_1\left(\mathcal{L}_{s,1}(\rho_s)\right) + \mathcal{K}_2\left(\mathcal{L}_{s,0}(\rho_s)\right).
\end{aligned}
\tag{6}
$$

III. REDUCED-MODEL FORMULAS

The aim of this Section is to provide explicit solutions to the zero-, first-, and second-order invariance equations (4)-(6) for the case of K-partite systems as introduced in Section II-A, i.e. for model (2). We immediately observe that the zero-order (4) is naturally solved by setting:

$$
\mathcal{L}_{s,0}(\rho_s) := -i\left[\tilde{\boldsymbol{H}}_B, \rho_s\right], \quad \mathcal{K}_0(\rho_s) := \left(\bigotimes_k \bar{\rho}_A^{(k)}\right) \otimes \rho_s.
\tag{7}
$$

At first order, let the Kraus map have the following structure inspired by [3]:

$$
\begin{aligned}
&\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \varepsilon\mathcal{K}_1(\rho_s) := \\
&(\boldsymbol{I} - i\varepsilon\boldsymbol{M})\left(\bar{\rho}_A \otimes \rho_s\right)\left(\boldsymbol{I} + i\varepsilon\boldsymbol{M}^\dagger\right) + \mathcal{O}\left(\varepsilon^2\right),
\end{aligned}
\tag{8}
$$

where $\boldsymbol{M} := \sum_k \boldsymbol{M}^{(k)}$, $\boldsymbol{M}^{(k)} := \boldsymbol{F}_1^{(k)} \otimes \boldsymbol{B}^\dagger + \boldsymbol{F}_2^{(k)} \otimes \boldsymbol{B}$ for any k. This would immediately imply that:

$$
\mathcal{K}_1(\rho_s) = -i\boldsymbol{M}\left(\bar{\rho}_A \otimes \rho_s\right) + i\left(\bar{\rho}_A \otimes \rho_s\right)\boldsymbol{M}.
\tag{9}
$$

The following assumption will be instrumental in establishing our main results.

Assumption 1: There exists $c_{B^\dagger} \in \mathbb{R}$ such that:

$$
\left[\tilde{\boldsymbol{H}}_B, \boldsymbol{B}^\dagger\right] = c_{B^\dagger} \boldsymbol{B}^\dagger.
\tag{10}
$$

---

[1] A Kraus map takes the form $\rho = \mathcal{K}(\rho_s) := \sum_\ell \boldsymbol{M}_\ell \rho_s \boldsymbol{M}_\ell^\dagger$ for some operators $\boldsymbol{M}_\ell$ in order to express any completely positive superoperator [10], and with $\sum_\ell \boldsymbol{M}_\ell \rho_s \boldsymbol{M}_\ell^\dagger = \boldsymbol{I}$ ensuring trace-preservation i.e. $\mathrm{Tr}\left(\mathcal{K}(\rho_s)\right) = \mathrm{Tr}(\rho_s) = 1$.

Theorem 1: Consider model (2). Let Assumption 1 hold. Then, the first-order invariance equation (5) is satisfied by the Lindbladian $\mathcal{L}_{s,1}(\rho_s) = \mathcal{L}_B(\rho_s)$ and by a map $\mathcal{K}_1$ of the form (9) where, for each k, $\boldsymbol{F}_1^{(k)}, \boldsymbol{F}_2^{(k)}$ respectively are the unique solutions of:

$$\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right) + \boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)} - i\,c_{B\dagger}\,\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)} = 0, \tag{11a}$$

$$\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right) + \boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)} + i\,c_{B\dagger}^*\,\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)} = 0. \tag{11b}$$

Furthermore, $\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \varepsilon\mathcal{K}_1(\rho_s)$ is a CPTP map up to second-order terms.

Proof: see Appendix A. ∎

Remark 1: The first-order (5) is also satisfied by the Lindbladian $\mathcal{L}_{s,1}(\rho_s) = \mathcal{L}_B(\rho_s) - i\,[\boldsymbol{H}_{s,1}, \rho_s]$ with $\boldsymbol{H}_{s,1} := \sum_k \mathrm{Tr}\left(\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}\right)\boldsymbol{B}^\dagger + \mathrm{Tr}\left(\boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)}\right)\boldsymbol{B}$, and by a map $\mathcal{K}_1$ of the form (9) where $\boldsymbol{F}_1^{(k)}, \boldsymbol{F}_2^{(k)}$ respectively are the unique solutions of:

$$\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right) + \mathcal{S}^{(k)}(\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}) - i\,c_{B\dagger}\,\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)} = 0, \tag{12a}$$

$$\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right) + \mathcal{S}^{(k)}(\boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)}) + i\,c_{B\dagger}^*\,\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)} = 0, \tag{12b}$$

Here, for an operator Q acting on $\mathcal{H}_A^{(k)}$, notation $\mathcal{S}^{(k)}(Q)$ denotes $Q - \mathrm{Tr}(Q)\bar{\rho}_A^{(k)}$. Furthermore, $\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \varepsilon\mathcal{K}_1(\rho_s)$ is a CPTP map up to second-order terms. The possibility of having alternative solutions to the first-order invariance equation hinges upon a gauge degree of freedom in the selection of the trace of terms $\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}$ and $\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}$. It appears that gauge choices are instrumental for positivity-preservation in the solution of the second-order invariance equation, as we consider next.

Theorem 2: Consider model (2). Let Assumption 1 hold. Assume furthermore that $\mathcal{L}_{s,1} = \mathcal{L}_B = 0$, with $\mathcal{K}_1$ selected according to Theorem 1. Then, the second-order invariance equation is satisfied by a Lindbladian:

$$\begin{aligned}
\mathcal{L}_{s,2}(\rho_s) = \sum_k &-i\,\Im\left(z_1^{(k)}\right)\left[\boldsymbol{B}\boldsymbol{B}^\dagger, \rho_s\right] - i\,\Im\left(z_2^{(k)}\right)\left[\boldsymbol{B}^\dagger\boldsymbol{B}, \rho_s\right] \\
&+ 2\,\Re\left(z_1^{(k)}\right)\mathcal{D}_{\boldsymbol{B}^\dagger}(\rho_s) + 2\,\Re\left(z_2^{(k)}\right)\mathcal{D}_{\boldsymbol{B}}(\rho_s) \\
&+ \sum_{k>k'}\left\{-i\,\delta^{(k,k')}\left[\left[\boldsymbol{B}, \boldsymbol{B}^\dagger\right], \rho_s\right]\right\},
\end{aligned} \tag{13}$$

with:

$$\delta^{(k,k')} = \frac{-2\Re\left(z_0^{(k)} z_0^{(k')\,*}\right)}{c_{B\dagger}}, \quad z_0^{(k)} = \mathrm{Tr}\left(\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}\right), \tag{14a}$$

$$z_1^{(k)} = \mathrm{Tr}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{A}^{(k)\,\dagger}\right), \quad z_2^{(k)} = \mathrm{Tr}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{A}^{(k)}\right), \tag{14b}$$

and by a map $\mathcal{K}_2$, obtained from formulas (23)-(??) such that $\mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \varepsilon\mathcal{K}_1(\rho_s) + \varepsilon^2\mathcal{K}_2(\rho_s)$ is a CPTP map up to third-order terms.

Proof: see Appendix B. ∎

Corollary 1: Under the same assumptions of Theorem 2, if $[\boldsymbol{B}, \boldsymbol{B}^\dagger] = c\boldsymbol{I}$ for some $c \in \mathbb{C}$, then the second-order invariance equation (6) is satisfied by the Lindbladian:

$$\begin{aligned}
\mathcal{L}_{s,2}(\rho_s) = \sum_k &\left\{-i\,\Im\left(z_1^{(k)} + z_2^{(k)}\right)\left[\boldsymbol{B}^\dagger\boldsymbol{B}, \rho_s\right]\right. \\
&\left.+ 2\,\Re\left(z_1^{(k)}\right)\mathcal{D}_{\boldsymbol{B}^\dagger}(\rho_s) + 2\,\Re\left(z_2^{(k)}\right)\mathcal{D}_{\boldsymbol{B}}(\rho_s)\right\}.
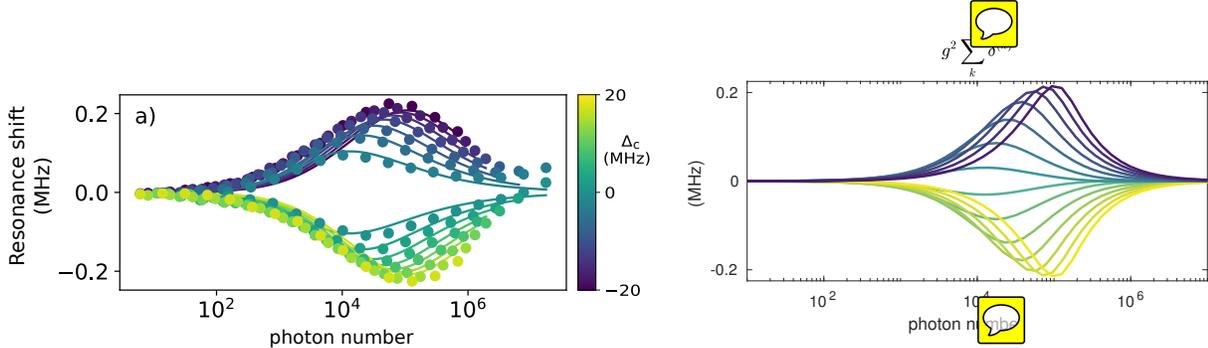\end{aligned} \tag{15}$$

