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A Symmetrical Property of the Spectral Trace in Banach Algebras

Abdelaziz Maouche

Department of Mathematics and Statistics, Faculty of Science
Sultan Qaboos University, Oman
maouche@squ.edu.om

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Abstract: Our aim in this paper is to extend a symmetrical property of the trace by M. Kennedy and H. Radjavi for bounded operators on a Banach space to the more general situation of Banach algebras. The main ingredients are Vesentini's result on subharmonicity of the spectral radius and the new spectral rank and trace defined on the socle of a Banach algebra by B. Aupetit and H. du T. Mouton.

Key words: Banach algebra, rank, spectral additivity, trace, subharmonic function.

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1. Preliminaries

In [5], the authors investigate the properties of bounded operators which satisfy a certain spectral additivity condition and use their results to study Lie and Jordan algebras of compact operators. As a first result, they obtain a symmetric trace condition on bounded operators (see [5, Lemma 3.12]). B. Aupetit and H. du T. Mouton proved that the spectral rank and trace as defined in [3] coincide with the classical notion of trace and rank in the case where $\mathcal{U} = \mathcal{L}(X)$, the Banach algebra of bounded linear operators on a Banach space X.

It is our aim to extend some results obtained in [5] to the general situation of Banach algebras, by replacing the classical trace by the spectral one defined in [3].

Let $\mathcal{U}$ be a semi-simple complex unital Banach algebra and $\Omega(\mathcal{U})$ its set of invertible elements. For $x \in \mathcal{U}$ we denote $\operatorname{Sp}(x) = \{\lambda : \lambda 1 - x \notin \Omega(\mathcal{U})\}$ and $\rho_{\mathcal{U}}(x) = \sup\{|\lambda| : \lambda \in \operatorname{Sp}(x)\}$ the spectrum and spectral radius of x. We denote by $\widehat{\operatorname{Sp}(x)}$ the full spectrum of x, i.e., the polynomially convex hull of $\operatorname{Sp}(x)$, that is the set obtained by filling the holes in $\operatorname{Sp}(x)$.

Let $\alpha \in \mathbb{C}$ and γ a small curve isolating α from the rest of the spectrum of a. By definition, the Riesz projection associated to a and α is given by

$$p(\alpha, a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda - a)^{-1} \mathrm{d}\lambda \,.$$

Using the identity

$$(\lambda - a)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda} a (\lambda - a)^{-1}$$

we obtain by integration

$$p(\alpha, a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\lambda} (\lambda - a)^{-1} \mathrm{d}\lambda = \frac{a}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\lambda} (\lambda - a)^{-1} \mathrm{d}\lambda \,. \quad (*)$$

Obviously $p(\alpha, a) = 0$ if $\alpha \notin \mathrm{Sp}(a)$. The Holomorphic Functional Calculus yields that the $p(\alpha, a)$ corresponding to different values of α are orthogonal projections (have zero product) and their sum is 1.

The next result is well known, we include it to illustrate the previous definition of Riesz projections and a kind of spectral additivity in a particular situation.

Proposition 1. *Let a, b be two elements of a Banach algebra $\mathcal{U}$ such that $ab = ba = 0$. Then*

$$\mathrm{Sp}(a + b) \setminus \{0\} = (\mathrm{Sp}(a)) \cup (\mathrm{Sp}(b)) \setminus \{0\} \,.$$

Moreover, if $\lambda_0 \neq 0$ is isolated in $\mathrm{Sp}(a+b)$ then the Riesz projection associated with $a + b$, a and b respectively satisfy the identity

$$p(\lambda_0, a + b) = p(\lambda_0, a) + p(\lambda_0, b) \,.$$

Proof. For $\lambda \neq 0$ it is easy to see from the identity,

$$\lambda - (a + b) = \frac{1}{\lambda} (\lambda - a)(\lambda - b) = \frac{1}{\lambda} (\lambda - b)(\lambda - a)$$

that $\lambda - (a + b)$ is invertible if and only if both $\lambda - a$ and $\lambda - b$ are invertible. Let γ be a circle centered at λ_0 which separates λ_0 from 0 and the rest of the spectrum of $a + b$. If $\lambda \in \gamma$, by the previous identity we have $\lambda \neq 0$, $\lambda - a$ and

