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# A Note on Some Isomorphic Properties in Projective Tensor Products

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*Abstract*: A Banach space $X$ is sequentially Right (resp. weak sequentially Right) if every Right subset of $X^*$ is relatively weakly compact (resp. weakly precompact). A Banach space $X$ has the $L$-limited (resp. the $wL$-limited) property if every $L$-limited subset of $X^*$ is relatively weakly compact (resp. weakly precompact). We study Banach spaces with the weak sequentially Right and the $wL$-limited properties. We investigate whether the projective tensor product of two Banach spaces $X$ and $Y$ has the sequentially Right property when $X$ and $Y$ have the respective property.

*Key words*: $R$-sets, $L$-limited sets, sequentially Right spaces, $L$-limited property.

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## 1. Introduction

A bounded subset $A$ of a Banach space $X$ is called a *Dunford-Pettis* (DP) (resp. *limited*) subset of $X$ if every weakly null (resp. $w^*$-null) sequence $(x_n^*)$ in $X^*$ tends to 0 uniformly on $A$; i.e.,

$$\lim_n \left( \sup\{|x_n^*(x)| : x \in A\} \right) = 0.$$

A sequence $(x_n)$ is DP (resp. limited) if the set $\{x_n : n \in \mathbb{N}\}$ is DP (resp. limited).

A subset $S$ of $X$ is said to be *weakly precompact* provided that every sequence from $S$ has a weakly Cauchy subsequence. Every DP (resp. limited) set is weakly precompact [37, p. 377], [1] (resp. [4, Proposition]).

An operator $T : X \to Y$ is called *weakly precompact (or almost weakly compact)* if $T(B_X)$ is weakly precompact and *completely continuous (or Dunford-Pettis)* if $T$ maps weakly convergent sequences to norm convergent sequences.

In [35] the authors introduced the *Right topology* on a Banach space $X$. It is the restriction of the Mackey topology $\tau(X^{**}, X)$ to $X$ and it is also the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$ compact subsets of $X^*$. Further, $\tau(X^{**}, X)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of $X^*$ [26].

A sequence $(x_n)$ in a Banach space $X$ is Right null if and only if it is weakly null and DP (see Proposition 1).

An operator $T : X \to Y$ is called *pseudo weakly compact (pwc)* (or *Dunford-Pettis completely continuous (DPcc)*) if it takes Right null sequences in $X$ into norm null sequences in $Y$ ([35], [25]). Every completely continuous operator $T : X \to Y$ is pseudo weakly compact. If $T : X \to Y$ is an operator with weakly precompact adjoint, then $T$ is a pseudo weakly compact operator ([18, Corollary 5]).

A subset $K$ of $X^*$ is called a *Right set (R-set)* if each Right null sequence $(x_n)$ in $X$ tends to 0 uniformly on $K$ [26]; i.e.,

$$\lim_n \big(\sup\{|x^*(x_n)| : x^* \in K\}\big) = 0.$$

A Banach space $X$ is said to be *sequentially Right (SR)* (has *property (SR)*) if every pseudo weakly compact operator $T : X \to Y$ is weakly compact, for any Banach space $Y$ [35]. Banach spaces with property $(V)$ are sequentially Right ([35, Corollary 15]).

A subset $A$ of a dual space $X^*$ is called an *L-limited set* if every weakly null limited sequence $(x_n)$ in $X$ converges uniformly on $A$ [39]; i.e.,

$$\lim_n \big(\sup\{|x^*(x_n)| : x^* \in A\}\big) = 0.$$

A Banach space $X$ has the *L-limited property* if every $L$-limited subset of $X^*$ is relatively weakly compact [39]. An operator $T : X \to Y$ is called *limited completely continuous (lcc)* if $T$ maps weakly null limited sequences to norm null sequences [40].

In this paper we introduce the weak sequentially Right ($wSR$) and $wL$-limited properties. A Banach space $X$ is said to have the *weak sequentially Right* ($wSR$) (resp. the *$wL$-limited*) *property* if every Right (resp. $L$-limited) subset of $X^*$ is weakly precompact. We obtain some characterizations of these properties with respect to some geometric properties of Banach spaces, such as the Gelfand-Phillips property, the Grothendieck property, and properties $(wV)$ and $(wL)$. We generalize some results from [39]. We also show that property $(SR)$ can be lifted from a certain subspace of $X$ to $X$.

We study whether the projective tensor product $X \otimes_\pi Y$ has the $(SR)$ (resp. the $L$-limited) property if $L(X, Y^*) = K(X, Y^*)$, and $X$ and $Y$ have the respective property. We prove that in some cases, if $X \otimes_\pi Y$ has the $(wSR)$ property, then $L(X, Y^*) = K(X, Y^*)$.

## 2. Definitions and notation

Throughout this paper, $X$, $Y$, $E$, and $F$ will denote Banach spaces. The unit ball of $X$ will be denoted by $B_X$ and $X^*$ will denote the continuous linear dual of $X$. An operator $T : X \to Y$ will be a continuous and linear function. We will denote the canonical unit vector basis of $c_0$ by $(e_n)$ and the canonical unit vector basis of $\ell_1$ by $(e_n^*)$. The set of all operators, weakly compact operators, and compact operators from $X$ to $Y$ will be denoted by $L(X,Y)$, $W(X,Y)$, and $K(X,Y)$. The projective tensor product of $X$ and $Y$ will be denoted by $X \otimes_\pi Y$.

A bounded subset $A$ of $X^*$ is called an *L-set* if each weakly null sequence $(x_n)$ in $X$ tends to 0 uniformly on $A$; i.e.,

$$\lim_n \big( \sup\{|x^*(x_n)| : x^* \in A\}\big) = 0.$$

A Banach space $X$ has *the Dunford-Pettis property* ($DPP$) if every weakly compact operator $T : X \to Y$ is completely continuous, for any Banach space $Y$. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the $DPP$. The reader can check [8], [9], and [10] for a guide to the extensive classical literature dealing with the $DPP$.

A Banach space $X$ has the *Dunford-Pettis relatively compact property (DPrcP)* if every Dunford-Pettis subset of $X$ is relatively compact [14]. Schur spaces have the $DPrcP$. The space $X$ does not contain a copy of $\ell_1$ if and only if $X^*$ has the $DPrcP$ if and only if every $L$-set in $X^*$ is relatively compact ([14, Theorem 1], [13, Theorem 2]).

The space $X$ has the *Gelfand-Phillips (GP)* property if every limited subset of $X$ is relatively compact. The following spaces have the Gelfand-Phillips property: Schur spaces; spaces with $w^*$-sequential compact dual unit balls (for example subspaces of weakly compactly generated spaces, separable spaces, spaces whose duals have the Radon-Nikodým property, reflexive spaces, and spaces whose duals do not contain $\ell_1$); dual spaces $X^*$ whith $X$ not containing $\ell_1$; Banach spaces with the separable complementation property, i.e., every separable subspace is contained in a complemented separable subspace (for example $L_1(\mu)$ spaces, where $\mu$ is a positive measure) [42, p. 31], [4, Proposition], [12, Theorem 3.1 and p. 384], [11, Proposition 5.2], [13, Corollary 5].

A series $\sum x_n$ in $X$ is said to be *weakly unconditionally convergent (wuc)* if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T : X \to Y$ is called *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

A bounded subset $A$ of $X^*$ is called a *$V$-subset* of $X^*$ provided that

$$\lim_n \big(\sup\{\, |x^*(x_n)| : x^* \in A\,\}\big) = 0)$$

for each wuc series $\sum x_n$ in $X$.