IV. APPLICATION

Fig. 1. Section IV. Shift of the resonator frequency: probe experiment (above left) versus reduced-order model (18) (above right) as a function of intraresonator photons $\langle\boldsymbol{N}\rangle$ and for different pump detunings Δ_c. Parameter values in reduced-order model (18): $g = 30\,\mathrm{kHz}$, $\Gamma_- = 10\,\mathrm{MHz}$, Δ_c ranges from $-20\,\mathrm{MHz}$ to $20\,\mathrm{MHz}$, $\Delta_q^{(k)}$ is uniformly distributed over k in the range $[-100\,\mathrm{MHz}, 100\,\mathrm{MHz}]$, and $\langle\boldsymbol{N}\rangle = \tilde{v}^2/\Delta_c^2$ where $\tilde{v}$ is in the range from 0 to 10 GHz.

Microwave superconducting resonators are an important component in various quantum devices, and in particular in the quantum electrodynamics circuits [5], [19] that are one of the most promising current technologies towards building a quantum computer [1]. Losses due to imperfections in amorphous materials constitute a dominant loss channel of such resonators [23], [12], and can be represented by a bath of two-level systems (TLSs). In many practical cases, strong microwave tones are applied with significant frequency detuning with respect to the resonance frequency [20] in order to activate a parametric interaction between the resonator mode and another circuit degree of freedom. Within this framework, the LKB team has performed a pump probe experiment on the microwave resonator (Figure): a strong "pump" drive, at a frequency far detuned from the resonator, is applied to essentially scramble the quantum behavior ("saturate") of the TLS bath, whereas a weak probe tone, assumed not to disturb the bath behavior, is used to retrieve the transmission spectrum of the resonator. The latter allows to extract induced detuning and damping rate.

Let $\mathcal{H}_Q^{(k)}$ and $\mathcal{H}_C$ respectively be the Hilbert space of the k-th TLS=qubit and the resonator, and $\mathcal{H}_Q := \bigotimes_k \mathcal{H}_Q^{(k)}$. Respectively denote with $\boldsymbol{\sigma}_+^{(k)}$ and $\boldsymbol{\sigma}_-^{(k)}$ the raising and lowering operator on the k-th qubit, and with $\boldsymbol{\sigma}_x^{(k)}$, $\boldsymbol{\sigma}_y^{(k)}$, and $\boldsymbol{\sigma}_z^{(k)}$ the Pauli operators on the k-th qubit. Let $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ be the annihilation and creation operators in the resonator mode. The experimental setup is modeled by the following system in Lindblad form:

$$\frac{d}{dt}\tilde{\rho} = -i[\mathbf{H}, \tilde{\rho}] + \Gamma_- \sum_k \mathcal{D}_{\boldsymbol{\sigma}_-^{(k)}}(\tilde{\rho}), \tag{16}$$

$$\begin{aligned}\mathbf{H} &= \omega_c\,\mathbf{a}^\dagger \mathbf{a} + \left(ve^{i\omega_r t} + v^* e^{-i\omega_r t}\right)\left(\boldsymbol{a}^\dagger + \boldsymbol{a}\right) \\ &+ \sum_k \left(\frac{\omega_q^{(k)}}{2}\boldsymbol{\sigma}_z^{(k)} + i\,g\,\boldsymbol{\sigma}_x^{(k)}\left(\mathbf{a}^\dagger - \mathbf{a}\right)\right).\end{aligned}$$

Here ω_c, ω_r, and $\omega_q^{(k)}$ are the frequencies of the resonator, the pump drive, and the k-th qubit respectively, v is the strength of the pump, g is the coupling strength between the resonator and each qubit, and Γ_- is the dissipation rate associated to $\boldsymbol{\sigma}_-^{(k)}$ on each qubit. The goal would be to obtain a reduced order model for (16) which matches the transmission spectrum of this experiment.

For each k, let $\Delta_q^{(k)} = \omega_q^{(k)} - \omega_r$ and $\Delta_c = \omega_c - \omega_r$. Under the assumption that $\omega_q^{(k)}, \omega_c, \omega_r \gg |\Delta_q^{(k')}|, |\Delta_c|, g, \Gamma_-, \Gamma_+$ for any k, k', we apply the standard rotating-wave approximation (i.e. first-order averaging) with $\boldsymbol{H}_0 := \omega_r\,\mathbf{a}^\dagger \mathbf{a} + \sum_k \frac{\omega_r}{2}\boldsymbol{\sigma}_z^{(k)}$ the Hamiltonian corresponding to the rotating change of frame and $H_1 = H - H_0$ the remaining Hamiltonian. The first-order RWA yields:

$$\frac{d}{dt}\rho_1^{rwa} = -i\left[\boldsymbol{H}_1^{rwa}, \rho_1^{rwa}\right] + \Gamma_- \sum_k \mathcal{D}_{\boldsymbol{\sigma}_-^{(k)}}(\rho_1^{rwa}),$$

where $\boldsymbol{H}_1^{rwa} = \Delta_c\,\mathbf{a}^\dagger \mathbf{a} + (v\boldsymbol{a} + v^*\boldsymbol{a}^\dagger) + \sum_k \left(\frac{\Delta_q^{(k)}}{2}\boldsymbol{\sigma}_z^{(k)} + ig\left(\boldsymbol{\sigma}_-^{(k)}\boldsymbol{a}^\dagger - \boldsymbol{\sigma}_+^{(k)}\boldsymbol{a}\right)\right)$ is the Jaynes-Cummings interaction Hamiltonian plus drive. We next apply a unitary coordinate change $\rho = \tilde{U}\rho_1^{rwa}\tilde{U}^\dagger$ on the resonator state, to center it around its well-known steady state under off-resonant drive, namely by a complex field amplitude displacement $\tilde{U} := \exp\left((v^*\boldsymbol{a}^\dagger - v\boldsymbol{a})/\Delta_c\right)$. This yields:

$$\frac{d}{dt}\rho = \sum_k \left\{\mathcal{L}_Q^{(k)}(\rho) + g\mathcal{L}_{int}^{(k)}(\rho)\right\} + (-i)\left[\Delta_c\,\boldsymbol{a}^\dagger\boldsymbol{a}, \rho\right], \tag{17}$$

where

$$\mathcal{L}_Q^{(k)}(\rho) := -i\left[\frac{\Delta_q^{(k)}}{2}\boldsymbol{\sigma}_z^{(k)} + \frac{g\,\tilde{v}}{\Delta_c}\boldsymbol{\sigma}_x^{(k)}, \rho\right] + \Gamma_-\mathcal{D}_{\boldsymbol{\sigma}_-^{(k)}}(\rho),$$

$$\mathcal{L}_{int}^{(k)}(\rho) := \left[\boldsymbol{\sigma}_-^{(k)}\boldsymbol{a}^\dagger - \boldsymbol{\sigma}_+^{(k)}\boldsymbol{a}, \rho\right].$$

The term with $v := i\tilde{v}$ for $\tilde{v} \in \mathbb{R}$ now expresses an effective, indirect drive on the TLSs.