$\lambda - b$ invertible. Since $b = (\lambda - a)\dfrac{b}{\lambda}$ we obtain $(\lambda - a)^{-1}b = \dfrac{b}{\lambda}$ on γ. Moreover, we have

$$\begin{aligned}(\lambda - (a+b))^{-1} &= \lambda(\lambda - b)^{-1}(\lambda - a)^{-1}\\ &= (\lambda - a)^{-1} + [\lambda(\lambda - b)^{-1} - 1](\lambda - a)^{-1}\\ &= (\lambda - a)^{-1} + b(\lambda - b)^{-1}(\lambda - a)^{-1}\\ &= (\lambda - a)^{-1} + (\lambda - b)^{-1}(\lambda - a)^{-1}b\\ &= (\lambda - a)^{-1} + \frac{b}{\lambda}(\lambda - b)^{-1}.\end{aligned}$$

Now, integrating this quantity on γ and multiplying by $\dfrac{1}{2\pi \mathrm{i}}$, we get

$$p(\lambda_0, a+b) = p(\lambda_0, a) + \frac{b}{2\pi\mathrm{i}}\int_\gamma \frac{1}{\lambda}(\lambda - b)^{-1}\mathrm{d}\lambda = p(\lambda_0, a) + p(\lambda_0, b)$$

by formula (*) applied to b. ∎

2. Trace and rank in Banach algebras

For each nonnegative integer m, let

$$\mathcal{F}_m = \{a \in \mathcal{U} \,:\, \#(\mathrm{Sp}(xa) \setminus \{0\}) \le m \text{ for all } x \in \mathcal{U}\}\,,$$

where the symbol $\#K$ denotes the number of distinct elements in a set $K \subset \mathbb{C}$. Following [3], we define the rank of an element a of $\mathcal{U}$ as the smallest integer m such that $a \in \mathcal{F}_m$, if it exists; otherwise the rank is infinite. In other words,

$$\mathrm{rank}(a) = \sup_{x \in \mathcal{U}} \#(\mathrm{Sp}(xa) \setminus \{0\}) \le \infty\,.$$

Of course,

$$\mathrm{rank}(a) = \sup_{x \in \mathcal{U}} \#(\mathrm{Sp}(ax) \setminus \{0\})\,.$$

A few elementary properties of the rank taken from [3], where more details and proofs are given, are listed below:

(a) $\#(\mathrm{Sp}(a) \setminus \{0\}) \le \mathrm{rank}(a)$ for a in $\mathcal{U}$.

(b) $\mathrm{rank}(xa) \le \mathrm{rank}(a)$ and $\mathrm{rank}(ax) \le \mathrm{rank}(a)$ for $a, x \in \mathcal{U}$; moreover, $\mathrm{rank}(ua) = \mathrm{rank}(au) = \mathrm{rank}(a)$ if u is invertible.

(c) If $a \in \mathcal{U}$ is a finite-rank element, then

$$\mathcal{E}(a) = \{x \in \mathcal{U} \,:\, \#(\mathrm{Sp}(xa) \setminus \{0\}) = \mathrm{rank}(a)\}$$

is a dense open subset of $\mathcal{U}$ (see [3, Theorem 2.2]).

It is known that the socle, denoted $\mathrm{Soc}(\mathcal{U})$, of a semisimple Banach algebra $\mathcal{U}$ coincides with the collection $\cup_{m=0}^{\infty} \mathcal{F}_m$ of finite-rank elements.

Following [3], if $a \in \mathrm{Soc}(\mathcal{U})$ we define the trace of a by

$$\mathrm{Tr(a)} = \sum_{\lambda \in \mathrm{Sp}(a)} \lambda \cdot m(\lambda, a)$$

where $m(\lambda, a)$ is the multiplicity of the spectral value λ.

More details on the trace and rank in Banach algebras are contained in [3], from which we recall the following results on the trace that will be used in the proof of our main result. For instance, it is shown in [3], formula (3) page 130, that the trace and rank satisfy

$$|\mathrm{Tr}(a)| \leq \rho(a) \cdot \mathrm{rank}(a)$$

where $\rho(a)$ is the usual spectral radius of a.

Proposition 2. (i) *Let* $a \in \mathrm{Soc}(\mathcal{U}),\ b,\ x,\ y \in (\mathcal{U})$. *Then we have*

$$\mathrm{Tr}(aL_xL_yb + bL_xL_ya) = \mathrm{Tr}(aL_yL_xb + bL_yL_xa)\,.$$

In particular $\mathrm{Tr}(x(ya)) = \mathrm{Tr}(y(xa))$.

