Distributed Coloring of Graphs with an Optimal Number of Colors

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Abstract

This paper studies sufficient conditions to obtain efficient distributed algorithms coloring graphs optimally (i.e. with the minimum number of colors) in the LOCAL model of computation. Most of the work on distributed vertex coloring so far has focused on coloring graphs of maximum degree Δ with at most $\Delta+1$ colors (or Δ colors when some simple obstructions are forbidden). When Δ is sufficiently large and $c \geq \Delta - k_\Delta + 1$, for some integer $k_\Delta \approx \sqrt{\Delta} - 2$, we give a distributed algorithm that given a c-colorable graph G of maximum degree Δ, finds a c-coloring of G in $\min\{O((\log \Delta)^{13/12} \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds, with high probability. The lower bound $\Delta - k_\Delta + 1$ is best possible in the sense that for infinitely many values of Δ, we prove that when $\chi(G) \leq \Delta - k_\Delta$, finding an optimal coloring of G requires $\Omega(n)$ rounds. Our proof is a light adaptation of a remarkable result of Molloy and Reed, who proved that for Δ large enough, for any $c \geq \Delta - k_\Delta$ deciding whether $\chi(G) \leq c$ is in P, while Embden-Weinert *et al.* proved that for $c \leq \Delta - k_\Delta - 1$, the same problem is NP-complete. Note that the sequential and distributed thresholds differ by one.

Our first result covers the case where the chromatic number of the graph ranges between $\Delta - \sqrt{\Delta}$ and $\Delta + 1$. Our second result covers a larger range, but gives a weaker bound on the number of colors: For any sufficiently large Δ, and $\Omega(\log \Delta) \leq k \leq \Delta/100$, we prove that every graph of maximum degree Δ and clique number at most $\Delta - k$ can be efficiently colored with at most $\Delta - \varepsilon k$ colors, for some absolute constant $\varepsilon > 0$, with a randomized algorithm running in $O(\log n / \log \log n)$ rounds with high probability.

2012 ACM Subject Classification Mathematics of computing → Graph coloring; Mathematics of computing → Graph algorithms; Theory of computation → Distributed algorithms

Keywords and phrases Graph coloring, distributed algorithm, maximum degree

Digital Object Identifier 10.4230/LIPIcs.STACS.2019.10

Related Version The full version of the paper [1] is available at `arXiv:1809.08140`.

Funding *Louis Esperet*: Partially supported by ANR Project GATO (ANR-16-CE40-0009-01) and LabEx PERSYVAL-Lab (ANR-11-LABX-0025).

Acknowledgements We thank David Harris for pointing out the updated version of [11] and for his kind remarks on earlier versions of the paper. We also thank two anonymous reviewers for their detailed comments and suggestions.

1 Introduction

The graph coloring problem plays an important role in distributed computing, since it is used as a subroutine in distributed algorithms for a large variety of problems (see the recent survey book of Barenboim and Elkin [2] for more details and further references). The central problem in distributed coloring is the $(\Delta+1)$-coloring problem, where a graph of maximum degree at most Δ has to be colored with at most $\Delta + 1$ colors (see [10] and [5] for the fastest deterministic and randomized algorithms to date and more on the history of the problem). The bound $\Delta + 1$ on the number of colors is best possible in general, but it follows from Brooks' Theorem that any connected graph of maximum degree Δ which is neither an odd cycle nor a complete graph can indeed be colored with Δ colors, instead of $\Delta + 1$, and there has been some work to find fast distributed algorithms coloring such graphs with Δ colors. The problem was first considered by Panconesi and Srinivasan [18], and it was recently proved in [12] that the Δ-coloring problem can be solved with a randomized algorithm running in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds when $\Delta \geq 4$, or $O((\log \log n)^2)$ rounds when Δ is a constant. On the other hand, it was proved in [4] that a randomized algorithm solving the Δ-coloring problem needs $\Omega(\log \log n)$ rounds. These results, as well as all the other algorithms mentioned in this paper, are proved in the LOCAL model of computation (see below for more details).

The main idea of Δ-coloring is that by forbidding some simple obstructions (complete graphs and odd cycles), we can save one color (compared with the easier $(\Delta + 1)$-coloring problem) while still having a fast algorithm, whether sequential or distributed. A natural question is: can we go further? Is there some small set of obstructions (that can be easily recognized locally, at least when Δ is sufficiently large), such that if we forbid these obstructions we can find fast distributed algorithms coloring graphs of maximum degree Δ with $\Delta - 1$ colors? Or $\Delta - 2$ colors? Or $\Delta - k$ colors, for some constant k?

The sequential version of this question turned out to have a very precise answer. For any Δ, let k_Δ be the maximum integer k such that $(k + 1)(k + 2) \leq \Delta$. It can be checked that $k_\Delta = \lfloor \sqrt{\Delta + 1/4} - 3/2 \rfloor$ and thus $\sqrt{\Delta} - 3 < k_\Delta < \sqrt{\Delta} - 1$. The following was proved by Embden-Weinert, Hougardy and Kreuter [8].

► **Theorem 1.1** ([8]). *For $3 \leq c \leq \Delta - k_\Delta - 1$, we cannot test for c-colorability of graphs with maximum degree Δ in polynomial time unless P = NP.*

The following strong converse was then proved by Molloy and Reed [17].

► **Theorem 1.2** ([17]). *For sufficiently large (but constant) Δ, and every $c \geq \Delta - k_\Delta$, there is a linear time deterministic algorithm to test whether graphs of maximum degree Δ are c-colorable. Furthermore, there is a polynomial time deterministic algorithm that will produce a c-coloring whenever one exists.*

Our main result will be to prove that a similar dichotomy occurs in the LOCAL model, with a slightly larger tractability threshold ($\Delta - k_\Delta + 1$ instead of $\Delta - k_\Delta$).

► **Theorem 1.3.** *For sufficiently large Δ, and any $c \geq \Delta - k_\Delta + 1$, there is a distributed randomized algorithm running w.h.p. in $\min\{O((\log \Delta)^{13/12} \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds, that takes a graph G with maximum degree Δ as input, and does the following: either some vertex outputs a certificate that G is not c-colorable, or the algorithm finds a c-coloring of G.*

Here, w.h.p. (with high probability) means with probability at least $1 - O(n^{-\alpha})$, for any fixed $\alpha > 0$. Note that the chromatic number of G can be smaller than the threshold $\Delta - k_\Delta + 1$, what matters is that the number c of available colors is at least this threshold. We will prove that the value of $\Delta - k_\Delta + 1$ is sharp, in the following sense.

