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Algorithmic Properties of Sparse Digraphs

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Abstract

The notions of bounded expansion [56] and nowhere denseness [58], introduced by Nešetřil and Ossona de Mendez as structural measures for undirected graphs, have been applied very successfully in algorithmic graph theory. We study the corresponding notions of directed bounded expansion and nowhere crownfulness on directed graphs, introduced by Kreutzer and Tazari [48]. The classes of directed graphs having those properties are very general classes of sparse directed graphs, as they include, on one hand, all classes of directed graphs whose underlying undirected class has bounded expansion, such as planar, bounded-genus, and H-minor-free graphs, and on the other hand, they also contain classes whose underlying undirected class is not even nowhere dense. We show that many of the algorithmic tools that were developed for undirected bounded expansion classes can, with some care, also be applied in their directed counterparts, and thereby we highlight a rich algorithmic structure theory of directed bounded expansion and nowhere crownful classes.

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1 Introduction

Structural graph theory has made a deep impact on the design of graph algorithms for hard problems. It provides a wealth of different tools for dealing with the intrinsic complexity of NP-hard problems on graphs and these methods have been applied very successfully in *algorithmic graph theory*, in *approximation theory*, *optimisation* and the design of *exact* and *parameterised algorithms* for problems on undirected graphs, see e.g. [11, 14, 16, 15, 17, 18, 28, 29, 66].

Concepts such as *tree width* or *excluded (topological) minors* as well as density based graph parameters such as *bounded expansion* or *nowhere denseness* capture important properties of graphs and make them applicable for algorithmic applications.

The notions of bounded expansion and nowhere denseness were introduced in [56] and [58] to capture structural sparseness of undirected graphs. Classes of bounded expansion are very general and properly generalise, for instance, planar graphs or more generally classes with excluded (topological) minors. But the concept goes far beyond excluded minor classes.

Starting with [56, 58], many algorithmic results for problems on classes of graphs excluding a fixed minor have been extended to the more general case of bounded expansion and nowhere dense classes of graphs, see e.g. [9, 13, 20, 21, 23, 24, 25, 33, 38, 43, 45, 49, 55, 61, 69, 71]. Furthermore, Demaine et al. [19] and Nadara et al. [53] analysed a range of real-world networks and showed that many of them indeed fall within the framework of bounded expansion. This shows that this concept captures many types of real world instances.

An important aspect of classes of bounded expansion and classes which are nowhere dense is that they can equivalently be defined in many different and seemingly unrelated ways: by the density of *bounded-depth minors*, by *low tree-depth colourings* [56], by *generalised colouring numbers* [73], by wideness properties such as *uniformly quasi-wideness* [57], by *sparse neighbourhood covers* [37, 38], *vc-density* [62], and many more. Each of these different aspects of the theory comes with its own set of algorithmic tools and many of the more advanced algorithmic results on bounded expansion classes mentioned above crucially rely on a combination of several of these techniques.

Developing a structural theory for directed graphs that yields classes of digraphs with a similarly broad algorithmic impact has so far not seen a comparable success as for the undirected case. The general goal is to identify structural parameters which define interesting and general classes of digraphs for which there is a comparably rich set of algorithmic tools. However, essentially all approaches, e.g. in [6, 7, 34, 40, 59, 67], of generalising even the well-understood and fairly basic concept of tree width to digraphs have failed to produce digraph parameters that come even near the wide spectrum of algorithmic applications that tree width has found. This even has led to claims that this programme cannot be successful and that such measures for digraphs cannot exist [35].

In this paper we exhibit examples of digraph parameters which we believe challenge this negative outlook on the potential of digraph parameters. Our main conceptual contribution is to give a positive example of a digraph parameter that satisfies the conditions of the programme outlined above: we identify a very general type of digraph classes which have a similar set of algorithmic tools available as their undirected counterparts. We believe that these classes give a positive answer to the question whether interesting graph parameters can successfully be generalised to the directed setting and we support this claim by algorithmic applications described below.

The classes of digraphs we study are classes of *directed bounded expansion* and *nowhere crownful classes* of digraphs which are modeled after the concepts of *bounded expansion* and *nowhere denseness* for undirected graphs, respectively. They were originally defined in [48], where basic properties of these classes were developed. In particular, it was shown that nowhere crownful classes can equivalently be defined in terms of *directed uniformly quasi-wideness*, analogous to its undirected counterpart, which easily implies fixed-parameter tractability of the directed dominating set problem on these classes. See Section 2 for details. The first improvement of these initial results appeared in [47], where structural properties of classes of digraphs of bounded expansion were studied. The main contribution of [47] was to establish their relation to a certain form of generalised colouring numbers, a concept which in the undirected setting has had huge algorithmic impact on the development of algorithms for nowhere dense and bounded expansion classes.

Our contributions.

These initial results are the starting point for our investigation in this paper. In addition to directed bounded expansion and nowhere crownful classes, we also define a new type of digraph classes which we call *bounded crownless expansion*.

Our main contributions are both structural and algorithmic. We show that classes of digraphs of directed bounded expansion, and especially classes of bounded crownless expansion, have structural properties very similar to their undirected counterpart. As a consequence, we are able to show that many of the algorithmic tools that were developed for undirected bounded expansion have their directed counterpart, resulting in a rich and diverse set of algorithmic techniques that can be applied in the design of algorithms for these classes. To the best of our knowledge, this is the first time that the generalisation of one of the widely studied and very general undirected graph parameter to the digraph setting has indeed led to a digraph concept with a similarly broad set of algorithmic tools as its undirected counterpart. We are therefore optimistic that classes of directed bounded expansion or crownless expansion will find a broad range of applications. We support this belief by providing several algorithmic results we describe next.

As a test case for these algorithmic techniques we use the directed variant of the (Distance-r) Dominating Set problem defined as follows. For a positive integer r, a *distance-r dominating set* in a digraph G is set $D \subseteq V(G)$ such that every $v \in V(G)$ is reachable by a directed path of length at most r from a vertex $d \in D$, i.e. $N_r^+(D) = V(G)$.

(Distance-r) Dominating Set is a common benchmark problem for the design of (parameterised or approximation) algorithms on graph classes with structural restrictions. It is NP-complete in general [42], and (under standard complexity theoretical assumptions) cannot be approximated better than up to a factor $\mathcal{O}(\log n)$ [64]. Better results can be achieved, e.g., on sparse graph classes, see e.g. [3, 4, 10, 12, 21, 22, 36, 39], but these classes do not contain classes of digraphs of bounded (crownless) expansion.

We study the complexity of the Directed (Distance-r) Dominating Set problem from the point of view of approximation, exact parameterised algorithms and kernelisation.

Approximation on directed bounded expansion. In [21], Dvořák proves a linear duality between distance-r dominating sets and r-scattered sets in classes of undirected bounded expansion. From this he derives an elegant polynomial-time constant-factor approximation algorithm on these classes of undirected graphs. Unfortunately, as we show in Section 3, no such duality holds in digraph classes of bounded directed expansion. In Theorem 7, we therefore use a different approach, inspired by recent results in [22], which is based on a combination of an LP-based approach and the characterisation of directed bounded expansion in terms of weak colouring numbers to obtain a constant-factor approximation algorithm for Directed-r Dominating Set on classes of directed bounded expansion.

Approximation on bounded crownless expansion. We then study classes of bounded crownless expansion. We first re-establish a polynomial duality between distance-r dominating sets and r-scattered sets on these classes. Towards this aim, we employ methods from stability theory, a branch of infinite model theory, developed in [50] in the digraph setting. The application of stability theory in this context is not straightforward. It is known that a class of (di)graphs which is closed under taking subgraphs is *stable*, if and only if, its underlying class of undirected graphs is nowhere dense [1]. However, classes of bounded crownless expansion in general are not nowhere dense and thus the stability theoretic techniques cannot be applied as such. Therefore, we have to carefully establish a situation in which stability is applicable, which then allows us to derive the polynomial duality theorem. As a consequence of this duality we also obtain a polynomial-time approximation algorithm for distance-r dominating sets (Corollary 11).

Parameterised complexity. We then study the parameterised complexity of the Distance-r Dominating Set problem. It is known that the problem is fixed-parameter tractable on nowhere crownful digraph classes [48] but the parameterised complexity of the problem on directed bounded expansion classes was still open. We first establish that classes of directed bounded expansion have *bounded directed neighbourhood depth*, a notion introduced in [26]. We then show that the methods developed in [26] can also be applied in the directed setting and establish that the Distance-r Dominating Set problem on classes of directed bounded expansion is fixed-parameter tractable (Theorem 14).

Kernelisation. Once fixed-parameter tractability is established, we turn our attention to the kernelisation problem for Distance-r Dominating Set. Recall that a kernelisation algorithm is a polynomial-time preprocessing algorithm that transforms a given instance into an equivalent one whose size is bounded by a function of the parameter only, independently of the overall input size. Fixed-parameter tractability implies the existence of a kernelisation algorithm, however, its output may be exponential or even larger in the parameter.