System (17) is in the form (2) with $\mathcal{L}_A^{(k)} := \mathcal{L}_Q^{(k)}$, $\boldsymbol{A}^{(k)} := i\boldsymbol{\sigma}_-^{(k)}$, $\boldsymbol{B} := \boldsymbol{a}$, $\tilde{H}_B := \Delta_c\boldsymbol{a}^\dagger\boldsymbol{a}$. The hypothesis of Theorems 1 and 2 are satisfied since $\mathcal{L}_B = 0$ and Assumption 1 holds with $c_{B\dagger} = \Delta_c$. Let $\mathcal{H}_s$ be a Hilbert space whose dimension matches the dimension of the resonator space $\mathcal{H}_C$, and ρ_s the density operator on $\mathcal{H}_s$. By virtue of Theorem 1 and Corollary 1, the reduced model is given in Lindbladian form as follows:

$$\frac{d}{dt}\rho_s = -i\left(\Delta_c + g^2\sum_k\delta^{(k)}\right)\left[\boldsymbol{a}^\dagger\boldsymbol{a}, \rho_s\right] \tag{18}$$

$$+ g^2\left(\sum_k\Gamma_{\boldsymbol{a}}^{(k)}\right)\mathcal{D}_{\boldsymbol{a}}(\rho_s) + g^2\left(\sum_k\Gamma_{\boldsymbol{a}^\dagger}^{(k)}\right)\mathcal{D}_{\boldsymbol{a}^\dagger}(\rho_s),$$

$$\delta^{(k)} = \Im\left(z_1^{(k)} + z_2^{(k)}\right),$$

$$\Gamma_{\boldsymbol{a}^\dagger}^{(k)} = 2\Re\left(z_1^{(k)}\right), \quad \Gamma_{\boldsymbol{a}}^{(k)} = 2\Re\left(z_2^{(k)}\right),$$

$$z_1^{(k)} = \mathrm{Tr}\left(-i\boldsymbol{F}_1^{(k)}\bar{\rho}_Q^{(k)}\boldsymbol{\sigma}_+^{(k)}\right), \quad z_2^{(k)} = \mathrm{Tr}\left(i\boldsymbol{F}_2^{(k)}\bar{\rho}_Q^{(k)}\boldsymbol{\sigma}_-^{(k)}\right)$$

and where, for each k, matrices $\boldsymbol{F}_1^{(k)}$, $\boldsymbol{F}_2^{(k)}$ satisfy equations (11). The solution of such equations can be computed directly since, on a qubit space $\mathcal{H}_Q^{(k)}$, one can always parameterize operators in terms of Pauli matrices. We immediately find: $z_1^{(k)} = \frac{W_1^{(k)}}{Z^{(k)}}$ and $z_2^{(k)} = \frac{W_2^{(k)}}{Z^{(k)}}$ where

$$W_1^{(k)} = -4g^2v^2\left(8ig^2v\right)$$

$$(\Gamma_- + i\Delta_c)\Delta_c \left((\Gamma_- + 2i\Delta_c)^2 + 4\Delta_q^2\right)),$$

$$W_2^{(k)} = 32ig^4v^4 + 2i(\Gamma_- - i\Delta_c)\Delta_c^4(\Gamma_2 + 4\Delta_q^2)\cdot$$
$$\cdot (\Gamma_- - 2i(\Delta_c + \Delta_q)) - 4g^2v^2\Delta_c\big(\Gamma_-^3 - 5i\Gamma_-^2\Delta_c +$$
$$4i\Delta_c(\Delta_c^2 + 2\Delta_c\Delta_q - \Delta_q^2) + 4\Gamma_-(-\Delta_c^2 + \Delta_q^2)\big),$$

$$Z^{(k)} = \left(8g^2v^2 + \Delta_c^2(\Gamma_-^2 + 4\Delta_q^2)\right)\left(8g^2v^2(i\Gamma_- + 2\Delta_c) + \Delta_c^2(i\Gamma_- + \Delta_c)\left((\Gamma_- - 2i\Delta_c)^2 + 4\Delta_q^2\right)\right).$$

Coefficients $g^2 \sum_k \delta^{(k)}$, $g^2 \sum_k \Gamma_a^{(k)}$, and $g^2 \sum_k \Gamma_{a^\dagger}^{(k)}$ appearing in our reduced-order model (18) can be visualized for different values of pump detuning Δ_c and intraresonator photon number $\langle N \rangle = \tilde{v}^2/(\kappa^2 + \Delta_c^2)$. As depicted in Figure 1, we can compare the frequency shift of the resonator $g^2 \sum_k \delta^{(k)}$ with experimental findings from the pump probe experiment. We find that, by properly calibrating the values of g and by selecting a proper density function of the TLSs, we are able to match the resonance shift in the trasmission spectrum observed in the pump probe experiment. Quantitative agreement between data and our model should hence enable to extract characteristics about the TLS bath, pending other experimental features that will have to be calibrated.

V. CONCLUSIONS

We have studied adiabatic elimination for open quantum systems in Lindblad form composed by a target subsystem weakly interacting with K strongly dissipative subsystems. The key novel features of our approach are twofold: the decomposition of the environment into K separately treated subsystems, and the presence of fast Hamiltonian dynamics on the target system. The time-scale separation between the uncoupled dynamics and the interaction allows model reduction via center manifold techniques and geometric singular perturbation theory. We have provided formulas for the first- and second-order expansion and shown that the asymptotic expansion of the center manifold retains a physical interpretation: the reduced model still evolves according to Lindbladian dynamics and can be mapped to the original model via Kraus map. Each strongly dissipative subsystem contributes linearly to the reduced model at first-order, and does the same at second-order if a specific commutation property about the interaction terms holds. We have successfully applied our proposed theory to the model of a microwave superconducting resonator subject to dielectric losses where our reduced-order model shows a trasmission spectrum whose shape matches experimental data. Future work will address the necessary conditions to satisfy the invariance equation at orders higher than two. The assumption about the commutator between the original Hamiltonian dynamics of the target system and the interaction terms might also be removed, yielding a full generalization of the proposed theory.

APPENDIX

A. Proof of Theorem 1

Let $\bar{\rho}_A^{[k]}$ denote $\bigotimes_{k' \neq k} \bar{\rho}_A^{(k)}$. By plugging (26) and (27) into the first-order invariance condition (5) and by making use of Assumption 1, condition (5) reads as:

$$\sum_k \bar{\rho}_A^{[k]} \otimes \Big\{ \Big(-i\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right) - i\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}$$
$$- c_{B^\dagger}\, \boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\Big) \otimes \boldsymbol{B}^\dagger \rho_s + \tag{19a}$$
$$\Big(-i\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right) - i\boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)}$$
$$+ c_{B^\dagger}^*\, \boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\Big) \otimes \boldsymbol{B}\rho_s + \tag{19b}$$
$$+ herm.conj. \ + \ \Big(\mathcal{L}_B(\rho_s) - \mathcal{L}_{s,1}(\rho_s)\Big) \otimes \bar{\rho}_A = 0 \tag{19c}$$

Case of Theorem 1. It can be immediately seen from (19) that one can select $\mathcal{L}_{s,1} := \mathcal{L}_B$ as long as each round parenthesis (19a),(19b) is set to zero. Taking the trace on (11a) and (11b) yields:

$$c_{B^\dagger}\,\mathrm{Tr}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right) = c_{B^\dagger}\,\mathrm{Tr}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right)^* = -i\,\mathrm{Tr}\left(\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}\right), \tag{20}$$

which solves the situation with the announced formulas.

Case of Remark 1. By taking the trace on equations (12), we observe that, for each k:

$$\mathrm{Tr}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right) = \mathrm{Tr}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right)^* = 0. \tag{21}$$

Then, by taking the partial trace over $\mathcal{H}_A$ in (19), we immediately have $\mathcal{L}_{s,1}(\rho_s) = \mathcal{L}_B(\rho_s) - i[\boldsymbol{H}_{s,1}, \rho_s]$ with $\boldsymbol{H}_{s,1}$ as in Theorem 1. Now, plugging $\mathcal{L}_{s,1}$ in (19) yields:

$$\sum_k \bar{\rho}_A^{[k]} \otimes \Big\{ \Big(-i\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right)$$
$$-i\mathcal{S}^{(k)}\left(\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}\right) - c_{B^\dagger}\, \boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\Big) \otimes \boldsymbol{B}^\dagger \rho_s + \tag{22a}$$
$$\Big(-i\mathcal{L}_A^{(k)}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right)$$
$$-i\mathcal{S}^{(k)}\left(\boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)}\right) + c_{B^\dagger}^*\, \boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\Big) \otimes \boldsymbol{B}\rho_s + \tag{22b}$$
$$herm.conj. \Big\} = 0$$

In order to solve (22) it is enough to set each round parenthesis of (22a) and (22b) to zero for each k — see equations (12).

It is immediate to see from (8) that $\mathcal{K}_0(\rho_s) + \varepsilon\mathcal{K}_1(\rho_s)$ is a completely positive map, as long as one can neglect the terms of order ε^2. One concludes that it is also trace-preserving at order ϵ by checking that $\mathrm{Tr}\left(\mathcal{K}_1(\rho_s)\right) = 0$, thanks to (20) for the case of Theorem 1 and thanks to (21) for the case of Remark 1.