► **Theorem 1.4.** *When $c \leq \Delta - k_\Delta - 1$ (for any value of Δ), and when $c = \Delta - k_\Delta$ (for infinitely many values of Δ), there exist arbitrarily large graphs G of maximum degree Δ for which $\chi(G) = c$, and such that any distributed algorithm coloring G with c colors takes $\Omega(n/\Delta)$ rounds.*

In the LOCAL model of computation, if the algorithm runs in r rounds, the color assigned to a vertex v is based only on the (subgraph induced by the) vertices at distance at most r from v. The fact that when $c \geq \Delta - k_\Delta + 1$, it can be decided whether G is c-colorable by only looking at each neighborhood was already proved by Molloy and Reed [17] (see Theorem 4.1). In this paper, we are mostly interested in *producing* such a coloring in a distributed way, and it is a priori unclear that it can be done in a small number rounds. For instance, in the LOCAL model it can be decided in a single round whether a graph has maximum degree at most two (and is therefore 3-colorable), but finding a 3-coloring of a path takes an unbounded number of rounds [15].

An interesting difference between Theorems 1.3 and 1.2 (besides the fact that the sequential and distributed thresholds are not the same), is that in the sequential result it is important that Δ is a constant. If Δ depends on n, then Molloy and Reed [17] proved that the tractability threshold is around $\Delta - \Theta(\log \Delta)$ colors. On the other hand, in the distributed setting there is no requirement on Δ.

It should be mentioned that efficient distributed coloring algorithms involving the chromatic number are not frequent. A rare example of such an algorithm involving a general class of graphs (not just paths or cycles, or line-graphs for instance) is the following result of Schneider and Wattenhofer [20]: when $\Delta = \Omega(\log^{1+1/\log^* n} n)$ and $\chi = O(\Delta/\log^{1+1/\log^* n} n)$, they find a randomized distributed algorithm coloring graphs of maximum degree Δ and chromatic number χ with at most $(1 - 1/O(\chi))\Delta$ colors w.h.p., and running w.h.p. in $O(\log \chi + \log^* n)$ rounds. Two significant differences with our result are the requirement on Δ and the fact that the number of colors in the resulting coloring is not best possible. We also note that in the setting of Theorem 1.2 and Theorem 1.5 below, the chromatic number is an additive factor away from the maximum degree, while the result of Schneider and Wattenhofer [20] mentioned above asks for a much larger (multiplicative) gap between χ and Δ.

Theorem 1.3 covers in particular the situation where $\chi(G) \geq \Delta - \sqrt{\Delta} + 1$ (and in this case, gives an efficient algorithm to obtain an optimal coloring of the graph). Recall that Brooks' theorem (and its algorithmic variants) colors graphs of maximum degree $\Delta \geq 3$ distinct from $K_{\Delta+1}$ (or equivalently, with clique number at most Δ) with at most Δ colors. Our next result generalizes the algorithmic versions of Brooks' theorem in the following direction.

▶ **Theorem 1.5.** *There exists $\Delta_0 > 0$ such that for every $\Delta \geq \Delta_0$ and $2^{59} \log \Delta \leq k \leq \frac{\Delta}{100}$, there exists a randomized distributed algorithm that given an n-vertex graph of maximum degree Δ, does the following: either some vertex outputs a clique of size more than $\Delta - k$ if such a clique exists, or the algorithm finds a coloring with at most $\Delta - 2^{-23}k$ colors. The round complexity is the minimum of $O(\log_\Delta n + \log_k \Delta) + 2^{O(\sqrt{\log \log n})}$ and $2^{O(\log \Delta + \sqrt{\log \log n})}$ w.h.p., and in particular it is $O(\log n / \log \log n)$ w.h.p.*

We start with some preliminaries on distributed computing, probability, and graph theory in Section 2. We then prove Theorem 1.5 in Section 3. It turns out that the proof of Theorem 1.5 contains several ingredients that will be reused in the proof of Theorem 1.3. In Section 4, we prove Theorem 1.4 and explain how to adapt the proof of Theorem 1.2 in [17] to prove Theorem 1.3. We conclude with some remarks in Section 5.

2 Preliminaries

2.1 Distributed computing

We consider the classical LOCAL model of computation, which is a distributed model in which the network corresponds to the graph under consideration, i.e. each vertex of the graph corresponds to a processor, with infinite computational power, and vertices can communicate with their neighbors in synchronous rounds (in this model there is no restriction on the size of the messages exchanged by two neighboring vertices during each round of communication). Each vertex knows the number n of vertices and its own id (a distinct integer between 1 and n). In this paper, the vertices also know the maximum degree Δ of the graph, and some number c of colors. Once the communication between the nodes is over, each vertex outputs a value (in our case, an integer between 1 and c corresponding to its color in a proper coloring of the graph, or some subset of its neighbors which cannot be colored with c colors). The complexity of the algorithm is the number of rounds of communication.

2.2 Vertex coloring

A *c-coloring* of a graph G is an assignment of integers from $\{1, \dots, c\}$ to the vertices of G such that any two adjacent vertices receive distinct colors. The chromatic number $\chi(G)$ of G is the least c such that G has a c-coloring.

In this paper it will be convenient to consider a slightly more general scenario, in which the colors available for each vertex are not necessarily the same. A *list-assignment* L for G is a collection of lists $L(v)$ of colors, one for each vertex v of G. Given a list-assignment L, an *L-list-coloring* of G is a coloring of G (i.e. any two adjacent vertices receive distinct colors, as before), with the additional constraint that each vertex v is colored with a color from its own list $L(v)$. A simple greedy algorithm shows that if for each vertex v, $|L(v)| \geq d_G(v) + 1$ (where $d_G(v)$ denotes the degree of v in G), then G has an L-list-coloring. This is a very useful generalisation of the fact that any graph of maximum degree Δ is $(\Delta + 1)$-colorable.

In this paper we will repeatedly use the following two important algorithmic results on list-coloring. The first result was proved in [3]

▶ **Theorem 2.1** ([3]). *Let G be a graph of maximum degree Δ and let L be a list-assignment such that for any vertex v, $|L(v)| - d_G(v) \geq 1$. Then an L-list-coloring of G can be found by a distributed randomized algorithm running in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds, w.h.p.*

The following stronger result was then proved in [7].