Starting with the groundbreaking work of Alber et al. [2], kernelisation for the Dominating Set and Distance-r Dominating Set problem on undirected graphs has received significant attention in the literature, see e.g. [8, 30, 31, 32]. In particular, Dominating Set admits polynomial kernels on graphs of bounded degeneracy [60]. The Distance-r Dominating Set problem admits a linear kernel on classes of bounded expansion [20], and an almost linear kernel on nowhere dense classes of graphs [45]. It is easy to observe that the result of [60] extends to digraphs of bounded degeneracy.

We show that the Distance-r Dominating Set problem admits a polynomial kernel on classes of bounded crownless expansion (Theorem 21). At a high level, our kernelisation algorithm follows the overall approach of [20] for undirected bounded expansion classes. Using our result above establishing the duality between distance-r dominating sets and r-scattered sets on bounded crownless expansion classes, the key property that remains to be established to apply the techniques from [20] are bounds on their distance-r neighbourhood complexity (the number of different intersections of r-balls with a given set). To establish these properties, we study the VC-dimension of set systems corresponding to r-neighbourhoods in digraphs of bounded directed expansion. In Section 4.2, we show that it is bounded on all classes of bounded crownless expansion which enables us to capture local separation properties in classes of bounded expansion. With this in place we can complete our kernelisation algorithm.

Steiner trees. As a further indication that digraphs of bounded expansion constitute a very useful notion, in Section 5 we consider the parameterised Directed Steiner Tree (Dst) problem, which is defined as follows. As input we are given a digraph G, a root $r \in V(G)$, a set $T \subseteq V(G) \setminus \{r\}$ of terminals and an integer k. The problem is to decide if there is a set $S \subseteq V(G) \setminus (\{r\} \cup T)$ of size at most k such that in $G[\{r\} \cup S \cup T]$ there is a directed path from r to every terminal T. The Steiner Tree problem is an intensively studied graph problem in computer science with many important applications. We refer to the textbook of Prömel and Steger [63] for background information. It is known for this parameterisation that both the directed and the undirected versions are W[2]-hard on general graphs [52], and even on graphs of degeneracy two [41]. On the positive side, Jones et al. [41] proved that the problem is fixed-parameter tractable on graphs excluding a topological minor when parameterised by the number of non-terminals. Their result is based on a preprocessing rule which allows to contract strongly connected subsets of terminal vertices to individual vertices. The authors furthermore show that if the subgraph induced by the terminals is required to be acyclic, then the problem becomes fixed-parameter tractable on graphs of bounded degeneracy. In this case, the strongly connected subsets of terminals have diameter 0. This suggests to consider the problem parameterised by the number k of non-terminals plus the maximal diameter s of a strongly connected component in the subgraph induced by the terminals. In fact, bounded expansion classes of digraphs are exactly those classes whose graphs have bounded degeneracy after bounded radius contractions. Therefore, the Steiner tree problem is fixed-parameter tractable on classes of bounded directed expansion under this parameterisation. On the other hand, it is straightforward to modify the example in [41] to show that the parameterisation $k + s$ cannot be replaced by taking only k as parameter: there exist classes of directed bounded expansion on which the directed Steiner tree problem parameterised by solution size k is W[2]-hard. Hence, we show that the results of Jones et al. [41] exactly identify classes of directed bounded expansion as those on which the Directed Steiner Tree problem parameterised by the number of non-terminal vertices and the maximal diameter of strongly connected components in the subgraph induced by the terminals is fixed parameter tractable (Theorem 23). At the time of writing, Jones et al. simply did not have the notions of bounded expansion available.

Connected dominating sets. Finally, we show that the restriction to classes of bounded crownless expansion is not sufficient to find efficient algorithms for the Strongly Connected Dominating Set problem and Strongly Connected Steiner Subgraph (Scss) problem, which is defined as the Steiner tree problem but here we need to find a set $S \subseteq V(G)$ of size at most k such that $G[S \cup T]$ is strongly connected. We prove that there exist classes of bounded crownless expansion on which the Strongly Connected Dominating Set problem and the Strongly Connected Steiner Subgraph problem remain W[1]-hard (Theorems 30 and 31).

Summary. The results reported above demonstrate that classes of bounded (crownless) expansion indeed exhibit a very rich set of algorithmic tools, broad enough so that even recent sophisticated algorithms for undirected bounded expansion can be extended to the digraph setting. We therefore believe that these concepts are new and interesting digraph parameters which hold the promise for further algorithmic applications. The hardness results for strongly connected dominating sets, on the other hand, indicate that for problems which in addition require control over strong connectivity, one may have to consider further restrictions, e.g. by combining directed expansion with directed treewidth. We leave this for future research.

2 Directed Minors and Directed Bounded Expansion

We refer to [5] for standard notation and background on digraph theory. Let G be a digraph, let $v \in V(G)$ and let $r \geq 1$ be an integer. The *r-out-neighbourhood* of v, denoted

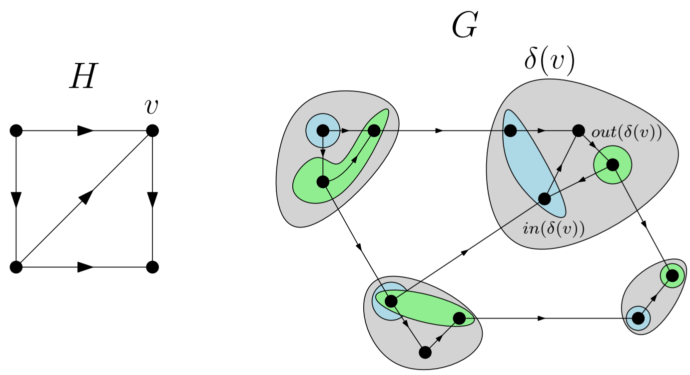

Figure 1 The graph H (left) is a directed minor of the graph G (right).

by $N^+_{G,r}(v)$, or just $N^+_r(v)$ if G is understood, is defined as the set of vertices u in G such that G contains a directed path of length at most r from v to u. We write $N^+(v)$ for $N^+_1(v) \setminus \{v\}$. The *r-in-neighbourhood* $N^-_{G,r}(v)$ and $N^-(v)$ are defined analogously. The *out-degree* of a vertex $v \in V(G)$ is $d^+(v) := |N^+(v)|$, its *in-degree* is $d^-(v) := |N^-(v)|$ and its *degree* is $d(v) := |N^+(v)| + |N^-(v)|$. The *minimum out-degree* of G is defined as $\delta^+(G) := \min\{d^+(v) \ : \ v \in V(G)\}$, *minimum in-degree* and *minimum degree* are defined analogously. A set $U \subseteq V(G)$ is *r-scattered* if there is no $v \in V(G)$ and $u_1, u_2 \in U$ with $u_1 \neq u_2$ and $u_1, u_2 \in N^+_r(v)$. If the arc relation of a digraph G is symmetric, i.e. if $(u, v) \in E(G)$ implies $(v, u) \in E(G)$, then we speak of an *undirected graph.* If G is a digraph, we write $\bar{G}$ for the *underlying undirected graph of* G, which has the same vertices as G and for each arc $(u, v) \in E(G)$ we have $(u, v) \in E(\bar{G})$ and $(v, u) \in E(\bar{G})$. Note that $|E(G)| \leq \left|E(\bar{G})\right| \leq 2\,|E(G)|$.

Directed minors. We are going to work with directed minors and directed topological minors. The following definition of directed minors is from [48]. A digraph H has a *directed model* in a digraph G if there is a function δ mapping vertices $v \in V(H)$ of H to sub-graphs $\delta(v) \subseteq G$ and arcs $e \in E(H)$ to arcs $\delta(e) \in E(G)$ such that

1. if $v \neq u$, then $\delta(v) \cap \delta(u) = \emptyset$;
2. if $e = (u, v)$ and $\delta(e) = (u', v')$ then $u' \in \delta(u)$ and $v' \in \delta(v)$. For $v \in V(H)$ let $\mathrm{in}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e=(u,v)\in E(H)} V(\delta(e))$ and $\mathrm{out}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e=(v,w)\in E(H)} V(\delta(e))$;
3. we require that for every $v \in V(H)$ (a) there is a directed path in $\delta(v)$ from every $u \in \mathrm{in}(\delta(v))$ to every $u' \in \mathrm{out}(\delta(v))$; (b) there is at least one source vertex $s_v \in \delta(v)$ that reaches (by a directed path in $\delta(v)$) every element of $\mathrm{out}(\delta(v))$; (c) there is at least one sink vertex $t_v \in \delta(v)$ that can be reached (by a directed path in $\delta(v)$) from every element of $\mathrm{in}(\delta(v))$.

A digraph H has a *directed model* in a digraph G if there is a function δ mapping vertices $v \in V(H)$ of H to sub-graphs $\delta(v) \subseteq G$ and arcs $e \in E(H)$ to arcs $\delta(e) \in E(G)$ such that

1. if $v \neq u$, then $\delta(v) \cap \delta(u) = \emptyset$;
2. if $e = (u, v)$ and $\delta(e) = (u', v')$ then $u' \in \delta(u)$ and $v' \in \delta(v)$.
3. Furthermore, we require that for each $v \in V(H)$ there are non-empty sets $\mathrm{in}(\delta(v))$ and $\mathrm{out}(\delta(v))$ such that $\mathrm{in}(\delta(v))$ contains the head of every arc $\delta((u, v))$ and $\mathrm{out}(\delta(v))$ contains the tail of every arc $\delta((v, u))$ and for every $s \in \mathrm{in}(\delta(v))$ and $t \in \mathrm{out}(\delta(v))$ there is a path in $\delta(v)$ from s to t.

