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Window of Chaotic Delayed Synchronization

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Abstract. We consider a general coupling of two chaotic dynamical systems and we obtain conditions that provide delayed synchronization. We consider four different couplings that satisfy those conditions. We define Window of Delayed Synchronization and we obtain it analytically. We use four different free chaotic dynamics in order to observe numerically the analytically predicted windows for the considered couplings.

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1. Introduction

The importance of synchronization is known for a long time and its application is wide [1], [2], [3], [4], [5]. There are different types of synchronization, but the most commonly considered in couplings of discrete dynamical systems is the complete synchronization [6], [7], [8]. Usually, the dynamical systems are supposed to be coupled to others using a linear symmetric coupling, but there are other types of couplings that can be considered. In previous papers, we extended the study of complete synchronization to other types of couplings [9], [10]. Now, we define another type of synchronization, the delayed synchronization. When the delayed synchronization takes place the coupled dynamical systems assume the same values but not at the same time as it happens in the complete synchronization. In the delayed synchronization, one of the coupled system repeats the values of the other after a time delay Δt. Obviously, not all the couplings admit delayed synchronization and even when they admit it they only synchronize with delay for some values of the coupling-strength constant, i.e. only some values of the coupling-strength constant correspond to exponentially stable solutions. Those values define the window of delayed synchronization of the coupling. We are going to define conditions for a coupling to admit delayed synchronization and for the couplings that satisfy those conditions we will be able to obtain the windows of delayed synchronization.

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2. Delayed synchronization

We consider couplings of two discrete one-dimensional chaotic dynamical systems of the following type

$$\begin{cases} x\,(t+1) = f\,(x\,(t)) + c \cdot [F_1(x(t)) + F_2(y(t))] \\ y\,(t+1) = g\,(y\,(t)) + c \cdot [G_1(x(t)) + G_2(y(t))] \end{cases} \tag{1}$$

with $c \in [0{,}1]$. The behavior of the free dynamical systems are chaotic in the sense that $\mu_0 = \lim\limits_{T\to+\infty} \frac{1}{T} \sum\limits_{t=0}^{T-1} \ln |f'|_{s(t)}$ is positive (the same is true of g)

As in the complete synchronization, when there is delayed synchronization the iterates of the dynamical systems of the coupling are identical. Nevertheless, unlike what happens in the complete synchronization the identity occurs for different times t as the following definition stands.

Definition 1. *We say that the coupling (1) admits delayed synchronization of Δt (with $\Delta t \in \mathbb{N}$) if, for each value of the coupling-strength constant c there is a function $s(t)$ such that $(x(t),y(t)) = (s(t+\Delta t),s(t))$ is a solution of (1). If, for each value of c, $s(t)$ can be a solution of a dynamical system with chaotic behavior we say that the complete synchronization has chaotic behavior.*

If the functions f, g, F_1, F_2, G_1 and G_2 satisfy some conditions it is possible to assure that the coupling (1) admits delayed synchronization, such as the following proposition determines.

Proposition 1. *If $f = g$, $F_1 \circ f^{(\Delta t)} + F_2 = 0$ and $G_1 \circ f^{(\Delta t)} + G_2 = 0$, with $f^{(\Delta t)} = f \circ f \circ ... \circ f$ (Δt times), then the coupling (1) admits delayed synchronization of Δt with chaotic behavior and the delayed synchronized solution corresponds to $y(t+1) = f(y(t))$*

Proof 1. *If $s(t)$ is a solution of $s(t+1) = f(s(t))$, then $(x(t),y(t)) = (s(t+\Delta t),s(t))$ satisfies (1), for any value of c. In fact, considering, in (1), $f = g$, $F_2 = -F_1 \circ f^{(\Delta t)}$, $G_2 = -G_1 \circ f^{(\Delta t)}$ and $(x(t),y(t)) = (s(t+\Delta t),s(t))$, we obtain*

$$\begin{cases} s(t+\Delta t+1) = f\,(s(t+\Delta t)) + c \cdot \left[F_1(s(t+\Delta t)) - \left(F_1 \circ f^{(\Delta t)}\right)(s\,(t))\right] \\ s\,(t+1) = f\,(s\,(t)) + c \cdot \left[G_1(s(t+\Delta t)) - \left(G_1 \circ f^{(\Delta t)}\right)(s\,(t))\right] \end{cases} \Leftrightarrow$$

$$\begin{cases} s(t+\Delta t+1) = f\,(s(t+\Delta t)) \\ s\,(t+1) = f\,(s\,(t)) \end{cases}$$

Both these equations are obviously verified, since $s(t+1) = f(s(t))$. So, we conclude that the coupling admit a delayed synchronization of Δt. Further, it is a synchronization with chaotic behavior since f corresponds to a chaotic dynamic. □

This proposition determines that a coupling of the following type

$$\begin{cases} x\,(t+1) = f\,(x\,(t)) + c \cdot \left[F_1(x(t)) - F_1(f^{(\Delta t)}(y(t))\right] \\ y\,(t+1) = f\,(y\,(t)) + c \cdot \left[G_1(x(t)) - G_1(f^{(\Delta t)}(y(t))\right] \end{cases} \tag{2}$$

admits a delayed synchronization of Δt with chaotic behavior.

