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Solving the dual Russian option problem by using change-of-measure arguments

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We apply the change-of-measure arguments of Shepp and Shiryaev [38] to study the dual Russian option pricing problem proposed by Shepp and Shiryaev [39] as an optimal stopping problem for a one-dimensional diffusion process with reflection. We recall the solution to the associated free-boundary problem and give a solution to the resulting one-dimensional optimal stopping problem by using the martingale approach of Beibel and Lerche [6] and [7].

1 Introduction

The articles of Shepp and Shiryaev [37] and [38] initiated a large research area of pricing of derivative securities of American type with payoffs depending on the running values of the maxima of the underlying processes. The original Russian (put) option pricing problem of maximising the expected value of the discounted running maximum of a geometric Brownian motion was proposed and explicitly solved in [37] as an optimal stopping problem for a two-dimensional continuous Markov process. It was further observed by Shepp and Shiryaev [38] that the original Russian option problem can be reduced by means of the change-of-measure arguments to an optimal stopping problem for a one-dimensional diffusion process with reflection. Building on the optimal stopping analysis of [37] and [38], Duffie and Harrison [11] derived rational economic values for the Russian options and then extended their arbitrage arguments to more

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general perpetual American lookback options. Gerber et al. [17] and Mordecki and Moreira [28] obtained closed-form solutions to the perpetual Russian option problems for diffusions with negative exponential jumps. Pedersen [30] and Guo and Shepp [21] obtained closed-form solutions to the problems of pricing of more complicated perpetual American lookback options with payoffs depending on the running maxima processes of the underlying geometric Brownian motions. Shepp et al. [40] proposed a barrier version of the Russian option in the same model, where the decision about stopping should be taken before the price process reaches a certain positive level. Asmussen et al. [1] derived explicit expressions for the prices of Russian options in models with Lévy processes of both positive and negative jumps, by means of reducing the initial problems to the first passage time problems and solving the latter problems by the martingale stopping and Wiener-Hopf factorisation. Avram et al. [2] studied exit problems for spectrally negative Lévy processes and applied the results to solving optimal stopping problems associated with perpetual Russian and American put options. Peskir [32] presented a solution to the Russian option problem for a geometric Brownian motion with a finite time horizon (see also Duistermaat et al. [12] for a numeric algorithm of solving the corresponding free-boundary problem and Ekström [13] for a study of asymptotic behaviour of the optimal stopping boundary near the expiration). The problems of pricing of perpetual American lookback and other options with more complicated structure of payoffs depending on the running maxima of the underlying processes were studied in Gapeev [15], Baurdoux and Kyprianou [5], Guo and Zervos [22], Ott [29], Kyprianou and Ott [25], and Rodosthenous and Zervos [36] among others (see also Gapeev [14] and Kitapbayev [24] for the finite-time horizon American lookback options on the running maxima of geometric Brownian motions). Along with the article of Dubins et al. [10], the papers [37] and [38] also made a crucial contribution to the optimal stopping problems arising in the proofs of maximal inequalities for the continuous-time processes further developed in Graversen and Peskir [18]-[19] and Peskir [31] among others (see also Peskir and Shiryaev [34; Chapter V] for an extensive overview of the optimal stopping problems related to maximal inequalities).

Shepp and Shiryaev [39] proposed and explicitly solved the dual Russian (call) option pricing problem of (2.4) as an optimal stopping problem for a two-dimensional continuous Markov process $(X, Y) = (X_t, Y_t)_{t \geq 0}$ defined in (2.1)–(2.3). The idea of writing this paper was to draw the attention of the readership to the article [39] in which an optimal stopping problem with positive exponential discounting rates was studied for one of the first times to the best of our knowledge. Other optimal stopping problems with positive exponential discounting rates were recently considered by Xia and Zhou [43], Battauz et al. [3]–[4], and De Donno et al. [9] among others. The introduction of positive exponential discounting rates into the optimal stopping problems implied the appearance of disconnected continuation regions or so-called double continuation regions for the underlying processes. In this paper, we present an explicit solution to the optimal stopping problem of (2.4) by means of reducing it to the optimal stopping problem of (2.8) for a one-dimensional continuous Markov process $Z = (Z_t)_{t \geq 0}$ solving the stochastic differential equation in (2.9) with the reflection term $L = (L_t)_{t \geq 0}$ from (2.10)–(2.11). This paper can therefore be considered as a companion one to [38] in the sense that it solves the dual Russian option problem of [39] by means of the change-of-measure arguments. We also give a proof of the result by using the martingale approach suggested by Beibel and Lerche [6] and [7].