A Banach space $X$ has *property* $(V)$ if every $V$-subset of $X^*$ is relatively weakly compact [33]. A Banach space $X$ has property $(V)$ if every unconditionally converging operator $T$ from $X$ to any Banach space $Y$ is weakly compact [33, Proposition 1]. $C(K)$ spaces and reflexive spaces have property $(V)$ ([33, Theorem 1, Proposition 7]). A Banach space $X$ has *property* $(wV)$ if every $V$-subset of $X^*$ is weakly precompact [41].

A Banach space $X$ has the *reciprocal Dunford-Pettis property* $(RDPP)$ if every completely continuous operator $T$ from $X$ to any Banach space $Y$ is weakly compact. The space $X$ has the $RDPP$ if and only if every $L$-set in $X^*$ is relatively weakly compact [28]. Banach spaces with property $(V)$ have the $RDPP$ [33]. A Banach space $X$ has *property* $(wL)$ if every $L$-set in $X^*$ is weakly precompact [19].

A topological space $S$ is called *dispersed (or scattered)* if every nonempty closed subset of $S$ has an isolated point. A compact Hausdorff space $K$ is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [34, Main theorem].

The Banach-Mazur distance $d(X, Y)$ between two isomorphic Banach spaces $X$ and $Y$ is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infinum is taken over all isomorphisms $T$ from $X$ onto $Y$. A Banach space $X$ is called an *$\mathcal{L}_\infty$-space* (resp. *$\mathcal{L}_1$-space*) [5, p. 7] if there is a $\lambda \geq 1$ so that every finite dimensional subspace of $X$ is contained in another subspace $N$ with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer $n$. Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$) spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) ([5, Proposition 1.26]). The dual of an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp. $\mathcal{L}_1$- space) ([5, Proposition 1.27]). The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the $DPP$ ([5, Corollary 1.30]).

## 3. The weak sequentially Right and $wL$-limited properties

The following result gives a characterization of Right null sequences.

Proposition 1. *A sequence $(x_n)$ in a Banach space $X$ is Right null if and only if it is weakly null and DP.*

*Proof.* Suppose that $(x_n)$ is a Right null sequence in $X$. Then $(x_n)$ is weakly null, since the Right topology is stronger than the weak topology.

Let $(x_n^*)$ be a weakly null sequence in $X^*$. Since $\{x_n^* : n \in \mathbb{N}\}$ is relatively weakly compact in $X^*$ and $(x_n)$ is Right null, $(x_n)$ converges uniformly on $\{x_n^* : n \in \mathbb{N}\}$. Therefore $\lim_n \sup_i |x_i^*(x_n)| = 0$, and thus $\lim_n |x_n^*(x_n)| = 0$. Hence $\{x_n : n \in \mathbb{N}\}$ is a DP set.

Suppose that $(x_n)$ is a weakly null DP sequence. Let $K$ be a relatively weakly compact subset of $X^*$. Suppose that $(x_n)$ does not converge uniformly on $K$. Let $\epsilon > 0$ and let $(x_n^*)$ be a sequence in $K$ so that $|x_n^*(x_n)| > \epsilon$ for all $n$. Without loss of generality suppose that $(x_n^*)$ converges weakly to $x^*$, $x^* \in X^*$. Since $(x_n^* - x^*)$ is weakly null in $X^*$ and $(x_n)$ is DP, $\lim_n (x_n^* - x^*)(x_n) = 0$. Thus $\lim_n x_n^*(x_n) = 0$, a contradiction. Hence $(x_n)$ converges uniformly to zero on $K$, and thus $(x_n)$ is Right null. ∎

A Banach space $X$ is sequentially Right if and only if every Right subset of $X^*$ is relatively weakly compact [26, Theorem 3.25]. A Banach space $X$ has the $L$-limited property if and only if every limited completely continuous operator $T : X \to Y$ is weakly compact, for every Banach space $Y$ [39, Theorem 2.8]. In the next theorem we give elementary operator theoretic characterizations of weak precompactness, relative weak compactness, and relative norm compactness for Right sets and $L$-limited sets. The argument contains the theorems in [26] and [39] just cited.

We say that a Banach space $X$ is *weak sequentially Right (wSR)* or has the *($wSR$) property* (resp. has the *$wL$-limited property*) if every Right (resp. $L$-limited) subset of $X^*$ is weakly precompact. If $\ell_1 \not\hookrightarrow X^*$, then $X$ is weak sequentially Right and has the $wL$-limited property, by Rosenthal's theorem ([8, Ch. XI]).

Theorem 2. *Let $X$ be a Banach space. The following assertions are equivalent:*

1. (i) *For every Banach space $Y$, every pseudo weakly compact operator $T : X \to Y$ has a weakly precompact (weakly compact, resp. compact) adjoint.*
   (ii) *Every pseudo weakly compact operator $T : X \to \ell_\infty$ has a weakly precompact (weakly compact, resp. compact) adjoint.*
   (iii) *Every Right subset of $X^*$ is weakly precompact (relatively weakly compact, resp. relatively compact).*

2. (i) *For every Banach space $Y$, every limited completely continuous operator $T : X \to Y$ has a weakly precompact (weakly compact, resp. compact) adjoint.*

   (ii) *Every limited completely continuous operator $T : X \to \ell_\infty$ has a weakly precompact (weakly compact, resp. compact) adjoint.*

   (iii) *Every $L$-limited subset of $X^*$ is weakly precompact (relatively weakly compact, resp. relatively compact).*

*Proof.* We will show that 1.(i)⇒1.(ii)⇒1.(iii)⇒1.(i) in the weakly precompact case as well as 2.(i)⇒2.(ii)⇒2.(iii)⇒2.(i) in the compact case. These two arguments are similar, and the arguments for the remaining implications of the theorem follow the same pattern.

1. (*weakly precompact*) (i)⇒(ii) is clear.

(ii)⇒(iii) Let $K$ be a Right subset of $X^*$ and let $(x_n^*)$ be a sequence in $K$. Define $T : X \to \ell_\infty$ by $T(x) = (x_i^*(x))$. Let $(x_n)$ be a Right null sequence in $X$. Since $K$ is a Right set,

$$\lim_n \|T(x_n)\| = \lim_n \sup_i |x_i^*(x_n)| = 0.$$

Therefore $T$ is pseudo weakly compact, and thus $T^* : \ell_\infty^* \to X^*$ is weakly precompact. Hence $(T^*(e_n^*)) = (x_n^*)$ has a weakly Cauchy subsequence.

(iii)⇒(i) Let $T : X \to Y$ be a pseudo weakly compact operator. Let $(x_n)$ be a Right null sequence in $X$. If $y^* \in B_{Y^*}$, $\langle T^*(y^*), x_n\rangle \le \|T(x_n)\| \to 0$. Then $T^*(B_{Y^*})$ is a Right subset of $X^*$. Therefore $T^*(B_{Y^*})$ is weakly precompact, and thus $T^*$ is weakly precompact.

2. (*compact*) (i)⇒(ii) is clear.

(ii)⇒(iii) Let $K$ be an $L$-limited subset of $X^*$ and let $(x_n^*)$ be a sequence in $K$. Define $T : X \to \ell_\infty$ as above and note that $T$ is limited completely continuous. Thus $T^* : \ell_\infty^* \to X^*$ is compact, and $(T^*(e_n^*)) = (x_n^*)$ has a norm convergent subsequence.