B. Proof of Theorem 2

Let $\bar{\rho}_A^{[k,k']}$ denote $\bigotimes_{k''\neq k,k'} \bar{\rho}_A^{(k'')}$. Let $\bar{\rho}_A^{(k,k')}$ denote $\bar{\rho}_A^{(k)} \otimes \bar{\rho}_A^{(k')}$. We use the following notation for operators: if the superscript of an operator respectively is (k) or (k,k'), then it respectively applies to $\mathcal{H}_A^{(k)}$ or $\mathcal{H}_A^{(k)} \otimes \mathcal{H}_A^{(k')}$ only, possibly leaving the remaining subsystems identical; superscripts (k,k) or $[k,k]$ are used interchangeably with superscripts (k) and $[k]$ respectively. Let $\mathcal{S}^{(k,k')}(Q)$ denote $Q - \mathrm{Tr}(Q)\bar{\rho}_A^{(k,k')}$. Let $\boldsymbol{B}_1 := \boldsymbol{B}^\dagger, \boldsymbol{B}_2 := \boldsymbol{B}, \boldsymbol{A}_1^{(k)} := \boldsymbol{A}^{(k)}, \boldsymbol{A}_2^{(k)} := \boldsymbol{A}^{(k)\,\dagger}$. For any k, k', let $\{\boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')}\}_{h,j\in\{1,2\}}$ be a family of four operators on $\mathcal{H}_A^{(k)} \otimes \mathcal{H}_A^{(k')}$ only, which we will define in the following. Define:

$$\boldsymbol{M} = \sum_k \sum_{j\in\{1,2\}} \boldsymbol{F}_j^{(k)} \otimes \boldsymbol{B}_j$$

$$\boldsymbol{N} := \sum_{k,k'} \sum_{h,j\in\{1,2\}} \boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')} \otimes \boldsymbol{B}_j^\dagger \boldsymbol{B}_h$$

$$\boldsymbol{W}_\mu^{(k)}(t) := [\boldsymbol{L}_{A,\mu}^{(k)}, \boldsymbol{F}_1^{(k)}] \otimes \boldsymbol{B}^\dagger + e^{2ic_{B^\dagger}t}[\boldsymbol{L}_{A,\mu}^{(k)}, \boldsymbol{F}_2^{(k)}] \otimes \boldsymbol{B}$$

$$\mathfrak{w}^{(k)}(t) := \boldsymbol{B}_1 + b^{(k)}e^{-2ic_{B^\dagger}t}\boldsymbol{B}_2$$

$$b^{(k)} := -\frac{1}{2c_{B^\dagger}} \mathrm{Tr}\left[\sum_\mu [\boldsymbol{L}_{A,\mu}, \boldsymbol{F}_2^{(k)}]\,\bar{\rho}_A^{(k)}\,[\boldsymbol{L}_{A,\mu}, \boldsymbol{F}_1^{(k)}]^\dagger\right]. \tag{23}$$

Let $b_1^{(k)} := b^{(k)\,\star}$ and $b_2^{(k)} := b^{(k)}$, Let $\mathcal{L}_A^{(k,k')}$ denote the operator $\mathcal{L}_A^{(k)} + \mathcal{L}_A^{(k')}$. Let $\delta_{hj} := h - j$. Let $f_1, f_2 \geq 0$ two constants which we will define in the following. Now define:

$$\mathfrak{g}(\rho_s) := \sum_k \int_0^{\pi/(2c_{B^\dagger})} \mathfrak{w}^{(k)}(t)\,\rho_s\,\mathfrak{w}^{(k)}(t)^\dagger\,dt$$

$$\mathfrak{f}(\rho_s) := \sum_{j\in\{1,2\}} f_j \boldsymbol{B}_j \rho_s \boldsymbol{B}_j^\dagger$$

$$\mathcal{G}(\rho_s) := \mathrm{Tr}_A\left[\int_0^{\pi/c_{B^\dagger}} \sum_{k,\mu} \boldsymbol{W}_\mu^{(k)}(t)(\bar{\rho}_A \otimes \rho_s)\boldsymbol{W}_\mu^{(k)}(t)^\dagger\,dt\right]$$

$$\mathcal{K}_2^Q(\rho_s) := \int_0^{+\infty} e^{\mathcal{L}_A(\cdot)t}\left(\mathcal{S}\left(\sum_{k,\mu} \boldsymbol{W}_\mu^{(k)}(t)(\bar{\rho}_A \otimes \rho_s)\boldsymbol{W}_\mu^{(k)}(t)^\dagger\right)\right)\,dt$$

$$+ \frac{c_{B^\dagger}\bar{\tau}}{\pi}\bar{\rho}_A \otimes \mathcal{G}(\rho_s) + c_{B^\dagger}\bar{\rho}_A \otimes \mathfrak{g}(\rho_s) + \bar{\rho}_A \otimes \mathfrak{f}(\rho_s)$$

$$\mathcal{K}(\rho_s) := \left(\boldsymbol{I} - i\varepsilon\boldsymbol{M} + \varepsilon^2\boldsymbol{N}\right)(\bar{\rho}_A \otimes \rho_s)\left(\boldsymbol{I} + i\varepsilon\boldsymbol{M}^\dagger + \varepsilon^2\boldsymbol{N}^\dagger\right)$$

$$+ \varepsilon^2\mathcal{K}_2^Q(\rho_s). \tag{24}$$

By collecting powers of ε in (24) and carrying out straightforward computations, we obtain the formulation $\mathcal{K}(\rho_s) = (\mathcal{K}_0 + \varepsilon\mathcal{K}_1 + \varepsilon^2\mathcal{K}_2)(\rho_s)$ where:

$$\mathcal{K}_0(\rho_s) = \bar{\rho}_A \otimes \rho_s,, \tag{25}$$

$$\mathcal{K}_1(\rho_s) = -i\sum_{k'}\sum_{j\in\{1,2\}} \bar{\rho}_A^{[k']} \otimes \boldsymbol{F}_j^{(k')}\bar{\rho}_A^{(k')} \otimes \boldsymbol{B}_j\rho_s\ + \ herm.\ conj., \tag{26}$$

$$\mathcal{K}_2(\rho_s) = \mathcal{K}_2^L(\rho_s) + \mathcal{K}_2^E(\rho_s) + \mathcal{K}_2^Q(\rho_s), \tag{27}$$

where:

$$\mathcal{K}_2^L(\rho_s) = \sum_{k,k'}\sum_{h,j\in\{1,2\}} \left\{\bar{\rho}_A^{[k,k']} \otimes \boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')}\bar{\rho}_A^{(k,k')} \otimes \boldsymbol{B}_j^\dagger \boldsymbol{B}_h\rho_s\right\} + herm.\ conj.,$$

$$\mathcal{K}_2^E(\rho_s) = \sum_{k,k'}\sum_{h,j\in\{1,2\}} \left\{\bar{\rho}_A^{[k,k']} \otimes \boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k,k')}\boldsymbol{F}_j^{(k')\,\dagger} \otimes \boldsymbol{B}_h\rho_s\boldsymbol{B}_j^\dagger\right\}$$

$$\mathcal{K}_2^Q(\rho_s) = \sum_k\sum_{h,j\in\{1,2\}} \left\{\bar{\rho}_A^{[k,k']} \otimes \bar{\mathcal{K}}_{hj}^{(k)}(\rho_s) \otimes \boldsymbol{B}_h\rho_s\boldsymbol{B}_j^\dagger\right\} \text{ }$$

$$+ \sum_{j\in\{1,2\}} \left(\bar{\tau}\,\mathrm{Tr}\left[\bar{\mathcal{F}}_{jj}(\bar{\rho}_A)^{(k)}\right]\bar{\rho}_A \otimes \boldsymbol{B}_j\rho_s\boldsymbol{B}_j^\dagger\right. \text{ }$$

$$+ \sum_k \bar{\rho}_A \otimes \left\{\frac{\pi}{2}\left(\boldsymbol{B}^\dagger\rho_s\boldsymbol{B} + |b_2^{(k)}|^2\boldsymbol{B}\rho_s\boldsymbol{B}^\dagger\right) - \right.$$

$$-\sum_{h\neq j} i\delta_{hj} b_h^{(k)} \boldsymbol{B}_h \rho_s \boldsymbol{B}_j^\dagger \Big\}+$$

$$+ \bar{\rho}_A \otimes \Big(f_1 \boldsymbol{B}^\dagger \rho_s \boldsymbol{B} + f_2 \boldsymbol{B}\rho_s \boldsymbol{B}^\dagger \Big), \tag{28}$$

and where:

$$\bar{\mathcal{K}}_{hj}^{(k)}(\rho_s) := \int_0^{+\infty} e^{t\, \mathcal{L}_A^{(k)}(\cdot)} \Big(\mathcal{S}^{(k)} \big(\bar{\mathcal{F}}_{hj}(t, \bar{\rho}_A)^{(k)} \big) \Big) \; dt$$

$$\bar{\mathcal{F}}_{hj}(t, \bar{\rho}_A)^{(k)} := \exp\big(2ic_{B^\dagger}\delta_{hj}\, t\big) \sum_\mu \Big[\boldsymbol{L}_{A,\mu}^{(k)}, \boldsymbol{F}_h^{(k)} \Big] \bar{\rho}_A^{(k)} \Big[\boldsymbol{L}_{A,\mu}^{(k')}, \boldsymbol{F}_j^{(k')} \Big]$$

$$\bar{\mathcal{F}}_{hj}(\bar{\rho}_A)^{(k)} := \bar{\mathcal{F}}_{hj}(t, \bar{\rho}_A)^{(k)}$$

$$f_1 := \sum_{k,k'} \mathrm{Tr} \Big[\boldsymbol{F}_2^{(k)} \bar{\rho}_A^{(k,k')} \boldsymbol{F}_2^{(k')} + \check{\delta}_{kk'} \Big(\bar{\mathcal{F}}_{22}^{(k)} + \frac{\pi}{2} |b^{(k)}|^2 \Big) \Big]$$