We consider four examples of these type:

• Commanded by the Past Coupling (CPC), corresponding to $\Delta t = 1$, $F_1 = 0$ and $G_1 = id$:

$$\begin{cases} x\,(t+1) = f\,(x\,(t)) \\ y\,(t+1) = f\,(y\,(t)) + c \cdot [x(t) - f(y(t))] \end{cases}$$

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(Note that, for the values of c such that the coupling synchronizes, we have $y(t+1) = x(t)$ and x is free, i.e. in that situation y is commanded by the past of the free x; that is why we name this coupling Commanded by the Past Coupling)

• Commanded by the Future Coupling (CFC), corresponding to $\Delta t = 1$, $F_1 = -f$ and $G_1 = 0$:

$$\begin{cases} x\left(t+1\right) = f\left(x\left(t\right)\right) + c \cdot \left[-f\left(x(t)\right) + f\left(f(y(t))\right)\right] \\ y\left(t+1\right) = f\left(y\left(t\right)\right) \end{cases}$$

(Note that, for the values of c such that the coupling synchronizes, we have $x(t) = y(t+1)$ and y is free, i.e. in that situation x is commanded by the future of the free y; that is why we name this coupling Commanded by the Future Coupling)

• Commanded by the Post-Future Coupling (CPFC), corresponding to $\Delta t = 2$, $F_1 = -f$ and $G_1 = 0$:

$$\begin{cases} x\left(t+1\right) = f\left(x\left(t\right)\right) + c \cdot \left[-f\left(x(t)\right) + f\left(f\left(f(y(t))\right)\right)\right] \\ y\left(t+1\right) = f\left(y\left(t\right)\right) \end{cases}$$

(Note that, for the values of c such that the coupling synchronizes, we have $x(t) = y(t+2)$ and y is free, i.e. in that situation x is commanded by the post-future of the free y; that is why we name this coupling Commanded by the Post-Future Coupling)

• Bidirectional Delayed Coupling (BDC), corresponding to $\Delta t = 1$, $F_1 = -f$ and $G_1 = id$:

$$\begin{cases} x\left(t+1\right) = f\left(x\left(t\right)\right) + c \cdot \left[-f\left(x(t)\right) + f\left(f(y(t))\right)\right] \\ y\left(t+1\right) = f\left(y\left(t\right)\right) + c \cdot \left[x(t) - f(y(t))\right] \end{cases}$$

(Note that, unlike the previous couplings, in this one the interaction between the dynamical systems of the coupling is bidirectional and, like that, none of them is free)

3. Window-of-delayed-synchronization

Even if a coupling admits delayed synchronization, only some values of the coupling-strength constant c correspond to an exponentially stable delayed synchronized solution $(x(t),y(t)) = (s(t+\Delta t),s(t))$.

Definition 2. *For a coupling that admits a delayed synchronization of Δt, the window-of-delayed-synchronization (WDS) of the coupling is the set of values of the coupling-strength constant c such that the delayed synchronized solution $(x(t),y(t)) = (s(t + \Delta t),s(t))$ is exponentially stable.*

For the coupling (2) it is possible to analytically obtain the window-of-delayed-synchronization.

Proposition 2. *The window-of-delayed-synchronization of the coupling (2) is*

$$WDS = \{c \in [0,1] : \mu_{ds} < 0\}$$

with

$$\mu_{ds} = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln \left| f' \circ f^{(\Delta t)} + c \cdot \left[F_1' \circ f^{(\Delta t)} - G_1' \circ f^{(\Delta t)} \cdot \left(f^{(\Delta t)} \right)' \circ f \right] \right|_{s(t)}$$

and $s\left(t+1\right) = f\left(s(t)\right)$, except eventually by some elements $\{c \in [0,1] : \mu_{ds} = 0\}$.