The paper is organised as follows. In Section 2, we introduce the setting and notation of the dual Russian option pricing optimal stopping problem of [39] and follow the change-of-measure arguments of [38] to reduce it to an optimal stopping problem for a one-dimensional diffusion process with reflection. In Section 3, we derive explicit solutions of the associated free-boundary problem for three different cases of relations between the parameters of the model. In Section 4, we verify that the solution of the free-boundary problem provides the solution of the initial optimal stopping problem of [39]. We also give a solution to the problem by means of the martingale approach of [6] and [7]. The main results of the paper are stated in Theorem 4.1.

2 Preliminaries

In this section, we recall the formulation of the dual Russian option problem proposed in [39] and the solution of the associated free-boundary problem.

2.1 The model

For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ and its natural filtration $(\mathcal{F}_t)_{t \geq 0}$. Let us define the process $X = (X_t)_{t \geq 0}$ by:

$$X_t = x \, \exp\Big(\big(r - \delta - \sigma^2/2\big) \, t + \sigma \, B_t \Big) \tag{2.1}$$

which solves the stochastic differential equation:

$$dX_t = (r - \delta) \, X_t \, dt + \sigma \, X_t \, dB_t \quad (X_0 = x) \tag{2.2}$$

where $x > 0$ is fixed, and $r > 0$, $\delta > 0$, and $\sigma > 0$ are some given constants. (Note that $r - \delta$ in (2.1)–(2.2) corresponds to μ in the notations of [39].) It is assumed that the process X describes the price of a risky asset on a financial market, where r is the riskless interest rate of a bank account, δ is the dividend rate paid to the asset holders, and σ is the volatility rate. Let us now define the associated with X *running minimum* process $Y = (Y_t)_{t \ge 0}$ by:

$$Y_t = y \wedge \min_{0 \le s \le t} X_s \tag{2.3}$$

for an arbitrary $0 < y \le x$. We study the optimal stopping problem with the value:

$$V_* = \inf_{\tau} E\big[e^{r\tau}\, Y_\tau\big] \tag{2.4}$$

where the infimum is taken over all stopping times τ with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$. The problem of (2.4) was introduced and solved in [39] as an optimal stopping problem for a two-dimensional continuous (time-homogeneous strong) Markov process $(X, Y) = (X_t, Y_t)_{t \ge 0}$. In the present paper, we derive a solution to the problem of (2.4), by means of reducing it to an optimal stopping problem for a one-dimensional Markov process, and by applying the martingale approach from [6] and [7] (see also [26] and [16] among others).

In order to reduce the problem of (2.4) to a one-dimensional optimal stopping problem, let us follow the arguments of [38] and introduce the probability measure $\widetilde{P}$ defined by:

$$\frac{d\widetilde{P}}{dP}\bigg|_{\mathcal{F}_t} = \exp\Big(\sigma\, B_t - \big(\sigma^2/2\big)\, t\Big) \equiv e^{-(r-\delta)t}\, \frac{X_t}{x} \tag{2.5}$$

for all $t \ge 0$, so that $\widetilde{P}$ is locally equivalent to P on the filtration $(\mathcal{F}_t)_{t \ge 0}$, according to [35; Appendix, Theorem 6.1]. Then, by virtue of Girsanov's theorem (see, e.g. [35; Chapter VIII, Theorem 1.12]), we obtain that the process $\widetilde{B} = (\widetilde{B}_t)_{t \ge 0}$ defined by $\widetilde{B}_t = B_t - \sigma t$, for all $t \ge 0$, is a standard Brownian motion under $\widetilde{P}$. It thus follows from the expressions in (2.1)–(2.2) that the process X admits the representation:

$$X_t = x\, \exp\Big(\big(r - \delta + \sigma^2/2\big)\, t + \sigma\, \widetilde{B}_t\Big) \tag{2.6}$$

and solves the stochastic differential equation:

$$dX_t = \big(r - \delta + \sigma^2\big)\, X_t\, dt + \sigma\, X_t\, d\widetilde{B}_t \quad (X_0 = x) \tag{2.7}$$

under $\widetilde{P}$. It therefore follows that the value in (2.4) takes the form $V_* = U_*/x$ with:

$$U_* = \inf_{\tau} \widetilde{E}\big[e^{\lambda\tau}\, Z_\tau\big] \tag{2.8}$$

where the infimum is taken over the class of stopping times τ with respect to $(\mathcal{F}_t)_{t \ge 0}$, and we set $\lambda = 2r - \delta$. Here, the process $Z = (Z_t)_{t \ge 0}$ is defined by $Z_t = Y_t/X_t$, for all $t \ge 0$, and satisfies the stochastic differential equation:

$$dZ_t = -(r - \delta)\, Z_t\, dt - \sigma\, Z_t\, d\widetilde{B}_t - dL_t \quad (Z_0 = z) \tag{2.9}$$

with $z = (y/x) \wedge 1$, where the process $L = (L_t)_{t \geq 0}$ defined by:

$$L_t = - \int_0^t \frac{dY_s}{X_s} \tag{2.10}$$

is non-decreasing which increases when $Z_t = 1$ and does not increase when $Z_t < 1$, for $t \geq 0$. Thus, we may conclude that the process L defined in (2.10) admits the representation:

$$L_t = - \int_0^t I(Z_s = 1)\, \frac{dY_s}{X_s} \tag{2.11}$$

for all $t \geq 0$, where $I(\cdot)$ denotes the indicator function.