▶ **Theorem 2.2** ([7]). *Let G be a graph of maximum degree Δ and let L be a list-assignment such that for any vertex v, $|L(v)| - d_G(v) \geq \epsilon\Delta$, for some $\epsilon > 0$. Then an L-list-coloring of G can be found by a distributed randomized algorithm running in $O(\log(1/\epsilon)) + 2^{O(\sqrt{\log \log n})}$ rounds, w.h.p.*

Note that Theorem 2.1 can be deduced from Theorem 2.2 by simply setting $\epsilon = 1/\Delta$.

The setting in which these two results will be applied is the following. Let G be a graph of maximum degree Δ with a subset S of vertices that are colored with at most c colors. We want to extend the c-coloring of S to a c-coloring of G (i.e. find a c-coloring of G that agrees with the original coloring on S).

Let $U = V(G) - S$ be the set of uncolored vertices, and for each vertex $u \in U$, let $L(u)$ be the subset of colors from $1, \dots, c$ that do not appear among the neighbors of u in S. Note that extending the c-coloring of S to a c-coloring of G is the same as finding an L-list-coloring of $G[U]$, the subgraph of G induced by U.

Let us denote the degree of a vertex $u \in U$ in $G[U]$ by $d_U(u)$. The following simple observation will be particularly useful in combination with Theorems 2.1 or 2.2.

▶ **Observation 2.3.** *If $u \in U$ has at least ℓ repeated colors in its neighborhood, then $|L(u)| - d_U(u) \geq c + \ell - d_G(u) \geq c + \ell - \Delta$.*

Note that this observation will sometimes be used without an explicit number of repeated colors (i.e. $\ell = 0$) and the statement above simply becomes $|L(u)| - d_U(u) \geq c - d_G(u)$.

2.3 Probabilistic tools

Consider a set X of independent random variables, and a set $B = B_1, \ldots, B_n$ of (typically bad) events, each depending on a subset of the variables from X. Consider the graph H with vertex-set B, with an edge between two events if the set of variables they depend on intersect. The graph H is called the *event dependency graph*. Let $d \geq 2$ be the maximum degree of H, and let p be the maximum probability of an event from B.

We will use the following algorithmic versions of the Lovász Local Lemma [6, 11].

▶ **Theorem 2.4** ([6]). *If $epd^2 < 1$, then there is a distributed randomized algorithm, running in H in $O(\log_{1/epd^2}(n))$ rounds w.h.p., that finds a value assignment to the variables of X such that no event from B holds.*

▶ **Theorem 2.5** ([11]). *If $2^{15}pd^8 < 1$, then there is a distributed randomized algorithm, running in H in $2^{O(\log d + \sqrt{\log \log n})}$ rounds w.h.p., that finds a value assignment to the variables of X such that no event from B holds.*

It should be noted that in each subsequent application of Theorem 2.4 or 2.5, the event dependency graph H will only be considered implicitly. The reason is that the variables of X will be associated to the vertices of some other graph G, and the events from B will correspond to connected subgraphs of G of constant radius. Thus, the outcomes of Theorems 2.4 and 2.5 will be computed in G directly (the round complexity is then simply multiplied by a constant, which does not change the asymptotic complexity).

We shall also use the following version of Talagrand's inequality (see the appendix in [17]).

▶ **Theorem 2.6** (Talagrand's Inequality). *Let X be a non-negative random variable whose value is determined by n independent trials $T_1,\ldots,T_n$ and satisfying the following for some $c,r \geq 0$:*

- *changing the outcome of any one trial changes the value of X by at most c.*
- *for any s, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify $X \geq s$.*

Then for any $t \geq 0$,

$$\mathbb{P}\left(|X - \mathbb{E}(X)| > t + 20c\sqrt{r\mathbb{E}(X)} + 64c^2r\right) \leq 4 \cdot \exp\left(-\frac{t^2}{8c^2r(\mathbb{E}(X) + t)}\right)$$

2.4 The dense decomposition

The graph decomposition described in this section is due to Reed [19] (see also [16, 17]). A somewhat similar (although not completely equivalent) decomposition was recently used by Harris, Schneider, and Su [14] (see also [5]) in the context of distributed $(\Delta + 1)$-coloring algorithms.

Consider a graph $G = (V, E)$ of maximum degree Δ. We call a vertex *d-dense* if its neighborhood has more than $\binom{\Delta}{2} - d\Delta$ edges (note that d might depend on Δ). A vertex v that is not d-dense is said to be *d-sparse*.

We say that $S, X_1, X_2, \ldots, X_t$ is a *d-dense decomposition of G* if each of the following holds:

1. $S, X_1, X_2, \ldots, X_t$ partition V;
2. every X_i has between $\Delta - 8d$ and $\Delta + 4d$ vertices;
3. there are at most $8d\Delta$ edges between X_i and $V - X_i$;
4. a vertex is adjacent to at least $\frac{3\Delta}{4}$ vertices of X_i if and only if it is in X_i;
5. every vertex in S is d-sparse.

The sets X_i are called the *dense components* and S is called the *sparse component*. Note that a simple consequence of (4) and (2) is that each dense component has diameter at most 2, provided that $d \leq \frac{\Delta}{8}$. The proof of the following result is given in the full version of the paper [1].

▶ **Lemma 2.7.** *A d-dense decomposition of G can be constructed in $O(1)$ rounds for every $d \leq \frac{\Delta}{100}$.*

3 Graphs with small clique number

In this section we prove Theorem 1.5. We will need the following two results, whose proofs are inspired from the proofs of Lemmas 10 and 16 in [17] (see also Section 10.3 in [16]).

▶ **Lemma 3.1.** *Let G be a graph of (sufficiently large) maximum degree Δ and let $\ell \geq 2^{54} \log \Delta$. Then there is a distributed randomized algorithm that finds a partial coloring of G with $\Delta/2$ colors in $\min\{O(\log_\Delta n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds w.h.p., such that for each uncolored vertex v with at least $\ell\Delta$ pairs of non-adjacent vertices in $N(v)$, there are more than $2^{-18}\ell$ repeated colors in $N(v)$.*

▶ **Lemma 3.2.** *Let $S, X_1, \ldots, X_t$ be a $2^{-4}k$-dense decomposition of a graph G of maximum degree $\Delta \geq 30k$ and clique number at most $\Delta - k$. Then there is a distributed randomized algorithm that extends any c-coloring of S with $c \geq \Delta - k/48$ colors to a c-coloring of G in $O(\log_k \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds, w.h.p.*

We now explain how these two results can be combined to provide a proof of Theorem 1.5. It should be mentioned that we have made no significant effort to optimize the various constants appearing throughout the proof, and have chosen instead to focus on making the proof as simple as possible. The proofs of Lemmas 3.1 and 3.2 can be found in the full version of the paper [1].