(iii)⇒(i) Let $T : X \to Y$ be a limited completely continuous operator. Let $(x_n)$ be a weakly null limited sequence in $X$. If $y^* \in B_{Y^*}$, $\langle T^*(y^*), x_n\rangle \le \|T(x_n)\| \to 0$. Then $T^*(B_{Y^*})$ is an $L$-limited subset of $X^*$. Therefore $T^*(B_{Y^*})$ is relatively compact, and thus $T^*$ is compact. ∎

COROLLARY 3. *If $X$ is weak sequentially Right (has the $wL$-limited, resp. the $L$-limited property), then every quotient space of $X$ has the same property.*

*Proof.* We only prove the result for the weak sequentially Right property. The proofs for the other properties are similar.

Suppose that $X$ is weak sequentially Right. Let $Z$ be a quotient space of $X$ and $Q : X \to Z$ be a quotient map. Let $T : Z \to E$ be a pseudo weakly compact operator. Then $TQ : X \to E$ is pseudo weakly compact, and thus $(TQ)^*$ is weakly precompact by Theorem 2. Since $Q^*T^*(B_E^*)$ is weakly precompact and $Q^*$ is an isomorphism, $T^*(B_E^*)$ is weakly precompact. Apply Theorem 2. ∎

COROLLARY 4. *Suppose $X$ is weak sequentially Right and $Y$ is a Banach space. Then an operator $T : X \to Y$ is pseudo weakly compact if and only if $T^* : Y^* \to X^*$ is weakly precompact.*

*Proof.* If $T : X \to Y$ is pseudo weakly compact, then $T^* : Y^* \to X^*$ is weakly precompact by Theorem 2, since $X$ is weak sequentially Right.

The converse follows from [18, Corollary 5]. ∎

COROLLARY 5. (i) *If $X$ is weak sequentially Right (resp. has the $wL$-limited property), then every pseudo weakly compact (resp. limited completely continuous) operator $T : X \to Y$ is weakly precompact.*

(ii) *If $X$ is an infinite dimensional space with the Schur property, then $X$ is not weak sequentially Right (resp. does not have the $wL$-limited property).*

(iii) *If $X$ is weak sequentially Right (resp. has the $wL$-limited property), then $\ell_1 \overset{c}{\not\hookrightarrow} X$.*

*Proof.* (i) Suppose $X$ is weak sequentially right (resp. has the $wL$-limited property). Let $T : X \to Y$ be pseudo weakly compact (resp. limited completely continuous). Then $T^*$ is weakly precompact by Theorem 2. Hence $T$ is weakly precompact, by [2, Corollary 2].

(ii) Since $X$ has the Schur property, the identity operator $i : X \to X$ is pseudo weakly compact (resp. limited completely continuous). Since $X$ is an infinite dimensional space with the Schur property, $i$ is not weakly precompact. Apply (i).

(iii) Apply Corollary 3 and (ii). ∎

COROLLARY 6. *A Banach space $X$ has the $L$-limited property if every separable subspace of $X$ has the same property.*

*Proof.* Let $T : X \to Y$ be a limited completely continuous operator. Then for every closed subspace $Z$ of $X$, $T|_Z$ is limited completely continuous. Let

$(x_n)$ be a sequence in $B_X$ and let $Z = [x_n : n \in \mathbb{N}]$ be the closed linear span of $(x_n)$. Since $Z$ is a separable subspace of $X$, $Z$ has the $L$-limited property. Since $T|_Z$ is limited completely continuous, it is weakly compact by Theorem 2. Then there is a subsequence $(x_{n_k})$ of $(x_n)$ so that $(T(x_{n_k}))$ is weakly convergent. Thus $T$ is weakly compact. Apply Theorem 2. ∎

Example. Corollary 6 cannot be reversed. Indeed, consider $\ell_1$ as a subspace of $\ell_\infty$. By [39, Theorem 2.11], $\ell_\infty$ has the $L$-limited property. However, $\ell_1$ does not have the $L$-limited property, by [39, Corollary 2.9] (or Corollary 5 (ii)).

Theorem 7. *The Banach space $X$ has the DPP if and only if every Right subset of $X^*$ is an $L$-set.*

*Proof.* Suppose $X$ has the $DPP$. Then every weakly null sequence $(x_n)$ is DP ([9, Theorem 1]). Therefore every Right subset of $X^*$ is an $L$-set.

Conversely, let $T : X \to Y$ be a pseudo weakly compact operator. Then $T^*(B_{Y^*})$ is a Right subset of $X^*$, hence an $L$-set. Therefore $T$ is completely continuous, and thus $X$ has the $DPP$ by [26, Proposition 3.17], [25, Theorem 1.5], [18, Theorem 10]. ∎

Corollary 8. *Suppose that $X$ has the DPP. Then the following are equivalent:*

(i) *$X$ does not contain a copy of $\ell_1$.*

(ii) *Every $L$-set in $X^*$ is relatively compact.*

(iii) *Every Right subset of $X^*$ is relatively compact.*

(iv) *$X^*$ has the Schur property.*

*Proof.* (i)⇔(ii) by [13, Theorem 2]. (ii)⇔(iii) by Theorem 7. (i)⇔(iv) by [9, p. 23]. ∎

Corollary 9. *$X^*$ has the Schur property if and only if every Right subset of $X^*$ is relatively compact.*

*Proof.* If $X^*$ has the Schur property, then $X$ has the $DPP$ and $X$ does not contain a copy of $\ell_1$ ([9, p. 23]). Hence every Right subset of $X^*$ is relatively compact by Corollary 8.

Conversely, let $(x_n^*)$ be a weakly Cauchy sequence in $X^*$. Then $(x_n^*)$ is a Right set, by the proof of [26, Corollary 3.26]. Thus $(x_n^*)$ is relatively compact, and $X^*$ has the Schur property. ∎

COROLLARY 10. (i) *Suppose $X$ has the DPP and $Y$ has the DPrcP. Then any operator $T : X \to Y$ is completely continuous.*

(ii) *The space $X$ has the DPP and the DPrcP if and only if $X$ has the Schur property.*

*Proof.* (i) Let $T : X \to Y$ be an operator. Since $Y$ has the $DPrcP$, $T$ is pseudo weakly compact. Then $T^*(B_{Y^*})$ is a Right set, thus an $L$-set in $X^*$ (by Theorem 7). Hence $T$ is completely continuous.

(ii) Suppose $X$ has the $DPP$ and the $DPrcP$. Then the identity operator $i : X \to X$ is completely continuous by (i). Hence $X$ has the Schur property. If $X$ has the Schur property, then $X$ has the $DPP$ and the $DPrcP$. ∎

Corollary 10 (i) generalizes [13, Corollary 6] when $Y$ is a dual space $E^*$ with $E$ not containing $\ell_1$ (since $E^*$ has the $DPrcP$ [14, Theorem 1]).

A bounded subset $A$ of $X^*$ is called *$w^*$- sequentially compact* if every sequence from $A$ has a subsequence which converges to a point in the $w^*$-topology of $X^*$.

The following theorem generalizes [39, Theorem 2.2 (b), (c)].