(ii) *Let* $a \in \mathrm{Soc}((\mathcal{U}))$ *be such that* $\mathrm{Tr}(au) = 0$, *for every* $u \in \mathrm{Soc}(\mathcal{U})$. *Then* $a = 0$.

(iii) *If* $a \in \mathrm{Soc}(\mathcal{U})$, *then* $\phi(x) = \mathrm{Tr}(ax)$ *is a bounded linear functional on* $(\mathcal{U})$.

Proof. For properties (i) and (ii), see [2, Corollary 1.2 and Corollary 1.3, p. 181]. For property (iii) see of [3, Theorem 3.3]. ∎

Theorem 1. ([3, Theorem 2.6]) *Let* $a \in \mathcal{U}$ *have finite rank and* $\lambda_1, \ldots,$ λ_n *be non-zero distinct elements of its spectrum with multiplicity* $m(\lambda_i, a)$. *If* p *denotes the Riesz projection associated with* a *and* $\lambda_1, \ldots, \lambda_n$ *that is,* $p = p(\lambda_1, a) + \cdots + p(\lambda_n, a)$, *then* $\mathrm{rank}(p) = m(\lambda_1, a) + \cdots + m(\lambda_n, a)$.

Another important result that we shall use in the proof of our main result is the following theorem.

THEOREM 2. ([3, THEOREM 3.1]) *Let f be an analytic function from a domain D of $\mathbb{C}$ into the socle of a semisimple Banach algebra $\mathcal{U}$. Then* $\mathrm{Tr}(f(\lambda))$ *is holomorphic on D.*

In what follows, an important tool will be the theory of subharmonic functions, based essentially on the celebrated result of E. Vesentini: if f is an analytic function from a domain D of the complex plane into a Banach algebra, then the functions $\lambda \mapsto \rho(f(\lambda))$ and $\lambda \mapsto \log \rho(f(\lambda))$ are subharmonic (see [1, Theorem 3.4.7]). We will require the following two fundamental results from the theory of subharmonic functions from ([1, Theorem A.1.3 and Theorem A.1.29]).

THEOREM 3. (MAXIMUM PRINCIPLE FOR SUBHARMONIC FUNCTIONS) *Let f be a subharmonic function on a domain D of $\mathbb{C}$. If there exists $\lambda_0 \in D$ such that $f(\lambda) \leq f(\lambda_0)$ for all $\lambda \in D$, then $f(\lambda) = f(\lambda_0)$ for all λ in D.*

We state here a special case of H. Cartan's Theorem (see [5] and the references given there).

THEOREM 4. (H. CARTAN'S THEOREM) *Let f be a subharmonic function on a domain D of $\mathbb{C}$. If $f(\lambda) = -\infty$ on an open disc in D, then $f(\lambda) = -\infty$ for all λ in D.*

To apply later H. Cartan's theorem, we shall need the maximum principle theorem for the full spectrum due to E. Vesentini, where ∂K means the boundary of the compact set K.

THEOREM 5. (SPECTRUM MAXIMUM PRINCIPLE) *Let f be a an analytic function on a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that there exists λ_0 of D such that $\mathrm{Sp} f(\lambda) \subset \mathrm{Sp} f(\lambda_0)$, for all $\lambda \in D$. Then $\partial \mathrm{Sp} f(\lambda_0) \subset \partial \mathrm{Sp} f(\lambda)$ and $\widehat{\mathrm{Sp} f(\lambda_0)} = \widehat{\mathrm{Sp} f(\lambda)}$, for all $\lambda \in D$. In particular, if $\mathrm{Sp} f(\lambda_0)$ has no interior points or if $\mathrm{Sp} f(\lambda)$ does not separate the plane for all $\lambda \in D$, then $\mathrm{Sp} f(\lambda)$ is constant on D.*

3. ELEMENTS WITH STABLE SPECTRUM

L. Harris and R. Kadison define spectrally additive elements in a C*-algebra as follows.

DEFINITION 1. An element a of a C*-algebra $\mathcal{U}$ is said to be spectrally additive (in $\mathcal{U}$) when $\mathrm{Sp}(a + b) \subseteq \mathrm{Sp}(a) + \mathrm{Sp}(b)$ for each b in $\mathcal{U}$.