Proof of Theorem 1.5. If G contains a clique on more than $\Delta - k$ vertices, it can be found in $O(1)$ rounds so we may assume in the remainder that G has clique number at most $\Delta - k$.

We start by using Lemma 2.7 to compute a $2^{-4}k$-dense decomposition $S, X_1, X_2, \ldots, X_t$ of G (note that we have $2^{-4}k \leq 2^{-4}\Delta/30 \leq \Delta/100$, as required). Let T be the vertices of S with degree at least $\Delta - 2^{-5}k$ in S. Since each vertex of $v \in T$ is $2^{-4}k$-sparse, $N(v)$ contains at least

$$\binom{\Delta - 2^{-5}k}{2} - \binom{\Delta}{2} + 2^{-4}k\Delta \geq 2^{-5}k\Delta$$

pairs of non-adjacent vertices in S.

Using Lemma 3.1 with $\ell = 2^{-5}k$, we then obtain a partial coloring of S with at most $\Delta/2 \le \Delta - 2^{-24}k$ colors in $\min\{O(\log_\Delta n), 2^{O(\log\Delta+\sqrt{\log\log n})}\}$ rounds w.h.p., such that each uncolored vertex of T has more than $2^{-23}k$ repeated colors in its neighborhood. Let U be the set of uncolored vertices of S, and for each vertex of $v \in U$, let $L(v)$ be the set of colors from $1, \ldots, \Delta - 2^{-24}k$ that do not appear in the neighborhood of v. We claim that

$$\text{for each } v \in U,\ |L(v)| - d_U(v) \ge 2^{-24}k, \tag{1}$$

where $d_U(v)$ denotes the number of neighbors of v in U, or equivalently the degree of v in $G[U]$.

To see why (1) holds, consider first the case $v \in U - T$. Observe that in this case v has degree at most $\Delta - 2^{-5}k$ in S, and thus (1) follows directly from Observation 2.3 with $c = \Delta - 2^{-24}k$, $\ell = 0$, and $d_S(v) \le \Delta - 2^{-5}k$ (which implies $c - d_S(v) \ge \Delta - 2^{-24}k - \Delta + 2^{-5}k \ge 2^{-24}k$).

Assume now that $v \in U \cap T$. Since each uncolored vertex of T has more than $2^{-23}d$ repeated colors in its neighborhood, (1) follows directly from Observation 2.3 with $c = \Delta - 2^{-24}k$ and $\ell = 2^{-23}k$ (which implies $c - \Delta + \ell = \Delta - 2^{-24}k - \Delta + 2^{-23}k = 2^{-24}k$). This concludes the proof of (1).

It follows from (1) that we can use Theorem 2.2 with $\epsilon = 2^{-24}k/\Delta$ to extend the partial coloring of S to all the vertices of S in $O(\log(\Delta/k)) + 2^{O(\sqrt{\log\log n})}$ rounds, w.h.p.

It remains to extend the coloring of S to the dense components $X_1, \ldots, X_t$. Using Lemma 3.2, the coloring of S can then be extended to $X_1, \ldots, X_t$ in $O(\log(\Delta/k)) + 2^{O(\sqrt{\log\log n})}$ rounds, w.h.p. It follows that the overall round complexity is the minimum of $O(\log_\Delta n + \log_k \Delta) + 2^{O(\sqrt{\log\log n})}$ and $2^{O(\log\Delta+\sqrt{\log\log n})}$. In particular, it is w.h.p. $O(\log n/\log\log n)$, for any value of Δ, which concludes the proof of Theorem 1.5. ◀

4 Graphs with chromatic number close to the maximum degree

In this section, we prove the main result of this paper.

We start with a sketch of the proof of Theorem 1.4 (the full proof is given in [1]), and then explain how Theorem 1.3 can be deduced from appropriate parts of the proof of Theorem 1.2 in [17]. It should be noted that our assumption that $c \ge \Delta - k_\Delta + 1$ makes the proof of Theorem 1.3 significantly easier than the proof of Theorem 1.2 in [17], where the main difficulty comes from the case $c = \Delta - k_\Delta$.

4.1 Reducers

A *stable set*, or *independent set*, is a set of pairwise non-adjacent vertices. A *c-reducer* in a graph G is a subset D of vertices consisting of a clique C with $c-1$ vertices and a disjoint stable set S such that every vertex of C is adjacent to all of S but none of $V(G) - D$. Given a graph G with a c-reducer $D = (C, S)$, the graph H obtained from G by removing C and identifying all the vertices of S into a single vertex is called the *reduction* of G with respect to D. Note that G is c-colorable if and only if H is c-colorable, and thus c-reductions preserve c-colorability and non-c-colorability.

Proof of Theorem 1.4 (Sketch). Let Δ be an integer, and assume that either (1) $c \le \Delta - k_\Delta - 1$, or (2) $c = \Delta - k_\Delta$ and $\Delta = (k_\Delta + 1)(k_\Delta + 2)$.

For $i \geq 1$, we define a graph G_i of maximum degree Δ and a subset C_i of G_i inductively as follows. G_1 is the complete graph on $c+1$ vertices, and C_1 is the set of vertices of G_1. For any $i \geq 2$, G_i is obtained from G_{i-1} by removing an arbitrary vertex v_{i-1} of C_{i-1}, adding a stable set S_i of size $\Delta - c + 2$ and a $(c-1)$-clique C_i such that (1) each neighbor of v_{i-1} in G_{i-1} is adjacent to exactly one vertex of S_i, and (2) each vertex of S_i is adjacent to all the vertices of C_i.

In order to make sure that the maximum degree of G_i is at most Δ, while performing (1) we split as evenly as possible the degree of v_{i-1} between the vertices of S_i (each edge between v_{i-1} and some neighbor u in G_{i-1} becomes an edge joining u and some vertex of S_i in G_i, and we want the degrees of the vertices of S to be as balanced as possible). Since $|S_i| = \Delta - c + 2$, each vertex of C_i has degree Δ in G_i. Each vertex of S_i must also have degree at most Δ so it can have up to $\Delta - c + 1$ neighbors in G_{i-1}. Since v_{i-1} has degree at most Δ, and $(\Delta - c + 2)(\Delta - c + 1) \geq \Delta$, the edges incident to v_{i-1} in G_{i-1} can be split among the vertices of S_i in such way that each vertex of S_i has degree at most Δ in G_i.