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Proof 2. *In order to simplify the notation, we use x_t instead of $x(t)$ and y_t instead of $y(t)$.*

Considering $u_t = f^{(\Delta t)}(y_t) - x_t$, or equivalently $x_t = f^{(\Delta t)}(y_t) - u_t$, we have around $x_t = f^{(\Delta t)}(y_t)$,

$$\begin{aligned}
u_{t+1} &= f^{(\Delta t)}(y_{t+1}) - x_{t+1} = \\
&= f^{(\Delta t)}\left(f\left(y_t\right) + c\cdot\left[G_1(f^{(\Delta t)}(y_t) - u_t) - G_1(f^{(\Delta t)}(y_t)\right]\right) - \\
&\quad -f\left(f^{(\Delta t)}(y_t) - u_t\right) - c\cdot\left[F_1(f^{(\Delta t)}(y_t) - u_t) - F_1(f^{(\Delta t)}(y_t))\right] \simeq \\
&\simeq f^{(\Delta t)}(f\left(y_t\right) + c\cdot\left[G_1(f^{(\Delta t)}(y_t)) - G_1'(f^{(\Delta t)}(y_t))\cdot u_t - G_1(f^{(\Delta t)}(y_t))\right]) - \\
&\quad - f\left(f^{(\Delta t)}(y_t)\right) + f'\left(f^{(\Delta t)}(y_t)\right)\cdot u_t - \\
&\quad - c\cdot\left[F_1(f^{(\Delta t)}(y_t)) - F_1'(f^{(\Delta t)}(y_t))\cdot u_t - F_1(f^{(\Delta t)}(y_t))\right] \simeq \\
&\simeq f^{(\Delta t)}\left(f\left(y_t\right)\right) - c\cdot\left(f^{(\Delta t)}\right)'\left(f\left(y_t\right)\right)\cdot G_1'(f^{(\Delta t)}(y_t))\cdot u_t - \\
&\quad -f\left(f^{(\Delta t)}(y_t)\right) + f'\left(f^{(\Delta t)}(y_t)\right)\cdot u_t + c\cdot F_1'(f^{(\Delta t)}(y_t))\cdot u_t
\end{aligned}$$

So, the linearization of the evolution of u_t is

$$u_{t+1} = \left(f'\left(f^{(\Delta t)}(y_t)\right) + c\cdot\left[F_1'(f^{(\Delta t)}(y_t)) - G_1'(f^{(\Delta t)}(y_t))\cdot\left(f^{(\Delta t)}\right)'\left(f\left(y_t\right)\right)\right]\right)\cdot u_t$$

Since for a solution with $x_t = f^{(\Delta t)}(y_t)$ we have $y(t+1) = f(y(t))$, we conclude that if

$$\mu_{ds} = \lim_{T\to+\infty}\frac{1}{T}\sum_{t=0}^{T-1}\ln\left|f'\circ f^{(\Delta t)} + c\cdot\left[F_1'\circ f^{(\Delta t)} - G_1'\circ f^{(\Delta t)}\cdot\left(f^{(\Delta t)}\right)'\circ f\right]\right|_{s(t)} < 0,$$

with $s(t+1) = f(s(t))$, then $u(t) = 0$ is an exponentially stable solution of the previous equation, i.e. $(x(t),y(t)) = (f^{(\Delta t)}(s(t)),s(t)) = (s(t+\Delta t),s(t))$ is an exponentially stable solution of (2) and the corresponding value of c belongs to the window-of-delayed-synchronization. If, instead of that, $\mu_{ds} > 0$, then $u(t) = 0$ is an unstable solution of the previous equation, i.e. $(x(t),y(t)) = (f^{(\Delta t)}(s(t)),s(t)) = (s(t+\Delta t),s(t))$ is an unstable solution of (2) and the corresponding value of c does not belong to the window-of-delayed-synchronization. So, we conclude that $WDS = \{c\in[0,1] : \mu_{ds} < 0\}$, except eventually by some elements $\{c\in[0,1] : \mu_{ds} = 0\}$. □

4. Applications

Now, we obtain the windows-of-delayed-synchronization for the four previously presented couplings applied to four different chaotic systems. Before obtaining them analytically, using the previous proposition, we visualize them numerically using the following procedure:

- we calculate the iterates $(x(t),y(t))$ for t sufficiently large, namely between $t = 100$ and $t = 200$, using random initial conditions [11].
- we use different random initial conditions for each considered value of c.
- we plot $y(t+\Delta t) - x(t)$ as a function of c.

Each graph that is obtained by this procedure show the window-of-complete-synchronization since it corresponds to the set of values of c for which the ordinate is only zero.