This same definition may be made for elements of a (unital) Banach algebra $\mathcal{U}$ (over $\mathbb{C}$). The concept of spectral additivity was studied in the context of Banach algebras with the aid of (purely algebraic) commutator results for 'Schurian algebras' by L. Harris and R. Kadison. It is proved there that a is spectrally additive in $\mathcal{U}$ if and only if $au - ua$ lies in the radical of $\mathcal{U}$ for each u in $\mathcal{U}$. In particular, if $\mathcal{U}$ is semi-simple, as is the case when $\mathcal{U}$ is a C*-algebra, then a is spectrally additive if and only if it lies in the center of $\mathcal{U}$.

Following [5], we introduce the notion of stable spectrum in a Banach algebra.

Definition 2. Let u be an element of a Banach algebra $\mathcal{U}$. We say that an element a has u-stable spectrum if $\rho(a + \lambda u) \leq \rho(a)$ for every complex number λ. A family of elements of $\mathcal{U}$ is said to have u-stable spectrum if each of its elements has u-stable spectrum.

Remark 1. (a) For elements a and u of a complex Banach algebra $\mathcal{U}$, the function $\lambda \mapsto a + \lambda u$ is analytic, so by Vesentini's result, the functions $\lambda \mapsto \rho(a + \lambda u)$ and $\lambda \mapsto \log \rho(a + \lambda u)$ are subharmonic.

(b) If a has u-stable spectrum, the Maximum Principle for subharmonic functions immediately implies that $\rho(a + \lambda u) = \rho(a)$ for all complex numbers.

(c) If a and u have sublinear spectrum, that is, $\mathrm{Sp}(a + \lambda u) \subseteq \mathrm{Sp}(a) + \lambda \mathrm{Sp}(u)$ for every complex number λ and u is quasi-nilpotent ($\rho(u) = 0$), then a has u-stable spectrum.

Lemma 1. *Let a and u be elements of a semi-simple complex Banach algebra $\mathcal{U}$. If a has u-stable spectrum, then u is quasi-nilpotent.*

Proof. By the above remark, $\rho(a + \lambda u) = \rho(a)$ for all λ in $\mathbb{C}$, so

$$\rho\big(\lambda^{-1} a + u\big) = |\lambda|^{-1} \rho(a)$$

for all non zero λ in $\mathbb{C}$. Thus, by subharmonicity of $\rho(\lambda^{-1} a + u)$, we get

$$\rho(u) = \limsup_{\lambda \to \infty} \rho\big(\lambda^{-1} a + u\big) = 0\,.$$

∎

Theorem 6. *Let a and u be elements of a semi-simple complex Banach algebra $\mathcal{U}$. Then a has u-stable spectrum if and only if $(\mu - a)^{-1} u$ is quasi-nilpotent for all $\mu \notin \widehat{\mathrm{Sp}(a)}$.*

Proof. By Remark 1, $\rho(a + \lambda u) = \rho(a)$ for all λ in $\mathbb{C}$, so for μ in $\mathbb{C}$ with $|\mu| > \rho(a)$, both $\mu - a$ and $\mu - a - \lambda u$ are invertible. Therefore,

$$\lambda^{-1}(\mu - a)^{-1}(\mu - a - \lambda u) = \lambda^{-1} - (\mu - a)^{-1}u$$

is invertible for all non-zero λ in $\mathbb{C}$. This means that the values of the analytic function $\mu \mapsto (\mu - a)^{-1}u$ for $\mu \notin \mathrm{Sp}(a)$, are quasi-nilpotent whenever $|\mu| > \rho(a)$. Consider the subharmonic function $\mu \mapsto \log(\rho(\mu - a)^{-1}u)$ defined for $\mu \notin \mathrm{Sp}(a)$. Since $\log(\rho(\mu - a)^{-1}u) = -\infty$ whenever $|\mu| > \rho(a)$, by H. Cartan's Theorem, $\log(\rho(\mu - a)^{-1}u) = -\infty$ for all $\mu \notin [\widehat{\mathrm{Sp}(a)}]$. In other words, $(\mu - a)^{-1}u$ is quasi-nilpotent for all $\mu \notin [\widehat{\mathrm{Sp}(a)}]$. ∎