2.2 The optimal stopping problem

In order to compute the value of (2.8), let us therefore consider the optimal stopping problem:

$$U_*(z) = \inf_{\tau} \widetilde{E}_z \big[e^{\lambda \tau} \, Z_\tau \big] \tag{2.12}$$

where the infimum is taken over stopping times τ of the one-dimensional continuous Markov process Z, and we recall that $\lambda = 2r - \delta$. Here, $\widetilde{E}_z$ denotes the expectation with respect to the probability measure $\widetilde{P}$ under the assumption that the process Z starts at $0 < z \leq 1$. (Note that $U_*(y/x)/x$ corresponds to $V_*(x, y)$ in the notations of [39].) Moreover, it follows from the result of [39; Section 2, Lemma] that if $0 < r \leq (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta < \sigma^2/2$ holds, then $U_*(z) = 0$, for all $0 < z \leq 1$. We further consider the case in which either $r - \delta > \sigma^2/2$ or $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds and will seek for the the optimal stopping time in the problem of (2.12) in the form:

$$\tau_* = \inf\{ t \geq 0 \,|\, Z_t \leq a_* \} \tag{2.13}$$

for some number $0 < a_* < 1$ to be determined. (Note that a_* in (2.13) corresponds to $1/\theta$ in the notations of [39].)

2.3 The free-boundary problem

It can be shown by means of standard arguments (see, e.g. [23; Chapter V, Section 5.1]) that the infinitesimal operator $\mathbb{L}$ of the process Z acts on an arbitrary twice continuously differentiable locally bounded function $F(z)$ according to the rule:

$$(\mathbb{L}F)(z) = -(r - \delta)\, z\, F'(z) + \frac{\sigma^2 z^2}{2}\, F''(z) \tag{2.14}$$

for all $0 < z \leq 1$ (see also [8; Appendix 1, Section 2]). In order to find explicit expressions for the unknown value function $U_*(z)$ from (2.12) and the unknown boundary a_* from (2.13), we may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [41; Chapter III, Section 8] and [34; Chapter IV, Section 8]). We formulate the associated free-boundary problem:

$$(\mathbb{L}U)(z) = -\lambda\, U(z) \quad \text{for} \quad a < z < 1 \tag{2.15}$$

$$U(a+) = a \quad (\textit{instantaneous stopping}) \tag{2.16}$$

$$U'(a+) = 1 \quad (\textit{smooth fit}) \tag{2.17}$$

$$U'(1-) = 0 \quad (\textit{normal reflection}) \tag{2.18}$$

$$U(z) = z \quad \text{for} \quad z < a \tag{2.19}$$

$$U(z) < z \quad \text{for} \quad a < z < 1 \tag{2.20}$$

for some number $0 < a < 1$ to be determined. Observe that the superharmonic characterisation of the value function (see, e.g. [41] and [34; Chapter IV, Section 9]) implies that $U_*(z)$ is the smallest function satisfying (2.15)-(2.16) and (2.19)-(2.20) with the boundary a_*.

3 Solution to the free-boundary problem

We now look for functions which solve the free-boundary problem stated in (2.15)–(2.17). For this purpose, we consider three separate cases based on the different relations between the parameters of the model.

3.1 The case $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$

Let us first assume that $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$ holds. Then, the general solution of the second-order ordinary differential equation in (2.15) has the form:

$$U(z) = C_1\, z^{\eta_1} + C_2\, z^{\eta_2} \tag{3.1}$$

where C_j, $j = 1, 2$, are some arbitrary constants, and η_j, $j = 1, 2$, are given by:

$$\eta_j = \frac{1}{2} + \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} + \frac{r-\delta}{\sigma^2}\right)^2 - \frac{2\lambda}{\sigma^2}} \equiv \frac{1}{2} + \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 - \frac{2r}{\sigma^2}} \tag{3.2}$$