We write $H \preccurlyeq G$ if H has a directed model in G and call H a *directed minor* of G. We call the sets $\delta(v)$ for $v \in V(H)$ the *branch-sets* of the model.

For $r \geq 0$, a digraph H is a *depth-r minor* of a digraph G, denoted as $H \preccurlyeq_r G$, if there exists a directed model δ of H in G in which for all $v \in V(H)$ and all $s \in \mathrm{in}(\delta(v))$ and $t \in \mathrm{out}(\delta(v))$ there is a path from s to t in $\delta(v)$ of length $\leq r$.

We write $H \preccurlyeq G$ if H has a directed model in G and call H a *directed minor* of G. We call the sets $\delta(v)$ for $v \in V(H)$ the *branch-sets* of the model.

For $r \geq 0$, a digraph H is a *depth-r minor* of a digraph G, denoted as $H \preccurlyeq_r G$, if there exists a directed model of H in G in which the length of all the paths in the branch sets of the model described in 3a)-c) above are bounded by r. Note that every subgraph of G is a depth-0 minor of G.

Directed topological minors. A digraph H is a *topological minor* of a digraph G if there is an injective function δ mapping vertices $v \in V(H)$ to vertices of $V(G)$ and arcs $e \in E(H)$ to directed paths in G such that if $e = (u, v) \in E(H)$, then $\delta(e)$ is a path from $\delta(u)$ to $\delta(v)$ in G which is internally vertex disjoint from all vertices $\delta(w)$ (for $w \in V(H)$) and all paths $\delta(e')$ (for $e' \in E(H)$, $e' \neq e$). For $r \geq 0$, H is a *topological depth-r minor* of G, written $H \preccurlyeq_r^{top} G$, if it is a topological minor and all paths $\delta(e)$ have length at most $2r$.

Grads, bounded expansion and crowns. Let G be a digraph and let $r \geq 0$. The *greatest reduced average density of rank r* (short *grad*) of G is

$$\nabla_r(G) \coloneqq \max\left\{\frac{|E(H)|}{|V(H)|} \ : \ H \preccurlyeq_r G\right\}$$

and its *topological greatest average density of rank r* (short *top-grad*) is

$$\widetilde{\nabla}_r(G) \coloneqq \max\left\{\frac{|E(H)|}{|V(H)|} \ : \ H \preccurlyeq_r^{top} G\right\}.$$

▶ **Definition 1.** *A class $\mathcal{C}$ of digraphs has* bounded expansion *if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $r \geq 0$ we have $\nabla_r(G) \leq f(r)$ (or equivalently, $\widetilde{\nabla}_r(G) \leq f(r)$) for all $G \in \mathcal{C}$.*

A *crown* of order q is a 1-subdivision of a clique of order q with all arcs oriented away from the subdivision vertices, that is, the digraph S_q with vertex set $\{v_1, \ldots, v_q\} \cup \{v_{ij} : 1 \leq i < j \leq q\}$ and arc set $\{(v_{ij}, v_i), (v_{ij}, v_j) : 1 \leq i < j \leq q\}$.

▶ **Definition 2.** *A class $\mathcal{C}$ of digraphs has* bounded crownless expansion *if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $r \geq 0$ we have $\nabla_r(G) \leq f(r)$ and $S_{f(r)} \not\preccurlyeq_r G$ for all $G \in \mathcal{C}$.*

Generalised colouring numbers. We next review the definition of generalised colouring numbers in the directed setting. Let G be a digraph. By $\Pi(G)$ we denote the set of all linear orders of $V(G)$. For $r \geq 0$, we say that u is *weakly r-reachable* from v with respect to an order $L \in \Pi(G)$ if there is a path P of length at most r, connecting u and v, *in either direction*, such that u is minimum among the vertices of P with respect to L. By $\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]$ we denote the set of vertices that are weakly r-reachable from v with respect to L. We define the *weak r-colouring number* $\mathrm{wcol}_r^{\rightleftarrows}(G)$ of G as

$$\mathrm{wcol}_r^{\rightleftarrows}(G) \coloneqq \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]\right|.$$

▶ **Theorem 3** ([47]). *A class $\mathcal{C}$ of digraphs has bounded expansion if, and only if, there is $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq f(r)$ for all $G \in \mathcal{C}$ and all $r \geq 1$.*

The next lemma shows that the weak r-colouring numbers are very useful to describe local separation properties in graphs of bounded expansion. The lemma is immediate by the definition of $\mathrm{WReach}_r^{\rightleftarrows}$.

▶ **Lemma 4.** *Let G be a digraph and let $r \geq 1$. Let P be a path of length at most r with endpoints u and v in either direction. Let L be an order of $V(G)$ and let z be the minimal vertex of P with respect to L. Then $z \in \mathrm{WReach}_r^{\rightleftarrows}[G, L, u] \cap \mathrm{WReach}_r^{\rightleftarrows}[G, L, v]$.*

We will also need an efficient algorithm to compute good weak reachability orders. We show in the full version that this is possible. All statements marked with $(\star)$ are proved in the full version [44].

▶ **Theorem 5** $(\star)$**.** *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. There exists a function $f : \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm which for an input graph $G \in \mathcal{C}$ and $r \in \mathbb{N}$ computes an order L with $|\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]| \leq f(r)$ for all $v \in V(G)$.*

3 Approximation of distance-r dominating sets and duality between distance-r dominating sets and r-scattered sets

In this section we study the duality between distance-r dominating sets and r-scattered sets and prove that for every fixed value $r \in \mathbb{N}$ the Distance-r Dominating Set problem admits a constant-factor approximation on every class of digraphs of bounded expansion. Given a digraph G, a set $U \subseteq V(G)$ is *r-scattered* if there is no $v \in V(G)$ and $u_1, u_2 \in U$ with $u_1 \neq u_2$ and $u_1, u_2 \in N_r^+(v)$. We write $\gamma_r(G)$ for the size of a minimum distance-r dominating set in a digraph G and $\alpha_{2r}(G)$ for the size of a maximum r-scattered set in G. Observe that in undirected graphs an r-scattered set corresponds to a distance-$2r$ independent set, which explains the index in the notation $\alpha_{2r}(G)$.

Clearly, every vertex $v \in V(G)$ can dominate at distance r at most one vertex of an r-scattered set. Hence we have $\alpha_{2r}(G) \leq \gamma_r(G)$ for every digraph G. In general, $\gamma_r(G)$ is not bounded in terms of $\alpha_{2r}(G)$. Dvořák proved in [21] that on classes of undirected graphs of bounded expansion $\gamma_r(G)$ is linearly bounded by $\alpha_{2r}(G)$, where the linear factor is the undirected weak colouring number $\mathrm{wcol}_{2r}(G)^2$, i.e., on undirected graphs the inequality $\gamma_r(G) \leq \mathrm{wcol}_{2r}(G)^2 \cdot \alpha_{2r}(G)$ holds. Furthermore, he derived an elegant linear time constant-factor approximation algorithm for the Distance-r Dominating Set problem.

As a first negative result we prove that no such duality theorem holds on digraphs of bounded expansion.

▶ **Theorem 6.** *There is a class of directed bounded expansion such that for every constant c we have $\gamma_1(G) \geq c$ for infinitely many $G \in \mathcal{C}$ and $\alpha_2(G) = 2$ for all $G \in \mathcal{C}$.*

Hence, we cannot follow the duality based approach to compute approximations for the Distance-r Dominating Set problem on classes of directed bounded expansion. Instead, we follow a very recent approach of Dvořák [22], which combines rounding of a linear program and a greedy choice based on the generalised colouring numbers. We consider the following linear programs. For each vertex $v \in V(G)$ we have one variable x_v.

Distance-r Dominating Set LP

- **Objective:** minimise $\gamma_r^\star = \sum_{v \in V(G)} x_v$
- **Subject to:** $\sum_{u \in N_r^-[v]} x_u \geq 1$ for all $v \in V(G)$
- **Constraints:** $x_v \geq 0$ for all $v \in V(G)$.

The dual linear program is the following program for r-Scattered Set.

r-Scattered Set LP

- **Objective:** maximise $\alpha^\star_{2r} = \sum_{v\in V(G)} x_v$
- **Subject to:** $\sum_{u\in N^-_r[v]} x_u \le 1$ for all $v \in V(G)$
- **Constraints:** $x_v \ge 0$ for all $v \in V(G)$.

Integer solutions for the Distance-r Dominating Set LP correspond to minimum size distance-r dominating sets in G, and analogously, integer solutions for the r-Scattered Set LP correspond to maximum size r-scattered sets in G. Observe that since the linear programs are dual to each other, for every graph G and every positive integer r we have

$$\alpha_{2r}(G) \le \alpha^\star_{2r}(G) = \gamma^\star_r(G) \le \gamma_r(G),$$

while in general $\gamma_r(G)$ is not functionally bounded by $\alpha_{2r}(G)$. Also note that $\alpha^\star_{2r}(G)$ and $\gamma^\star_r(G)$ can be determined exactly in polynomial time by solving the linear programs that define them.