We now make a couple of remarks on G_i. It can be observed that G_{i-1} is the reduction of G_i with respect to some c-reducer, and since G_1 is a clique on $c+1$ vertices and reductions preserve c-non-colorability, G_i is not c-colorable. It is also easy to see that any proper subgraph of G_i has chromatic number at most c and G_i has diameter at least $\frac{n}{2\Delta}$, where n denotes the number of vertices of G_i.

Let G be the graph obtained from G_i by deleting a single edge between a vertex of layer $i/2$ (i.e. a vertex that was added at step $i/2$) and a vertex of layer $i/2+1$. As a proper subgraph of G_i, G has maximum degree at most Δ and chromatic number at most c, and it can be checked that any ball of radius less than $\frac{n}{8\Delta}$ in G_i is isomorphic to a ball of the same radius in G. Since G_i is not c-colorable, it follows from a classical observation of Linial [15], that G cannot be colored optimally (i.e. with c colors) in less than $\frac{n}{8\Delta}$ rounds. ◀

4.2 Overview of the proof of Theorem 1.3

We start by considering the first part of the statement of Theorem 1.3: if G is not c-colorable, then some vertex is supposed to output a *certificate* that G is not c-colorable. In order to do so, we will use the following result of Molloy and Reed (Theorem 5 in [17]).

▶ **Theorem 4.1.** *For sufficiently large Δ, and for $c \geq \Delta - k_\Delta + 1$, if G has maximum degree at most Δ, and $\chi(G) > c$, then there is some vertex v in G such that the subgraph induced by $\{v\} \cup N(v)$ is not c-colorable.*

In the LOCAL model of computation, testing the c-colorability of all closed neighborhoods (i.e. all the balls of radius 1) in G can be done in a constant number of rounds, and any vertex finding a non c-colorable subgraph in its closed neighborhood can simply output this subgraph as a certificate of non c-colorability of G. It might be worth pointing that we heavily use the unbounded computational power of the nodes (and the unbounded bandwidth of the edges) in the LOCAL model here when $\Delta \gg \log n$. However, when $\Delta = O(\log n)$, all the closed neighborhoods have logarithmic size, so testing their c-colorability takes polynomial time (in n) in any classical model of computation. Moreover, when $\Delta = O(1)$ the same task can be performed in constant time in any classical model of computation.

We can now assume that G is c-colorable, and the goal is to find a c-coloring of G in $\min\{O((\log \Delta)^{13/12} \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds w.h.p. The high-level description of the proof is as follows: we set $d = 10^6 \sqrt{\Delta}$ and start by computing a d-dense decomposition $S, X_1, \ldots, X_t$ of G. We then delete all the sets X_i that are c-reducers or such that $\overline{G[X_i]}$

has a matching of size at least $100\sqrt{\Delta}$. These sets will be colored at the very end, once the rest of the graph will be colored, using a proof very similar to that of Lemma 3.2, in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ additional rounds (Lemmas 4.2 and 4.3). So we can assume that no set X_i is a c-reducer or has a large antimatching. Using this assumption, we then find a specific c-coloring in each set X_i, independently of the other sets X_j, with desirable properties (Lemma 4.4). Using this coloring of each set X_i, we will construct a new graph F from G by contracting the color classes from the dense sets into single vertices, and adding suitable edges at strategic places in the graph (Lemma 4.5). All these contractions and edge additions can be easily simulated in G, since they involve pairs of vertices at distance at most 4 apart. The final part will consist in coloring F with c colors, and from this coloring it will be easy to deduce a c-coloring of G. Note that because of the edge additions and contraction, the maximum degree of F is not bounded by Δ anymore, but it remains $O(\Delta)$. The coloring of F is then obtained by a very intricate semi-random process. Fortunately, for us it boils down to repeated applications of the Lovász Local Lemma (more precisely, $O((\log \Delta)^{13/12})$ successive applications), and we just need to make sure that Theorem 2.4 and 2.5 can be substituted everywhere in the proof (Lemma 4.6). With this high-level view in mind, we now proceed with the proof.

4.3 Proof of Theorem 1.3

Let $d = 10^6\sqrt{\Delta}$. We first compute a d-dense decomposition $S, X_1, \ldots, X_t$ of G in $O(1)$ rounds using Lemma 2.7.

A c-reducer $D = (C, S')$ is said to be *deletable* if there are fewer than c vertices in $G - D$ with a neighbor in S. Observe that if $D = (C, S')$ is a deletable c-reducer in G, then any c-coloring of $G - D$ can be extended to D (since there is a color which does not appear in the neighborhood of S' in $G - D$). It was observed in [17, Observation 8] that when $c \geq \Delta - k_\Delta + 1$, any c-reducer is deletable. It has the following consequence.

▶ **Lemma 4.2.** *Let X^r be the union of all the c-reducers X_i. Then there is a distributed randomized algorithm (running in G) that extends any c-coloring of $G - X^r$ to G in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds, w.h.p.*

Proof. For each c-reducer $X_i = (C_i, S_i)$, perform the reduction of G with respect to X_i (i.e. delete the clique C_i, and identify all the vertices of S_i into a single vertex v_i). Let R be the resulting graph, and let N be the set of newly created vertices in R. Note that the c-coloring of $G - X^r$ corresponds to a c-coloring of $R - N$, and our goal is simply to extend this coloring to R (once this is done, we only have to assign the color of v_i to all the vertices of the stable set S_i in G, and to color C_i with the $c - 1$ colors distinct from that of v_i, which can clearly be done in $O(1)$ rounds). Since each X_i we consider here is deletable, each vertex $v_i \in N$ has degree at most $c - 1$ in R. It follows from Observation 2.3 and Theorem 2.1 (similarly as in Section 3) that the c-coloring of $R - N$ can be extended to N by a distributed randomized algorithm running in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds w.h.p., as desired. ◀

We say that a dense set X_i is *hollow* if $\overline{G[X_i]}$ (the complement of $G[X_i]$) contains a matching of size at least $100\sqrt{\Delta}$. We now rephrase Lemma 16 from [17] for our convenience (the proof of Lemma 4.3 follows the same lines as that of Lemma 3.2).