so that $1 < \eta_2 < \eta_1$ in this case. (Note that η_j, $j = 1, 2$, in (3.2) correspond to $1 - \gamma_j$, $j = 1, 2$, in the notations of [39].) Hence, by applying the conditions from (2.16)–(2.18) to the function in (3.1), we get that the equalities:

$$C_1\, a^{\eta_1} + C_2\, a^{\eta_2} = a \tag{3.3}$$

$$C_1\, \eta_1\, a^{\eta_1} + C_2\, \eta_2\, a^{\eta_2} = a \tag{3.4}$$

$$C_1\, \eta_1 + C_2\, \eta_2 = 0 \tag{3.5}$$

should hold for some $0 < a < 1$. Thus, solving the system in (3.3)–(3.5), we obtain that the candidate value function has the form:

$$U(z;a_*) = a_* \left(\frac{\eta_2 - 1}{\eta_2 - \eta_1} \left(\frac{z}{a_*} \right)^{\eta_1} - \frac{\eta_1 - 1}{\eta_2 - \eta_1} \left(\frac{z}{a_*} \right)^{\eta_2} \right) \tag{3.6}$$

for $a_* < z \le 1$, with

$$a_* = \left(\frac{\eta_1(\eta_2 - 1)}{\eta_2(\eta_1 - 1)} \right)^{1/(\eta_1 - \eta_2)}. \tag{3.7}$$

3.2 The case $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$

Let us now assume that $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$ holds. Then, the general solution of the ordinary differential equation in (2.15) has the form:

$$U(z) = C_1 \, z^{\alpha} \, \ln z + C_2 \, z^{\alpha} \tag{3.8}$$

where C_j, $j = 1, 2$, are some arbitrary constants, and α is given by:

$$\alpha = \frac{1}{2} + \frac{r - \delta}{\sigma^2} \tag{3.9}$$

so that $\alpha > 1$ in this case. (Note that α in (3.9) corresponds to $1 - \gamma$ in the notations of [39].) Hence, by applying the conditions from (2.16)–(2.18) to the function in (3.8), we get that the equalities:

$$C_1 \, a^{\alpha} \, \ln a + C_2 \, a^{\alpha} = a \tag{3.10}$$

$$C_1 \, \alpha \, a^{\alpha} \, \ln a + C_1 \, a^{\alpha} + C_2 \, \alpha \, a^{\alpha} = a \tag{3.11}$$

$$C_1 + C_2 \, \alpha = 0 \tag{3.12}$$

should hold for some $0 < a < 1$. Thus, solving the system in (3.10)–(3.12), we obtain that the candidate value function has the form:

$$U(z;a_*) = a_* \left((1 - \alpha) \left(\frac{z}{a_*} \right)^{\alpha} \ln \left(\frac{z}{a_*} \right) + \left(\frac{z}{a_*} \right)^{\alpha} \right) \tag{3.13}$$

for $a_* < z \le 1$, with

$$a_* = \exp \left(\frac{1}{\alpha(1 - \alpha)} \right). \tag{3.14}$$

3.3 The case $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$

Let us finally assume that $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds. Then, the general solution of the ordinary differential equation in (2.15) has the form:

$$U(z) = C_1 \, z^{\alpha} \, \sin \big(\beta \ln z \big) + C_2 \, z^{\alpha} \, \cos \big(\beta \ln z \big) \tag{3.15}$$

where C_j, $j = 1, 2$, are some arbitrary constants, while α is given by (3.9) and β is defined as:

$$\beta = \sqrt{\frac{2\lambda}{\sigma^2} - \left(\frac{1}{2} + \frac{r-\delta}{\sigma^2}\right)^2} \equiv \sqrt{\frac{2r}{\sigma^2} - \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2}. \tag{3.16}$$

Hence, by applying the conditions from (2.16)–(2.18) to the function in (3.15), we get that the equalities:

$$C_1 \, a^\alpha \, \sin\big(\beta \ln a\big) + C_2 \, a^\alpha \, \cos\big(\beta \ln a\big) = a \tag{3.17}$$

$$(C_1\alpha - C_2\beta) \, a^\alpha \, \sin\big(\beta \ln a\big) + (C_1\beta + C_2\alpha) \, a^\alpha \, \cos\big(\beta \ln a\big) = a \tag{3.18}$$

$$C_1 \, \beta + C_2 \, \alpha = 0 \tag{3.19}$$

should hold for some $0 < a < 1$. Thus, solving the system in (3.17)–(3.19), we obtain that the candidate value function has the form:

$$U(z; a_*) = a_* \left(\frac{1-\alpha}{\beta} \left(\frac{z}{a_*}\right)^\alpha \sin\left(\beta \ln\left(\frac{z}{a_*}\right)\right) + \left(\frac{z}{a_*}\right)^\alpha \cos\left(\beta \ln\left(\frac{z}{a_*}\right)\right)\right) \tag{3.20}$$

for $a_* < z \le 1$, with

$$a_* = \exp\left(\frac{1}{\beta} \arctan\left(\frac{\beta}{\alpha - \alpha^2 - \beta^2}\right)\right). \tag{3.21}$$

(Note that $\arctan(\beta/(\alpha - \alpha^2 - \beta^2))$ in (3.21) corresponds to $-\phi$ in the notations of [39].)