Dvořák in [22] proved that $\gamma_r(G)$ is bounded linearly by $\gamma^\star_r(G)$ on classes of undirected graphs of bounded expansion. We are able to prove an analogous statement in digraphs of bounded expansion. Furthermore, the theorem is constructive and yields a polynomial-time approximation algorithm.

▶ **Theorem 7** ($\star$). *Let $\mathcal{C}$ be a class of directed bounded expansion and let $r \in \mathbb{N}$. Then there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm which on input $G \in \mathcal{C}$ computes a distance-r dominating set of G of size at most $f(r) \cdot \gamma^\star_r(G)$.*

We show next that for classes of bounded crownless expansion the values γ_r and α_{2r} are polynomially related. Thus, for such classes we can re-establish the duality between d-domination and r-scattered sets which we proved to fail in the general directed setting. Our proof is algorithmic in the sense that we apply the directed analogue of the algorithm of Dvořák [21] to the digraph G and prove that it finds both a distance-r dominating set and a polynomially smaller r-scattered set. Without requiring the duality to be polynomial we could have used standard Ramsey-type arguments. To establish a polynomial relation between the two parameters, we facilitate tools from stability theory, related to those developed in [50] and [46]. We first explain the stability theoretic tools used in the sequel.

Let T be a (rooted) binary tree, where each vertex (except the root) is marked as a left or right successor of its predecessor. We call w a *left (right) descendant* of v if the first successor on the unique v-w path in T is a left (right) successor.

Fix an enumeration $a_1, \ldots, a_\ell$ of a set $A \subseteq V(G)$. The *r-independence tree* of $(a_1, \ldots, a_\ell)$ is a binary tree which is constructed recursively as follows. We make a_1 the root of the tree. Assume that $a_1, \ldots, a_i$ have already been inserted into the tree. In order to insert the next element a_{i+1}, we follow a root-leaf path to find a position for it. Starting from the root a_1, at each point we are at some node a_j and we have to decide whether we continue along the left or to the right branch at a_j. If there is an element u such that $a_j, a_{i+1} \in N^+_r(u)$, we continue along the right branch at a_j, otherwise we follow the left branch. If there is no right successor (or left successor, respectively), we insert a_{i+1} as a right (or left) child of a_j.

▶ **Lemma 8** ($\star$). *Let T be a rooted binary tree and let $t \ge 1$ be an integer. Assume that no root-leaf path in T contains a sub-sequence $a_1, \ldots, a_t$ (of pairwise distinct elements) such that a_j is a right descendant of a_i for all $1 \le i < j \le t$. If T has height at most h, then T has at most h^{t+1} vertices.*

The following lemma is proved using the Finite Canonical Ramsey Theorem.

▶ **Lemma 9** ($\star$). *For all integers r, c, K there exists an integer N such that the following property holds. Let G be a digraph with maximum out-degree at most c and let S, T be subsets of vertices of G, such that $|T| \geq N$ and for each $t, t' \in T$ there exist a vertex $s = s(t, t') \in S$, a directed path $P_{s,t}$ of length at most r from s to t and a directed path $P_{s,t'}$ of length at most r from s to t'. Then G contains a crown of order K as a depth-r minor.*

We can now prove the polynomial duality theorem.

▶ **Theorem 10.** *Let G be a digraph with $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq c$ and $S_q \not\preccurlyeq_r G$. Then there exists $N = N(c, q, r)$ such that $\gamma_r(G) \in \mathcal{O}(\alpha_r(G)^N)$.*

Proof. The following algorithmic construction corresponds to the algorithm of Dvořák for undirected graphs [21]. Fix an order L witnessing that $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq c$. We compute a distance-$r$ dominating set D as follows. Initialise $D := \emptyset$, $A := \emptyset$ and $M := V(G)$. While there is a vertex $v \in M$, the set of non-dominated vertices, pick the smallest such vertex v with respect to L. Add v to A and $\mathrm{WReach}_{2r}^{\rightleftarrows}[G, L, v]$ to D. Mark all newly dominated vertices, that is, remove $N_r^+[\mathrm{WReach}_{2r}^{\rightleftarrows}[G, L, v]]$ from M. If $M = \emptyset$, return D. Clearly, D is a distance-r dominating set of G.

We now prove that we find a large r-scattered subset of A. Construct the undirected graph H with vertex set A such that two vertices $a, b \in A$ are connected in H if there is $u \in V(G)$ such that $a, b \in N_r^+(u)$. An independent set in H corresponds to an r-scattered subset of A in G.

We claim that every vertex $u \in V(G)$ satisfies $|N_r^+(u) \cap A| \leq c$. Fix $u \in V(G)$. Assume towards a contradiction that $|N_r^+(u) \cap A| > c$. For each $a \in N_r^+(u) \cap A$ fix a path P_{ua} of length at most r from u to a. For each path P_{ua}, denote by m_{ua} its minimal element with respect to L. Since $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq c$, we have $|\{m_{ua} : N_r^+(u) \cap A\}| \leq c$. Since we have more than c paths P_{ua}, there must be two paths P_{ua_1}, P_{ua_2}, $a_1 \neq a_2$, which have the same element m as their minimal element. Without loss of generality assume that $a_1 < a_2$. Since m is the smallest vertex on the path P_{ua_1}, the subpath of P_{ua_1} between m and a_1 certifies that m is weakly r-reachable from a_1. Hence, when a_1 was added to A, the element m was added to the set D. Now, the subpath of P_{ua_2} between m and a_2 shows that a_2 is at distance at most r from m, and hence a_2 is marked as dominated at this point. This again proves $a_2 \notin A$, a contradiction.

We now build the r-independence tree T of $a_1, \ldots, a_\ell$ (the enumeration of A with respect to L). Using Lemma 9, we conclude that there is $N'(c, r, q)$ such that T does not contain a path with $s = N'$ right descendants. Let $N := N' + 1$.

Hence, by Lemma 8, if we have $|A| > (m + N)^N$, then we find a sequence of length m with all left descendants. This set is r-scattered, which proves the theorem. ◀

Clearly, the r-independence tree of a sequence of vertices can be computed in polynomial time, which gives us the following corollary.

▶ **Corollary 11.** *Let $\mathcal{C}$ be a class of digraphs which has bounded crownless expansion. Then for every $r \in \mathbb{N}$, there is a polynomial-time algorithm which computes a distance-r dominating set D with $|D| \leq p(\gamma_r(G))$ for some polynomial p.*

4 Parameterised complexity of Distance-r Dominating Set

In this section we study the parameterised complexity of the Distance-r Dominating Set problem on classes of directed bounded expansion. We follow the approach of [26] and establish that digraphs of bounded expansion have bounded neighbourhood depth (which corresponds to having bounded semi-ladder index in that paper). We then show that a straight forward modification of the Semi-ladder-algorithm of [26] for the Distance-r Dominating Set problem on undirected graphs of bounded neighbourhood depth is an fpt-algorithm on digraphs of bounded expansion.

Let $\mathcal{F}$ be a family of subsets of some universe U. A *chain* in $\mathcal{F}$ is a family $\mathcal{H} \subseteq \mathcal{F}$ such that for all $X, Y \in \mathcal{H}$, we have either $X \subseteq Y$ or $Y \subseteq X$. The *depth* of $\mathcal{F}$ is the cardinality of the longest chain in $\mathcal{F}$. The *intersection closure* of $\mathcal{F}$ is the family of all sets of the form $X_1 \cap X_2 \cap \ldots \cap X_n$ for some $n \in \mathbb{N}$ and $X_1, X_2, \ldots, X_n \in \mathcal{F}$. For $n = 0$ we assume by convention that the intersection of an empty sequence of sets is equal to U, thus the intersection closure always contains the universe U.

▶ **Definition 12.** *Let G be a digraph and $r \in \mathbb{N}$. The* r-neighbourhood depth *of G, denoted* $\mathsf{depth}_r(G)$*, is the depth of the intersection closure of the family $\{N_r^+(v) : v \in V(G)\}$. We say that a graph class $\mathcal{C}$ has* bounded neighbourhood depth *if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ we have* $\mathsf{depth}_r(G) \leq f(r)$*.*

We show that classes of directed bounded expansion have bounded neighbourhood depth.

▶ **Lemma 13** ($\star$)**.** *Let $\mathcal{C}$ be a class of directed bounded expansion. Then $\mathcal{C}$ has bounded neighbourhood depth.*

4.1 Fixed-parameter tractability on bounded expansion classes

In this section we show that a straightforward modification of the so-called Semi-ladder-algorithm of [26] is an fpt-algorithm on digraphs of bounded neighbourhood depth.

We say that a set of vertices A r*-dominates* another set of vertices B if $B \subseteq N_r^+(A)$. The Semi-ladder-algorithm maintains two sets: $D, S \subset V(G)$. Initially, both are empty, and at each moment, D will have at most k elements. The algorithm proceeds in rounds, each consisting of two steps: first the S*-step* and then the D*-step*.