THEOREM 11. *If $(x_n^*)$ is a $w^*$-Cauchy sequence in $X^*$, then $\{x_n^* : n \in \mathbb{N}\}$ is an $L$-limited set.*

*Proof.* Supppse that $(x_n^*)$ is a $w^*$-Cauchy sequence in $X^*$ and $\{x_n^* : n \in \mathbb{N}\}$ is not an $L$-limited set. By passing to a subsequence if necessary, there is an $\epsilon > 0$ and a weakly null limited sequence $(x_n)$ in $X$ such that $|x_n^*(x_n)| > \epsilon$ for all $n$. Let $k_1 = 1$ and choose $k_2 > k_1$ so that $|x_{k_1}^*(x_{k_2})| < \epsilon/2$. We can do this since $(x_n)$ is weakly null. Continue inductively. Choose $k_n > k_{n-1}$ so that $|x_{k_{n-1}}^*(x_{k_n})| < \epsilon/2$ for all $n$. Then

$$|(x_{k_n}^* - x_{k_{n-1}}^*)(x_{k_n})| = |x_{k_n}^*(x_{k_n}) - x_{k_{n-1}}^*(x_{k_n})| > \epsilon/2.$$

This is a contradiction, since $(x_{k_n}^* - x_{k_{n-1}}^*)$ is $w^*$-null in $X^*$ and $(x_{k_n})$ is limited in $X$. ∎

A Banach space $X$ has the *Grothendieck property* if every $w^*$- convergent sequence in $X^*$ is weakly convergent [10, p. 179]. A space $X$ is *weakly sequentially complete* if every weakly Cauchy sequence in $X$ is weakly convergent.

Corollary 12. (i) *If $X$ has the $L$-limited property, then $X^*$ is weakly sequentially complete.*

(ii) *(*[39, Theorem 2.10]*) If $X$ has the $L$-limited property, then $X$ is a Grothendieck space.*

*Proof.* (i) Suppose that $X$ has the $L$-limited property. Let $(x_n^*)$ be a weakly Cauchy sequence in $X^*$. By Theorem 11, $\{x_n^* : n \in \mathbb{N}\}$ is an $L$-limited set, and thus relatively weakly compact. Hence $(x_n^*)$ is weakly convergent.

(ii) Let $(x_n^*)$ be a $w^*$- convergent sequence in $X^*$. By Theorem 11, $(x_n^*)$ is an $L$-limited set, thus relatively weakly compact. Hence $(x_n^*)$ is weakly convergent. ∎

Corollary 13. (i) *A Banach space $X$ with the Gelfand-Phillips property has the $wL$-limited property if and only if $X^*$ contains no copy of $\ell_1$.*

(ii) *A Banach space $X$ with the DPrcP has the $(wSR)$ property if and only if $X^*$ contains no copy of $\ell_1$.*

(iii) *If $X$ has the $wL$-limited property, then $c_0$ is not complemented in $X$.*

(iv) *(*[39, Corollary 2.9]*) A Banach space $X$ is reflexive if and only if it has the Gelfand-Phillips property and the $L$-limited property.*

(v) *(*[7, Corollary 17]*) A Banach space $X$ is reflexive if and only if it has the DPrcP and the $(SR)$ property.*

*Proof.* (i) Suppose that $X$ has the Gelfand-Phillips property and the $wL$-limited property. Then the identity operator $i : X \to X$ is limited completely continuous (since $X$ has the Gelfand-Phillips property) and $i^* : X^* \to X^*$ is weakly precompact by Theorem 2. Hence $X^*$ contains no copy of $\ell_1$, by Rosenthal's $\ell_1$ theorem. The converse follows by Rosenthal's $\ell_1$ theorem.

(ii) The proof is similar to that of (i).

(iii) Suppose that $X$ has the $wL$-limited property. Since $c_0$ is separable, it has the Gelfand-Phillips property [4, Proposition]. By (i), $c_0$ does not have the $wL$-limited property. Hence $c_0$ is not complemented in $X$ by Corollary 3.

(iv) If $X$ is reflexive, then it has the Gelfand-Phillips property [4, Proposition] and the $L$-limited property. Conversely, $X^*$ contains no copy of $\ell_1$ by (i) and $X^*$ is weakly sequentially complete by Corollary 12. Then $X^*$, thus $X$, is reflexive.

(v) Suppose $X$ is reflexive. Then $X$ has the $(SR)$ property and $X^*$ does not contain a copy of $\ell_1$. Hence $X^{**}$, thus $X$, has the $DPrcP$ ([13, Theorem 2]). Conversely, $X^*$ contains no copy of $\ell_1$ by (i) and $X^*$ is weakly sequentially complete by [26, Corollary 3.26]. Then $X$ is reflexive. ∎

Example. The converse of Corollary 12 (i) does not hold. Let $X$ be the first Bourgain-Delbaen space [5, p. 25]. Then $X$ has the Schur property and $X^*$ is weakly sequentially complete. Since $X$ has the Schur property, $X$ does not have the $L$-limited property (by Corollary 13 (iv)).

Corollary 14. (i) *If $X$ has property $(wV)$, then $X$ is weak sequentially Right.*

(ii) *If $X$ has the $L$-limited (resp. the $wL$-limited) property, then $X$ is sequentially Right (resp. weak sequentially Right).*

(iii) *If $X$ is sequentially Right (resp. weak sequentially Right), then it has the $RDPP$ (resp. property $(wL)$).*

(iv) *If $X$ is an infinite dimensional space with the $L$-limited property, then $X^*$ does not have the Schur property.*

*Proof.* (i) Suppose $X$ has property $(wV)$. Let $T : X \to Y$ be pseudo weakly compact. Then $T$ is unconditionally converging [35, Proposition 14]. Hence $T^*$ is weakly precompact [19, Theorem 1]. Apply Theorem 2.

(ii) Suppose $X$ has the the $L$-limited (resp. the $wL$-limited) property. Let $(x_n)$ be a weakly null limited sequence in $X$. Then $(x_n)$ is a weakly null DP sequence. Hence every Right subset of $X^*$ is $L$-limited, thus relatively weakly compact (resp. weakly precompact).

(iii) Suppose $X$ is sequentially Right (resp. weak sequentially Right). Every $L$-set in $X^*$ is a Right set, thus relatively weakly compact (resp. weakly precompact). Hence $X$ has the $RDPP$ [28] (resp. property $(wL)$).

(iv) Suppose that $X$ has the $L$-limited property. Then $X$ has the Grothendieck property, by Corollary 12 (ii). By the Jossefson-Nissezweig theorem, there is a $w^*$-null sequence $(x_n^*)$ in $X^*$ of norm one. Then $(x_n^*)$ is weakly null and not norm null, and $X^*$ does not have the Schur property. ∎

The fact that a space with property $(SR)$ has the $RDPP$ was obtained in [26, Corollary 3.3].

Example. The converse of Corollary 14 (i) is not true. Let $Y$ be the second Bourgain-Delbaen space [5, p. 25]. The space $Y$ is a non-reflexive $\mathcal{L}_\infty$-space with the $DPP$ that does not contain $c_0$ or $\ell_1$ and such that $Y^* \simeq \ell_1$. The space $Y$ is sequentially Right by Corollary 8. Since $Y$ does not contain $c_0$, the identity operator $i : Y \to Y$ is unconditionally converging ([8, p. 54]) and $i^* : Y^* \to Y^*$ is not weakly precompact (since $Y^* \simeq \ell_1$). Thus $Y$ does not have property $(wV)$ by [19, Theorem 1].

The converse of Corollary 14 (ii) (strong properties) is not true. The second Bourgain-Delbaen space $Y$ is sequentially Right and does not have the $L$-limited property (by Corollary 14 (iv)).