4 Main results and proofs

In this section, we show that the solution of the free-boundary problem from the previous section provides the solution of the initial optimal stopping problem of [39].

4.1 The verification theorem

Let us first verify that the solution of the free-boundary problem (2.14)–(2.20) coincides with the solution of the optimal stopping problem in (2.12). The following assertion was proved in [39] for the solution of the corresponding two-dimensional optimal stopping problem.

Theorem 4.1 *Let the processes X and Y be given by (2.1)–(2.3), so that the process $Z = Y/X$ admits the representation in (2.9). Then, the value function of the dual Russian optimal stopping problem in (2.12) has the form:*

$$U_*(z) = \begin{cases} U(z; a_*), & \text{if } \; a_* < z \le 1 \\ z, & \text{if } \; 0 < z \le a_* \end{cases} \tag{4.1}$$

and τ_* from (2.13) is an optimal stopping time which is finite ($\widetilde{P}$-a.s.), where we have:

(i) if $0 < r < (r-\delta-\sigma^2/2)^2/(2\sigma^2)$ and $r-\delta > \sigma^2/2$ holds, then $U(z; a_*)$ takes the expression of (3.6) and a_* is given by (3.7);

(ii) if $r = (r-\delta-\sigma^2/2)^2/(2\sigma^2)$ and $r-\delta > \sigma^2/2$ holds, then $U(z; a_*)$ takes the expression of (3.13) and a_* is given by (3.14);

(iii) if $r > (r-\delta-\sigma^2/2)^2/(2\sigma^2)$ holds, then $U(z; a_*)$ takes the expression of (3.20) and a_* is given by (3.21).

Proof: In order to verify the assertions stated above, let us show that the function defined in (4.1) coincides with the value function in (2.12), and that the stopping time τ_* in (2.13) is optimal with the boundary a_* specified above. For this purpose, let us denote by $U(z)$ the right-hand side of the expression in (4.1). Then, by means of straightforward calculations from the previous section, it is shown that the function $U(z)$ solves the system of (2.15) with (2.19)–(2.20) and satisfies the conditions of (2.16)–(2.18). Hence, by applying the change-of-variable formula from [33] to the process $e^{\lambda t}U(Z_t)$ (see also [34; Chapter II, Section 3.5] for a summary of the related results on the local time-space formula as well as further references), we obtain:

$$e^{\lambda t}\, U(Z_t) = U(z) + \int_0^t e^{\lambda s}\, (\mathbb{L}U + \lambda U)(Z_s)\, I(Z_s \neq a_*, Z_s \neq 1)\, ds + N_t \tag{4.2}$$

for all $t \geq 0$, where the process $N = (N_t)_{t \geq 0}$ defined by:

$$N_t = -\int_0^t e^{\lambda s}\, U'(Z_s)\, \sigma\, Z_s\, d\widetilde{B}_s \tag{4.3}$$

is a continuous local martingale with respect to the probability measure $\widetilde{P}_z$.

By using straightforward calculations and the arguments from the previous section, it is verified that the inequality $(\mathbb{L}V + \lambda V)(z) \equiv rz \geq 0$ holds for any $0 < z < 1$ such that $z \neq a_*$. Moreover, it is shown by means of standard arguments that the inequality in (2.20) holds, which together with the conditions of (2.16)–(2.19) imply that $U(z) \leq z$ holds, for all $0 < z \leq 1$. We also observe that the identity:

$$\int_0^t I(Z_s = 1)\, ds = 0 \quad (\widetilde{P}_z - \text{a.s.}) \tag{4.4}$$

holds, for all $t \geq 0$. This property follows from the fact that the couple of processes (X, Y) has a density with respect to the Lebesgue measure, so that $\widetilde{P}_z(Z_t = 1) = 0$, for all $t \geq 0$, and by virtue of Fubini's theorem, we get that the equality:

$$\widetilde{E}_z \int_0^\infty I(Z_s = 1)\, ds = \int_0^\infty \widetilde{E}_z\big[I(Z_s = 1)\big]\, ds = \int_0^\infty \widetilde{P}_z(Z_s = 1)\, ds = 0 \tag{4.5}$$

holds (see also [42; Chapter VIII, Section 2] for the proof related to the standard Russian put option). Thus, taking into account the fact that the time spent by the process Z at the point a_* is of the Lebesgue measure zero (see, e.g. [8; Chapter II, Section 1]), we may conclude that the indicator in (4.2) can be set to one. Hence, the expression in (4.2) yields that the inequalities:

$$e^{\lambda \tau} \, Z_\tau \geq e^{\lambda \tau} \, U(Z_\tau) \geq U(z) + N_\tau \tag{4.6}$$

hold, for any stopping time τ of the process Z started at $0 < z \leq 1$.