$$f_2 := \sum_{k,k'} \mathrm{Tr} \Big[\boldsymbol{F}_1^{(k)} \bar{\rho}_A^{(k,k')} \boldsymbol{F}_1^{(k')} + \check{\delta}_{kk'} \Big(\bar{\mathcal{F}}_{11}^{(k)} + \frac{\pi}{2} \Big) \Big]. \tag{29}$$

where $\check{\delta}_{kk'}$ denotes the Kronecker delta. For any k, k', let the family of operators $\{\boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')}\}_{h,j\in\{1,2\}}$ satisfy the following set of equations:

$$\mathcal{L}_A^{(k,k')} \Big(\boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')} \bar{\rho}_A^{(k,k')} \Big) + \mathcal{S}^{(k,k')} \Big(2ic_{B^\dagger}\, \delta_{hj}\, \boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')} \bar{\rho}_A^{(k,k')} -$$

$$- \boldsymbol{A}_j^{(k)\,\dagger} \boldsymbol{F}_h^{(k')} \bar{\rho}_A^{(k,k')} \Big) = 0, \quad \forall h,j \in \{1,2\}, \tag{30a}$$

$$\mathrm{Tr} \Big[2ic_{B^\dagger}\, \delta_{hj}\, \boldsymbol{U}_{\boldsymbol{B}_j^\dagger \boldsymbol{B}_h}^{(k,k')} \bar{\rho}_A^{(k,k')} \Big] =$$

$$= \mathrm{Tr} \Big[\boldsymbol{A}_j^{(k)\,\dagger} \boldsymbol{F}_h^{(k')} \bar{\rho}_A^{(k,k')} \Big], \quad \forall h \neq j \in \{1,2\}, \tag{30b}$$

$$\mathrm{Tr} \Big[\boldsymbol{U}_{\boldsymbol{B}\boldsymbol{B}^\dagger}^{(k,k')} \bar{\rho}_A^{(k,k')} \Big] = -\mathrm{Tr} \Big[\boldsymbol{F}_1^{(k)} \bar{\rho}_A^{(k,k')} \boldsymbol{F}_1^{(k')\,\dagger} + \check{\delta}_{kk'} \Big(\bar{\tau}\bar{\mathcal{F}}_{11}(\bar{\rho}_A)^{(k)} + \frac{\pi}{2} \Big) \Big], \tag{30c}$$

$$\mathrm{Tr} \Big[\boldsymbol{U}_{\boldsymbol{B}^\dagger \boldsymbol{B}}^{(k,k')} \bar{\rho}_A^{(k,k')} \Big] = -\mathrm{Tr} \Big[\boldsymbol{F}_2^{(k)} \bar{\rho}_A^{(k,k')} \boldsymbol{F}_2^{(k')\,\dagger} + \check{\delta}_{kk'} \Big(\bar{\tau}\bar{\mathcal{F}}_{22}(\bar{\rho}_A)^{(k)} + \frac{\pi}{2} |b^{(k)}|^2 \Big) \Big]. \tag{30d}$$

Equations (30) are always solvable, as proved in [3, Lemma 4]. We will then show that our definition of $\mathcal{K}_2$ in (23)-(30) indeed satisfies the second-order invariance condition (6). We start by observing that, thanks to assumptions $\mathcal{L}_B(\cdot) = 0$ and $\mathcal{L}_{s,1} = 0$, condition (6) reads as:

$$\mathcal{L}_A \big(\mathcal{K}_2(\rho_s) \big) + \mathcal{L}_{int} \big(\mathcal{K}_1(\rho_s) \big) - i \Big([\tilde{\boldsymbol{H}}_B, \mathcal{K}_2(\rho_s)] -$$

$$- \mathcal{K}_2([\tilde{\boldsymbol{H}}_B, \rho_s]) \Big) = \bar{\rho}_A \otimes \mathcal{L}_{s,2}(\rho_s). \tag{31}$$

Then, in order to compute the left-hand side of (31), we observe that the computation of term $-i[\tilde{\boldsymbol{H}}_B, \mathcal{K}_2(\rho_s)] - \mathcal{K}_2(\mathcal{L}_{s,0}(\rho_s))$ is simplified by the following set of properties directly implied by Assumption 1:

$$\Big[\tilde{\boldsymbol{H}}_B, \boldsymbol{B} \Big] = -c_{B^\dagger} \boldsymbol{B},$$

$$\Big[\tilde{\boldsymbol{H}}_B, \boldsymbol{B}_h \boldsymbol{B}_j^\dagger \Big] = -2\delta_{hj}\, c_{B^\dagger} \boldsymbol{B}_h \boldsymbol{B}_j^\dagger \quad \forall h,j \in \{1,2\},$$

$$- i \Big(\Big[\tilde{\boldsymbol{H}}_B, \; \boldsymbol{B}_h \rho_s \boldsymbol{B}_j^\dagger \Big] - \boldsymbol{B}_h \Big[\tilde{\boldsymbol{H}}_B, \; \rho_s \Big] \boldsymbol{B}_j^\dagger \Big) =$$

$$= 2ic_{B^\dagger}\, \delta_{hj}\, \boldsymbol{B}_h \rho_s \boldsymbol{B}_j^\dagger \quad \forall h,j \in \{1,2\}, \tag{32}$$

whereas the computation of term $\mathcal{L}_A(\mathcal{K}_2(\rho_s))$ is simplified by the following Claim.

Claim 1: $\mathcal{L}_A^{(k)} \Big(\bar{\mathcal{K}}_{hj}^{(k)}(\rho_s) \Big) + 2ic_{B^\dagger}\delta_{hj}\bar{\mathcal{K}}_{hj}^{(k)}(\rho_s) + \mathcal{S}^{(k)} \Big(\bar{\mathcal{F}}_{hj}(\bar{\rho}_A)^{(k)} \Big) = 0$.

Proof: Case $h = j$ is proved along the lines of [3, Lemma 1 and Lemma 4]. Case $(h,j) = (2,1)$ is the hermitian conjugate of case $(h,j) = (1,2)$ which we are now going to prove. Let $\mathcal{L}^\sharp(\cdot)$ denote the super-operator $\mathcal{L}_A^{(k)}(\cdot) - 2ic_{B^\dagger}Id(\cdot)$. Then, since $\mathcal{L}_A^{(k)}$ is strongly dissipative on $\mathcal{H}_A^{(k)}$, we have that:

$$\lim_{t\to+\infty} \exp(t\mathcal{L}^\sharp(\cdot))(\boldsymbol{X}) = \lim_{t\to+\infty} \exp(-i2c_{B^\dagger}t) \exp(t\,\mathcal{L}_A^{(k)}(\cdot))(\boldsymbol{X}) = 0, \tag{33}$$

for any operator $\boldsymbol{X}$ such that $\mathrm{Tr}_{\mathcal{H}_A^{(k)}}[\boldsymbol{X}] = 0$. First, we formulate $\bar{\mathcal{K}}_{12}^{(k)}$ as:

$$\bar{\mathcal{K}}_{12}^{(k)}(\rho_s) = \int_0^{+\infty} \mathfrak{K}_{12}(t, \bar{\rho}_A)^{(k)} \; dt$$

$$\mathfrak{K}_{12}(t, \bar{\rho}_A)^{(k)} := \exp\Big(t\mathcal{L}^\sharp(\cdot) \Big) \Big(\mathcal{S}^{(k)} \big(\bar{\mathcal{F}}_{12}(\bar{\rho}_A)^{(k)} \big) \Big)$$

Second, we observe that:

$$\mathcal{L}^\sharp \Big(\mathfrak{K}_{12}(t, \bar{\rho}_A)^{(k)} \Big) = \frac{d}{dt} \mathfrak{K}_{12}(t, \bar{\rho}_A)^{(k)}. \tag{34}$$

We conclude from (33) and (34) that:

$$\mathcal{L}^{\sharp}\left(\bar{\mathcal{K}}_{12}^{(k)}(\rho_s)\right) = \left[\mathfrak{K}_{12}(t, \bar{\rho}_A)^{(k)}\right]_0^{+\infty} = -\mathcal{S}^{(k)}\left(\bar{\mathcal{F}}_{12}(\bar{\rho}_A)^{(k)}\right).$$