We consider four examples of each coupling using four different maps f corresponding to chaotic dynamics . We choose relevant maps with different behavior in what respects to piecewise linearity and number of extrema [12], [13], [14]:

1. The tent map, which is a piecewise linear map with just one maximum:

$$f_1(x) = \begin{cases} 2x, x \in \left[0, \frac{1}{2}\right] \\ 2 - 2x, x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Its Lyapunov exponent is $\mu_0 = \ln 2$.

2. The logistic map, which is the polynomial interpolation of the tent map using the vertices $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$ as nodes:

$$f_2(x) = 4x\,(1-x)\,, x \in [0,1]$$

Its Lyapunov exponent is $\mu_0 = \ln 2$.

3. The 3-piecewise linear map, which is a piecewise map with two extrema (one maximum and one minimum):

$$f_3(x) = \begin{cases} 2.4 \cdot x, x \in [0, x_1] \\ 1.7 - 2.4 \cdot x, x \in [x_1, x_2] \\ 2.4 \cdot x - 1.4, x \in [x_2, 1] \end{cases}, \text{ with } x_1 = \frac{17}{48} \text{ and } x_2 = \frac{31}{48}$$

Its Lyapunov exponent is $\mu_0 = \ln 2.4$.

4. The cubic-like map, which is the polynomial interpolation of the 3-piecewise linear map using the vertices $x_0 = 0$, x_1, x_2 and $x_3 = 1$ as nodes:

$$f_4(x) = \frac{x(x-x_2)(x-1)}{x_1(x_1-x_2)(x_1-1)} \cdot 0.85 + \frac{x(x-x_1)(x-1)}{x_2(x_2-x_1)(x_2-1)} \cdot 0.15 + \frac{x(x-x_1)(x-x_2)}{(1-x_1)(1-x_2)}$$

Its Lyapunov exponent is $\mu_0 \simeq 0.715$.

These maps are shown in figure 1.

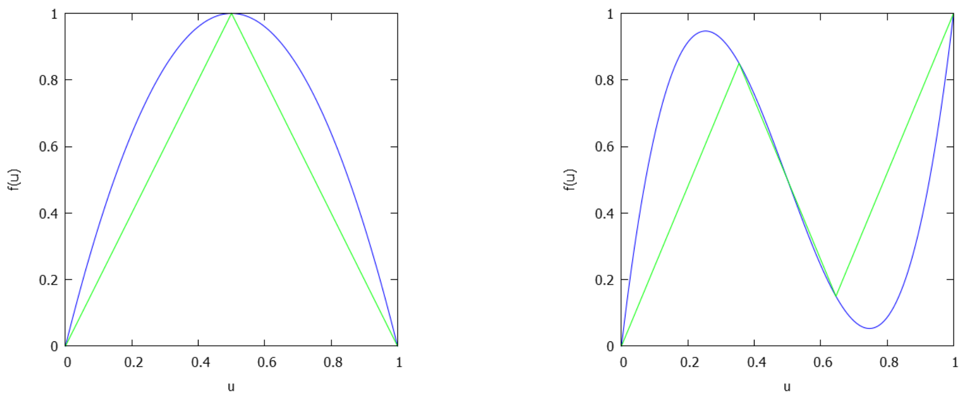

Figure 1: The tent map (left, green), the logistic map (left, blue), the 3-piecewise linear map (right, green) and the cubic-like map (right, blue)

4.1. Windows-of-delayed-synchronization for the Commanded by the Past Coupling

Since the Commanded by the Past Coupling admits a delayed synchronization of 1, we show in figure 2 the $y(t+1) - x(t)$ as a function of c graphs that the numerical procedure provides. We use the four free dynamics considered (corresponding to the tent, logistic,