Let $(\tau_k)_{k \in \mathbb{N}}$ be the localising sequence of stopping times for the process N from (4.3) such that $\tau_k = \inf\{t \geq 0 \,|\, |N_t| \geq k\}$, for each $k \in \mathbb{N}$. It therefore follows from the expression in (4.2) that the inequalities:

$$e^{\lambda (\tau \wedge \tau_k)} \, Z_{\tau \wedge \tau_k} \geq e^{\lambda (\tau \wedge \tau_k)} \, U(Z_{\tau \wedge \tau_k}) \geq U(z) + N_{\tau \wedge \tau_k} \tag{4.7}$$

hold, for any stopping time τ of the process Z and each $k \in \mathbb{N}$ fixed. Taking the expectation with respect to $\widetilde{P}_z$ in (4.6), by means of Doob's optional sampling theorem (see, e.g. [23; Chapter I, Theorem 3.22]), we get that the inequalities:

$$\widetilde{E}_z \big[e^{\lambda (\tau \wedge \tau_k)} \, Z_{\tau \wedge \tau_k} \big] \geq \widetilde{E}_z \big[e^{\lambda (\tau \wedge \tau_k)} \, U(Z_{\tau \wedge \tau_k}) \big] \geq U(z) + \widetilde{E}_z \big[N_{\tau \wedge \tau_k} \big] = U(z) \tag{4.8}$$

hold, for all $0 < z \leq 1$. Hence, letting k go to infinity and using Fatou's lemma, we obtain that the inequalities:

$$\widetilde{E}_z \big[e^{\lambda \tau} \, Z_\tau \big] \geq \widetilde{E}_z \big[e^{\lambda \tau} \, U(Z_\tau) \big] \geq U(z) \tag{4.9}$$

are satisfied, for any stopping time τ and all $0 < z \leq 1$. By virtue of the structure of the stopping time in (2.13), it is readily seen that the equalities in (4.9) hold with τ_* instead of τ when $0 < z \leq a_*$.

Let us now show that the stopping time τ_* in (2.13) is finite ($\widetilde{P}$-a.s.). For this purpose, we follow the schema of arguments of [6] and [42; Chapter VIII, Section 2]. We first observe that the expression:

$$\ln Z_t = \ln z \wedge \min_{0 \leq s \leq t} \bigg(\Big(r - \delta + \frac{\sigma^2}{2} \Big) (s - t) + \sigma \, (\widetilde{B}_s - \widetilde{B}_t) \bigg) \tag{4.10}$$

holds, for all $t \geq 0$. Then, we consider a sequence of first hitting times $(\zeta_k)_{k \in \mathbb{N}}$ such that $\zeta_k = \inf\{t \geq \zeta_{k-1} + 1 \,|\, \ln Z_t = 0\}$, for $k \in \mathbb{N}$, and $\zeta_0 = 0$. In this case, we see that:

$$\Big\{ \inf_{t \geq 0} \ln Z_t \leq \ln a_* \Big\} = \bigcup_{k \in \mathbb{N}} \Big\{ \inf_{\zeta_{k-1} \leq t < \zeta_k} \ln Z_t \leq \ln a_* \Big\} \tag{4.11}$$

holds, where we have $\ln a_* < 0$. Note that the events on the right-hand side of (4.11) are independent for different $k \in \mathbb{N}$, and their probabilities are equal and strictly positive. Hence, by virtue of the Borel-Cantelli lemma, we may conclude that:

$$\widetilde{P}_z \Big(\inf_{t \geq 0} \ln Z_t \leq \ln a_* \Big) = 1 \tag{4.12}$$

so that $\widetilde{P}_z(\tau_* < \infty) = 1$, for any $0 < z \le 1$ fixed.