∎

Furthermore, by considering $\mathcal{L}_{int}(\rho) = -i\sum_k \sum_{h \in \{1,2\}} [\boldsymbol{A}_h^{(k)\,\dagger} \otimes \boldsymbol{B}_h^{\dagger},\ \rho]$, we have that:

$$\mathcal{L}_{int}(\mathcal{K}_1(\rho_s)) = \sum_{k,k'} \bar{\rho}_A^{[k,k']} \otimes \sum_{j,h\in\{1,2\}} \boldsymbol{B}_j^{\dagger}\boldsymbol{B}_h\rho_s \otimes \left(-\boldsymbol{A}_j^{(k)\,\dagger}\boldsymbol{F}_h^{(k')}\bar{\rho}_A^{(k,k')}\right) + herm.\ conj.+$$

$$+ \boldsymbol{B}_h\rho_s\boldsymbol{B}_j^{\dagger} \otimes \left(\boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k,k')}\boldsymbol{A}_j^{(k')\,\dagger} + \boldsymbol{A}_h^{(k)}\bar{\rho}_A^{(k,k')}\boldsymbol{F}_j^{(k')\,\dagger}\right),$$

$$= \sum_{k,k'} \bar{\rho}_A^{[k,k']} \otimes \sum_{j,h\in\{1,2\}} \boldsymbol{B}_j^{\dagger}\boldsymbol{B}_h\rho_s \otimes \left(-\boldsymbol{A}_j^{(k)\,\dagger}\boldsymbol{F}_h^{(k')}\bar{\rho}_A^{(k,k')}\right) + herm.\ conj.+$$

$$+ \boldsymbol{B}_h\rho_s\boldsymbol{B}_j^{\dagger} \otimes \Big(-2ic_{B^{\dagger}}\,\delta_{hj}\,\boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k,k')}\boldsymbol{F}_j^{(k')\,\dagger}-$$

$$- \mathcal{L}_A^{(k,k')}\left(\boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k,k')}\boldsymbol{F}_j^{(k')\,\dagger}\right) +$$

$$+ \check{\delta}_{k,k'}\,\bar{\mathcal{F}}_{hj}(\bar{\rho}_A)^{(k)}\Big), \tag{35}$$

where, in the last equality, we first made use of formulas (11) and then we used the following formula generalized from [3, Lemma 6]:

$$\boldsymbol{F}_h^{(k)}\mathcal{L}_A^{(k,k')}\left(\bar{\rho}_A^{(k,k')}\boldsymbol{F}_j^{(k')\,\dagger}\right) + \mathcal{L}_A^{(k,k')}\left(\boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k,k')}\right)\boldsymbol{F}_j^{(k')\,\dagger} =$$

$$= \mathcal{L}_A^{(k,k')}\left(\boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k,k')}\boldsymbol{F}_j^{(k')\,\dagger}\right) - \check{\delta}_{k,k'}\,\bar{\mathcal{F}}_{hj}(\bar{\rho}_A)^{(k)}, \quad \forall h,j,\ \forall k,k'. \tag{36}$$

Finally, by making use of definitions (24)-(28), properties (32), simplification (35), and Claim 1, we can compute:

$$\mathcal{L}_A(\mathcal{K}_2(\rho_s)) - i\left([\tilde{\boldsymbol{H}}_B, \mathcal{K}_2(\rho_s)] - \mathcal{K}_2([\tilde{\boldsymbol{H}}_B, \rho_s])\right) + \mathcal{L}_{int}(\mathcal{K}_1(\rho_s)) =$$

$$= \sum_{k,k'} \bar{\rho}_A^{[k,k']} \otimes \sum_{h,j\in\{1,2\}} \Big\{\boldsymbol{B}_j^{\dagger}\boldsymbol{B}_h\rho_s \otimes \left(\mathcal{L}_A^{(k,k')}(\boldsymbol{U}_{\boldsymbol{B}_j^{\dagger}\boldsymbol{B}_h}^{(k,k')}\bar{\rho}_A^{(k,k')})+\right.$$

$$+ 2ic_{B^{\dagger}}\,\delta_{hj}\,\boldsymbol{U}_{\boldsymbol{B}_j^{\dagger}\boldsymbol{B}_h}^{(k,k')}\bar{\rho}_A^{(k,k')} - \boldsymbol{A}_j^{(k)\,\dagger}\boldsymbol{F}_h^{(k')}\bar{\rho}_A^{(k,k')}\Big) + herm.\ conj.\Big\}+$$

$$+ \sum_k \bar{\rho}_A^{[k]} \otimes \sum_{h,j\in\{1,2\}} \Big\{\boldsymbol{B}_h\rho_s\boldsymbol{B}_j^{\dagger} \otimes \Big(2c_{B^{\dagger}}\,|\delta_{hj}|b_h^{(k)}\bar{\rho}_A^{(k)}+$$

$$+ \mathrm{Tr}\left[\bar{\mathcal{F}}_{hj}(\bar{\rho}_A)^{(k)}\right]\bar{\rho}_A^{(k)}\Big)\Big\} =: \mathcal{E}(\rho_s). \tag{37}$$

Observe that the definition of $b_h^{(k)}$ in (23) implies that $2c_{B^{\dagger}}\,|\delta_{hj}|b_h^{(k)} + \mathrm{Tr}[\bar{\mathcal{F}}_{hj}(\bar{\rho}_A)^{(k)}] = 0$ whenever $h \neq j$. Furthermore, in the last equality of (35), we observed that $\mathrm{Tr}[\boldsymbol{F}_h^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{A}_h^{(k)\,\dagger} + \boldsymbol{A}_h^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{F}_h^{(k)\,\dagger}] = \mathrm{Tr}[\bar{\mathcal{F}}_{hh}(\bar{\rho}_A)^{(k)}]$ for any $h \in \{1,2\}$. The latter two observations and trace condition (30b) will be instrumental in the derivation of $\mathcal{L}_{s,2}$ in (38). Indeed, by recalling (31), Lindblad $\mathcal{L}_{s,2}$ can be obtained by taking the partial trace over $\mathcal{H}_A$ of $\mathcal{E}(\rho_s)$ in expression (37), as follows:

$$\mathcal{L}_{s,2}(\rho_s) = \sum_k \Big\{\boldsymbol{B}\boldsymbol{B}^{\dagger}\rho_s\ \mathrm{Tr}\left(-\boldsymbol{A}^{(k)\,\dagger}\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\right) + herm.conj.+$$

$$\boldsymbol{B}^{\dagger}\boldsymbol{B}\rho_s\ \mathrm{Tr}\left(-\boldsymbol{A}^{(k)}\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\right) + herm.conj.+$$

$$\boldsymbol{B}^{\dagger}\rho_s\boldsymbol{B}\ \mathrm{Tr}\left(\boldsymbol{F}_1^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{A}^{(k)\,\dagger} + \boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{F}_1^{(k)\,\dagger}\right) +$$

$$\boldsymbol{B}\rho_s\boldsymbol{B}^{\dagger}\ \mathrm{Tr}\left(\boldsymbol{F}_2^{(k)}\bar{\rho}_A^{(k)}\boldsymbol{A}^{(k)} + \boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)}\boldsymbol{F}_2^{(k)\,\dagger}\right)\Big\}+$$

$$\sum_{k\neq k'} \Big\{\boldsymbol{B}\boldsymbol{B}^{\dagger}\rho_s\ \mathrm{Tr}\left(-\boldsymbol{A}^{(k)\,\dagger}\bar{\rho}_A^{(k)}\right)\mathrm{Tr}\left(\boldsymbol{F}_1^{(k')}\bar{\rho}_A^{(k')}\right) + h.c.+$$

$$\boldsymbol{B}^{\dagger}\boldsymbol{B}\rho_s\ \mathrm{Tr}\left(-\boldsymbol{A}^{(k)}\bar{\rho}_A^{(k)}\right)\mathrm{Tr}\left(\boldsymbol{F}_2^{(k')}\bar{\rho}_A^{(k')}\right) + h.c.. \tag{38}$$

If we now apply definitions (14) and property (20), the expression of $\mathcal{L}_{s,2}$ in (38) simplifies to:

$$\mathcal{L}_{s,2}(\rho_s) = \sum_k \Big\{-z_1^{(k)}\,\boldsymbol{B}\boldsymbol{B}^{\dagger}\rho_s - z_2^{(k)}\,\boldsymbol{B}^{\dagger}\boldsymbol{B}\rho_s\ + herm.\ conj.+$$

$$+ \left(z_1^{(k)} + (z_1^{(k)\,*})\right)\,\boldsymbol{B}^{\dagger}\rho_s\boldsymbol{B}+$$

$$+ \left(z_2^{(k)} + (z_2^{(k)\,*})\right)\,\boldsymbol{B}\rho_s\boldsymbol{B}^{\dagger}\Big\}+$$

$$+ \sum_{k\neq k'} \frac{i}{c_{B^{\dagger}}}\Big\{\left(\boldsymbol{B}\boldsymbol{B}^{\dagger}\rho_s + \rho_s\boldsymbol{B}^{\dagger}\boldsymbol{B}\right)\,\left(z_0^{(k)\,*}z_0^{(k')}\right) - herm.conj.$$

$$+ \left(\boldsymbol{B}^{\dagger}\boldsymbol{B}\rho_s + \rho_s\boldsymbol{B}\boldsymbol{B}^{\dagger}\right)\,\left(-z_0^{(k)}z_0^{(k')\,*}\right)\Big\}$$

$$= \sum_k \Big\{-i\Im\left(z_1^{(k)}\right)\left[\boldsymbol{B}\boldsymbol{B}^{\dagger}, \rho_s\right] - i\Im\left(z_2^{(k)}\right)\left[\boldsymbol{B}^{\dagger}\boldsymbol{B}, \rho_s\right]+$$