S-step: Check whether D r-dominates $V(G)$. If so, terminate and output D as an r-dominating set of size at most k. Otherwise, pick any vertex u which is not r-dominated by D and add it to S.

D-step: Check whether some set of at most k vertices r-dominates S. If so, set D to be any such set and proceed to the next round. Otherwise, terminate and conclude that there is no r-dominating set of size at most k.

As in the undirected case one can easily implement each D-**step** using standard dynamic programming on subsets of S in time $\mathcal{O}(2^{|S|} \cdot |S| \cdot n)$. Since at each round the size of S grows by exactly 1, it is not hard to see that the ℓth round of the algorithm can be implemented in time $\mathcal{O}(2^{\ell} \cdot \ell n + km)$, and hence the time needed to execute it for L rounds is bounded by $\mathcal{O}(2^{L} \cdot Ln + kLm)$.

Clearly, the algorithm correctly decides whether a graph contains a distance-r dominating set of size at most k. It remains to show that it is in fact a fixed-parameter algorithm on classes of directed bounded expansion. We prove this by showing that the neighbourhood depth gives an upper bound on the number L of rounds executed by the algorithm.

▶ **Theorem 14** ($\star$)**.** *Let $\mathcal{C}$ be a class with bounded neighbourhood depth and let $r \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ there is a constant $L \in \mathbb{N}$, depending only on $k, r, \mathcal{C}$ and computable from k for fixed r and $\mathcal{C}$, such that the* Semi-ladder*-algorithm terminates after at most L rounds when applied to any $G \in \mathcal{C}$ and k. In particular, if G has n vertices and m edges, then the running time is bounded by $f(k) \cdot m$ for some computable function f.*

4.2 VC-dimension and neighbourhood complexity

Towards the goal of developing a kernelisation algorithm for the DISTANCE-r DOMINATING SET problem on classes of bounded crownful expansion, we first study the VC-dimension and neighbourhood complexity of radius-r balls in classes of directed bounded expansion.

Let $\mathcal{F} \subseteq 2^A$ be a family of subsets of a set A. For a set $X \subseteq A$, we denote $X \cap \mathcal{F} = \{X \cap F : F \in \mathcal{F}\}$. The set X is *shattered by* $\mathcal{F}$ if $X \cap \mathcal{F} = 2^X$. The *Vapnik-Chervonenkis dimension*, short *VC-dimension*, of $\mathcal{F}$ is the maximum size of a set X that is shattered by $\mathcal{F}$.

Note that if $\mathcal{F}$ has VC-dimension d, then also $B \cap \mathcal{F}$ for every subset $B \subseteq A$ of the ground set has VC-dimension at most d. The following theorem was first proved by Vapnik and Chervonenkis [72], and rediscovered by Sauer [68] and Shelah [70]. It is often called the Sauer-Shelah lemma in the literature.

▶ **Theorem 15.** *If $|A| \leq n$ and $\mathcal{F} \subseteq 2^A$ has VC-dimension d, then $|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i} \in \mathcal{O}(n^d)$.*

The study of the distance-r dominating set problem in context of bounded VC-dimension motivates the following definition. Let G be a digraph and $r \geq 1$. The *distance-r VC-dimension* of G is the VC-dimension of the set family $\{N_r^-(v) \colon v \in V(G)\}$ over the set $V(G)$. If $X \subseteq V(G)$, the *distance-r neighbourhood complexity of X in G*, denoted $\nu^-(G)$, is defined by

$$\nu^-(G, X) \coloneqq \left|\{N_r^-(v) \cap X \ : \ v \in V(G)\}\right|.$$

Analogously, one can define the *distance-r out-neighbourhood complexity* when using $N_r^+(v)$ and the *distance-r mixed neighbourhood complexity* when using $(N_r^+(v) \cup N_r^-(v))$ in the above definition and our proofs can be analogously carried out for these measures.

It was proved in [65] that a class $\mathcal{C}$ of undirected graphs has bounded expansion, if and only if, for every $r \geq 1$ there is a constant c_r such that for all $G \in \mathcal{C}$ and all $X \subseteq V(G)$ we have $\nu(G, X) \leq c_r \cdot |X|$ (where ν denotes the undirected neighbourhood complexity). The analogous statement for classes of directed graphs does not hold, not even for $r = 1$, as pointed out in [47]. However, we prove that the distance-r neighbourhood complexity of a digraph can be bounded in terms of its weak r-colouring numbers.

Using Lemma 4 we can well control the interaction of distance-r neighbourhoods with a set X. Let G be a digraph and let L be a linear order on $V(G)$ and let $r \geq 1$. Let $A \subseteq V(G)$ be enumerated as $a_1, \ldots, a_{|A|}$, consistently with the order. For $v \in V(G)$ let $D_r^-(v, A)$ denote the *distance-r vector* of v and A, that is, the vector $(d_1, \ldots, d_{|A|})$, where $d_i = \mathrm{dist}(a_i, v)$ if $0 \leq \mathrm{dist}(a_i, v) \leq r$, and ∞ otherwise. Here $\mathrm{dist}(a_i, v)$ is the length of a shortest path from a_i to v.

▶ **Lemma 16** (⋆). *Let G be a digraph, let $X \subseteq V(G)$ and let $r \geq 1$. Let $c \coloneqq \mathrm{wcol}_r^{\rightleftarrows}(G)$. Then the number of distinct distance-r vectors $D_r^-(v, X)$ is bounded by $((r+2) \cdot c \cdot |X|)^c$, and in particular, $\nu_r^-(G, X) \leq ((r+2) \cdot c \cdot |X|)^c$.*

▶ **Corollary 17.** *Let G be a digraph and $r \geq 1$. Then the distance-r VC-dimension of G is bounded by $(r+2) \cdot (2\mathrm{wcol}_r^{\rightleftarrows}(G))^2$.*

4.3 Kernelisation on classes of bounded crownful expansion

Recall that a kernelisation algorithm is a polynomial-time preprocessing algorithm that transforms a given instance into an equivalent one whose size is bounded by a function of the parameter only, independently of the overall input size. We are mostly interested in kernelisation algorithms whose output guarantees are polynomial in the parameter. In this section we prove that for every fixed value of $r \geq 1$, the distance-r dominating set problem admits a polynomial kernel on every class of bounded crownless expansion.

Our strategy follows on a high level that of Drange et al. [20] for kernelisation on classes of undirected bounded expansion. The first step is to compute a small domination core.

▶ **Definition 18** (r-domination core). *Let G be a digraph. A set $Z \subseteq V(G)$ is an* r-domination core *in G if every minimum-size set which r-dominates Z also r-dominates G.*

Clearly, the set $V(G)$ is an r-domination core. We will show how to iteratively remove vertices from this trivial core, to arrive at smaller and smaller domination cores, until finally, we arrive at a core of polynomial size in k. Observe that we do not require that every r-dominating set for Z is also an r-dominating set for G; there can exist dominating sets for Z which are not of minimum size and which do not dominate the whole graph.

▶ **Lemma 19** ($\star$). *There exists a polynomial p and a polynomial-time algorithm that, given an r-domination core $Z \subseteq V(G)$ with $|Z| > p(k)$, either correctly decides that G cannot be dominated by k vertices, or finds a vertex $z \in Z$ such that $Z \setminus \{z\}$ is still an r-domination core.*

Hence, by gradually reducing $|Z|$, we arrive at the following theorem.

▶ **Theorem 20.** *There exists a polynomial p and a polynomial-time algorithm that, given an instance (G, k) where $G \in \mathcal{C}$, either correctly decides that G cannot be dominated by k vertices, or finds an r-domination core $Z \subseteq V(G)$ with $|Z| \leq p(k)$.*

Now that it remains to dominate a subset Z, we may keep one representative from each equivalence class in the equivalence relation: $u \cong_{Z,r} v \Leftrightarrow N_r^+(u) \cap Z = N_r^+(v) \cap Z$. As before, there are only polynomially many equivalence classes, hence from a polynomial domination core we can construct a polynomial kernel.

▶ **Theorem 21.** *Let $\mathcal{C}$ be a class of bounded expansion. There is a polynomial time algorithm which on input G, k and r computes a subgraph $G' \subseteq G$ and a set $Z \subseteq V(G')$ such that G can be r-dominated by k vertices if, and only if, Z can be r-dominated by k vertices in G' and $|Z| \leq p(k)$.*

5 Steiner trees

▶ **Definition 22.** *The Directed Steiner Tree (DST) problem is defined as follows. The input is a tuple (G, r, T, k) where G is a digraph, $r \in V(G)$ is a vertex (a root), $T \subseteq V(G) \setminus \{r\}$ is a set of terminals and k is an integer. The problem is to decide if there is a set $S \subseteq V(G) \setminus (\{r\} \cup T)$ of size at most k such that in $G[\{r\} \cup S \cup T]$ there is a directed path from r to every terminal T.*

The DST problem has been widely studied in the area of approximation algorithms as it generalises several routing and domination problems. We are interested in the parameterised complexity of this problem. It follows from an algorithm by Nederlof [54] and Misra et al. [51], that the problem can be solved in time $2^{|T|} \cdot p(n)$, for some polynomial $p(n)$. In this paper, we are interested in the standard parameterisation in parameterised complexity, where as parameter we take the solution size, i.e. we take the number k of non-terminals as parameter. This models the case where we need to pay for any node we add to the solution and we want to keep the bound k on these nodes as small as possible without any restriction on the number of terminals to connect.