It remains us to show that the equalities are attained in (4.9) when τ_* replaces τ, for $a_* < z \le 1$. By virtue of the fact that the function $U(z; a_*)$ and the boundary a_* satisfy the conditions in (2.15) and (2.16), it follows from the expression in (4.2) and the structure of the stopping time in (2.13) that the equality:

$$e^{\lambda(\tau_* \wedge \tau_k)} \, U(Z_{\tau_* \wedge \tau_k}; a_*) = U(z) + N_{\tau_* \wedge \tau_k} \tag{4.13}$$

holds, for all $a_* < z \le 1$ and each $k \in \mathbb{N}$. Observe that the explicit form of the candidate value function in (3.6), (3.13), and (3.20) yields that the condition:

$$\widetilde{E}_z \Big[\sup_{t \ge 0} e^{\lambda(\tau_* \wedge t)} \, U(Z_{\tau_* \wedge t}; a_*) \Big] < \infty \tag{4.14}$$

holds, for all $a_* < z \le 1$. Hence, taking into account the property in (4.14) as well as the fact that $\widetilde{P}_z(\tau_* < \infty) = 1$, we conclude from the expression in (4.13) that the process $(N_{\tau_* \wedge t})_{t \ge 0}$ is a uniformly integrable martingale. Therefore, taking the expectation in (4.13) and letting n go to infinity, we apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$\widetilde{E}_z \big[e^{\lambda \tau_*} \, Z_{\tau_*} \big] = \widetilde{E}_z \big[e^{\lambda \tau_*} \, U(Z_{\tau_*}; a_*) \big] = U(z) \tag{4.15}$$

for all $a_* < z \le 1$. The latter, together with the inequalities in (4.9), implies the fact that $U(z)$ coincides with the value function $U_*(z)$ from (2.12). □

4.2 The martingale approach

Let us finally apply the martingale approach of [6] and [7] for solving the optimal stopping problem of (2.12). For this purpose, we first observe from the structure of the processes (X, Y) in (2.1)–(2.3) and of the reward in the optimal stopping problem of (2.12) that $U_*(z) = z\widetilde{U}_*(1) \equiv z\widetilde{U}_*$ holds for the value function with

$$\widetilde{U}_* = \inf_{\tau} \widetilde{E}_1 \big[e^{\lambda \tau} \, Z_\tau \, I(\tau < \infty) \big] \equiv \widetilde{E}_1 \big[e^{\lambda \tau_*} \, Z_{\tau_*} \, I(\tau_* < \infty) \big] \tag{4.16}$$

where the infimum is taken over all stopping times τ of the process Z, and we have $\widetilde{P}_1(\tau_* < \infty) = 1$. We now search for a function $H(z)$ such that the process $M = (M_t)_{t \ge 0}$ defined by:

$$M_t = e^{\lambda t} \, Z_t \, H(Z_t) \tag{4.17}$$

and started at $M_0 = 1$ is a continuous local martingale under the probability measure $\widetilde{P}$. In this case, by applying Itô's formula (see, e.g. [27; Theorem 4.4]) to the process $e^{\lambda t} Z_t H(Z_t)$, we obtain that the function $H(z)$ should solve the second-order ordinary differential equation:

$$\frac{\sigma^2 z^2}{2} \, H''(z) - (r - \delta - \sigma^2) \, z \, H'(z) = (r - \delta - \lambda) \, H(z) \tag{4.18}$$

as well as satisfy the boundary conditions:

$$H(1-) = 1 \quad \text{and} \quad H'(1-) = -1. \tag{4.19}$$

It is shown by means of straightforward computations, similar to the ones derived in the previous section, that the function $H(z)$ admits the representation:

$$H(z) = \frac{\eta_2}{\eta_2 - \eta_1}\, z^{\eta_1 - 1} - \frac{\eta_1}{\eta_2 - \eta_1}\, z^{\eta_2 - 1} \tag{4.20}$$

when $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$ holds, or

$$H(z) = -\alpha\, z^{\alpha-1} \ln z + z^{\alpha-1} \tag{4.21}$$

when $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$ holds, or

$$H(z) = -\frac{\alpha}{\beta}\, z^{\alpha-1} \sin\big(\beta \ln z\big) + z^{\alpha-1} \cos\big(\beta \ln z\big) \tag{4.22}$$

when $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, respectively. By means of straightforward computations, it can be deduced from the expressions in (4.20), or (4.21), or (4.22), that the first- and second-order derivatives $H'(z)$ and $H''(z)$ of the function $H(z)$ solving the equation in (4.18) and satisfying the conditions of (4.19) take the form:

$$H'(z) = \frac{\eta_2(\eta_1 - 1)}{\eta_2 - \eta_1}\, z^{\eta_1 - 2} - \frac{\eta_1(\eta_2 - 1)}{\eta_2 - \eta_1}\, z^{\eta_2 - 2} \tag{4.23}$$

and

$$H''(z) = \frac{\eta_2(\eta_1 - 1)(\eta_1 - 2)}{\eta_2 - \eta_1}\, z^{\eta_1 - 3} - \frac{\eta_1(\eta_2 - 1)(\eta_2 - 2)}{\eta_2 - \eta_1}\, z^{\eta_2 - 3} \tag{4.24}$$