The converse of Corollary 14 (iii) (strong properties) is not true. Let $J$ be the original James space [24]. Since $J$ is separable and 1-codimensional in $J^{**}$, all duals of $J$ are separable and $\ell_1$ fails to embed in any of them. Moreover, none of these spaces can be weakly sequentially complete. Thus $J$ and its duals are weak sequentially Right, but none of these spaces are sequentially Right by [26, Corollary 3.26], since their duals are not weakly sequentially complete. Since $J$ does not contain $\ell_1$, every completely continuous operator on $J$ is compact (by a result of Odell [37, p. 377]), and thus weakly compact. Hence $J$ has the $RDPP$.

The following theorem shows that the space $E$ has property $(SR)$ if some subspace of it has property $(SR)$.

LEMMA 15. *([23, Theorem 2.7]) Let $E$ be a Banach space, $F$ a reflexive subspace of $E$ (resp. a subspace not containing copies of $\ell_1$), and $Q : E \to E/F$ the quotient map. Let $(x_n)$ be a bounded sequence in $E$ such that $(Q(x_n))$ is weakly convergent (resp. weakly Cauchy). Then $(x_n)$ has a weakly convergent (resp. weakly Cauchy) subsequence.*

Let $E$ be a Banach space and $F$ be a subspace of $E^*$. Let

$$^{\perp}F = \{x \in E : y^*(x) = 0 \text{ for all } y^* \in F\}.$$

THEOREM 16. (i) *Let $E$ be a Banach space and $F$ be a reflexive subspace of $E^*$. If $^{\perp}F$ has property $(SR)$ (resp. the $L$-limited property), then $E$ has the same property.*

(ii) *Let $E$ be a Banach space and $F$ be a subspace of $E^*$ not containing copies of $\ell_1$. If $^{\perp}F$ has property $(wSR)$ (resp. the $wL$-limited property), then $E$ has the same property.*

*Proof.* We only prove (i) for the $(SR)$ property. The other proofs are similar.

Suppose that $^{\perp}F$ has property $(SR)$. Let $Q : E^* \to E^*/F$ be the quotient map and $i : E^*/F \to (^{\perp}F)^*$ be the natural surjective isomorphism ([31, Theorem 1.10.16]). It is known that $iQ : E^* \to (^{\perp}F)^*$ is $w^* - w^*$ continuous, since $iQ(x^*)$ is the restriction of $x^*$ to $^{\perp}F$ ([31, Theorem 1.10.16]). Then there is an operator $S :^{\perp} F \to E$ such that $iQ = S^*$.

Let $T : E \to G$ be a pseudo weakly compact operator. Then $TS :^{\perp} F \to G$ is pseudo weakly compact. Since ${}^{\perp}F$ has property $(SR)$, $TS$ has a weakly compact adjoint, by Theorem 2. Since $S^*T^* = iQT^*$ is weakly compact and $i$ is a surjective isomorphism, $QT^*$ is weakly compact. Let $(x_n^*)$ be a sequence in $B_{G^*}$. By passing to a subsequence, we can assume that $(QT^*(x_n^*))$ is weakly convergent. Hence $(T^*(x_n^*))$ has a weakly convergent subsequence by Lemma 15. Thus $E$ has property $(SR)$. ∎

The $w^* - w$ continuous operators from $X^*$ to $Y$ will be denoted by $L_{w^*}(X^*, Y)$.

THEOREM 17. *Let $X$ be a Banach space and $A$ be a bounded subset of $X^*$. The following are equivalent:*

(i) *$A$ is an $L$-limited set.*

(ii) *Every operator $T \in L_{w^*}(X^*, c_0)$ that is $w^*$-norm sequentially continuous maps $A$ into a relatively compact set.*

*Proof.* (i)⇒(ii) Let $T \in L_{w^*}(X^*, c_0)$ be an operator so that $T$ is $w^*$-norm sequentially continuous. Note that $T^* \in L_{w^*}(\ell_1, X)$, $(x_n) = (T^*(e_n^*))$ is a weakly null sequence in $X$, and $T(x^*) = (x^*(x_i))_i$. If $(x_n^*)$ is a $w^*$-null sequence in $X^*$ and $y \in B_{\ell_1}$, then

$$|\langle x_n^*, T^*(y)\rangle| \leq \|T(x_n^*)\| \to 0.$$

Hence $T^*(B_{\ell_1})$, thus $(x_n)$, is limited. Since $A$ is an $L$-limited set, $\sup_{x^* \in A} |x^*(x_n)| \to 0$. Therefore $T(A)$ is relatively compact in $c_0$, by the characterization of relatively compact subsets of $c_0$.

(ii)⇒(i) Let $(x_n)$ be a weakly null limited sequence in $X$. Define $T : X^* \to c_0$ by $T(x^*) = (x^*(x_n))_n$. Note that $T^*(b) = \sum b_n x_n$, $b = (b_n) \in \ell_1$, $T^*(\ell_1) \subseteq X$, and $T \in L_{w^*}(X^*, c_0)$. If $(x_n^*)$ is a $w^*$-null sequence in $X^*$, then

$$\|T(x_n^*)\| = \sup_i |x_n^*(x_i)| \to 0,$$

since $(x_i)$ is limited. Hence $T$ is $w^*$-norm sequentially continuous operator, and $T(A)$ is relatively compact in $c_0$. By the characterization of relatively compact subsets of $c_0$, $\sup_{x^* \in A} |x^*(x_n)| \to 0$, and thus $A$ is an $L$-limited subset of $X^*$. ∎

An operator $T : X \to Y$ is called *limited* if $T(B_X)$ is a limited subset of $Y$ ([4]). The operator $T$ is limited if and only if $T^* : Y^* \to X^*$ is $w^*$-norm sequentially continuous.

Corollary 18. *Let $X$ be a Banach space and $A$ be a bounded subset of $X^*$. The following are equivalent:*

(i) *$A$ is an $L$-limited set.*

(ii) *For every limited operator $S \in L_{w^*}(\ell_1, X)$, $S^*(A)$ is relatively compact.*

*Proof.* (i)⇒(ii) Let $S \in L_{w^*}(\ell_1, X)$ be a limited operator. Then $S^* \in L_{w^*}(X^*, c_0)$ and $S^*$ is $w^*$-norm sequentially continuous. By Theorem 17, $S^*(A)$ is relatively compact.

(ii)⇒(i) Let $T \in L_{w^*}(X^*, c_0)$ be a $w^*$-norm sequentially continuous operator and let $S = T^*$. Then $S \in L_{w^*}(\ell_1, X)$, $S$ is limited, and $S^*(A)$ is relatively compact. By Theorem 17, $A$ is an $L$-limited set. ∎

Corollary 19. *Suppose that $A$ is a bounded subset of $X^*$ such that for every $\epsilon > 0$, there is an $L$-limited subset $A_\epsilon$ of $X^*$ such that $A \subseteq A_\epsilon + \epsilon B_{X^*}$. Then $A$ is an $L$-limited set.*

*Proof.* Suppose that $A$ satisfies the hypothesis. Let $\epsilon > 0$ and $A_\epsilon$ as in the hypothesis. Let $T \in L_{w^*}(X^*, c_0)$ be an operator such that $T$ is $w^*$-norm sequentially continuous and $\|T\| \le 1$. Then $T(A) \subseteq T(A_\epsilon) + \epsilon B_{c_0}$, and $T(A_\epsilon)$ is relatively compact by Theorem 17. Then $T(A)$ is relatively compact [8, p. 5], and thus $A$ is an $L$-limited set by Theorem 17. ∎

## 4. The $(wSR)$ and $wL$-limited properties in projective tensor products

In this section we consider the $(SR)$ and $L$-limited properties in the projective tensor product $X \otimes_\pi Y$. We begin by noting that there are examples of Banach spaces $X$ and $Y$ such that $X \otimes_\pi Y$ has the $(SR)$ and $L$-limited properties. If $1 < q' < p < \infty$, then $L(\ell_p, \ell_{q'}) = K(\ell_p, \ell_{q'})$ ([36], [10, p. 247]). If $q$ is the conjugate of $q'$, then $\ell_p \otimes_\pi \ell_q$ is reflexive (by [38, Theorem 4.19], [10, p. 248]), and thus has the $(SR)$ and $L$-limited properties. Then the spaces $X = \ell_p$ and $Y = \ell_q$ are as desired.