In [41], Jones et al. show that DST with this parameterisation is fixed-parameter tractable on any class of digraphs such that the class of underlying undirected graphs excludes a fixed graph H as an undirected topological minor, as well as on any class of degenerate graphs if the set T of terminal vertices induces an acyclic graph. We immediately conclude the following.

▶ **Theorem 23** (⋆). *Let $\mathcal{C}$ be a class of digraphs of bounded expansion.* DST *is fixed-parameter tractable on $\mathcal{C}$ parameterised by the number k of non-terminals in the solution plus the maximal diameter s of the strongly connected components in the subgraph induced by the terminals.*

The proof of the theorem has the following immediate consequences.

▶ **Corollary 24.** *Let $\mathcal{C}$ be a class of digraphs closed under taking directed minors for which $\nabla_0(G) \leq c$ for a constant c for all $G \in \mathcal{C}$. Then* DST(G, r, T, k) *can be solved for all $G \in \mathcal{C}$, $r \in V(G)$, $T \subseteq V(G) \setminus \{r\}$ and k in time $2^{\mathcal{O}(k)} \cdot p(n)$, for some fixed polynomial $p(n)$.*

Note that this strictly generalises classes of undirected graphs excluding a fixed minor.

Another consequence of this is the following result, which immediately follows from the well-known observation in parameterised complexity (see e.g. [41, Lemma 7]), that for all functions $g(n) = o(\log n)$ there is a function $f(k)$ such that $f(k) \leq 2^{g(n)\cdot k}$, for all k and all n.

▶ **Corollary 25.** *Let $\mathcal{C}$ be a class of digraphs such that $\nabla_{|G|}(G) \cdot \log \nabla_{|G|}(G) \leq o(\log n)$ for all $G \in \mathcal{C}$. Then* DST *is fixed-parameter tractable on $\mathcal{C}$ with parameter k.*

Finally, the result also implies an fpt factor-2-approximation algorithm for the STRONGLY CONNECTED STEINER SUBGRAPH problem, SCSS, on classes of bounded directed expansion. In the SCSS we are given a digraph G, a number k, and a set T of terminals and we are asked to compute a set S of at most k non-terminals such that $G[T \cup S]$ is strongly connected.

▶ **Theorem 26** (⋆). *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. There is an fpt factor-2-approximation algorithm for* SCSS *on $\mathcal{C}$ parameterised by the number k of non-terminals in the solution plus the maximal diameter s of a strongly connected component in the subgraph of G induced by the terminal nodes.*

We close the section by showing that for bounded expansion classes, the parameterisation $k + s$ in Theorem 23 cannot be replaced by taking only k as parameter. This follows immediately from a result of [41] where it is shown that SET COVER can be reduced to DST on 2-degenerate graphs. It is straightforward to modify this example so that the resulting class of graphs has bounded directed expansion.

▶ **Theorem 27.** *The* DST*-problem restricted to classes of digraphs of bounded expansion parameterised by the solution size k is $W[2]$-hard.*

6 Hardness Results

In this section, we investigate the problems of DOMINATING SET and STEINER SUBGRAPH in classes of digraphs of bounded crownless expansion when we require strong connectivity for the graph induced by the output sets. We will show that both STRONGLY CONNECTED DOMINATING SET and STRONGLY CONNECTED STEINER SUBGRAPH are W[1]-hard.

The parameterised STRONGLY CONNECTED DOMINATING SET problem (SCDS) is the problem to decide whether a given digraph G contains a dominating set $D \subseteq V(G)$ of size at most k such that the digraph induced by D is strongly connected, where k is an input parameter. We prove that SCDS parameterised by solution size is W[1]-hard even in graphs of bounded crownless expansion. The proof is a reduction from the MULTICOLOURED CLIQUE problem, which is known to be W[1]-hard [27]. Given an integer k and a graph G whose vertex set is partitioned into k independent sets $V_1, V_2, \ldots, V_k$ called *colour classes*, the MULTICOLOURED CLIQUE problem asks whether there exists a k-vertex clique in G with exactly one vertex from every colour class.

The reduction. Let (G, k) be an instance of MULTICOLOURED CLIQUE and let $V_1, \ldots, V_k$ be the colour classes of G. We can assume that V_i is independent for each $1 \le i \le k$. We construct an instance (H, p) of STRONGLY CONNECTED DOMINATING SET, for a parameter p to be defined, starting from the graph G. For each class V_i we add two vertices s_i^1, s_i^2 and we connect each vertex $v \in V_i$ to both of these vertices with two edges vs_i^1 and vs_i^2.

For each $1 \le i, j \le k$ with $i \ne j$, let $E_{i,j}$ be the set of edges of G with one end in V_i and the other in V_j. For each $E_{i,j}$ we define $\mathcal{C}_{i,j}$ to be a set of $|E_{i,j}|$ directed cycles of length $2k+6$ such that they intersect in a single vertex $v_{i,j}$. Further, we index the set $\mathcal{C}_{i,j}$ by the elements of $E_{i,j}$. For each cycle C_e of $\bigcup_{1 \le i,j \le k} \mathcal{C}_{i,j}$ we denote with x_e the vertex of C_e such that the length of the path starting in x_e and ending in $v_{i,j}$ is $k+1$. Similarly we define y_e to be the vertex such that the length of the path starting in $v_{i,j}$ and ending in y_e is $k+1$. We further denote by z_e the vertex of C_e in $N^-(x_e)$.

For each $i < j$, with the exception of the pair $(1, k)$, we replace each edge $e = uv \in E_{i,j}$ with $u \in V_i$ and $v \in V_j$, with the two directed edges ux_e and $y_e v$. The pair $\{V_1, V_k\}$ is connected in the following way. For the pair $(1, k)$ we replace each $e' = u'v' \in E_{1,k}$ with $u' \in V_1$ and $v' \in V_k$ with the edges $v'x_e$ and $y_e u'$.

In addition, for each pair of classes $\{V_i, V_j\}$ we add two vertices $s_{i,j}^1, s_{i,j}^2$ and draw edges $z_e s_{i,j}^1$ and $z_e s_{i,j}^2$ for all $C_e \in \mathcal{C}_{i,j}$. We call the vertices of the set $\{s_{i,j}^l : l = 1, 2, 1 \le i, j \le k, i \ne j\} \cup \{s_i^l : 1 \le i \le k, l = 1, 2\}$ the *top vertices* of H.

Lastly, we add one vertex q and we draw qv directed edges for each v in one of the colour classes and an edge qv' for each vertex v' in one of the cycles $C \in \bigcup \mathcal{C}_{i,j}$. Since we want to be able to maintain strong connectivity when q is in D, we must add some edges directed towards q. In particular we add an edge vq for an arbitrary v in each cycle C_e with $e \in E_{1,2}$. This concludes the construction of (H, p).

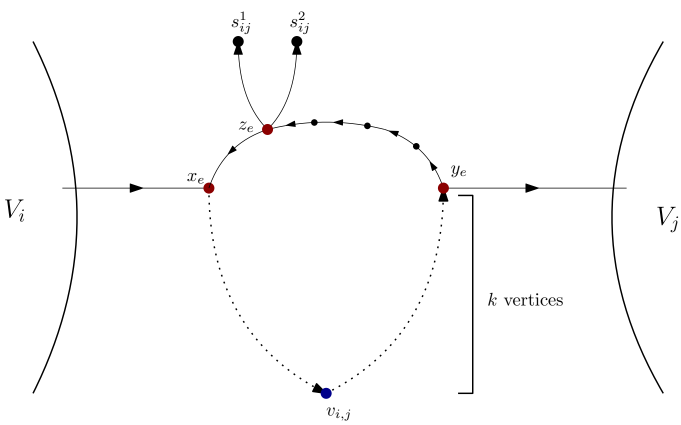

Figure 2 Each cycle C_e is connected to s_{ij}^1 and s_{ij}^2 with two edges incident to z_e. All cycles corresponding to edges of E_{ij} intersect in the vertex v_{ij}.

Before we proceed with our proof of the hardness, we will prove that the graph obtained is of bounded crownless expansion. We need the following easy lemma.

▶ **Lemma 28** ($\star$). *Let G be a graph of density $\frac{|E(G)|}{|V(G)|} = D$. Let G' be a graph with $|V(G')| = |V(G)| + t$ and $|E(G')| = |E(G)| + d$ edges. The density D' of G' is greater than D if and only if $D < d/t$.*

▶ **Lemma 29** ($\star$). *Let (G, k) be an instance of* Multicoloured Clique *and let (H, p) be the correspondent instance of* Strongly Connected Dominating Set *constructed as described above. We prove that $\widetilde{\nabla}_r(H) \le (r-1)/2$ and $S_{f(r)} \not\preccurlyeq_r H$.*

▶ **Theorem 30.** *There exists a class $\mathcal{C}$ of digraphs of bounded crownless expansion such that* Strongly Connected Dominating Set *parameterised by size of the solution is* W[1]*-hard on $\mathcal{C}$.*

Proof. Let (G, k) be an instance of Multicoloured Clique and let (H, p) be the correspondent instance of Strongly Connected Dominating Set with $p = k + \binom{k}{2}(6 + 2k) + 1$.