Corollary 1. *Let a and u be elements of a semi-simple complex Banach algebra $\mathcal{U}$, with* $\mathrm{Sp}(a)$ *without holes. Then a has u-stable spectrum if and only if* $\mathrm{Sp}(a + \lambda u) \subseteq \mathrm{Sp}(a) + \lambda \mathrm{Sp}(u)$ *for every λ in $\mathbb{C}$.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose $\mu \in \mathrm{Sp}(a + \lambda u)$, but that $\mu \notin \mathrm{Sp}(a)$. Then obviously λ is non-zero, and

$$\lambda^{-1}(\mu - a)^{-1}(\mu - a - \lambda u) = \lambda^{-1} - (\mu - a)^{-1}u$$

is not invertible. By Theorem 6, we get $(\mu - a)^{-1}u$ is quasi-nilpotent for all $\mu \notin \widehat{\mathrm{Sp}(a)}$; so it also holds for $\mu \notin \mathrm{Sp}(a)$ which yields a contradiction. ∎

The next result follows from [1, Theorem 3.4.14], as we can see in the following proof.

Lemma 2. *Let a and u be elements of a semi-simple complex Banach algebra $\mathcal{U}$. If a has u-stable spectrum and* $\mathrm{Sp}(a)$ *has no interior points, then* $\mathrm{Sp}(a + \lambda u) = \mathrm{Sp}(a)$ *for all λ in $\mathbb{C}$.*

Proof. ($\Leftarrow$) Clear by Corollary 1.

($\Rightarrow$) By Remark 1, $\rho(a + \lambda u) = \rho(a)$ for all λ in $\mathbb{C}$, so the analytic multifunction $\lambda \to \mathrm{Sp}(a + \lambda u)$ is bounded, and consequently by Liouville's theorem for analytic multivalued functions, (see [1, Theorem 3.4.14]), we have $\widehat{\mathrm{Sp}(a + \lambda u)} = \widehat{Sp(a)}$, where $\widehat{\mathrm{Sp}(x)}$ denotes the full spectrum of x. Since $\mathrm{Sp}(a)$ has no interior points, the result follows from Theorem 5 (see the proof in [1, Theorem 3.4.13]). ∎

Theorem 7. *Let a and u be elements of a semi-simple complex Banach algebra $\mathcal{U}$. If a has u-stable spectrum and* $\mathrm{Sp}(a)$ *has no interior points, then* $\mathrm{Sp}\big((1-\nu u)^{-1}a\big) = \mathrm{Sp}(a)$ *for all ν in $\mathbb{C}$.*

Proof. First suppose λ is non-zero, and that $\lambda \notin \mathrm{Sp}(a)$. By Lemma 2, we have $\mathrm{Sp}\big(\lambda^{-1}a + \nu u\big) = \mathrm{Sp}\big(\lambda^{-1}a\big)$, and by Lemma 1, u is quasi-nilpotent. These two facts imply that $1-\nu u$ and $1-\lambda^{-1}a-\nu u$ are both invertible, and hence that

$$\lambda(1-\nu u)^{-1}\big(1-\lambda^{-1}a-\nu u\big) = \lambda - (1-\nu u)^{-1}a$$

is invertible for all ν in $\mathbb{C}$. Therefore, $\lambda \notin \mathrm{Sp}\big((1-\nu u)^{-1}a\big)$ for all ν in $\mathbb{C}$. Now suppose $0 \notin \mathrm{Sp}(a)$. Then a is invertible, implying $(1-\nu u)^{-1}a$ is invertible, and hence by quasi-nilpotence of u, that $0 \notin \mathrm{Sp}\big((1-\nu u)^{-1}a\big)$ for all ν in $\mathbb{C}$. We have shown that $\mathrm{Sp}\big((1-\nu u)^{-1}a\big) \subseteq \mathrm{Sp}(a)$ for all ν in $\mathbb{C}$. Since $\mathrm{Sp}(a)$ has no interior points, the result follows from Theorem 5. ∎

We arrive at our main result which gives a symmetric spectral trace condition on stable elements, extending [5, Lemma 3.12] from the semi-simple algebra of bounded operators $\mathcal{B}(\mathcal{X})$ on a Banach space $\mathcal{X}$ to the more general situation of a semi-simple complex Banach algebra $\mathcal{U}$.