If $H \subseteq L(X, Y)$, $x \in X$ and $y^* \in Y^*$, let $H(x) = \{T(x) : T \in H\}$ and $H^*(y^*) = \{T^*(y^*) : T \in H\}$.

In the proofs of Theorems 23 and 25 we will need the following results.

THEOREM 20. *([20, Theorem 1]) Let $H$ be a subset of $K(X,Y)$ such that*

(i) *$H(x)$ is weakly precompact compact for all $x \in X$.*
(ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

*Then $H$ is weakly precompact.*

THEOREM 21. *([20, Theorem 3]) Suppose that $L(X,Y) = K(X,Y)$ and $H$ is a subset of $K(X,Y)$ such that:*

(i) *$H(x)$ is relatively weakly compact for all $x \in X$.*
(ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

*Then $H$ is relatively weakly compact.*

LEMMA 22. *Suppose $L(X,Y^*) = K(X,Y^*)$. If $(x_n)$ is a weakly null DP sequence in $X$ and $(y_n)$ is a DP sequence in $Y$, then $(x_n \otimes y_n)$ is a weakly null DP sequence in $X \otimes_\pi Y$.*

*Proof.* Suppose that $(x_n)$ is weakly null DP in $X$ and $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Let $T \in L(X,Y^*) \simeq (X \otimes_\pi Y)^*$ ([10, p. 230]). Since $T$ is completely continuous,

$$\langle T, x_n \otimes y_n \rangle \leq M\|T(x_n)\| \to 0.$$

Thus $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$.

Let $(A_n)$ be a weakly null sequence in $(X \otimes_\pi Y)^* \simeq L(X,Y^*)$ and let $x^{**} \in X^{**}$. Since the map $\gamma_{x^{**}} : L(X,Y^*) = K(X,Y^*) \to Y^*$, $\gamma_{x^{**}}(T) = T^{**}(x^{**})$ is linear and bounded, $(A_n^{**}(x^{**}))$ is weakly null in $Y^*$. Therefore

$$\langle x^{**}, A_n^*(y_n) \rangle = \langle A_n^{**}(x^{**}), y_n \rangle \to 0,$$

since $(y_n)$ is DP in $Y$. Hence $(A_n^*(y_n))$ is weakly null in $X^*$. Then

$$\langle A_n, x_n \otimes y_n \rangle = \langle A_n^*(y_n), x_n \rangle \to 0,$$

since $(x_n)$ is DP in $X$. Thus $(x_n \otimes y_n)$ is DP in $X \otimes_\pi Y$. ∎

THEOREM 23. *([7, Theorem 18]) Suppose that $L(X,Y^*) = K(X,Y^*)$. If $X$ and $Y$ are sequentially Right, then $X \otimes_\pi Y$ is sequentially right.*

*Proof.* Let $H$ be a Right subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$. We will use Theorem 20. We will verify the conditions (i) and (ii) of this theorem. Let $(T_n)$ be a sequence in $H$ and let $x \in X$. We prove that $\{T_n(x) : n \in \mathbb{N}\}$ is a Right subset of $Y^*$. Let $(y_n)$ be a Right null sequence in $Y$. Thus $(y_n)$ is weakly null and DP. For each $n$,

$$\langle T_n(x), y_n \rangle = \langle T_n, x \otimes y_n \rangle.$$

We show that $(x \otimes y_n)$ is Right null in $X \otimes_\pi Y$. If $T \in (X \otimes_\pi Y)^* \simeq L(X, Y^*)$ ([10, p. 230]), then

$$|\langle T, x \otimes y_n \rangle| = |\langle T(x), y_n \rangle| \to 0,$$

since $(y_n)$ is weakly null. Thus $(x \otimes y_n)$ is weakly null. Let $(A_n)$ be a weakly null sequence in $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$. Since the map $\phi_x : L(X, Y^*) \to Y^*$, $\phi_x(T) = T(x)$ is linear and bounded, $(A_n(x))$ is weakly null in $Y^*$. Therefore

$$|\langle A_n, x \otimes y_n \rangle| = |\langle A_n(x), y_n \rangle| \to 0,$$

since $(y_n)$ is DP in $Y$. Thus $(x \otimes y_n)$ is DP and $(x \otimes y_n)$ is Right null. Since $(T_n)$ is a Right set,

$$|\langle T_n, x \otimes y_n \rangle| = |\langle T_n(x), y_n \rangle| \to 0.$$

Thus $\{T_n(x) : n \in \mathbb{N}\}$ is a Right subset of $Y^*$, hence relatively weakly compact (by Theorem 2). We thus verified (i) of Theorem 20.

Let $y^{**} \in Y^{**}$. We show that $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a Right subset of $X^*$. Let $(x_n)$ be a Right null sequence in $X$. Thus $(x_n)$ is weakly null and DP. For each $n$,

$$\langle T_n^*(y^{**}), x_n \rangle = \langle y^{**}, T_n(x_n) \rangle.$$

It is enough to show that $(T_n(x_n))$ is weakly null in $Y^*$. Let $(y_n)$ be a Right null sequence in $Y$. By Lemma 22 and Proposition 1, $(x_n \otimes y_n)$ is Right null in $X \otimes_\pi Y$. Since $(T_n)$ is a Right set,

$$|\langle T_n, x_n \otimes y_n \rangle| = |\langle T_n(x_n), y_n \rangle| \to 0.$$

Therefore $(T_n(x_n))$ is a Right subset of $Y^*$, thus relatively weakly compact (by Theorem 2). By passing to a subsequence, we can assume that $(T_n(x_n))$ is weakly convergent. Let $y \in Y$. An argument similar to the one above shows that $(x_n \otimes y)$ is Right null in $X \otimes_\pi Y$. Then

$$|\langle T_n, x_n \otimes y \rangle| = |\langle T_n(x_n), y \rangle| \to 0,$$

since $(T_n)$ is a Right set. Hence $(T_n(x_n))$ is $w^*$-null. Since $(T_n(x_n))$ is also weakly convergent, $(T_n(x_n))$ is weakly null. Then $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a Right subset of $X^*$. Hence $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is relatively weakly compact (by Theorem 2). By Theorem 20, $H$ is weakly precompact. We can assume without loss of generality that $(T_n)$ is weakly Cauchy. Since $X$ and $Y$ are sequentially Right, $X^*$ and $Y^*$ are both weakly sequentially complete [26, Corollary 3.26], and thus $L(X, Y^*) = K(X, Y^*)$ is weakly sequentially complete, by [22, Theorem 3.10]. Then $(T_n)$ is weakly convergent. ∎

*Remark.* Theorem 23 can also be proved as follows. Let $H$ be a Right subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$ and let $(T_n)$ be a sequence in $H$. By the proof of Theorem 23, $\{T_n(x) : n \in \mathbb{N}\}$ and $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ are relatively weakly compact for all $x \in X$ and $y^{**} \in Y^{**}$. By Theorem 21, $H$ is relatively weakly compact.