$$+ 2\Re\left(z_1^{(k)}\right)\mathcal{D}_{\boldsymbol{B}^\dagger}(\rho_s) + 2\Re\left(z_2^{(k)}\right)\mathcal{D}_{\boldsymbol{B}}(\rho_s)\Big\} -$$
$$-i\sum_{k>k'}\left\{\delta^{(k,k')}\left(\boldsymbol{BB}^\dagger\rho_s + \rho_s\boldsymbol{B}^\dagger\boldsymbol{B}\right) - herm.conj.\right\},$$

which immediately reads as (13). Now, by first subtracting $\bar{\rho}_A \otimes \mathcal{L}_{s,2}(\rho_s)$ from $\mathcal{E}(\rho_s)$ in expression (37) and then using (30a), we conclude that:

$$\mathcal{E}(\rho_s) - \bar{\rho}_A \otimes \mathrm{Tr}\left[\mathcal{E}(\rho_s)\right] =$$
$$= \sum_{k,k'}\bar{\rho}_A^{[k,k']}\otimes\sum_{h,j\in\{1,2\}}\Big\{\boldsymbol{B}_j^\dagger\boldsymbol{B}_h\rho_s\otimes\left(\mathcal{L}_A^{(k,k')}\big(\boldsymbol{U}_{\boldsymbol{B}_j^\dagger\boldsymbol{B}_h}^{(k,k')}\bar{\rho}_A^{(k,k')}\big)+\right.$$
$$\left.+\mathcal{S}^{(k,k')}\big(2ic_{\boldsymbol{B}^\dagger}\,\delta_{hj}\,\boldsymbol{U}_{\boldsymbol{B}_j^\dagger\boldsymbol{B}_h}^{(k,k')}\bar{\rho}_A^{(k,k')} - \boldsymbol{A}_j^{(k)\,\dagger}\boldsymbol{F}_h^{(k')}\bar{\rho}_A^{(k,k')}\big)\right) + herm.\ conj.\Big\} = 0,$$

which immediately shows that (31) is satisfied as an identify.

We are now going to prove that $\mathcal{K}(\rho_s)$ in our definition (24) is indeed a CPTP mapping. Since terms

$$(\boldsymbol{I} - i\varepsilon\boldsymbol{M} + \varepsilon^2\boldsymbol{N})(\bar{\rho}_A\otimes\rho_s)(\boldsymbol{I} - i\varepsilon\boldsymbol{M} + \varepsilon^2\boldsymbol{N})^\dagger + \varepsilon^2\left(c_{\boldsymbol{B}^\dagger}\bar{\rho}_A\otimes\mathfrak{g}(\rho_s) + \bar{\rho}_A\otimes mathfrakf(\rho_s)\right)$$

already retain the Kraus map form, what remains to prove is complete positivity of $\mathcal{K}_2^Q(\rho_s)$.

Claim 2: There exists $\bar{\tau} > 0$ such that $\mathcal{K}_2^Q(\rho_s)$ is a completely positive mapping.

Proof: Consider an Hilbert basis $\{|n\rangle\}_{1\le n\le d}$ of $\mathcal{H}_A\otimes\mathcal{H}_B$. Let $\tilde{\mathcal{H}}$ be any Hilbert space of any dimension. Let:

$$\bar{\mathcal{K}}_{\bar{\tau}}(\boldsymbol{X}) := \int_0^{+\infty}e^{\mathcal{L}_A(\cdot)t}\left(\mathcal{S}\big(\boldsymbol{W}_\mu^{(k)}(t)\,\boldsymbol{X}\,\boldsymbol{W}_\mu^{(k)\,\dagger}(t)\big)\right)dt+$$
$$+\bar{\tau}\frac{c_{\boldsymbol{B}^\dagger}}{\pi}\bar{\rho}_A\otimes\mathrm{Tr}_A\left[\int_0^{\pi/c_{\boldsymbol{B}^\dagger}}\boldsymbol{W}_\mu^{(k)}(t)\,\boldsymbol{X}\,\boldsymbol{W}_\mu^{(k)\,\dagger}(t)\,dt\right]\tag{40}$$

For each n and ν, select any $|\phi_n\rangle, |\psi_\nu\rangle \in \tilde{\mathcal{H}}$ and define:

$$|\Phi\rangle := \sum_{n=1}^d|n\rangle\otimes|\phi_n\rangle, \quad |\Psi\rangle := \sum_{\nu=1}^d|\nu\rangle\otimes|\psi_\nu\rangle,\tag{41}$$

We are then going to prove that there exists $\bar{\tau} > 0$ such that $\langle\Psi|\bar{\mathcal{K}}(|\Phi\rangle\langle\Phi|)|\Psi\rangle \ge 0$. Standard computation give:

$$\langle\Psi|\bar{\mathcal{K}}(|\Phi\rangle\langle\Phi|)|\Psi\rangle \ge 0 = \sum_{n',\nu',n,\nu}z_{n',\nu'}^\star\,M_{n',\nu',n,\nu}\,z_{n,\nu},$$

where $z_{n,\mu} := \langle\phi_n|\psi_\nu\rangle$ and

$$M_{n',\nu',n,\nu} := \int_0^{+\infty}\mathfrak{m}_{n',\nu',n,\nu}(t) - \mathfrak{r}_{n',\nu',n,\nu}(t)\,dt + \bar{\tau}\frac{c_{\boldsymbol{B}^\dagger}}{\pi}R_{n',\nu',n,\nu}$$
$$R_{n',\nu',n,\nu} := \int_0^{\pi/c_{\boldsymbol{B}^\dagger}}\mathfrak{r}_{n',\nu',n,\nu}(t)\,dt$$
$$\mathfrak{m}_{n',\nu',n,\nu}(t) := \langle\nu'|\,e^{t\mathcal{L}_A(\cdot)}\left(\boldsymbol{W}_\mu^{(k)}(t)\,|n'\rangle\langle n|\,\boldsymbol{W}_\mu^{(k)\,\dagger}(t)\right)|\nu\rangle$$
$$\mathfrak{r}_{n',\nu',n,\nu}(t) := \langle\nu'|\left(\bar{\rho}_A\otimes\mathrm{Tr}_A\left[\boldsymbol{W}_\mu^{(k)}(t)\,|n'\rangle\langle n|\,\boldsymbol{W}_\mu^{(k)\,\dagger}(t)\right]\right)|\nu\rangle.\tag{42}$$

Since $\bar{\rho}_A\otimes\mathrm{Tr}_A\left[\boldsymbol{W}_\mu^{(k)}(t)\boldsymbol{X}\boldsymbol{W}_\mu^{(k)\,\dagger}(t)\right]$ is a completely-positive superoperator on $\boldsymbol{X}$, the $d^2\times d^2$ Hermitian matrix $\mathfrak{r}_{n',\nu',n,\nu}(t)$ is non-negative, and therefore, for any vector $z\in\mathbb{C}^{d^2}$:

$$z^\dagger R z = 0\tag{43}$$
$$\implies z^\dagger\cdot\mathfrak{r}(t)\cdot z = 0\ \forall t\in\left[0,\frac{\pi}{c_{\boldsymbol{B}^\dagger}}\right].\tag{44}$$

Now take a d^2 vector z such that (43) is satisfied. We then have:

$$\langle\Psi|\bar{\mathcal{K}}(|\Phi\rangle\langle\Phi|)|\Psi\rangle \ge 0 = \int_0^{+\infty}\sum_{n',\nu',n,\nu}z_{n',\nu'}^\star\,\mathfrak{m}_{n',\nu',n,\nu}(t)\,z_{n',\nu'}\ \ dt,$$

Since the propagator $e^{t\mathcal{L}_A(\cdot)}$ is a completely-positive mapping of the form $e^{t\mathcal{L}_A(\cdot)}(\boldsymbol{X}) = \sum_\theta\boldsymbol{H}_\theta(t)\boldsymbol{X}\boldsymbol{H}(t)_\theta^\dagger$ for some operators $\boldsymbol{H}_\theta(t)$, we then have [3, Lemma 1]:

$$\int_0^{+\infty}\sum_{n',\nu',n,\nu}z_{n',\nu'}^\star\,\mathfrak{m}_{n',\nu',n,\nu}(t)\,z_{n',\nu'}dt =$$
$$\int_0^{+\infty}\sum_{n',\nu',n,\nu}z_{n',\nu'}^\star\,\langle\nu'|\,\boldsymbol{H}_\theta(t)\boldsymbol{W}_\mu^{(k)}(t)\,|n'\rangle\langle n|\,\boldsymbol{W}_\mu^{(k)\,\dagger}(t)\boldsymbol{H}_\theta^\dagger(t)\,|\nu\rangle\,z_{n,\nu}dt =$$

$$= \int_0^{+\infty} \left| \sum_{n',\nu',n,\nu} \langle n| \boldsymbol{W}_\mu^{(k)\,\dagger}(t) \boldsymbol{H}_\theta^\dagger(t) |\nu\rangle \; z_{n,\nu} \right|^2 dt \tag{45}$$

$$\geq 0,$$

and we thus conclude that $z^\dagger R z = 0$ implies $z^\dagger M z \geq 0$. Assume now that $z^\dagger R z = 0$ and $z^\dagger M z = 0$. Inequality (45) then implies that for any $t \geq 0$

$$\sum_{n',\nu',n,\nu} \langle n| \boldsymbol{W}_\mu^{(k)\,\dagger}(t) \boldsymbol{H}_\theta^\dagger(t) |\nu\rangle \; z_{n,\nu} = 0$$

and therefore $M z = 0$. Then, by virtue of [3, Lemma 2], we conclude that there exists $\bar\tau > 0$ such that $\bar{\mathcal{K}}_{\bar\tau}$ is completely positive. $\blacksquare$