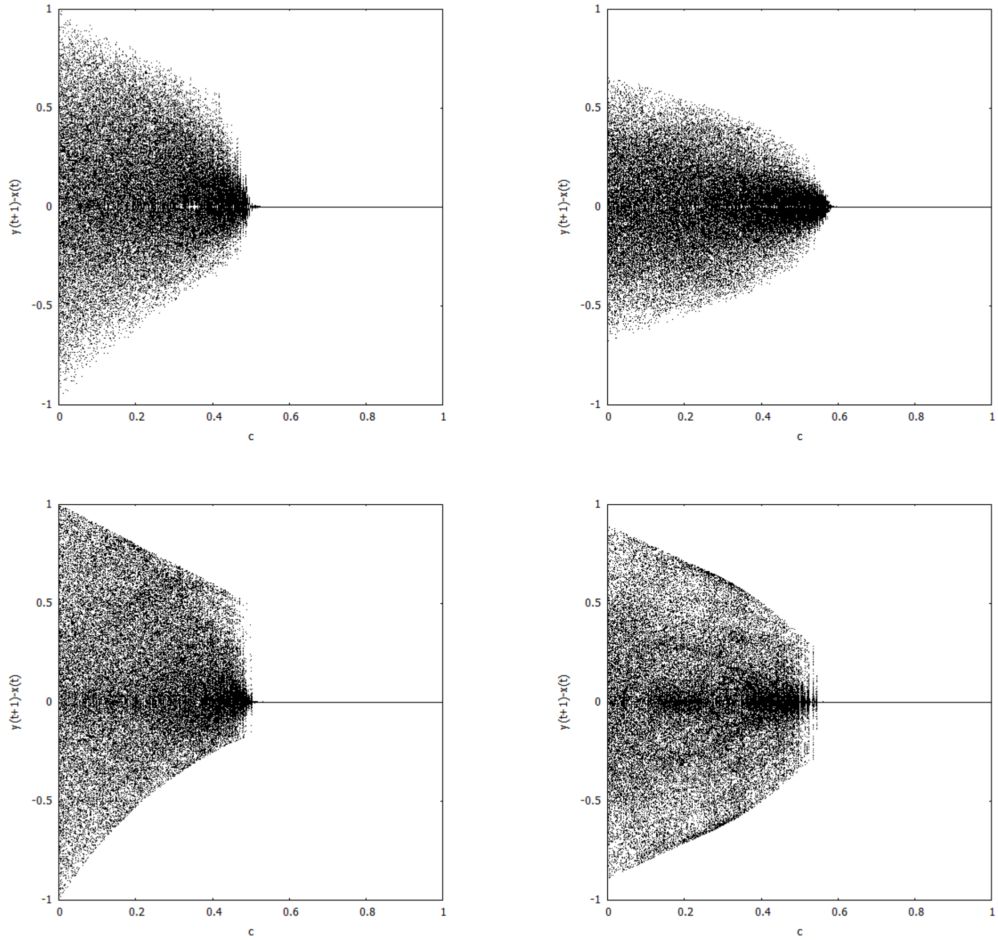

Figure 2: Graphs of the iterates $y(t+1)-x(t)$ as a function of c for the CPC of the tent map (at the top left), the logistic map (at the bottom left), the 3-piecewise linear map (at the top right) and the cubic-like map (at the bottom right)

3-piecewise linear and cubic-like maps) and we verify that all of corresponding couplings have non-empty windows-of-delayed-synchronization.

Along with this, since $\Delta t = 1$, $F_1 = 0$ and $G_1 = id$, we obtain

$$\mu_{ds} < 0 \Leftrightarrow \lim_{T\to+\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln \left| f' - cf' \right|_{f(s(t))} < 0 \Leftrightarrow \mu_0 + \ln|1-c| < 0 \Leftrightarrow$$
$$\Leftrightarrow 1 - e^{-\mu_0} < c < 1 + e^{-\mu_0} \Rightarrow$$
$$\Rightarrow WDS = \left]1 - e^{-\mu_0}, 1\right]$$

Given the values of the Lyapunov exponents for the chosen maps f, Proposition 2 provides the following windows-of-delayed-synchronization:

· tent and logistic maps: $WDS =]0.5,1]$

· 3-piecewise linear map: $WDS =]0.58(3),1]$

· cubic-like map: $WDS \simeq]0.511,1]$

These results confirm the ones that the numerical approach provides and that the figure 2 shows.

4.2. Windows-of-delayed-synchronization for the Commanded by the Future Coupling

Since the Commanded by the Future Coupling admits a delayed synchronization of 1, we show in figure 3 the $y(t+1)-x(t)$ as a function of c graphs that the numerical procedure provides. We use the four free dynamics considered (corresponding to the tent, logistic, 3-piecewise linear and cubic-like maps) and we verify that all of corresponding couplings have non-empty windows-of-delayed-synchronization.