▶ **Lemma 4.3.** *Let X^h be the union of the all the hollow sets X_i. Then any c-coloring of $G - X^h$ can be extended to G by a distributed randomized algorithm running w.h.p. in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds.*

We temporarily delete from G all the X_i that are c-reducers or hollow. These sets of vertices will be colored at the very end using Lemmas 4.2 and 4.3. Let H be the graph obtained from G by removing the dense components from Lemmas 4.2 and 4.3. Note that the restriction of the decomposition $S, X_1, \dots, X_t$ to H is still a d-dense decomposition of H, and for convenience we keep denoting it in this way (even if some sets X_i have disappeared). It follows from our construction that no dense set X_i in H is a c-reducer or is such that $\overline{H[X_i]}$ contains a matching of size at least $100\sqrt{\Delta}$.

Given a subset Y of vertices from some dense component X_i, an *external neighbor* of Y is a vertex outside of X_i with a neighbor in Y. Recall that a coloring of a graph G partitions the vertex-set of G into stable sets, which are called the *color classes* associated to the coloring. Given a c-coloring of X_i, we define C_i as the set of vertices of X_i whose color class is a singleton. We say that a c-coloring of X_i is *nice* if:

(1) C_i is a clique of size at least $\Delta - 2 \cdot 10^6\sqrt{\Delta}$,

(2) each vertex from any color class of size at least 3 is adjacent to all the vertices of C_i, and

(3) if $\{x, y\}$ is a color class of size 2, then either there is $z \in C_i$ such that x, y are both adjacent to all the vertices of $C_i - \{z\}$, or one of x, y is adjacent to all the vertices of C_i and the other is adjacent to all but at most $\frac{\Delta}{4} + 10^7\sqrt{\Delta}$ vertices of C_i.

Note that the unique c-coloring of a c-reducer is nice. Lemma 4.3 now allows us to use the following result of [17]. The proof heavily uses the crucial property that no dense set X_i contains a large antimatching.

▶ **Lemma 4.4** (Lemmas 19, 20, 21, and 25 in [17]). *Each dense set X_i of H has a nice c-coloring such that:*

(a) *If a color class is not the unique largest colour class in X_i, then it has at most $\frac{\Delta}{2} + 10\sqrt{\Delta}$ external neighbors.*

(b) *Every color class of X_i has at most $c - \sqrt{\Delta} + 3$ external neighbors.*

(c) *If there is a colour class of X_i with more than $c - 10^8\sqrt{\Delta}$ external neighbor, then $|C_i| \geq c - 2 \cdot 10^8$ and each vertex of C_i has at most $3 \cdot 10^8$ external neighbors.*

(d) *If there is a colour class of X_i with more than $c - 2\sqrt{\Delta} + 3$ external neighbours then $|C_i| = c - 1$ and each vertex of C_i has at most 5 external neighbors.*

(e) *If there is a colour class of X_i with more than $c - 2\Delta^{3/4}$ external neighbors then $|C_i| \geq c - 5\Delta^{1/4}$ and each vertex of C_i has at most $8\Delta^{1/4}$ external neighbors.*

We stress that the union of the c-colorings of each of the dense components X_i is not necessarily a c-coloring of the union of the dense components: there might be some edges between vertices of different sets X_i having the same color. It should be noted that parts (b)–(e) of this result, as stated here, look a bit different from their counterparts from Lemma 25 in [17]. Indeed, each of properties (b)–(e) in Lemma 25 from [17] starts by the precondition *"If X_i is not a reducer or a near-reducer"*. We assumed earlier that X_i is not a c-reducer, so this part of the precondition can certainly be omitted in our case. A *c-near-reducer* is a subgraph D which is the union of a clique C of size $c - 1$ and a stable set S' of size $\Delta - c + 1$, such that each vertex of C is adjacent to every vertex of S' (in particular each vertex of C has at most one neighbor outside D). Note that each vertex of S' has at most $\Delta - c + 1$ neighbors outside D, and thus S' has at most $(\Delta - c + 1)^2$ neighbors outside D. Since $c \geq \Delta - k_\Delta + 1$ and $\sqrt{\Delta} - 3 < k_\Delta = \left\lfloor \sqrt{\Delta + 1/4} - 3/2 \right\rfloor \leq \sqrt{\Delta + 1/4} - 3/2$, S' has at most

$$(\Delta - c + 1)^2 \leq k_\Delta^2 \leq \Delta - 3k_\Delta - 2 \leq c - 2k_\Delta - 3 \leq c - 2\sqrt{\Delta} + 3$$

neighbors outside D. In particular, in our case (i.e. when $c \geq \Delta - k_\Delta + 1$), any dense set X_i which is a c-near-reducer satisfies Lemma 4.4(a)–(e), so we can indeed remove the preconditions from Lemma 25 in [17]. Note also that since each dense set X_i has diameter at most 2, a nice coloring of each X_i with the additional properties of Lemma 4.4 can be found in $O(1)$ rounds.

Based on the nice c-coloring of each of the dense components X_i resulting from Lemma 4.4, we now construct (locally) a new graph F from H, which will be easier to color with a semi-random procedure, and such that any c-coloring of F can be turned (locally and efficiently) into a c-coloring of H.

▶ **Lemma 4.5** (Lemma 12 as used in the proof of Theorem 43 in [17]). *We can construct locally in H in $O(1)$ rounds a graph F of maximum degree at most $10^9\Delta$ (such that a c-coloring of H can be deduced from any c-coloring of F in $O(1)$ rounds) and find a partition of the vertices of F into $S, B, A_1, \ldots, A_t$ such that:*

(a) *Every A_i is a clique with $c - 10^8\sqrt{\Delta} \leq |A_i| \leq c$.*
(b) *Every vertex of A_i has at most $10^8\sqrt{\Delta}$ neighbors in $F - A_i$.*
(c) *There is a set $\mathrm{All}_i \subseteq B$ of $c - |A_i|$ vertices which are adjacent to all of A_i. Every other vertex of $F - A_i$ is adjacent to at most $\frac{3}{4}\Delta + 10^8\sqrt{\Delta}$ vertices of A_i.*
(d) *Every vertex of S either has fewer than $\Delta - 3\sqrt{\Delta}$ neighbors in S or has at least $900\Delta^{3/2}$ non-adjacent pairs of neighbors within S.*
(e) *Every vertex of B has fewer than $c - \sqrt{\Delta} + 9$ neighbors in $F - \bigcup_j A_j$.*
(f) *If a vertex $\in B$ has at least $c - \Delta^{3/4}$ neighbors in $F - \bigcup_j A_j$, then there is some i such that: v has at most $c - \sqrt{\Delta} + 9$ neighbors in $F - A_i$ and every vertex of A_i has at most $30\Delta^{1/4}$ neighbors in $F - A_i$.*
(g) *For every A_i, every two vertices outside of $A_i \cup \mathrm{All}_i$ which have at least $2\Delta^{9/10}$ neighbors in A_i are joined by an edge of F.*

There is one subtlety in the application of Lemma 12 from [17]: the statement of Lemma 12 there start with the precondition *"For any minimum counterexample"*. Here we avoid this precondition in the same way Molloy and Reed avoid it in their application of Lemma 12 in the algorithmic proof of Theorem 43 from [17] (by starting to remove deletable reducers and hollow sets).