We claim that if a multicoloured clique K_k exists in G then there is strongly connected dominating set D of size at most p in H. For each uv in K_k, let u' and v' be the corresponding vertices in H and let C_{uv} be the cycle of $\mathcal{C}_{i,j}$ that they are connected to. Take S to be the union of $V(C_{uv})$ and the vertices u' and v' over all $uv \in E(K_k)$. Further, we take $D = S \cup \{q\}$. The size of D is equal to $k + \binom{k}{2}(6 + 2k) + 1$ which in turn is equal to the parameter p of the instance (H, p). It is easy to see that D is a dominating set. The vertex q dominates the vertices included in the colour classes. What is left to check are the top vertices. By taking a vertex in each class V_i, we ensure that s_i^1 and s_i^2 are dominated. In addition, for each edge uv of K_k, with $u \in V_i$ and $v \in V_j$, we add the vertices of the cycle C_{uv} which include the vertex $z_{C_{uv}}$. Hence, the vertices $s_{i,j}^1, s_{i,j}^2$ are dominated. It is easy to see that D is strongly connected. For every $i < j$, $u \in V_i$ and $v \in V_j$, there is a directed path starting in u and ending in v. The vertices on the cycles are connected through the cycles to the same paths. The path starting in V_k and ending in V_i ensures the strong connectivity for these vertices. Since we have added at least one cycle $C \in \mathcal{C}_{1,2}$, strong connectivity is preserved for $q \in D$.

We will now prove that the existence of a strongly connected dominating set D of size at most p in H implies the existence of a multicoloured clique K_k in G. We know that for each pair i, j with $1 \le i, j \le k$, D needs to dominate $s_{i,j}^1$ and $s_{i,j}^2$. Hence D must contain at least a vertex z_e for some $C_e \in \mathcal{C}_{i,j}$. Hence, since D is strongly connected $G[D]$ must contain the cycle C_e. In addition, for each pair of cycles C_e and $C_{e'}$ there must be a path connecting them. It follows that D contains edges ux_e and y_eu for some u, v with $u \in V_i$ and $v \in V_j$. Let us assume that G does not contain a multicoloured clique K_k, we will prove that the $|D| > p$. We know that for each pair i, j, D contains at least cycle C_e plus two vertices u, v. Since there does not exist a multicoloured clique in G, there exists at least three vertices $u, v, w \in D$ in distinct colour classes, such that the corresponding vertices u', v', w' of G are such that $u'v' \in E(G)$ and $u'w' \notin E(G)$. Hence, there exists at least one class V_i such that D contains two vertices in that class. Further, D must dominate all the vertices contained in cycles that are not in D and the vertex q. As a consequence, the vertex q must be in D and $|D| \ge k + \binom{k}{2}(6 + 2k) + 2$ which is larger than the parameter p. ◀

Similarly, we show that Strongly Connected Steiner Subgraph is hard.

▶ **Theorem 31** ($\star$). *There exists a class $\mathcal{C}$ of digraphs of bounded crownless expansion such that* Strongly Connected Steiner Subgraph *parameterised by size of the solution is* W[1]*-hard on $\mathcal{C}$.*

References

1 Hans Adler and Isolde Adler. Interpreting nowhere dense graph classes as a classical notion of model theory. *European Journal of Combinatorics*, 36:322–330, 2014.

2 Jochen Alber, Michael R Fellows, and Rolf Niedermeier. Polynomial-time data reduction for dominating set. *Journal of the ACM (JACM)*, 51(3):363–384, 2004.

3 Saeed Akhoondian Amiri, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz. Distributed Domination on Graph Classes of Bounded Expansion. In *Proceedings of the 30th on Symposium on Parallelism in Algorithms and Architectures, SPAA*, pages 143–151, 2018.

4 Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM (JACM)*, 41(1):153–180, 1994.

5 Jørgen Bang-Jensen and Gregory Z Gutin. *Digraphs: theory, algorithms and applications*. Springer Science & Business Media, 2008.

6 János Barát. Directed path-width and monotonicity in digraph searching. *Graphs and Combinatorics*, 22(2):161–172, 2006.

7 Dietmar Berwanger, Anuj Dawar, Paul Hunter, and Stephan Kreutzer. DAG-width and parity games. In *Annual Symposium on Theoretical Aspects of Computer Science*, pages 524–536. Springer, 2006.

8 Hans L Bodlaender, Fedor V Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M Thilikos. (Meta) kernelization. In *Foundations of Computer Science, 2009. FOCS'09. 50th Annual IEEE Symposium on*, pages 629–638. IEEE, 2009.

9 Marthe Bonamy, Łukasz Kowalik, Michał Pilipczuk, and Arkadiusz Socała. Linear kernels for outbranching problems in sparse digraphs. *Algorithmica*, pages 1–30, 2015.

10 Hervé Brönnimann and Michael T Goodrich. Almost optimal set covers in finite VC-dimension. *Discrete & Computational Geometry*, 14(4):463–479, 1995.

11 Julia Chuzhoy and Shi Li. A Polylogarithmic Approximation Algorithm for Edge-Disjoint Paths with Congestion 2. *J. ACM*, 63(5):45:1–45:51, November 2016. `doi:10.1145/2893472`.

12 Anuj Dawar, Martin Grohe, Stephan Kreutzer, and Nicole Schweikardt. Approximation schemes for first-order definable optimisation problems. In *21st Annual IEEE Symposium on Logic in Computer Science, 2006*, pages 411–420. IEEE, 2006.

13 Anuj Dawar and Stephan Kreutzer. Domination Problems in Nowhere-Dense Classes. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2009, December 15-17, 2009, IIT Kanpur, India*, pages 157–168, 2009.

14 E. Demaine and M. Hajiaghayi. The bidimensionality theory and its algorithmic applications. *The Computer Journal*, pages 332–337, 2008.

15 E. D. Demaine, M. Hajiaghayi, and K. Kawarabayashi. Algorithmic Graph Minor Theory: Decomposition, Approximation, and Coloring. In *46th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 637–646, 2005.

16 Erik D Demaine, Fedor V Fomin, Mohammadtaghi Hajiaghayi, and Dimitrios M Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs. *Journal of the ACM (JACM)*, 52(6):866–893, 2005.

17 Erik D. Demaine and Mohammad Taghi Hajiaghayi. Fast Algorithms for Hard Graph Problems: Bidimensionality, Minors, and Local Treewidth. In *Graph Drawing*, pages 517–533, 2004. URL: `http://springerlink.metapress.com/openurl.asp?genre=article{&}issn=0302-9743{&}volume=3383{&}spage=517`.

18 Erik D. Demaine and Mohammad Taghi Hajiaghayi. Bidimensionality: new connections between FPT algorithms and PTASs. In *Symp. on Discrete Algorithms (SODA)*, pages 590–601, 2005. `doi:10.1145/1070432.1070514`.

19 Erik D. Demaine, Felix Reidl, Peter Rossmanith, Fernando Sánchez Villaamil, Somnath Sikdar, and Blair D. Sullivan. Structural Sparsity of Complex Networks: Random Graph Models and Linear Algorithms. *CoRR*, abs/1406.2587, 2014. URL: `http://arxiv.org/abs/1406.2587`, `arXiv:1406.2587`.

20 Pål Grønås Drange, Markus Sortland Dregi, Fedor V. Fomin, Stephan Kreutzer, Daniel Lokshtanov, Marcin Pilipczuk, Michal Pilipczuk, Felix Reidl, Fernando Sánchez Villaamil, Saket Saurabh, Sebastian Siebertz, and Somnath Sikdar. Kernelization and Sparseness: the Case of Dominating Set. In *STACS*, volume 47 of *LIPIcs*, pages 31:1–31:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

21 Zdeněk Dvořák. Constant-factor approximation of the domination number in sparse graphs. *European Journal of Combinatorics*, 34(5):833–840, 2013.

22 Zdeněk Dvořák. On distance r-dominating and 2r-independent sets in sparse graphs. *arXiv preprint arXiv:1710.10010*, 2017.

23 Zdeněk Dvořák, Daniel Král, and Robin Thomas. Testing first-order properties for subclasses of sparse graphs. *Journal of the ACM (JACM)*, 60(5):36, 2013.

24 Eduard Eiben, Mithilesh Kumar, Amer E. Mouawad, Fahad Panolan, and Sebastian Siebertz. Lossy Kernels for Connected Dominating Set on Sparse Graphs. In *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France*, pages 29:1–29:15, 2018.

25 Kord Eickmeyer, Archontia C Giannopoulou, Stephan Kreutzer, O-joung Kwon, Michał Pilipczuk, Roman Rabinovich, Sebastian Siebertz, et al. Neighborhood complexity and kernelization for nowhere dense classes of graphs. *arXiv preprint arXiv:1612.08197*, 2016.