when $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$ holds, or

$$H'(z) = -\alpha\,(\alpha - 1)\, z^{\alpha-2} \ln z - z^{\alpha-2} \tag{4.25}$$

and

$$H''(z) = -\alpha\,(\alpha - 1)\,(\alpha - 2)\, z^{\alpha-3} \ln z - (\alpha^2 - 2)\, z^{\alpha-3} \tag{4.26}$$

when $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$ holds, or

$$H'(z) = -\frac{\alpha^2 - \alpha + \beta^2}{\beta}\, z^{\alpha-2} \sin\big(\beta \ln z\big) - z^{\alpha-2} \cos\big(\beta \ln z\big) \tag{4.27}$$

and

$$H''(z) = -\frac{(\alpha^2 - \alpha + \beta^2)(\alpha - 2) - \beta^2}{\beta}\, z^{\alpha-3} \sin\big(\beta \ln z\big) - (\alpha^2 + \beta^2 - 2)\, z^{\alpha-3} \cos\big(\beta \ln z\big) \tag{4.28}$$

when $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, respectively. It is shown by verifying the equation $H'(a_*) = 0$ and the inequality $H''(z) < 0$, for all $z \in (0, 1)$, for the functions in (4.23)-(4.24), or (4.25)-(4.26), or (4.27)-(4.28), that the function $H(z)$ in (4.20), or (4.21), or (4.22), attains its maximum at the point a_* given by (3.7), or (3.14), or (3.21), respectively (see also [42; Chapter VIII, Section 2] for the proof related to the standard Russian put option). Hence, it follows from the structure of the expressions in (4.20), or (4.21), or (4.22), that the process M defined in (4.17) is a continuous local martingale bounded above, and thus, it is a submartingale. Therefore, we obtain that the inequalities:

$$\begin{aligned} &\widetilde{E}_1\big[e^{\lambda\tau}\, Z_\tau\, I(\tau < \infty)\big] = \widetilde{E}_1\big[(1/H(Z_\tau))\, M_\tau\, I(\tau < \infty)\big] \\ &\ge (1/H(a_*))\, \widetilde{E}_1\big[M_\tau\, I(\tau < \infty)\big] \ge 1/H(a_*) \end{aligned} \tag{4.29}$$

hold, for any finite stopping time τ of the process Z. By virtue of the fact that $\widetilde{P}_1(\tau_* < \infty) = 1$, we may therefore summarise the arguments above to conclude that the expressions:

$$\widetilde{U}_* = \widetilde{E}_1\big[(1/H(Z_{\tau_*}))\, M_{\tau_*}\, I(\tau_* < \infty)\big] = (1/H(a_*))\, \widetilde{E}_1\big[M_{\tau_*}\, I(\tau_* < \infty)\big] = 1/H(a_*) \tag{4.30}$$

hold and formulate the following assertion.

Corollary 4.2 *Let the assumptions of Theorem 4.1 above hold. Then, the value function of the dual Russian optimal stopping problem in (2.12) admits the representation* $U_*(z) = z\widetilde{U}_*(1) \equiv z\widetilde{U}_*$ *with* $\widetilde{U}_*$ *being of the form of (4.30), where* $H(z)$ *and* a_* *take the expressions of (4.20) and (3.7) when* $0 < r < (r-\delta-\sigma^2/2)^2/(2\sigma^2)$ *and* $r-\delta > \sigma^2/2$ *holds, or (4.21) and (3.14) when* $r = (r-\delta-\sigma^2/2)^2/(2\sigma^2)$ *and* $r-\delta > \sigma^2/2$ *holds, or (4.22) and (3.21) when* $r > (r-\delta-\sigma^2/2)^2/(2\sigma^2)$ *holds, respectively.*

Remark 4.3 *Note that it can also be deduced from the arguments of the previous section that the function* $zH(z)$ *should solve the second-order ordinary differential equation in (2.15), so that the function* $H(z)$ *from (4.20), or (4.21), or (4.22), equivalently admits the representation:*

$$H(z) = \frac{U(z; a_*)}{zU(1; a_*)} \tag{4.31}$$

for all $0 < z < 1$*. Here, the function* $U(z; a_*)$ *is the solution of the equation in (2.15) with the boundary conditions in (2.16)-(2.17) which takes the expressions of (3.6) when* $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ *and* $r - \delta > \sigma^2/2$ *holds, or (3.13) when* $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ *and* $r - \delta > \sigma^2/2$ *holds, or (3.20) when* $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ *holds, respectively.*

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