We explain briefly how the graph F is constructed in [17] to stress that the construction can indeed be performed locally in H (and then in G).

The construction starts by doing the following for each colored dense component X_i. Recall that C_i was defined above as the set of vertices of X_i whose color class is a singleton, and it follows from the definition of a nice coloring that C_i is a clique of size at least $\Delta - 2 \cdot 10^6\sqrt{\Delta}$. Now, each color class of size at least 2 (i.e. each color class which is not a singleton) in X_i is contracted into a single vertex, and vertices and edges are added inside X_i to make it into a clique D_i of size precisely c. It can be proved using Lemma 4.4 that the maximum degree does not increase too much and that each clique D_i is not much larger than C_i (see Lemma 29 in [17]).

A significant issue when trying to find a c-coloring of H (or rather the current modification of H) is that given a clique D_i, there might be vertices outside D_i that have many neighbors (say more than $\frac{3\Delta}{4}$) in C_i. Each such vertex must be in $D_j - C_j$, for some $j \neq i$. Consider such a vertex $v \in D_j - C_j$, with many neighbors in C_i. We need to make sure that the color of v will be used by one of the few non-neighbors of v in D_i, and one way to do it is, for some vertices $w \in D_i$, to construct a set R_w of vertices with many neighbors in C_i such that

$\{w\} \cup R_w$ is a stable set and every vertex with many neighbors in C_i lies in such a set R_w. We then contract each set $\{w\} \cup R_w$ into a single vertex (this will force that all these vertices have the same color at the end), and denote by A_i the set C_i after the removal of the vertices w for which some set R_w was defined. We also set $\text{All}_i = D_i - A_i$. Again it can be proved that the maximum degree does not increase too much and each A_i is not too small compared to C_i (see Lemma 30 in [17]).

A second issue (related to the issue described above) is that we need to prevent that many different external neighbors of A_i are all colored with the same color, and their neighborhoods cover A_i (this would prevent this color from being used in A_i). The way it is solved in [17] is by adding an edge between every pair of external neighbors of A_i having at least $\Delta^{9/10}$ neighbors in A_i. It is proved (see Lemma 31 in [17]) that it does not increase the maximum degree too much and is enough to deduce properties (a)–(g) of Lemma 4.5 (the issue raised in this paragraph is in particular related to property (g)).

To sum up, F has been obtained from H by identifying (or adding edges between) pairs of vertices at distance at most 4, since each dense component has diameter at most 2 and any two vertices that have been identified or joined by an edge have a neighbor in the same dense component. Moreover, each modification has been carried out independently by each dense set X_i (even if the modifications had some impact outside of X_i), so F can be simulated by H (and then by G) with at most a small multiplicative loss on the round complexity. It is also clear that a c-coloring of H can be obtained from any c-coloring of F in $O(1)$ rounds.

It remains to show how to efficiently color F with c colors.

▶ **Lemma 4.6.** *The graph F described in Lemma 4.5 can be colored with c colors in $\min\{O((\log \Delta)^{13/12} \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds, w.h.p.*

We will be rather brief here (the proof of the corresponding sequential statement, Lemma 13 in [17], takes 20 pages). Consider some $1 \leq i \leq t$. Since $\text{All}_i \cup A_i$ forms a clique of size c, we need to make sure that the colors that do not appear in All_i do not appear either on too many external neighbors of A_i. A key property of the construction of F (see properties (c) and (g) in Lemma 4.5) is that for any color x, there is at most one vertex $v \notin \text{All}_i \cup A_i$ having at least $2\Delta^{9/10}$ neighbors in A_i that is colored x, and moreover v has at most $\frac{3}{4}\Delta + o(\Delta)$ neighbors in A_i. The goal will be to maintain this property throughout the whole process, namely that all of the time, at most $\frac{3}{4}\Delta + o(\Delta)$ vertices of A_i have a neighbor colored x outside of $\text{All}_i \cup A_i$ (let us call this event $E(i, x)$).

The starting point will be to color S (the d-sparse vertices, see property (d) of Lemma 4.5) randomly as in the proof of Lemma 3.1, i.e. with the property that many colors are repeated in the neighborhoods of the high degree vertices, but also with the additional property that $E(i, x)$ still holds for any i, x after the coloring.

We then proceed to extend the coloring to B. Recall that by property (e) of Lemma 4.5, each vertex of B has at most $c - \Omega(\sqrt{\Delta})$ neighbors in $F - \bigcup_j A_j$. It turns out that it is a bit too high to extend randomly the coloring of S to B while maintaining property $E(i, x)$, so instead we color the remaining vertices in this order:

1. We first color the set B_H of vertices of B with at most $c - \Delta^{3/4}$ neighbors in $F - \bigcup_j A_j$ (coloring these vertices will preserve $E(i, x)$).
2. We then color the sets A_i such that each vertex of A_i has at most $30\Delta^{1/4}$ neighbors outside of $\text{All}_i \cup A_i$.
3. We color $B_L = B - B_H$, using property (f) of Lemma 4.5 (which implies that property $E(i, x)$ can now be preserved while coloring these vertices).
4. Finally we color the sets A_i that have not been colored yet.

The proofs that desirable properties are maintained during the coloring of the vertices of S and B and the A_i are fairly similar to the proof of Lemma 3.1 (see the full version of the paper [1]), in the sense that they boil down to the estimation of the expectation of some random variables, the proof that these random variables are highly concentrated, and then some application of the Lovász Local Lemma.

We should note two important differences, though.