LEMMA 24. *Suppose $L(X, Y^*) = K(X, Y^*)$. If $(x_n)$ is a weakly null limited sequence in $X$ and $(y_n)$ is a limited sequence in $Y$, then $(x_n \otimes y_n)$ is a weakly null limited sequence in $X \otimes_\pi Y$.*

*Proof.* By Lemma 22, $(x_n \otimes y_n)$ is a weakly null. Let $(A_n)$ be a $w^*$-null sequence in $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$. Then $(A_n^*(x))$ is a $w^*$-null sequence in $Y^*$. If $x \in X$, then $\langle A_n(x), y_n\rangle = \langle A_n^*(y_n), x\rangle \to 0$, since $(y_n)$ is limited in $Y$. Hence $(A_n^*(y_n))$ is $w^*$-null in $X^*$. Since $(x_n)$ is limited,

$$\langle A_n, x_n \otimes y_n\rangle = \langle A_n^*(y_n), x_n\rangle \to 0.$$

Thus $(x_n \otimes y_n)$ is limited in $X \otimes_\pi Y$. ∎

THEOREM 25. ([7, Theorem 25]) *Suppose that $L(X, Y^*) = K(X, Y^*)$. If $X$ and $Y$ have the $L$-limited property, then $X \otimes_\pi Y$ has the $L$-limited property.*

*Proof.* The proof is similar to the proof of Theorem 23 and uses Lemma 24. ∎

*Remark.* Theorem 25 can also be proved with a method similar to the one in the previous remark.

The fact that the $(SR)$ and $L$-limited properties are inherited by quotients, immediately implies the following result.

Corollary 26. (i) *Suppose that $L(X^*,Y^*) = K(X^*,Y^*)$, and $X^*$ and $Y$ are sequentially Right. Then the space $N_1(X,Y)$ of all nuclear operators from $X$ to $Y$ is sequentially Right.*

(ii) *Suppose that $L(X^*,Y^*) = K(X^*,Y^*)$, and $X^*$ and $Y$ have the L-limited property. Then the space $N_1(X,Y)$ of all nuclear operators from $X$ to $Y$ has the L-limited property.*

*Proof.* It is known that $N_1(X,Y)$ is a quotient of $X^* \otimes_\pi Y$ ([38, p. 41]). (i) Apply Theorem 23. (ii) Apply Theorem 25. ∎

*Observation* 1. If $T : Y \to X^*$ be an operator such that $T^*|_X$ is (weakly) compact, then $T$ is (weakly) compact. To see this, let $T : Y \to X^*$ be an operator such that $T^*|_X$ is (weakly) compact. Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net $(x_\alpha)$ in $B_X$ which is $w^*$- convergent to $x^{**}$. Then $(T^*(x_\alpha)) \overset{w^*}{\to} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively (weakly) compact set. Then $(T^*(x_\alpha)) \to T^*(x^{**})$ (resp. $(T^*(x_\alpha)) \overset{w}{\to} T^*(x^{**})$). Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively (weakly) compact. Therefore $T^*(B_{X^{**}})$ is relatively (weakly) compact, and thus $T$ is (weakly) compact.

It follows that if $L(X,Y^*) = K(X,Y^*)$, then $L(Y,X^*) = K(Y,X^*)$.

The following result improves Corollaries 19 and 21 of [7].

Corollary 27. *If $X$ is sequentially Right and $Y^*$ has the Schur property (or $Y$ is sequentially Right and $X^*$ has the Schur property), then $X \otimes_\pi Y$ is sequentially Right.*

*Proof.* Since $Y^*$ has the Schur property, every Right set in $Y^*$ is relatively compact (by Corollary 9). Let $T : X \to Y^*$ be an operator. Then $T$ is pseudo weakly compact (since $Y^*$ has the Schur property), hence compact (by Theorem 2). Apply Theorem 23. ∎

Theorem 28. *Suppose that $L(X,Y^*) = K(X,Y^*)$. The following statements are equivalent:*

1. (i) *$X$ and $Y$ are sequentially Right and at least one of them does not contain $\ell_1$.*
   (ii) *$X \otimes_\pi Y$ is sequentially Right.*
2. (i) *$X$ and $Y$ have the L-limited property and at least one of them does not contain $\ell_1$.*
   (ii) *$X \otimes_\pi Y$ has the L-limited property.*

*Proof.* We only prove 1. The other proof is similar.

(i)⇒(ii) by Theorem 23.

(ii)⇒(i) Suppose that $X \otimes_\pi Y$ is sequentially Right. Then $X$ and $Y$ are sequentially Right, since the sequentialy Right property is inherited by quotients [26, Proposition 3.8]. We will show that $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. Suppose that $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$. Hence $L_1 \hookrightarrow X^*$ ([32, Theorem 3.4], [8, p. 212]). Also, the Rademacher functions span $\ell_2$ inside of $L_1$, and thus $\ell_2 \hookrightarrow X^*$. Similarly $\ell_2 \hookrightarrow Y^*$. Then $c_0 \hookrightarrow K(X, Y^*)$ ([15, Theorem 3], [21, Corollary 21]). Thus $\ell_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$ ([3, Theorem 4], [8, Theorem 10, p. 48]), a contradiction with Corollary 5 (iii). ∎

*Observation* 2. If $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$, then $\ell_2 \hookrightarrow X^*$ and $\ell_2 \hookrightarrow Y^*$, and $c_0 \hookrightarrow K(X, Y^*)$ ([15, Theorem 3], [21, Corollary 21]). More generally, if $\ell_1 \hookrightarrow X$ and $\ell_p \hookrightarrow Y^*$, $p \geq 2$, then $c_0 \hookrightarrow K(X, Y^*)$ ([15], [21]). Thus $\ell_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$ ([3, Theorem 4], [8, Theorem 10, p. 48]). Hence $X \otimes_\pi Y$ is not weak sequentially Right (and does not have the $wL$-limited property), by Corollary 5 (iii).

COROLLARY 29. *Suppose that $L(X, Y^*) = K(X, Y^*)$.*

1. *If $X \otimes_\pi Y$ is weak sequentially Right, then $X$ and $Y$ are weak sequentially Right and at least one of them does not contain $\ell_1$.*
2. *If $X \otimes_\pi Y$ has the $wL$-limited property, then $X$ and $Y$ have the $wL$-limited property and at least one of them does not contain $\ell_1$.*

*Proof.* We only prove 1. The other proof is similar. If $X \otimes_\pi Y$ is weak sequentially Right, then $X$ and $Y$ are weak sequentially Right, since the weak sequentially Right property is inherited by quotients (by Corollary 3). Apply Observation 2. ∎

COROLLARY 30. ([7, Theorem 22]) *Suppose that $X$ and $Y$ have the $DPP$. The following statements are equivalent:*

(i) *$X$ and $Y$ are sequentially Right and at least one of them does not contain $\ell_1$.*

(ii) *$X \otimes_\pi Y$ is sequentially Right.*

*Proof.* (i)⇒(ii) Suppose that $X$ and $Y$ have the $DPP$. Without loss of generality suppose that $\ell_1 \not\hookrightarrow X$. Then $X^*$ has the Schur property [9]. Apply Corollary 27.

(ii)⇒(i) by Observation 2. ∎

By Corollary 30, the space $C(K_1) \otimes_\pi C(K_2)$ is sequentially Right if and only if either $K_1$ or $K_2$ is dispersed.