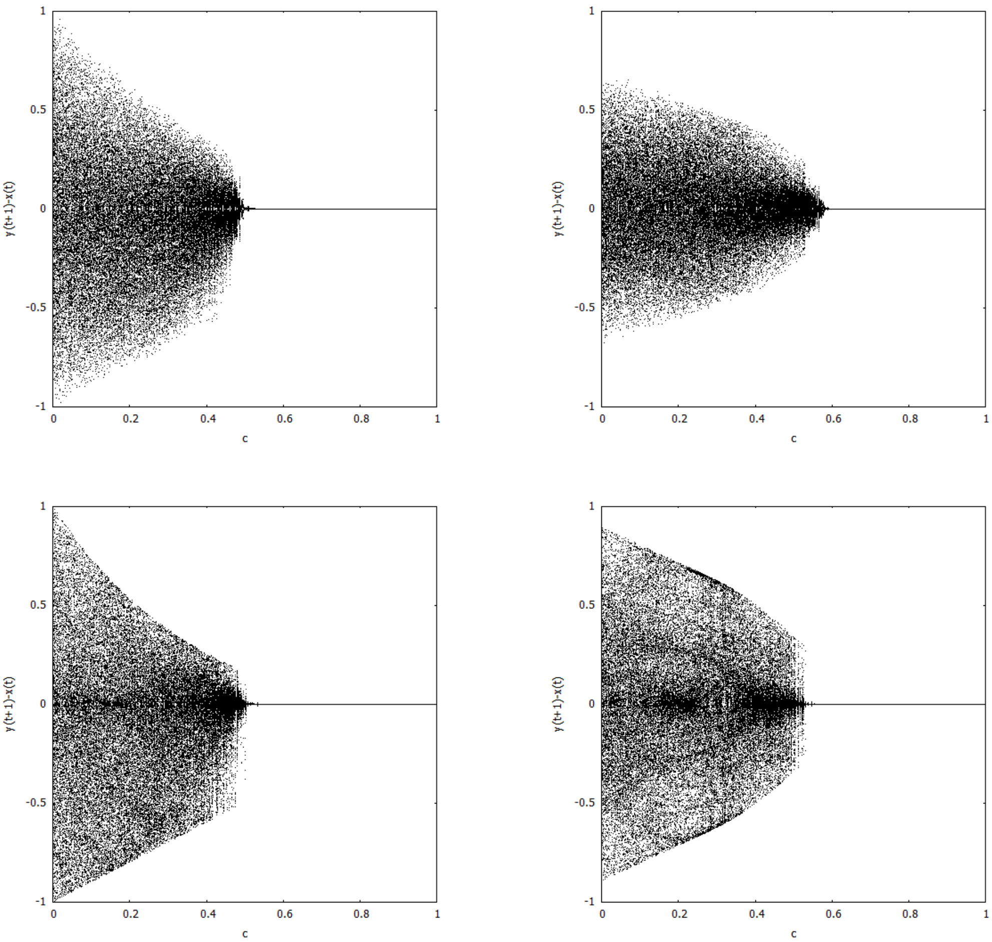

Figure 3: Graphs of the iterates $y(t+1)-x(t)$ as a function of c for the CFC of the tent map (at the top left), the logistic map (at the bottom left), the 3-piecewise linear map (at the top right) and the cubic-like map (at the bottom right)

Along with this, since $\Delta t = 1$, $F_1 = -f$ and $G_1 = 0$, we obtain

$$\mu_{ds} < 0 \Leftrightarrow \lim_{T\to+\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ln\left|f' - cf'\right|_{f(s(t))} < 0 \Leftrightarrow \mu_0 + \ln|1-c| < 0 \Leftrightarrow$$

$$\Leftrightarrow 1 - e^{-\mu_0} < c < 1 + e^{-\mu_0} \Rightarrow$$

$$\Rightarrow WDS = \left]1 - e^{-\mu_0}, 1\right]$$

Given the values of the Lyapunov exponents for the chosen maps f, Proposition 2 provides the following windows-of-delayed-synchronization:

· tent and logistic maps: $WDS =]0.5,1]$
· 3-piecewise linear map: $WDS =]0.58(3),1]$
· cubic-like map: $WDS \simeq]0.511,1]$

These results confirm the ones that the numerical approach provides and that the figure 3 shows.

4.3. Windows-of-delayed-synchronization for the Bidirectional Delayed Coupling

Since the Bidirectional Delayed Coupling admits a delayed synchronization of 1, we show in figure 4 the $y(t+1) - x(t)$ as a function of c graphs that the numerical procedure provides. We use the four free dynamics considered (corresponding to the tent, logistic, 3-piecewise linear and cubic-like maps) and we verify that all of corresponding couplings have non-empty windows-of-delayed-synchronization.