- The first is that instead of a single random partial coloring, followed by a greedy procedure completing the coloring, the process for coloring S, B_H, and B_L here involves multiple rounds (more specifically, at most $O((\log \Delta)^{13/12})$ rounds) of random partial coloring and a careful study of all the random variables throughout the process.
- The second is that while coloring the A_i, the partial random coloring procedure is a bit different than in the proof of Lemma 3.1. Recall that each A_i is a clique, so assigning each vertex a color uniformly at random, and then uncoloring pairs of vertices with the same color would be extremely unpractical. Instead, each A_i is colored with a permutation of the $|A_i|$ colors not appearing on All_i, taken uniformly at random among all the possible permutations. A consequence is that instead of using Talagrand's Inequality to prove the concentration of random variables around their expectation, McDiarmid's Inequality has to be used instead (see [17]), but the resulting bounds are of a similar order of magnitude.

It can be checked that in all the applications of the Lovász Local Lemma in [17], bad events correspond to subgraphs of H of bounded radius, and the probabilities of the bad events are smaller than any fixed polynomial function of the maximum degree of the event dependency graph (these probabilities are typically of order $\exp(-d^\alpha)$ or $\exp(-\beta \log^2 d)$, where $\alpha, \beta > 0$ and d is the maximum degree of the event dependency graph), so in particular Theorem 2.4 and 2.5 can be substituted everywhere in the proof, and since the semi-random process involves at most $O((\log \Delta)^{13/12})$ successive applications of the Lovász Local Lemma[1], the c-coloring of F can be obtained in $\min\{O((\log \Delta)^{13/12} \log n), 2^{O(\log \Delta + \sqrt{\log \log n})}\}$ rounds, w.h.p.

We find it necessary to insist on a technical (but important) detail here. Theorems 2.4 and 2.5 use the so-called *variable setting* of the Local Lemma, which covers most applications of the original Local Lemma but not all of them. In particular we have to be careful here since the coloring of the A_i involved random permutations of colors assigned to a given set of vertices, instead of colors chosen uniformly at random for each vertex, and it is not clear at first sight whether the former can be handled in the variable setting. It turns out that it can, since in the proof of Lemmas 39 and 40 in [17] the graph under consideration has one vertex for each uncolored A_i, and an edge between two vertices if the corresponding sets A_i are adjacent in H (since each set A_i is a clique, this graph can be simulated within H). The variable associated to each vertex is the random permutation of colors assigned to the corresponding set A_i, so this is indeed an instance of the variable setting of the Local Lemma, and we can use Theorems 2.4 and 2.5.

[1] In the proof of Molloy and Reed [17] the authors use $O(\Delta^\lambda)$ successive applications of the Local Lemma (for any fixed constant $\lambda > 0$), but the proof can easily be optimized to work with only $O((\log \Delta)^{13/12})$ applications of the Local Lemma. The bound $\log^{13/12} \Delta$ comes from the proof of the concentration of Z'_C, page 175 of [17], which dominates the other related bounds on the number of iterations in the proof of Lemma 34 of [17]. Note that the authors of [17] were aiming at a polynomial complexity, so it did not make much sense for them to replace the polynomial number of iterations by a polylogarithmic number of iterations, at the cost of tedious computations.

Now that F has been colored with c colors, we obtain a c-coloring of H in $O(1)$ rounds using Lemma 4.5, and it remains to color the dense components X_i that are c-reducers, or such that $\overline{G[X_i]}$ contains a matching of size at least $100\sqrt{\Delta}$ (recall that these dense components had been removed from the graph at the beginning of the procedure). It follows from Lemmas 4.2 and 4.3 that the c-coloring of H can be extended to the remaining dense components of G w.h.p. in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds, which concludes the proof of Theorem 1.3. ◀

4.4 Summary of our contributions

We now make a brief summary of our contributions (to make clear what we added and subtracted from the proof of Molloy and Reed [17]).

In [17], c-reducers are dealt with slightly differently: some are simply removed as we do here, but some are reduced as in the definition of c-reduction of Section 4.1 (i.e. by removing the clique and contracting the stable set into a single vertex). This operation can create new c-reducers, and thus c-reducers have to be reduced sequentially until no c-reducer appears in the graph (the fact that it has to be done sequentially is essentially the proof of Theorem 1.4). For c-near-reducers, the situation is slightly more complicated (see Lemma 27 in [17]) but again inherently sequential. It is fortunate that in our case (i.e. when $c \geq \Delta - k_\Delta + 1$), we do not need to worry about these cases, as explained after Lemma 4.4. So our contribution is simply to have checked that the initial d-dense decomposition can be computed locally (see Lemma 2.7), that the construction of F can be performed locally, that all the applications of the Local Lemma can be also carried out locally in the phase where the c-coloring of F is obtained, and that the resulting coloring of H can be extended to G locally and efficiently (see Lemmas 4.2 and 4.3).

5 Concluding remarks

Note that using recent results of Ghaffari *et al.* [11], the randomized algorithms in Theorem 1.3 and 1.5 can be replaced by deterministic algorithms with a round complexity of $2^{O(\log \Delta + \sqrt{\log n})}$. An interesting question is whether the dependency in Δ can be significantly reduced (the same question can be asked for Theorem 1.5 and 1.3). It seems to us that techniques that have been developed so far, such as Theorem 1.8 in [11] or the ad-hoc techniques from [9], do not work well in our case.

When the maximum degree Δ is a constant, the list-coloring problem where every vertex v has a list of at least $d(v) + 1$ colors can be solved in $O(\log^* n)$ rounds [13, 15], which is much faster than the round complexity of Theorems 2.1 and 2.2. In this case it is interesting to use a slightly faster version of Theorem 2.5 from [11], with round complexity $\exp(\exp(O(\sqrt{\log \log \log n})))$, or $\exp(\exp(\exp(O(\sqrt{\log \log \log \log n}))))$, or more generally $\exp^{(i)}(O(\sqrt{\log^{(i+1)} n}))$ for any $1 \leq i \leq \log^* n - 2 \log^* \log^* n$. It is not difficult to see that in this case this round complexity dominates the other parts of the algorithms used in this paper. It follows that the round complexity in Theorem 1.3 and 1.5 in the bounded degree case can be replaced by $\exp^{(i)}(O(\sqrt{\log^{(i+1)} n}))$ for any $1 \leq i \leq \log^* n - 2 \log^* \log^* n$. Moreover, any improvement on the round complexity of the distributed Lovász Local Lemma under some criterion would immediately yield an improved complexity in Theorems 1.5 and 1.3 in the case of bounded degree graphs.

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