26 Grzegorz Fabiański, Michał Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. Progressive Algorithms for Domination and Independence. *CoRR*, abs/1811.06799, 2018. `arXiv:1811.06799`.

27 Michael R Fellows, Danny Hermelin, Frances Rosamond, and Stéphane Vialette. On the parameterized complexity of multiple-interval graph problems. *Theoretical Computer Science*, 410(1):53–61, 2009.

28 Fedor V. Fomin, Daniel Lokshtanov, Venkatesh Raman, and Saket Saurabh. Bidimensionality and EPTAS. In *SODA*, pages 748–759, 2011. URL: `http://www.siam.org/proceedings/soda/2011/SODA11_058_fominf.pdf`.

29 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. Bidimensionality and Kernels. In *SODA*, pages 503–510, 2010. URL: `http://www.siam.org/proceedings/soda/2010/SODA10_043_fominf.pdf`.

30 Fedor V Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M Thilikos. Bidimensionality and kernels. In *Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms*, pages 503–510. SIAM, 2010.

31 Fedor V Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M Thilikos. Linear kernels for (connected) dominating set on H-minor-free graphs. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pages 82–93. Society for Industrial and Applied Mathematics, 2012.

32 Fedor V Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M Thilikos. Linear kernels for (connected) dominating set on graphs with excluded topological subgraphs. In *30th International Symposium on Theoretical Aspects of Computer Science*, page 92, 2013.

33 Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Sebastian Ordyniak, Felix Reidl, Peter Rossmanith, Fernando Sanchez Villaamil, and Somnath Sikdar. Kernelization using structural parameters on sparse graph classes. In *European Symposium on Algorithms*, pages 529–540. Springer, 2013.

34 Robert Ganian, Petr Hliněný, Joachim Kneis, Alexander Langer, Jan Obdržálek, and Peter Rossmanith. On digraph width measures in parameterized algorithmics. In *International Workshop on Parameterized and Exact Computation*, pages 185–197. Springer, 2009.

35 Robert Ganian, Petr Hlinený, Joachim Kneis, Daniel Meister, Jan Obdrzálek, Peter Rossmanith, and Somnath Sikdar. Are there any good digraph width measures? *J. Comb. Theory, Ser. B*, 116:250–286, 2016. `doi:10.1016/j.jctb.2015.09.001`.

36 Martin Grohe. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.

37 Martin Grohe, Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Konstantinos Stavropoulos. Colouring and covering nowhere dense graphs. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 325–338. Springer, 2015.

38 Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 89–98. ACM, 2014.

39 Sariel Har-Peled and Kent Quanrud. Approximation algorithms for polynomial-expansion and low-density graphs. *SIAM Journal on Computing*, 46(6):1712–1744, 2017.

40 Paul Hunter and Stephan Kreutzer. Digraph measures: Kelly decompositions, games, and orderings. *Theoretical Computer Science*, 399(3):206–219, 2008.

41 M. Jones, D. Lokshtanov, M.S. Ramanujan, S. Saurabh, and O. Suchy. Parameterized Complexity of Directed Steiner Tree on Sparse Graphs. In *Proceedings of European Symposium on Algorithms (ESA 2013)*, 2013.

42 Richard M Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.

43 Wojciech Kazana and Luc Segoufin. Enumeration of first-order queries on classes of structures with bounded expansion. In *Proceedings of the 32nd ACM SIGMOD-SIGACT-SIGAI symposium on Principles of database systems*, pages 297–308. ACM, 2013.

44 Stephan Kreutzer, Irene Muzi, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz. Algorithmic Properties of Sparse Digraphs. *CoRR*, abs/1707.01701, 2017.

45 Stephan Kreutzer, Roman Rabinovich, and Sebastian Siebertz. Polynomial Kernels and Wideness Properties of Nowhere Dense Graph Classes. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1533–1545. SIAM, 2017.

46 Stephan Kreutzer, Roman Rabinovich, and Sebastian Siebertz. Polynomial Kernels and Wideness Properties of Nowhere Dense Graph Classes. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, pages 1533–1545, 2017. `doi:10.1137/1.9781611974782.100`.

47 Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Grischa Weberstädt. Structural Properties and Constant Factor-Approximation of Strong Distance-r Dominating Sets in Sparse Directed Graphs. In *34th Symposium on Theoretical Aspects of Computer Science (STACS 2017)*, volume 66 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 48:1–48:15, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. `doi:10.4230/LIPIcs.STACS.2017.48`.

48 Stephan Kreutzer and Siamak Tazari. Directed nowhere dense classes of graphs. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pages 1552–1562. SIAM, 2012.

49 Daniel Lokshtanov, Amer E Mouawad, Fahad Panolan, MS Ramanujan, and Saket Saurabh. Reconfiguration on sparse graphs. In *Workshop on Algorithms and Data Structures*, pages 506–517. Springer, 2015.

50 Maryanthe Malliaris and Saharon Shelah. Regularity lemmas for stable graphs. *Transactions of the American Mathematical Society*, 366(3):1551–1585, 2014.

51 N. Misra, G. Philip, V. Raman, S. Saurabh, and S. Sikdar. FPT algorithms for connected feedback vertex set. In *WALCOM 2010*, pages 269–280, 2010.

52 Daniel Mölle, Stefan Richter, and Peter Rossmanith. Enumerate and expand: Improved algorithms for connected vertex cover and tree cover. *Theory of Computing Systems*, 43(2):234–253, 2008.

53 Wojciech Nadara, Marcin Pilipczuk, Roman Rabinovich, Felix Reidl, and Sebastian Siebertz. Empirical Evaluation of Approximation Algorithms for Generalized Graph Coloring and Uniform Quasi-Wideness. In *17th International Symposium on Experimental Algorithms, SEA*, pages 14:1–14:16, 2018.

54 J. Nederlof. Fast polynomial-space algorithms using Möbius inversion: Improving on Steiner tree and related problems. In *Proc. of ICALP 2009*, pages 713–725, 2009.

55 Jaroslav Nešetřil and Patrice Ossona de Mendez. Linear time low tree-width partitions and algorithmic consequences. In *Proceedings of the thirty-eighth annual ACM Symposium on Theory of Computing (STOC)*, pages 391–400. ACM, 2006.

56 Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion I. Decompositions. *European Journal of Combinatorics*, 29(3):760–776, 2008.

57 Jaroslav Nešetřil and Patrice Ossona de Mendez. First order properties on nowhere dense structures. *The Journal of Symbolic Logic*, 75(03):868–887, 2010.

58 Jaroslav Nešetřil and Patrice Ossona de Mendez. On nowhere dense graphs. *European Journal of Combinatorics*, 32(4):600–617, 2011.

59 Jan Obdržálek. DAG-width: connectivity measure for directed graphs. In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, pages 814–821. Society for Industrial and Applied Mathematics, 2006.

60 Geevarghese Philip, Venkatesh Raman, and Somnath Sikdar. Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond. *ACM Transactions on Algorithms (TALG)*, 9(1):11, 2012.

61 MichałPilipczuk and Sebastian Siebertz. Kernelization and approximation of distance-r independent sets on nowhere dense graphs. *arXiv preprint arXiv:1809.05675*, 2018.

62 MichałPilipczuk, Sebastian Siebertz, and Szymon Torunczyk. On the number of types in sparse graphs. *arXiv preprint arXiv:1705.09336*, 2017.

63 Hans Jürgen Prömel and Angelika Steger. *The Steiner tree problem: a tour through graphs, algorithms, and complexity.* Springer Science & Business Media, 2012.

64 Ran Raz and Shmuel Safra. A sub-constant error-probability low-degree test, and a sub-constant error-probability PCP characterization of NP. In *Proceedings of the twenty-ninth annual ACM symposium on Theory of computing*, pages 475–484. ACM, 1997.

65 Felix Reidl, Fernando Sánchez Villaamil, and Konstantinos Stavropoulos. Characterising bounded expansion by neighbourhood complexity. *Eur. J. Comb.*, 75:152–168, 2019.

66 N. Robertson and P.D. Seymour. Graph minors XIII. The disjoint paths problem. *Journal of Combinatorial Theory, Series B*, 63:65–110, 1995.

67 Mohammad Ali Safari. D-width: A more natural measure for directed tree width. In *International Symposium on Mathematical Foundations of Computer Science*, pages 745–756. Springer, 2005.

68 Norbert Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.

69 Luc Segoufin and Alexandre Vigny. Constant Delay Enumeration for FO Queries over Databases with Local Bounded Expansion. In Michael Benedikt and Giorgio Orsi, editors, *20th International Conference on Database Theory (ICDT 2017)*, volume 68 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 20:1–20:16, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. `doi:10.4230/LIPIcs.ICDT.2017.20`.

70 Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.

71 Sebastian Siebertz. Reconfiguration on Nowhere Dense Graph Classes. *Electr. J. Comb.*, 25(3):P3.24, 2018.

72 VN Vapnik and A Ya Chervonenkis. On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities. *Measures of Complexity*, 16(2):11, 1971.

73 Xuding Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Mathematics*, 309(18):5562–5568, 2009.