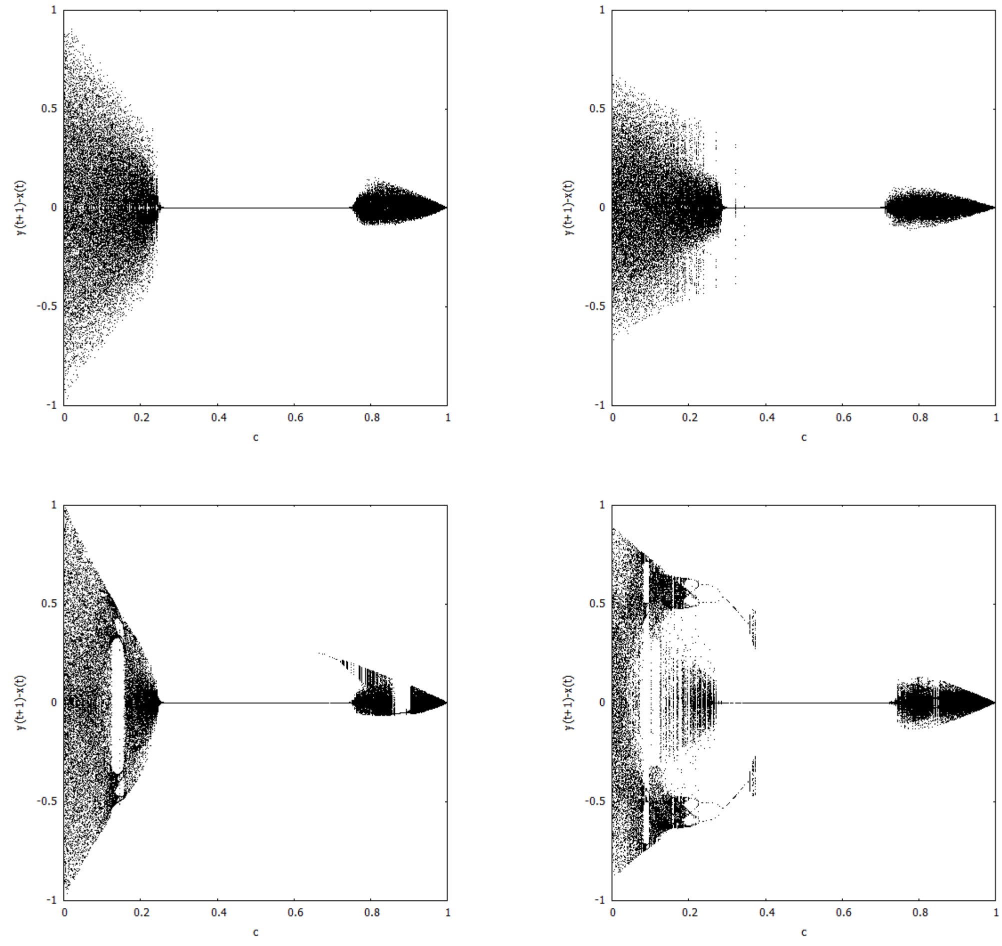

Figure 4: Graphs of the iterates $y(t+1) - x(t)$ as a function of c for the BDC of the tent map (at the top left), the logistic map (at the bottom left), the 3-piecewise linear map (at the top right) and the cubic-like map (at the bottom right)

Along with this, since $\Delta t = 1$, $F_1 = -f$ and $G_1 = id$, we obtain

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$$\mu_{ds} < 0 \Leftrightarrow \lim_{T\to+\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ln \left| f' - cf' - cf' \right|_{f(s(t))} < 0 \Leftrightarrow \mu_0 + \ln|1-2c| < 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{1-e^{-\mu_0}}{2} < c < \frac{1+e^{-\mu_0}}{2} \Rightarrow$$

$$\Rightarrow WDS = \left]\frac{1-e^{-\mu_0}}{2}, \frac{1+e^{-\mu_0}}{2}\right[$$

Given the values of the Lyapunov exponents for the chosen maps f, Proposition 2 provides the following windows-of-delayed-synchronization:

· tent and logistic maps: $WDS =]0.25,0.75[$
· 3-piecewise linear map: $WDS =]0.291(6),0.708(3)[$
· cubic-like map: $WDS \simeq]0.2557,0.7443[$

These results confirm the ones that the numerical approach provides and that the figure 4 shows.

4.4. Windows-of-delayed-synchronization for the Commanded by the Post-Future Coupling

Since the Commanded by the Post-Future Coupling admits a delayed synchronization of 2, we show in figure 5 the $y(t+2) - x(t)$ as a function of c graphs that the numerical procedure provides. We use the four free dynamics considered (corresponding to the tent, logistic, 3-piecewise linear and cubic-like maps) and we verify that all of corresponding couplings have non-empty windows-of-delayed-synchronization.