Theorem 8. *Let a and b two elements of a semi-simple complex unital Banach algebra $\mathcal{U}$. If a is b-stable and one of a or b is of finite-rank, then* $\mathrm{Tr}(a^n b) = \mathrm{Tr}(ab^n) = 0$ *for all $n \geq 1$.*

Proof. First suppose that a is of finite rank. Since b is quasi-nilpotent by Lemma 1, the function $\nu \mapsto ((1-\nu b)^{-1}a)$ is entire. Moreover, $\mathrm{Sp}\big((1-\nu b)^{-1}a\big) = \mathrm{Sp}(a)$ for all ν in $\mathbb{C}$ by Theorem 7. Then, taking n-th powers, the function

$$\nu \longmapsto \big((1-\nu b)^{-1}a\big)^n$$

is also entire, and

$$\mathrm{Sp}\big((1-\nu b)^{-1}a\big)^n = \mathrm{Sp}(a^n)$$

for all ν in $\mathbb{C}$. Clearly,

$$\mathrm{rank}\Big(\big((1-\nu b)^{-1}a\big)^n\Big) \leq \mathrm{rank}(a)\,,$$

so

$$\mathrm{Tr}\Big(\big((1-\nu b)^{-1}a\big)^n\Big) = \mathrm{Tr}(a^n)$$

for all $\nu \in \mathbb{C}$ by Theorem 2, property (3) of Proposition 2 and Liouville's theorem for entire functions.

For $|\nu| < ||b||^{-1}$, we may expand $\big((1-\nu b)^{-1}a\big)^{-1}$ as a power series in ν,

$$(1-\nu b)^{-1}a = \sum_{k\geq 0} b^k a \nu^k.$$

Hence

$$\big((1-\nu b)^{-1}a\big)^n = \left(\sum_{k\geq 0} b^k a \nu^k\right)^n.$$

The coefficient of ν^k in the above expansion is $b^k a$, and for $n \geq 1$, the coefficient of ν is $ba^n + aba^{n-1} + \cdots + a^{n-1}ba$. But we may expand the constant function $\mathrm{Tr}\big(\big((1-\nu b)^{-1}a\big)^n\big)$ as a power series in ν, and the linearity of the trace implies that for $n = 1$, the coefficient of ν^k in this expansion is $\mathrm{Tr}(b^k a)$, and for $n \geq 1$, that the coefficient of ν is

$$\mathrm{Tr}\big(ba^n + aba^{n-1} + \cdots + a^{n-1}ba\big) = n\mathrm{Tr}(a^n b)\,.$$

Comparing the coefficients on the left and right hand side of the equation

$$\mathrm{Tr}\big(\big((1-\nu b)^{-1}a\big)^n\big) = \mathrm{Tr}(a^n)$$

therefore gives $\mathrm{Tr}(a^n b) = 0$ for all $n \geq 1$ and $\mathrm{Tr}(ab^k) = 0$ for all $k \geq 1$.

Now suppose that b is of finite rank. The function $(1-\nu a)^{-1}b$ is analytic, with quasi-nilpotent values by Theorem 6, for $\frac{1}{\nu} \notin \widehat{\mathrm{Sp}(a)}$. Taking n-th powers, the function $\nu \mapsto \big((1-\nu a)^{-1}b\big)^n$ is also analytic for all $\frac{1}{\nu} \notin \mathrm{Sp}(a)$. As above, for $|\nu| < ||a||^{-1}$, we may expand $((1-\nu a)^{-1}b)^n$ as a power series in ν,

$$\big((1-\nu a)^{-1}b\big)^n = \left(\sum_{k\geq 0} a^k b \nu^k\right)^n.$$

For $n = 1$, the coefficient of ν^k in the above expansion is $a^k b$, and for $n \geq 1$, the coefficient of ν is $ab^n + bab^{n-1} + \cdots + b^{n-1}a$. Proceeding as before, we may expand the constant function $\mathrm{Tr}\big(\big((1-\nu a)^{-1}b\big)^n\big)$ as a power series in ν, and linearity of the trace implies that for $n = 1$, the coefficient of ν^k in this expansion is $\mathrm{Tr}(a^k b)$, and for $n \geq 1$, that coefficient of ν is

$$\mathrm{Tr}\big(ab^n + bab^{n-1} + \cdots + b^{n-1}ab\big) = n\mathrm{Tr}(ab^n)\,.$$

Comparing the coefficients of the left and right hand side of the equation

$$\mathrm{Tr}\big(\big((1-\nu a)^{-1}b\big)^n\big) = 0$$

hence gives $\mathrm{Tr}(a^k b) = 0$ for all $k \geq 1$, and $\mathrm{Tr}(ab^n) = 0$ for all $n \geq 1$. ∎

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