We are now going to prove trace-preservation of $\mathcal{K}(\rho_s)$. Since $\mathrm{Tr}[\mathcal{K}_1(\rho_s)] = 1$ and $\mathrm{Tr}[\mathcal{K}_1(\rho_s)] = 0$ for all ρ_s, what remains to prove is $\mathrm{Tr}[\mathcal{K}_2(\rho_s)] = 0$ for all ρ_s. First, by a subsequent application of formulas (30b), (36), and (11), we have:

$$\sum_{k,k'} \mathrm{Tr}\left[\boldsymbol{U}_{BB}^{(k,k')} \bar\rho_A^{(k,k')} + \bar\rho_A^{(k,k')} \boldsymbol{U}_{B^\dagger B^\dagger}^{(k,k')\,\dagger} \right] =$$

$$= \sum_{k,k'} \mathrm{Tr}\left[\boldsymbol{U}_{BB}^{(k',k)} \bar\rho_A^{(k,k')} + \bar\rho_A^{(k,k')} \boldsymbol{U}_{B^\dagger B^\dagger}^{(k,k')\,\dagger} \right] =$$

$$= \sum_{k,k'} \frac{1}{2ic_{B^\dagger}} \mathrm{Tr}\left[\boldsymbol{F}_2^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{A}^{(k')\,\dagger} + \boldsymbol{A}^{(k)\,\dagger} \bar\rho_A^{(k,k')} \boldsymbol{F}_1^{(k')\,\dagger} \right] =$$

$$= \sum_{k,k'} \mathrm{Tr}\left[-\boldsymbol{F}_2^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{F}_1^{(k')} \right] + \frac{1}{2ic_{B^\dagger}} \mathrm{Tr}\left[\check\delta_{kk'} \bar{\mathcal{F}}_{21}(\bar\rho_A)^{(k)} \right] =$$

$$= \sum_{k,k'} \mathrm{Tr}\left[-\boldsymbol{F}_2^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{F}_1^{(k')} \right] + i\,\check\delta_{kk'}\, b^{(k)}. \tag{46}$$

Secondly, it is straightforward to prove from (30c) and (30d) that $\sum_{k,k'} \mathrm{Tr}[\boldsymbol{U}_{BB^\dagger}^{(k,k')} \bar\rho_A^{(k,k')}]$ and $\sum_{k,k'} \mathrm{Tr}[\boldsymbol{U}_{B^\dagger B}^{(k,k')} \bar\rho_A^{(k,k')}]$ are real. Indeed:

$$\sum_{k,k'} \mathrm{Tr}\left[\boldsymbol{U}_{BB^\dagger}^{(k,k')} \bar\rho_A^{(k,k')} \right] = \sum_k \mathrm{Tr}\left[-\boldsymbol{F}_1^{(k)} \bar\rho_A^{(k)} \boldsymbol{F}_1^{(k)\,\dagger} + \bar\tau \bar{\mathcal{F}}_{11}(\bar\rho_A)^{(k)} + \frac{\pi}{2} \right] +$$

$$+ \sum_{k>k'} \mathrm{Tr}\left[-\boldsymbol{F}_1^{(k)} \bar\rho_A^{(k)} \boldsymbol{F}_1^{(k')\,\dagger} - \boldsymbol{F}_1^{(k')} \bar\rho_A^{(k)} \boldsymbol{F}_1^{(k)\,\dagger} \right] =$$

$$= \sum_{k,k'} \mathrm{Tr}\left[\bar\rho_A^{(k,k')} \boldsymbol{U}_{BB^\dagger}^{(k,k')\,\dagger} \right]$$

$$\sum_{k,k'} \mathrm{Tr}\left[\boldsymbol{U}_{B^\dagger B}^{(k,k')} \bar\rho_A^{(k,k')} \right] = \sum_k \mathrm{Tr}\left[-\boldsymbol{F}_2^{(k)} \bar\rho_A^{(k)} \boldsymbol{F}_2^{(k)\,\dagger} + \bar\tau \bar{\mathcal{F}}_{22}(\bar\rho_A)^{(k)} + \frac{\pi}{2} |b^{(k)}|^2 \right] +$$

$$+ \sum_{k>k'} \mathrm{Tr}\left[-\boldsymbol{F}_2^{(k)} \bar\rho_A^{(k)} \boldsymbol{F}_2^{(k')\,\dagger} - \boldsymbol{F}_2^{(k')} \bar\rho_A^{(k)} \boldsymbol{F}_2^{(k)\,\dagger} \right] =$$

$$= \sum_{k,k'} \mathrm{Tr}\left[\bar\rho_A^{(k,k')} \boldsymbol{U}_{B^\dagger B}^{(k,k')\,\dagger} \right]. \tag{47}$$

Finally, thanks to observations (46), (47) and the definition of f_1, f_2 in (29), we compute the trace of $\mathcal{K}_2$ from (28) as follows:

$$\mathrm{Tr}[\mathcal{K}_2(\rho_s)] = \sum_{k,k'} \Bigg\{ \mathrm{Tr}\left[\boldsymbol{B}\boldsymbol{B}^\dagger \rho_s \right] \mathrm{Tr}\left[\boldsymbol{U}_{BB^\dagger}^{(k,k')} \bar\rho_A^{(k,k')} + \bar\rho_A^{(k,k')} \boldsymbol{U}_{B^\dagger B}^{(k,k')\,\dagger} + \boldsymbol{F}_1^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{F}_1^{(k)\,\dagger} \right] +$$

$$+ \mathrm{Tr}\left[\boldsymbol{B}^\dagger \boldsymbol{B} \rho_s \right] \mathrm{Tr}\left[\boldsymbol{U}_{B^\dagger B}^{(k,k')} \bar\rho_A^{(k,k')} + \bar\rho_A^{(k,k')} \boldsymbol{U}_{BB^\dagger}^{(k,k')\,\dagger} + \boldsymbol{F}_2^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{F}_2^{(k)\,\dagger} \right] +$$

$$+ \mathrm{Tr}\left[\boldsymbol{B}\boldsymbol{B} \rho_s \right] \mathrm{Tr}\left[\boldsymbol{U}_{BB}^{(k,k')} \bar\rho_A^{(k,k')} + \bar\rho_A^{(k,k')} \boldsymbol{U}_{B^\dagger B^\dagger}^{(k,k')\,\dagger} + \boldsymbol{F}_2^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{F}_1^{(k')\,\dagger} \right] +$$

$$+ \mathrm{Tr}\left[\boldsymbol{B}^\dagger \boldsymbol{B}^\dagger \rho_s \right] \mathrm{Tr}\left[\boldsymbol{U}_{B^\dagger B^\dagger}^{(k,k')} \bar\rho_A^{(k,k')} + \bar\rho_A^{(k,k')} \boldsymbol{U}_{BB}^{(k,k')\,\dagger} + \boldsymbol{F}_1^{(k)} \bar\rho_A^{(k,k')} \boldsymbol{F}_2^{(k')\,\dagger} \right] \Bigg\} +$$

$$+ \mathrm{Tr}\left[\boldsymbol{B}^\dagger \rho_s \boldsymbol{B} \right] \mathrm{Tr}\left[f_1 + \sum_k \left(\bar\tau \bar{\mathcal{F}}_{11}^{(k)} + \frac{\pi}{2} \right) \right]$$

$$+ \mathrm{Tr}\left[\boldsymbol{B} \rho_s \boldsymbol{B}^\dagger \right] \mathrm{Tr}\left[f_2 + \sum_k \left(\bar\tau \bar{\mathcal{F}}_{22}^{(k)} + \frac{\pi}{2} |b^{(k)}|^2 \right) \right]$$

$$+ \mathrm{Tr}\left[\boldsymbol{B} \rho_s \boldsymbol{B} \right] \mathrm{Tr}\left[\sum_k -ib^{(k)} \right] + \mathrm{Tr}\left[\boldsymbol{B}^\dagger \rho_s \boldsymbol{B}^\dagger \right] \mathrm{Tr}\left[\sum_k ib^{(k)} \right]$$

$$= 0,$$

for all ρ_s.

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