Along with this, since $\Delta t = 2$, $F_1 = -f$ and $G_1 = 0$, we obtain

$$\mu_{sd} < 0 \Leftrightarrow \lim_{T\to+\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ln \left| f' - cf' \right|_{(f\circ f)(s(t))} < 0 \Leftrightarrow \mu_0 + \ln|1-c| < 0 \Leftrightarrow$$

$$\Leftrightarrow 1 - e^{-\mu_0} < c < 1 + e^{-\mu_0} \Rightarrow$$

$$\Rightarrow WDS = \left]1 - e^{-\mu_0}, 1\right]$$

Given the values of the Lyapunov exponents for the chosen maps f, Proposition 2 provides the following windows-of-delayed-synchronization:

· tent and logistic maps: $WDS =]0.5,1]$
· 3-piecewise linear map: $WDS =]0.58(3),1]$
· cubic-like map: $WDS \simeq]0.511,1]$

These results confirm the ones that the numerical approach provides and that the figure 5 shows.

5. Conclusions

We defined delayed synchronization and obtained conditions that assure that a coupling admit it. The delayed synchronized solution is only an exponentially stable one for some values of the coupling-strength constant c, defining an window-of-delayed-synchronization. The windows that we obtained numerically for several couplings that admit delayed synchronization, using several chaotic maps, were confirmed analytically. In fact, we were succeeded in obtaining an analytical expression for the window-of-delayed-synchronization.

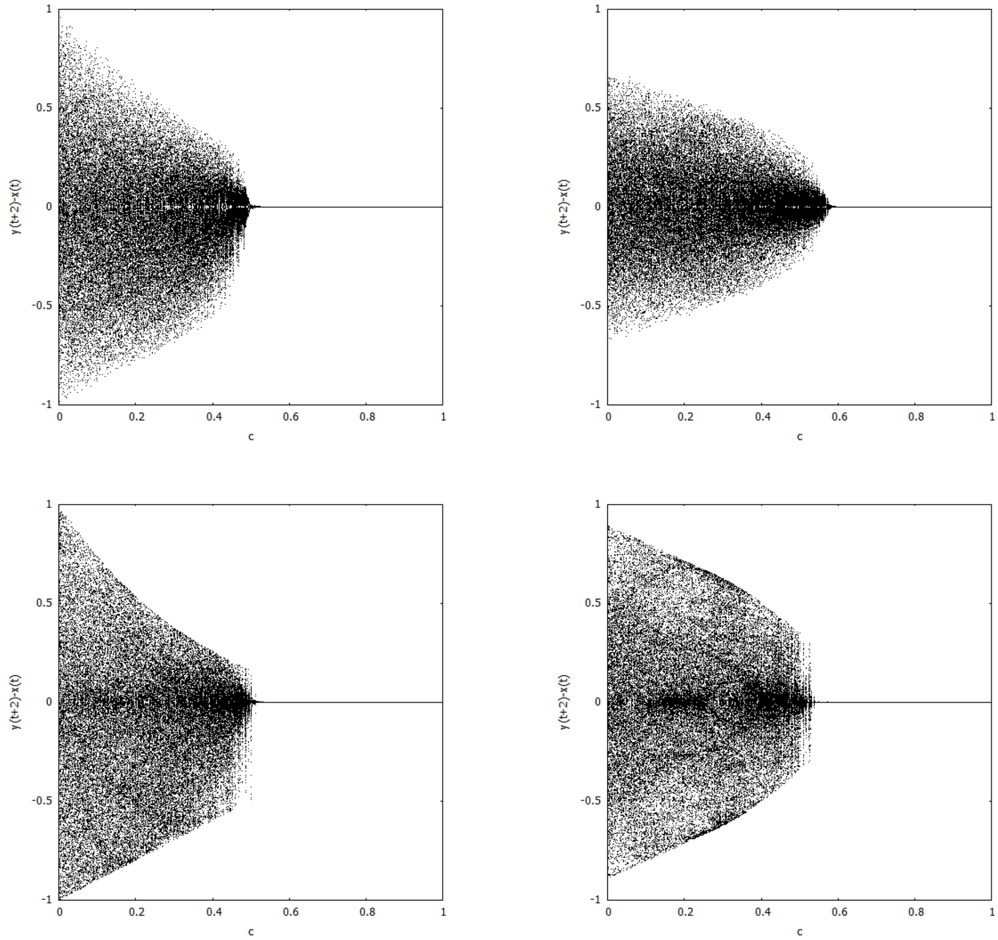

Figure 5: Graphs of the iterates $y(t+2) - x(t)$ as a function of c for the CPFC of the tent map (at the top left), the logistic map (at the bottom left), the 3-piecewise linear map (at the top right) and the cubic-like map (at the bottom right)

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