Next we present some results about the necessity of the condition $L(X, Y^*) = K(X, Y^*)$. It is implicit in [6] that a Banach space X has all bilinear forms weakly sequentially continuous if and only if every operator $S : X \to X^*$ transforms weakly null sequences into $L$-sets. Emmanuelle shows in [13] that a Banach space X does not contain $\ell_1$ if and only if every $L$-set in $X^*$ is relatively compact. Then, it is easy to see that if $X$ and $Y$ are not containing $\ell_1$, then $L(X, Y^*) = K(X, Y^*)$ if and only if every operator $T : X \to Y^*$ transforms weakly null sequences into $L$-sets (for more details see [6]).

A Banach space $X$ has the *approximation property* if for each norm compact subset $M$ of $X$ and $\epsilon > 0$, there is a finite rank operator $T : X \to X$ such that $\|Tx - x\| < \epsilon$ for all $x \in M$. If in addition $T$ can be found with $\|T\| \leq 1$, then $X$ is said to have the *metric approximation property*. $C(K)$ spaces, $c_0$, $\ell_p$, $1 \leq p < \infty$, $L_p(\mu)$ ($\mu$ any measure), $1 \leq p < \infty$, and their duals have the metric approximation property [10, p. 238].

A separable Banach space $X$ has an *unconditional compact expansion of the identity (u.c.e.i)* if there is a sequence $(A_n)$ of compact operators from $X$ to $X$ such that $\sum A_n(x)$ converges unconditionally to $x$ for all $x \in X$ [17]. In this case, $(A_n)$ is called an (u.c.e.i.) of $X$.

A sequence $(X_n)$ of closed subspaces of a Banach space $X$ is called an *unconditional Schauder decomposition* of $X$ if every $x \in X$ has a unique representation of the form $x = \sum x_n$, with $x_n \in X_n$, for every $n$, and the series converges unconditionally [30, p. 48].

The space $X$ has (Rademacher) *cotype* $q$ for some $2 \leq q \leq \infty$ if there is a constant $C$ such that for for every $n$ and every $x_1, x_2, \ldots, x_n$ in $X$,

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leq C \left(\int_0^1 \|r_i(t)x_i\|^q dt\right)^{1/q},$$

where $(r_n)$ are the Radamacher functions. A Hilbert space has cotype 2 [8, p. 118]. $\mathcal{L}_p$-spaces have cotype 2, if $1 \leq p \leq 2$ [8, p. 118].

THEOREM 31. *Assume one of the following holds:*

(i) *If $T : X \to Y^*$ is an operator which is not compact, then there is a sequence $(T_n)$ in $K(X, Y^*)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$.*
(ii) *$X$ is an $\mathcal{L}_\infty$-space and $Y^*$ is a subspace of an $\mathcal{L}_1$-space.*
(iii) *$X = C(K)$, $K$ a compact Hausdorff space, and $Y^*$ is a space with cotype 2.*
(iv) *Either $X$ or $Y^*$ has an (u.c.e.i.).*
(v) *$X$ has the DPP and $\ell_1 \hookrightarrow Y$.*
(vi) *$X$ and $Y$ have the DPP.*

*If $X \otimes_\pi Y$ is weak sequentially Right, then $L(X, Y^*) = K(X, Y^*)$.*

*Proof.* Suppose that $X \otimes_\pi Y$ is weak sequentially Right. Then $X$ and $Y$ are weak sequentially Right.

(i) Let $T : X \to Y^*$ be a noncompact operator. Let $(T_n)$ be a sequence as in the hypothesis. By the Uniform Boundedness Principle, $\{\sum_{n \in A} T_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $K(X, Y^*)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since $T$ is noncompact). Hence $c_0 \hookrightarrow K(X, Y^*)$ ([3, Theorem 5]), $\ell_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$ ([3, Theorem 4]), and we have a contradiction with Corollary 5 (iii).

Suppose (ii) or (iii) holds. It is known that any operator $T : X \to Y^*$ is 2-absolutely summing ([8, p. 189]), hence it factorizes through a Hilbert space. If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$ (by [16, Remark 3]), a contradiction.

(iv) If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$ (by [27, Theorem 6]), a contradiction.

(v) Suppose that $X$ has the $DPP$ and $\ell_1 \hookrightarrow Y$. By Observation 1, $\ell_1 \not\hookrightarrow X$. Then $X^*$ has the Schur property ([9, Theorem 3]). Let $T : Y \to X^*$ be an operator. Then $T$ is pseudo weakly compact (since $X^*$ has the Schur property), and thus weakly precompact (by Corollary 5 (i)). Then $L(Y, X^*) = K(Y, X^*)$. Hence $L(X, Y^*) = K(X, Y^*)$, by Observation 1.

(vi) Suppose that $X$ and $Y$ have the $DPP$. Then $L(X, Y^*) = K(X, Y^*)$, either by (v) if $\ell_1 \hookrightarrow Y$, or since $Y^*$ has the Schur property ([9, Theorem 3]) if $\ell_1 \not\hookrightarrow Y$ (by an argument similar to the one in (v)). ∎

Assumption (i) of the previous theorem is satisfied, for instance, if $X^*$ (or $Y^*$) has an (u.c.e.i.).

Examples. By Theorem 31, the space $\ell_p \otimes \ell_q$, where $1 < p \le q' < \infty$ and $q$ and $q'$ are conjugate, is not weak sequentially Right, since the natural inclusion map $i : \ell_p \to \ell_{q'}$ is not compact.

The space $C(K) \otimes_\pi \ell_p$, with $K$ not dispersed and $1 < p \le 2$, is not weak sequentially Right (by Observation 2, since $\ell_1 \hookrightarrow C(K)$ and $\ell_2 \hookrightarrow \ell_p^*$).

For $1 < p_1, p_2 < \infty$, $L_{p_1}[0,1] \otimes_\pi L_{p_2}[0,1]$ is not weak sequentially Right by Corollary 5 (iii), since $\ell_1 \overset{c}{\hookrightarrow} L_{p_1}[0,1] \otimes_\pi L_{p_2}[0,1]$ ([38, Corollary 2.26]).

Theorem 32. (i) *Suppose $Y^*$ is complemented in a Banach space $Z$ which has an unconditional Schauder decomposition $(Z_n)$, and $L(X, Z_n) = K(X, Z_n)$ for all $n$. If $X \otimes_\pi Y$ is weak sequentially Right, then $L(X, Y^*) = K(X, Y^*)$.*

(ii) *Suppose either $X^*$ or $Y^*$ has the metric approximation property. If $X \otimes_\pi Y$ is sequentially Right, then $W(X, Y^*) = K(X, Y^*)$.*

*Proof.* (i) Let $T : X \to Y^*$ be a noncompact operator, $P_n : Z \to Z_n$, $P_n(\sum z_i) = z_n$, and let $P$ be the projection of $Z$ onto $Y^*$. Define $T_n : X \to Y^*$ by $T_n(x) = PP_nT(x)$, $x \in X$, $n \in \mathbb{N}$. Note that $P_nT$ is compact since $L(X, Z_n) = K(X, Z_n)$. Then $T_n$ is compact for each $n$. For each $z \in Z$, $\sum P_n(z)$ converges unconditionally to $z$; thus $\sum T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$. Then $\sum T_n$ is wuc and not unconditionally converging. Hence $c_0 \hookrightarrow K(X, Y^*)$ ([3, Theorem 5]), and we obtain a contradiction.

(ii) Since $X \otimes_\pi Y$ is sequentially Right, $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$ is weakly sequentially complete ([26, Corollary 3.26]). Under assumption (ii), [29, Corollary 2.4] implies $W(X, Y^*) = K(X, Y^*)$.

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